Sel. Math. New Ser. (2018) 24:3835–3873
https://doi.org/10.1007/s00029-018-0417-z

Selecta Mathematica
New Series

Some phenomena in tautological rings of manifolds

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Published online: 5 June 2018

Abstract We prove several basic ring-theoretic results about tautological rings of manifolds W, that is, the rings of generalised Miller–Morita–Mumford classes for fibre bundles with fibre W. Firstly we provide conditions on the rational cohomology of W which ensure that its tautological ring is finitely-generated, and we show that these conditions cannot be completely relaxed by giving an example of a tautological ring which fails to be finitely-generated in quite a strong sense. Secondly, we provide conditions on torus actions on W which ensure that the rank of the torus gives a lower bound for the Krull dimension of the tautological ring of W. Lastly, we give extensive computations in the tautological rings of $\mathbb{CP}^2$ and $S^2 \times S^2$.

Mathematics Subject Classification 57S25 · 57R22 · 57R20 · 55R40 · 55R40 · 55R10

1 Introduction

1.1 Recollections on tautological rings

A smooth fibre bundle $\pi : E \to B$ with closed d-dimensional fibre W equipped with an orientation of the vertical tangent bundle $T_\pi E$ has characteristic classes defined as follows. For each characteristic class $c \in H^k(BSO(d))$ of oriented d-dimensional vector bundles, we may form

The author was partially supported by EPSRC Grant EP/M027783/1.

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$$\kappa_c(\pi) := \int_\pi c(T_\pi E) \in H^{k-d}(B),$$

the *generalised Mumford–Morita–Miller class* (or *κ-class*) associated to c, by evaluating c on the vector bundle $T_\pi E$ and integrating the result along the fibres of the map π. This construction may in particular be applied to the universal such fibre bundle, whose base space is the classifying space $B\mathrm{Diff}^+(W)$ of the topological group of orientation-preserving diffeomorphisms of W, to give universal characteristic classes $\kappa_c \in H^*(B\mathrm{Diff}^+(W))$. If c has degree d then κ_c is a degree zero cohomology class, and may be identified with the characteristic number $\int_W c(TW)$ of W.

If we work in cohomology with rational coefficients then $H^*(BSO(d);\mathbb{Q})$ is generated by the Pontrjagin and Euler classes, and in this case we define the *tautological ring*

$$R^*(W) \subset H^*(B\mathrm{Diff}^+(W);\mathbb{Q})$$

to be the subring generated by all classes κ_c. Our goal is to describe some quantitative and qualitative properties of these rings, for certain manifolds W.

Before doing so, we introduce some variants. The topological group $\mathrm{Diff}^+(W,*)$ of diffeomorphisms of W which fix a marked point $* \in W$ has a homomorphism to $GL_d^+(\mathbb{R})$ by sending a diffeomorphism φ to its differential $D\varphi_*$ at the marked point. On classifying spaces this gives a map

$$s : B\mathrm{Diff}^+(W,*) \longrightarrow BGL_d^+(\mathbb{R}) \simeq BSO(d)$$

and for each $c \in H^*(BSO(d);\mathbb{Q})$ we may also form $s^*c \in H^*(B\mathrm{Diff}^+(W,*);\mathbb{Q})$. We let the *tautological ring fixing a point* $R^*(W,*) \subset H^*(B\mathrm{Diff}^+(W,*);\mathbb{Q})$ be the subring generated by all the classes κ_c and s^*c.

Finally, if $B\mathrm{Diff}^+(W,D^d)$ is the classifying space of the group of diffeomorphisms of W which are the identity near a marked disc $D^d \subset W$, then we let the *tautological ring fixing a disc* $R^*(W,D^d) \subset H^*(B\mathrm{Diff}^+(W,D^d);\mathbb{Q})$ be the subring generated by all the classes κ_c. The inclusions of diffeomorphism groups

$$B\mathrm{Diff}^+(W) \longleftarrow B\mathrm{Diff}^+(W,*) \longleftarrow B\mathrm{Diff}^+(W,D^d)$$

therefore yield $\mathbb{Q}$-algebra homomorphisms

$$R^*(W) \longrightarrow R^*(W,*) \longrightarrow R^*(W,D^d)$$

whose composition is surjective.

These rings have been studied by Grigoriev [22], and by Galatius, Grigoriev, and the author [21], mainly for the manifolds $W = \#^g S^n \times S^n$ with n odd. This is the natural generalisation of the case of oriented surfaces, i.e. $n = 1$, which has been studied in great detail: see e.g. [17,28,31,34]. Our purpose here is to explain to what extent those results apply to more general manifolds. We will only consider even-dimensional manifolds. For odd-dimensional manifolds the classes κ_c have odd degree and so anticommute and are nilpotent, and tautological rings in this situation seem to have a different flavour.

1.2 Finiteness

Our first result concerns conditions under which the rings $R^*(W)$ and $R^*(W, *)$ are suitably finite.

Theorem A *Let W be a closed smooth oriented $2n$-manifold, and assume that either*

(H1) $H^*(W; \mathbb{Q})$ is non-zero only in even degrees, or
(H2) $H^*(W; \mathbb{Q})$ is non-zero only in degrees 0, $2n$ and odd degrees, and $\chi(W) \neq 0$.

Then

(i) $R^*(W)$ is a finitely-generated $\mathbb{Q}$-algebra, and
(ii) $R^*(W, *)$ is a finitely-generated $R^*(W)$-module.

The result under hypothesis (H2) generalises a theorem of Grigoriev [22], and proceeds by establishing the same basic source of relations among κ-classes found by Grigoriev. In the case $2n = 2$ this source of relations had been established by the author [36], using ideas of Morita [29,30]. As the later results of [22] and the results of [21] are deduced almost entirely from this basic source of relations, the same results largely follow assuming only hypothesis (H2). For example, for $g > 1$, k odd, and $n \geq k$, it follows that

$$\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_{n-1}}] \longrightarrow R^*(\#^g S^k \times S^{2n-k})/\sqrt{0}$$

is surjective, which was obtained in [21] only in the case $k = n$. We give details of this in Sect. 4.1.

The result under hypothesis (H1) is entirely new and its method of proof is novel. We consider a fibre bundle $W \to E \xrightarrow{\pi} B$ as determining a parametrised spectrum over B, and hence its rational "cochains" as giving a parametrised $H\mathbb{Q}$-module spectrum over B. We then use the notion of Schur-finiteness from the theory of motives to obtain a Cayley–Hamilton-type trace identity for endomorphisms of this cochain object, which establishes concrete relations among κ-classes. Later we shall describe some explicit calculations done using these relations.

1.3 Krull dimension

Our second main result is a general technique, continuing on from our work with Galatius and Grigoriev [21, §4], for estimating the Krull dimension (for which we write Kdim) of the rings $R^*(W)$ from below in terms of torus actions on W. The general statement is Theorem 3.1, but the hypotheses of that theorem are somewhat involved: we state here one of its corollaries with hypotheses which are easy to verify.

Corollary B *Let a k-torus T act effectively on W, and suppose that either*

(i) $\chi(W) \neq 0$ *and the fixed set* W^T *is connected, or*
(ii) the fixed set W^T *is discrete and non-empty.*

Then $\mathrm{Kdim}(R^*(W)) \geq k$.

For example, if W^{2n} is a quasitoric manifold then case (ii) gives the estimate $\mathrm{Kdim}(R^*(W)) \geq n$. As another example, if $W = \#^g S^n \times S^n$ with n odd then it is a consequence of the localisation theorem in equivariant cohomology (which we shall discuss in Sect. 3.1) that *any* torus action on W has connected fixed set, so by case (i) restricting the $SO(n) \times SO(n)$-action on W constructed in [21, §4] to a maximal torus (which has rank $n-1$) we obtain $\mathrm{Kdim}(R^*(W)) \geq n-1$ for $g > 1$, which recovers the calculation of that paper. This example admits many variants: the construction of [21, §4] can be easily modified to give a $SO(k) \times SO(2n-k)$-action on $\#^g S^k \times S^{2n-k}$, so for any odd k and any n we have

$$\mathrm{Kdim}(R^*(\#^g S^k \times S^{2n-k})) \geq n-1.$$

We shall say more about this example in Sect. 4.1.

1.4 Examples

In the last section of the paper we exhibit several phenomena in tautological rings by calculations for specific manifolds. The following result is complementary to Theorem A, and shows that the hypotheses of that theorem cannot be completely removed.

Theorem C *There are closed smooth manifolds W for which $R^*(W)/\sqrt{0}$ is not finitely-generated as a $\mathbb{Q}$-algebra. There are examples of any dimension $4k+2 \geq 6$, and in dimensions $4k+2 \geq 14$ such manifolds can also be assumed to be simply-connected.*

To show the effectiveness of the relations between κ-classes arising in the proof of Theorem A, we apply them to the simplest manifold whose tautological ring is not yet known, namely $\mathbb{CP}^2$. These relations, along with relations associated to the Hirzebruch $\mathcal{L}$-classes coming from index theory, give the following.

Theorem D *The ring $R^*(\mathbb{CP}^2)$ has Krull dimension 2. The ring $R^*(\mathbb{CP}^2, D^4)$ is a vector space of dimension at most 7 over $\mathbb{Q}$.*

In fact, we show that the ring $R^*(\mathbb{CP}^2)/\sqrt{0}$ is equal to either

$$\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/(4\kappa_{p_1^2} - 7\kappa_{ep_1}) \cap (\kappa_{p_1^2} - 2\kappa_{ep_1}, 316\kappa_{ep_1}^3 - 343\kappa_{p_1^4}),$$

whose variety is the union of a line and a plane, or

$$\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/(4\kappa_{p_1^2} - 7\kappa_{ep_1}),$$

whose variety is a plane. It would be interesting to determine which case occurs, and very interesting if it is the first case.

Finally, we give a calculation which shows that the lower bound of Corollary B is not always sharp. The 3-torus cannot act effectively on $S^2 \times S^2$, and yet

Theorem E *The ring $R^*(S^2 \times S^2)$ has Krull dimension 3 or 4.*

The lower bound on the Krull dimension comes from a 1-parameter family of 2-torus actions, to which the method of proof of Corollary B is applied. The upper bound comes from the relations between κ-classes which we found in the proof of Theorem A.

2 Tautological relations and finite generation

Unless specified, all cohomology in this paper will be taken with $\mathbb{Q}$ coefficients.

In this section we describe some techniques for obtaining relations between tautological classes, which for some manifolds W suffice to establish that $R^*(W)$ is finitely-generated. The techniques we will introduce are perhaps more important than any particular application that can be made, but Theorem A will be a consequence.

2.1 Integrality

One consequence of conclusion (ii) of Theorem A is that $R^*(W, *)$ is integral over $R^*(W)$. In fact, this integrality statement implies the two finiteness statements, as follows.

Proposition 2.1 *Suppose that W is a closed smooth oriented d-manifold such that $R^*(W, *)$ is integral over $R^*(W)$. Then*

(i) $R^*(W)$ is a finitely-generated $\mathbb{Q}$-algebra, and
(ii) $R^*(W, *)$ is a finitely-generated $R^*(W)$-module.

For the sake of clarity, we will first formulate and prove a purely algebraic statement of which this proposition is a consequence.

Lemma 2.2 *Let $\pi : B \to E$ be a homomorphism of $\mathbb{Q}$-algebras, $g : E \to B$ be a homomorphism of B-modules (where E is made into an B-module via π), and $C \subset E$ be a finitely-generated subalgebra, generated by $\{c_i\}_{i \in I}$. Let $R \subset B$ be the subalgebra generated by $g(C)$. If each c_i is integral over $\pi(R) \subset E$, with*

$$c_i^{n_i} = \sum_{j=0}^{n_i - 1} \pi(a_{i,j}) c_i^j$$

for some $a_{i,j} \in R$, then R is generated by the finitely-many elements

$$\{a_{i,j}\}_{i,j \in I} \cup \left\{ g\left(\prod c_i^{m_i}\right) \mid m_i < n_i \right\}.$$

Proof By definition R is generated by the elements $g(\prod c_i^{k_i})$, so we must show that these lie in the subring generated by the indicated elements. By assumption we may write $c_i^{k_i}$ as a $\mathbb{Q}[\pi(a_{i,j})]$-linear combination of terms c_i^j with $j < n_i$, and so we may write $\prod c_i^{k_i}$ as a $\mathbb{Q}[\pi(a_{i,j})]$-linear combination of terms $\prod c_i^{m_i}$ with $m_i < n_i$. As g is B- and hence R-linear, we may therefore write $g(\prod c_i^{k_i})$ as a $\mathbb{Q}[a_{i,j}]$-linear combination of terms $g(\prod c_i^{m_i})$ with $m_i < n_i$. Thus $g(\prod c_i^{k_i})$ lies in the subring generated by the indicated elements. □

Proof of Proposition 2.1 The universal fibre bundle with fibre W may be identified with the natural projection

$$p : B\mathrm{Diff}^+(W, *) \cong E\mathrm{Diff}^+(W) \times_{\mathrm{Diff}^+(W)} W \longrightarrow B\mathrm{Diff}^+(W).$$

This gives a $\mathbb{Q}$-algebra homomorphism

$$p^* : H^*(B\mathrm{Diff}^+(W); \mathbb{Q}) \longrightarrow H^*(B\mathrm{Diff}^+(W, *); \mathbb{Q})$$

by pullback and an $H^*(B\mathrm{Diff}^+(W); \mathbb{Q})$-module homomorphism

$$p_! : H^*(B\mathrm{Diff}^+(W, *); \mathbb{Q}) \to H^{*-d}(B\mathrm{Diff}^+(W); \mathbb{Q})$$

by fibre integration. As we have described in the introduction, taking the differential at the marked point gives a map $s : B\mathrm{Diff}^+(W, *) \to BGL_d^+(\mathbb{R}) \simeq BSO(d)$, which classifies the vertical tangent bundle of the universal fibre bundle p.

Applying Lemma 2.2 with $B = H^*(B\mathrm{Diff}^+(W); \mathbb{Q})$, $E = H^*(B\mathrm{Diff}^+(W, *); \mathbb{Q})$, $\pi = p^*$, $g = p_!$, and $C = \mathrm{Im}(s^*)$ (using that $H^*(BSO(d); \mathbb{Q})$ is finitely-generated and that $C \subset R^*(W, *)$ so by assumption consists of elements which are integral over $R = R^*(W)$) shows that $R^*(W) \subset H^*(B\mathrm{Diff}^+(W); \mathbb{Q})$ is finitely-generated, proving the first part.

For the second part, as $H^*(BSO(d))$ is a finitely-generated $\mathbb{Q}$-algebra, we know that $R^*(W, *)$ is a finitely-generated $R^*(W)$-algebra, so under the integrality assumption it follows that $R^*(W, *)$ is in fact finitely-generated as a $R^*(W)$-module. □

Thus in order to prove Theorem A we shall actually show that $R^*(W, *)$ is integral over $R^*(W)$.

2.2 Outline

To motivate the proof of Theorem A let us first explain its proof under hypothesis (H1) and an additional assumption: that the universal smooth oriented fibre bundle $W \to$

$E \xrightarrow{\pi} B = B\mathrm{Diff}^+(W)$ satisfies the Leray–Hirsch property in rational cohomology, i.e. that $\pi_1(B)$ acts trivially on $H^*(W)$ and the Serre spectral sequence for $\pi : E \to B$ collapses. (The proof of Theorem A under hypothesis (H1) is a technical device which allows the following argument to be made without this additional assumption.)

Under this assumption, $H^*(E)$ is a free finitely-generated $H^*(B)$-module, say with basis $\bar{x}_1, \ldots, \bar{x}_k \in H^*(E)$ lifting a basis $x_1, \ldots, x_k$ for $H^*(W)$. Furthermore, as W has all its cohomology in even degrees, $H^{ev}(E)$ is a free finitely-generated module over the commutative ring $H^{ev}(B)$, with basis the $\bar{x}_i$. For $x \in H^{ev}(E)$ the map

$$- \cdot x : H^{ev}(E) \longrightarrow H^{ev}(E)$$

is a $H^{ev}(B)$-module map, so has a characteristic polynomial $\chi_x(z) \in H^{ev}(B)[z]$, and by the Cayley–Hamilton theorem (for finite modules over a commutative ring, alias the determinantal trick) we have $\chi_x(x) = 0 \in H^{ev}(E)$. Furthermore, the coefficients of the characteristic polynomial $\chi_x(z)$ may be expressed as polynomials in the elements

$$\mathrm{Tr}(- \cdot x^i : H^{ev}(E) \to H^{ev}(E)) \in H^{ev}(B),$$

which make sense as $H^{ev}(E)$ is a finite free $H^{ev}(B)$-module. The following lemma relates such traces to fibre-integration and the Euler class of the vertical tangent bundle.

Lemma 2.3 *For any $x \in H^{ev}(E)$ we have*

$$\mathrm{Tr}(- \cdot x : H^{ev}(E) \to H^{ev}(E)) = \int_\pi e(T_\pi E) \cdot x \in H^{ev}(B).$$

We apply the above discussion to $x = c(T_\pi E)$ for $c \in H^*(BSO(2n))$ a characteristic class of oriented $2n$-dimensional vector bundles. Then the polynomial $\chi_x(z)$ is monic, has coefficients in the subring generated by the $\kappa_{ec^i} = \int_\pi e(T_\pi E) \cdot c(T_\pi E)^i$, and satisfies $\chi_x(c(T_\pi E)) = 0$. Thus we deduce that $c(T_\pi E) = s^*c$ is integral over $R^*(W)$ and hence that $R^*(W, *)$ is integral over $R^*(W)$. Theorem A in the case we are considering follows by applying Proposition 2.1. It remains to prove this lemma.

Proof of Lemma 2.3 Rational cohomology classes are determined by their evaluations against rational homology classes, and any rational homology class is carried on a map from a smooth oriented manifold. So we may assume that $\pi : E \to B$ is a fibre bundle over a smooth oriented manifold, still satisfying the Leray–Hirsch property.

The pairing

$$\begin{aligned} \langle -, - \rangle : H^*(E) \otimes_{H^*(B)} H^*(E) &\longrightarrow H^{*-d}(B) \\ a \otimes b &\longmapsto \int_\pi a \cdot b \end{aligned}$$

is non-singular, as in the basis $\bar{x}_i$ its matrix X agrees modulo the ideal $H^{*>0}(B)$ of $H^*(B)$ with that of the intersection form of W in the basis x_i, so $\det(X) \in H^*(B)$ is a unit modulo $H^{*>0}(B)$, and hence is a unit as the ideal $H^{*>0}(B)$ is nilpotent. Let us write $\bar{x}_i^\vee$ for the dual $H^*(B)$-module basis of $H^*(E)$ with respect to this pairing, characterised by $\langle \bar{x}_i, \bar{x}_j^\vee \rangle = \delta_{ij}$. Then for any $x \in H^{ev}(E)$ we have

$$\mathrm{Tr}(- \cdot x : H^{ev}(E) \to H^{ev}(E)) = \sum_i \langle \bar{x}_i \cdot x, \bar{x}_i^\vee \rangle = \int_\pi \left(\left(\sum_i \bar{x}_i \cdot \bar{x}_i^\vee \right) \cdot x \right),$$

so to establish the claimed formula we must show that $\sum_i \bar{x}_i \cdot \bar{x}_i^\vee = e(T_\pi E) \in H^{ev}(E)$.

The diagonal map $\Delta : E \to E \times_B E$ is a map of smooth oriented manifolds, whose normal bundle is identified with $T_\pi E$. Thus the Euler class $e(T_\pi E) \in H^d(E)$ may be described as $\Delta^* \Delta_!(1)$. It is therefore enough to show that

$$\Delta_!(1) = \sum_i \bar{x}_i \otimes \bar{x}_i^\vee \in H^*(E \times_B E) = H^*(E) \otimes_{H^*(B)} H^*(E).$$

This is the parametrised analogue of the classical formula [32, Theorem 11.11] for the Poincaré dual of the diagonal, and we shall prove it in the same way. For any $b \in H^*(B)$ we calculate

$$\begin{aligned} \int_{E \times_B E} \Delta_!(1) \cdot ((b \cdot \bar{x}_j^\vee) \otimes \bar{x}_k) &= \int_E b \cdot \bar{x}_j^\vee \cdot \bar{x}_k \\ &= \int_B b \int_\pi \bar{x}_j^\vee \cdot \bar{x}_k = \delta_{jk} \int_B b \end{aligned}$$

and

$$\begin{aligned} \int_{E \times_B E} \left(\sum_i \bar{x}_i \otimes \bar{x}_i^\vee \right) \cdot ((b \cdot \bar{x}_j^\vee) \otimes \bar{x}_k) &= \sum_i \int_{E \times_B E} (b \cdot \bar{x}_j^\vee \cdot \bar{x}_i) \otimes (\bar{x}_k \cdot \bar{x}_i^\vee) \\ &= \sum_i \int_B b \int_{\pi \times_B \pi} (\bar{x}_j^\vee \cdot \bar{x}_i) \otimes (\bar{x}_k \cdot \bar{x}_i^\vee) \\ &= \sum_i \delta_{ij} \delta_{ki} \int_B b = \delta_{jk} \int_B b. \end{aligned}$$

As the classes $(b \cdot \bar{x}_j^\vee) \otimes \bar{x}_k$ generate $H^*(E \times_B E)$ as a $\mathbb{Q}$-module, it follows from Poincaré duality for $E \times_B E$ that $\Delta_!(1) = \sum_i \bar{x}_i \otimes \bar{x}_i^\vee$, as required. □

2.3 Parametrised spectra and Schur functors

The technical device we shall use to attempt the argument of the previous section without the Leray–Hirsch assumption is to consider a fibre bundle as a parametrised manifold over its base, and make the argument in the parametrised setting. In order to do so, we shall suppose that B is a connected CW-complex, and work in a symmetric monoidal category $(\mathsf{Sp}_{/B}, \wedge_B, S^0_B)$ of parametrised spectra over B. For concreteness we take the category developed by May–Sigurdsson [33].[1] We will write $r : B \to \{*\}$ for the unique map; then, as $\mathsf{Sp}_{/*} = \mathsf{Sp}$, by [33, Theorem 11.4.1] there are right and left adjoint functors

$$r^* : \mathsf{Sp} \longrightarrow \mathsf{Sp}_{/B} \quad \text{and} \quad r_! : \mathsf{Sp}_{/B} \longrightarrow \mathsf{Sp},$$

(apart from this map, the notation $(-)_!$ will always denote Gysin maps). The functor r^* is strong monoidal.

Our argument applies more generally than to oriented fibre bundles: for now, we let $\pi : E \to B$ be a Hurewicz fibration (later we will add a finiteness hypothesis to the fibres of π). This defines a parametrised spectrum $\Sigma^\infty_B E \in \mathsf{Sp}_{/B}$; we shall abuse notation and continue to call it E. Note that $r_!(E) \in \mathsf{Sp}$ is the suspension spectrum $\Sigma^\infty E_+$.

The ring spectrum $H\mathbb{Q}$ has a 2-periodic version

$$HP\mathbb{Q} = \bigvee_{i \in \mathbb{Z}} \Sigma^{2i} H\mathbb{Q},$$

and we write $\pi_*(HP\mathbb{Q}) = \mathbb{Q}[t^{\pm 1}]$ with $t \in \pi_2(HP\mathbb{Q})$. The constant parametrised spectra $H_B\mathbb{Q} := r^*(H\mathbb{Q})$ and $HP_B\mathbb{Q} := r^*(HP\mathbb{Q})$ define ring objects in $\mathsf{Sp}_{/B}$, and the main objects we will consider are the function objects

$$C := F_B(E, H_B\mathbb{Q}) \quad \text{and} \quad CP := F_B(E, HP_B\mathbb{Q}).$$

These are again ring objects, using the fibrewise diagonal map on E and the multiplication on $H_B\mathbb{Q}$ and $HP_B\mathbb{Q}$; we write μ for the multiplication on either object. The map $E \to *$ gives ring maps $H_B\mathbb{Q} \to C$ and $HP_B\mathbb{Q} \to CP$, making them $H_B\mathbb{Q}$- and $HP_B\mathbb{Q}$-modules respectively.

Let us write $(H_B\mathbb{Q}\text{-}\mathsf{mod}, \otimes, H_B\mathbb{Q})$ for the homotopy category of $H_B\mathbb{Q}$-module spectra, with derived smash product of $H_B\mathbb{Q}$-modules as the symmetric monoidal structure and unit $H_B\mathbb{Q}$; similarly write $(HP_B\mathbb{Q}\text{-}\mathsf{mod}, \otimes, HP_B\mathbb{Q})$ for the homotopy category of $HP_B\mathbb{Q}$-module spectra. We have $C \in H_B\mathbb{Q}\text{-}\mathsf{mod}$ and $CP \in HP_B\mathbb{Q}\text{-}\mathsf{mod}$, and we can calculate

$$\begin{aligned}[\Sigma^d H_B\mathbb{Q}, C]_{H_B\mathbb{Q}\text{-}\mathsf{mod}} &= [\Sigma^d S^0_B, C]_{\mathsf{Sp}_{/B}} \\ &= [E, \Sigma^{-d} H_B\mathbb{Q}]_{\mathsf{Sp}_{/B}} = [E, r^*(\Sigma^{-d} H\mathbb{Q})]_{\mathsf{Sp}_{/B}} \\ &= [\Sigma^\infty E_+, \Sigma^{-d} H\mathbb{Q}]_{\mathsf{Sp}} = H^{-d}(E)\end{aligned}$$

[1] However, our arguments are not model-dependent and can be applied in the ∞-categorical formalism of Ando, Blumberg, Gepner, Hopkins, and Rezk [2,3], and presumably even in more naïve models of parametrised spectra.

and

$$\begin{aligned}[HP_B\mathbb{Q}, CP]_{HP_B\mathbb{Q}\text{-}\mathsf{mod}} &= [S^0_B, CP]_{\mathsf{Sp}_{/B}} \\ &= [E, HP_B\mathbb{Q}]_{\mathsf{Sp}_{/B}} = [E, r^*(HP\mathbb{Q})]_{\mathsf{Sp}_{/B}} \\ &= \left[\Sigma^\infty E_+, \bigvee_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Q}\right]_{\mathsf{Sp}} = \bigoplus_{i\in\mathbb{Z}} H^{2i}(E).\end{aligned}$$

Both $H_B\mathbb{Q}$-mod and $HP_B\mathbb{Q}$-mod are $\mathbb{Q}$-linear tensor categories (i.e. categories enriched in $\mathbb{Q}$-modules, equipped with a symmetric monoidal structure which is an enriched functor) which are idempotent complete (the retract associated to an endomorphism $e : X \to X$ which is idempotent up to homotopy may be taken to be the homotopy colimit of a diagram $X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots$ of modules over the appropriate ring object).

We must now recall a little representation theory of symmetric groups; we need nothing beyond Lecture 4 of [18]. To each partition λ of n there is associated an irreducible representation S^λ of Σ_n, with character χ_λ. This character takes rational (in fact, integer) values, so we may form the element

$$d_\lambda := \frac{\dim S^\lambda}{n!} \sum_{\sigma\in\Sigma_n} \chi_\lambda(\sigma)\cdot\sigma \in \mathbb{Q}[\Sigma_n],$$

which is central (as χ_λ is a class function) and idempotent (the coefficient $\frac{\dim S^\lambda}{n!}$ is chosen to make this so). For any object X in a $\mathbb{Q}$-linear tensor category $(\mathsf{D}, \otimes, \mathbb{1})$, the action of the nth symmetric group Σ_n on $X^{\otimes n}$ yields a map of $\mathbb{Q}$-algebras

$$e : \mathbb{Q}[\Sigma_n] \longrightarrow \mathrm{Hom}_{\mathsf{D}}(X^{\otimes n}, X^{\otimes n}),$$

so $e(d_\lambda)$ is an idempotent endomorphism of $X^{\otimes n}$ in D; if D is idempotent complete then we write $S_\lambda(X)$ for the corresponding retract of $X^{\otimes n}$ in D: this defines the *Schur functor* $S_\lambda(-)$ on D. In this paper the trivial and sign representations will play the most prominent role, and we write

$$\wedge^n X := S_{(1^n)}(X) \quad \text{and} \quad \mathrm{Sym}^n(X) := S_{(n)}(X),$$

or, if we wish to emphasise the ambient category, $\wedge^n_{\mathsf{D}}$ and $\mathrm{Sym}^n_{\mathsf{D}}$.

The categories $H_B\mathbb{Q}$-mod and $HP_B\mathbb{Q}$-mod are idempotent complete $\mathbb{Q}$-linear tensor categories, so there are defined Schur functors on both categories. Furthermore, let us write $(\mathsf{V}_{\mathbb{Q}}, \otimes_{\mathbb{Q}}, \mathbb{Q})$ for the symmetric monoidal category of graded $\mathbb{Q}$-modules, and $(\mathsf{V}_{\mathbb{Q}[t^{\pm 1}]}, \otimes_{\mathbb{Q}[t^{\pm 1}]}, \mathbb{Q}[t^{\pm 1}])$ for the symmetric monoidal category of graded $\mathbb{Q}[t^{\pm 1}]$-modules (where t has degree 2). These are also idempotent complete $\mathbb{Q}$-linear tensor categories. Taking homotopy groups defines functors

$$\pi_*(-) : H\mathbb{Q}\text{-mod} \longrightarrow \mathsf{V}_{\mathbb{Q}}$$
$$\pi_*(-) : HP\mathbb{Q}\text{-mod} \longrightarrow \mathsf{V}_{\mathbb{Q}[t^{\pm 1}]}$$

which are strong monoidal (by the Künneth theorem, as every graded $\mathbb{Q}$- or $\mathbb{Q}[t^{\pm 1}]$-module is free). Taking derived homotopy fibres at $b \in B$ defines functors

$$(-)_b : H_B\mathbb{Q}\text{-mod} \longrightarrow H\mathbb{Q}\text{-mod}$$
$$(-)_b : HP_B\mathbb{Q}\text{-mod} \longrightarrow HP\mathbb{Q}\text{-mod}$$

which are strong monoidal and reflect isomorphisms (by definition, cf. [33, Definition 12.3.4], as B is assumed path-connected). Finally,

$$- \otimes_{\mathbb{Q}} \mathbb{Q}[t^{\pm 1}] : \mathsf{V}_{\mathbb{Q}} \longrightarrow \mathsf{V}_{\mathbb{Q}[t^{\pm 1}]}$$

is also strong monoidal. In particular, all of the above functors preserve Schur functors.

Lemma 2.4 *Let $X \in H_B\mathbb{Q}\text{-mod}$ or $HP_B\mathbb{Q}\text{-mod}$ be such that for each fibre X_b we have $S_\lambda(\pi_*(X_b)) = 0$. Then $S_\lambda(X) \simeq *$.*

Proof As taking homotopy groups preserves Schur functors, we have $\pi_*(S_\lambda(X_b)) = S_\lambda(\pi_*(X_b))$ which vanishes by assumption. Thus $S_\lambda(X_b) \simeq *$, so as taking derived fibres preserves Schur functors it follows that $S_\lambda(X)_b \simeq *$. Thus the map from $S_\lambda(X)$ to the terminal object is an equivalence on derived fibres, and hence an equivalence, as taking derived fibres reflects isomorphisms. □

2.4 Duals, trace, and transfer

We recall the framework of categorical traces, from [15]. If $(\mathsf{C}, \otimes, \mathbb{1})$ is a symmetric monoidal category and $X \in \mathsf{C}$ is an object, a *strong dual* of X is an object $X^\vee \in \mathsf{C}$ and morphisms

$$\varepsilon : X^\vee \otimes X \longrightarrow \mathbb{1} \qquad \eta : \mathbb{1} \longrightarrow X \otimes X^\vee$$

such that the compositions $(X \otimes \varepsilon) \circ (\eta \otimes X)$ and $(\varepsilon \otimes X^\vee) \circ (X^\vee \otimes \eta)$ are the identity maps of X and $X^\vee$ respectively. If $f : X \to Y$ is a map of objects having strong duals, then the dual of f is

$$f^\vee : Y^\vee \xrightarrow{Y^\vee \otimes \eta} Y^\vee \otimes X \otimes X^\vee \xrightarrow{Y^\vee \otimes f \otimes X^\vee} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\varepsilon \otimes X^\vee} X^\vee.$$

If $f : X \to X$ is an endomorphism of X, the *trace of f* is the composition

$$\mathrm{Tr}(f) : \mathbb{1} \xrightarrow{\eta} X \otimes X^\vee \cong X^\vee \otimes X \xrightarrow{f^\vee \otimes X} X^\vee \otimes X \xrightarrow{\varepsilon} \mathbb{1}.$$

This agrees with the perhaps more obvious choice

$$\mathrm{Tr}(f) : \mathbb{1} \xrightarrow{\eta} X \otimes X^\vee \xrightarrow{f \otimes X^\vee} X \otimes X^\vee \cong X^\vee \otimes X \xrightarrow{\varepsilon} \mathbb{1},$$

but the first definition is that of [15]. Generalising this more obvious choice, if $f : A \otimes X \to B \otimes X$ is a morphism then the *trace of f over X* is the composition

$$\mathrm{Tr}^X(f) : A \xrightarrow{A \otimes \eta} A \otimes X \otimes X^\vee \xrightarrow{f \otimes X^\vee} B \otimes X \otimes X^\vee \cong B \otimes X^\vee \otimes X \xrightarrow{B \otimes \varepsilon} B.$$

If X is in addition equipped with a comultiplication $d : X \to X \otimes X$ then the *transfer of f* is

$$\tau(f) : \mathbb{1} \xrightarrow{\eta} X \otimes X^\vee \cong X^\vee \otimes X \xrightarrow{f^\vee \otimes d} X^\vee \otimes X \otimes X \xrightarrow{\varepsilon \otimes X} X.$$

First, consider the symmetric monoidal category given by the homotopy category of $(\mathsf{Sp}_{/B}, \wedge_B, S^0_B)$.

Lemma 2.5 *If $\pi : E \to B$ is a Hurewicz fibration and its fibre has the homotopy type of a finite CW-complex, then its associated parametrised spectrum $\Sigma^\infty_B E$ is a strongly dualisable object in the homotopy category of parametrised spectra.*

Proof This follows from Theorem 15.1.1 of [33]. □

Suppose then that $\pi : E \to B$ is a Hurewicz fibration and its fibre has the homotopy type of a finite CW-complex. Recall that we abuse notation by writing E for $\Sigma^\infty_B E$. The fibrewise suspension of the fibrewise diagonal map $\Delta : E \to E \times_B E$ gives a comultiplication on the object E, we may thus form

$$\mathrm{trf}_\pi = \tau(\mathrm{Id}_E) : S^0_B \longrightarrow E.$$

On applying $r_! : \mathsf{Sp}_{/B} \to \mathsf{Sp}$ this gives a map of spectra $\Sigma^\infty B_+ \to \Sigma^\infty E_+$, which on cohomology gives a map

$$\mathrm{trf}^*_\pi : H^*(E) \longrightarrow H^*(B),$$

the *Becker–Gottlieb transfer*. See [10] or [33, Section 15.3] for this construction of the Becker–Gottlieb transfer. When $\pi : E \to B$ is an oriented smooth fibre bundle, by [9, Theorem 4.3] we have the identity

$$\mathrm{trf}^*_\pi(-) = \int_\pi e(T_\pi E) \cdot - : H^*(E) \longrightarrow H^*(B). \tag{2.1}$$

In particular for $c \in H^*(BSO(2n))$ we have that $\mathrm{trf}^*_\pi(c(T_\pi E)) = \kappa_{ec}$ is a tautological class.

Let us now consider the symmetric monoidal categories $(H_B\mathbb{Q}\text{-}\mathsf{mod}, \otimes, H_B\mathbb{Q})$ and $(HP_B\mathbb{Q}\text{-}\mathsf{mod}, \otimes, HP_B\mathbb{Q})$.

Corollary 2.6 *If $\pi : E \to B$ is a Hurewicz fibration and its fibre has the homotopy type of a finite CW-complex, then $C = F_B(E, H_B\mathbb{Q})$ is a dualisable object of $H_B\mathbb{Q}$-mod, and $CP = F_B(E, HP_B\mathbb{Q})$ is a dualisable object of $HP_B\mathbb{Q}$-mod.*

Proof The functor

$$F_B(-, H_B\mathbb{Q}) : \mathsf{Ho}(\mathsf{Sp}_{/B}) \longrightarrow H_B\mathbb{Q}\text{-}\mathsf{mod}$$

has a monoidality given by the adjoint of the morphism

$$X \wedge_B Y \wedge_B (F_B(X, H_B\mathbb{Q}) \wedge_{H_B\mathbb{Q}} F_B(Y, H_B\mathbb{Q})) \longrightarrow H_B\mathbb{Q} \wedge_B H_B\mathbb{Q} \longrightarrow H_B\mathbb{Q}$$

given by evaluation and product. This is a strong monoidality: the induced morphism

$$F(X_b, H\mathbb{Q}) \wedge_{H\mathbb{Q}} F(Y_b, H\mathbb{Q}) \longrightarrow F(X_b \wedge Y_b, H\mathbb{Q})$$

on derived fibres is a weak equivalence (by the Künneth theorem, as every $\pi_*(H\mathbb{Q}) = \mathbb{Q}$-module is free). As $E \in \mathsf{Ho}(\mathsf{Sp}_{/B})$ is strongly dualisable by Lemma 2.5, so is $C = F_B(E, H_B\mathbb{Q})$, because strong monoidal functors preserve (strong) duals. The argument for CP is identical. □

2.5 Schur-finiteness and trace identities

Deligne has introduced [13, §1] the notion of *Schur-finiteness* of an object X in an idempotent complete $\mathbb{Q}$-linear tensor category to be the property that $S_\lambda(X)$ is trivial for some partition $\lambda \vdash n$. In this section we consider this notion applied to the category $(HP_B\mathbb{Q}\text{-}\mathsf{mod}, \otimes, HP_B\mathbb{Q})$ and so consider an $HP_B\mathbb{Q}$-module X such that $S_\lambda(X) \simeq *$, and let us in addition suppose that X is dualisable in $HP_B\mathbb{Q}$-mod. Let us write $X^\vee$ for the dual of X, with duality structure given by $\eta : HP_B\mathbb{Q} \to X \otimes X^\vee$ and $\varepsilon : X^\vee \otimes X \to HP_B\mathbb{Q}$.

Given an endomorphism $f : X \to X$, we may form the endomorphism

$$X^{\otimes n} \xrightarrow{X \otimes f^{\otimes n-1}} X^{\otimes n} \xrightarrow{e(d_\lambda)} X^{\otimes n}$$

and take the trace over the last $(n-1)$ copies of X, i.e. apply the construction $\mathrm{Tr}^{X^{\otimes n-1}}(-)$ described in the previous section, to obtain an endomorphism of X. This endomorphism is null because the idempotent $e(d_\lambda) : X^{\otimes n} \to X^{\otimes n}$ factors through $S_\lambda(X)$ which is contractible by assumption.

We now translate this into formulas. We have $d_\lambda = \frac{\dim S^\lambda}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \sigma$ so the essential calculation is to describe the endomorphism of X obtained from $\sigma \circ (X \otimes f^{\otimes n-1}) : X^{\otimes n} \to X^{\otimes n}$ by taking the trace over the last $(n-1)$ copies of X. This is a universal construction in idempotent complete $\mathbb{Q}$-linear tensor categories, and has been worked out by Abramsky [4, Proposition 3]. In the notation of that proposition, one takes $A_1 = B_1 = X$ and $U_2 = \cdots = U_n = X$, then $f_1 = \mathrm{Id}_X$ and

$f_2 = \cdots = f_n = f$, and $\pi = \sigma$. The trace of $\sigma \circ (X \otimes f^{\otimes n-1})$ over the last $(n-1)$ copies of X is then given by

$$\left(\prod_{l \in \mathcal{L}(\sigma)} s_l\right) \cdot (p_\sigma^{-1} \circ g_1).$$

Here $p_\sigma^{-1} = \mathrm{Id}_X$, and g_1 is given by composing the f_i along the cycle in σ starting at 1: as $f_1 = \mathrm{Id}_X$, if this cycle is $(1, p_2, \ldots, p_k)$ then this gives $g_1 = f^{\circ k-1}$; $\mathcal{L}(\sigma)$ is the set of cycles in the permutation σ which do not contain 1, and for such a cycle $l = (p_1, p_2, \ldots, p_k)$ we have $s_l := \mathrm{Tr}(f_{p_k} \circ \cdots \circ f_{p_1}) = \mathrm{Tr}(f^{\circ k})$.

Applying this discussion to the identity $0 = \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot (\sigma \circ (X \otimes f^{\otimes n-1}))$ gives the identity

$$0 = \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \mathrm{Tr}(f^{\circ l(\gamma_2)}) \cdots \mathrm{Tr}(f^{\circ l(\gamma_{q(\sigma)})}) \cdot f^{\circ l(\gamma_1)-1} \in [X, X]_{HP_B\mathbb{Q}\text{-mod}} \tag{2.2}$$

where $\sigma = \gamma_1 \cdot \gamma_2 \cdots \gamma_{q(\sigma)}$ is a decomposition into disjoint cycles, with 1 being in the support of γ_1, and $l(\gamma_i)$ denotes the length of the cycle γ_i.

We originally learnt this idea from the thesis of del Padrone [16] (see [16, Proposition 2.2.4] for a closely related result).

2.6 Proof of Theorem A under the first hypothesis

In this case we will work with the periodic chains CP. For each $b \in B$ we have

$$\pi_0(CP_b) = \bigoplus_{i \in \mathbb{Z}} H^{-2i}(E_b) \cong \bigoplus_{i \in \mathbb{Z}} H^{-2i}(W),$$
$$\pi_1(CP_b) = \bigoplus_{i \in \mathbb{Z}} H^{-2i-1}(E_b) \cong \bigoplus_{i \in \mathbb{Z}} H^{-2i-1}(W)$$

and so by 2-periodicity we have an isomorphism $\pi_*(CP_b) \cong H^{-*}(W) \otimes_{\mathbb{Q}} \mathbb{Q}[t^{\pm 1}]$ of graded $\mathbb{Q}[t^{\pm 1}]$-modules. Here the right-hand side is to be interpreted as the tensor product of graded $\mathbb{Q}$-modules, which in degree k is

$$\bigoplus_{-i+2n=k} H^{-i}(W) \otimes \mathbb{Q}\{t^n\}.$$

Thus we have $\wedge^\ell_{\mathbb{Q}[t^{\pm 1}]}(\pi_*(CP_b)) \cong \wedge^\ell_{\mathbb{Q}}(H^{-*}(W)) \otimes_{\mathbb{Q}} \mathbb{Q}[t^{\pm 1}]$.

Under hypothesis (H1) the cohomology $H^*(W)$ is concentrated in even degrees, so if it has total degree k then we have

$$\wedge^{k+1}_{\mathbb{Q}[t^{\pm 1}]}(\pi_*(CP_b)) \cong \wedge^{k+1}_{\mathbb{Q}}(H^{-*}(W)) \otimes_{\mathbb{Q}} \mathbb{Q}[t^{\pm 1}] = 0$$

and so it follows from Lemma 2.4 that $\wedge^{k+1} CP \simeq *$.

As $\pi : E \to B$ is a fibre bundle with compact fibres Corollary 2.6 applies to it, so CP is dualisable in $HP_B\mathbb{Q}\text{-mod}$ and hence the discussion of the previous section applies. Thus, as CP is a ring object in $HP_B\mathbb{Q}\text{-mod}$, for any

$$x \in H^{2p}(E) \subset \bigoplus_{i\in\mathbb{Z}} H^{-2i}(E) = [HP_B\mathbb{Q}, CP]_{HP_B\mathbb{Q}\text{-mod}}$$

multiplication by x yields an endomorphism $\hat{x} : CP \to CP$, and in this case composing the map (2.2) with $1 \in [HP_B\mathbb{Q}, CP]_{HP_B\mathbb{Q}\text{-mod}}$ gives the identity

$$0 = \sum_{\sigma\in\Sigma_{k+1}} \mathrm{sign}(\sigma)\cdot \mathrm{Tr}(\hat{x}^{\circ l(\gamma_2)})\cdots \mathrm{Tr}(\hat{x}^{\circ l(\gamma_{q(\sigma)})})\cdot x^{\circ l(\gamma_1)-1} \tag{2.3}$$

in $[HP_B\mathbb{Q}, CP]_{HP_B\mathbb{Q}\text{-mod}}$, because the partition $\lambda = (1^{k+1})$ corresponds to the sign representation.

Corollary 2.7 *The polynomial*

$$\rho_x(z) := \frac{(-1)^k}{k!} \sum_{\sigma\in\Sigma_{k+1}} \mathrm{sign}(\sigma)\cdot \mathrm{trf}_\pi^*(x^{l(\gamma_2)})\cdots \mathrm{trf}_\pi^*(x^{l(\gamma_{q(\sigma)})})\cdot z^{l(\gamma_1)-1} \in H^{2*}(B)[z]$$

is monic of degree k and satisfies $\rho_x(x) = 0 \in H^{2*}(E)$.

Proof Let $E^\vee$ be a dual of $E \in \mathsf{Ho}(\mathsf{Sp}_{/B})$, so $CP^\vee := F_B(E^\vee, HP_B\mathbb{Q})$ is a dual in $HP_B\mathbb{Q}\text{-mod}$ of CP. The Becker–Gottlieb transfer trf_π^* is the map on cohomology induced by the composition

$$S^0_B \xrightarrow{\eta} E\wedge_B E^\vee \cong E^\vee \wedge_B E \xrightarrow{E^\vee\wedge_B\Delta} E^\vee\wedge_B E\wedge_B E \xrightarrow{\varepsilon\wedge_B E} E.$$

Applying $F_B(-, HP_B\mathbb{Q})$, this is

$$CP \xrightarrow{\eta\wedge CP} CP^\vee\otimes CP\otimes CP \xrightarrow{CP^\vee\otimes\mu} CP^\vee\otimes CP \cong CP\otimes CP^\vee \xrightarrow{\varepsilon} HP_B\mathbb{Q}.$$

If $t \in \bigoplus_{i\in\mathbb{Z}} H^{-2i}(E) = [HP_B\mathbb{Q}, CP]_{HP_B\mathbb{Q}\text{-mod}}$ then composing the previous map with t gives the class $\mathrm{trf}_\pi^*(t) \in \bigoplus_{i\in\mathbb{Z}} H^{-2i}(B) = [HP_B\mathbb{Q}, HP_B\mathbb{Q}]_{HP_B\mathbb{Q}\text{-mod}}$. The commutative diagram

$$\begin{tikzcd}
HP_B\mathbb{Q} \arrow[r, "t"] \arrow[d, "\eta"] & CP \arrow[r, "\eta\wedge CP"] & CP^\vee\otimes CP\otimes CP \arrow[r, "CP^\vee\otimes\mu"] & CP^\vee\otimes CP \arrow[r, "\cong"] & CP\otimes CP^\vee \arrow[d, "\varepsilon"] \\
CP^\vee\otimes CP \arrow[rrru, "CP^\vee\otimes\hat{t}"'] & & & & HP_B\mathbb{Q}
\end{tikzcd}$$

shows that $\mathrm{trf}^*_\pi(t)$ is the trace of $\hat{t}^\vee : CP^\vee \to CP^\vee$, which is the same as the trace of $\hat{t} : CP \to CP$: thus

$$\mathrm{trf}^*_\pi(t) = \mathrm{Tr}(\hat{t}) \in [HP_B\mathbb{Q}, HP_B\mathbb{Q}]_{HP_B\mathbb{Q}\text{-mod}} = \bigoplus_{i\in\mathbb{Z}} H^{-2i}(B).$$

In particular we have $\mathrm{Tr}(\hat{x}^{\circ i}) = \mathrm{trf}^*_\pi(x^i)$. Substituting this into (2.3) therefore shows that

$$\sum_{\sigma\in\Sigma_{k+1}} \mathrm{sign}(\sigma)\cdot \mathrm{trf}^*_\pi(x^{l(\gamma_2)})\cdots \mathrm{trf}^*_\pi(x^{l(\gamma_{q(\sigma)})})\cdot x^{l(\gamma_1)-1} = 0$$

in $H^{2pk}(E) \subset [HP_B\mathbb{Q}, CP]_{HP_B\mathbb{Q}\text{-mod}}$. The coefficient of x^k is the sum over the $k!$-many $(k+1)$-cycles $\sigma \in \Sigma_{k+1}$ of $\mathrm{sign}(\sigma) = (-1)^k$, which is $k!(-1)^k$. Thus after dividing by this coefficient we see that $\rho_x(x) = 0$ as required. □

Applying this to the universal fibre bundle $p : B\mathrm{Diff}^+(W, *) \to B\mathrm{Diff}^+(W)$ and the cohomology class s^*c, and using the identity $\mathrm{trf}^*_p((s^*c)^i) = \kappa_{c^i e}(p)$ from (2.1), one obtains a monic polynomial $\rho_c(z) \in R^*(W)[z]$ such that $\rho_c(s^*c) = 0 \in R^*(W, *)$ and hence that $R^*(W, *)$ is integral over $R^*(W)$. Theorem A under hypothesis (H1) follows by applying Proposition 2.1.

2.7 Proof of Theorem A under the second hypothesis

In this case we will work with the non-periodic chains C. We shall first prove the following generalisation of a theorem of Grigoriev [22].

Theorem 2.8 *Let W be a manifold of dimension $2n$ having rational cohomology only in degrees 0, $2n$, and odd degrees, and let $d := \dim_{\mathbb{Q}} H^{odd}(W)$. Let $\pi : E \to B$ be a smooth oriented fibre bundle with fibre W. Let $a, b \in H^*(E)$ satisfy $\pi_!(a) = \pi_!(b) = 0$, and a have even degree. Then*

$$\pi_!(a^2)^{\lceil \frac{d+1}{2} \rceil} = 0 \quad \textit{and} \quad \pi_!(ab)^{d+1} = 0.$$

To begin with, we prove the following extension of Corollary 2.6, which is the appropriate form of Poincaré duality in our setting.

Lemma 2.9 *If $\pi : E \to B$ is a Hurewicz fibration over a CW-complex, its fibre F has the homotopy type of a finite Poincaré complex of dimension n, and $\pi_1(B)$ acts trivially on $H^n(F)$, then an orientation of F determines an identification of the dual of C with $\Sigma^n C$ in $H_B\mathbb{Q}$-mod.*

Proof This is proved in Section 3.1 of [23] in dual form, where, passing to rational coefficients, the equivalence is expressed as $D^{fw}_E : \Sigma^n F_B(E, H_B\mathbb{Q}) \xrightarrow{\sim} E \wedge_B H_B\mathbb{Q}$. The domain of this morphism is $\Sigma^n C$ and as $C = F_B(E, H_B\mathbb{Q}) = F_{H_B\mathbb{Q}\text{-mod}}(E \wedge_B H_B\mathbb{Q}, H_B\mathbb{Q})$ we recognise $E \wedge_B H_B\mathbb{Q}$ as the dual of C. □

We now consider a smooth oriented fibre bundle $\pi : E \to B$ as in the statement of Theorem 2.8, with B a CW-complex; this satisfies the hypotheses of the previous lemma. Fibre integration $\pi_! : H^*(E) \to H^{*-2n}(B)$ is realised in the category $H_B\mathbb{Q}\text{-mod}$ by the morphism $\pi_! : C \to \Sigma^{-2n} H_B\mathbb{Q}$ dual to the unit $\iota : H_B\mathbb{Q} \to C$, using the self-duality of C described in Lemma 2.9. We may thus define an $H_B\mathbb{Q}$-module D' by the homotopy fibre sequence

$$D' \longrightarrow C \xrightarrow{\pi_!} \Sigma^{-2n} H_B\mathbb{Q}.$$

The composition $H_B\mathbb{Q} \xrightarrow{\iota} C \xrightarrow{\pi_!} \Sigma^{-2n} H_B\mathbb{Q}$ is null, as it represents the class

$$\pi_!(1) = 0 \in H^{-2n}(B) = [H_B\mathbb{Q}, \Sigma^{-2n} H_B\mathbb{Q}]_{H_B\mathbb{Q}\text{-mod}},$$

so ι lifts to a map $\iota' : H_B\mathbb{Q} \to D'$ and we can define an $H_B\mathbb{Q}$-module D by the homotopy cofibre sequence

$$H_B\mathbb{Q} \xrightarrow{\iota'} D' \longrightarrow D.$$

Lemma 2.10 *If W only has rational cohomology in degree 0, $2n$, and odd degrees, and $d := \dim_{\mathbb{Q}} H^{odd}(W)$ then* $\mathrm{Sym}^{d+1}(D) \simeq *$.

Proof The $H\mathbb{Q}$-module spectrum D_b is obtained by forming the homotopy fibre sequence $D'_b \to F(W, H\mathbb{Q}) \xrightarrow{\pi_!} \Sigma^{-2n} H\mathbb{Q}$ and then the homotopy cofibre sequence $H\mathbb{Q} \xrightarrow{\iota'_b} D'_b \to D_b$. Now $\pi_*(F(W, H\mathbb{Q})) = H^{-*}(W)$, and the map

$$(\pi_!)_* : \pi_*(F(W, H\mathbb{Q})) = H^{-*}(W) \longrightarrow \pi_*(\Sigma^{-2n} H\mathbb{Q})$$

realises capping with the fundamental class so is surjective. By the associated long exact sequence we have

$$\pi_0(D'_b) = \mathbb{Q}\{1\}$$
$$\pi_{odd}(D'_b) = H^{-odd}(W)$$

and the remaining even homotopy groups are zero. Now the map

$$(\iota'_b)_* : \mathbb{Q} = \pi_0(H\mathbb{Q}) \longrightarrow \pi_0(D'_b)$$

realises the unit so is an isomorphism, and it follows that $\pi_{odd}(D_b) = H^{-odd}(W)$ and the even homotopy groups of D_b vanish. As the $(d+1)$-st symmetric power of the graded $\mathbb{Q}$-module $H^{-odd}(W)$ vanishes, the rest follows from Lemma 2.4. $\square$

The map

$$D' \otimes D' \longrightarrow C \otimes C \xrightarrow{\mu} C \xrightarrow{\pi_!} \Sigma^{-2n} H_B\mathbb{Q}$$

is null when precomposed with $D' \otimes H_B\mathbb{Q} \overset{D'\otimes\iota'}{\to} D' \otimes D'$ or $H_B\mathbb{Q} \otimes D' \overset{\iota'\otimes D'}{\to} D' \otimes D'$, so taking homotopy cofibres of these two maps gives a morphism

$$\phi : D \otimes D \longrightarrow \Sigma^{-2n} H_B\mathbb{Q}.$$

Proof of Theorem 2.8 If $a \in H^{-k}(E) = [\Sigma^k H_B\mathbb{Q}, C]_{H_B\mathbb{Q}\text{-mod}}$ is such that $\pi_!(a) = 0$ then it lifts to a map to D' and hence determines a map $\bar{a} : \Sigma^k H_B\mathbb{Q} \to D$. Similarly if $b \in H^{-\ell}(E)$ satisfies $\pi_!(b) = 0$ then it gives a $\bar{b} : \Sigma^\ell H_B\mathbb{Q} \to D$. The class $\pi_!(a \cdot b)$ may therefore be represented by

$$\Sigma^k H_B\mathbb{Q} \otimes \Sigma^\ell H_B\mathbb{Q} \xrightarrow{\bar{a}\otimes\bar{b}} D \otimes D \xrightarrow{\phi} \Sigma^{-2n} H_B\mathbb{Q}.$$

Hence the class $\pi_!(a \cdot b)^N$ may be written as

$$(\Sigma^k H_B\mathbb{Q})^{\otimes N} \otimes (\Sigma^\ell H_B\mathbb{Q})^{\otimes N} \xrightarrow{\bar{a}^N \otimes \bar{b}^N} D^{\otimes N} \otimes D^{\otimes N} \xrightarrow{\phi^N} (\Sigma^{-2n} H_B\mathbb{Q})^{\otimes N},$$

as the degree k of $\bar{a}$ is even so no sign is incurred in rearranging the factors. As k is even, the map $\bar{a}^N : (\Sigma^k H_B\mathbb{Q})^{\otimes N} \to D^{\otimes N}$ factors through $\mathrm{Sym}^N(D)$ which is contractible as long as $N \geq d+1$. Hence $\pi_!(a \cdot b)^{d+1} = 0$.

Similarly, if $a = b$ then we can choose $\bar{b} = \bar{a} : \Sigma^k H_B\mathbb{Q} \to D$ in which case the map $\bar{a}^N \otimes \bar{a}^N : (\Sigma^k H_B\mathbb{Q})^{\otimes N} \otimes (\Sigma^k H_B\mathbb{Q})^{\otimes N} \to D^{\otimes N} \otimes D^{\otimes N}$ factors through $\mathrm{Sym}^{2N}(D)$, which is contractible as long as $2N \geq d+1$. Hence $\pi_!(a^2)^{\lceil \frac{d+1}{2} \rceil} = 0$. □

Now that we have Theorem 2.8, the entirety of Section 5 of [22] goes through with only notational changes, as this only uses the statement of Grigoriev's theorem. In particular, for $p \in H^*(BSO(2n))$ of even degree and $\chi = \chi(W) \neq 0$, the analogue of [22, Example 5.19] gives the relation

$$\left(p - \frac{\kappa_{ep}}{\chi} - \frac{e\kappa_p}{\chi} + \frac{\kappa_{e^2}\kappa_p}{\chi^2}\right)^{d+1} = 0 \in R^*(W, *).$$

From this it is clear that $R^*(W, *)$ is a finite $R^*(W)$-module, as the monomials in $\mathbb{Q}[p_1, p_2, \ldots, p_{n-1}, e]$ where no variable occurs with exponent larger than d give a finite set of module generators. Thus $R^*(W, *)$ is integral over $R^*(W)$, so by Proposition 2.1 the algebra $R^*(W)$ is finitely-generated. This proves Theorem A under hypothesis (H2).

2.8 Tautological relations

Under either hypothesis we have established more than Theorem A, as we have produced explicit relations in $R^*(W, *)$. Under hypothesis (H2) these relations are equal to those obtained by Grigoriev, and under hypothesis (H1) they are given by Corollary 2.7 as

$$0 = \sum_{\sigma \in \Sigma_{k+1}} \mathrm{sign}(\sigma) \cdot \kappa_{ec^{l(\gamma_2)}} \cdots \kappa_{ec^{l(\gamma_{q(\sigma)})}} \cdot c^{l(\gamma_1)-1} \in R^*(W, *)$$

for each $c \in H^*(BSO(2n))$, where $k = \dim_{\mathbb{Q}} H^*(W)$. These may of course be pushed forward to obtain relations in $R^*(W)$.

More generally, the trace identity technique of Sect. 2.5 may be used to find relations among tautological classes for *any* manifold. Recall that given a fibre bundle $W \to E \xrightarrow{\pi} B$ we have formed an associated object $CP \in HP_B\mathbb{Q}\text{-mod}$. Let us write $d_{ev} = \dim_{\mathbb{Q}} H^{ev}(W)$ and $d_{odd} = \dim_{\mathbb{Q}} H^{odd}(W)$. The first ingredient is the following consequence of a calculation of Deligne.

Lemma 2.11 *If λ is a partition whose Young diagram contains the rectangle $(d_{ev} + 1) \times (d_{odd} + 1)$, then $S_\lambda(CP) \simeq *$.*

Proof By Lemma 2.4 it is enough to verify that $S_\lambda(\pi_*(CP_b)) = 0$ for all $b \in B$. But we have shown that $\pi_*(CP_b) \cong H^{-*}(W) \otimes \mathbb{Q}[t^{\pm 1}]$ as graded $\mathbb{Q}[t^{\pm 1}]$-modules so $S_\lambda(\pi_*(CP_b))$ vanishes if $S_\lambda(H^{-*}(W))$ does, where the latter Schur functor is taken in $\mathsf{V}_{\mathbb{Q}}$. By [13, Corollary 1.9] a ($\mathbb{Z}/2$-)graded vector space is annihilated by $S_\lambda(-)$ under the given assumption on its (super)dimension. □

In particular, for a given manifold W we may take λ to be the partition of $n = (d_{ev}+1) \cdot (d_{odd}+1)$ with Young diagram equal to the rectangle $(d_{ev}+1) \times (d_{odd}+1)$, so that we have $S_\lambda(CP) \simeq *$ and hence by (2.2) we have the relation

$$0 = \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \kappa_{ec^{l(\gamma_2)}} \cdots \kappa_{ec^{l(\gamma_{q(\sigma)})}} \cdot c^{l(\gamma_1)-1} \in R^*(W, *).$$

It is a simple exercise with the Murnaghan–Nakayama rule to show that the character χ_λ vanishes on all n-cycles if both $d_{ev} > 0$ and $d_{odd} > 0$. As d_{ev} cannot be zero, because $H^0(W) \neq 0$, it follows that this relation is a monic polynomial in c (after perhaps scaling by a rational number) if and only if $d_{odd} = 0$. (This accounts for why we restricted to manifolds with only even rational cohomology in the first case of Theorem A.)

3 Torus actions

In this section we suppose that we have a smooth action of the torus $T = (S^1)^k$ on a d-dimensional orientable manifold W. We write W^T for the fixed set of this action. The Borel construction gives a smooth fibre bundle

$$W \longrightarrow W/\!\!/T \xrightarrow{\pi} BT, \tag{3.1}$$

and the action of T on the tangent bundle $TW \to W$ gives a vector bundle $T^T W := TW/\!\!/T \to W/\!\!/T$, which is the vertical tangent bundle of the smooth fibre bundle π. Following the usual notation of equivariant cohomology we write

$$H_T^* = H^*(BT; \mathbb{Q}) = \mathbb{Q}[x_1, x_2, \ldots, x_k] \quad \text{and} \quad H_T^*(W) = H^*(W/\!\!/T; \mathbb{Q}).$$

As (3.1) is a smooth fibre bundle, there is a ring homomorphism $\rho : R^*(W) \to H_T^*$, and we denote by $R_T^* \leq H_T^*$ its image. Pulling back π along itself gives a smooth fibre bundle over $W /\!\!/ T$ with canonical section, and so a ring homomorphism $\rho_* : R^*(W, *) \to H_T^*(W)$, and we denote by $R_T^*(*) \leq H_T^*(W)$ its image.

Our goal in this section is to describe conditions on the manifold W and the action of T on W which allow us to estimate the Krull dimension of $R^*(W)$ as $\mathrm{Kdim}(R^*(W)) \geq k$. We will regularly use the following standard piece of commutative algebra: when one ring is integral over another they have the same Krull dimension, by the "going up" and "going down" theorems [5, Ch. 5]. Our most general result is as follows.

Theorem 3.1 *Let T act smoothly and effectively on a connected closed orientable manifold W. Let $V_1, V_2, \ldots, V_p$ be an enumeration of the T-representations arising as normal spaces to points on W^T, and let B_i denote the Euler characteristic of the subspace of W^T consisting of those path components having normal representation V_i.*

If some $y_i \in \mathbb{Q}[y_1, y_2, \ldots, y_p]$ is integral over the subring generated by

$$\sum_{i=1}^{p} B_i y_i^n, \quad n = 1, 2, 3, \ldots,$$

then H_T^* is integral over R_T^*. In particular $\mathrm{Kdim}(R^*(W)) \geq k$.

It is perhaps not clear when the hypothesis of this theorem is likely to hold. The following lemma, which we learnt from [8], gives a simple criterion.

Lemma 3.2 *Suppose that we have discarded the B_i which are zero, and that this is not all of them. If the remaining numbers $B_1, B_2, \ldots, B_p$ have all partial sums non-zero, then $\mathbb{Q}[y_1, y_2, \ldots, y_p]$ is finite over the subring generated by*

$$\sum_{i=1}^{p} B_i y_i^n, \quad n = 1, 2, 3, \ldots, \tag{3.2}$$

and so every y_i is integral over this subring.

Proof Write $B \leq \mathbb{Q}[y_1, y_2, \ldots, y_p]$ for the subring generated by the $\sum_{i=1}^{p} B_i y_i^n$, and B^+ for the subset of positive-degree elements.

Claim. *If $\sqrt{(B^+)} = (y_1, y_2, \ldots, y_p)$ then $\mathbb{Q}[y_1, \ldots, y_p]$ is finite over B.*

Our proof of this claim follows the discussion at [26]. Under the assumption the quotient ring $\mathbb{Q}[y_1, y_2, \ldots, y_p]/(B^+)$ has every y_i nilpotent, so is a finite $\mathbb{Q}$-module; let $z_1, z_2, \ldots, z_m \in \mathbb{Q}[y_1, y_2, \ldots, y_p]$ be lifts of these finitely-many generators, which can be taken to be homogeneous as the ideal (B^+) is homogeneous. We claim that these generate $\mathbb{Q}[y_1, y_2, \ldots, y_p]$ as a B-module; let $M \subset \mathbb{Q}[y_1, y_2, \ldots, y_p]$ be the B-submodule that they generate.

As the z_i are homogeneous, and B is generated by homogeneous elements, M is a graded submodule of $\mathbb{Q}[y_1, y_2, \ldots, y_p]$ with the monomial-length grading. Suppose

$p \in \mathbb{Q}[y_1, y_2, \dots, y_p]$ is an element of minimal grading which does not lie in M. Then we may write

$$p = \sum_{i=1}^{m} U_i z_i + \sum V_j b_j$$

with $U_i \in \mathbb{Q}$, $V_j \in \mathbb{Q}[y_1, y_2, \dots, y_p]$, and $b_j \in (B^+)$. But the b_j have strictly positive degree, so the V_j have strictly smaller degree than p so must lie in M, and hence p does too, which proves the claim.

In order to prove the lemma we must therefore show that $(y_1, y_2, \dots, y_p) = (0, 0, \dots, 0)$ is the only simultaneous solution to the equations $\sum_{i=1}^{p} B_i y_i^n = 0$ for $n \in \mathbb{N}$. If $(y_1, y_2, \dots, y_p) \in \mathbb{Q}^p$ is a solution, then grouping terms with $y_i = y_j$ together we obtain *distinct* rational numbers $\bar{y}_i$ solving the equations

$$\sum_{i=1}^{q} \bar{B}_i \bar{y}_i^n = 0$$

where each $\bar{B}_i$ is a partial sum of the B_i, and hence non-zero by assumption. But this means that the vector $(\bar{B}_1 \bar{y}_1, \dots, \bar{B}_q \bar{y}_q)$ is in the kernel of the (transposed) Vandermonde matrix associated to $(\bar{y}_1, \dots, \bar{y}_q)$, so as the $\bar{y}_i$ are all distinct it follows that $(\bar{B}_1 \bar{y}_1, \dots, \bar{B}_q \bar{y}_q) = 0$, and as the $\bar{B}_i$ are all non-zero it follows that $\bar{y}_i = 0$ as required. □

The following corollary, whilst not so powerful as Theorem 3.1, is often easier to apply as one does not need to classify the normal representations at the fixed set.

Corollary 3.3 *Let the path components $X_1, X_2, \dots, X_\ell$ of the fixed set W^T have Euler characteristics $A_1, A_2, \dots, A_\ell$. If some $x_i \in \mathbb{Q}[x_1, x_2, \dots, x_\ell]$ is integral over the subring generated by*

$$\sum_{i=1}^{\ell} A_i x_i^n, \quad n = 1, 2, 3, \dots,$$

then H_T^* is integral over R_T^*. In particular $\mathrm{Kdim}(R^*(W)) \geq k$.

Proof Consider the ring homomorphism $\phi : \mathbb{Q}[x_1, x_2, \dots, x_\ell] \to \mathbb{Q}[y_1, y_2, \dots, y_p]$ defined by sending x_i to y_j if the normal representation at every point of X_i is V_j. Then

$$\phi\left(\sum_{i=1}^{\ell} A_i x_i^n\right) = \sum_{i=1}^{\ell} A_i \phi(x_i)^n = \sum_{j=1}^{p} B_j y_j^n$$

so ϕ sends the subring $A \subset \mathbb{Q}[x_1, x_2, \dots, x_\ell]$ generated by the $\sum_{i=1}^{\ell} A_i x_i^n$ onto the subring $B \subset \mathbb{Q}[y_1, y_2, \dots, y_p]$ generated by the $\sum_{i=1}^{p} B_i y_i^n$.

If x_i is integral over A then there is a polynomial $q(x) = \sum a_i x^i$ with coefficients in A such that $q(x_i) = 0$. Then $q'(y) = \sum \phi(a_i) y^i$ is a polynomial over B such that $q'(\phi(x_i)) = 0$, so $y_j = \phi(x_i)$ is integral over B, and hence Theorem 3.1 applies. □

Example 3.4 There are several standard conditions which oblige a torus action on a manifold W to have connected fixed-set. For example

(i) Let W have dimension $2n$, and suppose that all its cohomology apart from $H^0(W;\mathbb{Q})$ and $H^{2n}(W;\mathbb{Q})$ lies in odd degrees, and that there is some cohomology in odd degrees. Then W^T is connected (by the localisation theorem in equivariant cohomology, which we will describe in the following section).
(ii) If W has trivial even-dimensional rational homotopy groups, then W^T is empty or connected [25, Theorem IV.5].

In such cases $\chi(W^T) = \chi(W)$, so if this is non-zero then the hypotheses of Corollary 3.3 are satisfied.

Example 3.5 Suppose that the action of T^k on W has isolated fixed points, or more generally that all A_i are equal and non-zero. Then the subring generated by the $\sum_{i=1}^{\ell} A_i x_i^n$ is the subring of symmetric polynomials in $\mathbb{Q}[x_1, x_2, \ldots, x_\ell]$, and every x_i is integral over this so the hypotheses of Corollary 3.3 are satisfied.

This immediately implies that if W^{2n} is a quasitoric manifold (that is, the "toric manifolds" of [14]) then $\mathrm{Kdim}(R^*(W)) \geq n$, as such manifolds by definition have an action of an n-torus with isolated fixed points. Slightly more subtly, if G/K is a homogeneous space of rank zero (i.e. $\mathrm{rk}(G) = \mathrm{rk}(K)$) then a common maximal torus T of G and K acts on G/K with fixed points given by the finite set $(W_G(T) \cdot K)/K \subset G/K$, where $W_G(T) := N_G(T)/T$ denotes the (finite) Weyl group of G, so $\mathrm{Kdim}(R^*(G/K)) \geq \mathrm{rk}(G)$.

3.1 The localisation theorem

We now prepare for the proof of Theorem 3.1. Let $X_1, X_2, \ldots, X_\ell$ be the components of the fixed set W^T, with $d_i := \dim(X_i)$, let ν_{X_i} be the normal bundle of X_i in W, and let ν_i be the T-representation which arises as each fibre of ν_{X_i}. Let us write $A_i := \chi(X_i)$, and write $V_1, V_2, \ldots, V_p$ for an enumeration of the T-representations ν_i which arise. Then we have $B_i = \sum_{j \text{ s.t. } \nu_j = V_i} A_j$.

Let us write $\rho_i : H_T^*(W) \to H_T^*(X_i)$ for the restriction map in equivariant cohomology, and $\pi_! : H_T^*(W) \to H_T^{*-d}$ and $(\pi_i)_! : H_T^*(X_i) \to H_T^{*-d_i}$ for the fibre integration maps. As the T-action on X_i is trivial we have $X_i /\!\!/ T = BT \times X_i$, and so the fibre integration map $(\pi_i)_!$ is simply given by slant product with the fundamental class of X_i. As T acts on the normal bundle $\nu_{X_i} \to X_i$, there is an induced vector bundle $\nu_{X_i}^T := \nu_{X_i} /\!\!/ T \to X_i /\!\!/ T$.

Let $S \subset H_T^*$ be the multiplicative subset of nonzero elements. The localisation theorem in equivariant cohomology (of Borel [11, XII.§3], Hsiang [24] and Quillen [35, Section 4]) says that the map

$$\bigoplus_i \rho_i : S^{-1}H_T^*(W) \longrightarrow \bigoplus_i S^{-1}H_T^*(X_i)$$

is an isomorphism. Even more is true: Atiyah and Bott have shown [1, eq (3.8)] that the class $e(\nu_{X_i}^T) \in S^{-1}H_T^{d-d_i}(X_i)$ is a unit and that we have a commutative diagram

$$\begin{tikzcd} S^{-1}H_T^*(W) \arrow[r, "\bigoplus_i \rho_i", "\sim"'] \arrow[d, "\pi_!"] & \bigoplus_i S^{-1}H_T^*(X_i) \arrow[r, "\bigoplus_i e(\nu_{X_i}^T)^{-1}", "\sim"'] & \bigoplus_i S^{-1}H_T^{*+d_i-d}(X_i) \arrow[d, "\sum_i (\pi_i)_!"] \\ S^{-1}H_T^{*-d} \arrow[rr, equal] & & S^{-1}H_T^{*-d}. \end{tikzcd} \tag{3.3}$$

See [6, p. 366] for a textbook exposition of the localisation theorem.

3.2 Proof of Theorem 3.1

Using the diagram (3.3) to compute

$$\kappa_{ep_I} = \pi_!(e(T^T W)p_I(T^T W)) \in S^{-1}H_T^*,$$

which we know lies in the subring H_T^*, gives

$$\begin{aligned}\kappa_{ep_I} &= \sum_{i=1}^{\ell} (\pi_i)_! \left(\frac{e(TX_i \oplus \nu_{X_i}^T)p_I(TX_i \oplus \nu_{X_i}^T)}{e(\nu_{X_i}^T)} \right) \\ &= \sum_{i=1}^{\ell} (\pi_i)_! (e(TX_i)p_I(TX_i \oplus \nu_{X_i}^T))\end{aligned}$$

and in $H_T^*(X_i) = H_T^* \otimes H^*(X_i)$ we have

$$p_I(TX_i \oplus \nu_{X_i}^T) = p_I(\nu_i) \otimes 1 + \text{terms with a nontrivial } H^*(X_i) \text{ component.}$$

When we multiply by $e(TX_i)$ and integrate over X_i the latter terms do not contribute, so as $\int_{X_i} e(TX_i) = \chi(X_i) = A_i$ we get

$$\kappa_{ep_I} = \sum_{i=1}^{\ell} A_i p_I(\nu_i) \in H_T^*.$$

Grouping these terms by the representation types V_j instead gives

$$\kappa_{ep_I} = \sum_{i=1}^{p} B_i p_I(V_i) \in H_T^*. \tag{3.4}$$

Applying this to $p_I = p_j^n$ we find that

$$\sum_{i=1}^{p} B_i p_j(V_i)^n \in R_T^* \text{ for all } j \text{ and } n.$$

Applying the hypothesis of the theorem for each j, we find that there exists an i such that all $p_j(V_i)$ lie in a common integral extension $R_T^* \subseteq R' \subseteq H_T^*$. On the one hand R' is integral over R_T^*. On the other hand by a theorem of Venkov [37] the ring H_T^* is finite over the subring generated by the $p_j(V_i)$ (because V_i is a faithful representation of T, by the standard lemma given below), and hence is finite (and so integral) over R'. It follows that H_T^* is integral over R_T^*, so in particular they have the same Krull dimension, namely k.

Finally, $R^*(W) \to R_T^*$ is surjective and so $\mathrm{Kdim}(R^*(W)) \geq \mathrm{Kdim}(R_T^*) = k$.

Lemma 3.6 *If T acts effectively and smoothly on a connected closed manifold W, then any T-representation arising as the normal space to a point on W^T is faithful.*

Proof We may choose a T-invariant Riemannian metric on W, so the exponential map $\exp : TW \to W$ is equivariant; the restriction of the exponential map to a fibre $T_x W \to W$ is a diffeomorphism when restricted to a neighbourhood of $0 \in T_x W$.

If the action of T on the normal space V to W^T at x had a non-trivial kernel $\{e\} < T' \leq T$ then the T'-action on $T_x W = T(W^T) \oplus V$ is trivial. By exponentiating, it follows that T' fixes an open neighbourhood of $x \in W$. Thus the fixed set $W^{T'}$ is a submanifold of W which contains an open subset; as W is connected it follows that it is the whole of W. This contradicts the action being effective. □

3.3 An extension

The discussion so far gives a technique more general Theorem 3.1, but difficult to formalise in a single result. It is best described through an example.

Proposition 3.7 *Let T act effectively on W with two fixed components X_1 and X_2. Suppose that $\chi(X_1) = -\chi(X_2) \neq 0$ but that the normal T-representations ν_1 and ν_2 at X_1 and X_2 have all Pontrjagin classes distinct (when they are non-zero). Then* $\mathrm{Kdim}(R^*(W)) \geq k$.

Proof We have that

$$\tfrac{1}{\chi(X_1)}\kappa_{ep_j^n} = p_j(\nu_1)^n - p_j(\nu_2)^n \in R_T^* \leq H_T^*$$

for all j and n, and $p_j(\nu_1) - p_j(\nu_2) \neq 0 \in R_T^*$. Hence

$$p_j(\nu_1) = \frac{1}{2}\left(p_j(\nu_1) - p_j(\nu_2) + \frac{p_j(\nu_1)^2 - p_j(\nu_2)^2}{p_j(\nu_1) - p_j(\nu_2)}\right) \in R_T^*[(p_j(\nu_1) - p_j(\nu_2))^{-1}].$$

Therefore after inverting the finite set

$$S := \{p_j(\nu_1) - p_j(\nu_2),\ j = 1, 2, \ldots\}$$

of non-zero elements in $R^*_T \leq H^*_T = \mathbb{Q}[x_1, x_2, \ldots, x_k]$, we find that the $p_j(\nu_1)$ lie in $S^{-1}R^*_T$, and hence by Venkov's theorem [37] that $S^{-1}H^*_T$ is a finite $S^{-1}R^*_T$-module. As $S^{-1}H^*_T$ still has Krull dimension k (there is a maximal ideal $\mathfrak{m}$ of H^*_T not containing the product of the finitely-many elements in S—as the intersection of all maximal ideals is zero—whence $(S^{-1}H^*_T)_{S^{-1}\mathfrak{m}} \cong (H^*_T)_{\mathfrak{m}}$ so $S^{-1}\mathfrak{m}$ is a maximal ideal of $S^{-1}H^*_T$ of height k), it follows that $S^{-1}R^*_T$ has Krull dimension k and so $\mathrm{Kdim}(R^*_T) \geq k$. Hence $\mathrm{Kdim}(R^*(W)) \geq k$. □

4 Examples

4.1 Manifolds with mostly odd cohomology

Let W be a $2n$-dimensional manifold whose cohomology is only non-trivial in degrees 0, $2n$, and odd degrees, let $d = \dim_{\mathbb{Q}} H^{odd}(W)$, and suppose $\chi(W) = 2 - d \neq 0$. Then by Theorem A the $\mathbb{Q}$-algebra $R^*(W)$ is finitely-generated and $R^*(W, *)$ is a finite $R^*(W)$-module.

Furthermore, by our method of proof, Grigoriev's theorem holds for these manifolds (our Theorem 2.8). Therefore the results of Sections 2 and 3 of [21] hold for W as well, as Grigoriev's theorem was the only external input. So if $d > 2$ then

$$\mathbb{Q}[\kappa_{ep_1}, \ldots, \kappa_{ep_{n-1}}] \longrightarrow R^*(W)/\sqrt{0}$$

is surjective. Hence $\mathrm{Kdim}(R^*(W)) \leq n - 1$.

By Example 3.4 (i), if $T = (S^1)^k$ acts on such a manifold W then the fixed set W^T is connected, so $\mathrm{Kdim}(R^*(W)) \geq k$. The construction of [21, Section 4.1] can be mimicked to obtain an action of $SO(k) \times SO(2n-k)$ on $\#^g S^k \times S^{2n-k}$ for any k, and the calculation of the characteristic classes κ_{ep_i} for the associated bundle is entirely analogous.

We obtain the following generalisation of the results of [21].

Corollary 4.1 *For k odd and $g > 1$ we have*

$$\mathbb{Q}[\kappa_{ep_1}, \ldots, \kappa_{ep_{n-1}}] \xrightarrow{\sim} R^*(\#^g S^k \times S^{2n-k})/\sqrt{0}$$

and

$$R^*(\#^g S^k \times S^{2n-k})/\sqrt{0} \xrightarrow{\sim} R^*(\#^g S^k \times S^{2n-k}, *)/\sqrt{0}.$$

Furthermore $(2 - 2g) \cdot c = \kappa_{ec} \in R^*(\#^g S^k \times S^{2n-k}, *)/\sqrt{0}$, *so*

$$R^*(\#^g S^k \times S^{2n-k}, D^{2n})/\sqrt{0} = \mathbb{Q},$$

and hence $R^*(\#^g S^k \times S^{2n-k}, D^{2n})$ *is a finite-dimensional* $\mathbb{Q}$*-vector space.*

As in [21] results can be obtained for $g = 0$ or 1 too, but we shall not write them out here.

4.2 Quasitoric manifolds

A quasitoric manifold W^{2n} has by definition a smooth action of $T = (S^1)^n$ with isolated fixed points, so has $\mathrm{Kdim}(R^*(W)) \geq n$ by Corollary 3.3. Furthermore, the integral cohomology of W is supported in even degrees, so its rational cohomology is too, and therefore by Theorem A the $\mathbb{Q}$-algebra $R^*(W)$ is finitely-generated and $R^*(W, *)$ is a finite $R^*(W)$-module.

4.3 Non-finite generation

We shall give some examples of manifolds W for which $R^*(W)$, and in fact even $R^*(W)/\sqrt{0}$, is not finitely-generated. We shall do so by constructing actions of a torus T on W and showing that the tautological subring $R_T^* \leq H_T^*$ is not finitely-generated. As H_T^* is an integral domain the natural surjection $R^*(W) \to R_T^*$ factors through $R^*(W)/\sqrt{0}$, which therefore shows that $R^*(W)/\sqrt{0}$ is not finitely-generated.

Before attempting this method there is an important observation to be made.

Observation 4.2 Let $T = (S^1)^k$ act on W satisfying the hypotheses of Theorem 3.1; then that theorem shows that the inclusion $R_T^* \hookrightarrow H_T^*$ is integral.

As H_T^* is Noetherian, and $H^*(BT; H^*(W))$ is a finitely-generated H_T^*-module, it follows from the Serre spectral sequence for the Borel construction that $H_T^*(W)$ is a finitely-generated H_T^*-module and hence is integral over H_T^*.

Therefore the morphism $R_T^* \to H_T^* \to H_T^*(W)$ is integral, so $R_T^* \to R_T^*(*)$ is integral too. It then follows from applying Lemma 2.2 as in the proof of Proposition 2.1 that $R_T^* \to R_T^*(*)$ is finite and R_T^* is a finitely-generated $\mathbb{Q}$-algebra.

So to pursue the programme we have suggested one should only try to use torus actions which *do not* satisfy the hypotheses of Theorem 3.1. The following allows us to construct manifolds with torus actions having prescribed normal representations and Euler characteristics of its fixed sets.

Construction 4.3 Fix a positive odd integer n and an even integer k. Let $\Sigma(k)^{2n}$ be the $2n$-manifold of Euler characteristic k obtained as $\#^g S^n \times S^n$ (if k is non-positive) or $\coprod^g S^{2n}$ (if k is positive). Let $H(k)^{2n+1}$ be the manifold with boundary $\Sigma(k)^{2n}$ given by $\natural^g S^n \times D^{n+1}$ or $\coprod^g D^{2n+1}$ respectively.

Let T be a torus, and suppose we are given even integers $B_1, B_2, \ldots, B_p$ and distinct faithful complex T-representations $V_1, V_2, \ldots, V_p$, which are all of the same dimension and which have no trivial subrepresentations. Then we can form the manifold

$$M(i) = M(B_i, V_i) := H(B_i)^{2n+1} \times \mathbb{S}(V_i) \cup_{\Sigma(B_i)\times\mathbb{S}(V_i)} \Sigma(B_i)^{2n} \times \mathbb{D}(V_i).$$

which has a T-action on the right-hand factors. We may then let M be the disjoint union $M = M(1) \sqcup M(2) \sqcup \cdots \sqcup M(p)$.

As V_i is a representation having no trivial subrepresentations, T acts freely on $\mathbb{S}(V_i)$ and its only fixed point on $\mathbb{D}(V_i)$ is 0. Thus $M(i)^T = \Sigma(B_i)^{2n} \times \{0\}$, and the normal representation at these fixed points is given by V_i.

Each V_i may be written as a sum $L_1 \oplus \cdots \oplus L_m$ of 1-dimensional complex T-representations; if a unit vector $v \in \mathbb{S}(V_i)$ is written in components as $(l_1, \ldots, l_m)$ with all l_j non-zero, then a $t \in T$ which stabilises it must act trivially on each L_j, so must act trivially on V_i, so t must be the identity as V_i is a faithful T-representation. Thus such a $v \in \mathbb{S}(V_i)$ must lie in a free orbit, so in particular each path component of $M(i)$ has a free orbit. If one prefers a connected manifold, such free orbits in two different path components have tubular neighbourhoods T-equivariantly diffeomorphic to $T \times D^{2n+2m-\mathrm{rk}(T)}$, which can therefore be cut out and the remaining pieces glued together T-equivariantly along the common boundaries $T \times S^{2n+2m-\mathrm{rk}(T)-1}$. Doing this enough times yields a connected T-manifold with the same fixed-point data, and hence by localisation with the same characteristic classes.

Lemma 4.4 *The T-manifold M so obtained has $\kappa_{p_I} = 0$ and*

$$\kappa_{ep_I} = \sum_{i=1}^{p} B_i \cdot p_I(V_i) \in H_T^*.$$

Proof The second statement follows from (3.4). An analogous calculation shows that

$$\kappa_{p_I} = \sum_{i=1}^{p} (\pi_i)_! \left(\frac{p_I(TX_i \oplus \nu_{X_i}^T)}{e(\nu_{X_i}^T)} \right).$$

The bundle $\nu_{X_i} \to X_i$ is trivial, so the equivariant bundle $\nu_{X_i}^T$ is isomorphic to the pullback of V_i to $X_i /\!\!/ T = BT \times X_i$. Thus the total Pontrjagin class satisfies

$$p(TX_i \oplus \nu_{X_i}^T) = p(V_i) \otimes p(TX_i) = p(V_i) \otimes 1 \in H_T^* \otimes H^*(X_i)$$

as TX_i is stably trivial, and so $p_j(TX_i \oplus \nu_{X_i}^T) = p_j(V_i) \otimes 1$. Hence

$$\frac{p_I(TX_i \oplus \nu_{X_i}^T)}{e(\nu_{X_i}^T)} = \frac{p_I(V_i)}{e(V_i)} \otimes 1$$

which pushes forward to zero (as $\dim(X_i) = 2n > 0$), so $\kappa_{p_I} = 0$. □

We now give our example.

Example 4.5 Let $T = (S^1)^2$ and V_1 be the 2-dimensional complex T-representation with weights $\{x_1 + x_2, x_2\}$, and V_2 be the 2-dimensional complex T-representation with weights $\{x_1, x_2\}$. Construction 4.3 with $B_1 = 2$ and $B_2 = -2$ yields a T-manifold

W (which may be chosen to have any dimension at least 6 and congruent to 2 modulo 4) having $\kappa_{p_I} = 0$ and

$$\kappa_{ep_I} = 2(p_I(V_1) - p_I(V_2)) \in H_T^* = \mathbb{Q}[x_1, x_2].$$

For the chosen representations the total Pontrjagin classes are

$$\begin{aligned} p(V_1) &= (1 - (x_1 + x_2)^2)(1 - x_2^2) \\ p(V_2) &= (1 - x_1^2)(1 - x_2^2). \end{aligned}$$

Let us consider the image of the tautological subring $R_T^* \leq H_T^* = \mathbb{Q}[x_1, x_2]$ in the quotient $\mathbb{Q}[x_1, x_2]/(x_2^2)$. Here $p_2(V_1) = p_2(V_2) = 0$ and

$$\begin{aligned} p_1(V_1) &= -(2x_1x_2 + x_1^2) \\ p_1(V_2) &= -x_1^2, \end{aligned}$$

so the only non-zero κ_{ep_I} in this quotient ring are

$$\kappa_{ep_1^i} = 2(-1)^i((2x_1x_2 + x_1^2)^i - (x_1^2)^i) = 4i(-1)^i x_1^{2i-1}x_2,$$

so the image of R_T^* in $\mathbb{Q}[x_1, x_2]/(x_2^2)$ is the subring $S := \mathbb{Q}\langle x_1x_2, x_1^3x_2, x_1^5x_2, \ldots\rangle$. The ring S is an infinite-dimensional $\mathbb{Q}$-vector space, as the $x_1^{2i-1}x_2$ all have different degrees and are non-zero as they are not divisible by x_2^2. On the other hand, multiplication of any two positive-degree elements in S is zero, as each positive-degree element is divisible by x_2 so a product is divisible by x_2^2. Thus S is infinitely-generated, so R_T^* is too, and hence $R^*(W)/\sqrt{0}$ is too.

Let us record some observations about this example.

Remark 4.6 If we suppose that $n \geq 5$ is odd and the T-manifolds $M(2, V_1)$ and $M(-2, V_2)$ are glued along a free orbit as suggested above, then the $(2n+4)$-manifold M obtained is simply-connected and has the same integral homology as

$$(S^2 \times S^{2n+2})\#(S^2 \times S^{2n+2})\#(S^3 \times S^{2n+1})\#(S^n \times S^{n+4})\#(S^n \times S^{n+4}).$$

Remark 4.7 Although this tautological ring is not finitely-generated, Proposition 3.7 applies to this torus action and gives $\mathrm{Kdim}(R^*(W)) \geq 2$. (Specifically, we have

$$p_1(V_1) - p_1(V_2) = -(2x_1x_2 + x_2^2) \qquad p_2(V_1) - p_2(V_2) = x_2^2(2x_1x_2 + x_2^2)$$

so after inverting $s := x_2(2x_1 + x_2) \neq 0 \in R_T^*$ the subring $s^{-1}R_T^* \leq s^{-1}H_T^*$ contains $p_1(V_1)$, $p_2(V_1)$, $p_1(V_2)$, and $p_2(V_2)$.)

Remark 4.8 Choosing $* \in X_2$ gives a map $R^*(W, *) \to H_T^*$, whose image is generated by the $\kappa_{ep_I} = 2(p_I(V_1) - p_I(V_2))$ along with the characteristic classes of the representation V_2, which are $e(V_2) = x_1 x_2$, $p_1(V_2) = -(x_1^2 + x_2^2)$, and $p_2(V_2) = x_1^2 x_2^2$. Rearranging a little shows that this is the subring generated by $e(V_2)$ and the $p_j(V_i)$, so is finitely generated. (Similarly if we choose $* \in X_1$.) This raises the interesting possibility that $R^*(W, *)$ might be finitely-generated in more generality than $R^*(W)$ is.

4.4 The complex projective plane

Let us consider the manifold $\mathbb{CP}^2$, whose cohomology is supported in even degrees. Thus by Theorem A the $\mathbb{Q}$-algebra $R^*(\mathbb{CP}^2)$ is finitely-generated and $R^*(\mathbb{CP}^2, *)$ is a finite $R^*(\mathbb{CP}^2)$-module. We will explain estimates on the generators for these algebras, using the relations developed in Sect. 2.5. The computations were done with assistance from `Maple`™.

The trace identity technique of Sect. 2.5 gives the relation

$$c^3 = \kappa_{ec} c^2 - \frac{\kappa_{ec}^2 - \kappa_{ec^2}}{2!} c + \frac{\kappa_{ec}^3 - 3\kappa_{ec}\kappa_{ec^2} + 2\kappa_{ec^3}}{3!} \in R^*(\mathbb{CP}^2, *)$$

for any $c \in H^*(BSO(4)) = \mathbb{Q}[p_1, e]$. In particular, for $c = e$ and $c = p_1$ we obtain

$$e^3 = \kappa_{e^2} e^2 - \frac{\kappa_{e^2}^2 - \kappa_{e^3}}{2!} e + \frac{\kappa_{e^2}^3 - 3\kappa_{e^2}\kappa_{e^3} + 2\kappa_{e^4}}{3!} \tag{4.1}$$

$$p_1^3 = \kappa_{ep_1} p_1^2 - \frac{\kappa_{ep_1}^2 - \kappa_{ep_1^2}}{2!} p_1 + \frac{\kappa_{ep_1}^3 - 3\kappa_{ep_1}\kappa_{ep_1^2} + 2\kappa_{ep_1^3}}{3!} \tag{4.2}$$

We may partially polarise the relation by taking $c = e + t \cdot p_1$, expanding and collecting coefficients of powers of t. The coefficients of 1 and of t^3 simply give the relations (4.1) and (4.2). The coefficient of t gives

$$\begin{aligned} &- \kappa_{e^3 p_1} - (1/2)\kappa_{e^2}^2 \kappa_{ep_1} - \kappa_{e^2 p_1} e - \kappa_{ep_1} e^2 + (1/2)\kappa_{e^2}^2 p_1 + \kappa_{e^2}\kappa_{e^2 p_1} \\ &\quad - (1/2)\kappa_{e^3} p_1 + (1/2)\kappa_{ep_1}\kappa_{e^3} + \kappa_{e^2}\kappa_{ep_1} e - 2\kappa_{e^2} e p_1 + 3e^2 p_1 = 0 \end{aligned} \tag{4.3}$$

and the coefficient of t^2 gives

$$\begin{aligned} &(1/2) e \kappa_{ep_1}^2 - (1/2)\kappa_{ep_1}^2 \kappa_{e^2} - \kappa_{e^2 p_1^2} - (1/2) e \kappa_{ep_1^2} + \kappa_{ep_1}\kappa_{e^2 p_1} \\ &\quad + (1/2)\kappa_{ep_1^2}\kappa_{e^2} - p_1 \kappa_{e^2 p_1} - p_1^2 \kappa_{e^2} + p_1 \kappa_{ep_1}\kappa_{e^2} - 2 e p_1 \kappa_{ep_1} + 3 e p_1^2 = 0. \end{aligned} \tag{4.4}$$

(More generally, one could fully polarise this relation, by writing $c = u + t \cdot v + s \cdot w$, expanding out and taking the coefficient of ts: this gives a trilinear form in the variables (u, v, w) which vanishes for all $u, v, w \in H^*(BSO(4)) = \mathbb{Q}[p_1, e]$, and there is no reason to take these to be linear terms. However, we will not pursue this here.)

The relations (4.1), (4.2), (4.3) and (4.4), multiplied by monomials in $\mathbb{Q}[p_1, e]$ and pushed forward, show that certain $\kappa_{e^a p_1^b} \in R^*(\mathbb{CP}^2)$ are decomposable. Specifically

$$\begin{aligned}
\kappa_{xp_1^3} &\text{ is decomposable for any monomial } x \neq 1, e, p_1\\
\kappa_{xep_1^2} &\text{ is decomposable for any monomial } x \neq 1, e, p_1\\
\kappa_{xe^2p_1} &\text{ is decomposable for any monomial } x \neq 1, e, p_1\\
\kappa_{xe^3} &\text{ is decomposable for any monomial } x \neq 1, e, p_1.
\end{aligned}$$

Writing $\equiv$ to mean "equal modulo decomposables", there are further relations:

(i) Pushing (4.2) forward gives $\kappa_{p_1^3} \equiv \frac{3}{2}\kappa_{ep_1^2}$.
(ii) Pushing (4.2) multiplied by p_1 forward gives $\kappa_{p_1^4} \equiv \kappa_{ep_1^3}$.
(iii) Pushing (4.1) forward gives that κ_{e^3} is decomposable, and in fact that $\kappa_{e^3} = \kappa_{e^2}^2$.
(iv) Pushing (4.1) multiplied by p_1 forward gives $\kappa_{e^3 p_1} \equiv \kappa_{e^4}$.
(v) Pushing (4.3) multiplied by p_1 forward gives $\kappa_{e^2 p_1^2} \equiv \kappa_{e^3 p_1}$.
(vi) Pushing (4.4) forward gives $2\kappa_{e^2 p_1} \equiv \kappa_{ep_1^2}$.
(vii) Pushing (4.4) multiplied by p_1 forward gives $\kappa_{ep_1^3} \equiv \kappa_{e^2 p_1^2}$.

Using these relations we find that the five classes

$$\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{p_1^4}, \kappa_{ep_1}, \kappa_{e^2} \in R^*(\mathbb{CP}^2)$$

generate.

As described in [21, Section 2], it follows from work of Atiyah [7] that for each Hirzebruch class $\mathcal{L}_i$ the associated class $\kappa_{\mathcal{L}_i} \in R^*(W)$ is pulled back via the natural map

$$\phi : B\mathrm{Diff}^+(W) \longrightarrow B\mathrm{Aut}(H, \lambda),$$

where $H = H^n(W; \mathbb{Z})/\text{torsion}$ and $\lambda : H \otimes H \to \mathbb{Z}$ is the intersection form of W.

For $W = \mathbb{CP}^2$ the bilinear form $(H, \lambda) = (\mathbb{Z}, (1))$ has automorphism group $\mathbb{Z}/2$, which has trivial rational cohomology. Thus the classes $\kappa_{\mathcal{L}_i} \in R^*(\mathbb{CP}^2)$ are zero. The first few are

$$\begin{aligned}
7\kappa_{e^2} - \kappa_{p_1^2} &= 0\\
-13\kappa_{e^2p_1} + 2\kappa_{p_1^3} &= 0\\
-19\kappa_{e^4} + 22\kappa_{e^2p_1^2} - 3\kappa_{p_1^4} &= 0\\
127\kappa_{e^4p_1} - 83\kappa_{e^2p_1^3} + 10\kappa_{p_1^5} &= 0\\
8718\kappa_{e^6} - 27635\kappa_{e^4p_1^2} + 12842\kappa_{e^2p_1^4} - 1382\kappa_{p_1^6} &= 0\\
-7978\kappa_{e^6p_1} + 11880\kappa_{e^4p_1^3} - 4322\kappa_{e^2p_1^5} + 420\kappa_{p_1^7} &= 0\\
-68435\kappa_{e^8} + 423040\kappa_{e^6p_1^2} - 407726\kappa_{e^4p_1^4} + 122508\kappa_{e^2p_1^6} - 10851\kappa_{p_1^8} &= 0\\
11098737\kappa_{e^8p_1} - 29509334\kappa_{e^6p_1^3} + 20996751\kappa_{e^4p_1^5} - 5391213\kappa_{e^2p_1^7} + 438670\kappa_{p_1^9} &= 0.
\end{aligned}$$

The first Hirzebruch relation allows us to remove κ_{e^2} from the list of generators. The second Hirzebruch relation, with the relations $\kappa_{p_1^3} \equiv \frac{3}{2}\kappa_{ep_1^2} \equiv 3\kappa_{e^2 p_1}$ proved above, shows that $\kappa_{p_1^3}$ is decomposable. This proves the

Lemma 4.9 *The classes $\kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}$ generate $R^*(\mathbb{CP}^2)$.*

Let the ideal I of $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}]$ be generated by those relations implied by (4.1)–(4.4) for $\kappa_{e^a p_1^b}$ for $a + b \leq 9$, and the Hirzebruch relations listed above.[2] Generators for this ideal can be computed to be

$$\begin{aligned}
&(4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1})\kappa_{p_1^4}\\
&(4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1})(21\kappa_{ep_1} + 8\kappa_{p_1^2})\\
&(4\kappa_{p_1^2} - 7\kappa_{ep_1})(316\kappa_{ep_1}^3 - 343\kappa_{p_1^4})\\
&(4\kappa_{p_1^2} - 7\kappa_{ep_1})(1264\kappa_{p_1^2}\kappa_{ep_1}^2 + 2212\kappa_{ep_1}^3 - 5145\kappa_{p_1^4}).
\end{aligned}$$

This ideal is not radical, and $\sqrt{I}$ is generated by

$$\begin{aligned}
&(4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1})\\
&(4\kappa_{p_1^2} - 7\kappa_{ep_1})(316\kappa_{ep_1}^3 - 343\kappa_{p_1^4}).
\end{aligned}$$

Corollary 4.10 *There is a surjection from*

$$\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/((4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1}), (4\kappa_{p_1^2} - 7\kappa_{ep_1})(316\kappa_{ep_1}^3 - 343\kappa_{p_1^4}))$$

to $R^*(\mathbb{CP}^2)/\sqrt{0}$.

One can see that this quotient ring contains $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^4}]$ as a subring and is integral over it, so it has Krull dimension 2. It follows that $\mathrm{Kdim}(R^*(\mathbb{CP}^2)) \leq 2$.

4.4.1 Fixing a point

It follows from Lemma 4.9 that $R^*(\mathbb{CP}^2, *)$ is generated by $e, p_1, \kappa_{p_1^2}, \kappa_{p_1^4}$ and κ_{ep_1}. Adding to the ideal I above the relations (4.1)–(4.4) gives an ideal J of $\mathbb{Q}[e, p_1, \kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}]$ which is rather complicated, but its radical is generated by the relations

[2] The threshold $a + b \leq 9$ is not significant, and one could try to go further, but we have checked that adding those relations with $a + b = 10$ does not change the ideal I.

$$(4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1})$$
$$1264\kappa_{p_1^2}\kappa_{ep_1}^3 - 2212\kappa_{ep_1}^4 - 1372\kappa_{p_1^2}\kappa_{p_1^4} + 2401\kappa_{p_1^4}\kappa_{ep_1}$$
$$10\kappa_{p_1^2}\kappa_{ep_1} - 28\kappa_{p_1^2}p_1 - 21\kappa_{ep_1}^2 - 14\kappa_{ep_1}e + 63\kappa_{ep_1}p_1$$
$$3\kappa_{p_1^2}\kappa_{ep_1} - 28\kappa_{p_1^2}e - 7\kappa_{ep_1}^2 + 42\kappa_{ep_1}e + 7\kappa_{ep_1}p_1$$
$$45\kappa_{p_1^2}\kappa_{ep_1} - 112\kappa_{ep_1}^2 - 84\kappa_{ep_1}e + 182\kappa_{ep_1}p_1 + 196e^2 - 196p_1^2$$
$$15\kappa_{p_1^2}\kappa_{ep_1} - 35\kappa_{ep_1}^2 + 14\kappa_{ep_1}e + 35\kappa_{ep_1}p_1 + 196e^2 - 196ep_1$$
$$316\kappa_{ep_1}^4 + 1264\kappa_{ep_1}^3 e - 1264\kappa_{ep_1}^3 p_1 - 343\kappa_{p_1^4}\kappa_{ep_1} - 1372\kappa_{p_1^4}e + 1372\kappa_{p_1^4}p_1$$
$$12263\kappa_{p_1^2}\kappa_{ep_1}^2 - 19446\kappa_{ep_1}^3 + 168\kappa_{ep_1}^2 e - 168\kappa_{ep_1}^2 p_1 - 4116\kappa_{ep_1}e^2 + 16464e^3 - 5488\kappa_{p_1^4}$$

the last of which shows that the generator $\kappa_{p_1^4}$ may be eliminated from the ring $\mathbb{Q}[e, p_1, \kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}]/\sqrt{J}$. One may also deduce from these relations that κ_{ep_1} and $\kappa_{p_1^2}$ are integral over $\mathbb{Q}[e, p_1]$, so that $R^*(\mathbb{CP}^2, *)/\sqrt{0}$ is finite over $\mathbb{Q}[e, p_1]$.

4.4.2 Fixing a disc

As passing from $R^*(\mathbb{CP}^2, *)$ to $R^*(\mathbb{CP}^2, D^4)$ in particular kills e and p_1, we deduce from the above that

Corollary 4.11 *$R^*(\mathbb{CP}^2, D^4)$ is a finite-dimensional $\mathbb{Q}$-vector space.*

In fact, setting $K = J + (e, p_1)$ and simplifying, we find that K is generated by

$$\begin{array}{rl} \kappa_{p_1^2}^4 & \kappa_{p_1^2}^2(105\kappa_{ep_1} - 11\kappa_{p_1^2}) \\ \kappa_{p_1^2}(245\kappa_{ep_1}^2 - 52\kappa_{p_1^2}^2) & 1029\kappa_{ep_1}^3 - 52\kappa_{p_1^2}^3 \\ 245\kappa_{p_1^4} - 29\kappa_{p_1^2}^3 & \end{array}$$

and $R^*(\mathbb{CP}^2, D^4)$ is a quotient of $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}]/K$ so

Corollary 4.12 $\dim_{\mathbb{Q}} R^*(\mathbb{CP}^2, D^4) \leq 7$.

4.4.3 Lower bounds via torus actions

Consider the standard toric action of the torus $T = (S^1)^2$ on $\mathbb{CP}^2$, via

$$\begin{aligned} S^1 \times S^1 \times \mathbb{CP}^2 &\longrightarrow \mathbb{CP}^2 \\ (\xi_1, \xi_2, [z_0 : z_1 : z_2]) &\longmapsto [z_0 : \xi_1 z_1 : \xi_2 z_2] \end{aligned}$$

This gives a ring homomorphism $\phi : R^*(\mathbb{CP}^2) \to H_T^* = \mathbb{Q}[x_1, x_2]$. It is an elementary exercise to compute, by equivariant localisation, the classes

$$\begin{aligned}\phi(\kappa_{p_1^2}) &= 7x_1^2 - 7x_1x_2 + 7x_2^2\\ \phi(\kappa_{ep_1}) &= 4x_1^2 - 4x_1x_2 + 4x_2^2\\ \phi(\kappa_{p_1^4}) &= 23x_1^6 - 69x_1^5x_2 + 135x_1^4x_2^2\\ &\quad - 155x_1^3x_2^3 + 135x_1^2x_2^4 - 69x_1x_2^5 + 23x_2^6\end{aligned}$$

and by eliminating variables to find that the unique relation between these is $\phi(7\kappa_{ep_1} - 4\kappa_{p_1^2}) = 0$. Thus ϕ gives a surjection

$$R^*(\mathbb{CP}^2)/\sqrt{0} \longrightarrow \mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/(7\kappa_{ep_1} - 4\kappa_{p_1^2})$$

and hence $\mathrm{Kdim}(R^*(\mathbb{CP}^2)) \geq 2$. Combining this with the above gives

Corollary 4.13 $\mathrm{Kdim}(R^*(\mathbb{CP}^2)) = 2$.

The fixed point $[1 : 0 : 0]$ of the T-action gives an extension of ϕ to a ring homomorphism $\hat{\phi} : R^*(\mathbb{CP}^2, *)/\sqrt{0} \to H_T^* = \mathbb{Q}[x_1, x_2]$. At this fixed point we have

$$\begin{aligned}\hat{\phi}(s^*e) &= x_1x_2\\ \hat{\phi}(s^*p_1) &= x_1^2 + x_2^2\end{aligned}$$

which shows that the image of $\hat{\phi}$ is isomorphic to

$$\begin{aligned}&\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}, e, p_1]/(\kappa_{p_1^2} - 7p_1 + 7e, \kappa_{ep_1} - 4p_1 + 4e,\\ &\quad 17e^3 - 66e^2p_1 + 69ep_1^2 - 23p_1^3 + \kappa_{p_1^4}),\end{aligned}$$

or in other words $\mathbb{Q}[e, p_1]$.

Remark 4.14 In [19,20] there is given an analysis of S^1-actions on simply-connected 4-manifolds, from which it is possible to deduce—through a very laborious consideration of cases and analysis of fixed-point data—that for any circle action on $\mathbb{CP}^2$ we have $4\kappa_{e^2} = \kappa_{ep_1}$ and so by the first Hirzebruch relation we have $4\kappa_{p_1^2} - 7\kappa_{ep_1} = 0$. Alternatively, this may be proved using Hsiang's splitting theorem for the S^1-equivariant cohomology of $\mathbb{CP}^2$ [25, Theorem VI.1].

4.4.4 The tautological variety

We find it quite revealing to consider the (reduced) tautological ring $R^*(\mathbb{CP}^2)/\sqrt{0}$ by considering its associated variety $\mathbf{V}_{\mathbb{CP}^2}$. The choice of generators $\kappa_{p_1^2}, \kappa_{p_1^4}$, and κ_{ep_1}

of $R^*(\mathbb{CP}^2)$ presents $\mathbf{V}_{\mathbb{CP}^2}$ as a subvariety of $\mathbf{A}^3$, and it follows from Corollary 4.10 that $\mathbf{V}_{\mathbb{CP}^2}$ is contained in the union of the plane

$$\mathbf{P} := \{4\kappa_{p_1^2} - 7\kappa_{ep_1} = 0\}$$

and the line

$$\mathbf{L} := \{\kappa_{p_1^2} - 2\kappa_{ep_1} = 0, 316\kappa_{ep_1}^3 - 343\kappa_{p_1^4} = 0\}.$$

Furthermore, it follows from the calculation of Sect. 4.4.3 that $\mathbf{V}_{\mathbb{CP}^2}$ contains $\mathbf{P}$, so the variety $\mathbf{V}_{\mathbb{CP}^2}$ is either $\mathbf{P}$ or $\mathbf{P} \cup \mathbf{L}$. It would be extremely interesting if $\mathbf{L} \subset \mathbf{V}_{\mathbb{CP}^2}$, but no method for showing this seems to be available. (Each circle action on $\mathbb{CP}^2$ gives a homomorphism $R^*(\mathbb{CP}^2)/\sqrt{0} \to \mathbb{Q}[x_1]$ and hence a morphism $\mathbf{A}^1 \to \mathbf{V}_{\mathbb{CP}^2}$, but by Remark 4.14 all such morphisms have image in $\mathbf{P}$.)

Similarly, by the calculation of Sect. 4.4.1 the four elements e, p_1, $\kappa_{p_1^2}$, and κ_{ep_1} generate $R^*(\mathbb{CP}^2, *)/\sqrt{0}$, which presents the associated variety $\mathbf{V}_{(\mathbb{CP}^2,*)}$ as a subvariety of $\mathbf{A}^4$. Eliminating the variable $\kappa_{p_1^4}$ from the radical ideal described in Sect. 4.4.1 shows that $\mathbf{V}_{(\mathbb{CP}^2,*)}$ is contained in the union of the plane

$$\{4\kappa_{p_1^2} - 7\kappa_{ep_1} = 0, \kappa_{ep_1} - 4p_1 + 4e = 0\}$$

and the lines

$$\{\kappa_{p_1^2} - 2\kappa_{ep_1} = 0, e = 0, \kappa_{ep_1} - 7p_1 = 0\}$$
$$\{\kappa_{p_1^2} - 2\kappa_{ep_1} = 0, 2\kappa_{ep_1} - 7e = 0, 5\kappa_{ep_1} - 7p_1 = 0\}.$$

It follows from the calculation of Sect. 4.4.3 that the plane is contained in $\mathbf{V}_{(\mathbb{CP}^2,*)}$.

4.5 The manifold $S^2 \times S^2$

The cohomology of $S^2 \times S^2$ is supported in even degrees. Thus by Theorem A the algebra $R^*(S^2 \times S^2)$ is finitely-generated and $R^*(S^2 \times S^2, *)$ is a finite $R^*(S^2 \times S^2)$-module.

The trace identity technique of Sect. 2.5 gives the relation

$$c^4 = \kappa_{ec}c^3 - \frac{\kappa_{ec}^2 - \kappa_{ec^2}}{2}c^2 + \frac{\kappa_{ec}^3 - 3\kappa_{ec}\kappa_{ec^2} + 2\kappa_{ec^3}}{6}c$$
$$- \frac{\kappa_{ec}^4}{24} + \frac{\kappa_{ec}^2\kappa_{ec^2}}{4} - \frac{\kappa_{ec^2}^2}{8} - \frac{\kappa_{ec}\kappa_{ec^3}}{3} + \frac{\kappa_{ec^4}}{4} \in R^*(S^2 \times S^2, *)$$

for any $c \in H^*(BSO(4)) = \mathbb{Q}[p_1, e]$. Partially polarising via $c = e + t \cdot p_1$ as in Sect. 4.4, we obtain the relations

$$(1/8)\kappa_{ep_1^2}^2 + (1/24)\kappa_{ep_1}^4 - (1/6)p_1\kappa_{ep_1}^3 - (1/4)\kappa_{ep_1}^2\kappa_{ep_1^2}$$

$$
\begin{aligned}
&+ (1/2)p_1^2\kappa_{ep_1}^2 - (1/3)p_1\kappa_{ep_1^3} + (1/3)\kappa_{ep_1}\kappa_{ep_1^3} - p_1^3\kappa_{ep_1}\\
&- (1/2)p_1^2\kappa_{ep_1^2} - (1/4)\kappa_{ep_1^4} + p_1^4 + (1/2)p_1\kappa_{ep_1}\kappa_{ep_1^2} = 0 \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
&- p_1^2\kappa_{e^2p_1} - (1/6)\kappa_{ep_1}^3 e + \kappa_{ep_1}\kappa_{e^2p_1^2} - p_1\kappa_{e^2p_1^2} + (1/6)\kappa_{ep_1}^3\kappa_{e^2}\\
&+ (1/3)\kappa_{ep_1^3}\kappa_{e^2} - p_1^3\kappa_{e^2} + (1/2)\kappa_{ep_1^2}\kappa_{e^2p_1} - (1/2)\kappa_{ep_1}^2\kappa_{e^2p_1}\\
&+ 4ep_1^3 - (1/3)e\kappa_{ep_1^3} - (1/2)\kappa_{ep_1}\kappa_{ep_1^2}\kappa_{e^2} + (1/2)p_1\kappa_{ep_1^2}\kappa_{e^2}\\
&- (1/2)p_1\kappa_{ep_1}^2\kappa_{e^2} + p_1^2\kappa_{ep_1}\kappa_{e^2} - ep_1\kappa_{ep_1^2} + (1/2)e\kappa_{ep_1}\kappa_{ep_1^2}\\
&+ ep_1\kappa_{ep_1}^2 - 3ep_1^2\kappa_{ep_1} + p_1\kappa_{ep_1}\kappa_{e^2p_1} - \kappa_{e^2p_1^3} = 0 \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
&6e^2p_1^2 + (1/4)\kappa_{ep_1}^2\kappa_{e^2}^2 + (1/2)p_1^2\kappa_{e^2}^2 - (1/2)e^2\kappa_{ep_1^2}\\
&+ (1/2)e^2\kappa_{ep_1}^2 - p_1\kappa_{e^3p_1} + \kappa_{ep_1}\kappa_{e^3p_1} + (1/4)\kappa_{ep_1^2}\kappa_{e^3} - (1/4)\kappa_{ep_1}^2\kappa_{e^3}\\
&- (1/2)p_1^2\kappa_{e^3} - e\kappa_{e^2p_1^2} + \kappa_{e^2}\kappa_{e^2p_1^2} - (1/4)\kappa_{ep_1^2}\kappa_{e^2}^2\\
&- (1/2)e\kappa_{ep_1}^2\kappa_{e^2} - 3ep_1^2\kappa_{e^2} - 3e^2p_1\kappa_{ep_1}\\
&+ (1/2)p_1\kappa_{ep_1}\kappa_{e^3} + (1/2)\kappa_{e^2p_1}^2 + (1/2)e\kappa_{ep_1^2}\kappa_{e^2}\\
&- (1/2)p_1\kappa_{ep_1}\kappa_{e^2}^2 + e\kappa_{ep_1}\kappa_{e^2p_1} + p_1\kappa_{e^2}\kappa_{e^2p_1}\\
&- 2ep_1\kappa_{e^2p_1} - \kappa_{ep_1}\kappa_{e^2}\kappa_{e^2p_1} + 2ep_1\kappa_{ep_1}\kappa_{e^2} - (3/2)\kappa_{e^3p_1^2} = 0 \qquad (4.7)
\end{aligned}
$$

$$
\begin{aligned}
&(1/2)\kappa_{e^2p_1}\kappa_{e^3} - (1/2)\kappa_{e^2}^2\kappa_{e^2p_1} - e^2\kappa_{e^2p_1} - (1/6)\kappa_{e^2}^3p_1\\
&+ (1/6)\kappa_{e^2}^3\kappa_{ep_1} + \kappa_{e^2}\kappa_{e^3p_1} - e^3\kappa_{ep_1} + (1/3)\kappa_{ep_1}\kappa_{e^4} - (1/3)p_1\kappa_{e^4}\\
&- e\kappa_{e^3p_1} + (1/2)p_1\kappa_{e^2}\kappa_{e^3} - (1/2)\kappa_{ep_1}\kappa_{e^2}\kappa_{e^3} + (1/2)e\kappa_{ep_1}\kappa_{e^3}\\
&- ep_1\kappa_{e^3} + e\kappa_{e^2}\kappa_{e^2p_1} + ep_1\kappa_{e^2}^2 - (1/2)e\kappa_{ep_1}\kappa_{e^2}^2 - 3e^2p_1\kappa_{e^2}\\
&+ e^2\kappa_{ep_1}\kappa_{e^2} - \kappa_{e^4p_1} + 4e^3p_1 = 0 \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
&(1/8)\kappa_{e^3}^2 + (1/24)\kappa_{e^2}^4 - (1/6)e\kappa_{e^2}^3 + (1/2)e^2\kappa_{e^2}^2 - (1/4)\kappa_{e^2}^2\kappa_{e^3}\\
&+ (1/2)e\kappa_{e^2}\kappa_{e^3} - (1/4)\kappa_{e^5} - e^3\kappa_{e^2} - (1/2)e^2\kappa_{e^3} - (1/3)e\kappa_{e^4}\\
&+ (1/3)\kappa_{e^2}\kappa_{e^4} + e^4 = 0 \qquad (4.9)
\end{aligned}
$$

Modulo decomposables in $R^*(S^2 \times S^2)$, when multiplied by monomials in $\mathbb{Q}[e, p_1]$ and fibre integrated these give the relations

$$
\begin{aligned}
\kappa_{xp_1^4} &\text{ is decomposable for any monomial } x \neq e\\
\kappa_{xep_1^3} &\text{ is decomposable for any monomial } x \neq e\\
\kappa_{xe^2p_1^2} &\text{ is decomposable for any monomial } x \neq e\\
\kappa_{xe^3p_1} &\text{ is decomposable for any monomial } x \neq 1, e\\
\kappa_{xe^4} &\text{ is decomposable for any monomial } x \neq e.
\end{aligned}
$$

This shows that all generators apart from

$$\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^2}, \kappa_{e^2 p_1}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}$$

are decomposable. The Hirzebruch relations of the previous section hold here as well, as the bilinear form associated to $S^2 \times S^2$ is $(\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ which also has finite automorphism group. The first two Hirzebruch relations $\kappa_{e^2} = \frac{1}{7}\kappa_{p_1^2}$ and $\kappa_{e^2 p_1} = \frac{2}{13}\kappa_{p_1^3}$ allow us to remove two of these generators, and so we find that

Lemma 4.15 *The classes $\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}$ generate $R^*(S^2 \times S^2)$.*

Consider the ideal I of $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}]$ of relations implied by (4.5)–(4.9) for $\kappa_{e^a p_1^b}$ for $a + b \leq 9$, and the Hirzebruch relations of the previous section. It is quite complicated, but it is easy to compute (in `Macaulay2`) that it has codimension 3.

Corollary 4.16 $\mathrm{Kdim}(R^*(S^2 \times S^2)) \leq 4$.

4.5.1 Fixing a point

It follows from Lemma 4.9 that $R^*(S^2 \times S^2, *)$ is generated by $e, p_1, \kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}$. Adding to the ideal I above the relations (4.5)–(4.9) gives an ideal J of $\mathbb{Q}[e, p_1, \kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}]$ which has codimension 5.

4.5.2 Fixing a disc

Passing from $R^*(S^2 \times S^2, *)$ to $R^*(S^2 \times S^2, D^4)$ in particular kills e and p_1, and we may compute the radical of the ideal $K := J + (e, p_1)$, giving the following.

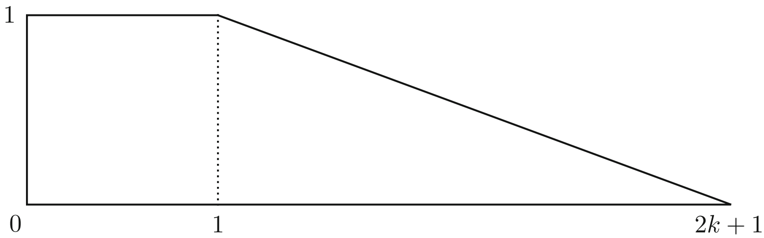
Corollary 4.17 *$R^*(S^2 \times S^2, D^4)/\sqrt{0}$ is a quotient of*

$$\frac{\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}]}{(\kappa_{p_1^3}, \kappa_{p_1^2}, \kappa_{e^3}^2 - 2\kappa_{e^5}, \kappa_{ep_1}\kappa_{e^3} - 2\kappa_{e^3 p_1}, \kappa_{ep_1}^2 - 2\kappa_{ep_1^2})} \cong \mathbb{Q}[\kappa_{ep_1}, \kappa_{e^3}]$$

so has Krull dimension at most 2.

4.5.3 Lower bounds via torus actions

We will use a family of almost-complex torus actions $\phi_k : T^2 \to \mathrm{Diff}(S^2 \times S^2)$ defined for $k \in \mathbb{N}$. These actions are well-known among symplectic geometers: we learnt their construction from work of Karshon [27], suggested to us by Ivan Smith. In that paper these actions are constructed as the toric varieties associated to the Delzant polytopes



and it follows from [27, Lemma 3], and the fact that $k = 0$ yields $S^2 \times S^2$, that all the manifolds so obtained are diffeomorphic to $S^2 \times S^2$. In toric geometry the above polytope should be considered as lying in the dual $\mathfrak{t}^*$ of the Lie algebra of T, having integral basis $\{x_1, x_2\}$ which we identify with the cartesian coordinates in the figure above. The T^2-fixed points correspond to the vertices of the polytope, and the weights at each fixed point are given by the pair of elements of $\mathfrak{t}^*$ given by the two primitive integral vectors associated to the edges incident at that vertex (cf. [12, Example 7.3.19]). For the polytope above the weights are therefore

$$\{x_1, x_2\}, \{x_1, -x_2\}, \{-x_1, 2kx_1 - x_2\}, \{-x_1, x_2 - 2kx_1\}.$$

It follows that at the four fixed points of the action ϕ_k the Euler class is

$$x_1x_2, -x_1x_2, x_1(x_2 - 2kx_1), x_1(2kx_1 - x_2)$$

and the Pontrjagin class $p_1 = c_1^2 - 2c_2$ is

$$x_1^2 + x_2^2, x_1^2 + x_2^2, (4k^2 + 1)x_2^2 - 4kx_1x_2 + x_1^2, (4k^2 + 1)x_2^2 - 4kx_1x_2 + x_1^2$$

We may thus compute the map

$$\psi_k : \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3p_1}, \kappa_{e^5}] \longrightarrow R^*(S^2 \times S^2) \xrightarrow{\phi_k} H_T^* = \mathbb{Q}[x_1, x_2]$$

by equivariant localisation, giving

$$\begin{aligned}
\kappa_{p_1^2} &= 0 \\
\kappa_{p_1^3} &= 0 \\
\kappa_{ep_1} &= 8k^2x_2^2 - 8kx_2x_1 + 4x_2^2 + 4x_1^2 \\
\kappa_{ep_1^2} &= 32k^4x_2^4 - 64k^3x_2^3x_1 + 16k^2x_2^4 + 48k^2x_2^2x_1^2 - 16kx_2^3x_1 - 16kx_2x_1^3 + 4x_2^4 \\
&\quad + 8x_2^2x_1^2 + 4x_1^4 \\
\kappa_{e^3} &= 8k^2x_2^4 - 8kx_2^3x_1 + 4x_2^2x_1^2 \\
\kappa_{e^3p_1} &= 32k^4x_2^6 - 64k^3x_2^5x_1 + 8k^2x_2^6 + 48k^2x_2^4x_1^2 - 8kx_2^5x_1 - 16kx_2^3x_1^3 + 4x_2^4x_1^2 \\
&\quad + 4x_2^2x_1^4 \\
\kappa_{e^5} &= 32k^4x_2^8 - 64k^3x_2^7x_1 + 48k^2x_2^6x_1^2 - 16kx_2^5x_1^3 + 4x_2^4x_1^4
\end{aligned}$$

By eliminating x_1, x_2, and k from the above, one finds generators for the ideal $U := \cap_{k\in\mathbb{N}} \mathrm{Ker}(\psi_k)$ of $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}]$ to be

$$\begin{aligned}
&\kappa_{p_1^2}\\
&\kappa_{p_1^3}\\
&\kappa_{ep_1^2}\kappa_{e^3} - \kappa_{e^3}^2 - \kappa_{ep_1}\kappa_{e^3 p_1} + 4\kappa_{e^5}\\
&\kappa_{e^3}^3 - \kappa_{ep_1}\kappa_{e^3}\kappa_{e^3 p_1} + \kappa_{ep_1}^2\kappa_{e^5} + 4\kappa_{e^3 p_1}^2 - 4\kappa_{ep_1^2}\kappa_{e^5} - 4\kappa_{e^3}\kappa_{e^5},
\end{aligned}$$

so this ideal has codimension 4. There is a surjection

$$R^*(S^2 \times S^2) \longrightarrow \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}]/U,$$

and hence the Krull dimension of $R^*(S^2 \times S^2)$ is bounded below by $7 - \mathrm{codim}(U) = 3$.

Corollary 4.18 $\mathrm{Kdim}(R^*(S^2 \times S^2)) \geq 3$.

Note that each ideal $\mathrm{Ker}(\psi_k)$ of $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}]$ has codimension 5, so each particular torus action only gives 2 as a lower bound for $\mathrm{Kdim}(R^*(S^2 \times S^2))$: it is only by considering the countably-many such actions that we are able to improve this lower bound to 3.

Acknowledgements I am grateful to Søren Galatius for an enlightening discussion of the ideas in Sect. 2 of this paper, and to Jens Reinhold and Dexter Chua for spotting several errors. I would also like to thank the anonymous referee for their useful suggestions.

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