



Families of nested graphs with compatible symmetric-group actions

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Abstract

For fixed positive integers n and k , the Kneser graph $KG_{n,k}$ has vertices labeled by k -element subsets of $\{1, 2, \dots, n\}$ and edges between disjoint sets. Keeping k fixed and allowing n to grow, one obtains a family of nested graphs, each of which is acted on by a symmetric group in a way which is compatible with these inclusions and the inclusions of each symmetric group into the next. In this paper, we provide a framework for studying families of this kind using the FI-module theory of Church et al. (Duke Math J 164(9):1833–1910, 2015), and show that this theory has a variety of asymptotic consequences for such families of graphs. These consequences span a range of topics including enumeration, concerning counting occurrences of subgraphs, topology, concerning Hom-complexes and configuration spaces of the graphs, and algebra, concerning the changing behaviors in the graph spectra.

Keywords FI-modules · Representation stability · Graph theory · Kneser graphs

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1 Introduction

1.1 Motivation

Let FI denote the category whose objects are the finite sets $[n] := \{1, \dots, n\}$, and whose morphisms are injections. In their seminal work, Church et al. [10] introduced the notion of an *FI-module* to formalize the connection between a large number of seemingly unrelated phenomena in topology and representation theory. Formally, an *FI-module* is a functor from FI to the category of real vector spaces. Noting that the endomorphisms in FI are permutations, one may imagine an *FI-module* as a series of representations of the symmetric groups \mathfrak{S}_n , with n increasing, which are compatible in some sense.

Recently there has been a push in the literature to use the same philosophy underlying *FI-modules* to study combinatorial objects. For instance, in his recent work [21] Gadish studies what he calls *FI-posets* and *FI-arrangements*. In this work, we will be mostly focused on *FI-graphs*, functors from FI to the category of graphs. For us, a *graph* is a finite 1-dimensional simplicial complex. Given a graph G , we write $V(G)$ for the set of vertices of G and $E(G)$ for the set of edges. Note that $V(G)$ and $E(G)$ are, by how we have defined graph, both necessarily finite. Just as with the work of Gadish, we will discover that a relatively simple combinatorial condition on *FI-graphs* will allow us to conclude a plethora of interesting structural properties of the graphs which comprise it.

Throughout this paper we will often denote *FI-graphs* by G_\bullet , and use G_n as a shorthand for its evaluation on $[n]$. The *transition maps* of G_\bullet are the graph morphisms induced by the morphisms of FI which are not permutations. We say that an *FI-graph* G_\bullet is *vertex-stable of stable degree $\leq d$* if for all $n \geq d$, every vertex of G_n appears in the image of some transition map. Some common examples of vertex-stable *FI-graphs* include:

- The complete graphs K_n ;
- The Kneser graphs $K_{G_{n,r}}$, for each fixed r . These are the graphs whose vertices are r -element subsets of $[n]$, and whose edges indicate disjointness;
- The Johnson graphs $J_{n,r}$, for each fixed r . These are the graphs whose vertices are r -element subsets of $[n]$, and whose edges indicate that the intersection of the two subsets has size $r - 1$.

Other examples of vertex-stable *FI-graphs* are given at the end of Sect. 3.1. While it is straightforward to verify that the above examples are vertex-stable, one might also observe that they have a variety of other symmetries. The main structure theorem of vertex-stable *FI-graphs* is that the condition of vertex-stability automatically yields several other symmetries.

Theorem A *Let G_\bullet be a vertex-stable *FI-graph*. Then for all $n \gg 0$:*

1. *The transition maps originating from G_n are injective;*
2. *The transition maps originating from G_n have induced images (see Definition 2.1);*
3. *Every edge of G_{n+1} is the image of some edge of G_n under some transition map;*

4. For any fixed $r \geq 1$ and any collection of vertices $\{v_1, \dots, v_r\}$ of G_{n+1} , there exists a collection of r vertices of G_n , $\{w_1, \dots, w_r\}$ which map to $\{v_1, \dots, v_r\}$ under some transition map.

One should note two recurring themes in the above theorem. Firstly, many of the results in this work (indeed, many of the results in the theory of FI-modules) are only true asymptotically. Secondly, while one can prove the existence of certain behaviors in general, it is usually quite difficult to make such existential statements effective (see Theorem 3.31 for an instance where this is not the case). This is a consequence of the methods used to prove such statements. In this work, the main proof techniques which will be employed fall under what one might call a Noetherian method. Namely, we rephrase what needs to be proven in terms of finite generation of some associated module. We then prove that this module is a submodule of something which is easily seen to be finitely generated, and apply standard Noetherianity arguments to conclude that the original module was finitely generated. It is an interesting question to ask which, if any, of our results can be made effective through more combinatorial means.

Following the proof of Theorem A, we spend the majority of the body of the paper illustrating various applications. These applications come in three flavors: enumerative, topological, and algebraic.

1.2 Enumerative applications

We begin by asking the following question: Given a vertex-stable FI-graph G_\bullet , is it possible to count the occurrences of some fixed substructure in G_n , as a function of n ?

If G is a graph, then an *induced subgraph* of G is a graph obtained from G by deleting some subset of the vertices and any edges involving those vertices, and a *subgraph* of G is a graph obtained from G by deleting some subset of the vertices, any edges involving those vertices, and some subset of the remaining edges. For a graph H , there could be multiple ways to realise it as a subgraph of G , by deleting different vertices and/or edges. This gives an instance of the above question. Can we count the number of times a given graph H occurs in G_n as a function of n ? We answer this question in the affirmative.

Theorem B *Let G_\bullet be a vertex-stable FI-graph of stable degree $\leq d$, and let H be a graph. Then there exists a polynomial $p_H(x) \in \mathbb{Q}[x]$ of degree $\leq |V(H)| \cdot d$ such that for all $n \gg 0$ the function*

$$n \mapsto \text{the number of subgraphs of } G_n \text{ isomorphic to } H$$

agrees with $p_H(n)$.

Remark 1.1 For a fixed pair of graphs G and H , the number of subgraphs of G isomorphic to H is not the number of graph injections from H to G . Indeed, usually one is concerned with counting the number of such injections *up to composition with automorphisms of H* . Because H is independent of n , the above theorem remains true regardless of how the counting problem is interpreted.

To convince themselves of this theorem, one should consider the case of the complete graphs K_n . In this case, one can count the number of occurrences of H by first choosing $|V(H)|$ vertices, and then counting the number of copies of H in the induced $K_{|V(H)|}$ subgraph. We will see in Sect. 3.1 that FI-graphs are fairly diverse, and therefore one should not expect the general case to be quite this straightforward. However, the idea that one should begin by choosing $|V(H)|$ vertices of G_n remains relevant. From this point one proceeds by applying the fourth part of Theorem A.

Another interesting enumerative consequence of vertex-stability involves counting degrees of vertices. Recall that in a given graph G , the *degree* of a vertex v is the number of edges adjacent to v . We usually write $\Delta(G)$ for the maximum degree of a vertex in G , and $\delta(G)$ for the minimum degree.

Theorem C *Let G_\bullet be a vertex-stable FI-graph of stable degree $\leq d$. Then the functions*

$$n \mapsto \Delta(G_n) \quad \text{and} \quad n \mapsto \delta(G_n)$$

each agree with a polynomial of degree at most d for all $n \gg 0$.

While Theorem C appears very similar to Theorem B, there is one subtle difference. In the case of Theorem B, one reduces to the case of FI-modules by considering the family of symmetric group representations induced by the symmetric group action on copies of H inside G_n . It is unclear, however, whether such an approach can work to prove Theorem C, as the maximum and minimum degrees of G_n cannot in any obvious way be realized as the dimension of some symmetric group representation. The proof of Theorem C is therefore a bit more subtle, and can be considered more traditionally combinatorial than that of Theorem B.

To conclude our enumerative applications, we consider the question of counting walks in G_n . Recall that for a fixed integer $r \geq 0$ and a graph G , a *walk of length r* in G is an $(r + 1)$ -tuple of vertices of G , (v_0, \dots, v_r) , such that for all $0 \leq i \leq r - 1$, $\{v_i, v_{i+1}\} \in E(G)$. We say that a walk (v_0, \dots, v_r) is *closed* if $v_r = v_0$.

Theorem D *Let G_\bullet be a vertex-stable FI-graph of stable degree $\leq d$. Then the functions*

$$n \mapsto |\{\text{walks in } G_n \text{ of length } r\}| \quad \text{and} \quad n \mapsto |\{\text{closed walks in } G_n \text{ of length } r\}|$$

each agree with a polynomial of degree at most $(r + 1)d$ whenever $n \gg 0$.

1.3 Topological applications

In this paper we will be primarily concerned with two topological applications of the theory of vertex-stable FI-graphs. Our major results will prove that certain natural topological spaces associated to vertex-stable FI-graphs will be *representation stable* in the sense of Church and Farb [12] (see Definition 2.19).

Remark 1.2 In the language of [12], representation stability is a property of sequences of symmetric group representations. In this paper, we expand this definition to sequences of topological spaces with symmetric group actions, by asserting that the homology groups of spaces are representation stable in the original sense. This use of the terminology is not standard in the literature.

The first of our applications is related to the so-called Hom-complexes. Let H and G be two graphs. A **multi-homomorphism** from H to G is a map of sets,

$$\alpha : V(H) \rightarrow \mathcal{P}(V(G)) - \emptyset$$

such that for all edges $\{x, y\} \in E(H)$, and all choices of $v \in \alpha(x)$ and $w \in \alpha(y)$, one has $\{v, w\} \in E(G)$. The Hom-complex of H and G , denoted $\mathcal{H}om(H, G)$, the polyhedral complex whose cells are indexed by multi-homomorphisms between H and G , such that the closure of any cell given by subset inclusion (See Definition 2.4 for details). These complexes first rose to popularity through the work of Babson and Koslov [4,5], which expanded upon famous work of Lovász [28]. For instance, it is shown in those works that the topological connectivity of the space $\mathcal{H}om(K_2, G)$ can be used to bound the chromatic number of G .

Theorem E *Let G_\bullet be a vertex-stable FI-graph. Then for any graph H , the functor*

$$n \mapsto \mathcal{H}om(H, G_n)$$

is representation stable (see Definition 2.19). In particular, if $i \geq 0$ is fixed, then the function

$$n \mapsto \dim_{\mathbb{R}}(H_i(|\mathcal{H}om(H, G_n)|; \mathbb{R}))$$

eventually agrees with a polynomial of degree at most $|V(H)| \cdot d(i + 1)$.

While this result might seem somewhat technical, it has one particularly notable consequence about counting graph homomorphisms into FI-graphs.

Corollary F *Let G_\bullet denote a vertex-stable FI-graph of stable degree at most d . Then for any graph H the function*

$$n \mapsto |\text{Hom}(H, G_n)|$$

agrees with a polynomial of degree at most $|V(H)| \cdot d$ for all $n \gg 0$.

Remark 1.3 The algebraic theory of graph homomorphisms implies that there are very concrete connections between counting homomorphisms into a graph, counting injective homomorphisms into a graph, and counting induced homomorphisms into a graph (see, for instance, [29, Chapter 5]). In particular, Corollary F, Theorem D, and Theorem B are not independent of each other, and can be in certain cases deduced from one another. Our presentation of the material was chosen to stress the interpretation that the polynomial behavior of homomorphisms can be thought of as a consequence of the fact that a certain family of topological spaces exhibits representation stability.

It is a well known fact that n -colorings of vertices of a graph H are in bijection with $\text{Hom}(H, K_n)$, where K_n is the complete graph on n vertices. The above theorem can therefore be thought of as an extension of the theorem which posits the existence of the *chromatic polynomial*.

Remark 1.4 The idea of treating the chromatic polynomial as an “FI phenomenon” was conveyed to the first author by John Wiltshire-Gordon and Jordan Ellenberg. This observation was a large part of the motivation for the present work.

Following our treatment of the Hom-complex, we next turn our attention to configuration spaces of graphs. Given a topological space X , the n -stranded configuration space of X is the topological space of n distinct points on X ,

$$\text{Conf}_n(X) := \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j, i \neq j\}.$$

Configuration spaces are in many ways the prototypical topological application of FI-module theory. In fact, one of the results which eventually inspired the study of FI-modules was Church’s proof that configuration spaces of manifolds are often representation stable [7]. It is unfortunately true, however, that if G is any graph then the family of topological spaces $\{\text{Conf}_n(G)\}_n$ cannot be representation stable. In fact, they are extremely unstable in this sense, exhibiting factorial growth in their Betti numbers (see the discussion following Theorem 2.10). In this paper we therefore adapt a different approach, recently used by Lütgehetmann [27]. We consider the spaces $\text{Conf}_m(G_\bullet)$, where m is fixed and G_\bullet is a vertex-stable FI-graph.

Theorem G *Let G_\bullet be a vertex-stable FI-graph with stable degree at most d whose transition maps are all injective and whose constituent graphs G_n are all connected. Then for any $m \geq 1$ the functor*

$$n \mapsto \text{Conf}_m(G_n)$$

is representation stable (see Definition 2.19). In particular, if $i \geq 0$ is fixed, then the function

$$n \mapsto \dim_{\mathbb{R}}(H_i(\text{Conf}_m(G_n); \mathbb{R}))$$

eventually agrees with a polynomial of degree at most $2dm$.

Remark 1.5 Theorem A implies that the transition maps of any vertex-stable FI-graph are eventually injective. Because the content of the previous theorem is asymptotic, we may always replace our FI-graph with a new FI-graph whose transition maps are injective and agrees with our original graph for all $n \gg 0$. In particular, the assumption that the transition maps of our FI-graph must be injective is not particularly restrictive.

The condition that G_n be connected is also not necessary, although the eventual conclusion is a bit less clean if it is not assumed. The most general version of Theorem G is proven as Theorem 4.12 below.

This theorem was proven for a particular FI-graph (see Example 3.9) by Lütgehetmann [27], although he did not use this language. His approach in that work is very topological, and sharpens certain bounds that we discover in this work, although it is limited to that example. Our approach is much more combinatorial in nature, and has the benefit of proving the above theorem for all vertex-stable FI-graphs.

1.4 Algebraic applications

Our final kind of application involves studying the spectrum of vertex-stable FI-graphs. For any graph G , let $\mathbb{R}V(G)$ denote the real vector space with basis indexed by the vertices of G . Then there are many natural endomorphisms of $\mathbb{R}V(G)$ which are of interest in algebraic graph theory. Perhaps the most significant is the *adjacency matrix* of G . This is the matrix A_G defined on vertices $v \in V(G)$ by

$$A_G v = \sum_{\{w,v\} \in E(G)} w$$

The adjacency matrix of any graph is a real symmetric matrix, and therefore its eigenvalues must be real. This justifies the hypotheses of the following theorem.

Theorem H *Let G_\bullet be a vertex-stable FI-graph, and let A_n denote the adjacency matrix of G_n . We may write the distinct eigenvalues of A_n as,*

$$\lambda_1(n) > \lambda_2(n) > \dots > \lambda_{r(n)}(n),$$

for some function $r(n)$. Then for all $n \gg 0$

1. The function $r(n)$ is constant. In particular, the number of distinct eigenvalues of A_n is eventually constant;
2. For any i the function

$$n \mapsto \lambda_i(n)$$

agrees with an function which is algebraic over the field $\mathbb{Q}(n)$;

3. For any i the function

$$n \mapsto \text{the multiplicity of } \lambda_i(n)$$

agrees with a polynomial.

Remark 1.6 The proof of the above theorem will appear in upcoming work of the authors and David Speyer [36]. It is included in this paper for completeness's sake. Hints toward the proof are given in Sect. 4.3.

Further note that the most general version of Theorem H allows one to work with matrices other than the adjacency matrix. For instance, one reaches the same conclusion working with the Laplacian matrix (see Definition 2.6).

Table 1 A summary of the quantities shown in this paper to be eventually polynomial, and bounds on the degree of that polynomial in terms of the stable degree d of the FI-graph

Quantity	Max degree	Reference
Number of subgraphs isomorphic to H	$ V(H) \cdot d$	Theorem B
Min and max vertex degrees	d	Theorem C
Number of walks and closed walks	$(r + 1)d$	Theorem D
$\dim_{\mathbb{R}}(H_i(\mathcal{J}\text{Com}(H, G_n) ; \mathbb{R}))$	$ V(H) \cdot d(i + 1)$	Theorem E
Number of homomorphisms from H to G_n	$ V(H) \cdot d$	Corollary F
$\dim_{\mathbb{R}}(H_i(\text{Conf}_m(G_n); \mathbb{R}))$	$2dm$	Theorem G

See the referenced results for definitions and notation

Perhaps the simplest example one can call upon to illustrate this theorem is the complete graph. In this instance the eigenvalues of the adjacency matrix A_n are -1 and $n - 1$, with multiplicities $n - 1$ and 1 respectively. Hence the number of distinct eigenvalues of A_n becomes constantly 2 beginning at $n = 2$, and the multiplicities of these eigenvalues are given by polynomials.

Table **1** summarizes these results.

1.5 Outline

The overall structure of the present work is as follows. We begin by recalling necessary background. This ranges from graph theory (Sect. **2.1**) to the configuration spaces of graphs (Sect. **2.2**) to the theory of FI-modules and representation stability (Sect. **2.3**). Our hope is that this background will be sufficient so that readers from a large variety of fields can better follow the work in the body of the paper.

Following this, we turn our attention to the basic definitions and examples from the theory of FI-graphs (Sect. **3.1**). We then describe the phenomenon of vertex-stability and its major structural consequences (Sect. **3.2**). This third section is then capped off by a more technical chapter which solves the question of when the transition maps of a vertex-stable FI-graph must begin to have induced image (Sect. **3.3**). The fourth section is dedicated to proving the applications detailed above, as well as various smaller consequences that one might find interesting.

To conclude the work, we consider generalization of the theory of FI-graphs in two distinct directions. Firstly, we consider what would happen if instead of FI, one considered functors from certain other categories into the category of graphs (Sect. **5.1**). In particular, we argue that virtually everything described in the paper will have some analog for FI^m -graphs and $\text{VI}(q)$ -graphs (see Definition **5.1**). Secondly, we consider higher dimensional analogs of FI-graphs. Namely, we consider general FI-simplicial-complexes and show that certain structural facts will continue to work in this context (Sect. **5.2**).

1.6 Future directions

In an upcoming paper of the authors and David Speyer [36], we classify finitely generated FI-sets and investigate the behavior of relations between FI-sets, proving for instance the present Theorem H.

Other forthcoming work concerns the behavior of random walks on FI-graphs. We show that expected hitting times of simple random walks on FI-graphs eventually agree with algebraic functions, and give bounds for the mixing times of these walks in terms of the relative sizes of vertex and edge orbits.

It would be interesting to investigate which graph theoretic properties stabilize as one moves along an FI-graph, particularly global properties which do not follow from our observations on local structure. Example 3.16 provides an example of an FI-graph for which the existence of a Hamiltonian cycle need not stabilize. Another particularly interesting question concerns the chromatic number. The examples considered in this paper whose chromatic number have been computed each have chromatic number eventually agreeing with a polynomial, though it is unknown whether this is something one should expect for all FI-graphs. A result in this direction would be particularly relevant to the Johnson graphs, whose chromatic number is still not known.

Recent work of Bahran has applied the theory of FI-graphs to questions in finite group theory [2].

2 Background

2.1 Graph theory

For the purposes of this paper, we will only consider finite graphs with no multi-edges or self-loops. Graphs will be permitted to be disconnected.

Definition 2.1 A **graph** is a finite 1-dimensional simplicial complex. Given a graph G , we will write $V(G)$ to denote its vertex set, and $E(G)$ to denote its edge set. Both $V(G)$ and $E(G)$ are necessarily finite. If $v \in V(G)$, then $\mu(v)$ will be used to denote its **degree**, which is the number of edges having v as one of their endpoints. The **minimum degree** of a vertex of G will be denoted $\delta(G)$, while the **maximum degree** will be written $\Delta(G)$.

A **homomorphism** of graphs $\phi : G \rightarrow G'$ is a map of sets $\phi : V(G) \rightarrow V(G')$ such that if $\{x, y\} \in E(G)$, then $\{\phi(x), \phi(y)\} \in E(G')$. The category of graphs and graph homomorphisms will be denoted **Graph**.

A **subgraph** of a graph G is a graph G' with inclusions $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. We say that a subgraph G' is **induced** if for all $x, y \in V(G')$, $\{x, y\} \in E(G')$ whenever $\{x, y\} \in E(G)$.

In this work, we will be applying the theory of FI-modules to the study of certain natural families of graphs. Our applications will be grouped into three categories: enumerative, topological, and algebraic.

To begin, we review some elementary facts and notations from enumerative graph theory. Much of what follows can be found in any standard text in graph theory (see, for instance, [3]).

Definition 2.2 Let G and H be graphs. We write $\eta_H(G)$ to denote the total number of distinct subgraphs of G which are isomorphic to H . We will also write $\eta_H^{ind}(G)$ to denote the total number of distinct induced subgraphs of G which are isomorphic to H .

Remark 2.3 When one speaks of computing the number of copies of H inside G , one is usually talking about counting the number of graph injections from H to G up to composition of automorphisms of H . This is the perspective we take in this work.

The question of determining whether $\eta_H(G) > 0$ is known as the subgraph isomorphism problem. It is known, for general choices of H and G , that the subgraph isomorphism problem is NP-complete [8,26]. The analogous induced subgraph isomorphism problem is also known to be NP-complete, although it is also known to be solvable in polynomial time in many instances [38]. In this paper, we will be concerned with computing these two counting invariants across the members of certain families of graphs (see Theorem 4.1).

After enumerative considerations, we next turn our attention to topological applications of the FI-graph structure. Our first application is related to so called Hom-complex construction. Interest in these complexes originates from work of Lovász [28], wherein similar spaces were used to resolve the Kneser conjecture. Babson and Koslov later showed that the the spaces used in Lovász's work were specific examples of Hom-complexes [4,5]. Following this, there has been some amount of interest in various topological aspects of these spaces (see [16,17] for some examples). For instance, it is known that every simplicial complex can be realized as a subdivision of the Hom-complex of some pair of graphs [17]. In this paper, we will approach the Hom-complex from the perspective of representation stability.

Definition 2.4 Let H, G be graphs. A **multi-homomorphism** from H to G is a map of sets

$$\alpha : V(H) \rightarrow \mathcal{P}(V(G)) - \emptyset$$

between the vertices of H and the power set of the vertices of G , such that if $\{x, y\} \in E(H)$ then for all $x' \in \alpha(x)$ and all $y' \in \alpha(y)$, $\{x', y'\} \in E(G)$. The **Hom-complex of H and G** , $\mathcal{H}om(H, G)$, is the polyhedral complex whose cells are in bijection with multi-homomorphisms from H to G . Given two multi-homomorphisms α and τ , we have that the cell corresponding to α is contained in the closure of the cell corresponding to τ if and only if $\alpha(x) \subseteq \tau(x)$ for all $x \in H$.

Remark 2.5 In the literature, the Hom-complex is sometimes defined to be the order complex of the poset of multi-homomorphisms and inclusions [16,17]. This simplicial definition of the Hom-complex can be realized as a subdivision of our chosen polyhedral complex construction (See the discussion following [4, Definition 1.2]). We use the definition of the Hom-complex as a polyhedral complex provided by [4], as it makes the arguments which follow a bit easier.

We will later construct large families of graphs G_n , indexed by the natural numbers, such that for any graph H , the complexes $\mathcal{H}om(H, G_n)$ are representation stable in the sense of Church and Farb (see Theorem 4.9 and Definition 2.19).

Following this, we will spend some time proving facts about configuration spaces of graphs. The background for this material is detailed in the next section.

The final type of application we will concern ourselves with relates to spectral properties of graphs. More specifically, we will concern ourselves with eigenspaces and eigenvalues of adjacency and Laplacian matrices.

Definition 2.6 Let G be a graph. The **adjacency matrix** of G , A_G , is the matrix whose columns and rows are labeled by vertices of G and whose entries are defined by

$$(A_G)_{(v,w)} := \begin{cases} 1 & \text{if } \{v, w\} \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

The **Laplacian matrix** of G , L_G , is the difference $D_G - A_G$, where D_G is the diagonal matrix whose entries display the degrees of the vertices of G .

The collection of eigenvalues of A_G will be referred to as the **spectrum** of G .

There are many things that one may immediately observe from the fact that A_G and L_G are real and symmetric. For instance:

1. The matrices A_G and L_G are diagonalizable.
2. The eigenvalues of A_G and L_G are real. Therefore, they can be ordered as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{|V(G)|}$.

In our work we will be largely concerned with the following two questions: Given certain natural families of graphs G_n , indexed by the natural numbers, how many distinct eigenvalues can A_{G_n} and L_{G_n} have (as a function of n), and how do the multiplicities of these eigenvalues change with n ? For instance, the adjacency matrix of the complete graph K_n , with $n \geq 2$, has distinct eigenvalues $n - 1$ and -1 with multiplicities 1 and $n - 1$, respectively. In other words, so long as n is sufficiently large, the complete graph K_n has a fixed number of distinct eigenvalues, and the corresponding eigenspaces have dimensions which are polynomial in n . One of the main motivations for this paper is proving a framework which explains such behavior.

For references on graph spectra, see [3,9,14,15].

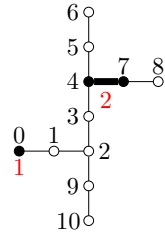
2.2 Configuration spaces of graphs

Definition 2.7 Let G be a graph. Then the **m -stranded configuration space** of G is the topological space

$$\text{Conf}_m(G) := \{(x_1, \dots, x_m) \in G^m \mid x_j \neq x_i, i \neq j.\}$$

Configuration spaces of various topological spaces have a long history including work of McDuff [33], and Church et al. [10], among many others. Much of the literature is focused on the configuration spaces of manifolds. Recently, some attention has been

Fig. 1 The cell $(v_0, e_{4,7})$



given to the configuration spaces of graphs, due to their connections with robotics [23]. Much of the newly emerging literature seems to indicate that these configuration spaces are heavily influenced by the combinatorics of the graph (see [1,13,19,22,23,27,35], for a small sampling). For instance, the following theorem of Abrams puts a very natural cellular structure on $\text{Conf}_m(G)$, which depends highly on the vertices of G of degree at least 3. Cellular models have also been proposed by Światkowski [37], Ghrist [23], Lütgehetmann [27], and Wiltshire-Gordon [42].

Definition 2.8 Let G be a graph. The m th subdivision of G is the graph $G^{(m)}$ obtained from G by adding $m - 1$ vertices of degree 2 to every edge of G .

Theorem 2.9 [1] Let G be a graph, and let $\text{DConf}_m(G)$ denote the sub-complex of the cubical complex G^m comprised of cells of the form

$$\sigma_1 \times \cdots \times \sigma_m$$

where σ_i is either an edge or vertex of G , and for each $i \neq j$,

$$\partial(\sigma_i) \cap \partial(\sigma_j) = \emptyset.$$

Then $\text{DConf}_m(G^{(m)})$ is homotopy equivalent to $\text{Conf}_m(G^{(m)})$.

The original work of Abrams is more precise than the above, but this is sufficient for what follows. We note that for any graph G , $\text{Conf}_m(G^{(m)})$ is identical to $\text{Conf}_m(G)$. We observe that $\text{DConf}_m(G)$ is the largest subcomplex of G^m which avoids the diagonals $x_i = x_j$. Abrams’ theorem therefore states that this complex will contain the same topological information as $\text{Conf}_m(G)$ so long as there are enough vertices in G such that every coordinate in a given configuration can fit on a single edge using only vertices.

It is often convenient to visualize the cells of $\text{DConf}_m(G^{(m)})$ as living on the graph $G^{(m)}$. In such a visualization, we bolden the vertices and edges appearing in the cell on the graph $G^{(m)}$, and label the position in which they appear in the cell. For instance, Fig. 1 shows a cell of $\text{DConf}_2(G^{(2)})$ for a particular choice of G .

Among the many incredible theoretical properties of configuration spaces of graphs is the precise computation of their Euler characteristic. The following result is due to Gal, and provides a large part of the motivation for this work.

Theorem 2.10 [22] *Let G be a graph, and let $\epsilon(t)$ denote the exponential generating function*

$$\epsilon(t) = \sum_{m \geq 0} \frac{\chi(\text{Conf}_m(G))}{m!} t^m.$$

Then

$$\epsilon(t) = \frac{\prod_{v \in V(G)} (1 - (1 - \mu(v))t)}{(1 - t)^{|E(G)|}}.$$

A theorem of Ghrist [23] and Światkowski [37] implies that $H_i(\text{Conf}_m(G)) = 0$ for all graphs G and all i larger than the number of vertices of G of degree at least 3. In particular, this is independent of m . It follows from this fact, as well as the theorem of Gal, that the Betti numbers of $\text{Conf}_m(G)$ should be expected to grow in m like $m!$. Such growth precludes $\text{Conf}_m(G)$ from being representation stable (see Definition 2.19 for the definition of representation stable, and Theorem 2.18 to see why the above precludes $\text{Conf}_m(G)$ from having this property). Looking again at the theorem of Gal, we see that the Euler characteristic of $\text{Conf}_m(G)$, as a function of m , looks like $m!$ multiplied by a polynomial in invariants of G . In other words, the extreme growth in the Euler characteristic seems to be primarily influenced by the number of points being configured, rather than the the graph G itself.

One guiding philosophy of the present work is that if we fix the number of points begin configured, and instead allow the graph itself to vary, then the collection of spaces $\text{Conf}_m(G_n)$ will be representation stable in the sense of Definition 2.19.

This philosophy has also appeared in recent work of Lütgehetmann [27]. Theorem G extends the main theorem of that work.

2.3 FI-modules and representation stability

The main tool we introduce in this paper are objects we refer to as FI-graphs. Before working through the technical details of that construction, we must first discuss a key auxiliary concept: FI-modules.

Definition 2.11 Let FI denote the category whose objects are the finite sets $[n] := \{1, \dots, n\}$ and whose maps are injections. An FI-module is a (covariant) functor from FI to the category of \mathbb{R} -vector spaces.

Remark 2.12 FI-modules can be put into more concrete terms. Observe that for any set $[n]$, the endomorphisms in FI are precisely the permutations on n letters, \mathfrak{S}_n . Therefore, if V is an FI-module, each of the vector spaces $V([n])$ is actually a representation of the symmetric group \mathfrak{S}_n . An FI-module may be thought of as a sequence of vector spaces

$$V_0 \rightarrow V_1 \rightarrow \dots$$

such that each V_n is a representation of \mathfrak{S}_n , and each of the forward maps $V_n \rightarrow V_{n+1}$ is a linear map which is equivariant with respect to the action of \mathfrak{S}_n , where $\mathfrak{S}_n \leq \mathfrak{S}_{n+1}$ as the subgroup of permutations which fix the element $n + 1$.

More precisely, Remark 3.3.1 of [10] says that such a sequence of spaces and maps forms an FI-module if and only if for any n and k , any element of \mathfrak{S}_{n+k} which fixes all of $[n]$ acts trivially on the image of V_n in V_{n+k} .

Remark 2.13 Note that most works in the literature allow FI-modules to be valued in any module category over a commutative ring. For our purposes, we will mostly consider FI-modules which are valued in vector spaces over \mathbb{R} . In certain areas of the paper, such as Lemma 4.6, we consider **FI-modules over \mathbb{Z}** , i.e. functors from FI to the category of abelian groups. Most of the definitions and theorems in this section work equally well in this case.

For an FI-module V , we will often write $V_n := V([n])$ and $f_* := V(f)$. One should note that for any n , the endomorphisms of $[n]$ in FI are precisely the permutations on n letters, \mathfrak{S}_n . Functoriality therefore implies that, for each n , V_n is a representation of \mathfrak{S}_n .

Just as with the study of vector spaces, it is often reasonable to restrict one's attention to those objects which are finitely generated in the appropriate sense. Before describing how such a condition can be applied to FI-modules, we note that the category of FI-modules and natural transformations is abelian. Indeed, one may define the usual abelian operations point-wise. In fact, one may very naturally define constructions such as direct sums and products, tensor products, symmetric products, etc. for FI-modules.

Definition 2.14 An FI-module V is said to be **finitely generated in degree $\leq d$** if there is a finite set

$$\{v_i\} \subseteq \sqcup_{n=0}^d V_n$$

which no proper submodule of V contains. Equivalently, the set $\{v_i\}$ generates V if, for all n , the vector space V_n is spanned by the images of the v_i under the various maps f_* induced by V from injections of sets.

Perhaps the most remarkable thing about finitely generated FI-modules is that they exhibit a Noetherian property. The following was first proven by Snowden [39], and later reproven by Church et al. [10].

Theorem 2.15 [10,39] *Let V be a finitely generated FI-module. Then every submodule of V is also finitely generated.*

We will use the above Noetherian property to deduce various somewhat surprising combinatorial facts about FI-graphs.

As one might expect, if V is an FI-module generated in degree $\leq d$, then it is not necessarily the case that submodules of V are also generated in degree $\leq d$. Despite this, one may still conclude certain things about submodules of V based on properties of V . For this reason, we introduce the following.

Definition 2.16 We say that a finitely generated FI-module V is **d -small** if V is a subquotient of an FI-module which is finitely generated in degree $\leq d$.

Proposition 2.17 [10] *If V is finitely generated in degree $\leq d$ and W is finitely generated in degree $\leq e$, then*

1. *The FI-module $V \oplus W$ is generated in degree $\leq \max\{d, e\}$, where $(V \oplus W)_n = V_n \oplus W_n$;*
2. *The FI-module $V \otimes W$ is generated in degree $\leq d + e$, where $(V \otimes W)_n = V_n \otimes W_n$.*

The following list of properties are proven throughout [10].

Theorem 2.18 [10] *Let V be an FI-module. If V is finitely generated then for all $n \gg 0$ and all injections $f : [n] \rightarrow [n + 1]$,*

1. *The function f_* is injective;*
2. *The vector space V_{n+1} is spanned as an \mathfrak{S}_{n+1} -representation by $f_*(V_n)$;*
3. *The \mathfrak{S}_n -representation V_n admits a decomposition of the form*

$$V_n = \bigoplus_{\lambda, |\lambda| \leq d} m_\lambda V(\lambda)_n$$

where the coefficient m_λ is independent of n and d is some constant independent of n (see [10] for details on the representations $V(\lambda)$). In particular, the multiplicity of the trivial representation in V_n is eventually independent of n .

4. *If V is d -small, then there exists a polynomial $p_V(X) \in \mathbb{Q}[X]$ of degree $\leq d$ such that for all $n \gg 0$, $p_V(n) = \dim_{\mathbb{Q}} V_n$.*

The above will be used extensively in what follows.

The notion of representation stability was first introduced by Church and Farb in their seminal work [12]. From these beginnings the field has seen a boom in the literature and has been proven to be applicable to a large collection of subjects. For the purposes of this paper, we state the following definition, which is a modernized version of the original definition of Church and Farb.

Definition 2.19 Let X_\bullet denote a functor from FI to the category of topological spaces. Then we say that X_\bullet is **representation stable** if for all $i \geq 0$ the FI-module over \mathbb{Z}

$$H_i(X_\bullet; \mathbb{Z})$$

is finitely generated.

Note that this definition describes what is meant by an FI-space being representation stable, not an FI-module.

Remark 2.20 Again we note that this definition is not standard in the literature. Observe that being representation stable in the above sense implies that the FI-module $H_i(X_\bullet; \mathbb{R})$ is finitely generated in at most the same degree. This is much more similar, in spirit, to the original work of Church and Farb [12].

It was famously proven by Church [7], and later reexamined by Church et al. [10], that if M is a compact orientable manifold with boundary of dimension at least two then

$$n \mapsto \text{Conf}_n(M)$$

is representation stable. We have already seen, however, that an analogous statement cannot be true if we replace M with a graph (see the discussion following Theorem 2.10). We therefore change our approach and instead consider the functors

$$n \mapsto \text{Conf}_m(G_n) \tag{2.1}$$

where m is fixed, and G_\bullet is a particularly nice FI-graph (see the statement of Theorem G). The main theorem of this paper can be restated to say that in this case the functor (2.1) is representation stable. Our approach will be largely combinatorial, and we will use structural facts about FI-graphs as well as the cellular model of Theorem 2.9. This is in contrast to the work of Lütgehetmann, which proves that $n \mapsto \text{Conf}_m(G_n)$ is representation stable for a particular choice of G_\bullet (see Example 3.9) using very topological methods. We will find that our method provides a stronger bound on the degree of the polynomial encoding the Betti numbers in this case, while Lütgehetmann's method provides bounds on the degree of generation of the FI-modules $H_i(\text{Conf}_m(G_\bullet))$.

Remark 2.21 For future use, we now explicitly point out the properties of FI-modules over \mathbb{Z} which will be used in the sequel. These are:

1. FI-modules over \mathbb{Z} satisfy the Noetherian property. That is, submodules of finitely generated modules are once again finitely generated [11, Theorem A];
2. If V is a finitely generated FI-module over \mathbb{Z} , then the function $n \mapsto \text{rank}(V_n)$ agree with a polynomial for $n \gg 0$ [11, Theorem B];
3. If V is a finitely generated FI-module over \mathbb{Z} , then transition maps of V are eventually injective [11, Lemma 2.15].
4. If V and W are finitely generated FI-modules over \mathbb{Z} , then both $V \otimes W$ and $V \oplus W$ are also finitely generated [10, Proposition 2.61].

Note that the primary differences between FI-modules and FI-modules over \mathbb{Z} involve the behaviors of the \mathfrak{S}_n -representations V_n . This can be thought of as a consequence of the fact that representation theory, and the representation theory of the symmetric groups specifically, is generally much less well behaved over \mathbb{Z} .

3 FI-graphs

3.1 Definitions and examples

The primary objective of this section is to provide a framework through which one can study families of graphs in the spirit of Kneser graphs and their generalizations. Recall that, for any fixed integers $n \geq k$, one defines the Kneser graph $KG_{n,k}$ as

the graph whose vertices are labeled by k -element subsets of $[n]$, and whose edges connect disjoint sets.

It is clear that for each n , elements of \mathfrak{S}_n act on $KG_{n,k}$ by graph automorphisms. What is perhaps more subtle, is that if $f : [n] \hookrightarrow [m]$ is any injection, then there is an induced map of graphs

$$KG(f) : KG_{n,k} \rightarrow KG_{m,k}$$

Looking back at the definition of FI-modules, one is therefore motivated to make the following definition.

Definition 3.1 An FI-graph is a functor from the category FI to the category **Graph** of (simple) graphs. We will usually denote an FI-graph by $G_\bullet : \text{FI} \rightarrow \mathbf{Graph}$. We will use $G(f)$ to denote the induced maps of G_\bullet .

Remark 3.2 We may consider FI-graphs in a more concrete fashion, similar to how FI-modules were treated in Remark 2.12. An FI-graph may be thought of as a sequence of graphs

$$G_0 \rightarrow G_1 \rightarrow \dots$$

such that each G_n carries a vertex action by \mathfrak{S}_n , and each of the forward maps $G_n \rightarrow G_{n+1}$ is a graph theoretic homomorphism which is equivariant with respect to the action of \mathfrak{S}_n , where $\mathfrak{S}_n \leq \mathfrak{S}_{n+1}$ as the subgroup of permutations which fix $n + 1$.

As in Remark 2.12, such a sequence of graphs and maps may not be an FI-graph, but following Remark 3.3.1 of [10], the sequence of graphs and maps forms an FI-graph if and only if for any n and k , any element of \mathfrak{S}_{n+k} which fixes all of $[n]$ acts trivially on the image of G_n in G_{n+k} . Example 3.15 shows the behavior that may occur without this assumption.

While the above definition captures the core of the above discussion, it is still a bit too general for our purposes. For instance, if

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$$

is any chain of graphs, then we may define an FI-graph by setting the \mathfrak{S}_n -action to be trivial for each n , and having the transition maps be the given inclusions. An arbitrary chain of graphs like the above can become rather complicated, and there won't necessarily be any way to gather meaningful information above the invariants of any G_n from those that came before it. What is needed is some notion of finite generation for FI-graphs. For this purpose, we define the following.

Definition 3.3 Let G_\bullet be an FI-graph. We say that G_\bullet is **vertex-stable with stable degree** $\leq d$ if for all $n \geq d$, and every vertex $v \in V(G_{n+1})$ there exists some vertex $w \in V(G_n)$ and some injection $f : [n] \hookrightarrow [n + 1]$ such that $G(f)(w) = v$.

That is, an FI-graph is vertex-stable with stable degree $\leq d$ if for each $n > d$, every vertex in G_n is in the image of one of the transition maps. Informally, no ‘new’ vertices appear after the graph G_d , up to symmetric group actions.

We will find that this fairly simple combinatorial condition is sufficient to prove a plethora of facts about the graphs G_n . Before we delve into these details, we first introduce the various examples which motivated this paper. In most of these examples, vertices are labeled by elements of $[n] = \{1, 2, \dots, n\}$ or by sets or tuples (unordered or ordered, respectively) of such elements. The symmetric group \mathfrak{S}_n acts on such vertices by acting on each element individually. It is a result of the upcoming work [36] that all vertex-stable FI-graphs arise from such constructions, slightly generalized.

Example 3.4 For any fixed $k \geq 0$, the Kneser graphs $KG_{\bullet,k}$ form a vertex-stable FI-graph with stable degree k (or stable degree 1 if $k = 0$). The same can therefore be said about the complete graphs $K_{\bullet} = KG_{\bullet,1}$.

More generally, if n, k , and r are fixed integers, then we define the generalized Kneser graph $KG_{n,k,\leq r}$ to have vertices labeled by subsets of $[n]$ of size k and edges connecting subsets whose intersection has size at most r . In particular, $KG_{n,k} = KG_{n,k,0}$. The generalized Kneser graphs $KG_{\bullet,k,\leq r}$ form a vertex-stable FI-graph for each fixed k and r , again with stable degree k .

Rather than putting edges between subsets whose intersection is of at most a certain size, we could instead require that the intersection have exactly that size—for instance, let $KG_{n,k,r}$ be the graph whose vertices are subsets of size k and with an edge between two vertices if their subsets have an intersection of exactly size r . The graphs $KG_{\bullet,k,r}$ also form a vertex-stable FI-graph with stable degree k .

We could generalize this example further, allowing edges to correspond to intersections of various specified sizes.

Example 3.5 For any fixed $k \geq 0$, we can define a variant of the Kneser graph, which we denote $KG_{n,\leq k}$. The vertices of $KG_{n,\leq k}$ will be labeled by subsets of $[n]$ of size at most k , and the edges will connect disjoint subgraphs, just as was the case with the Kneser graph. Because self-loops are forbidden, we do not connect the empty set to itself.

Note that for each n , the symmetric group action on $KG_{n,\leq k}$ is not transitive. Despite this, the collection $KG_{\bullet,\leq k}$ still forms a vertex-stable FI-graph with stable degree k . It will be useful to consider the orbits of vertices under the symmetric group actions. Our examples tend to have few orbits for the sake of being simple examples, but this is not a restriction on general FI-graphs.

Example 3.6 For any fixed $k \geq 0$, the complete bipartite graphs $K_{\bullet,k}$ form a vertex-stable FI-graph with stable degree 1. Here, our transition maps and permutations fix the vertices in the part of size k . It follows that the series of star graphs, $\text{Star}_{\bullet} = K_{\bullet,1}$ form a vertex-stable FI-graph.

Example 3.7 For any fixed $n, k \geq 0$, define the Johnson graph $J_{n,k}$ as that whose vertices are labeled by subsets of $[n]$ with size k , and whose edges connect subsets with intersection size $k - 1$. Then $J_{\bullet,k}$ naturally forms a vertex-stable FI-graph with stable degree k .

In the notation of Example 3.4, the Johnson graph $J_{n,k}$ is the generalized Kneser graph $K G_{n,k,k-1}$.

Example 3.8 Recall that the n -cube graph Q_n is defined to be the 1-skeleton of the n -dimensional hypercube. This collection cannot be endowed with the structure of a vertex-stable FI-graph, as its number of vertices grows too fast (see Theorem 4.3). There is, however, a variation of the n -cube graph which can be endowed with the structure of a finitely generated FI-graph.

For fixed $n, k \geq 0$, let $Q_{n,k}$ denote the graph whose vertices are ordered k -tuples of elements of $[n]$, where two vertices are connected if they differ in only one coordinate. This graph is sometimes called the k -lattice graph of characteristic n . The cubic lattice graph of characteristic n is notable in that it can be entirely characterized by certain simple combinatorial properties (see [30]). For our purposes, we simply note that for any fixed k the family $Q_{\bullet,k}$ can be endowed with the structure of a vertex-stable FI-graph. Indeed, let $n > k$, and let (i_1, \dots, i_k) be a vertex of $Q_{n+1,k}$. Because $k < n$, we know that there is some integer $l \in [n]$ such that $l \neq i_j$ for any j . Then (i_1, \dots, i_k) is in the image of the transition map induced by the injection $f : [n] \hookrightarrow [n + 1]$ given by,

$$f(x) = \begin{cases} n + 1 & \text{if } x = l, \\ x & \text{otherwise.} \end{cases}$$

This FI-graph has stable degree k .

Example 3.9 Our next example appears in earlier work of Lütgehetmann [27]. Let G, H be any pair of pointed graphs. Then we can construct a new graph by wedging G with H n times, producing the graph

$$G_n := G \bigvee H^{\vee n}$$

Then we may endow G_n with the structure of an FI-graph by having the symmetric group act by permuting the factors of H . This FI-graph has stable degree 1.

The examples thus far have been quite regular, in the sense that for each n , the construction of the vertices and edges of the graph G_n has been the same. It is worth examining how this can be varied, particularly because results later in this section will limit how wild such variation can be.

Example 3.10 Let G_\bullet be an FI-graph, and modify it by removing all edges from each G_i , for $i = 1$ to $k - 1$.

Example 3.11 Let G_\bullet be an FI-graph, and modify it by replacing each G_i by the empty graph, for $i = 1$ to $k - 1$.

While Examples 3.10 and 3.11 remove vertices and edges from graphs in the first few degrees, this cannot necessarily be done in later degrees. The transition maps are permitted to map pairs of vertices not connected by an edge to pairs of vertices

connected by an edge, but not the reverse. Two vertices joined by an edge may not map to the same vertex, because there cannot be an edge from this vertex to itself. For instance, if G_n contains a complete graph on k vertices then G_{n+1} also contains a complete graph on k vertices.

Disjoint unions of FI-graphs are FI-graphs, and it is possible to increase the number of copies from a certain point onwards.

Example 3.12 Fix a positive integer k , and let G_\bullet be any FI-graph with stable degree at most k . We create a new FI-graph H_\bullet as follows. For $i < k$, the graph H_i is equal to G_i . For $i \geq k$, the graph H_i is a disjoint union of two copies of G_i . For concreteness, color vertices and edges in one of these subgraphs red and in the other, blue. The action of \mathfrak{S}_n preserves the color of vertices. Transition maps preserve the color of vertices and take uncolored vertices to red vertices. This FI-graph has stable degree k .

Example 3.12 did not need the two graphs to be the same—the new graphs introduced from degree k could have been the respective components of any FI-graph.

It is also possible to decrease the number of components. This does require the use of transition maps which are not injective.

Example 3.13 Let G_\bullet be any FI-graph. Fix a positive integer k , and create a new FI-graph H_\bullet as follows. For $i < k$, the graph H_i is a disjoint union of two copies of G_i . Color vertices and edges in one of these subgraphs red and in the other, blue. For $i \geq k$, the graph H_i is equal to G_i . The action of \mathfrak{S}_n preserves the color of vertices. Transition maps preserve the color of vertices if their image is in G_i with $i < k$, and forget colors otherwise.

An FI-graph may be modified by changing the times at which the various ‘types’ of edges begin to appear, as in the following variant of the Kneser graph.

Example 3.14 Let the vertex set of G_n be indexed by subsets of $[n]$ of size r , and let a_0 to a_r be $r + 1$ fixed positive integers. In G_n , there is an edge between two vertices if and only if $n \geq a_k$, where k is the number of elements the two vertices have in common.

That is, Example 3.14 is describing a sequence of graphs where edges between disjoint sets are present from the a_0 th graph onwards, edges between sets with a single element in common are present from the a_1 th graph onwards, and so on.

Example 3.14 could be generalized further by taking the vertices to be ordered r -tuples, in which case there would be more edge orbits—two tuples may be compared not just by how many elements they have in common, but also by which positions these overlaps occupy. For instance, if the vertices are ordered pairs, then there are five orbits of edges rather than three in the unordered case—between pairs of vertices $((a, b), (a, c)), ((a, b), (c, b)), ((a, b), (b, c)), ((a, b), (c, a))$, and $((a, b), (c, d))$.

There are also examples whose vertices are in between sets (unordered) and tuples (ordered)—we could start with vertices indexed by r -tuples, choose a subgroup H of the symmetric group \mathfrak{S}_r , and identify two tuples if the action of H takes one to the other. For instance, we might care about the order of an r -tuple up to cyclic

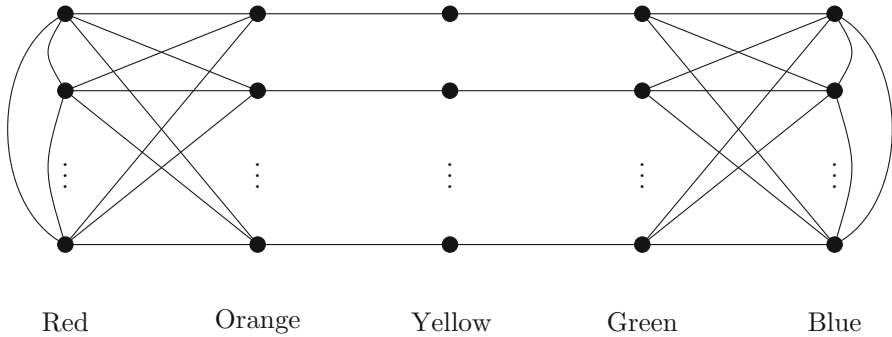


Fig. 2 The FI-graph of Example 3.16. The key feature is that there are n ‘bridges’ from the right side of the graph to the left, and any Hamiltonian path must cross each of them

permutation. It turns out that all finitely-generated FI-graphs may be constructed in essentially this way—see Theorem A of [36].

The next example fails to be an FI-graph in a subtle way. If it was an FI-graph, it would violate Theorem 3.31.

Example 3.15 For each $i \neq 2$, let G_i be the complete graph on the vertex set $[i]$, with the natural symmetric group action where the group \mathfrak{S}_i permutes the set $[i]$. Let G_2 have vertex set $\{1, 2, 3\}$, with edges 13 and 23 and the nonidentity element of \mathfrak{S}_2 interchanging the vertices 1 and 2 and fixing 3. Transition maps from G_n to G_{n+1} are obtained by including G_n into G_{n+1} , using the same symbols for the vertex labels of each graph, and then acting by any element of \mathfrak{S}_{n+1} . We shall not attempt to define further transition maps, because this cannot be done in a consistent manner, as we shall now see.

From Remark 3.2, we know that for G_\bullet to be an FI-graph, the transposition (3 4) would need to fix the image of G_2 in G_4 , because it fixes 1 and 2. But this isn’t the case, so G_\bullet can’t be an FI-graph. This is perhaps a surprising failure, because transition maps from each G_i to the next graph G_{i+1} can be defined naturally, and it is only longer-range maps which fail. Attempts to define such transition maps in a consistent way will run into difficulties deciding where to send the vertex 3 from G_2 . This example illustrates the consistency condition required by Remark 3.2.

If rather than FI we were working over a category where maps from $[2]$ to $[n]$ were instead sequences of maps from $[2]$ to $[3]$ to $[4]$ and so on, then this construction would not fail in this way, and so over such a category, the analogue of Theorem 3.31 is false.

Example 3.16 Consider the FI-graph defined as follows. Let G_n have five orbits of n vertices, indexed by the colors red, orange, yellow, green, and blue. At the moment, these orbits are isomorphic, though they will differ once we introduce the edges. The symmetric group action and the transition maps both preserve the colors. The graph G_n has edges between

- Each red vertex and each red or orange vertex
- Each yellow vertex and the orange and green vertices of equal label, and

- Each green or blue vertex and each blue vertex.

See Fig. 2 for a schematic of this graph.

This particular FI-graph is of theoretical interest, as it provides an example of a global property which does not stabilize in n . In particular, we claim that G_n admits a Hamiltonian cycle if and only if n is even.

When n is even, consider the path which starts at the top left of Fig. 2 and ‘snakes’ downward by moving all the way to the right, takes one step down, moves all the way to the left, takes a step down, and repeats. The initial and final vertices of this path are adjacent, so this is a Hamiltonian cycle.

When n is odd, each time a non-backtracking path passes through a yellow vertex, it switches between the left and right pieces of the graph, comprised of red and orange or green and blue vertices, respectively. There is no other way to move between the two sides, and there are an odd number of yellow vertices, so any path passing through each vertex once must end on the opposite side to which it started.

3.2 Vertex-stability and its consequences

While it is clearly the case that the examples of Sect. 3.1 are vertex-stable, one might also note that these cases seem to have much more structure than this. For instance, it is natural to go a step further and make the following definitions:

- Definition 3.17**
1. An FI-graph is **eventually injective** if for $n \gg 0$, the transition maps of G_\bullet are injective;
 2. An FI-graph is **eventually induced** if for $n \gg 0$, the image of any transition map is an induced subgraph;
 3. An FI-graph is **edge-stable** with edge-stable degree $\leq k$ if for $n \geq k$ and any $\{x, y\} \in E(G_n)$ there is an edge $\{v, w\} \in V(G_k)$ and an injection $f : [k] \hookrightarrow [n]$ such that $G(f)(v) = x$ and $G(f)(w) = y$;
 4. An FI-graph is **r -vertex-stable** if for all $n \gg 0$, and any collection of r vertices of $V(G_{n+1})$, $\{x_1, \dots, x_r\}$, there is a collection of vertices of G_n , $\{v_1, \dots, v_r\}$, and an injection $f : [n] \hookrightarrow [n+1]$, such that $G(f)(v_i) = x_i$ for each i .

These stability properties may occur at quite different times, and at different times to vertex-stability. Example 3.13 is injective only from degree k onwards, Example 3.12 is vertex-stable and edge-stable from degree k onwards, and Example 3.14 is vertex-stable in degree r , but edge-stable only once the degree is greater than all of a_0 through a_r .

The Kneser graphs $KG_{\bullet,k}$ (Example 3.4) are vertex-stable in degree k , edge-stable in degree $2k$, and r -vertex-stable in degree rk . In contrast, the lattice graphs $Q_{\bullet,k}$ (Example 3.8) are vertex-stable in degree k , edge-stable in degree $k+1$, and r -vertex-stable in degree rk .

It is left to the reader to verify that all of the examples of the previous section satisfy each of the above conditions. Somewhat miraculously, it turns out that this is not a coincidence.

Theorem 3.18 *Let G_\bullet be a vertex-stable FI-graph. Then:*

1. G_\bullet is r -vertex-stable for all $r \geq 1$;
2. G_\bullet is edge-stable;
3. G_\bullet is eventually injective and induced.

It is worth noting that vertex-stability is strictly stronger than edge-stability, as shown by the following example.

Example 3.19 For each n , let G_n be the union of the complete graph K_n and n isolated vertices. The symmetric group \mathfrak{S}_n acts naturally on the complete graph and fixes each of the other vertices. This FI-graph is edge-stable in degree 2, but is not vertex-stable.

Edge stability may happen either before or after 2-vertex-stability, because edge-stability includes only pairs of vertices which are connected by edges, but it is possible for edges to not appear until long after any pair of vertices are contained in the image of some transition map. Example 3.14 is 2-vertex-stable in degree $2r$, but is not edge-stable until the degree equal to the maximum of the a_i .

Before we prove Theorem 3.18, it will be useful to us to rephrase the above properties in terms of finite generation of certain FI-modules.

Definition 3.20 Let G_\bullet denote an FI-graph, and let $r \geq 1$ be fixed. We write

$$\mathbb{R} \binom{V(G_\bullet)}{r}$$

to denote the FI-module whose evaluation at $[n]$ is the \mathbb{R} vector space with basis indexed by collections of r vertices of G_n . We will often write $\mathbb{R}V(G_\bullet) := \mathbb{R} \binom{V(G_\bullet)}{1}$. Note that the image of a collection of r vertices under a transition map may not be a collection of r vertices if this transition map is not injective on vertices. In this case we simply declare the map to be zero on this collection. Similarly, we define $\mathbb{R}E(G_\bullet)$ to be the FI-module whose evaluation at $[n]$ is the \mathbb{R} vector space with basis indexed by the edges of $E(G_n)$.

Remark 3.21 The modules $\mathbb{R} \binom{V(G_\bullet)}{r}$ can also be constructed in the following fashion. Observe that if G_\bullet is an FI-graph, then $\binom{V(G_\bullet)}{r}$ is an FI-set, i.e. a functor from FI to the category of finite sets. There is a functor from the category of finite sets to the category of \mathbb{R} vector spaces given by **linearization**. Specifically, this is the functor which sends a set to the \mathbb{R} vector space with a basis indexed by the elements of the set. The module $\mathbb{R} \binom{V(G_\bullet)}{r}$ can therefore be realized as a composition of the functor $\binom{V(G_\bullet)}{r}$ with linearization. This perspective is pervasive through the sequel work [36], where FI-sets are a more primary focus. In this work we will not dive too deeply into this idea.

Lemma 3.22 Let G_\bullet be an FI-graph.

1. G_\bullet is vertex-stable with stable degree $\leq d$ if and only if $\mathbb{R}V(G_\bullet)$ is finitely generated in degree $\leq d$.
2. G_\bullet is eventually injective if and only if the transition maps of $\mathbb{R}V(G_\bullet)$ are eventually injective.

3. G_\bullet is edge-stable with edge-stable degree $\leq d$ if and only if $\mathbb{R}E(G_\bullet)$ is finitely generated in degree $\leq d$.
4. G_\bullet is r -vertex-stable if and only if $\mathbb{R}\binom{V(G_\bullet)}{r}$ is finitely generated.

Proof All of these assertions follow from the relevant definitions. □

Remark 3.23 Note that this lemma is critically dependent on the assumption that G_n has finitely many vertices and edges for each n . For instance, consider the collection of infinite graphs

$$V(G_n) := \mathbb{N}, \quad E(G_n) := \{\{1, 2\}, \{2, 3\}, \dots, \{n - 1, n\}\}$$

We can introduce an FI-structure on G_\bullet by having the symmetric group act trivially. Then it is clear that $\mathbb{R}V(G_n)$ is not finitely generated, despite G_\bullet being “vertex-stable” in some sense. Also note that the collection G_\bullet is not edge-stable in this case, seemingly violating Theorem 3.18.

This lemma is the key piece in the proof of Theorem 3.18.

Proof of Theorem 3.18 To begin, Lemma 3.22 implies that we must show that $\mathbb{R}\binom{V(G_\bullet)}{r}$ is finitely generated. We note that there is a surjection of FI-modules

$$\mathbb{R}V(G_\bullet)^{\otimes r} \rightarrow \mathbb{R}\binom{V(G_\bullet)}{r}$$

Indeed, this is induced by the assignments

$$x_1 \otimes \cdots \otimes x_r \mapsto \begin{cases} \{x_1, \dots, x_r\} & \text{if } x_i \neq x_j \text{ for } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

By assumption $\mathbb{R}V(G_\bullet)$ is finitely generated, whence the same is true of $(\mathbb{R}V(G_\bullet))^{\otimes r}$ by Proposition 2.17. This concludes the proof of the first statement.

The second statement follows from the Noetherian property as well as the inclusion

$$\mathbb{R}E(G_\bullet) \hookrightarrow \mathbb{R}\binom{V(G_\bullet)}{2}.$$

Eventual injectivity follows from Theorem 2.18.

By definition, G_\bullet is eventually induced if and only if for n large enough and for $\{x, y\} \notin E(G_n)$ any pair of nonadjacent vertices of G_n , then the images of x and y under any injection $f : [n] \hookrightarrow [n + 1]$, $f_*(x)$ and $f_*(y)$, are not connected by an edge in G_{n+1} . For each n , let \mathcal{O}_n denote the set of \mathfrak{S}_n -orbits of pairs of vertices in G_n . Note that \mathcal{O}_n may be partitioned into two subsets, depending on whether or not pairs in the orbit correspond to edges or not. Further note that the transition maps of G_\bullet will send an “edge” orbit to an edge orbit. On the other hand, the third part of Theorem 2.18 implies that $|\mathcal{O}_n|$ is eventually independent of n , as it is equal to the multiplicity of the trivial representation in $\mathbb{R}\binom{V(G_n)}{2}$.

For similar reasons the orbits of pairs corresponding to edges must stabilize as well. Note that even once the number of orbits of pairs of orbits has stabilized (that is, that $|\mathcal{O}_k|$ is constant for all $k \geq n$), it may not be the case that the edge orbits have already stabilized at the same graph n . Rather, this value of $|\mathcal{O}_k|$ gives a finite upper bound on the number of times that the number of edge orbits may increase, which shows that this number eventually stabilizes. However, there is no bound on how long this may take, as can be seen by considering Example 3.14 and taking any of the a_k to be arbitrarily large.

Once the edge orbits have stabilized, non-edged orbits will eventually map exclusively into non-edged orbits, as desired. \square

Remark 3.24 Proposition 2.17 and the above proof together imply that $\mathbb{R}^{(V(G_\bullet))}_r$ is generated in degree $\leq rd$, where d is the generating degree of $\mathbb{R}V(G_\bullet)$. In particular, $\mathbb{R}E(G_\bullet)$ is $2d$ -small.

Remark 3.25 It is possible to prove one part of Theorem 3.18 directly. Consider any set of k vertices v_1 through v_k in G_n , for $n \geq kr$. Each v_i is in the image of a transition map from G_k to G_n , and each of these transition maps is induced by an injection from $[k]$ to $[n]$. Let f_1 through f_k be these injections. Take f to be an injection from $[kr]$ to n whose image includes the image of each f_i . Then each f_i factors through f , so each v_i is in the image of the transition map induced by f . This completes the proof.

The proof of r -vertex-stability in Theorem 3.18 relies on the tensor product of finitely generated FI-modules being finitely generated. The proof of this fact may be made explicit, and this is what lies behind the proof given above.

An application of Theorem 3.18 is the following construction of new vertex-stable FI-graphs from existing vertex-stable FI-graphs.

Definition 3.26 Let G be a graph. The **line graph** of G , $L(G)$, is the graph whose vertices are labeled by the edges of G such that two vertices are connected if and only if the corresponding edges of G share an end point.

Line graphs have been studied extensively. One avenue of research is the question of how much of the graph G can be determined by studying its line graph. A celebrated theorem of Whitney [41] implies that the line graph almost always uniquely determines the original graph. Indeed, the only exception to this is the fact that $L(K_3) = L(K_{3,1})$. Algebraically, one is also interested in the question of deciding when a line graph is determined by its spectrum (See, for instance, [25] or Chapter 1.3 of [14]).

Corollary 3.27 Let G_\bullet denote a vertex-stable FI-graph. Then the collection of line graphs $L(G_\bullet)$ can be endowed with the structure of a vertex-stable FI-graph.

Proof This result follows immediately from Theorem 3.18 and the definition of the line graph. \square

Remark 3.28 We note that the line graph $L(K_n)$ is isomorphic to the Johnson graph $J_{n,2}$. The line graphs of the complete bipartite graphs $K_{n,m}$ have been studied (see,

for instance, [32] or the references in [14]), and are sometimes referred to as the **rook graphs**, as they can be thought of as encoding legal rook moves on an $m \times n$ chess board.

3.3 Determining when the induced property begins

Theorem 3.18 implies that all FI-graphs are eventually induced. In this section we consider the question of bounding when this behavior begins. To begin we impose the following technical condition on the FI-graph G_\bullet . We will see this condition return again when we consider configuration spaces of graphs.

Definition 3.29 We say an FI-graph G_\bullet is **torsion-free** if for all injections $f : [n] \hookrightarrow [m]$ the transition map $G(f)$ is injective.

Most of the examples in Sect. 3.1 are torsion-free. Example 3.13 is not torsion-free.

Remark 3.30 We say an FI-module is **torsion-free** if all of its transition maps are injective. The above definition is intended to emulate this.

Theorem 3.18 insists that vertex-stability implies edge-stability. In particular, at some point the transition maps of a vertex-stable FI-graph will contain every edge in the union of their respective images. It is therefore natural for one to guess that it will be at this point that the image of these transition maps must be induced. We do indeed find this to be the case for torsion-free FI-modules.

Theorem 3.31 *Let G_\bullet be a torsion-free vertex-stable FI-graph with edge-stable degree $\leq d_E$. Then for any $n \geq d_E$ and any injection $f : [n] \hookrightarrow [n + 1]$ the image of G_n under the transition map $G(f)$ is an induced subgraph of G_{n+1} .*

While it might seem natural for there to be some kind of pigeon-hole or counting argument for the above theorem, such an argument has thus far eluded the authors. Just like much of the rest of this work, we instead prove Theorem 3.31 through the algebra of FI-modules. To begin, we must rephrase the eventually induced property in the language of FI-modules.

Definition 3.32 The **coinvariants functor** Φ from FI-modules to graded $\mathbb{R}[x]$ -modules is defined by

$$\Phi(V)_n := V_n \otimes_{\mathfrak{S}_n} \mathbb{R}$$

Multiplication by x is induced by the action of the transition maps.

In the setting of FI-graphs and their associated FI-modules, the coinvariants functor takes a particularly nice form.

Recall that we define $\mathbb{R}^{\binom{V(G_\bullet)}{2}}$ to be the FI-module encoding pairs of vertices of G_\bullet . The coinvariants of $\mathbb{R}^{\binom{V(G_\bullet)}{2}}$ can be constructed in the following way. We define

Φ to be the graded $\mathbb{R}[x]$ -module for which Φ_n is the free \mathbb{R} vector space with basis indexed by the orbits of the symmetric group action on pairs of vertices of G_n . For each n we may define $\iota_n : [n] \hookrightarrow [n + 1]$ to be the standard inclusion. Then $G(\iota_n)$ induces a map between the orbits of pairs of vertices of G_n and those of G_{n+1} . Multiplication by x in the module Φ will be defined by this map.

Lemma 3.33 *Let V be a finitely generated FI-module. If V is torsion-free as an FI-module, then $\Phi(V)$ is torsion-free as a $\mathbb{R}[x]$ -module.*

Proof This follows from the fact that coinvariants are exact over fields of characteristic 0. □

This lemma is the key piece needed to prove Theorem 3.31.

Proof of Theorem 3.31 Let G_\bullet be a torsion-free vertex-stable FI-graph, and assume that G_\bullet has edge-stable degree $\leq d_E$. Assume by way of contradiction that there is some $n \geq d_E$ such that the image of G_n under any transition map $G(f) : G_n \rightarrow G_{n+1}$ is not an induced subgraph. This implies that there is some pair of vertices $\{v_1, v_2\}$ in G_n , which are not connected by an edge, while $G(f)(\{v_1, v_2\})$ is an edge of G_{n+1} . On the other hand, because $n \geq d_E$, there must be some transition map $G(h)$, as well as some edge $e \in E(G_n)$ such that $G(h)(e) = G(f)(\{v_1, v_2\})$. We may apply some element of \mathfrak{S}_{n+1} to conclude the following: The transition map $G(f)$ must map some non-edge of G_n , as well as some edge of G_n , to the same \mathfrak{S}_{n+1} orbit on the pairs of vertices of G_{n+1} . In particular, this would imply that the coinvariants of $\mathbb{R}^{\binom{V(G_\bullet)}{2}}$ has torsion. This contradicts Lemma 3.33. □

4 Applications

In the following sections we prove the variety of applications of the primary structure theorem that were claimed in the introduction. Many of these proofs ultimately take the same form: one encodes the invariant or homology groups as the graded pieces of a finitely generated FI-module (over \mathbb{Z}). Finite generation in these cases is usually proven by embedding the FI-module into a larger FI-module which is known to be finitely generated, and then applying the Noetherian property. These “bigger” FI-modules which we embed into are almost always $\mathbb{R}^{\binom{V(G_\bullet)}{j}}$ (Remark 3.24), as well as tensor products of the modules $\mathbb{R}^{\binom{V(G_\bullet)}{j}}$ (Proposition 2.17).

4.1 Enumerative consequences of vertex-stability

We begin this section by revisiting the invariants η_H and η_H^{ind} for some fixed graph H . In particular, if G_\bullet is a vertex-stable FI-graph, we consider the functions

$$n \mapsto \eta_H(G_n) \text{ and } n \mapsto \eta_H^{ind}(G_n).$$

Our primary result in this direction is the following.

Theorem 4.1 *Let G_\bullet be a vertex-stable graph with stable degree $\leq d$. Then for any graph H there exists polynomials $p_H(X), p_H^{ind}(X) \in \mathbb{Q}[X]$ of degree $\leq d \cdot |V(H)|$ such that for all $n \gg 0$,*

$$p_H(n) = \eta_H(G_n) \text{ and } p_H^{ind}(n) = \eta_H^{ind}(G_n)$$

Proof We will count the number of graph injections from H to G_\bullet . This quantity is a constant multiple of $\eta_H(G_\bullet)$, and it is therefore sufficient to count. Let V_\bullet^H denote the FI-module whose evaluation at $[n]$ is the \mathbb{R} vector space with basis indexed by the distinct graph theoretic injections of H into G_n . To make sure the transition maps are well defined, we will set $V_n^H = 0$ before the point where the transition maps of G_\bullet are both induced and injective. We therefore see that V_\bullet^H can be realized as a submodule

$$V_\bullet^H \hookrightarrow (\mathbb{R}V(G_\bullet))^{\otimes |V(H)|}$$

via the point-wise assignment,

$$\phi \mapsto \bigotimes_{x \in V(H)} (\phi(x)).$$

Proposition 2.17 implies that V_\bullet^H is $(d \cdot |V(H)|)$ -small, and Theorem 2.18 implies the existence of our desired polynomial. The proof for the induced case is the same. \square

Example 4.2 Looking the FI-graph K_\bullet of complete graphs, the above result is clear. H cannot appear in K_n when $n < |V(H)|$. If we call γ_H the number of copies of H in $K_{|V(H)|}$, then

$$p_H(n) = \binom{n}{|V(H)|} \gamma_H$$

The content of Theorem 4.1 is that this behavior is common to all vertex-stable FI-graphs. The examples of the previous section illustrate that vertex-stable FI-graphs can be fairly diverse, and so this might come as a bit of a surprise.

Fix $k \geq 2$ and let $KG_{\bullet,k}$ be the FI-graph which encodes the Kneser graphs. In this case we may easily count the number of triangles which appear in $KG_{n,k}$. Indeed, to form a triangle, one needs to provide three mutually disjoint subsets of $[n]$ of size k . It follows that

$$p_{K_3}(n) = \frac{\binom{n}{3k} \binom{3k}{k,k,k}}{6}.$$

Note that if we take the usual convention that $\binom{n}{k} = 0$ whenever $n < k$, then the above polynomial agrees with $\eta_{K_3}(KG_{n,k})$ for all $n \geq 0$. If we instead try to count the number of occurrences of the six-vertex graph which looks like the letter H , things get considerably more complicated. Despite the seeming drastic increase in difficulty, Theorem 4.1 assures us that the value of $\eta_H(KG_{n,k})$ must (eventually) agree with a polynomial, and that this polynomial will have degree at most $6k$.

As an immediate corollary to the above, we find that vertex-stable FI-graphs have very controlled growth in their vertices and edges, as well as in the degrees of their vertices.

Corollary 4.3 *Let G_\bullet be a vertex-stable FI-graph. Then the following functions are each equal to a polynomial for $n \gg 0$:*

1. $n \mapsto |V(G_n)|$;
2. $n \mapsto |E(G_n)|$;
3. $n \mapsto \delta(G_n)$;
4. $n \mapsto \Delta(G_n)$.

If the stable degree of G_\bullet is at most d , then these polynomials have degree at most d in the first, third, and fourth cases, and degree at most $2d$ in the second case ($n \mapsto |E(G_n)|$).

Proof The first two statements follow from a direct application of Theorem 4.1 with H being an isolated vertex and a single edge, respectively.

For the final two statements, we prove a more general statement. Fix $m \gg 0$, and let $v \in V(G_m)$. Then every vertex in the \mathfrak{S}_m -orbit of v , which we denote $\mathcal{O}_v(m)$, has the same degree. Let $\mathbb{R}\mathcal{O}_v(\bullet)$ denote the submodule of $\mathbb{R}V(G_\bullet)$ generated by v , and for $n \geq m$ let $\mu(\mathcal{O}_v(n))$ denote the degree of any (and therefore all) vertices in $\mathcal{O}_v(n)$. We will prove that the map

$$n \mapsto \mu(\mathcal{O}_v(n))$$

is equal to a polynomial. To see that this implies the final two statements of our corollary, note that the final part of Theorem 2.18 implies that the total number of distinct orbits of $V(G_n)$ is eventually independent of n . Because non-equal polynomials are only permitted to be equal at finitely many points, the above implies that there is a well-defined polynomial which outputs the smallest (or largest) degree of any vertex of G_n when n is large enough.

To prove our more general claim, we need a bit of notation. We will write $\mathbb{R}E(\mathcal{O}_v(\bullet))$ for the submodule of $\mathbb{R}E(G_\bullet)$ whose n -th piece is spanned by edges whose both end points are in $\mathcal{O}_v(n)$. We will also write $N(\mathcal{O}_v(n))$ to denote the subgraph of G_n comprised of all vertices and edges that one may encounter by beginning at a vertex in $\mathcal{O}_v(n)$ and moving along any single edge adjacent to it. Put another way, $N(\mathcal{O}_v(n))$ is the neighborhood graph on the vertex set $\mathcal{O}_v(n)$. By setting $N(\mathcal{O}_v(n)) = \emptyset$ whenever $n < m$, we see that $N(\mathcal{O}_v(\bullet))$ is actually a vertex-stable FI-graph. Therefore, by the second part of this corollary,

$$n \mapsto |E(N(\mathcal{O}_v(n)))|$$

is eventually equal to a polynomial. On the other hand, we may count the set $|E(N(\mathcal{O}_v(n)))|$ in the following alternative way,

$$|E(N(\mathcal{O}_v(n)))| = \mu(\mathcal{O}_v(n)) \cdot |\mathcal{O}_v(n)| - |E(\mathcal{O}_v(n))|.$$

In other words, if we sum the degrees of all vertices in $\mathcal{O}_v(n)$, then we would have counted each edge in $E(\mathcal{O}_v(n))$ exactly twice. Because $\mathbb{R}\mathcal{O}_v(\bullet)$ is a submodule of $\mathbb{R}V(G_\bullet)$, we know that its dimension is eventually equal to a polynomial. A similar statement can also be made about $|E(\mathcal{O}_v(n))|$. Solving for $\mu(\mathcal{O}_v(n))$, we find that it is equal to a rational function for n sufficiently large. However, the only rational functions which can take integral values at all sufficiently large integers are polynomials. This concludes the proof. \square

Another consequence of vertex-stability concerns finite walks in the graph G_n . Recall that a **walk of length r** in a graph G is a tuple of vertices of G , (v_0, \dots, v_r) , such that for each i , $\{v_i, v_{i+1}\}$ is an edge of G . We say that a walk is **closed** if $v_r = v_0$.

Theorem 4.4 *Let G_\bullet be a vertex-stable FI-graph with stable degree $\leq d$. Then the following functions are each equal to a polynomial, of degree $\leq (r + 1)d$ in the first case and of degree $\leq rd$ in the second, for $n \gg 0$ and any fixed $r \geq 0$:*

1. $n \mapsto |\{\text{walks in } G_n \text{ of length } r\}|$;
2. $n \mapsto |\{\text{closed walks in } G_n \text{ of length } r\}|$.

Proof Our strategy here is similar to the strategy of much of the rest of the paper. Encode the objects we hope to count as the dimension of some vector space, and use the Noetherian property to prove that the collection of all these vector spaces form a finitely generated FI-module. Let $W_r(G_\bullet)$ denote the FI-module for which $W_r(G_n)$ is the formal vector space spanned by walks of length r in G_n . Similarly define $W_r^c(G_\bullet)$ for closed walks. Note that these FI-modules may not be well defined if the transition maps of G_\bullet are not injective. While injectivity may not be the case for small n , Theorem 3.18 implies that it certainly will be the case for $n \gg 0$. Therefore, we simply define $W_r(G_n)$, and $W_r^c(G_n)$ to be zero before injectivity takes effect. To prove that $W_r(G_\bullet)$ is finitely generated, we simply note that there is an embedding,

$$W_r(G_\bullet) \hookrightarrow \mathbb{R}V(G_\bullet)^{\otimes(r+1)}$$

defined on points by

$$(v_0, \dots, v_r) \mapsto v_0 \otimes v_1 \otimes \dots \otimes v_r$$

The module $\mathbb{R}V(G_\bullet)^{\otimes(r+1)}$ is finitely generated in degree $\leq (r + 1) \cdot d$ by Proposition 2.17. The Noetherian property concludes the proof. Note that in the case of closed walks, the fact that $v_0 = v_r$ removes a degree of freedom in our choices of vertices. In particular, we may embed $W_r^c(G_\bullet)$ into $\mathbb{R}V(G_\bullet)^{\otimes r}$. By consequence, the dimension of $W_r^c(G_\bullet)$ is bounded by a polynomial of degree $\leq rd$, as required. \square

The above work illustrates how certain invariants of G_n can grow with n . We also find, however, there are some invariants which must eventually stabilize.

Proposition 4.5 *Let G_\bullet denote a vertex-stable FI-graph. Then the following invariants are independent of n for $n \gg 0$:*

1. The diameter of G_n ;

2. The **girth** (i.e. the size of the smallest cycle) of G_n ;

Proof For both statements, it suffices to show that the relevant invariant is eventually weakly decreasing in n . If $n \gg 0$, and $u, v \in V(G_n)$, then by Theorem 3.18 there exists $x, y \in V(G_{n-1})$ and an injection $f : [n - 1] \hookrightarrow [n]$ such that $u = G(f)(x), v = G(f)(y)$. In particular, if P is any path in G_{n-1} connecting x and y , then $G(f)(P)$ is a path in G_n connecting u to v . This shows that the shortest path between u and v cannot be longer than the shortest path between x and y . By definition, the diameter of G_n cannot be bigger than the diameter of G_{n-1} . A similar argument works for girth. \square

4.2 Topological consequences of vertex-stability

In this section we consider a collection of topological applications of vertex-stability. Our first applications are simple consequences of the work in the previous section, as well as facts from our background sections.

Lemma 4.6 *Let H be a graph, G_\bullet a vertex-stable FI-graph of stable degree $\leq d$, and $\mathcal{C}_{n,i}^H$ denote the free \mathbb{Z} -module with basis indexed by the i -cells of $\mathcal{H}om(H, G_n)$. Then $\mathcal{C}_{\bullet,i}^H$ can be endowed with the structure of a finitely generated FI-module over \mathbb{Z} which is $(d(|V(H)| + i))$ -small.*

Proof We first recall the definition of the Hom-complex $\mathcal{H}om(H, G_n)$. The cells of $\mathcal{H}om(H, G_n)$ are multi-homomorphisms, where α is contained in the closure of τ if and only if $\alpha(x) \subseteq \tau(x)$ for all x . It is clear that the transition maps of G_\bullet induce the transition maps of $\mathcal{C}_{\bullet,i}^H$, turning this collection of abelian groups into an FI-module over \mathbb{Z} .

We have that i -cells of the complex correspond to multi-homomorphisms α satisfying

$$\sum_{x \in V(H)} |\alpha(x)| = |V(H)| + i$$

The data of an i -cell can therefore be encoded as an $|V(H)|$ -tuple

$$(\alpha_x)_{x \in V(H)}$$

such that:

- Each α_x is a non-empty subset of $V(G_n)$;
- $\sum_{x \in V(H)} |\alpha_x| = |V(H)| + i$;
- If $\{x, y\} \in E(H)$ then for all $v \in \alpha_x$ and $w \in \alpha_y, \{v, w\} \in E(G_n)$.

Just as we have done previously, such as in the proof of Theorem 3.18, we may realize $\mathcal{C}_{\bullet,i}^H$ as a submodule of

$$\bigoplus_{r_1 + \dots + r_{|V(H)|} = |V(H)| + i} \left(\bigotimes_k \mathbb{R} \binom{V(G_\bullet)}{r_k} \right)$$

The Noetherian property, as well as previous discovered facts about the modules $\mathbb{R}(\binom{V(G_\bullet)}{j})$ (see the proof of Theorem 3.18) imply our lemma. \square

Lemma 4.6 is the main tool we will need in proving that Hom-complexes of vertex-stable FI-graphs are representation stable in the sense of Church and Farb. Before we get to this theorem, we observe the following consequence of the above in terms of counting homomorphisms into G_\bullet .

Corollary 4.7 *Let H be any graph, and let G_\bullet be a vertex-stable FI-graph of stable degree $\leq d$. Then for $n \gg 0$, then the function*

$$n \mapsto |\text{Hom}(H, G_\bullet)|$$

agrees with a polynomial of degree $\leq d \cdot |V(H)|$.

Proof This follows from Theorem 2.18, Lemma 4.6, and the fact that the module $\mathcal{C}_{n,0}^H$ from Lemma 4.6 has basis indexed by $\text{Hom}(H, G_n)$. \square

Remark 4.8 It is easily seen that homomorphisms $\text{Hom}(H, K_n)$ are in bijection with vertex colorings of H for which no adjacent vertices are of the same color. The above corollary therefore recovers the existence of the so-called **chromatic polynomial**. Note that the chromatic polynomial exists for all $n \geq 0$, while the above only guarantees it for $n \gg 0$. One can recover the fact that the chromatic polynomial exists for $n \geq 0$ by showing that the collection of vector spaces $\mathbb{R} \text{Hom}(H, K_\bullet)$ can be endowed with the structure of an $\text{FI}_\#$ -module (see [10]).

Note that a similar idea, i.e. using FI-module techniques to recover the chromatic polynomial, was conveyed to the authors by John Wiltshire-Gordon and Jordan Ellenberg. This alternative technique was very similar in spirit, but used FA-modules instead of FI-modules. Here, FA is the category of finite sets and all maps (see, for instance, [42]).

Theorem 4.9 *Let H be a graph, G_\bullet a vertex-stable FI-graph of stable degree $\leq d$, and let $i \geq 0$ be a fixed integer. Then the FI-module over \mathbb{Z}*

$$H_i(\mathcal{H}om(H, G_\bullet))$$

is $(d(|V(H)| + i))$ -small.

Proof Recall the groups $\mathcal{C}_{n,i}^H$ from the Lemma 4.6. Standard polyhedral homology informs us that there is a complex,

$$\mathcal{C}_{n,\star}^H : \dots \rightarrow \mathcal{C}_{n,i}^H \xrightarrow{\partial} \mathcal{C}_{n,i-1}^H \rightarrow \dots \rightarrow \mathcal{C}_{n,0}^H \rightarrow 0$$

with homology isomorphic to $H_\star(\mathcal{H}om(H, G_n))$. Lemma 4.6 tells us that for each fixed i the groups $\mathcal{C}_{\bullet,i}^H$ form a finitely generated FI-module over \mathbb{Z} . It isn't hard to show that the action of the transition maps of $\mathcal{C}_{\bullet,i}^H$ commute with the differentials ∂ . It follows that there is a complex of FI-modules over \mathbb{Z}

$$\mathcal{C}_{\bullet,\star}^H : \dots \rightarrow \mathcal{C}_{\bullet,i}^H \xrightarrow{\partial} \mathcal{C}_{\bullet,i-1}^H \rightarrow \dots \rightarrow \mathcal{C}_{\bullet,0}^H \rightarrow 0$$

whose homology agrees with the FI-modules $H_i(\mathcal{H}om(H, G_\bullet))$. The Noetherian property and Lemma 4.6 imply our result. \square

To conclude this section, we review some fundamental concepts and definitions which will be used in the proof of Theorem G.

For the remainder of this section, we fix a vertex-stable, torsion-free FI-graph G_\bullet as well as a positive integer m . We will assume that G_\bullet has stable degree $\leq d$ and edge-stable degree $\leq d_E$.

To begin, we note that the necessary edge subdivisions of Theorem 2.9 can be accomplished in a way consistent with the FI-module structure of G_\bullet .

Proposition 4.10 *There exists an FI-graph, $G_\bullet^{(m)}$, for which $G_n^{(m)}$ is the m -th subdivision of G_n and for any injection of sets $f : [n] \hookrightarrow [r]$ one has*

$$G^{(m)}(f)(x) = G(f)(x)$$

for all $x \in V(G_n)$. If G_\bullet has stable degree $\leq d$ and edge-stable degree $\leq d_E$, then $G_\bullet^{(m)}$ has stable degree $\leq \max\{d, d_E\}$ and edge-stable degree $\leq d_E$.

Proof The existence of $G_\bullet^{(m)}$ follows from the definition of the m -th subdivision. If $x, y \in V(G_n)$ are connected via an edge, then the same must be true of $G(f)(x)$ and $G(f)(y)$, for any f . Therefore, if we enumerate the subdivision vertices of $\{x, y\}$ (resp. $\{G(f)(x), G(f)(y)\}$) as v_1, \dots, v_{m-1} (resp. w_1, \dots, w_{m-1}), where v_1 is adjacent to x (resp. w_1 to $G(f)(x)$), then we set $G(f)(v_i) = w_i$. The statement on stable degrees follows from the fact that subdivision creates new vertices and edges within existing edges. \square

This proposition will prove to be critical for us, as it essentially asserts, with Theorem 2.9, that there exists a combinatorial model of $\text{Conf}_m(G_\bullet)$ which interacts nicely with the FI-graph structure of G_\bullet .

We are now ready to provide the main novel computational construction of this section. Recall that we have fixed a vertex-stable torsion-free FI-graph G_\bullet .

Definition 4.11 Fix integers $m, n, i \geq 0$. We write $\mathcal{K}_{n,m,i}$ to denote the free \mathbb{Z} -module with basis vectors indexed by the i -dimensional cells of the cubical complex $\text{DConf}_m(G_n^{(m)})$. Given any injection of sets $f : [n] \hookrightarrow [r]$, Proposition 4.10 implies that the transition map $G(f)$ induces a transition map $G_n^{(m)} \rightarrow G_r^{(m)}$, which, in turn, induces a map

$$f_* : \mathcal{K}_{n,m,i} \rightarrow \mathcal{K}_{r,m,i}.$$

This procedure equips the family $\{\mathcal{K}_{n,m,i}\}_n$ with the structure of an FI-module over \mathbb{Z} .

Having observed the FI-module structure on the families $\mathcal{K}_{\bullet,m,i}$, the strategy of our proof of Theorem G becomes clear. We begin by proving that, for all choices of m and i , the FI-module $\mathcal{K}_{\bullet,m,i}$ is finitely generated. In fact, we will prove that $\mathcal{K}_{\bullet,m,i}$ is

$(\max\{d, d_E\}(m - i) + d_E i)$ -small, where d is the stable degree of G_\bullet . Following this one notes that the action of FI on the collection $\{K_{n,m,i}\}_n$ commutes with the usual differentials

$$\partial_{n,i,m} : K_{n,m,i} \rightarrow K_{n,m,i-1}.$$

This implies that the collection of complexes

$$\cdots \rightarrow K_{n,m,i} \rightarrow K_{n,m,i-1} \rightarrow \cdots \rightarrow K_{n,m,0} \rightarrow 0$$

can be pieced together to form a complex of FI-modules. The Noetherian property is then sufficient for us to prove the main theorem.

We observe that this approach has the downside that it cannot be used to estimate the generating degree of the FI-module over \mathbb{Z} , $H_i(\text{Conf}_m(G_\bullet))$. It is the belief of the authors that proving a result of this kind will require a deeper topological understanding of the spaces $\text{Conf}_m(G_n)$ as n -varies. This seems like a rich avenue for future research, as surprisingly little is thus far understood about these spaces.

Theorem 4.12 *Assume that G_\bullet has stable degree $\leq d$ and edge-stable degree $\leq d_E$. Then for all choices of $m, i \geq 0$, the FI-module over \mathbb{Z} , $\mathcal{K}_{\bullet,m,i}$, is $(\max\{d, d_E\}(m - i) + d_E i)$ -small. In the case wherein G_n is connected for $n \gg 0$, the module $\mathcal{K}_{\bullet,m,i}$ is $d_E m$ -small.*

Proof Note that it follows from definition that $\mathbb{R}V(G_\bullet)$ is generated in degrees $\leq d$, while $\mathbb{R}E(G_\bullet)$ is generated in degrees $\leq d_E$. Propositions 2.17 and 4.10 imply that the FI-module

$$Q_i := \bigoplus_{f:[n] \rightarrow \{V, E\}, |f^{-1}(E)|=i} \bigotimes_{j=1}^n Q_{f,j}$$

is generated in degrees $\leq \max\{d, d_E\}(m - i) + d_E i$, where,

$$Q_{f,j} = \begin{cases} \mathbb{R}E(G_\bullet^{(m)}) & \text{if } f(j) = E \\ \mathbb{R}V(G_\bullet^{(m)}) & \text{otherwise.} \end{cases}$$

We have that $\mathcal{K}_{\bullet,m,i}$ is a submodule of Q_i , whence it is $(\max\{d, d_E\}(m - i) + d_E i)$ -small.

In the case where G_n is connected for $n \gg 0$, it is a fact that $d_E = \max\{d, d_E\}$. This concludes the proof. \square

Example 4.13 Let G_\bullet be the FI-graph of Example 3.9. Then the above implies that the Betti numbers of $H_i(\text{Conf}_m(G_n))$ eventually agree with a polynomial of degree $\leq m$. This bound is sharp for $m \geq 2$, and $i = 1$ in the case wherein G is a single point, and H is an edge. Namely, the case where $G_\bullet = \text{Star}_\bullet$ (see [23] for this computation).

4.3 Algebraic consequences of vertex-stability

In this section, we consider adjacency and Laplacian matrices associated to an FI-graph. We focus on properties of the eigenspaces associated to these matrices.

To begin, note that we may view the adjacency and Laplacian matrices of a graph G as linear endomorphisms of $\mathbb{R}V(G)$. Given an FI-graph G_\bullet , it is unfortunately not the case that the collections A_{G_\bullet} and L_{G_\bullet} can be considered as endomorphisms of the FI-module $\mathbb{R}V(G_\bullet)$. Despite this, we will find that these matrices have some surprising interactions with the FI-module structures. To begin, we have the following key observation.

Lemma 4.14 *Let G_\bullet be an FI-graph. Then for each n the matrices A_{G_n} and L_{G_n} commute with the action of \mathfrak{S}_n . In particular, the eigenspaces of these matrices are sub-representations of $\mathbb{R}V(G_n)$.*

Proof For a fixed vertex $v \in V(G_n)$, we write $N(v)$ to denote the collection of vertices adjacent to v . Then,

$$A_{G_n}v = \sum_{w \in N(v)} w.$$

Therefore if $\sigma \in \mathfrak{S}_n$,

$$A_{G_n}\sigma(v) = \sum_{w \in N(\sigma(v))} w = \sum_{w' \in N(v)} \sigma(w') = \sigma(A_{G_n}v).$$

The same proof works for the Laplacian matrix.

The second half of the lemma follows from linear algebra. If two matrices commute, then they preserve each others' eigenspaces. □

As an immediate consequence of Lemma 4.14, we obtain the following:

Proposition 4.15 *Let G_\bullet be a vertex-stable FI-graph. Then there is a constant c , independent of n , such that the number of distinct eigenvalues of the adjacency matrix (resp. the Laplacian) of G_n is bounded by c for all n .*

Proof This follows from Lemma 4.14 as well as the third part of Theorem 2.18. In particular, we see that the number of distinct eigenvalues are both bounded from above by the (eventually constant) number of irreducible factors of the \mathfrak{S}_n -representation $\mathbb{R}V(G_n)$. □

Remark 4.16 Proposition 4.15 can be used in certain cases to prove that certain families of graphs *cannot* be endowed with the structure of a vertex-stable FI-graph. For example, the cycle graphs C_n and the wheel graphs W_n have n distinct eigenvalues.

In fact Proposition 4.15 is the first piece of evidence describing a much more robust structure. The following theorem follows as a consequence of upcoming work of David Speyer and the authors [36].

Theorem 4.17 [36] *Let G_\bullet denote a vertex-stable FI-graph. Then there exist constants c_A, c_L such that for all $n \gg 0$, A_{G_n} (resp. L_{G_n}) has c_A (resp. c_L) distinct eigenvalues. For $i = 1, \dots, c_A$ (resp. $i = 1, \dots, c_L$) and $n \gg 0$, let $\lambda_i^A(n)$ (resp. $\lambda_i^L(n)$) denote the i -th largest eigenvalue of A_{G_n} (resp. L_{G_n}). Then the for all i and all $n \gg 0$ the functions*

$$n \mapsto \text{the multiplicity of } \lambda_i^A(n), \quad n \mapsto \text{the multiplicity of } \lambda_i^L(n)$$

each agree with a polynomial.

Example 4.18 We illustrate the above theorem with some examples. Let $G_\bullet = K_\bullet$ denote the FI-graph of complete graphs. Then for $n \geq 1$, the \mathfrak{S}_n -representation $\mathbb{R}V(G_n)$ is isomorphic to the usual permutation representation on \mathbb{R}^n . This decomposes into a pair of irreducible representations

$$\mathbb{R}^n \cong \mathbb{R} \oplus S_n,$$

where \mathbb{R} is the trivial representation, and S_n is the standard irreducible $(n - 1)$ -dimensional representation of \mathfrak{S}_n . We note that the decomposition $\mathbb{R}^n \cong \mathbb{R} \oplus S_n$ agrees with the eigenspace decomposition of $\mathbb{R}V(G_n)$ with respect to both the adjacency matrix and the Laplacian matrix. The trivial representation is the eigenspace for $n - 1$ (resp. 0), while S_n is the eigenspace for -1 (resp. $-n$). It is easy to see that the collection S_\bullet actually forms a submodule of the FI-module $\mathbb{R}V(G_n)$, and is therefore finitely generated. This implies that $\dim_{\mathbb{R}} S_n$ agrees with a polynomial for $n \gg 0$, which implies the same about the eigenvalue multiplicities in question.

Next let $G_\bullet = \text{Star}_\bullet = K_{\bullet,1}$. For simplicity we only work with the eigenspaces for the adjacency matrix, although the Laplacian is not much different. For $n \geq 1$, the distinct eigenvalues of G_n are $\pm\sqrt{n}$ and 0. We may decompose the representation $\mathbb{R}V(G_n)$ as

$$\mathbb{R}V(G_n) = \mathbb{R} \oplus \mathbb{R} \oplus S_n$$

where S_n is as in the previous example, and \mathbb{R} is once again the trivial representation. As before, this decomposition of $\mathbb{R}V(G_n)$ as a representation corresponds exactly to its decomposition in terms of eigenspaces.

Of course, one should not expect these eigenspaces to be irreducible as \mathfrak{S}_n -representations in general. For example if we instead consider $G_\bullet = K_{\bullet,m}$, where $m > 1$, then the eigenspaces of the adjacency matrix are not all irreducible as \mathfrak{S}_n -representations. Generally, one will find it much easier to use the fact that the adjacency and Laplacian matrices preserve isotypic components of $\mathbb{R}V(G_n)$ to work with these submodules instead of the eigenspaces themselves. Indeed, the proof of the previous theorem involves working with a filtration of the FI-module $\mathbb{R}V(G_\bullet)$ by its isotypic pieces.

5 Generalizations and alterations

In this final section, we briefly discuss how the work of the previous sections can be generalized and altered to prove facts about different families of graphs and other simplicial complexes. We begin by considering graphs over categories other than FI, and then move on to higher dimensional analogues to the previous work. Note that these sections are intended to be more motivation for further study, and should by no means be considered exhaustive.

5.1 Other categories

The representation theory of categories has seen a recent explosion in the literature, largely motivated by its connections with representation stability. In this section we consider representations of the categories $VI(q)$, where q is a power of a prime, and FI^m , where m is a positive integer. These categories can be seen discussed in [20,24,31,34,40].

Definition 5.1 Let m be a fixed positive integer, and let q be a power of a fixed prime p . The category $VI(q)$ is that whose objects are free vector spaces over the finite field \mathbb{F}_q , and whose morphisms are injective linear maps. The category FI^m is defined to be the categorical product of FI with itself m times. That is, it is the category whose objects are m -tuples of non-negative integers (n_1, \dots, n_m) , and whose morphisms are m -tuples of injective maps $(f_1, \dots, f_m) : [n_1] \times \dots \times [n_m] \hookrightarrow [n'_1] \times \dots \times [n'_m]$, where $f_i : [n_i] \rightarrow [n'_i]$.

One may think of $VI(q)$ as an analog of FI, where the relevant acting groups are the finite general linear groups $GL(n, q)$. Similarly, FI^m is the analog of FI where the relevant acting groups are $\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_m}$. Just as with FI, a module over either of these categories will be defined to be a functor from the category to \mathbb{R} vector spaces. Definitions such as finite generation carry over in the obvious way

The following facts can be found in [20,24,31,34,40].

Theorem 5.2 Let \mathcal{C} denote either the category FI^m or $VI(q)$. Then:

1. [20,31] If $\mathcal{C} = FI^m$, and V is a finitely generated \mathcal{C} -module, then there exists a polynomial $p_V(x_1, \dots, x_m) \in \mathbb{Q}[x_1, \dots, x_m]$ such that for all (n_1, \dots, n_m) with $\sum_i n_i \gg 0$,

$$\dim_{\mathbb{R}}(V_{n_1, \dots, n_m}) = p_V(n_1, \dots, n_m)$$

2. [20,40] If V, W are finitely generated \mathcal{C} -modules, then the same is true of $V \otimes W$.
3. [24] If V is a finitely generated $VI(q)$ -module, then there exists a polynomial $p_V(x) \in \mathbb{Q}[x]$ such that for all $n \gg 0$

$$p_V(q^n) = \dim_{\mathbb{R}} V(\mathbb{F}_q^n).$$

4. If V is a finitely generated \mathcal{C} -module, then the transition maps of V are all eventually injective.

5. [34,40] If V is a finitely generated \mathcal{C} -module, then all submodules of V are also finitely generated.

As one can see, these two categories have very similar properties to FI-modules. Indeed, it is sufficient for us to recover virtually everything that was proven in previous sections.

Definition 5.3 Let \mathcal{C} denote either the category FI^m or $\text{VI}(q)$. Then a \mathcal{C} -**graph** is a functor $G_\bullet : \mathcal{C} \rightarrow \mathbf{Graph}$. We say that G_\bullet is **vertex-stable** if the associated \mathcal{C} -module $\mathbb{R}V(G_\bullet)$ is finitely generated.

Borrowing notation and proofs from the previous sections, we conclude the following.

Theorem 5.4 Let \mathcal{C} denote either $\text{VI}(q)$ or FI^m , and let G_\bullet be a vertex-stable \mathcal{C} -graph. Then:

1. The \mathcal{C} -module $\mathbb{R}E(G_\bullet)$ is finitely generated;
2. For any $r \geq 1$, the \mathcal{C} -module $\mathbb{R}\binom{V(G_\bullet)}{r}$ is finitely generated;
3. If $\mathcal{C} = \text{FI}^m$, and H is any fixed graph, then there exist polynomials $p_H(x_1, \dots, x_m)$, $p_H^{ind}(x_1, \dots, x_m) \in \mathbb{R}[x]$ such that for all $\mathbf{n} := (n_1, \dots, n_m)$ with $\sum_i n_i \gg 0$

$$p_H(\mathbf{n}) = \eta_H(G_{\mathbf{n}}), \text{ and } p_H^{ind}(\mathbf{n}) = \eta_H^{ind}(G_{\mathbf{n}})$$

4. If $\mathcal{C} = \text{VI}(q)$, and H is any fixed graph, then there exist polynomials $p_H(x)$, $p_H^{ind}(x) \in \mathbb{Q}[x]$ such that for all $n \gg 0$

$$p_H(q^n) = \eta_H(G_{\mathbb{F}_q^n}), \text{ and } p_H^{ind}(q^n) = \eta_H^{ind}(G_{\mathbb{F}_q^n})$$

5. If $\mathcal{C} = \text{FI}^m$, and $r \geq 1$ is fixed, then there exist polynomials $p_r(x)$, $p_r^c(x) \in \mathbb{Q}[x_1, \dots, x_m]$ such that for all $\mathbf{n} := (n_1, \dots, n_m)$ with $\sum_i n_i \gg 0$

$$\begin{aligned} p_r(\mathbf{n}) &= |\{\text{number of walks in } G_{\mathbf{n}} \text{ of length } r\}|, \text{ and } p_r^c(\mathbf{n}) \\ &= |\{\text{number of closed walks in } G_{\mathbf{n}} \text{ of length } r\}|. \end{aligned}$$

6. If $\mathcal{C} = \text{VI}(q)$, and $r \geq 1$ is fixed, then there exist polynomials $p_r(x)$, $p_r^c(x) \in \mathbb{Q}[x]$ such that for all $n \gg 0$

$$\begin{aligned} p_r(q^n) &= |\{\text{number of walks in } G_n \text{ of length } r\}|, \text{ and } p_r^c(q^n) \\ &= |\{\text{number of closed walks in } G_n \text{ of length } r\}| \end{aligned}$$

7. For any fixed i , and any fixed graph H the \mathcal{C} -module over \mathbb{Z} , $H_i(\text{Hom}(H, G_\bullet))$, is finitely generated;
8. If G_\bullet is torsion-free then for any fixed m, i the \mathcal{C} -module over \mathbb{Z} , $H_i(\text{Conf}_m(G_n))$, is finitely generated.

To conclude this section, we consider various natural examples of FI^m and $\text{VI}(q)$ graphs. The reader should keep in mind Theorem 5.4 while reading what follows.

Example 5.5 Recall that for fixed m , we considered the vertex-stable FI-graph $K_{\bullet, m}$. While this yielded various results, it is perhaps more correct to allow m to vary, and consider the vertex-stable FI²-graph K_{\bullet_1, \bullet_2} . More generally, we can consider the complete r -partite graph $K_{\bullet_1, \dots, \bullet_r}$ as a vertex-stable FI ^{r} -graph.

If G, H are any graphs, then there are multiple ways one can define the product of G and H . One such method is with the tensor (or categorical) product $G \times H$. The graph $G \times H$ is that whose vertex set is given by $V(G \times H) = V(G) \times V(H)$ and for which $\{(x_1, y_1), (x_2, y_2)\} \in E(G \times H)$ if and only if $\{x_1, x_2\}, \{y_1, y_2\} \in E(G) \cup E(H)$. If G_{\bullet} and H_{\bullet} are two vertex-stable FI-graphs, then we may define the FI²-graph $G_{\bullet} \times H_{\bullet}$ by the assignments

$$(G_{\bullet} \times H_{\bullet})_{n_1, n_2} = G_{n_1} \times H_{n_2}.$$

It is clear that this family is vertex-stable as an FI²-graph. Note that a similar statement will hold for many of the other common graph products such as strong products and Cartesian products (see any standard reference on algebraic graph theory for definitions of these products such as [3]).

Turning our attention to VI(q), one is immediately reminded of the Grassmann graphs $J_q(n, k)$. The vertices of $J_q(n, k)$ are k -dimensional subspaces of \mathbb{F}_q^n , and two vertices form an edge if and only if the intersection of the corresponding subspaces is non-trivial. Note that one may think of $J_q(n, k)$ is a “ q -version” of the Johnson graph $J(n, k)$. In fact, many of the FI-graphs we previously studied will have associated VI(q)-graphs. For instance, we may define $KG_q(n, k)$ to be the graph whose vertices are subspaces of \mathbb{F}_q^n of dimension k , and for which two vertices are connected if and only if their corresponding subspaces have trivial intersection.

5.2 FI-simplicial-complexes

In this section, we generalize the work of the previous sections to higher dimensional simplicial complexes.

Definition 5.6 Let X be a (compact) simplicial complex. We will write $V_i(X)$ for the set of i -simplices of X . A **simplicial map** between simplicial complexes X, Y is a continuous morphism $f : X \rightarrow Y$ such that $f(V_i(X)) \subseteq V_i(Y)$.

An **FI-simplicial-complex** is a (covariant) functor from FI to the category of simplicial complexes and simplicial maps. Given an FI-simplicial complex X_{\bullet} , we write $\mathbb{R}V_0(X_{\bullet})$ for the FI-module whose evaluation at $[n]$ is the vector space with basis indexed by $\mathbb{R}V_0(X_n)$. We similarly define the FI-modules $\mathbb{R}V_i(X_{\bullet})$, and $\binom{V_i(X_{\bullet})}{r}$ for all $i, r \geq 0$.

We say that an FI-simplicial complex X_{\bullet} is **vertex-stable with stable degree $\leq d$** if the FI-module $\mathbb{R}V_0(X_{\bullet})$ is finitely generated in degree $\leq d$.

The following theorem is proven in the exact same way as Theorem 3.18.

Theorem 5.7 *Let X_{\bullet} be a vertex-stable FI-simplicial-complex with stable degree $\leq d$. Then:*

1. For all i the FI-modules $\mathbb{R}V_i(X_\bullet)$ are $(d(i + 1))$ -small;
2. For all i, r the FI-modules $\binom{V_i(X_\bullet)}{r}$ are $(rd(i + 1))$ -small;

From the perspective of representation stability, the above reveals something a bit striking about FI-simplicial-complexes. Note that the following was also true about graphs, though in that case it is less interesting.

Corollary 5.8 *Let X_\bullet be a vertex-stable FI-simplicial-complex with stable degree $\leq d$. Then for all $i \geq 0$ the FI-module over \mathbb{Z}*

$$H_i(X_\bullet)$$

is $d(i + 1)$ -small.

Proof This follows from the first part of Theorem 5.7, as well as the usual complex for computing simplicial homology and the Noetherian property of FI-modules. \square

Another interesting corollary of Theorem 5.7 relates to counting colorings of a simplicial complex T . Just as in the graph case, the colorings we will consider are intimately linked with simplicial maps into a certain FI-simplicial-complex. To begin, we therefore note the following.

Theorem 5.9 *Let X_\bullet denote a vertex-stable FI-simplicial-complex. Then for any simplicial complex T , the FI-module,*

$$\mathbb{R} \operatorname{Hom}(T, X_\bullet),$$

whose evaluation on $[n]$ is the real vector space with basis indexed by $\operatorname{Hom}(T, X_n)$, is finitely generated. In particular, for $n \gg 0$, the function

$$n \mapsto |\operatorname{Hom}(T, X_n)|$$

agrees with a polynomial of degree $\leq |V_0(T)| \cdot d$, where d is the vertex degree of X_\bullet .

Proof The FI-module $\mathbb{R} \operatorname{Hom}(T, X_\bullet)$ can be realized as a submodule of $(\mathbb{R}V_0(X_\bullet))^{\otimes |V_0(T)|}$. This module is finitely generated by Proposition 2.17, so the Noetherian property implies our result. The bound on the degree of the polynomial follows immediately from the fact that we have embedded our FI-module into the vector space $(\mathbb{R}V_0(X_n))^{\otimes |V_0(T)|}$, whose dimension agrees with a polynomial of the required degree. \square

Definition 5.10 Let (r, s) be a pair of positive integers. An (r, s) -coloring of a simplicial complex T is a map of sets $f : V_0(T) \rightarrow [r]$ such that if $\{v_0, \dots, v_i\} \in V_i(T)$, then at most s of the vertices v_0, \dots, v_i share the same color.

Colorings of simplicial complexes have recently seen interest in the literature, and seem to have deep connections with Stanley–Reisner theory [6,18].

Corollary 5.11 *Let T be a simplicial complex, and let $s \geq 1$ be an integer. If we write $V_{\bullet}^{T,s}$ to denote the FI-module whose evaluation on $[r]$ is the real vector space with basis indexed by (r, s) -colorings of T , then $V_{\bullet}^{T,s}$ is finitely generated and $|V_0(T)|$ -small. In particular, for $r \gg 0$, the function*

$$r \mapsto |\{(r, s)\text{-colorings of } T\}|$$

agrees with a polynomial of degree $\leq |V_0(T)|$.

Proof Let X_{\bullet} denote the FI-simplicial-complex for which $V_0(X_r) = [r] \times V_0(T)$, and for which the i -cells are given by all possible collections of pairs of size i . It is clear that X_{\bullet} is vertex-stable with vertex degree 1, whence $\mathbb{R} \operatorname{Hom}(T, X_{\bullet})$ is finitely generated. Given an element of $V_r^{T,s}$, we may associate a morphism $f \in \operatorname{Hom}(T, X_r)$ by assigning

$$f(v) = (i_v, v)$$

where i_v is the color of the vertex $v \in V_0(T)$. This defines an injective map of FI-modules

$$V_{\bullet}^{T,s} \hookrightarrow \mathbb{R} \operatorname{Hom}(T, X_{\bullet})$$

as desired. From the proof of Theorem 5.9, we see that $\mathbb{R} \operatorname{Hom}(T, X_{\bullet})$ is $|V_0(T)|$ -small, and so we conclude the same about $V_{\bullet}^{T,s}$. \square

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