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On classical upper bounds for slice genera

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Abstract
We introduce a new link invariant called the algebraic genus, which gives an upper bound for the topological slice genus of links. In fact, the algebraic genus is an upper bound for another version of the slice genus proposed here: the minimal genus of a surface in the four-ball whose complement has infinite cyclic fundamental group. We characterize the algebraic genus in terms of cobordisms in three-space, and explore the connections to other knot invariants related to the Seifert form, the Blanchfield form, knot genera and unknotting. Employing Casson-Gordon invariants, we discuss the algebraic genus as a candidate for the optimal upper bound for the topological slice genus that is determined by the S-equivalence class of Seifert matrices.

Keywords Slice genus · Seifert form · Casson-Gordon invariants · Algebraic unknotting number

Mathematics Subject Classification 57M25 · 57M27

1 Introduction

In this paper we introduce the notion of the *algebraic genus* of a link L in S^3, denoted by $g_{\text{alg}}(L)$. The main interest in g_{alg} is that it provides an upper bound for the $\mathbb{Z}$-*slice genus* $g_{\mathbb{Z}}(L)$ of a link L—the smallest genus of an oriented connected properly embedded locally flat surface F in the 4–ball B^4 with oriented boundary $L \subset \partial B^4 = S^3$ and $\pi_1(B^4 \setminus F) \cong \mathbb{Z}$.

Theorem 1 *For all links L, $g_{\mathbb{Z}}(L) \leq g_{\text{alg}}(L)$.*

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Postponing a more conceptual definition to Sect. 2, we let the *algebraic genus* $g_{\text{alg}}(L)$ of a link L with $r > 0$ components be defined by

$$\min\left\{ \frac{m-r+1}{2} - n \;\middle|\; \begin{array}{l} L \text{ admits an } m \times m \text{ Seifert matrix of the form} \\ \begin{pmatrix} A & * \\ * & * \end{pmatrix}, \text{ where } A \text{ is a top-left } 2n \times 2n \\ \text{submatrix with } \det(tA - A^\top) = t^n \end{array} \right\}.$$

We will establish (see Theorem 2) that the algebraic genus can be characterized as the *3–dimensional cobordism distance* to the set of knots with Alexander polynomial 1, and that it is related to a number of known knot invariants such as the algebraic unknotting number (see Theorem 3). However, in our opinion, what makes the algebraic genus most worth considering are the following two questions. We conjecture that both of them have a positive answer.

Question A *Does $g_{\text{alg}}(L) = g_{\mathbb{Z}}(L)$ hold for all links L, i.e. is the inequality in Theorem 1 an equality?*

Question B *Is the algebraic genus the best upper bound for the topological slice genus of a link L determined by the S-equivalence class of the Seifert matrices of L? More precisely, is it true for all links L that $g_{\text{alg}}(L) = \max\{g_{\text{top}}(L') \mid$ The Seifert matrices of L' are S-equivalent to those of $L\}$?*

1.1 The disk embedding theorem and other context for the above questions

Freedman's celebrated disk embedding theorem [15,16] implies that a locally-flat 2–sphere S in S^4 is unknotted (i.e. bounds an embedded locally flat 3–ball) if and only if its complement satisfies $\pi_1(S^4 \setminus S) \cong \mathbb{Z}$ [16, Theorem 11.7A]. This makes the study of surfaces with that fundamental group condition rather natural.

In the relative case of disks bounding knots, Freedman established the following [15] [16, Theorem 11.7B], which in fact is the only consequence of the disk embedding theorem that we will use in this text.

A knot K has Alexander polynomial 1 if and only if it bounds a properly embedded locally flat disk in the 4–ball. (1)

In terms of the invariants we introduce in this text, (1) may be written as

$$g_{\mathbb{Z}}(K) = 0 \Leftrightarrow g_{\text{alg}}(K) = 0.$$

This gives a positive answer to the simplest case of Question A.

A positive answer to Question A in general would show that $g_{\mathbb{Z}}$ is a *classical* link invariant in the sense of [6]: a link invariant is classical if it only depends on the S-equivalence class of Seifert matrices of L. Such a simple—in particular 3–dimensional—characterization of $g_{\mathbb{Z}}$ would a priori be surprising. For example, we note that such a characterization is impossible for the more extensively studied *topological slice genus* $g_{\text{top}}(L)$ of a link L—the smallest genus of an oriented connected properly embedded locally flat surface F in the 4–ball B^4 with oriented boundary $L \subset \partial B^4 = S^3$. This is because there are pairs K, K' of knots with the same Seifert form such that K is topologically slice while K' is not, i.e. $g_{\mathrm{top}}(K) = 0$ and $g_{\mathrm{top}}(K') > 0$. Such examples of knots K and K' were first found by Casson and Gordon using what are now known as *Casson-Gordon invariants* [7,8].

In Sect. 1.6, we will see how Gilmer's lower bounds derived from Casson-Gordon invariants [18] can be used to obtain a partial answer to Question B. Positive answers to both Questions A and B would yield a rather satisfying understanding of the possible slice genera of links with a given S-equivalence class: the maximal upper bound in terms of Seifert forms is attained and it is equal to a version of the slice genus with a natural condition on π_1. This fits with the following important point about invariants that depend on more than just the S-equivalence class such as the Casson-Gordon invariants and L^2-signatures (as used by Cochran, Orr, and Teichner [11]). Namely, these obstructions involve subtle questions concerning the extension of representations of π_1 of knot complements to π_1 of the complements of surfaces in B^4 bounding the knot; an issue that completely disappears when the latter complement has cyclic π_1.

1.2 The algebraic genus via 3–dimensional cobordism distance

Rather than in terms of Seifert matrices, $g_{\mathrm{alg}}(L)$ can also be characterized as the smallest genus of a cobordism in 3–space between L and a knot with Alexander polynomial 1:

Theorem 2 *For all links L with r components, $g_{\mathrm{alg}}(L)$ equals the smallest genus among Seifert surfaces for links L' with $r + 1$ components such that the first r components form L and the last component forms a knot with Alexander polynomial 1.*

This characterization of $g_{\mathrm{alg}}(L)$ is the reason for naming the invariant "algebraic genus", in parallel to the algebraic unknotting number u_{alg} (see Sect. 1.4): for a knot K, both $g_{\mathrm{alg}}(K)$ and $u_{\mathrm{alg}}(K)$ can be defined either purely in terms of the Seifert form, or as a 3–dimensional distance (using the genus of Seifert surfaces and unknotting, respectively) to knots that have Alexander polynomial 1. The name "algebraic slice genus", on the other hand, would be more fitting for Taylor's invariant (see Sect. 1.5).

1.3 The algebraic genus and other knot invariants

We summarize the relation between g_{alg} and other knot invariants in Fig. 1. By g and g_{smooth} we respectively denote the three-dimensional genus and the smooth slice genus, neither of which is classical (i.e. determined by the S-equivalence class).

Some of the considered invariants, notably the algebraic unknotting number and Taylor's invariant, are currently only defined for knots. One might expect those invariants and their relations to g_{alg} to generalize to multi-component links, for which g_{alg} is naturally defined. However, such generalizations need not be straight-forward. This is the reason we consider knots rather than multi-component links in Sects. 1.4, 1.5 and 1.6.

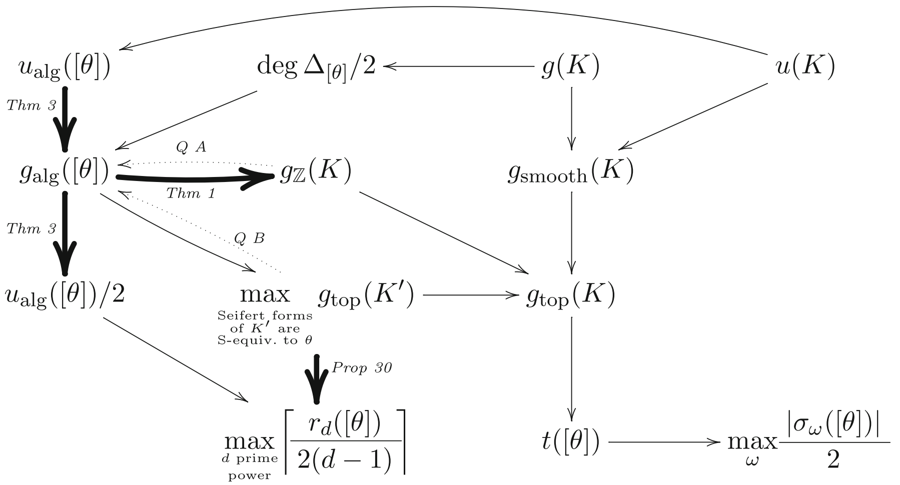

Fig. 1 Diagrammatic summary of relations of g_{alg} to other knot invariants, in homage to [6, Figure 2]. Arrows $a \to b$ indicate that the inequalities $a \geq b$ hold for all knots. A dotted arrow means that the status of the respective inequality is open. If there is no directed path from a to b, it means that $a \geq b$ is known to be false for some knots. Thick arrows indicate original results of this article. We write $[\theta]$ (the S-equivalence class of a Seifert form θ of K) rather than K as an argument to indicate that an invariant is classical, i.e. depends only on $[\theta]$

1.4 The algebraic genus and the algebraic unknotting number

The algebraic unknotting number u_{alg} of a knot, introduced by Murakami [31], is the optimal classical lower bound for the unknotting number u. We prove the following inequality between u_{alg} and g_{alg}:

Theorem 3 *For all knots K, $g_{\text{alg}}(K) \leq u_{\text{alg}}(K) \leq 2g_{\text{alg}}(K)$.*

Let Δ_K denote the Alexander polynomial of K. We understand its *degree* $\deg(\Delta_K(t))$ to be the breadth of Δ_K; e.g. 2 for the trefoil. Then, using that $2g_{\text{alg}}(K) \leq \deg(\Delta_K(t))$, the above theorem yields

Corollary 4 *For all knots, $u_{\text{alg}}(K) \leq \deg(\Delta_K(t))$.*

This answers a question of Borodzik and Friedl, who had previously shown

$$u_{\text{alg}}(K) \leq \deg(\Delta_K(t)) + 1$$

and asked whether that bound could be sharpened. We use their characterization of u_{alg} in terms of the Blanchfield pairing (cf. [5, Theorem 2], [6, Lemma 2.3]) for the proof of the second inequality of Theorem 3.

1.5 The algebraic genus and Taylor's invariant

The algebraic genus can be understood as a measure of how much a knot fails to have Alexander polynomial 1. Indeed, $g_{\text{alg}}(K) = 0$ if and only if $\Delta_K = 1$. Taylor introduced a knot invariant $t(K)$ that generalizes algebraic sliceness [37]: a knot K is algebraically slice, i.e. has a metabolic Seifert form, if and only if $t(K) = 0$. Explicitly, if $\theta : \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \to \mathbb{Z}$ is a Seifert form of a knot K, then $t(K)$ is defined as n minus the maximal rank of a totally isotropic subgroup of $\mathbb{Z}^{2n}$. Taylor's invariant provides a lower bound for the topological slice genus, which is indeed the optimal classical bound. In particular, Taylor's bound subsumes the bounds given by the Levine-Tristram signatures σ_ω.

Since $t(K) \leq g_{\mathrm{top}}(K) \leq g_{\mathrm{alg}}(K)$, it would be of interest to relate $t(K)$ and $g_{\mathrm{alg}}(K)$. It appears that aside from $t(K) \leq g_{\mathrm{alg}}(K)$ the two invariants are rather independent. However, we can prove the following: if a genus 2 fibered knot K is algebraically slice (i.e. $t(K) = 0$), then $g_{\mathrm{alg}}(K)$ (and thus also the topological slice genus of K) is at most 1; compare Proposition 25. In contrast, such results are not available for knots with Alexander polynomials of higher degree. For example, there exist algebraically slice knots K with monic Alexander polynomial of degree 6 and $g_{\mathrm{alg}}(K) = 3$; compare Example 28.

1.6 Towards optimality of the algebraic genus as slice genus bound

Taylor's lower bound to the slice genus is known to be the best classical lower bound for knots (see Sect. 1.5); that is to say, every Seifert form θ is realized by a knot whose slice genus equals $t(\theta)$. Question B is the analogous question about classical *upper* bound for the topological slice genus. As a first step towards determining the best classical upper bound for the topological slice genus of knots—for which the algebraic genus is a candidate—we prove in Proposition 30 that every Seifert form θ of a knot is realized by a knot K with

$$g_{\mathrm{top}}(K) \geq \max_{\substack{d \text{ prime} \\ \text{power}}} \left\lceil \frac{r_d(\theta)}{2(d-1)} \right\rceil.$$

Here, $r_d(\theta)$ denotes the minimum number of generators of the first integral homology group of the d–fold branched cover of a knot K' realizing θ (note that r_d only depends on θ). The relevant knots in the proof are constructed via infection, following Livingston [27].

1.7 Calculations of the algebraic genus and its role as upper bound for the slice genus

It is a virtue of g_{alg} that upper bounds for it can be explicitly calculated using Seifert matrix manipulation. Previous work by Baader, Liechti, McCoy and the authors [2, 3,12,13,23,24] used this method (without any focus on the algebraic genus itself) to determine upper bounds for the topological slice genus of various classes of links. Due to Theorem 1, all of those results in fact give upper bounds for the $\mathbb{Z}$–slice genus.

Although no general algorithm is known, in many cases a combination of calculable upper and lower bounds for the algebraic genus determines it completely. For example, the algebraic genus has been calculated for all prime knots with 11 crossings or less [23].

1.8 Structure of the paper

In Sect. 2, the algebraic genus is defined, first examples are given and basic results are proven, as well as a result about the stable algebraic genus. Section 3 contains the proof for the alternative three-dimensional characterization of g_{alg} given in Theorem 2. Theorems 1 and 3 on the $\mathbb{Z}$–slice genus and the algebraic unknotting number are proven in Sects. 4 and 5, respectively. Section 6 is concerned with the algebraic genus of knots with monic Alexander polynomial, and contains the proof of Propostion 25. In Sect. 7, optimality of slice genus bounds is discussed and Proposition 30 is proven. The paper concludes with the short Sect. 8, in which previously known results are reformulated in terms of the algebraic genus.

2 The algebraic genus—basic definitions and properties

2.1 Definitions

We consider links, by which we mean smooth oriented non-empty closed 1-dimensional submanifolds of S^3. We define the algebraic genus of a link L using the Seifert form defined on $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{2g+r-1}$, where F is a genus $g \geq 0$ Seifert surface with boundary the $r > 0$ component link L. By a *Seifert surface* for a link L, we mean an oriented connected embedded surface in S^3 with boundary L. The *genus* $g(L)$ of L is the minimum genus of a Seifert surface of L.

Let us start with some notations on bilinear forms (which we will readily use for Seifert forms). For integers $g \geq 0$ and $r \geq 1$, let θ be a bilinear form on an abelian group $H \cong \mathbb{Z}^{2g+r-1}$ such that its antisymmetrization, denoted by $\theta - \theta^\top$, satisfies the following: the radical $\mathrm{rad}_{\theta-\theta^\top}$ of $\theta - \theta^\top$—the subgroup of elements that pair to 0 with all other elements—is isomorphic to $\mathbb{Z}^{r-1}$ as a group and the form induced by $\theta - \theta^\top$ on $H/\,\mathrm{rad}_{\theta-\theta^\top}$ has determinant 1. These are precisely the bilinear forms that arise as Seifert forms of genus g Seifert surfaces of links with r components. If M is a matrix representing such a form θ, we call

$$t^{-g} \cdot \det(t \cdot M - M^\top) \in \mathbb{Z}[t^{\pm 1}]$$

the *Alexander polynomial* of θ, and denote it by Δ_θ. This is independent of the choice of basis and hence indeed a well-defined; in fact, it is invariant under S-equivalence. We call a subgroup $U \subseteq H \cong \mathbb{Z}^{2g+r-1}$ *Alexander-trivial* if $\det(t \cdot M - M^\top)$ is a unit in $\mathbb{Z}[t^{\pm 1}]$ for a matrix M representing $\theta|_U$. One obtains

$$\det(M - M^\top) = 1$$

by substituting $t = 1$. It follows that U is a summand of $H \cong \mathbb{Z}^{2g+r-1}$, and the rank of U is even and at most $2g$. Furthermore, U is Alexander-trivial if and only if $\theta|_U$ is the Seifert form of a knot K with $\Delta_K = \Delta_{\theta|_U} = 1$.

Suppose $2d$ is the maximal rank of an Alexander-trivial subgroup for a bilinear form θ. We define $\widetilde{g_{\mathrm{alg}}}(\theta)$ to be $\widetilde{g_{\mathrm{alg}}}(\theta) = g - d$ and we define $g_{\mathrm{alg}}(\theta)$ to be the minimum $\widetilde{g_{\mathrm{alg}}}(\eta)$, where η ranges over all forms that are S-equivalent to θ.

Definition 5 For all links L, we define the *algebraic genus* $g_{\mathrm{alg}}(L)$ *of* L to be

$$g_{\mathrm{alg}}(L) = \min \left\{ g_{\mathrm{alg}}(\theta) \;\middle|\; \begin{array}{l} \theta \text{ is the Seifert form of} \\ \text{a Seifert surface for } L \end{array} \right\}.$$

Clearly, if L is a link and θ some fixed Seifert form of L, then

$$g_{\mathrm{alg}}(\theta) \leq g_{\mathrm{alg}}(L) \leq \widetilde{g_{\mathrm{alg}}}(\theta). \tag{2}$$

We will prove in Proposition 10 that $g_{\mathrm{alg}}(\theta) = g_{\mathrm{alg}}(L)$. But whether the second inequality of (2) is an equality remains an open question.

Note that reversing the orientation of all components of L or taking the mirror image of L does not change $g_{\mathrm{alg}}(L)$ since the Alexander-trivial subgroups with respect to θ, $\theta^\top$, and $-\theta$ are the same.

2.2 More on Alexander-trivial subgroups

In practice, establishing that a subgroup $U \subseteq H$ is Alexander-trivial may be done by finding a basis of U with respect to which $\theta|_U$ is given by a matrix M of the form

$$\begin{pmatrix} 0 & \mathbb{1} + P \\ L & Q \end{pmatrix}, \tag{3}$$

where 0, $\mathbb{1}$, P, L and Q denote square matrices of half the dimension of M that are zero, the identity, lower triangular with zeros on the diagonal, upper triangular with zeros on the diagonal, and arbitrary, respectively. For this we note that, if a $2n \times 2n$ matrix is of the form (3), then $\det(t \cdot M - M^\top) = t^n$. The following lemma implies that Alexander-triviality of a subgroup can always be established by finding such a basis.

Lemma 6 *If a Seifert form θ on $H \cong \mathbb{Z}^{2g}$ has Alexander polynomial 1, then there exists a basis of $\mathbb{Z}^{2g}$ with respect to which θ is given by a matrix of the form* $\begin{pmatrix} 0 & \mathbb{1} + P \\ P^\top & 0 \end{pmatrix}$, *where* 0, $\mathbb{1}$, *and* P *denote $g \times g$ matrices that are zero, identity, and upper triangular with zeros on the diagonal, respectively.*

For general θ, there is no basis such that P is the zero matrix, since the rank of a matrix of θ is an invariant of the form. Note that a significantly stronger statement holds: there are knots which have Alexander-polynomial 1, yet do not admit a Seifert matrix as above with P the zero matrix [17].

Proof of Lemma 6 By a calculation provided in [12, Lemma 6 and Remark 7] there is a basis such that the corresponding $(2g \times 2g)$ matrix is of the form $M = \begin{pmatrix} 0 & \mathbb{1}+P \\ P^\top & Q \end{pmatrix}$, where Q is some $g \times g$ matrix. The statement follows by applying the following base change

$$\begin{pmatrix} \mathbb{1} & 0 \\ -N & \mathbb{1} \end{pmatrix} M \begin{pmatrix} \mathbb{1} & -N^\top \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}+P \\ P^\top & Q - N(\mathbb{1}+P) - P^\top N^\top \end{pmatrix},$$

where N is the unique $g \times g$ matrix that satisfies the equation $Q = N + NP + (NP)^\top$. To be explicit, N is inductively given as follows. Set $N_{11} = Q_{11}$. For the induction step, we fix $\ell \in \{2, 3, \ldots, 2g-1\}$ and assume N_{ij} is defined whenever $i + j \leq \ell$, and thus we can set

$$N_{ij} = Q_{ij} - \sum_{k=1}^{j-1}(N_{ik}P_{kj}) - \sum_{k=1}^{i-1}(N_{jk}P_{ki}) \quad \text{whenever } i + j = \ell + 1.$$

□

2.3 Examples

Example 7 We consider the 12–crossing alternating knot K_{12a908} and one of its Seifert matrices

$$M = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & -2 \\ 1 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -2 \end{pmatrix}$$

as provided by KnotInfo [10]. There exists an Alexander-trivial subgroup of rank 4 in $\mathbb{Z}^6$ with respect to the bilinear form represented by M. Indeed,

$$B = \begin{pmatrix} 3 & 0 & -2 & 1 \\ -5 & 1 & 2 & 0 \\ 2 & -1 & -1 & 0 \\ -2 & 0 & -1 & 2 \\ 6 & 0 & 0 & -2 \\ 1 & 0 & -1 & 1 \end{pmatrix} \quad \Rightarrow \quad B^\top \cdot M \cdot B = \begin{pmatrix} 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -5 & 8 \\ 2 & 0 & 5 & -8 \end{pmatrix},$$

which shows that the columns of B are the basis of an Alexander-trivial subgroup of rank 4 in $\mathbb{Z}^6$ (note that $B^\top MB$ is of the form (3)). Furthermore, no Seifert form of K_{12a908} has an Alexander-trivial subgroup of full rank, since the Alexander polynomial of K_{12a908} is different from 1 (it is $4t^3 - 22t^2 + 55t - 73 + 55t^{-1} - 22t^{-2} + 4t^{-3}$).

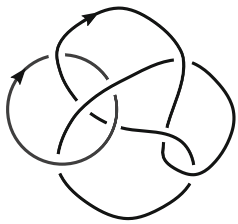

Fig. 2 The link L8n2 (diagram from KnotInfo [10])

Thus, $g_{\text{alg}}(K_{12a908}) = 1$ by Definition 5. In fact, $|\sigma(K_{12a908})/2| = 1$, where $\sigma(K)$ denotes the signature of the knot K, and so

$$\left|\frac{\sigma(K_{12a908})}{2}\right| = g_{\text{top}}(K_{12a908}) = g_{\mathbb{Z}}(K_{12a908}) = g_{\text{alg}}(K_{12a908}) = 1.$$

The genus of K_{12a908} is 3 (since the degree of the Alexander polynomial of K_{12a908} is 6), the smooth slice genus is 2 (by an argument based on Donaldson's diagonalization theorem; compare [23]), and the algebraic unknotting number is 2 [4,6]. Therefore, there is no immediate way via the smooth slice genus or the algebraic unknotting number to find that $g_{\text{top}}(K_{12a908}) = 1$; while the above calculation of g_{alg} is quite explicit.

Example 8 Let us calculate the algebraic genus of the two-component link L shown below in Fig. 2. Seifert's algorithm gives a Seifert surface of genus 2 with a Seifert matrix M:

$$M = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The standard basis vectors e_1, e_2, e_4, e_5 form an Alexander-trivial subgroup for M, and so $g_{\text{alg}}(M) = 0$. Consequently, all links with the Seifert matrix M are topologically (weakly) slice. The particular link L turns out to be actually smoothly slice, and has three-genus 1; note M can be destabilized by removing e_4 and e_5.

2.4 Basic properties of the algebraic genus

Definition 5 is made such that for all links L the inequality

$$g_{\text{top}}(L) \leq g_{\text{alg}}(L) \tag{4}$$

follows immediately from Freedman's (1) and the following proposition, which is proven in detail in [2, Proof of Prop. 3] (compare also [12, Proposition 2]). For the sake of completeness, we nevertheless include a concise version of the proof below. Proposition 9 will be used in the proof of Theorem 1, which subsumes (4).

Proposition 9 *Let L be a link, F a genus g Seifert surface for L, and θ the Seifert form on $H_1(F;\mathbb{Z})$. If U is an Alexander-trivial subgroup of $H_1(F;\mathbb{Z})$ of rank $2d$, then there exists a separating simple closed curve K on F with the following properties:*

- *the curve K (viewed as a knot in S^3) has Alexander polynomial 1,*
- *$F\backslash K = F_1 \sqcup F_2$ such that $\overline{F_1}$ is of genus d with boundary K.*

Sketch of the proof As discussed at the beginning of the section, U is a summand, i.e. $H_1(F;\mathbb{Z}) = U \oplus V$. One may choose a separating simple closed curve K' on F such that $F\backslash K' = F_1' \sqcup F_2'$, $\partial F_1' = K'$, and $\operatorname{rk} H_1(F_1';\mathbb{Z}) = 2d$. It turns out that there is a group automorphism $\varphi' : H_1(F;\mathbb{Z}) \to H_1(F;\mathbb{Z})$ with $\varphi'(U) = H_1(F_1';\mathbb{Z})$ that preserves the intersection form $\theta - \theta^\top$ and that maps homology classes given by components of ∂F to homology classes given by components of ∂F. The group automorphism φ' is realized as the action of a diffeomorphism φ of F since the mapping class group of F surjects onto the symplectic group. Take $K = \varphi(K')$ and $F_i = \varphi(F_i')$. Then $\overline{F_1}$ is a genus d Seifert surface of K and the corresponding Seifert form is given by $\theta|_U$, hence $\Delta_K = 1$. □

The next proposition shows that $g_{\mathrm{alg}}(L)$ only depends on the S-equivalence class of Seifert forms of L.

Proposition 10 *For all links L,*

$$g_{\mathrm{alg}}(L) = \min\left\{ \widetilde{g_{\mathrm{alg}}}(\theta) \;\middle|\; \begin{array}{c} \theta \text{ is a bilinear form that is S-equivalent} \\ \text{to a Seifert form (and thus all) for } L \end{array} \right\}.$$

Proof Clearly $\geq$ holds, since the minimum ranges over a larger class of bilinear forms. For the other direction, we recall that any bilinear form S-equivalent to a Seifert form, can be stabilized to become a Seifert form. Indeed, let θ_1 be a Seifert form for L and θ_2 any bilinear form S-equivalent to θ. There exists a bilinear form θ that arises as the stabilization of both θ_1 and θ_2. Since stabilizations of a Seifert form can be realized geometrically by a stabilization of the corresponding Seifert surface, the bilinear form θ arises as a Seifert form. Now, the statement follows from the following lemma about stabilizations. □

Lemma 11 *Let θ be a bilinear form arising as Seifert form of a link, and let η be obtained from θ by a stabilization. Then $\widetilde{g_{\mathrm{alg}}}(\eta) \leq \widetilde{g_{\mathrm{alg}}}(\theta)$.*

Proof For the bilinear form θ on $\mathbb{Z}^{2g+r-1}$, let $U \subseteq \mathbb{Z}^{2g+r-1}$ be an Alexander-trivial subgroup of maximal rank, and denote this rank by $2d$. We view the stabilization η of θ as a bilinear form on $\mathbb{Z}^{2g+r-1} \oplus \mathbb{Z}^2$ such that with respect to the standard basis η is given by

$$\left(\begin{array}{ccc|cc} & & & & 0 \\ & M & & & 0 \\ & & & v & \vdots \\ & & & & 0 \\ \hline & v^\top & & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right),$$

where M represents θ and v is some element of $\mathbb{Z}^{2g+r-1}$. The statement follows from the fact that $U \oplus \mathbb{Z}^2$ is an Alexander-trivial subgroup of $\mathbb{Z}^{2g+r-1} \oplus \mathbb{Z}^2$ with respect to η. □

2.5 Lower bounds for the algebraic genus of links

For knots, prominent lower bounds for the algebraic genus come from the ranks of branched covers (see Fig. 1), and from Taylor's invariant; the latter statement is evident from the definitions, because an Alexander-trivial subgroup of rank $2d$ contains a totally isotropic subgroup of rank d. In this subsection we investigate what can be proved for multi-component links.

For this purpose, let $\eta(L)$ denote the nullity of a link L, defined as $\dim(\mathrm{rad}_{(\theta+\theta^\top)\otimes\mathbb{Q}})$ for any Seifert form θ of L (i.e. interpret $\theta + \theta^\top$ as a bilinear form over the rationals and take the dimension of its radical). Let $r_2(L)$ be the minimum number of generators of $H_1(M_2(L); \mathbb{Z})$, the first integral homology group of the double branched covering $M_2(L)$.

Proposition 12 *If L is an r–component link, then:*

$$|\sigma(L)| + \eta(L) - r + 1 \leq 2g_{\mathrm{alg}}(L), \tag{i}$$
$$r_2(L) \leq 2g_{\mathrm{alg}}(L). \tag{ii}$$

Note that both lower bounds for $2g_{\mathrm{alg}}(L)$ are classical, and additive with respect to the connected sum along arbitrary components.

Proof (i) Of course, (i) follows because $2g_{\mathrm{top}}(L)$ is greater than or equal to the left hand side, and less than or equal to the right hand side, but there is also a more direct reason: pick a Seifert form $\theta : \mathbb{Z}^{2g+r-1} \times \mathbb{Z}^{2g+r-1} \to \mathbb{Z}$ of L with $\widetilde{g_{\mathrm{alg}}}(\theta) = g_{\mathrm{alg}}(L)$. Denote by $n_\pm$ the indices of inertia of the symmetrization of θ, so that $n_+ - n_- = \sigma(L)$ and $n_+ + n_- + \eta(L) = 2g + r - 1$. There is an Alexander-trivial subgroup of rank $2(g - g_{\mathrm{alg}}(L))$, which implies that both n_+ and n_- are greater than or equal to $g - g_{\mathrm{alg}}(L)$. Hence $|\sigma(L)| \leq 2g_{\mathrm{alg}} + r - 1 - \eta(L)$.

(ii) Note that if A is a matrix for θ, then $H_1(M_2(L); \mathbb{Z})$ is isomorphic to the cokernel of $A + A^\top$. By the classification of finite abelian groups, there is a prime p such that $\dim_{\mathbb{Z}/p} H_1(M_2(L); \mathbb{Z}/p) = r_2(L)$. If U is an Alexander-trivial subgroup of rank $2d$, and B is a matrix of $\theta|_U$, then $\det(B + B^\top) = \pm\Delta_{\theta|_U}(-1) = \pm 1$. So $B + B^\top$ has full rank $2d$ over $\mathbb{Z}/p$, and thus $A + A^\top$ has rank at least $2d$ over $\mathbb{Z}/p$. Thus $\mathrm{coker}(A + A^\top) \otimes \mathbb{Z}/p \cong H_1(M_2(L); \mathbb{Z}/p)$ has dimension at most $2g - 2d = 2\widetilde{g_{\mathrm{alg}}}(\theta)$ over $\mathbb{Z}/p$. This means that $r_2(L) \leq 2\widetilde{g_{\mathrm{alg}}}(\theta)$. Since r_2 is invariant under S-equivalence, this implies (ii). □

Remark 13 Consider a Seifert form θ of a knot of dimension $2g$ with maximal r_2, i.e. $r_2(\theta) = 2g$. Clearly, that condition is equivalent to the existence of an odd prime p modulo which $\theta + \theta^\top$ vanishes ($\theta + \theta^\top$ cannot vanish modulo 2, since its determinant is odd). Such a form may be realized as Seifert form of a knot with topological slice genus g (Proposition30). Moreover, all knots K admitting such a Seifert form have some peculiar properties: for example, the unknotting number of K is bounded below by $u_{\mathrm{alg}}(\theta) = 2g$; and all $2g$ Alexander ideals are non-trivial over $\mathbb{Z}$, since they are sent to $p\mathbb{Z}$ by the substitution $t = -1$.

2.6 The stable algebraic genus

If θ, ζ are Seifert forms with respective Alexander-trivial subgroups U, V, then $\theta \oplus \zeta$ has the Alexander-trivial subgroup $U \oplus V$. This implies that g_{alg} is subadditive with respect to the connected sum of links:

$$g_{\mathrm{alg}}(L\#L') \leq g_{\mathrm{alg}}(L) + g_{\mathrm{alg}}(L')$$

for all links L and L', where the connected sum is taken along arbitrary components. Let us construct an example demonstrating that g_{alg} is in general not additive. Take θ to be the Seifert form given by

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

for which we have $g_{\mathrm{alg}}(\theta) = 1$, since $\Delta_\theta \neq 1$. On the other hand, the form $\theta \oplus \theta$ admits an Alexander-trivial subgroup U of rank 2 generated by $(1, 0, 0, 1)$ and $(0, 1, 0, 0)$. Indeed

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

and thus $g_{\mathrm{alg}}(\theta \oplus \theta) = 1$.

Following Livingston's definition of the stable slice genus [28], one may define the *stable algebraic genus* $\widehat{g_{\mathrm{alg}}}(L)$ of a link L as

$$\widehat{g_{\mathrm{alg}}}(L) = \lim_{n\to\infty} \frac{g_{\mathrm{alg}}(L^{\#n})}{n}.$$

Similarly one can define $\widehat{g_{\mathrm{top}}}(L)$ and $\widehat{g_{\mathbb{Z}}}(L)$ for links L with a distinguished component along which the connected sums are taken. It is now an immediate consequence of Theorem 1 that for all links L with distinguished component,

$$\widehat{g_{\mathrm{alg}}}(L) \geq \widehat{g_{\mathbb{Z}}}(L) \geq \widehat{g_{\mathrm{top}}}(L).$$

These inequalities give some motivation for studying the stable algebraic genus. We will refrain from doing so here, with the exception of the following proposition which results from strengthening a result of Baader [1] by recasting his argument algebraically, making connections to r_2, and generalizing to multi-component links.

Note that the lower bounds for $2g_{\mathrm{alg}}$ in Proposition 12, being additive, are also lower bounds for $2\widehat{g_{\mathrm{alg}}}$. The following proposition shows that taken together they characterize knots whose stable algebraic genus is strictly less than their genus.

Proposition 14 *For all r–component links L, $\widehat{g_{\mathrm{alg}}}(L) < g(L)$ holds if and only if*

$$\max\{|\sigma(L)| + \eta(L), r_2(L)\} \ < \ 2g(L) + r - 1.$$

In particular for all knots K, $\widehat{g_{\mathrm{alg}}}(K) < g(K)$ holds if and only if

$$\max\{|\sigma(K)|, r_2(K)\} < 2g(K).$$

Proof The 'only if' part of the statement were discussed in the paragraph preceding the proposition. Let us now prove the 'if' part. For this, fix a Seifert matrix A of L of size $(2g + r - 1) \times (2g + r - 1)$, where g is the genus of L. Note that $g > 0$. Indeed, $g = 0$ implies $A = A^\top$, whence $A + A^\top$ is the zero matrix modulo 2, which implies $r_2(L) = r - 1$, contradicting the hypotheses. We denote by $Q_n : \mathbb{Z}^{(2g+r-1)n} \to \mathbb{Z}$ the quadratic form defined by $A^{\oplus n}$, i.e. $Q_n(v) = v^\top A^{\oplus n} v$, and let $\mathcal{F} \subset \mathbb{Z}$ be the union of the images of Q_n for all $n \geq 1$.

In a first step, we prove that $\mathcal{F}$ is equal to the subgroup $\mathcal{G}$ of $\mathbb{Z}$ generated by $A_{ij} + A_{ji}$ and A_{kk} for $1 \leq i < j \leq 2g + r - 1$ and $1 \leq k \leq 2g + r - 1$. To show '$\mathcal{F} \subset \mathcal{G}$', we notice that by definition elements $x \in \mathcal{F}$ are sums of elements of the form

$$v^\top A v = \sum_{k=1}^{2g+r-1} v_k^2 A_{kk} + \sum_{1 \leq i < j \leq 2g+r-1} v_i v_j (A_{ij} + A_{ji}),$$

for $v \in \mathbb{Z}^{2g+r-1}$, which are $\mathbb{Z}$–linear combinations of $A_{ij} + A_{ji}$ and A_{kk}. To show '$\mathcal{F} \supset \mathcal{G}$', it is sufficient to show that $\mathcal{F}$ is a subgroup of $\mathbb{Z}$ and contains all $A_{ij} + A_{ji}$ and A_{kk}. Clearly, $\mathcal{F}$ is non-empty as $0 \in \mathcal{F}$. If $x_1, x_2 \in \mathcal{F}$, and $x_\ell = Q_{n_\ell}(v_\ell)$, then

$$x_1 + x_2 = Q_{n_1+n_2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{F}.$$

This proves that $\mathcal{F}$ is closed under addition. Now, let a non-zero integer $x \in \mathcal{F}$ be given. We are going to show that $-x \in \mathcal{F}$, which will complete the proof that $\mathcal{F} \subset \mathbb{Z}$ is a subgroup. Since $|\sigma(L)| + \eta(L) < 2g + r - 1$, the form Q_1 is indefinite, and so there exists $y \in \mathcal{F}$ with the opposite sign of x. We first consider the case $x > 0$ and $y < 0$. Since $\mathcal{F}$ is closed under addition and thus also under multiplication with positive integers, we find

$$x \cdot y, (-y - 1) \cdot x \in \mathcal{F} \quad \Rightarrow \quad x \cdot y + (-y - 1) \cdot x = -x \in \mathcal{F},$$

If instead, we have $x < 0$ and $y > 0$, then we similarly find

$$(-x) \cdot y, (y - 1) \cdot x \in \mathcal{F} \quad \Rightarrow \quad (-x) \cdot y + (y - 1) \cdot x = -x \in \mathcal{F}.$$

Finally, let us check that all $A_{ij} + A_{ji}$ and A_{kk} are in $\mathcal{F}$. Firstly, $Q_1(e_k) = A_{kk}$, where e_k denotes k–th standard basis vector. Secondly,

$$Q_3 \begin{pmatrix} e_i + e_j \\ -e_i \\ -e_j \end{pmatrix} = A_{ij} + A_{ji}.$$

As a second step, we prove that $\mathcal{F} = \mathcal{G} = \mathbb{Z}$. Note that by definition $\mathcal{G}$ contains every entry of $A + A^\top$, i.e. $A_{ij} + A_{ji}$ and $2A_{kk}$ for $1 \le i < j \le 2g + r - 1$ and $1 \le k \le 2g + r - 1$. Assume towards a contradiction that $\mathcal{G} \ne \mathbb{Z}$. This would imply that the greatest common divisor of the entries of $A + A^\top$ is non-trivial. Now recall that $A + A^\top$ is a presentation matrix of the first integral homology group of the double branched covering of L. Therefore, that homology group would be the sum of $2g + r - 1$ groups of the form $\mathbb{Z}/a_i d$ for some $a_i \in \mathbb{Z}$, in contradiction to $r_2(L) < 2g + r - 1$.

To finish, pick two vectors $v_1, v_2 \in \mathbb{Z}^{2g+r-1}$ with $v_1^\top(A - A^\top)v_2 = 1$. This is possible since $g > 0$. Since $\mathcal{F} = \mathbb{Z}$, there exist two positive integers n_1 and n_2 and another two vectors w_1, w_2 with $w_i \in \mathbb{Z}^{(2g+r-1)n_i}$ such that $Q_{n_1}(w_1) = -v_1^\top A v_2$ and $Q_{n_2}(w_2) = v_1^\top A(v_2 - v_1)$. Then

$$u_1 = \begin{pmatrix} v_1 \\ w_1 \\ w_2 \end{pmatrix} \in \mathbb{Z}^{(2g+r-1)(1+n_1+n_2)}, \quad u_2 = \begin{pmatrix} v_2 \\ w_1 \\ 0 \end{pmatrix} \in \mathbb{Z}^{(2g+r-1)(1+n_1+n_2)}$$

generate an Alexander-trivial subgroup since

$$\begin{pmatrix} u_1^\top \\ u_2^\top \end{pmatrix} A^{\oplus(1+n_1+n_2)} \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & (v_2 - v_1)^\top A v_2 \end{pmatrix}.$$

This implies that $g_{\mathrm{alg}}(L^{\#(n_1+n_2+1)}) < (n_1 + n_2 + 1)g(L) = g(L^{\#(n_1+n_2+1)})$, and therefore $\widehat{g_{\mathrm{alg}}}(L) < g(L)$. □

Remark 15 Livingston [28] constructed a family of knots K_i with $\widehat{g_{\mathrm{top}}}(K_i) < 1$ and $\lim_{i\to\infty} \widehat{g_{\mathrm{top}}}(K_i) = 1$. That shows that the hypotheses of Proposition 14 cannot give a lower bound for the difference $g(K) - \widehat{g_{\mathrm{alg}}}(K)$.

3 Three-dimensional characterizations of g_{alg}

Definition 16 A *3D-cobordism* between two links L_1 and L_2 with r_1 and r_2 components, respectively, is a Seifert surface for a link with $r_1 + r_2$ components such that the link given by the first r_1 components is L_1 and the link given by the other components is L_2^{rev}, i.e. L_2 with reversed orientation.

For context, recall that a cobordism between two links L_0 and L_1 is an oriented connected smooth embedded surface C in $S^3 \times [0, 1]$ such that

$\partial C = L_0 \times \{0\} \sqcup L_1^{\text{rev}} \times \{1\}$. 3D-cobordisms correspond to cobordisms C such that the projection $S^3 \times [0, 1] \to S^3$ to the first factor restricts to an embedding on C. Up to isotopy, this is equivalent to the projection to the second factor being a Morse function all of whose critical points have index 1.

The following proposition provides the proof of Theorem 2.

Proposition 17 *For all links L, the following holds.*

(i) *Let F be a Seifert surface of L. Let $K \subset F$ be a simple closed curve such that $F \backslash K = F_1 \sqcup F_2$ with $\partial F_2 = L \sqcup K$ and $\partial F_1 = K$, and Δ_K equals 1. Then $g_{\text{alg}}(L)$ is the minimal genus of such a surface F_2.*

(ii) *The algebraic genus of L equals the minimum genus among all 3D-cobordisms between L and a knot with Alexander polynomial 1.*

These should be viewed in light of similar characterizations for the algebraic unknotting number u_{alg} of a knot K, which can be defined purely algebraically using the Seifert form, but is most quickly defined as the smallest number of crossing changes needed to turn K into an Alexander polynomial 1 knot.

Proof We note that the first statement of Proposition 17 is an immediate consequence of Proposition 9. Indeed, let h be the smallest genus among surfaces $F_2 \subset F_1 \sqcup F_2 = F \backslash K$ as in (i). By the definition of $g_{\text{alg}}(L)$, there exists a Seifert surface F of some genus g such that $g_{\text{alg}}(L) = g - d$, where $2d$ is the rank of an Alexander-trivial subgroup in $H_1(F; \mathbb{Z})$; thus, by Proposition 9, $h \leq g_{\text{alg}}(L)$. On the other hand, for any surface F_1 as in (i), $H_1(F_1; \mathbb{Z}) \subset H_1(F; \mathbb{Z})$ is an Alexander-trivial subgroup with respect to the Seifert form on the Seifert surface F, thus $h \geq g_{\text{alg}}(L)$.

By (i), the second statement of Proposition 17 follows, if we establish the following: given a 3D-cobordism between L and a knot K with Alexander polynomial 1 of some genus g, there exists a Seifert surface F_1 for K and a 3D-cobordism F_2 between L and a knot K of genus g such that F_1 and F_2 precisely intersect in K. This is established in the following Lemma. □

Lemma 18 *Let two links L and K, a Seifert surface C for K, and a genus g 3D-cobordism F' between L and K be given. Then C can be stabilized to a Seifert surface C' such that there exists a 3D-cobordism F with genus g between L and K with $C' \cap F = K$.*

Proof The surface F' defines a framing $N_{F'}(K)$ of K; i.e. a disjoint union of embedded annuli, given as a small closed neighborhood of K in F'. We first modify F' such that that the induced framing on K agrees with the framing induced by C: for every component K_j of K take a properly embedded interval J_j in F' with one boundary point on K_j and the other on L, which is possible since F' is connected (by the definition of a Seifert surface). By inserting full twists along J_j into F' if necessary, we get a new genus g 3D-cobordism F'' that induces the correct framing on K, which we denote by $N_{F''}(K)$.

Next, we observe that the cobordism F'' can be viewed as arising by adding 1–handles $H_1, \ldots, H_k$ to $N_{F''}(K)$. More precisely, the following is true. Let K' be $(\partial N_{F''}(K)) \backslash K$; in other words, K' is the parallel copy of K that forms the other part

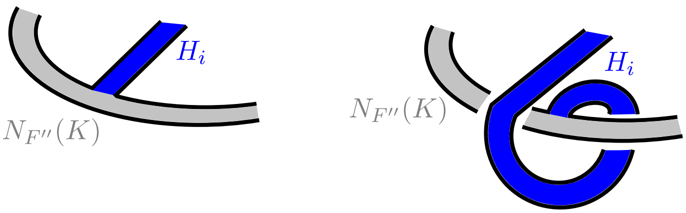

Fig. 3 The Seifert surface F'' viewed as the union of the framing $N_{F''}(K)$ (gray) and the 1–handles $H_1, \ldots, H_k$ (blue). The depicted local move (left-to-right) changes the algebraic intersection number between the core of H_i and C by ± 1 without changing the genus of F'' or the isotopy class of K and L (color figure online)

of the boundary of $N_{F''}(K)$. For some non-negative integer k, there exist pairwise disjoint disks $H_1, \ldots, H_k$ in S^3 such that

$$F'' = N_{F''}(K) \cup H_1 \cup \cdots \cup H_k,$$

where the H_i are pairwise disjoint disks in S^3 such that $H_i \cap N_{F''}(K)$ consists of two closed intervals contained in K'. Let I_i denote the core of the handle H_i; i.e. a properly embedded interval in H_i such that its two boundary points lie in the interior of $K' \cap H_i$, one in each component.

Now, we study the intersection between F'' and C. Since F'' and C induce the same framing on K, we may isotope them such that $N_{F''}(K) \cap C = K$. We also arrange that the cores I_i intersect C transversely.

We now modify F'' such that the algebraic intersection number between I_i and C becomes 0. Indeed, by modifying F'' as depicted in Fig. 3, we change the algebraic intersection number between I_i and C by ± 1. Thus, by modifying F'' several times as described in Fig. 3, we obtain a genus g 3D-cobordism F such that the corresponding cores I_i have algebraic intersection number 0 with C. We note that F is still a genus g 3D-cobordism between K and L since the operation described in Fig. 3 does not change the isotopy type of L or K (however, in general, it does change the isotopy type of $L \cup K$).

In a last step, we show that C can be stabilized such that it no longer intersects F. This is done by inductively doing stabilizations on C to reduce the geometric intersection between the cores I_i and C to 0. Indeed, if $C \cap I_i$ is non-empty, then we find two consecutive (on I_i) intersection points $x, y \in C \cap I_i$ of opposite orientation. The subinterval of I_i connecting x and y defines a stabilization of C that intersects I_i in two fewer points than C. Inductively, we find a stabilization C' of C which does not intersect any I_i and so, it can be isotoped (rel K) away from F except for the intersection at K. □

4 The $\mathbb{Z}$–slice genus

In this section, we establish Theorem 1, which states that $g_{\mathbb{Z}}(L) \leq g_{\mathrm{alg}}(L)$ for all links L. We recall from the introduction:

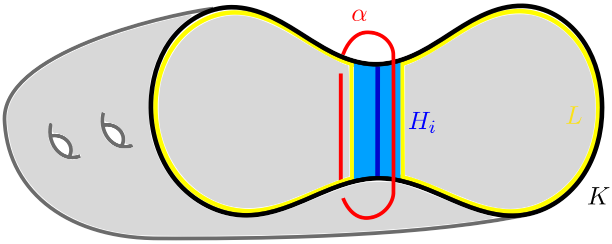

Fig. 4 The Seifert surface F for the link L (yellow) given by attaching 2–dimensional 1–handles H_i (blue) to the Seifert surface C (gray) for the knot K (black). In the case depicted, we have $g = 2 = d$ and $r = 2$ (color figure online)

Definition 19 Let the $\mathbb{Z}$-*slice genus* $g_{\mathbb{Z}}(L)$ of a link L denote the smallest genus of an oriented connected properly embedded locally flat surface F in the 4–ball B^4 with boundary $L \subset S^3$ and $\pi_1(B^4 \backslash F) \cong \mathbb{Z}$.

Proof of Theorem 1 Given an r–component link L, let F be a Seifert surface such that $g_{\text{alg}}(L) = g - d$, where g denotes the genus of F and $2d$ is the rank of an Alexander-trivial subgroup V of $H_1(F; \mathbb{Z})$. By Proposition 9, there exists a separating curve K with Alexander polynomial 1 on F such that F can be written as the following union of surfaces:

$$F = C \cup H_1 \cup \cdots \cup H_{2(g-d)+(r-1)};$$

where C is a Seifert surface for K of genus d and the H_i are closed disks that are pairwise disjoint and each disk intersects C in two closed intervals that lie in $K = \partial C$. In other words, F is given by attaching $2(g - d) + (r - 1)$ many 1–handles to C; compare Fig. 4. Compare also with the proof of Lemma 18, where we started with a similar setup.

By (1), K bounds a properly embedded locally flat disk D in B^4 such that its complement has fundamental group $\mathbb{Z}$. We may arrange that the disk D meets S^3 transversely and is smooth close to S^3.

Let S be the following locally flat surface of genus $g_{\text{alg}}(L)$ in the 4–ball B^4_2 of radius 2:

$$S = D \cup H_1 \cup \cdots \cup H_{2(g-d)+(r-1)} \cup \left(\bigcup_{t \in [1,2]} L_t \right),$$

where L_t denotes the link in the 3–sphere of radius t obtained by stretching L by t. In particular, S is a witness for $g_{\text{top}}(L) \leq g_{\text{alg}}(L)$; i.e. we have established (4). To get the stronger statement $g_{\mathbb{Z}}(L) \leq g_{\text{alg}}(L)$, it suffices to establish the following claim.

Claim 20 *The fundamental group of $B^4_2 \backslash S$ is isomorphic to $\mathbb{Z}$.*

Briefly said, it turns out that the inclusion $B^4 \backslash D \to B^4_2 \backslash S$ induces a surjection on π_1, which implies the claim since $H_1(B^4_2 \backslash S; \mathbb{Z}) \cong \mathbb{Z}$ by an appropriate version of Alexander duality. We provide a more detailed argument.

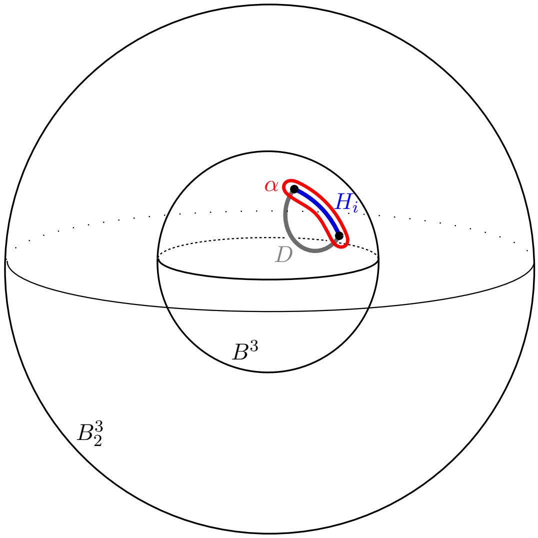

Fig. 5 A knot S in the 3–ball B^3_2 of radius 2 is given by attaching a 1–handle $H_i \subset S^2$ (blue) to the interval $D \subset B^3$ (gray) along two points (a 0–dimensional attaching sphere) (black). The knot complement $B^3_2 \backslash N(S)$ of S can be obtained by attaching a 3–dimensional 2–handle to the solid torus $B^3 \backslash N(D)$. The attaching sphere for this 2–handle is the curve $\alpha \subset S^2$ (red) (color figure online)

For this, we consider the topological 4–manifold with boundary $B^4_2 \backslash N(S)$, where $N(S)$ denotes an open tubular neighborhood of S, rather than $B^4_2 \backslash S$. The main point is that $B^4_2 \backslash N(S)$ (as a topological manifold with boundary) can be obtained from $B^4 \backslash N(D)$ by attaching $2(g-d)+r-1$ many 2–handles: one 4–dimensional 2–handle $\widetilde{H_i}$ corresponding to each H_i; compare [19, Proposition 6.2.1]. In Fig. 5,

we illustrate the situation one dimension lower: for a knot in the 3–ball rather than a surface in the 4–ball.

We describe the attaching spheres for the handles $\widetilde{H_i}$ in more detail. Let I_i be a core of the handle H_i; i.e. a properly embedded interval in H_i such that its two boundary points lie in the interior of $C \cap H_i \subset K$, one in each component; compare Fig. 4. Choose closed disks D_i in S^3 such that each D_i intersects $\overline{F \backslash C} = \bigcup_{j=1}^{2(g-d)+r-1} H_i$ only in the interior of D_i, and such that the intersection is I_i. Let α_i be the boundary curve of D_i; i.e. a curve that wraps once 'around' H_i while staying close to I_i; see Fig. 4. We leave it to the reader to check that, indeed, $B^4_2 \backslash N(S)$ (as a topological manifold with boundary) is obtained from $B^4 \backslash N(D)$ by attaching $2(g-d)+r-1$ many 2–handles, one along each α_i; compare [19, Proof of Proposition 6.2.1].

In particular, we have that $B^4_2 \backslash N(S)$ deformation retracts to the topological space X obtained by gluing $2(g-d)+r-1$ many disks along α_i to $B^4 \backslash N(D)$. Therefore, we have

$$\pi_1(B^4_2 \backslash N(S)) \cong \pi_1(X) \cong \frac{\pi_1(B^4 \backslash N(D))}{\langle [\alpha_1], \ldots, [\alpha_{2(g-d)+(r-1)}] \rangle}$$

by the Seifert-van Kampen Theorem, where $[\alpha_i]$ denotes the homotopy class of α_i in $\pi_1(B^4 \backslash N(D))$ (a base point may be appropriately chosen). However, note that α_i is null-homologous in $S^3 \backslash K$, i.e. the algebraic linking number of α_i and K is zero; since α_i is clearly homologous to a meridian of K plus an oppositely oriented meridian of

K. In particular, the α_i are also null-homologous in $B^4\backslash N(D)$. Since $\pi_1(B^4\backslash D)\cong\mathbb{Z}$, we have $\pi_1(B^4\backslash N(D))\cong H_1(B^4\backslash N(D);\mathbb{Z})$, and so the α_i are also null-homotopic in $B^4\backslash N(D)$. With this we conclude

$$\pi_1(B^4_2\backslash N(S))\cong\frac{\pi_1(B^4\backslash N(D))}{\langle[\alpha_1],\ldots,[\alpha_{2(g-d)+(r-1)}]\rangle}\cong\pi_1(B^4\backslash N(D))\cong\mathbb{Z}.$$

□

5 Algebraic genus and algebraic unknotting number

In this section, we relate the algebraic genus and the algebraic unknotting number of knots as follows:

Theorem 3 *For all knots K, $g_{\mathrm{alg}}(K)\le u_{\mathrm{alg}}(K)\le 2g_{\mathrm{alg}}(K)$.*

We consider knots rather than all links since, a priori, the invariant u_{alg} is only a knot invariant (rather than an invariant of links) and, for now, we do not know of a generalization of u_{alg} to links such that Theorem 3 holds.

We prove the two inequalities of Theorem 3 using different interpretations of u_{alg}. Using that $u_{\mathrm{alg}}(K)$ is equal to the minimum number of crossing changes needed to transform K into a knot with Alexander polynomial 1 [14,36], the first inequality $g_{\mathrm{alg}}\le u_{\mathrm{alg}}$ is an immediate consequence of the following proposition (which might be of independent interest):

Proposition 21 *Let L_1, L_2 be two links related by a crossing change. Then $|g_{\mathrm{alg}}(L_1)-g_{\mathrm{alg}}(L_2)|\le 1$.*

Proof Applying Seifert's algorithm to diagrams of L_1 and L_2 that differ by one crossing change, one finds two Seifert surfaces, say of genus g. A good choice of basis for the first homology of these Seifert surfaces yields $2g\times 2g$ Seifert matrices M_i for L_i such that $M_2=M_1\pm e_{11}$, where e_{11} denotes the square $2g\times 2g$ matrix with top-left entry 1 and all other entries equal to zero. By stabilizing M_1, we may assume that $g_{\mathrm{alg}}(L_1)=\widetilde{g_{\mathrm{alg}}}(M_1)$; i.e. the maximal Alexander-trivial subgroup of $\mathbb{Z}^{2g}$ with respect to M_1 is of rank $2g-2g_{\mathrm{alg}}(L_1)$. Let us apply the same stabilizations to M_2, so that the property $M_2=M_1\pm e_{11}$ is retained.

Now consider the following $(2g+2)\times(2g+2)$ matrix obtained as a stabilization of M_2:

$$\widetilde{M}_2=\left(\begin{array}{ccc|cc} & & & \mp 1 & 0\\ & M_2 & & 0 & 0\\ & & & \vdots & \vdots\\ & & & 0 & 0\\ \hline 0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 0 & 0\end{array}\right).$$

A change of basis turns $\widetilde{M_2}$ into

$$\left(\begin{array}{ccc|cc} & & & \mp 1 & 1 \\ & M_2 \mp e_{11} & & 0 & 0 \\ & & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|cc} & & & \mp 1 & 1 \\ & M_1 & & 0 & 0 \\ & & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right);$$

indeed, the latter is obtained from $\widetilde{M_2}$ by adding the second-to-last column to the first column and, correspondingly, adding the second-to-last row to the first row. Since there is an Alexander-trivial subgroup of rank $2g - 2g_{\text{alg}}(L_1)$ with respect to M_1, the same holds for $\widetilde{M_2}$. Consequently,

$$g_{\text{alg}}(L_2) \leq \widetilde{g_{\text{alg}}}(\widetilde{M_2}) \leq g + 1 - (g - g_{\text{alg}}(L_1)) = g_{\text{alg}}(L_1) + 1,$$

where the two inequalities are immediate from the definition of g_{alg}. This gives $g_{\text{alg}}(L_2) - g_{\text{alg}}(L_1) \leq 1$, and by switching the roles of L_1 and L_2 also $g_{\text{alg}}(L_1) - g_{\text{alg}}(L_2) \leq 1$, which concludes the proof. □

We point out that the stabilizations in the first paragraph in the above proof are necessary as it remains an open question whether $\widetilde{g_{\text{alg}}}(\theta) = \widetilde{g_{\text{alg}}}(\widetilde{\theta})$ holds for all S-equivalent Seifert forms θ and $\widetilde{\theta}$.

To tackle the second inequality of Theorem 3, we use Friedl and Borodzik's knot invariant n, which they show to be equal to u_{alg} [5,6]. Let us briefly give the necessary definitions. Let $\Lambda = \mathbb{Z}[t^{\pm 1}]$ be a ring with involution $a \mapsto \overline{a}$ given by the linear extension of $t \mapsto t^{-1}$, and let Ω be its quotient field. For a Hermitian $m \times m$ matrix A over Λ that is invertible over Ω, denote by $\lambda(A)$ the Hermitian form

$$\Lambda^m / A\Lambda^m \times \Lambda^m / A\Lambda^m \to \Omega/\Lambda, \qquad (a, b) \mapsto \overline{a}^\top A^{-1} b.$$

Suppose V is a $2g \times 2g$ Seifert matrix of a knot K of the following kind:

$$\begin{pmatrix} B & C + \mathbb{1} \\ C^\top & D \end{pmatrix}, \qquad \begin{array}{c} B, C, D \text{ are } g \times g, \\ B, D \text{ are symmetric.} \end{array} \tag{5}$$

Note that any Seifert matrix of K is congruent to one of this kind. Then the Blanchfield pairing is isometric to $\lambda(\widetilde{V})$, where $\widetilde{V}$ is the following Hermitian matrix over Λ (see [21] and formulas (2.3) and (2.4) in [6]):

$$\begin{pmatrix} B & -t\mathbb{1} + (1-t)C \\ -t^{-1}\mathbb{1} + (1-t^{-1})C^\top & x \cdot D \end{pmatrix},$$

where we use the shorthand $x = (1-t) + (1-t^{-1}) = (1-t) \cdot (1-t^{-1})$. For a knot K, $n(K)$ is defined as the minimal size of a Hermitian matrix A over Λ such that $\lambda(A)$

is isometric to the Blanchfield pairing of K, and the integral matrix $A(1)$ is congruent to a diagonal matrix.

We will use the following classical result on integral forms; see e.g. [30].

Lemma 22 *An indefinite odd unimodular symmetric integral form can be represented by a diagonal matrix.* □

Here, an integral form θ is called even if $\theta(v, v) \in 2\mathbb{Z}$ for all v, and odd otherwise. Note the sum of two even forms is even, and the sum of an even and an odd form is odd.

Lemma 23 *Any matrix of the kind (5) is congruent to another matrix of the kind (5) with B representing an odd form.*

Proof Suppose B is even. Then we distinguish two cases, depending on whether D is even as well. If it is, a simple change of basis yields the following congruent matrix:

$$\begin{pmatrix} B + C + C^\top + \mathbb{1} + D & C + D + \mathbb{1} \\ C^\top + D & D \end{pmatrix},$$

and $B + C + C^\top + \mathbb{1} + D$ is odd, because B, D, and $C + C^\top$ are even, while $\mathbb{1}$ is odd. If, on the other hand, D is odd, then again, a simple change of basis gives the congruent matrix:

$$\begin{pmatrix} D & -C^\top \\ -C - \mathbb{1} & B \end{pmatrix}.$$

□

Before addressing $u_{\mathrm{alg}} \leq 2g_{\mathrm{alg}}$, let us warm up by directly proving Corollary 4 (which also follows from Theorem 3); i.e. we show that for all knots K, we have

$$u_{\mathrm{alg}}(K) = n(K) \leq \deg \Delta_K.$$

Proof of Corollary 4 It is well-known that the S-equivalence class of Seifert forms of K contains a Seifert matrix V of size $\deg \Delta_K$; indeed, every S-equivalence class has a non-singular representative [38], whose dimension must equal the degree of the Alexander polynomial.

After a basis transformation, we may assume that V is of the kind (5). By Lemma 23 we may assume B represents an odd form. Thus

$$\widetilde{V}(1) = \begin{pmatrix} B & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

is odd as well; and furthermore symmetric, unimodular and indefinite, and thus congruent over $\mathbb{Z}$ to a diagonal matrix by Lemma 22. Since $\lambda(\widetilde{V})$ represents the Blanchfield pairing, this concludes the proof. □

Proof of the second inequality of Theorem 3 Let V be a Seifert matrix of K of size $2g$ with an Alexander-trivial subgroup of rank $2g-2g_{\mathrm{alg}}(K)$. Such a V exists by the definition of g_{alg}. Using Lemma 6, one may change the basis such that the first $2g-2g_{\mathrm{alg}}(K)$ basis vectors generate the Alexander-trivial subgroup, and such that V appears as follows:

$$\begin{array}{c} g-g_{\mathrm{alg}}(K) \\ g-g_{\mathrm{alg}}(K) \\ g_{\mathrm{alg}}(K) \\ g_{\mathrm{alg}}(K) \end{array} \begin{pmatrix} 0 & \mathbb{1}+U & E & F \\ U^\top & 0 & G & H \\ E^\top & G^\top & B & \mathbb{1}+C \\ F^\top & H^\top & C^\top & D \end{pmatrix}.$$

Here, all upper-case letters denote square matrices, whose sizes are indicated to the left of the matrix. The matrix U is upper triangular with zeros on the diagonal; and the matrices B and D are symmetric. Furthermore, we may assume B represents an odd form by applying Lemma 23 to the lower right $2g_{\mathrm{alg}}(K) \times 2g_{\mathrm{alg}}(K)$ submatrix of V; note that the involved basis change does not affect the upper left quadratic submatrix of size $2(g - g_{\mathrm{alg}}(K))$.

If one swaps the second and third column and second and third row of V, one obtains a matrix V' of the kind (5), so λ of the following matrix $\widetilde{V}'$ is isometric to the Blanchfield pairing:

$$\begin{pmatrix} 0 & E & -t\mathbb{1}+(1-t)U & (1-t)F \\ E^\top & B & (1-t)G^\top & -t\mathbb{1}+(1-t)C \\ -t^{-1}\mathbb{1}+(1-t^{-1})U^\top & (1-t^{-1})G & 0 & xH \\ (1-t^{-1})F^\top & -t^{-1}\mathbb{1}+(1-t^{-1})C^\top & xH^\top & xD \end{pmatrix}.$$

Again swapping the second and third column and row now gives a matrix W_1 equal to

$$\begin{pmatrix} 0 & -t\mathbb{1}+(1-t)U & E & (1-t)F \\ -t^{-1}\mathbb{1}+(1-t^{-1})U^\top & 0 & (1-t^{-1})G & xH \\ E^\top & (1-t)G^\top & B & -t\mathbb{1}+(1-t)C \\ (1-t^{-1})F^\top & xH^\top & -t^{-1}\mathbb{1}+(1-t^{-1})C^\top & xD \end{pmatrix}.$$

Since $\det(-t\mathbb{1}+(1-t)U) = (-t)^{g_{\mathrm{alg}}}$ is a unit in Λ, there is an inverse $S = (-t\mathbb{1} + (1-t)U)^{-1}$ over Λ. Now let the transformation matrix T be

$$\begin{pmatrix} \mathbb{1} & 0 & -(1-t^{-1})\overline{S}^\top G & -x\overline{S}^\top H \\ 0 & \mathbb{1} & -SE & -(1-t)SF \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}.$$

Note that $\det(T) = 1$. One may compute $W_2 = \overline{T}^\top W_1 T$ to be the block sum of the quadratic matrix of size $2(g - g_{\mathrm{alg}}(K))$

$$W_3 = \begin{pmatrix} 0 & -t\mathbb{1}+(1-t)U \\ -t^{-1}\mathbb{1}+(1-t^{-1})U^\top & 0 \end{pmatrix}$$

and another quadratic matrix W_4 of size $2g_{\mathrm{alg}}(K)$, which we do not write out for aesthetic reasons. The first block W_3 can be split off because it has determinant 1. In other words, since $\lambda(W_2)$ is isometric to the Blanchfield pairing, and $W_2 = W_3 \oplus W_4$, we find $\lambda(W_4)$ to be isometric to the Blanchfield pairing as well. If W_4 evaluates at $t = 1$ to an integral matrix that is congruent to a diagonal matrix, then we have proven that $u_{\mathrm{alg}}(K) = n(K) \leq 2g_{\mathrm{alg}}(K)$, as desired. Note that

$$W_1(1) = \begin{pmatrix} 0 & -\mathbb{1} & E(1) & 0 \\ -\mathbb{1} & 0 & 0 & 0 \\ E(1) & 0 & B(1) & -\mathbb{1} \\ 0 & 0 & -\mathbb{1} & 0 \end{pmatrix}$$

and

$$T(1) = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & E(1) & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} \quad \Rightarrow \quad W_4(1) = \begin{pmatrix} B(1) & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

So $W_4(1)$ is indeed congruent to a diagonal matrix by Lemma 22. □

Example 24 There is no obvious way in which Theorem 3 could be sharpened, since each of the two inequalities in that theorem may be an equality. One need not look far for examples. On the one hand, $g_{\mathrm{alg}}(K) = u_{\mathrm{alg}}(K)$ occurs e.g. for knots K with $|\sigma(K)| = 2g(K)$, such as 2–stranded torus knots. On the other hand, the algebraic unknotting number may exceed the 3–genus of a knot, which is an upper bound for the algebraic genus; e.g. $u_{\mathrm{alg}}(7_4) = 2$, $g(7_4) = g_{\mathrm{alg}}(7_4) = 1$, or $u_{\mathrm{alg}}(9_{49}) = 3$, $g(9_{49}) = g_{\mathrm{alg}}(9_{49}) = |\sigma(9_{49})|/2 = 2$.

6 The algebraic genus of fibered knots

By the definitions (compare Sect. 1 and Definition 5), we have $t(K) \leq g_{\mathrm{alg}}(K)$ for all knots, where t denotes Taylor's invariant. A priori, g_{alg} can be arbitrarily larger than Taylor's invariant t. In particular, for knots K with Alexander polynomial of degree 4, $g_{\mathrm{alg}}(K) \leq 2 = \frac{\deg \Delta_K(t)}{2}$; and one would suspect that 2 can be attained independently of the value of $t(K)$. However, it turns out that if additionally $t(K) = 0$, i.e. K is algebraically slice, and $\Delta_K(t)$ is monic, then $g_{\mathrm{alg}}(K)$ is at most 1. In fact, we show the following.

Proposition 25 *If a knot K is algebraically slice and has monic Alexander polynomial of degree* 4*, then* $g_{\mathrm{alg}}(K) = 1$.

To the authors this was surprising; for example, since fibered knots are known to have monic Alexander polynomial, this yields the following.

Corollary 26 *Algebraically slice, genus* 2*, fibered knots have topological slice genus at most* 1. □

By recalling the definition of an algebraically slice knot (one, and thus all, Seifert matrices are metabolic) and the fact that every knot has a Seifert matrix of size $\deg\Delta_K(t) \times \deg\Delta_K(t)$ up to S-equivalence (which is implied by the fact that all Seifert matrices are S-equivalent to one with non-zero determinant as proven by Trotter [38]), Proposition 25 follows from the following lemma.

Lemma 27 *Let A be an integral metabolic 4×4 matrix. Suppose A and $A - A^\top$ are invertible. Then there is a subgroup of $\mathbb{Z}^4$ of rank two restricted to which A has the form*

$$\begin{pmatrix} 0 & 1 \\ 0 & * \end{pmatrix}$$

Before providing the proof, which consists of an elementary calculation, we provide an example that shows that no analog statement holds for 6×6 Seifert matrices.

Example 28 Let K be a knot with the following Seifert matrix of det 1:

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & -3 & -6 & 0 & 0 & 0 \\ -3 & -1 & -3 & 0 & 0 & 0 \\ -6 & -3 & -7 & 0 & 0 & 0 \end{pmatrix}.$$

By definition K is algebraically slice; however, $M + M^\top$ is the zero matrix modulo 3, whence $g_{\text{alg}}(K) = 3 = \frac{\deg\Delta_K(t)}{2}$ by Proposition 12(ii).

Proof of Lemma 27 By assumption, A is of the form

$$\left(\begin{array}{c|c} 0 & U \\ \hline V & * \end{array}\right)$$

for 2×2 matrices U, V. Invertibility of A is inherited by U and V, so A is congruent to

$$\left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & U^{-t} \end{array}\right) \cdot \left(\begin{array}{c|c} 0 & U \\ \hline V & * \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & U^{-1} \end{array}\right) = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline V' & * \end{array}\right). \tag{6}$$

Let

$$V' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $a = 0$, the subgroup generated by the first and third basis vector is of the desired form with respect to the matrix (6), and similarly if $d = 0$. For any invertible 2×2 matrix T, the matrix (6) is congruent to:

$$\left(\begin{array}{c|c} T^\top & 0 \\ \hline 0 & T^{-1} \end{array}\right) \cdot \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline V' & * \end{array}\right) \cdot \left(\begin{array}{c|c} T & 0 \\ \hline 0 & T^{-\top} \end{array}\right) = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline T^{-1}V'T & * \end{array}\right).$$

So one may try to decrease $|a|$ by replacing V' by $T^{-1}V'T$. Assume this is no longer possible. Taking

$$T = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$$

replaces a by $a \mp c$. So our assumption yields that $|a| \leq |c|/2$, and similarly $|a| \leq |b|/2$. Taking

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

switches a and d, so we also have $|a| \leq |d|$. We have $\det V' = \pm 1$, and $\det(V' - \mathbb{1}) = \pm 1$. Thus

$$\begin{aligned} &|\det V' - \det(V' - \mathbb{1})| \leq 2 \Rightarrow \\ &|(ad - bc) - (ad - bc - a - d + 1)| \leq 2 \Rightarrow \\ &|a + d| \leq 3. \end{aligned}$$

Therefore,

$$\begin{aligned} bc &= ad \pm 1 \Rightarrow \\ |bc| &\leq |a||(a + d) - a| + 1 \\ &\leq |a|^2 + |a + d||a| + 1 \\ &\leq |bc|/4 + 3|c|/2 + 1 \Rightarrow \\ 0 &\leq -3|bc| + 6|c| + 4 \Rightarrow \\ 0 &\leq (6 - 3|b|)|c| + 4. \end{aligned}$$

This implies that if $|b| \geq 3$, then $|c| \leq 1$, which in turn implies $|a| = 0$ by our assumption. Since one may switch the role of b and c, the only remaining case is $|b|, |c| = 2$ and $|a|, |d| = 1$. But these values contradict that $\det V' = 1$. □

7 On the optimality of slice genus bounds

The algebraic genus $g_{\text{alg}}(L)$ of a link L is an upper bound for the topological slice genus of L that depends only on the S-equivalence class of Seifert forms for L. Question B asks if it is the best bound with that property. In this section, we pursue this and related questions on the optimality of slice genus bounds. To make the dependency on the Seifert form more precise, let us fix for each Seifert form θ the following sets of links:

$$\begin{array}{l}
\mathcal{E}_\theta = \left\{ \begin{array}{c}\text{links that admit a Seifert surface with}\\ \text{Seifert form isometric to } \theta\end{array} \right\}, \\
\cap \\
\mathcal{S}_\theta = \left\{ \begin{array}{c}\text{links that admit a Seifert surface with}\\ \text{Seifert form S-equivalent to } \theta\end{array} \right\}, \\
\cap \\
\mathcal{C}_\theta = \left\{ \begin{array}{c}\text{links that admit a Seifert surface with}\\ \text{Seifert form algebraically concordant to } \theta\end{array} \right\}.
\end{array}$$

Written in this notation, the statement of Proposition 10 is that

$$\forall\, L, L' \in \mathcal{S}_\theta \quad \Rightarrow \quad g_{\text{alg}}(L) = g_{\text{alg}}(L'),$$

the inequality (4) says that for all Seifert forms θ,

$$\max_{L' \in \mathcal{S}_\theta} g_{\text{top}}(L') \le g_{\text{alg}}(\theta), \tag{7}$$

and Question B asks whether the inequality (7) is in fact an equality. This question is about the slice genus. A corresponding qualitative question about sliceness would be: which Seifert forms guarantee the sliceness of a link? More specifically, given a Seifert form θ, does the following hold:

$$g_{\text{alg}}(\theta) > 0 \quad \Rightarrow \quad \max_{L \in \mathcal{E}_\theta} g_{\text{top}}(L) > 0? \tag{8}$$

The rest of this section pursues Question B for knots. Except for Remark 35, where answers for related questions are provided, we only consider knots rather than links in the rest of this section.

Note that for the Seifert form of a knot, $g_{\text{alg}}(\theta) = 0 \Leftrightarrow \Delta_\theta = 1$. Livingston [27] proved (8) for all Seifert forms θ of knots satisfying a technical condition on the Alexander polynomial of θ. For this, he used Casson-Gordon obstructions to sliceness, which involve the d–fold branched covers of a knot for prime powers d. For non-prime powers, no such obstructions are available, which is precisely the reason that Casson-Gordon obstructions do not solve (8) for all Seifert forms of knots. The proof of (8) for Seifert forms of knots was completed by Kim using L^2–invariants—for Seifert forms of links, it appears to be open.

Theorem 29 ([20]) *Let θ be the Seifert form of a knot with Alexander polynomial Δ_θ not equal to* 1. *Then θ is realized as the Seifert form of a knot that is not topologically slice.*

Now, the strategy to attack the quantitative question must be to construct for a given Seifert form θ a knot K realizing θ with $g_{\text{top}}(K) = g_{\text{alg}}(K)$; or, to obtain partial results, with $g_{\text{top}}(K)$ as high as possible. To this end, one needs lower bounds for the topological slice genus. To the best knowledge of the authors, there are only three such bounds (disregarding those which are in fact only obstructions to sliceness):

- the Seifert form bounds (such as Levine-Tristram signatures), subsumed by the bound coming from Taylor's invariant $t(\theta)$ [37];
- the bounds from Casson-Gordon invariants [18];
- and bounds coming from L^2–signatures [9].

In what follows, we will apply the first two bounds of that list to the problem, and obtain partial results. A fully affirmative answer would in all probability require stronger lower bounds for the topological slice genus than the ones at our disposal.

Since Taylor's bound is determined by the Seifert form, its only contribution to Question B is

$$t(\theta) \leq \max_{K \in \mathcal{S}_\theta} g_{\text{top}}(K)$$

for Seifert forms θ of knots. In fact, Taylor's bound is rather the solution to the opposite problem—it is the optimal *lower* bound determined by the Seifert form. The corresponding question for Seifert forms coming from links with more than one component appears to be open.

Let us now apply Casson-Gordon obstructions to Question B. We briefly fix our notations for branched covers and recall some of their well-known properties (cf. e.g. [32]). Let K be a knot with Seifert form θ. For a positive integer d, we write $M_d(K)$ for the d–fold branched covering of S^3 along K. The homology group $H_1(M_d(K); \mathbb{Z})$ is one of the oldest knot invariants. If d is a prime power, then $H_1(M_d(K); \mathbb{Z})$ is a finite group. If d is an odd prime power, that group is equal to $G \oplus G$ for some group G. Denote by $r_d(K)$ the minimum number of generators of $H_1(M_d(K); \mathbb{Z})$. Then

$$0 \leq r_d(K) \leq \deg(\Delta_K(t)), \tag{9}$$

and $r_d(K)$ is even if d is odd. While the order of $H_1(M_d(K); \mathbb{Z})$ is determined by Δ_K, this is not the case for $r_d(K)$, which is, however, determined by the S-equivalence class of θ: e.g. if M is a matrix for θ, and $P = (M^\top - M)^{-1} M^\top$, then $P^d - (P - \mathbb{1})^d$ is a presentation matrix of $H_1(M_d(K); \mathbb{Z})$. Thus, we also write $r_d(\theta)$ instead of $r_d(K)$. Our result is now the following:

Proposition 30 *Every Seifert form θ of a knot is realized by a knot K with*

$$g_{\text{top}}(K) \geq \max_{\substack{d \text{ prime} \\ \text{power}}} \left\lceil \frac{r_d(\theta)}{2(d-1)} \right\rceil.$$

Of course, this implies

$$\max_{K' \in \mathcal{S}_\theta} g_{\text{top}}(K') \geq \max_{\substack{d \text{ prime} \\ \text{power}}} \left\lceil \frac{r_d(\theta)}{2(d-1)} \right\rceil.$$

It also recovers the inequality $2g_{\text{alg}}(L) \geq r_2(L)$, which was proved earlier on in Proposition 12(ii).

Before proving Proposition 30, we fix our notation of Casson-Gordon invariants. The first integral homology group of $M_d(K)$ admits a linking form $\beta_d : M_d(K) \times M_d(K) \to \mathbb{Q}/\mathbb{Z}$, which is determined by θ. We write $H_1(M_d(K);\mathbb{Z})^*$ for the group of characters χ of $H_1(M_d(K))$, i.e. homomorphisms $H_1(M_d(K);\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. The linking form then induces a dual form β_d^* on $H_1(M_d(K);\mathbb{Z})^*$. Casson and Gordon associate to (K, d, χ) an invariant

$$\tau(K, d, \chi) \in W(\mathbb{C}(t)) \otimes \mathbb{Q},$$

where $W(\cdot)$ denotes the Witt group [7,8]. One may take signatures of $\tau(K, d, \chi)$, who obstruct the topological sliceness of K. We will only need the ordinary signature, which we denote by $\sigma\tau(K, d, \chi)$. Gilmer showed how this knot invariant induces lower bounds for the topological slice genus:

Lemma 31 ([18]) *For a knot K and a prime power d, the form β^* on $H_1(M_d(K);\mathbb{Z})^*$ decomposes as a direct sum of forms β_1 and β_2 on G_1 and G_2, respectively; such that G_1 may be generated by $2(d-1)g_{\text{top}}(K)$ elements, and β_2 admits a metabolizer H in which all non-trivial characters $\chi \in H$ of prime power order satisfy*

$$\left|\sigma\tau(K, d, \chi) + \sum_{j=1}^{d-1} \sigma_{j/d}(K)\right| \leq 2d g_{\text{top}}(K). \tag{10}$$

Note that Gilmer proved this statement in the smooth category, but it is known to carry over to the topological category by the the work of Freedman.

The Proof of Proposition 30 strongly relies on the techniques Livingston used for proving his partial resolution of (8). We will need the following construction:

Lemma 32 ([26, Theorem 4.6]) *For every Seifert form θ of a knot, every positive number C, and every prime power d, there is a knot K realizing θ with $\sigma\tau(K, d, \chi) > C$ for all non-trivial characters $\chi \in H_1(M_d(K);\mathbb{Z})^*$.* □

The knot K in Lemma 32 is constructed from an arbitrary knot K' that realizes θ by *infection*, i.e. satellite operations that do not change the Seifert form, but affect the Casson-Gordon invariants to an extent dependent on the signatures of the pattern knots (as was determined by Litherland [25]).

Proof of Proposition 30 Let a prime power d be fixed, and set

$$C = d \cdot \dim\theta + \sum_{j=1}^{d-1} \sigma_{j/d}(K).$$

By Lemma 32, there is a knot K realizing θ with $\sigma\tau(K, d, \chi) > C$ for all non-trivial $\chi \in H_1(M_d(K);\mathbb{Z})^*$. Since $2g_{\text{top}}(K) \leq \dim\theta$, no non-trivial character χ satisfies (10). Therefore, in the notation of Lemma 31, G_2 is trivial, and thus $H_1(M_d(K);\mathbb{Z})^* = G_1$ cannot be generated by less than $2(d-1)g_{\text{top}}(K)$ elements. The statement of Proposition 30 follows. □

Remark 33 Let us discuss how to apply Proposition 30. First of all, to determine

$$\max_{\substack{d \text{ prime} \\ \text{power}}} \left\lceil \frac{r_d(\theta)}{2(d-1)} \right\rceil, \tag{11}$$

for the Seifert form θ of a knot, it is not necessary to calculate $r_d(\theta)$ for all prime powers d; indeed, Livingston showed that (11) is 0 if and only if Δ_θ is the product of n–th cyclotomic polynomials with n divisible by three distinct primes [27]. On the other hand, a short calculation yields that (11) can only be greater than 1 if $r_d(K) > 2(d-1)$ for some d, and because of (9) this can only happen for $d \leq \deg \Delta_\theta/2$. These are the cases for which Theorem 30 goes beyond Kim's Theorem 29.

One can also ask if Proposition 30 can show that a Seifert form θ is realized by a knot K without topological genus defect, i.e. a knot with $g_{\text{top}}(K) = g(K)$. One checks that if $\dim \theta > 2$, this can only be accomplished by $d = 2$, namely if $r_2(\theta) \in \{\dim \theta, \dim \theta - 1\}$.

Example 34 Let us give a concrete example of a Seifert form for which Question B is open. Namely, take θ to be a Seifert form of the knot K, which is 10_{103} in Rolfsen's table, given by the following matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -2 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ -1 & -2 & -1 & -2 & -2 & 1 \end{pmatrix}.$$

The first and second standard basis vector generate an isotropic subgroup of rank 2, so $t(\theta) \leq 1$. One computes the signature to be 2, and so $t(\theta) \geq 1 \Rightarrow t(\theta) = 1$. On the other hand, the first and third standard basis vector generate an Alexander-trivial subgroup of rank 2, so $g_{\text{alg}}(\theta) \leq 2$, and moreover $u_{\text{alg}}(\theta) = 3$ [4], which implies $g_{\text{alg}}(\theta) \geq 2 \Rightarrow g_{\text{alg}}(\theta) = 2$. Now, the smooth slice genus of K happens to be 1, and so $g_{\text{top}}(K) = 1$. But can θ be realized by another knot K' with $g_{\text{top}}(K') = 2$? If $r_2(\theta)$ were at least 3, this would follow from Proposition 30, but $M_d(\theta) = \mathbb{Z}/15 \oplus \mathbb{Z}/5$, whence $r_2(\theta) = 2$. So we do not know whether $\max_{K \in \mathcal{S}_\theta} g_{\text{top}}(K)$ is 1 or 2.

Remark 35 We end this section by providing answers to some related questions on optimality of classical slice genus bounds. All of these answers are rather immediate from standard results, but we provide them for completeness.

The algebraic concordance class of a Seifert form yields no upper bound for the topological slice genus:

$$\max_{L \in \mathcal{C}_\theta} g_{\text{top}}(L) = \infty. \tag{12}$$

The Seifert form gives only trivial upper bounds for the smooth slice genus:

$$\max_{L\in\mathcal{E}_\theta} g_{\text{smooth}}(L) = \text{rk}(\theta - \theta^\top)/2 \quad \text{and} \tag{13}$$

$$\max_{L\in\mathcal{S}_\theta} g_{\text{smooth}}(L) = \infty. \tag{14}$$

Proof of (12),(13),(14) Let ζ be the (algebraically slice) Seifert form given by the matrix

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Then for any $k \geq 0$, the form $\theta_k = \theta \oplus \zeta^{\oplus k}$ is algebraically concordant to θ. Moreover, the first homology of the double branched covering of a knot with Seifert form θ_k has minimum number of generators at least $2k$. Applying Proposition 30 settles (12).

Statement (13) may be proven as in [29]: it is Rudolph's result [33] that every Seifert form θ may be realized as the Seifert form of a quasipositive Seifert surface F. As a consequence of the slice-Bennequin inequality, which Rudolph proved [35] building on Kronheimer and Mrowka's resolution of the Thom Conjecture [22], the smooth slice genus of ∂F equals its three-genus, which is $g(F) = \text{rk}(\theta - \theta^\top)/2$. This shows (14) as well, since $\mathcal{S}_\theta$ contains Seifert forms of arbitrarily high dimension. □

8 Reformulation of previously known results in terms of g_{alg}

By finding a separating curve with Alexander polynomial 1 on a minimum genus Seifert surface and using Freedman's Theorem (1), one can show that g_{top} is smaller than the three-dimensional genus. This was used by Rudolph to show that $g_{\text{top}}(T) < g(T)$ for most torus knots; in fact even $g_{\text{top}}(T) \leq \frac{9}{10} g(T)$ [34]. Baader used this idea to show that if a minimal genus Seifert surface for a knot K contains an embedded annulus with framing ± 1, then $\widehat{g_{\text{top}}}(K) = g(K)$ if and only if $|\sigma(K)| = 2g(K)$ [1], a result that we generalized in Proposition 14.

It turns out that the existence of separating Alexander polynomial 1 knots on Seifert surfaces is completely determined by the Seifert form; compare Proposition 9. The following results by Baader, Liechti, McCoy and the authors were proven using some version of this. We present them rewritten in the language of g_{alg}, while suppressing the inequalities $g_{\text{top}} \leq g_{\mathbb{Z}} \leq g_{\text{alg}}$:

- For all knots K [12]: $\quad g_{\text{alg}}(K) \leq \dfrac{\deg \Delta_K}{2}.$
- For prime homogeneous knots K that are not positive or negative [3]:

$$\widehat{g_{\text{alg}}}(K) \leq g(K) - \frac{1}{3}.$$

– There is an infinite family of 2–bridge knots K satisfying [13]

$$g_{\mathrm{alg}}(K) < g_{\mathrm{smooth}}(K) = g(K).$$

– The algebraic genus of torus links satisfies [2]

$$\frac{1}{2} \le \lim_{p,q\to\infty} \frac{g_{\mathrm{alg}}(T_{p,q})}{g(T_{p,q})} \le \frac{3}{4}$$

and for $p \ge q \ge 3$, and $(p, q) \notin \{(3, 3), (4, 3), (5, 3), (6, 3), (4, 4)\}$:

$$\frac{1}{2} \le \frac{g_{\mathrm{alg}}(T_{p,q})}{g(T_{p,q})} \le \frac{6}{7} = \frac{g_{\mathrm{alg}}(T_{8,3})}{g(T_{8,3})}.$$

– For all prime knots with up to 11 crossings, one has [23]

$$g_{\mathrm{top}}(K) = \min\{g_{\mathrm{alg}}(K), g_{\mathrm{smooth}}(K)\}.$$

– For positive braid knots K with $\sigma(K) < 2g(K)$ one has [24]

$$g_{\mathrm{alg}}(K) < g(K).$$

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References

1. Baader, S.: On the stable 4-genus of knots with indefinite Seifert form. Commun. Anal. Geom. **24**(2), 301–305 (2016). arXiv:1408.6091
2. Baader, S., Feller, P., Lewark, L., Liechti, L.: On the topological 4-genus of torus knots. Trans. Am. Math. Soc. **370**(4), 2639–2656 (2018). arXiv:1509.07634
3. Baader, S., Lewark, L.: The stable 4-genus of alternating knots. Asian J. Math. (6), 1183–1190 (2017). arXiv:1505.03345
4. Borodzik, M., Friedl, S.: Knotorious world wide web page. http://www.mimuw.edu.pl/~mcboro/knotorious.php, retrieved November 8 (2016)
5. Borodzik, M., Friedl, S.: On the algebraic unknotting number. Trans. Lond. Math. Soc. **1**(1), 57–84 (2014). arXiv:1308.6105
6. Borodzik, M., Friedl, S.: The unknotting number and classical invariants, I. Algebr. Geom. Topol. **15**(1), 85–135 (2015). arXiv:1203.3225
7. Casson, A. J., McA, C.: Gordon: on slice knots in dimension three, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII, Am. Math. Soc., Providence, RI (1978) pp. 39–53
8. Casson, A. J., Gordon, C. McA.: Cobordism of classical knots, À la recherche de la topologie perdue, Progr. Math., vol. 62, Birkhäuser Boston, Boston, MA, With an appendix by P. M. Gilmer, pp. 181–199 (1986)
9. Cha, J.C.: Topological minimal genus and L^2-signatures. Algebr. Geom. Topol. **8**(2), 885–909 (2008)
10. Cha, J. C., Livingston, C.: KnotInfo: Table of Knot Invariants. www.indiana.edu/~knotinfo, retrieved on November 8 (2016)

11. Cochran, T. D., Orr, K. E., Teichner, P.: Knot concordance, Whitney towers and L^2-signatures, Ann. of Math. (2) **157**(2), pp. 433–519 (2003). arXiv:math/9908117
12. Feller, P.: The degree of the Alexander polynomial is an upper bound for the topological slice genus. Geom. Topol. **20**(3), 1763–1771 (2016). arXiv:1504.01064
13. Feller, P., McCoy, D.: On 2-bridge knots with differing smooth and topological slice genera. Proc. Am. Math. Soc. **144**(12), 5435–5442 (2016). arXiv:1508.01431
14. Fogel, M.: The algebraic unknotting number, Ph.D. thesis, University of California, Berkeley (1993)
15. Freedman, M.H.: The topology of four-dimensional manifolds. J. Differ. Geom. **17**(3), 357–453 (1982)
16. Freedman, M.H., Quinn, F.: Topology of 4-manifolds, Princeton Mathematical Series, vol. 39. Princeton University Press, Princeton, NJ (1990)
17. Garoufalidis, S., Teichner, P.: On knots with trivial Alexander polynomial. J. Differ. Geom. **67**(1), 167–193 (2004)
18. Gilmer, P.M.: On the slice genus of knots. Invent. Math. **66**(2), 191–197 (1982)
19. Gompf, R.E., Stipsicz, A.I.: 4-manifolds and Kirby Calculus, Graduate Studies in Mathematics, vol. 20. American Mathematical Society, Providence, RI (1999)
20. Kim, T.: An infinite family of non-concordant knots having the same Seifert form. Comment. Math. Helv. **80**(1), 147–155 (2005). arXiv:math/0402425
21. Ko, K.H.: A Seifert-matrix interpretation of Cappell and Shaneson's approach to link cobordisms. Math. Proc. Cambridge Philos. Soc. **106**(3), 531–545 (1989)
22. Kronheimer, P.B., Mrowka, T.S.: Gauge theory for embedded surfaces, I. Topology **32**(4), 773–826 (1993)
23. Lewark, L., McCoy, D.: On calculating the slice genera of 11- and 12-crossing knots, accepted for publication by Exp. Math. (2016) arXiv:1508.01098v2
24. Liechti, L.: Positive braid knots of maximal topological 4-genus. Math. Proc. Camb. Philos. Soc. **161**(3), 559–568 (2016). arXiv:1511.03883
25. Litherland,R. A.: Signatures of iterated torus knots, topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, : Lecture Notes in Math., vol. 722. Springer, Berlin **1979**, 71–84 (1977)
26. Livingston,C.: Examples in Concordance, a shortened version of this article has been published as [27] (2001). arXiv:math/0101035v2
27. Livingston, C.: Seifert forms and concordance. Geom. Topol. **6**, 403–408 (2002). arXiv:math/0101035v3
28. Livingston, C.: The stable 4-genus of knots. Algebr. Geom. Topol. **10**(4), 2191–2202 (2010). arXiv:0904.3054
29. Livingston, C., Naik, S.: Ozsváth-Szabó and Rasmussen invariants of doubled knots. Algebr. Geom. Topol. **6**, 651–657 (2006). arXiv:math/0505361
30. Milnor, J., Husemöller,D.: Symmetric bilinear forms, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 73, Springer (1973)
31. Murakami,H.: Algebraic unknotting operation, Proceedings of the Second Soviet-Japan Joint Symposium of Topology (Khabarovsk, 1989), vol. 8, Questions Answers Gen. Topology, vol. 1, pp. 283–292 (1990)
32. Rolfsen,D.: Knots and Links, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Houston, TX, Corrected reprint of the 1976 original (1990)
33. Rudolph, L.: Constructions of quasipositive knots and links, I, Knots, Braids and Singularities (Plans-sur-Bex, : Monogr. Enseign. Math., vol. 31, Enseignement Math. Geneva **1983**, 233–245 (1982)
34. Rudolph, L.: Some topologically locally-flat surfaces in the complex projective plane. Comment. Math. Helv. **59**(4), 592–599 (1984)
35. Rudolph,L.: Quasipositivity as an obstruction to sliceness, Bull. Amer. Math. Soc. (N.S.) **29**(1), 51–59 (1993). arXiv:math/9307233
36. Saeki, O.: On algebraic unknotting numbers of knots. Tokyo J. Math. **22**(2), 425–443 (1999)
37. Taylor,L. R.: On the genera of knots, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, : Lecture Notes in Math., vol. 722. Springer, Berlin **1979**, 144–154 (1977)
38. Trotter, H.F.: Homology of group systems with applications to knot theory. Ann. Math. **76**(2), 464–498 (1962)