Sel. Math. New Ser. (2018) 24:3175–3216
https://doi.org/10.1007/s00029-018-0410-6

Selecta Mathematica
New Series

Deligne–Lusztig constructions for division algebras and the local Langlands correspondence, II

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Published online: 29 March 2018

Abstract In 1979, Lusztig proposed a cohomological construction of supercuspidal representations of reductive p-adic groups, analogous to Deligne–Lusztig theory for finite reductive groups. In this paper we establish a new instance of Lusztig's program. Precisely, let X be the Deligne–Lusztig (ind-pro-)scheme associated to a division algebra D over a non-Archimedean local field K of positive characteristic. We study the $D^\times$-representations $H_\bullet(X)$ by establishing a Deligne–Lusztig theory for families of finite unipotent groups that arise as subquotients of $D^\times$. There is a natural correspondence between quasi-characters of the (multiplicative group of the) unramified degree-n extension of K and representations of $D^\times$ given by $\theta \mapsto H_\bullet(X)[\theta]$. For a broad class of characters θ, we show that the representation $H_\bullet(X)[\theta]$ is irreducible and concentrated in a single degree. After explicitly constructing a Weil representation from θ using χ-data, we show that the resulting correspondence matches the bijection given by local Langlands and therefore gives a geometric realization of the Jacquet–Langlands transfer between representations of division algebras.

Mathematics Subject Classification 22E50 · 20G05 · 11S37

Contents

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1 Introduction

Deligne–Lusztig theory [8] gives a geometric description of the irreducible representations of finite groups of Lie type. In [12], Lusztig suggests an analogue of Deligne–Lusztig theory for p-adic groups G. For a maximal unramified torus $T \subset G$, he introduces a certain set which has a natural action of $T \times G$. Conjecturally, this set has an algebro-geometric structure and one should be able to define ℓ-adic homology groups functorial for the $T \times G$ action. By [2, 12], when G is a division algebra, one can realize Lusztig's set X as an (ind-pro-)scheme and define corresponding ℓ-adic homology groups $H_i(X, \overline{\mathbb{Q}}_\ell)$. One therefore obtains a correspondence $\theta \mapsto H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ between characters of T and representations of G. In this paper, we study this correspondence and, after describing a Weil representation associated to θ, give a description from the perspective of the local Langlands and Jacquet–Langlands correspondences.

Let K be a non-Archimedean local field of positive characteristic with ring of integers $\mathcal{O}_K$ and residue field $\mathbb{F}_q = \mathcal{O}_K/\pi$ for a fixed uniformizer π, and let $L \supset K$ be the unramified extension of degree n with ring of integers $\mathcal{O}_L$. The *level* of a smooth character $\theta \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ is the smallest integer h such that θ is trivial on U_L^h, where $U_L^0 := \mathcal{O}_L^\times$ and $U_L^h := 1 + \pi^h \mathcal{O}_L$ for $h \geq 1$. We say that θ is *primitive* if for all $1 \neq \gamma \in \mathrm{Gal}(L/K)$, the smooth characters θ and θ/θ^γ have the same level. Equivalently, the restriction of θ to U_L^{h-1}/U_L^h has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer. There is a canonical choice of Langlands–Shelstad χ-datum associated to the maximal torus $L^\times \hookrightarrow \mathrm{GL}_K(L) \cong \mathrm{GL}_n(K)$, and using this, one can associate a smooth irreducible n-dimensional $\mathcal{W}_K$-representation σ_θ to a primitive character $\theta \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$.[1] The representation σ_θ corresponds via local Langlands to an irreducible supercuspidal representation π_θ of $\mathrm{GL}_n(K)$, which in turn corresponds via Jacquet–Langlands to an irreducible representation ρ_θ of $D^\times$ where D is a division algebra of dimension n^2 over K.

Main Theorem *Let $\theta \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a primitive character of level h. Then*

$$H_i(X, \overline{\mathbb{Q}}_\ell)[\theta] = \begin{cases} \rho_\theta & \text{if } i = r_\theta := (n-1)(h-1), \\ 0 & \text{otherwise.} \end{cases}$$

[1] Let $\xi \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be the rectifying character determined by $\xi(\pi) = -1$ and $\xi|_{\mathcal{O}_L^\times} \equiv 1$. Viewing $\theta \cdot \xi$ as a character of $\mathcal{W}_L$ via local class field theory, the representation σ_θ is isomorphic to $\mathrm{Ind}_{\mathcal{W}_L}^{\mathcal{W}_K}(\theta \cdot \xi)$.

Pictorially,

$$
\begin{array}{ccccc}
\theta & & \theta & & \mathfrak{X} \\
\Big\downarrow & & \downarrow & & \downarrow \chi\text{-datum} \\
\text{Deligne–Lusztig construction} \Big\downarrow & & \sigma_\theta & & \mathcal{G}_K(n) \\
\Big\downarrow & & \downarrow & & \downarrow \text{Local Langlands} \\
\Big\downarrow & & \pi_\theta & & \mathcal{A}_K(n) \\
\Big\downarrow & & \downarrow & & \downarrow \text{Jacquet–Langlands} \\
H_{r_\theta}(X, \overline{\mathbb{Q}}_\ell)[\theta] & \cong & \rho_\theta & & \mathcal{A}'_K(n)
\end{array}
$$

where

$$
\begin{aligned}
\mathfrak{X} &:= \{\text{primitive characters } L^\times \to \overline{\mathbb{Q}}_\ell^\times\} \\
\mathcal{G}_K(n) &:= \{\text{smooth irreducible dimension-}n\text{ representations of the Weil group } \mathcal{W}_K\} \\
\mathcal{A}_K(n) &:= \{\text{supercuspidal irreducible representations of } \mathrm{GL}_n(K)\} \\
\mathcal{A}'_K(n) &:= \{\text{smooth irreducible representations of } D^\times\}
\end{aligned}
$$

1.1 What is known

The only progress on Deligne–Lusztig constructions X is in the context of division algebras. For two relatively prime integers $k, n \geq 1$, let $D_{k/n}$ denote the division algebra over K of invariant k/n. (Note that the Brauer group of K is $\mathbb{Q}/\mathbb{Z}$, so $D_{k/n} \cong D_{k'/n}$ if $k \equiv k'$ modulo n.) In the next two sections, we will pick an embedding $L \hookrightarrow D_{k/n}$ and set $G = D_{k/n}^\times$, $T = L^\times$.

Let G^1 and T^1 denote the norm-1 elements of G and T, and let X^1 be the associated Deligne–Lusztig construction. In [12], Lusztig proves that when $k = 1$, the virtual G^1-representations $\sum(-1)^i H_i(X^1, \overline{\mathbb{Q}}_\ell)[\theta]$ are (up to a sign) irreducible and mutually nonisomorphic. We remark that his argument can be modified to prove the same conclusion for $\sum(-1)^i H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$.

Our paper focuses on the much subtler issue of describing the individual homology groups $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ and their vanishing behavior. Analogous to the behavior of classical Deligne–Lusztig varieties, one expects $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ to vanish outside a single degree, at least for "sufficiently generic" characters θ. Additionally, one hopes to get a description of the irreducible representations arising from these homology groups.

There exists a unipotent group scheme $U_{h,k}^{n,q}$ over $\mathbb{F}_p$ such that $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ is isomorphic to a subquotient of G. The study of $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ reduces to the study of certain subschemes $X_h \subset U_{h,k}^{n,q}$ endowed with a left action by U_L^1/U_L^h and a right action by $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$. When $k = 1$, these definitions were established in [4] for $h \leq 2$ and in [2] for $h > 2$. We remark that $U_{2,1}^{n,q}(\mathbb{F}_{q^n})$ is isomorphic to a subquotient of G even if K has characteristic zero, but this fails when $h > 2$ (see Remark 2.2). The definitions of X, X_h, and $U_{h,k}^{n,q}$ can be generalized to arbitrary k, and we do so in this paper.

In [4, Sections 4–6], Boyarchenko and Weinstein study the representations $H_c^i(X_2, \overline{\mathbb{Q}}_\ell)$ when $k = 1$ (see Theorem 4.7 of *op. cit.*). This comprises one of the main ingredients in studying the cohomology of the Lubin–Tate tower. In [3], they specialize this result to the primitive case to give an explicit and partially geometric description of local Langlands correspondences. Roughly speaking, the Weil representation in classical constructions is replaced by the cohomology of X_2. In [2], Boyarchenko uses the representations $H_c^i(X_2, \overline{\mathbb{Q}}_\ell)$ to prove that for any smooth character $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$ of level ≤ 2, the representation $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ vanishes outside a single degree and gives a description of this representation (see Theorem 5.3 of *op. cit.*). Moreover, he shows that if θ is primitive, then $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ is irreducible in the nonvanishing degree.

In contrast to the Lubin–Tate setting, we need to understand the cohomology of X_h for all h to understand high-depth representations arising in Deligne–Lusztig constructions. Outside of the case for $k = 1$, $n = 3$, and $h = 3$ (see [2, Theorem 5.20]), this was completely open.

In [6], we study X_h for arbitrary h, assuming $n = 2$ and θ is primitive. We prove that the representation $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ is irreducible and nonvanishing in a single degree. In addition we prove a character formula in the form of a branching rule for representations of the finite unipotent group $U_{h,1}^{2,q}(\mathbb{F}_{q^2})$, a subquotient of the quaternion algebra. Using this, we are able to give an explicit description of the representation $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$.

In this paper, we generalize this work to arbitrary n and arbitrary k. We take a more conceptual approach that allows us to bypass many of the computations needed in [6]. As a corollary, we obtain a geometric realization of the Jacquet–Langlands transfer between representations of division algebras.

Remark 1.1 In the special case that $n = 2$, the Deligne–Lusztig constructions we study in this paper and its prequel [6] are cut out by equations that resemble the equations defining certain covers of affine Deligne–Lusztig varieties. This was observed by Ivanov in [10, Section 3.6]. $\diamondsuit$

1.2 Outline of this paper

Let $h, k, n \geq 1$ be integers with $(k, n) = 1$. In Sect. 2, we introduce the unipotent group scheme $U_{h,k}^{n,q}$ together with a certain subgroup scheme $H \subset U_{h,k}^{n,q}$, both of which are defined over $\mathbb{F}_{q^n}$. These group schemes have the property that $H(\mathbb{F}_{q^n}) \cong U_L^1/U_L^h$ and $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ is isomorphic to an analogous finite subquotient of $D_{k/n}^\times$ (see Remark 2.2). We then define a certain $\mathbb{F}_{q^n}$-scheme $X_h \subset U_{h,k}^{n,q}$, whose relation to the Deligne–Lusztig construction X is as follows: X can be identified with a set $\widetilde{X}$ endowed with an ind-pro-scheme structure

$$\widetilde{X} = \bigsqcup_{m \in \mathbb{Z}} \varprojlim_h \widetilde{X}_h^{(m)},$$

where each $\widetilde{X}_h^{(m)}$ is isomorphic to the disjoint union of $q^n - 1$ copies of $X_h(\overline{\mathbb{F}}_q)$. This decomposition naturally realizes $\widetilde{X}$ as an increasing union of $\overline{\mathbb{F}}_q$-(pro-)schemes. Roughly speaking, the action of $T \times G$ on $\widetilde{X}$ has two behaviors: there is an action on each $\widetilde{X}_h^{(m)}$, and there is an action permuting these pieces. In order to understand the $(T \times G)$-representations arising from $H_i(X, \overline{\mathbb{Q}}_\ell)$, one must understand these two actions. The former is captured by the action of $H(\mathbb{F}_{q^n}) \times U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ on X_h; the latter was studied by Boyarchenko [2, Proposition 5.19].

Let $\mathcal{T}$ denote the set of primitive characters of $H(\mathbb{F}_{q^n})$. Let $\mathcal{G}$ denote the set of irreducible representations of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ whose central character has trivial $\mathrm{Gal}(L/K)$-stabilizer. In Sect. 4, we give a correspondence $\chi \mapsto \rho_\chi$ from $\mathcal{T}$ to $\mathcal{G}$. This construction matches that of [7].

In Sect. 5 we study the geometry of X_h using a combinatorial notion known as *juggling sequences*. We prove in Theorem 5.4 and Corollary 5.5 that the varieties X_h are smooth affine varieties of dimension $(n-1)(h-1)$ defined by the vanishing of polynomials whose monomials are indexed by juggling sequences. By studying the combinatorics of these objects, we are able to prove structural lemmas crucial to the analysis of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$.

Section 6 is concerned with combining the general algebro-geometric results of Sect. 3, the representation-theoretic results of Sect. 4, and the combinatorial results of Sect. 5. The main result of this section is Theorem 6.4, but the heart of its proof is in Proposition 6.1, where we calculate certain cohomology groups by inducing on linear fibrations. In Theorem 6.4, we prove that the correspondence $\chi \mapsto \rho_\chi$ is bijective and that every representation $\rho \in \mathcal{G}$ appears in $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ with multiplicity one. In addition, we prove a character formula (Proposition 6.2) for the representations $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ using the Deligne–Lusztig fixed point formula [8, Theorem 3.2].

Section 7 is devoted to understanding two connections. The first, explained in Sect. 7.1, is to unravel the relationship between the results of Sect. 6 and the representations of division algebras arising from Deligne–Lusztig constructions $\widetilde{X}$. The second, explained in Sect. 7.2, is to describe $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ from the perspective of the local Langlands and Jacquet–Langlands correspondences. We use Theorem 6.4, the trace formula established in Proposition 6.2, and a criterion of Henniart described in [3, Proposition 1.5(b)].

Theorem (7.12, 7.13) *Let $\theta \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a primitive character of level h and let ρ_θ be the $D_{k/n}^\times$-representation corresponding to θ under the local Langlands and Jacquet–Langlands correspondences. Then*

$$H_i(X, \overline{\mathbb{Q}}_\ell)[\theta] = \begin{cases} \rho_\theta & \text{if } i = (n-1)(h-1), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if X and X' are the Deligne–Lusztig constructions associated to $D_{k/n}$ and $D_{k'/n}$, then the Jacquet–Langlands transfer of $H_{(n-1)(h-1)}(X, \overline{\mathbb{Q}}_\ell)[\theta]$ is isomorphic to $H_{(n-1)(h-1)}(X', \overline{\mathbb{Q}}_\ell)[\theta]$.

Using the techniques developed in this paper, we have evidence to support that for nonprimitive characters $\theta \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ of level h with restriction $\chi : U_L^1 \to \overline{\mathbb{Q}}_\ell^\times$, the cohomology groups $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ are irreducible and concentrated in a single non-middle degree. This implies that the homology groups $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ are also concentrated in a single degree, though it not expected that these representations are irreducible in general. We plan to investigate this in a future paper.

2 Definitions

Fix a non-Archimedean local field K with residue field $\mathbb{F}_q$ and fix a uniformizer π. Fix an integer $n \geq 1$ and let L be the unramified degree-n extension of K. For any integer $k \geq 1$ with $(k, n) = 1$, we denote by $D := D_{k/n}$ the rank-n division algebra of Hasse invariant k/n over K. Fix an integer l such that $lk \equiv 1$ modulo n. Then we may write $D = L\langle\Pi\rangle/(\Pi^n - \pi)$, where $\Pi \cdot a = \varphi^l(a) \cdot \Pi$ and $\varphi \in \mathrm{Gal}(L/K)$ is the arithmetic q-Frobenius, and this specifies an embedding $L \hookrightarrow D$. The ring of integers (i.e. the unique maximal order) of D is $\mathcal{O}_D = \mathcal{O}_L\langle\Pi\rangle/(\Pi^n - \pi)$, where $\mathcal{O}_L$ is the ring of integers of L. We write $U_L^0 := \mathcal{O}_L^\times$ and $U_D^0 := \mathcal{O}_D^\times$, and for $h \in \mathbb{Z}_{>0}$, we write $U_L^h := 1 + P_L^h$ and $U_D^h := 1 + P_D^h$, where $P_L := \pi \cdot \mathcal{O}_L$ and $P_D := \Pi \cdot \mathcal{O}_D$.

From now until Sect. 7, we assume that K has positive characteristic. In Sect. 2.1, we construct a ring scheme $\mathcal{R}_{h,k,n,q}$ over $\mathbb{F}_p$ with the property that $\mathcal{R}_{h,k,n,q}(\mathbb{F}_{q^n})$ is a quotient of $\mathcal{O}_D$. We then focus our attention on a unipotent group scheme $U_{h,k}^{n,q} \subset \mathcal{R}_{h,k,n,q}^\times$ with the property that $U_{h,k}^{n,q}(\mathbb{F}_{q^n}) \cong U_D^1/U_D^{n(h-1)+1}$. In Sect. 2.2, we define a $\mathbb{F}_{q^n}$-subscheme $X_h \subset U_{h,k}^{n,q}$ endowed with commuting actions of $H(\mathbb{F}_{q^n})$ and $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$. These actions are described in Sect. 2.3.

2.1 The unipotent group scheme $U_{h,k}^{n,q}$

Definition 2.1 If A is an $\mathbb{F}_p$-algebra, let $A\langle\tau\rangle$ be the twisted polynomial ring with the commutation relation $\tau \cdot a = a^{q^l} \cdot \tau$, and define

$$\mathcal{R}_{h,k,n,q}(A) := A\langle\tau\rangle/(\tau^{n(h-1)+1}).$$

The functor $A \mapsto \mathcal{R}_{h,k,n,q}(A)$ defines a ring scheme representable by $\mathbb{A}^{n(h-1)+1}$ over $\mathbb{F}_p$. We write

$$\mathcal{R}_{h,k,n,q}(A) = \left\{a_0 + a_1\tau + \cdots + a_{n(h-1)}\tau^{n(h-1)} : a_i \in A\right\},$$

and consider the following subgroup schemes of $\mathcal{R}_{h,k,n,q}^\times$:

$$U_{h,k}^{n,q}(A) := \left\{1 + \sum_{i=1}^{n(h-1)} a_i\tau^i \in \mathcal{R}_{h,k,n,q}(A)\right\},$$

$$H(A) := \left\{1 + \sum_{i=1}^{h-1} a_{ni}\tau^{ni} \in U_{h,k}^{n,q}(A)\right\}.$$

The q-Frobenius φ induces a morphism $\mathcal{R}_{h,k,n,q}$ by acting on the coefficients of τ. Note that $H(\mathbb{F}_{q^n})$ is commutative since $\mathbb{F}_{q^n} = (\overline{\mathbb{F}}_q)^{\varphi^n}$, but H is not a commutative group scheme.

Remark 2.2 Note that $\mathcal{R}_{h,k,n,q}^{\times}(\mathbb{F}_{q^n}) \cong \mathbb{F}_{q^n}^{\times} \ltimes U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ and we have natural isomorphisms

$$\mathcal{R}_{h,k,n,q}^{\times}(\mathbb{F}_{q^n}) \cong \mathcal{O}_D^{\times}/U_D^{n(h-1)+1},\ U_{h,k}^{n,q}(\mathbb{F}_{q^n}) \cong U_D^1/U_D^{n(h-1)+1},\ H(\mathbb{F}_{q^n}) \cong U_L^1/U_L^h. \tag{2.1}$$

These are induced by the ring isomorphism

$$\mathcal{R}_{h,k,n,q}(\mathbb{F}_{q^n}) \to \mathcal{O}_D/P_D^{n(h-1)+1},\quad \sum_{i=0}^{n(h-1)} a_i\tau^i \mapsto \sum_{i=0}^{n(h-1)} a_i\Pi^i = \sum_{j=0}^{n-1} A_j\Pi^j, \tag{2.2}$$

where we write

$$A_0 := a_0 + a_n\pi + \cdots + a_{n(h-1)}\pi^{h-1},$$
$$A_j := a_j + a_{n+j}\pi + \cdots + a_{n(h-2)+j}\pi^{h-2},\quad 1 \le j \le n-1.$$

Note that we crucially used that $L = \mathbb{F}_{q^n}[\![\pi]\!]$. We remark that when $h \le 2$, the morphism in (2.2) defines an isomorphism of multiplicative monoids even when K has characteristic 0, and therefore the isomorphisms in (2.1) hold regardless of the characteristic of K.

The center $Z(U_{h,k}^{n,q}(\mathbb{F}_{q^n}))$ of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ is a subgroup of $H(\mathbb{F}_{q^n})$ and can be described explicitly:

$$Z\left(U_{h,k}^{n,q}(\mathbb{F}_{q^n})\right) = \left\{1 + \sum a_{ni}\tau^{ni} \in H(\mathbb{F}_{q^n}) : a_{n(h-1)} \in \mathbb{F}_{q^n} \text{ and } a_{ni} \in \mathbb{F}_q \text{ for } 1 \le i \le h-2\right\}.$$

$\Diamond$

Definition 2.3 We say that a character $\chi\colon H(\mathbb{F}_{q^n}) \cong U_L^1/U_L^h \to \overline{\mathbb{Q}}_\ell^{\times}$ is *primitive* if its restriction to $U_L^{h-1}/U_L^h \cong \mathbb{F}_{q^n}$ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer.

2.2 The varieties X_h

Definition 2.4 For any $\mathbb{F}_p$-algebra A, let $M_h(A)$ denote the ring of all $n \times n$ matrices $(b_{ij})_{i,j=1}^n$ with $b_{ii} \in A[\![\pi]\!]/(\pi^h)$, $b_{ij} \in A[\![\pi]\!]/(\pi^{h-1})$ for $i < j$, and $b_{ij} \in \pi A[\![\pi]\!]/(\pi^h)$ for $i > j$. The determinant can be viewed as a multiplicative map $\det\colon M_h(A) \to A[\![\pi]\!]/(\pi^h)$.

For any integer m, let $[m]$ denote the unique integer with $1 \le [m] \le n$ such that $m \equiv [m]$ modulo n. Let A be any $\mathbb{F}_p$-algebra. The q-Frobenius morphism φ on A induces a ring endomorphism on $A[\![\pi]\!]/(\pi^m)$ given by $\sum_{i=0}^{m-1} a_i\pi^i \mapsto \sum_{i=0}^{m-1} \varphi(a_i)\pi^i$ for any positive integer m. Consider the injective morphism of sets

$$\iota_{h,k}\colon \mathcal{R}_{h,k,n,q}(A) \to M_h(A)$$

given by defining $\iota_{h,k}\left(\sum a_i \tau^i\right)$ to be

$$\begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} \\ \pi\varphi^{[l]}(A_{n-1}) & \varphi^{[l]}(A_0) & \varphi^{[l]}(A_1) & \cdots & \varphi^{[l]}(A_{n-2}) \\ \pi\varphi^{[2l]}(A_{n-2}) & \pi\varphi^{[2l]}(A_{n-1}) & \varphi^{[2l]}(A_0) & \cdots & \varphi^{[2l]}(A_{[n-3]}) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \pi\varphi^{[(n-1)l]}(A_1) & \pi\varphi^{[(n-1)l]}(A_2) & \cdots & \pi\varphi^{[(n-1)l]}(A_{n-1}) & \varphi^{[(n-1)l]}(A_0) \end{pmatrix} \tag{2.3}$$

where we write

$$\begin{aligned} A_0 &= a_0 + a_n\pi + \cdots + a_{n(h-1)}\pi^{h-1}, \\ A_j &= a_j + a_{n+j}\pi + \cdots + a_{n(h-2)+j}\pi^{h-2}, \quad j = 1, \ldots, n-1. \end{aligned} \tag{2.4}$$

Although $\iota_{h,k}$ does not preserve the ring structure, it does satisfy a weak multiplicative property that we explicate in Sect. 2.3.

In Sect. 7.1, we describe how to extend the results of [2, Sects. 4.2, 4.3] to division algebras of arbitrary invariant. In particular, we show that the Deligne–Lusztig construction X described in [12] can be identified with a certain set $\widetilde{X}$ which can be realized as the $\overline{\mathbb{F}}_q$-points of an ind-pro-scheme

$$\widetilde{X} := \bigsqcup_{m\in\mathbb{Z}} \varprojlim_h \widetilde{X}_h^{(m)},$$

where each $\widetilde{X}_h^{(m)}$ is a finite-type $\mathbb{F}_p$-scheme and $\widetilde{X}_h^{(m)} \cong \widetilde{X}_h^{(0)}$ for all $m \in \mathbb{Z}$. By Lemma 7.3, for any $\mathbb{F}_p$-algebra A,

$$\widetilde{X}_h^{(0)}(A) \cong \left\{x = \iota_{h,k}\left(\sum a_i\tau^i\right) : a_i \in A, \det(x) \text{ is fixed by } \varphi\right\} =: \widetilde{X}_h'^{(0)}(A).$$

Definition 2.5 For any $\mathbb{F}_p$-algebra A, define

$$X_h(A) := U_{h,k}^{n,q}(A) \cap \iota_{h,k}^{-1}\left(\widetilde{X}_h'^{(0)}(A)\right).$$

Remark 2.6 Observe that $\widetilde{X}_h'^{(0)}$ is a disjoint union of $q^n - 1$ copies of X_h. ♢

2.3 Group actions

We first prove the following lemma.

Lemma 2.7 *Let A be an $\mathbb{F}_{q^n}$-algebra. The map $\iota_{h,k}$ has the following weak multiplicativity property:*

$$\iota_{h,k}(xy) = \iota_{h,k}(x)\iota_{h,k}(y) \quad \textit{for all } x \in U_{h,k}^{n,q}(A) \textit{ and all } y \in U_{h,k}^{n,q}(\mathbb{F}_{q^n}). \tag{2.5}$$

Moreover, for $y \in U_{h,k}^{n,q}(\mathbb{F}_{q^n})$, the determinant of $\iota_{h,k}(y)$ is fixed by φ.

Proof Observe from Eq. (2.3) that

$$\iota_{h,k}\left(\sum a_i\tau^i\right) = \iota_{h,k}(A_0) + \iota_{h,k}(A_1)\varpi + \cdots + \iota_{h,k}(A_{n-1})\varpi^{n-1}, \tag{2.6}$$

where we write $\varpi = \left(\begin{smallmatrix} 0 & 1_{n-1} \\ \pi & 0 \end{smallmatrix}\right)$ and

$$A_j = \begin{cases} a_0 + a_n\tau^n + \cdots + a_{n(h-1)}\tau^{n(h-1)} & \text{if } j = 0, \\ a_j + a_{n+j}\tau^n + \cdots + a_{n(h-2)+j}\tau^{n(h-2)} & \text{if } j > 0. \end{cases}$$

Note that $\sum a_i\tau^i = A_0 + A_1\tau + \cdots + A_{n-1}\tau^{n-1}$. For any $a \in H(A)$, we have

$$\varpi \cdot \iota_{h,k}(a) = \operatorname{diag}\left(\varphi^{[l]}(a), \ldots, \varphi^{[(n-1)l]}(a), a\right) \cdot \varpi,$$
$$\iota_{h,k}(\varphi^{[l]}(a)) = \operatorname{diag}\left(\varphi^{[l]}(a), \ldots, \varphi^{[(n-1)l]}(a), \varphi^n(a)\right),$$

and therefore we see that if $a \in H(\mathbb{F}_{q^n})$, then

$$\varpi \cdot \iota_{h,k}(a) = \iota_{h,k}(\varphi^{[l]}(a)) \cdot \varpi.$$

This proves Eq. (2.5). Using Eq. (2.6) together with the observation that under the isomorphism $H(\mathbb{F}_{q^n}) \cong U_L^1/U_L^h$, we have $\det(\iota_{h,k}(a)) = \mathrm{Nm}_{L/K}(a)$ for $a \in H(\mathbb{F}_{q^n})$. The second assertion of the lemma follows. $\square$

It follows from Lemma 2.7 that after base-changing to $\mathbb{F}_{q^n}$, the variety X_h is stable under right-multiplication by $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$. For this reason, *from now on,* we consider X_h as a variety over $\mathbb{F}_{q^n}$. We denote by $x \cdot g$ the action of $g \in U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ on $x \in X_h$.

The conjugation action of $\zeta \in \mathbb{F}_{q^n}^\times$ on $U_{h,k}^{n,q}(A)$ stabilizes $X_h(A)$. This extends the right $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$-action on X_h to an action of the semidirect product $\mathbb{F}_{q^n}^\times \ltimes U_{h,k}^{n,q}(\mathbb{F}_{q^n}) \cong \mathcal{R}_{h,k,n,q}^\times(\mathbb{F}_{q^n})$.

We now describe a left action of $H(\mathbb{F}_{q^n})$ on X_h. We can identify $H(\mathbb{F}_{q^n})$ with the set $\iota_{h,k}(H(\mathbb{F}_{q^n}))$. Note that by the weak multiplicativity property, the map $\iota_{h,k}$ is a group homomorphism on $H(\mathbb{F}_{q^n})$, and since $\iota_{h,k}$ is injective, we have $H(\mathbb{F}_{q^n}) \cong \iota_{h,k}(H(\mathbb{F}_{q^n}))$ as groups. Explicitly, this isomorphism is given by

$$1 + \sum_{i=1}^{h-1} a_{ni}\tau^{ni} \mapsto \operatorname{diag}\left(1 + \sum a_{ni}\pi^i, 1 + \sum a_{ni}^{q^l}\pi^i, \ldots, 1 + \sum a_{ni}^{q^{(n-1)l}}\pi^i\right).$$

Observe that we may remove the brackets in the exponent since $\varphi^n(A_0) = A_0$. From Eq. (2.6), it is clear that the left-multiplication action of $\iota_{h,k}(H(\mathbb{F}_{q^n}))$ on $M_h(A)$ stabilizes $\iota_{h,k}(X_h(A))$, and we therefore obtain an action[2] of $g \in H(\mathbb{F}_{q^n}) \cong U_L^1/U_L^h$

[2] Warning: This is *not* the same as the action induced by left-multiplication of $H(\mathbb{F}_{q^n}) \subset H(A)$ on $U_{h,k}^{n,q}(A)$. For example, if $x = \iota_{h,k}(x_0, \ldots, x_{n-1}) \in X_h(\overline{\mathbb{F}}_q)$ and $x_0 \notin \mathbb{F}_{q^n}$, then for $g := 1 + a_n\tau^n \in$

on $x \in X_h$, which we denote by $g * x$. The actions of $H(\mathbb{F}_{q^n})$ and $\mathcal{R}^\times_{h,k,n,q}(\mathbb{F}_{q^n})$ commute.

3 General principles: some algebraic geometry

In this section, we prove some general algebro-geometric results that will allow us to compute certain cohomology groups via an inductive argument. We generalize the techniques of [2] from $\mathbb{G}_a$ to the group scheme $H \subset U_{h,k}^{n,q}$ defined in Sect. 2.1.

We begin by recalling some results of [2, Section 2.2]. Let G be an algebraic group over $\mathbb{F}_{q^n}$, suppose that $Y \subset G$ is a (locally closed) subvariety defined over $\mathbb{F}_{q^n}$, and set $X = L_{q^n}^{-1}(Y)$, where $L_{q^n} \colon G \to G$ is the Lang map $g \mapsto \mathrm{Fr}_{q^n}(g)g^{-1}$. Let $G_0 \subset G$ be any connected subgroup defined over $\mathbb{F}_{q^n}$ and let $\eta \colon G_0(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ be a homomorphism. Write $V_\eta = \mathrm{Ind}_{G_0(\mathbb{F}_{q^n})}^{G(\mathbb{F}_{q^n})}(\eta)$.

Consider the right-multiplication action of $G_0(\mathbb{F}_{q^n})$ on G and form the quotient $Q := G/(G_0(\mathbb{F}_{q^n}))$. The Lang map $L_{q^n} \colon G \to G$ is invariant under right multiplication by $G_0(\mathbb{F}_{q^n})$ and thus it factors through a morphism $\alpha \colon Q \to G$. On the other hand, the quotient map $G \to Q$ is a right $G_0(\mathbb{F}_{q^n})$-torsor, so the character η yields a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{E}_\eta$ of rank 1 on Q.

Lemma 3.1 [2, Lemma 2.1] *There is a natural* Fr_{q^n}*-equivariant vector-space isomorphism*

$$\mathrm{Hom}_{G(\mathbb{F}_q)}\left(V_\eta, H_c^i(X, \overline{\mathbb{Q}}_\ell)\right) \cong H_c^i\left(\alpha^{-1}(Y), \mathcal{E}_\eta|_{\alpha^{-1}(Y)}\right) \quad \textit{for all } i \geq 0.$$

As in [2], we now make two further assumptions under which the right-hand side of the isomorphism in Lemma 3.1 can be described much more explicitly. This will allow us to calculate certain cohomology groups via an inductive argument. These two assumptions are:

1. The quotient morphism $G \to G/G_0$ admits a section $s \colon G/G_0 \to G$.
2. There is an algebraic group morphism $f \colon G_0 \to H$ defined over $\mathbb{F}_{q^n}$ such that $\eta = \chi \circ f$ for a character $\chi \colon H(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$.

Let $\mathcal{L}_\chi$ be the local system on H defined by χ via the Lang map $L_{q^n} \colon H \to H$. The following lemma is proved in [2].

Lemma 3.2 [2, Lemma 2.2] *There is an isomorphism* $\gamma \colon (G/G_0) \times G_0 \xrightarrow{\simeq} Q$ *such that* $\gamma^*\mathcal{E}_\eta \cong (f \circ \mathrm{pr}_2)^*\mathcal{L}_\chi$ *and* $\alpha \circ \gamma = \beta$*, where* $\mathrm{pr}_2 \colon (G/G_0) \times G_0 \to G_0$ *is the second projection and* $\beta \colon (G/G_0) \times G_0 \to G$ *is given by* $\beta(x, h) = s(\mathrm{Fr}_{q^n}(x)) \cdot h \cdot s(x)^{-1}$.

Combining Lemmas 3.1 and 3.2, we obtain the following proposition.

$H(\mathbb{F}_{q^n})$ has the property that $g * x = \iota_{h,k}(x_0 + a_n x_0 \pi, \ldots)$ but left-multiplication gives $g \cdot x = \iota_{h,k}(x_0 + a_n x_0^{q^n} \pi, \ldots) \in U_{h,k}^{n,q}(\overline{\mathbb{F}}_q)$.

Proposition 3.3 [2, Proposition 2.3] *Assume that we are given the following data:*

- *an algebraic group G with a connected subgroup $G_0 \subset G$ defined over $\mathbb{F}_{q^n}$;*
- *a section $s\colon G/G_0 \to G$ of the quotient morphism $G \to G/G_0$;*
- *an algebraic group homomorphism $f\colon G_0 \to H$;*
- *a character $\chi\colon H(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$;*
- *a locally closed subvariety $Y \subset G$.*

Set $X = L_{q^n}^{-1}(Y)$, where L_{q^n} is the Lang map $g \mapsto \mathrm{Fr}_{q^n}(g)g^{-1}$ on G. Then for each $i \geq 0$, we have a Fr_{q^n}-compatible vector space isomorphism

$$\mathrm{Hom}_{G(\mathbb{F}_{q^n})}\left(\mathrm{Ind}_{G_0(\mathbb{F}_{q^n})}^{G(\mathbb{F}_{q^n})}(\chi \circ f), H_c^i(X, \overline{\mathbb{Q}}_\ell)\right) \cong H_c^i\left(\beta^{-1}(Y), P^*\mathcal{L}_\chi\right).$$

Here, $\mathcal{L}_\chi$ is the local system on H corresponding to χ, the morphism $\beta\colon (G/G_0) \times G_0 \to G$ is given by $\beta(x, h) = s(\mathrm{Fr}_{q^n}(x)) \cdot h \cdot s(x)^{-1}$, and the morphism $P\colon \beta^{-1}(Y) \to H$ is the composition $\beta^{-1}(Y) \hookrightarrow (G/G_0) \times G_0 \xrightarrow{\mathrm{pr}_2} G_0 \xrightarrow{f} H$.

Our goal now is to prove the following crucial proposition. This is the proposition that gives us an inductive technique for calculating the cohomology groups appearing in Sect. 6.

Proposition 3.4 *Let q be a power of p and let $n \in \mathbb{N}$. Let S_2 be a scheme of finite type over $\mathbb{F}_{q^n}$, put $S = S_2 \times \mathbb{G}_a$ and suppose that a morphism $P\colon S \to H$ has the form*

$$P(x, y) = g\left(f(x)^{q^{j_1}} y^{q^{j_2}} - f(x)^{q^{j_3}} y^{q^{j_4}}\right) \cdot P_2(x)$$

where

- *$j_1, \ldots, j_4$ are non-negative integers,*
- *$j_1 - j_2 = j_3 - j_4$ and $j_2 - j_4$ is not divisible by n,*
- *$f\colon S_2 \to \mathbb{G}_a$, $P_2\colon S_2 \to H$ are two morphisms defined over $\mathbb{F}_{q^n}$, and*
- *$g\colon \mathbb{G}_a \to H$ is the morphism $z \mapsto 1 + z\tau^{n(h-1)}$.*

Let $S_3 \subset S_2$ be the subscheme defined by $f = 0$ and let $P_3 = P_2|_{S_3}\colon S_3 \to H$. If $\chi\colon H(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ is primitive, then for all $i \in \mathbb{Z}$,

$$H_c^i\left(S, P^*\mathcal{L}_\chi\right) \cong H_c^{i-2}\left(S_3, P_3^*\mathcal{L}_\chi\right)(-1)$$

as vector spaces equipped with an action of Fr_{q^n}, where the Tate twist (-1) means that the action of Fr_{q^n} on $H_c^{i-2}(S_3, P_3^*\mathcal{L}_\chi)$ is multiplied by q^n.

Proof Let $\mathrm{pr}\colon S = S_2 \times \mathbb{G}_a \to S_2$ be the first projection, let $\iota\colon S_3 \to S_2$ be the inclusion map, and let $\eta\colon S \to H$ be the morphism $(x, y) \mapsto g(\eta_0(x, y))$, where $\eta_0\colon S \to \mathbb{G}_a$ is the morphism $(x, y) \mapsto f(x)^{q^{j_1}} y^{q^{j_2}} - f(x)^{q^{j_3}} y^{q^{j_4}}$. We then have the following commutative diagram, where $(*)$ is a Cartesian square

$$
\begin{array}{ccccccccccc}
 & & & & S & \xrightarrow{\eta_0} & \mathbb{G}_a & & & & \\
 & & & & \| & & \downarrow g & & & & \\
 & & S_3 \times \mathbb{G}_a & \xrightarrow{\iota} & S_2 \times \mathbb{G}_a & \xrightarrow{\eta} & H & \xrightarrow{(-,1)} & & & \\
 & & \downarrow \mathrm{pr} & (*) & \downarrow \mathrm{pr} & & & & H \times H & \xrightarrow{m} & H \\
\mathbb{V}(f) & = & S_3 & \xrightarrow{\iota} & S_2 & \xrightarrow{P_2} & H & \xrightarrow{(1,-)} & & &
\end{array}
$$

The sheaf $\mathcal{L}_\chi$ is a multiplicative local system on H, and hence

$$P^*\mathcal{L}_\chi \cong (\eta^*\mathcal{L}_\chi) \otimes \mathrm{pr}^*(P_2^*\mathcal{L}_\chi).$$

Thus, by the projection formula,

$$R\,\mathrm{pr}_!(P^*\mathcal{L}_\chi) \cong P_2^*\mathcal{L}_\chi \otimes R\,\mathrm{pr}_!(\eta^*\mathcal{L}_\chi) \quad \text{in } D_c^b(S_2, \overline{\mathbb{Q}}_\ell).$$

We now claim that

$$R\,\mathrm{pr}_!(\eta^*\mathcal{L}_\chi) \cong \iota_!(\overline{\mathbb{Q}}_\ell)[-2](-1) \quad \text{in } D_c^b(S_2, \overline{\mathbb{Q}}_\ell),$$

where $\overline{\mathbb{Q}}_\ell$ denotes the constant local system of rank 1 on S_2. It is clear that once we have established this, the desired conclusion follows. We therefore spend the rest of the proof proving this.

The restriction of η to $\mathrm{pr}^{-1}(S_3) \subset S_2$ is constant, so the restriction of the pullback $\eta^*\mathcal{L}_\chi$ to $\mathrm{pr}^{-1}(S_3)$ is a constant local system of rank 1. Thus, by proper base change with respect to the Cartesian square $(*)$, we have the following isomorphisms in $D_c^b(S_2, \overline{\mathbb{Q}}_\ell)$:

$$\iota^* R\,\mathrm{pr}_!(\eta^*\mathcal{L}_\chi) \cong R\,\mathrm{pr}_!(\iota^*\eta^*\mathcal{L}_\chi) = R\,\mathrm{pr}_!(\overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[-2](-1).$$

To complete the proof, we need to show that $R\,\mathrm{pr}_!(\eta^*\mathcal{L}_\chi)$ vanishes outside $S_3 \subset S_2$. Let ψ denote the restriction of χ to $g(\mathbb{G}_a)(\mathbb{F}_{q^n}) \cong \mathbb{G}_a(\mathbb{F}_{q^n})$ and let $\mathcal{L}_\psi$ denote the corresponding Artin–Schreier sheaf on $\mathbb{G}_a$. Since $\eta = g \circ \eta_0$,

$$\eta^*\mathcal{L}_\chi \cong \eta_0^*\mathcal{L}_\psi.$$

We now calculate the stalk of $R\,\mathrm{pr}_!(\eta_0^*\mathcal{L}_\psi)$ for any $x \in S_2(\overline{\mathbb{F}}_q) \smallsetminus S_3(\overline{\mathbb{F}}_q)$. By proper base change,

$$R^i\,\mathrm{pr}_!(\eta_0^*\mathcal{L}_\psi)_x \cong H_c^i(\mathbb{G}_a, f_x^*\mathcal{L}_\psi),$$

where $f_x \colon \mathbb{G}_a \to \mathbb{G}_a$ is given by $y \mapsto f(x)^{q^{j_1}} y^{q^{j_2}} - f(x)^{q^{j_3}} y^{q^{j_4}}$. Fix an auxiliary nontrivial additive character $\psi_0 \colon \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$, and for any $z \in \overline{\mathbb{F}}_p$, define

$$\mathcal{L}_z := m_z^*\mathcal{L}_{\psi_0}, \quad \text{where } m_z \colon \mathbb{G}_a \to \mathbb{G}_a \text{ is the map } x \mapsto xz,$$

where $\mathcal{L}_{\psi_0}$ is the Artin–Schreier sheaf on $\mathbb{G}_a$ corresponding to ψ_0. Then there exists a unique $z \in \mathbb{F}_{q^n}$ such that $\mathcal{L}_\psi = \mathcal{L}_z$, and since ψ has nontrivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer by assumption, so must z. By [2, Corollary 6.5], we have $f_x^*\mathcal{L}_\psi \cong \mathcal{L}_{f_x^*(z)}$, where

$$f_x^*(z) = f(x)^{q^{j_1}/q^{j_2}} z^{1/q^{j_2}} - f(x)^{q^{j_3}/q^{j_4}} z^{1/q^{j_4}} = f(x)^{q^{j_1-j_2}}(z^{q^{-j_2}} - z^{q^{-j_4}}).$$

But $z^{q^{-j_2}} - z^{q^{-j_4}} \neq 0$ since by assumption $z \neq 0$ and $j_2 - j_4$ is not divisible by n. Thus $f_x^*\mathcal{L}_\psi$ is a nontrivial local system on $\mathbb{G}_a$ and by [1, Lemma 9.4], $H_c^i(\mathbb{G}_a, f_x^*\mathcal{L}_\psi) = 0$ for all $i \geq 0$. □

Proposition 3.5 *Let $j_1, \ldots, j_4, f, g, S, P_2, S_2, S_3, P_3$ be as in Proposition 3.4 and suppose that $P\colon S = S_2 \times \mathbb{A}^1 \to H$ has the form*

$$P(x, y) = g\left(f(x)^{q^{j_1}} y^{q^{j_2}} - f(x)^{q^{j_3}} y^{q^{j_4}} + \alpha(x, y)^{q^n} - \alpha(x, y)\right) \cdot P_2(x)$$

for some morphism $\alpha\colon S \to \mathbb{G}_a$ defined over $\mathbb{F}_{q^n}$. If $\chi : H(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ is primitive, then for all i,

$$H_c^i(S, P^*\mathcal{L}_\chi) \cong H_c^{i-2}(S_3, P_3^*\mathcal{L}_\chi)(-1)$$

as vector spaces equipped with an action of Fr_{q^n}, where the Tate twist (-1) means that the action of Fr_{q^n} on $H_c^{i-2}(S_3, P_3^*\mathcal{L}_\chi)$ is multiplied by q^n.

Proof Let $P'(x, y) = g(f(x)^{q^{j_1}} y^{q^{j_2}} - f(x)^{q^{j_3}} y^{q^{j_4}}) \cdot P_2(x)$. Then $P^*\mathcal{L}_\chi$ and $(P')^*\mathcal{L}_\chi$ are isomorphic since the pullback of $\mathcal{L}_\chi|_{g(\mathbb{G}_a)}$ by the map $1 + z\tau^{n(h-1)} \mapsto 1 + z^{q^n}\tau^{n(h-1)}$ is trivial. Then by Proposition 3.4, the desired conclusion holds. □

The following proposition is extremely useful in the context of applying the inductive argument described by the above propositions. We will use it in several of the technical lemmas in Sect. 5 and in the proof of the main proposition and theorem of Sect. 6.

Proposition 3.6 *Suppose that $S \hookrightarrow R$ is a finite map of polynomial rings over $k = \overline{\mathbb{F}}_q$. Assume that* Frac *$R$ is finite Galois over* Frac *S with Galois group G a p-group. Then:*

(a) *R is stable under G and $R^G = S$.*
(b) *As multiplicative monoids, $((R \smallsetminus \{0\})/k^\times)^G = (S \smallsetminus \{0\})/k^\times$.*
(c) *If $(f) \subset R$ is an ideal such that $(\sigma f) = (f)$ for all $\sigma \in G$, then $f \in S$.*

Proof First observe that since S and R are polynomial rings, they are normal and therefore integrally closed. Since $S \hookrightarrow R$ is a finite map, R is the integral closure of S in Frac R. Thus R is G-stable. It is clear that $S \subset R^G$ and that R^G is integral over S. But since S is integrally closed, we necessarily have $S = R^G$. This proves (a).

To see (b), consider the short exact sequence

$$1 \to k^\times \to \mathrm{Frac}\, R^\times \to \mathrm{Frac}\, R^\times/k^\times \to 1$$

and take G-invariants to get a long exact sequence

$$1 \to k^\times \to \mathrm{Frac}\, S^\times \to (\mathrm{Frac}\, R^\times/k^\times)^G \to H^1(G, k^\times) \to \cdots$$

Since G acts trivially on $k^\times$, we have $H^1(G, k^\times) = \mathrm{Hom}(G, k^\times)$, which is trivial since G is a p-group. Thus $(\mathrm{Frac}\, R^\times/k^\times)^G = \mathrm{Frac}\, S^\times/k^\times$ and $((R \smallsetminus \{0\})/k^\times)^G = (S \smallsetminus \{0\})/k^\times$.

Now we prove (c). If $f = 0$, then we are done, so for the rest of the proof we may assume $f \neq 0$. Necessarily $\sigma f = f$ up to a unit in R, and thus their images in the quotient $(R \smallsetminus \{0\})/k^\times$ are equal. Thus the image of f is in $((R \smallsetminus \{0\})/k^\times)^G = (S \smallsetminus \{0\})/k^\times$, and so $f \in S$. □

4 Representations of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$

Let $\mathcal{T}$ denote the set of all primitive characters of $H(\mathbb{F}_{q^n})$ and let $\mathcal{G}$ be the set of irreducible representations of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ whose central character has trivial $\mathrm{Gal}(L/K)$-stabilizer.

In this section, we show that $\mathcal{G}$ can be parametrized by $\mathcal{T}$ and explicitly describe such a parametrization. There are two main cases of behavior, depending on the parameters n and h.

Definition 4.1 Given a pair of positive integers (n, h), we say that:

- (n, h) is in *Case 1* if $(n - 1)(h - 1)$ is even.
- (n, h) is in *Case 2* if $(n - 1)(h - 1)$ is odd.

Consider the subset of $\mathbb{Z}$ given by

$$\mathcal{A}' := \{ni : 1 \leq i \leq h - 1\} \cup \left\{i : n \nmid i,\ \tfrac{n(h-1)}{2} < i < n(h - 1)\right\}. \tag{4.1}$$

and define a subgroup scheme H' of $U_{h,k}^{n,q}$ by setting

$$H'(A) := \left\{1 + \sum_{i \in \mathcal{A}'} a_i \tau^i \in U_{h,k}^{n,q}(A)\right\} \quad \text{for any } \mathbb{F}_{q^n}\text{-algebra } A.$$

We now specialize to the setting where $A = \mathbb{F}_{q^n}$. If (n, h) is in Case 1, set $H^+(\mathbb{F}_{q^n}) := H'(\mathbb{F}_{q^n})$, and if (n, h) is in Case 2, define

$$H^+(\mathbb{F}_{q^n}) := \left\{1 + a_{n(h-1)/2}\tau^{n(h-1)/2} + \sum_{i \in \mathcal{A}'} a_i \tau^i \in U_{h,k}^{n,q}(\mathbb{F}_{q^n}) : a_{n(h-1)/2} \in \mathbb{F}_{q^{n/2}}\right\}.$$

Notice that

$$\left[H^+(\mathbb{F}_{q^n}) : H'(\mathbb{F}_{q^n})\right] = \begin{cases} 1 & \text{if } (n, h) \text{ is in Case 1,} \\ q^{n/2} & \text{if } (n, h) \text{ is in Case 2,} \end{cases} \tag{4.2}$$

$$\left[U_{h,k}^{n,q}(\mathbb{F}_{q^n}) : H^+(\mathbb{F}_{q^n})\right] = q^{n(n-1)(h-1)/2}. \tag{4.3}$$

One can think of $H'(\mathbb{F}_{q^n})$ and $H^+(\mathbb{F}_{q^n})$ as enlargements of $H(\mathbb{F}_{q^n})$ by the "deeper half" of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$. We will also need the analogous enlargements of $Z(U_{h,k}^{n,q}(\mathbb{F}_{q^n}))$:

$$H_0'(\mathbb{F}_{q^n}) := \left\{1 + \sum_{i=1}^{n(h-1)} a_i\tau^i \in H'(\mathbb{F}_{q^n}) : 1 + \sum_{i=1}^{h-1} a_{ni}\tau^{ni} \in Z\left(U_{h,k}^{n,q}(\mathbb{F}_{q^n})\right)\right\},$$

$$H_0^+(\mathbb{F}_{q^n}) := \left\{1 + \sum_{i=1}^{n(h-1)} a_i\tau^i \in H^+(\mathbb{F}_{q^n}) : 1 + \sum_{i=1}^{h-1} a_{ni}\tau^{ni} \in Z\left(U_{h,k}^{n,q}(\mathbb{F}_{q^n})\right)\right\}.$$

These subgroups of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ fit into the picture

$$\begin{array}{ccccc} H(\mathbb{F}_{q^n}) & \hookrightarrow & H'(\mathbb{F}_{q^n}) & \hookrightarrow & H^+(\mathbb{F}_{q^n}) \\ \uparrow & & \uparrow & & \uparrow \\ Z\left(U_{h,k}^{n,q}(\mathbb{F}_{q^n})\right) & \hookrightarrow & H_0'(\mathbb{F}_{q^n}) & \hookrightarrow & H_0^+(\mathbb{F}_{q^n}) \end{array}$$

For $\chi \in \mathcal{T}$, define an extension $\chi^\sharp$ of χ to $H'(\mathbb{F}_{q^n})$ by setting

$$\chi^\sharp\left(1 + \sum_{i\in\mathcal{A}'} a_i\tau^i\right) := \chi\left(1 + \sum_{n|i} a_i\tau^i\right).$$

Fix any extension $\widetilde{\chi}$ of $\chi^\sharp$ to $H^+(\mathbb{F}_{q^n})$. Note that in Case 1, necessarily $\widetilde{\chi} = \chi^\sharp$. In Case 2, there are $q^{n/2}$ choices of $\widetilde{\chi}$.

Lemma 4.2 *If $\rho \in \mathcal{G}$ has central character ω and ω has trivial* $\mathrm{Gal}(L/K)$*-stabilizer, then the restriction of ρ to $H_0'(\mathbb{F}_{q^n})$ contains the character*

$$\omega^\sharp\colon H_0'(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times, \quad 1 + \sum_{i\in\mathcal{A}'} a_i\tau^i \mapsto \omega\left(1 + \sum_{n|i} a_i\tau^i\right).$$

Furthermore, the restriction of ρ to $H_0^+(\mathbb{F}_{q^n})$ contains every extension of $\omega^\sharp$ to $H_0^+(\mathbb{F}_{q^n})$.

Proof First let ψ be the restriction of ω to $\{1+a\tau^{n(h-1)} : a \in \mathbb{F}_{q^n}\} \cong \mathbb{F}_{q^n}$ and observe that the assumption on the stabilizer of ω implies that ψ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer.

We will first show that if the restriction of ρ to $H_0'(\mathbb{F}_{q^n})$ contains $\omega^\sharp$, then the restriction of ρ to $H_0^+(\mathbb{F}_{q^n})$ contains every extension of $\omega^\sharp$ to $H_0^+(\mathbb{F}_{q^n})$. This assertion is trivial if we are in Case 1 since $H_0'(\mathbb{F}_{q^n}) = H_0^+(\mathbb{F}_{q^n})$, so let us assume we are in Case 2.

Let $\nu := n(h-1)/2$. Let $\widetilde{\omega}$ be any extension of $\omega^\sharp$ to $H_0^+(\mathbb{F}_{q^n})$. To prove that $\rho|_{H_0^+(\mathbb{F}_{q^n})}$ contains $\widetilde{\omega}$, it is enough to prove that the orbit of $\widetilde{\omega}$ under $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$-conjugacy contains every extension of $\omega^\sharp$ to $H_0^+(\mathbb{F}_{q^n})$. Indeed, for any $b \in \mathbb{F}_{q^n}$, consider the element $g := 1 + b\tau^\nu \in U_{h,k}^{n,q}(\mathbb{F}_{q^n})$. Then writing $h = 1 + a\tau^\nu + \sum_{i\in\mathcal{A}'} a_i\tau^i \in H_0^+(\mathbb{F}_{q^n})$, we have

$$\begin{aligned}
\widetilde{\omega}\left(ghg^{-1}\right) &= \widetilde{\omega}\left(\left(1+b\tau^\nu\right)\left(1 + a\tau^\nu + \sum_{i\in\mathcal{A}'} a_i\tau^i\right)\left(1 - b\tau^\nu + b^{q^{l\nu}+1}\tau^{n(h-1)}\right)\right) \\
&= \widetilde{\omega}\left(1 + a\tau^\nu + \left(ba^{q^{l\nu}} - ab^{q^{l\nu}}\right)\tau^{n(h-1)} + \sum_{i\in\mathcal{A}'} a_i\tau^i\right) \\
&= \widetilde{\omega}\left(1 + a\tau^\nu + \sum_{i\in\mathcal{A}'} a_i\tau^i\right)\cdot\psi\left(ba^{q^{l\nu}} - ab^{q^{l\nu}}\right).
\end{aligned}$$

Note that for any m not divisible by n, since ψ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer,

$$\#\left\{\psi_b\colon \mathbb{F}_{q^n} \to \overline{\mathbb{Q}}_\ell^\times \text{ such that } b \in \mathbb{F}_{q^n}\right\} = q^n, \tag{4.4}$$

where $\psi_b(a) := \psi\left(ba^{q^{n-m}} - ab^{q^m}\right)$. Indeed, if $b \neq 0$ and $\psi(ba^{q^{n-m}} - ab^{q^m}) = 1$ for all $a \in \mathbb{F}_{q^n}$, then it follows that $\psi(x) = \psi(x^{q^m})$ for all $x \in \mathbb{F}_{q^n}$, which contradicts the assumption on the $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer of ψ. By assumption, $a \in \mathbb{F}_{q^{n/2}}$ and $l\nu = n/2$ modulo n. Since every character of $\mathbb{F}_{q^{n/2}}$ extends to a character of $\mathbb{F}_{q^n}$, then by (4.4) in the special case $m = n/2$, it follows that

$$\#\left\{\psi_b\colon \mathbb{F}_{q^{n/2}} \to \overline{\mathbb{Q}}_\ell^\times \text{ such that } b \in \mathbb{F}_{q^n}\right\} = q^{n/2},$$

where $\psi_b(a) := \psi\left(a(b - b^{q^{n/2}})\right)$. Thus the orbit of $\widetilde{\omega}$ under $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$-conjugacy contains every extension of $\omega^\sharp$ to $H_0^+(\mathbb{F}_{q^n})$.

It now remains to show that the restriction of ρ to $H_0'(\mathbb{F}_{q^n})$ contains $\omega^\sharp$. Define

$$\begin{aligned}
&I := \{i : n(h-1)/2 < i \leq n(h-1),\ n \nmid i\} \\
&r_1 := \max(I), \quad r_i := \max(I \smallsetminus \{r_1, \ldots, r_{i-1}\}), \quad \text{for } 2 < i \leq \#I.
\end{aligned}$$

We prove the lemma by extending ω to each step of the chain

$$Z\left(U_{h,k}^{n,q}(\mathbb{F}_{q^n})\right) \subset G_1 \subset G_2 \subset \cdots \subset G_{\#I} = H_0'(\mathbb{F}_{q^n}),$$

where

$$G_{d_0} := \left\{ 1 + \sum_{n|i} a_i \tau^i + \sum_{i \geq d_0} a_{r_i} \tau^{r_i} \in H_0'(\mathbb{F}_{q^n}) \right\}, \quad \text{for } 1 \leq d_0 \leq \#I.$$

Consider the following extension of ω to G_1:

$$\omega_1 \colon G_1 \to \overline{\mathbb{Q}}_\ell^\times, \quad 1 + \sum_{n|i} a_i \tau^i + a_{r_1} \tau^{r_1} \mapsto \omega \left(1 + \sum_{n|i} a_i \tau^i \right).$$

For any $b \in \mathbb{F}_{q^n}$, consider the element $g_1 := 1 + b\tau^{n(h-1)-r_1} \in U_{h,k}^{n,q}(\mathbb{F}_{q^n})$. Then for any $g := 1 + \sum_{n|i} a_i \tau^i + a\tau^{r_1} \in G_1$,

$$\begin{aligned} \omega^\sharp(g_1 g g_1^{-1}) &= \omega^\sharp \left(1 + \sum_{n|i} a_i \tau^i + \left(b a^{q^{l(n(h-1)-r_1)}} - a b^{q^{lr_1}} \right) \tau^{n(h-1)} \right) \\ &= \omega \left(1 + \sum_{n|i} a_i \tau^i \right) \cdot \psi \left(b a^{q^{-lr_1}} - a b^{q^{lr_1}} \right). \end{aligned}$$

Since ψ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer and lr_1 is not divisible by n, it follows from Eq. (4.4) that the orbit of ω_1 under the conjugation action of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ contains every extension of ω to G_1, and so the restriction of ρ to G_1 must contain ω_1. Applying the above argument to each G_{d_0} inductively proves that the restriction of ρ to $H_0'(\mathbb{F}_{q^n})$ contains $\omega^\sharp$. □

Theorem 4.3 *For any $\chi \in \mathcal{T}$, the representation*

$$\rho_\chi := \mathrm{Ind}_{H^+(\mathbb{F}_{q^n})}^{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}(\widetilde{\chi})$$

is irreducible with dimension $q^{n(n-1)(h-1)/2}$. Moreover, $\mathcal{G} = \{\rho_\chi : \chi \in \mathcal{T}\}$.

Proof The dimension follows from Eq. (4.3). To prove irreducibility, we use Mackey's criterion. First note that it is clear that $H'(\mathbb{F}_{q^n})$ centralizes $\chi^\sharp$ and $H^+(\mathbb{F}_{q^n})$ centralizes $\widetilde{\chi}$. We must show that these are exactly the centralizers of these characters.

Let i be an integer such that $n \nmid i$ and $i \leq n(h-1)/2$. Then for any $a, b \in \mathbb{F}_{q^n}$,

$$\begin{aligned} &\widetilde{\chi}\Big((1 + b\tau^i)(1 + a\tau^{n(h-1)-i})(1 + b\tau^i)^{-1}\Big) \\ &= \widetilde{\chi}\Big((1 + b\tau^i)(1 + a\tau^{n(h-1)-i})(1 - b\tau^i + \cdots)^{-1}\Big) \\ &= \widetilde{\chi}\Big(1 + a\tau^{n(h-1)-i} + \left(b a^{q^{li}} - a b^{q^{l(n(h-1)-i)}}\right)\tau^{n(h-1)}\Big) \\ &= \widetilde{\chi}\Big(1 + a\tau^{n(h-1)-i}\Big) \cdot \psi\Big(b a^{q^{li}} - a b^{q^{-li}}\Big). \end{aligned} \tag{4.5}$$

If $i < n(h-1)/2$, then since li is not divisible by n and ψ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer, it follows from (4.5) that if $b \neq 0$, then $1 + b\tau^i$ does not centralize $\widetilde{\chi}$. Now assume we are in Case 2 and that $i = n(h-1)/2$. If $b \in \mathbb{F}_{q^n} \smallsetminus \mathbb{F}_{q^{n/2}}$ and $a \in \mathbb{F}_{q^{n/2}}$, then (4.5) simplifies to

$$\widetilde{\chi}(1 + a\tau^i) \cdot \psi\left(a\left(b - b^{q^{li}}\right)\right) = \widetilde{\chi}(1 + a\tau^i) \cdot \psi\left(a\left(b - b^{q^{n/2}}\right)\right).$$

Every character of $\mathbb{F}_{q^{n/2}}$ has exactly $q^{n/2}$ extensions to $\mathbb{F}_{q^n}$, and since ψ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer, it follows that $\psi(a(b - b^{q^{n/2}})) = 1$ for all $a \in \mathbb{F}_{q^{n/2}}$ if and only if $b \in \mathbb{F}_{q^{n/2}}$. Hence $1 + b\tau^i$ does not centralize $\widetilde{\chi}$ and this completes the proof. □

5 Juggling sequences and the varieties X_h

We give a description of X_h in terms of juggling sequences that will be crucial in understanding the cohomology groups $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$. In this section, we also include some technical lemmas that will be used in the proof of Theorem 6.4. As usual, for any integer m, let $[m]$ be the unique integer with $1 \leq [m] \leq n$ such that $m \equiv [m]$ modulo n.

5.1 Juggling sequences

We recall the combinatorial notion of a juggling sequence [5].

Definition 5.1 A *juggling sequence* of *period n* is a sequence $(j_1, \ldots, j_n)$ of nonnegative integers satisfying the following condition:

The integers $i + j_i$ are all distinct modulo n.

For a juggling sequence $j = (j_1, \ldots, j_n)$, define $|j| := \sum_{i=1}^n j_i$.

The following lemmas are straightforward.

Lemma 5.2 (Properties of juggling sequences) *Let $j = (j_1, \ldots, j_n)$ be a juggling sequence.*

(a) *There exists a unique permutation $\sigma_j \in S_n$ such that*

$$(j_1, \ldots, j_n) \equiv (\sigma_j(1) - 1, \ldots, \sigma_j(n) - n) \mod n.$$

(b) *Let $c = (12 \cdots n) \in S_n$ and define $c \cdot j := (j_{c(1)}, \ldots, j_{c(n)})$. Then $\sigma_{c \cdot j} = c^{-1}\sigma_j c$. In particular, the map $j \mapsto \operatorname{sgn} \sigma_j$ is invariant under the action of the subgroup $\langle c \rangle \subset S_n$.*

Lemma 5.3 *Let $m \geq 1$ be an integer, let j be a juggling sequence of period n with $|j| = mn$, and let $e_i \in \mathbb{Z}^n$ denote the n-tuple with a 1 in the ith coordinate and 0's elsewhere.*

(a) *If j has a coordinate labelled mn, then $j = (mn) \cdot e_1$ up to the action of $\langle c \rangle$.*
(b) *Let $r \leq mn$ be a positive integer with $n \nmid r$. If j consists of coordinates labelled only by 0, r, and $mn - r$, then $j = r \cdot e_1 + (mn - r) \cdot e_{[r]+1}$ up to the action of $\langle c \rangle$.*

5.2 The varieties X_h

We coordinatize $U_{h,k}^{n,q} = \mathbb{A}^{n(h-1)}$ in the following way. Let

$$\mathcal{A} := \{0, 1, \ldots, n(h-1)\}. \tag{5.1}$$

Then every element of $U_{h,k}^{n,q}$ is of the form $\sum_{i \in \mathcal{A}} x_i \tau^i$, where we set $x_0 := 1$.

Lemma 5.4 *The scheme $X_h \subset U_{h,k}^{n,q}$ is defined by the vanishing of the polynomials*

$$g_{mn} := \sum_j (-1)^{\mathrm{sgn}(\sigma_j)} x_{j_1}^{q^{[l]}} x_{j_2}^{q^{[2l]}} \cdots x_{j_{n-1}}^{q^{[(n-1)l]}} \left(x_{j_n}^{q^n} - x_{j_n} \right), \quad 1 \leq m \leq h-1,$$

where $x_0 := 1$ and the sum ranges over juggling sequences $j = (j_1, \ldots, j_n) \in \mathcal{A}^n$ with $|j| = mn$.

Proof Let $A = \iota_{h,k}(\sum x_i \tau^i)$ (see Eq. (2.3)) and let $A_{r,s}$ denote the (r, s)th entry of A. Then if we set $x_i = 0$ for $i \notin \mathcal{A}$,

$$A_{r,s} = \sum_{i \in \mathbb{Z}} x_{ni+s-r}^{q^{[(r+k-1)l]-1}} \pi^i.$$

For $1 \leq m \leq h-1$, let c_m denote the coefficient of π^m in

$$\det A = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}\, \sigma} \prod_{r=1}^n A_{r,\sigma(r)}.$$

Then

$$c_m = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}\, \sigma} \sum_{|i|=m} \prod_{r=1}^n x_{ni_r+\sigma(r)-r}^{q^{[(r+k-1)l]-1}},$$

where $i = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n$. Then setting $j_r := ni_r + \sigma(r) - r$ defines a juggling sequence $j = (j_1, \ldots, j_n) \in \mathcal{A}^n$ with

$$|j| = \sum_{r=1}^n j_r = \sum_{r=1}^n ni_r + \sigma(r) - r = mn.$$

It is clear that every juggling sequence $j \in \mathcal{A}^n$ arises in this way, and we therefore have

$$c_m = \sum_j (-1)^{\mathrm{sgn}\, \sigma_j} x_{j_1} x_{j_2}^{q^{[l]}} \cdots x_{j_n}^{q^{[(n-1)l]}},$$

where the sum ranges over juggling sequences $j \in \mathcal{A}^n$ with $|j| = mn$.

Recall that X_h is defined by the equations $c_m^q - c_m$ for $1 \leq m \leq h-1$. Let $c = (12\cdots n) \in S_n$ and let j be any juggling sequence with $|j| = mn$. By Lemma 5.2, $c^k \cdot j$ is a juggling sequence such that $|c \cdot j| = mn$ and $\operatorname{sgn}(\sigma_{c^k \cdot j}) = \operatorname{sgn}(\sigma)$. Moreover, $j' := (j'_1, \ldots, j'_n) := c^k \cdot j = (j_{[k+1]}, j_{[k+2]}, \ldots, j_{[k]})$ has the property that

$$\left(x_{j_1} x_{j_2}^{q^{[l]}} x_{j_3}^{q^{[2l]}} \cdots x_{j_n}^{q^{[(n-1)l]}}\right)^q = x_{j'_1}^{q^n} x_{j'_2}^{q^{[l]}} x_{j'_3}^{q^{[2l]}} \cdots x_{j'_n}^{q^{[(n-1)l]}}.$$

Thus we may arrange the monomials in $c_m^q - c_m$ so that we obtain:

$$c_m^q - c_m = \sum_j (-1)^{\operatorname{sgn}\sigma_j} x_{j_1}^{q^{[l]}} x_{j_2}^{q^{[2l]}} \cdots x_{j_{n-1}}^{q^{[(n-1)l]}} \left(x_{j_n}^{q^n} - x_{j_n}\right).$$

□

Corollary 5.5 *X_h is smooth integral affine scheme of pure dimension $(n-1)(h-1)$ over $\mathbb{F}_p$.*

Proof By Lemma 5.4, we know that

$$X_h = \operatorname{Spec}\left(\mathbb{F}_p[x_0, x_1, \ldots, x_{n(h-1)}]/(g_0, g_n, g_{2n}, \ldots, g_{(h-1)n})\right),$$

where $g_0 := 1 - x_0$ and g_{ni} for $1 \leq i \leq h-1$ is as in the lemma. Let $J = \frac{d\mathbf{g}}{d\mathbf{x}}$ be the corresponding Jacobian matrix and consider the $h \times h$ square submatrix

$$M := \left(\frac{\partial g_{nr}}{\partial x_{ns}}\right)_{0 \leq r,s \leq h-1}.$$

Obviously

$$\frac{\partial g_0}{\partial x_{ns}} = \begin{cases} -1 & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since we are working in characteristic p, for any $1 \leq r \leq h-1$, we have

$$\frac{\partial g_{nr}}{\partial x_i} = -\sum_j (-1)^{\operatorname{sgn}(\sigma_j)} x_{j_1}^{q^{[l]}} x_{j_2}^{q^{[2l]}} \cdots x_{j_{n-1}}^{q^{[(n-1)l]}},$$

where the sum ranges over juggling sequences $j = (j_1, \ldots, j_n) \in \mathcal{A}^n$ such that $|j| = nr$ and $j_n = i$. It follows that if $h-1 \geq s \geq r \geq 1$, then

$$\frac{\partial g_{nr}}{\partial x_{ns}} = \begin{cases} -1 & \text{if } s = r, \\ 0 & \text{if } s > r. \end{cases}$$

This implies that M is lower-triangular with -1 along the diagonal and hence is invertible at every point in X_h. It follows then that X_h has dimension $n(h-1)+1-h = (n-1)(h-1)$. □

5.3 Technical lemmas

This section contains technical lemmas that will be used in the proof of Theorem 6.4. We recommend the reader to return to this section during or after Sect. 6.

Recall the definitions of $\mathcal{A}$ and $\mathcal{A}'$ from Eqs. (5.1) and (4.1). The first two lemmas are straightforward computations.

Lemma 5.6 *For any elements*

$$s(x) := \sum_{i \in \mathcal{A} \smallsetminus \mathcal{A}'} x_i \tau^i \quad \text{and} \quad y := 1 + \sum_{i \in \mathcal{A}'} y_i \tau^i$$

in $U_{h,k}^{n,q}(\overline{\mathbb{F}}_q)$, we have $s(x) \cdot y = \sum_{i \in \mathcal{A}} a_i \tau^i$ where

$$a_i = \begin{cases} x_i + \sum\limits_{\substack{j \equiv i \pmod n \\ 1 < j < i}} x_j y_{i-j}^{q^{lj}} & \text{if } i \in \mathcal{A} \smallsetminus \mathcal{A}', \\ y_i + \sum\limits_{\substack{j \in \mathcal{A} \smallsetminus \mathcal{A}' \\ 1 \leq j < i}} x_j y_{i-j}^{q^{lj}} & \text{if } i \in \mathcal{A}'. \end{cases}$$

Lemma 5.7 *Suppose $1 + \sum_{i \in \mathcal{A}'} x_i \tau^i = L_{q^n}(1 + \sum_{i \in \mathcal{A}'} y_i \tau^i) \in H'(\overline{\mathbb{F}}_q)$. Then*

$$x_i = y_i^{q^n} - y_i + \delta_i,$$

where δ_i is some polynomial in y_j for $j < i$.

Lemma 5.8 *Let $s(x) := \sum_{i \in \mathcal{A} \smallsetminus \mathcal{A}'} x_i \tau^i \in U_{h,k}^{n,q}$ and for any integer m with $1 \leq m \leq h-1$, let g_{mn} be as in Lemma 5.4. Suppose that for any $y, y' \in H'$ with $L_{q^n}(y) = L_{q^n}(y')$,*

$$g_{mn}(s(x) \cdot y) = 0 \Longleftrightarrow g_{mn}(s(x) \cdot y') = 0.$$

If $L_{q^n}(y) = 1 + \sum_{i \in \mathcal{A}'} x_i \tau^i$, then $g_{mn}(s(x) \cdot y)$ is a polynomial in x_i for $i \in \mathcal{A}$ with $i \leq mn$.

This is a corollary of Proposition 3.6.

Proof For $i \in \mathcal{A}'$, let x_i be the polynomials determined by $L_{q^n}(1 + \sum_{i \in \mathcal{A}'} y_i \tau^i) = 1 + \sum_{i \in \mathcal{A}'} x_i \tau^i$. For $i \in \mathcal{A} \smallsetminus \mathcal{A}'$, define $y_i := x_i$. Consider the rings

$$R = \overline{\mathbb{F}}_q[y_i : i \in \mathcal{A}] \supset S = \overline{\mathbb{F}}_q[x_i : i \in \mathcal{A}]$$

and their fraction fields

$$E = \operatorname{Frac} R = \overline{\mathbb{F}}_q(y_i : i \in \mathcal{A}) \supset F = \operatorname{Frac} S = \overline{\mathbb{F}}_q(x_i : i \in \mathcal{A}).$$

It is clear that $S \hookrightarrow R$ is a finite map of polynomial rings.

We now show that E/F is a Galois extension of degree $q^{n\#\mathcal{A}'}$. For every $\zeta = 1 + \sum_{i \in \mathcal{A}'} \zeta_i \tau^i \in H'(\mathbb{F}_{q^n})$, the assignment

$$y_i \mapsto y_i', \quad \text{for } i \in \mathcal{A}', \text{ where } y\zeta = 1 + \sum_{i \in \mathcal{A}'} y_i' \tau^i$$

defines an automorphism of E fixing F. Indeed, $L_{q^n}(y\zeta) = \mathrm{Fr}_{q^n}(y\zeta) \cdot (y\zeta)^{-1} = L_{q^n}(y)$ since $\zeta \in H'(\mathbb{F}_{q^n}) = H'(\overline{\mathbb{F}}_q)^{\mathrm{Fr}_{q^n}}$. On the other hand, $[E : F] = q^{n\#\mathcal{A}'}$ since by Lemma 5.7, each y_i for $i \in \mathcal{A}'$ satisfies a separable degree-q^n polynomial. It follows that $\#\operatorname{Aut}(E/F) \geq |H'(\mathbb{F}_{q^n})| = q^{n\#\mathcal{A}'} = [E : F]$, and so E/F is Galois.

We are now in a position to apply Proposition 3.6. Fix $1 \leq m \leq h-1$. For each $\sigma \in \operatorname{Gal}(E/F)$,

$$\sigma(g_{mn}(s(x) \cdot y)) = g_{mn}(s(x) \cdot y'), \quad \text{for some } y' \in H' \text{ with } L_{q^n}(y') = L_{q^n}(y).$$

Hence by assumption, we know that for each $\sigma \in \operatorname{Gal}(E/F)$,

$$g_{mn}(s(x) \cdot y) = 0 \iff \sigma(g_{mn}(s(x) \cdot y)) = 0.$$

By the Nullstellensatz, this implies that the ideal generated by $g_{mn}(s(x) \cdot y)$ in R is equal to the ideal generated by $\sigma(g_{mn}(s(x) \cdot y))$ for all $\sigma \in \operatorname{Gal}(E/F)$. Thus by Proposition 3.6, we have that in fact $g_{mn}(s(x) \cdot y) \in S$. Finally, since $g_{mn}(s(x) \cdot y) \in R$ is a polynomial in x_i and y_i for $i \leq mn$ by Lemma 5.4, it follows by Lemma 5.7 that $g_{mn}(s(x) \cdot y) \in S$ is a polynomial in x_i for $i \leq mn$. $\square$

To prove Proposition 6.1, we will need a more precise result than Lemma 5.8.

Lemma 5.9 *Let $s(x), y \in U_{h,k}^{n,q}(\overline{\mathbb{F}}_q)$ be as in Lemma 5.6 and let $a = (a_0, a_1, \ldots, a_{n(h-1)})$ where $s(x) \cdot y = \sum_{i \in \mathcal{A}} a_i \tau^i$. Let $L_{q^n}(y) := 1 + \sum_{i \in \mathcal{A}'} x_i \tau^i$ and assume that for any $y, y' \in H'(\overline{\mathbb{F}}_q)$ with $L_{q^n}(y) = L_{q^n}(y')$, we have $g_{mn}(s(x) \cdot y) = 0$ if and only if $g_{mn}(s(x) \cdot y') = 0$.*

(a) *For any $1 \leq m \leq h-1$, the polynomial $g_{mn}(a)$ is a polynomial in x_i for $1 \leq i \leq mn$ and*
$$g_{mn}(a) = x_{mn} + (\textit{polynomial in } x_i \textit{ for } i < mn).$$

(b) *Let $I := \{i : n(h-1)/2 < i \leq n(h-1),\ n \nmid i\}$ and define*
$$r_1 := \max(I), \quad r_i := \max(I \smallsetminus \{r_1, \ldots, r_{i-1}\}), \quad \textit{for } 2 < i \leq \#I.$$
Pick a positive integer $d_0 \leq \#I$ and set
$$t_{d_0}(x) := x^{q^{l(n(h-1)-r_{d_0})-n}} + x^{q^{l(n(h-1)-r_{d_0})-2n}} + \cdots + x^{q^{n-[lr_{d_0}]}}.$$

If $x_{r_i} = x_{(h-1)n-r_i} = 0$ *for* $1 \leq i \leq d_0 - 1$*, then the contribution of* $x_{r_{d_0}}$ *to* $g_{(h-1)n}(a)$ *occurs in the expression*

$$x_{n(h-1)-r_{d_0}}^{q^n} x_{r_{d_0}}^{q^{n-[lr_{d_0}]}} - x_{n(h-1)-r_{d_0}}^{q^{[lr_{d_0}]}} x_{r_{d_0}} + \left(x_{n(h-1)-r_{d_0}} t_{d_0}(x_{r_{d_0}})\right)^{q^n} - x_{n(h-1)-r_{d_0}} t_{d_0}(x_{r_{d_0}}).$$

Proof We first prove (a). By Lemma 5.4, $g_{mn}(a)$ is a polynomial in a_i for $i \leq mn$, and by Lemma 5.6, y_{mn} only appears in a_i for $i \geq mn$. Therefore by Lemma 5.3(a), the contribution of y_{mn} to $g_{mn}(a)$ must come from the juggling sequence $(0, \dots, 0, mn)$, and hence we have

$$\begin{aligned} g_{mn}(a) &= y_{mn}^{q^n} - y_{mn} + (\text{polynomial in } x_i, y_i \text{ for } i < mn) \\ &= x_{mn} + (\text{polynomial in } x_i, y_i \text{ for } i < mn) && \text{(by Lemma 5.7)} \\ &= x_{mn} + (\text{polynomial in } x_i \text{ for } i < mn) && \text{(by Lemma 5.8).} \end{aligned}$$

We now prove (b). By Lemma 5.6 and the vanishing assumption, $y_{r_{d_0}}$ only appears in a_i for $i = r_{d_0}$ and $i = (h-1)n$. Furthermore, any juggling sequence $j = (j_1, \dots, j_n)$ wherein $y_{r_{d_0}}$ contributes to $g_{(h-1)n}$ nontrivially must have the following criteria:

- $j_n \neq 0$
- For $1 \leq i \leq d_0 - 1$, the numbers r_i and $(h-1)n - r_i$ do not appear in j.

It therefore follows from Lemma 5.3 that the only terms in $g_{(h-1)n}$ involving $y_{r_{d_0}}$ occur exactly in the summands corresponding to the juggling sequences

$$\begin{aligned} &(h-1)n \cdot e_n \longleftrightarrow 1 \in S_n, \\ &r_{d_0} \cdot e_{n-\bar{r}_{d_0}} + ((h-1)n - r_{d_0}) \cdot e_n \longleftrightarrow (n - \bar{r}_{d_0}, n) \in S_n, \\ &((h-1)n - r_{d_0}) \cdot e_{\bar{r}_{d_0}} + r_{d_0} \cdot e_n \longleftrightarrow (\bar{r}_{d_0}, n) \in S_n. \end{aligned}$$

By Lemma 5.4, this exactly corresponds to the following summands in $g_{(h-1)n}(a)$:

$$\left(a_{(h-1)n}^{q^n} - a_{(h-1)n}\right) - a_{r_{d_0}}^{q^{[l(n-\bar{r}_{d_0})]}} \left(a_{(h-1)n-r_{d_0}}^{q^n} - a_{(h-1)n-r_{d_0}}\right) - a_{(h-1)n-r_{d_0}}^{q^{[l\bar{r}_{d_0}]}} \left(a_{r_{d_0}}^{q^n} - a_{r_{d_0}}\right).$$

Thus by Lemma 5.6, we see that the only terms involving $y_{r_{d_0}}$ occur in the expression

$$\begin{aligned} &\left(\left(x_{n(h-1)-r_{d_0}} y_{r_{d_0}}^{q^{l(n(h-1)-r_{d_0})}}\right)^{q^n} - x_{n(h-1)-r_{d_0}} y_{r_{d_0}}^{q^{l(n(h-1)-r_{d_0})}}\right) \\ &\quad - y_{r_{d_0}}^{q^{n-[l\bar{r}_{d_0}]}} \left(x_{n(h-1)-r_{d_0}}^{q^n} - x_{n(h-1)-r_{d_0}}\right) - x_{n(h-1)-r_{d_0}}^{q^{[l\bar{r}_{d_0}]}} \left(y_{r_{d_0}}^{q^n} - y_{r_{d_0}}\right). \end{aligned} \tag{5.2}$$

By Lemma 5.7, $x_{r_{d_0}} = y_{r_{d_0}}^{q^n} - y_{r_{d_0}} + \delta_{r_{d_0}}$. By Lemma 5.8, the terms in $\delta_{r_{d_0}}$ will contribute elsewhere to a polynomial that can be written in terms of x_i for $i \in \mathcal{A}$ with $i < r_{d_0}$. (The condition $i < r_{d_0}$ can be seen from the proof of Lemma 5.8, proceeds by showing that y_i is a polynomial in x_j for $j \leq i$.) Thus the contribution of $x_{r_{d_0}}$ in (5.2) simplifies to

$$\begin{aligned} & x_{n(h-1)-r_{d_0}}^{q^n}\left(y_{r_{d_0}}^{q^{l((h-1)n-r_{d_0})+n}} - y_{r_{d_0}}^{q^{n-[lr_{d_0}]}}\right) \\ & - x_{n(h-1)-r_{d_0}}\left(y_{r_{d_0}}^{q^{l(n(h-1)-r_{d_0})}} - y_{r_{d_0}}^{q^{n-[lr_{d_0}]}}\right) - x_{n(h-1)-r_{d_0}}^{q^{[lr_{d_0}]}}\left(y_{r_{d_0}}^{q^n} - y_{r_{d_0}}\right) \\ & = x_{n(h-1)-r_{d_0}}^{q^n} t_{d_0}(x_{r_{d_0}})^{q^n} + x_{n(h-1)-r_{d_0}}^{q^n} x_{r_{d_0}}^{q^{n-[lr_{d_0}]}} - x_{n(h-1)-r_{d_0}} t_{d_0}(x_{r_{d_0}}) - x_{n(h-1)-r_{d_0}}^{q^{[lr_{d_0}]}} x_{r_{d_0}}, \end{aligned}$$

where the last equality holds modulo terms without $x_{r_{d_0}}$. □

6 The representations $H_c^\bullet(X_h)[\chi]$

In this section, we prove the irreducibility of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ and its vanishing outside a single degree. The key proposition, which we prove in Sect. 6.1, is:

Proposition 6.1 *For any $\chi \in \mathcal{T}$,*

$$\dim \operatorname{Hom}_{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}\left(\rho_\chi, H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right) = \delta_{i,(n-1)(h-1)},$$

where $\rho_\chi \in \mathcal{G}$ is the representation described in Theorem 4.3. Moreover, Fr_{q^n} *acts on the cohomology group $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ via multiplication by $(-q^n)^{(n-1)(h-1)}$.*

Recall that $\mathbb{F}_{q^n}^\times \ltimes U_{h,k}^{n,q}(\mathbb{F}_{q^n}) \cong \mathcal{R}_{h,k,n,q}^\times(\mathbb{F}_{q^n})$ and that $\mathbb{F}_{q^n}^\times$ acts on X_h by conjugation. For any $z \in \mathbb{F}_{q^n}^\times$ and any $g, h \in H(\mathbb{F}_{q^n})$, let (z, h, g) denote the map $X_h \to X_h$ given by $x \mapsto z(h * x \cdot g)z^{-1}$. We prove the following proposition in Sect. 6.2.

Proposition 6.2 *If $\zeta \in \mathbb{F}_{q^n}^\times$ has trivial* $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$*-stabilizer, then for any $g \in H(\mathbb{F}_{q^n})$,*

$$\operatorname{Tr}\left((\zeta, 1, g)^*; H_c^{(n-1)(h-1)}\left(X_h, \overline{\mathbb{Q}}_\ell\right)[\chi]\right) = (-1)^{(n-1)(h-1)}\chi(g).$$

From the multiplicity-one statement of Proposition 6.1, the nonvanishing statement of Proposition 6.2, and a counting argument coming from Theorem 4.3, one obtains the following two results, which we prove simultaneously in Sect. 6.3.

Proposition 6.3 *The parametrization*

$$\mathcal{T} \to \mathcal{G}, \quad \chi \mapsto \rho_\chi$$

described in Theorem 4.3 is a bijection.

Theorem 6.4 *For any $\chi \in \mathcal{T}$, the $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$-representation $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ is irreducible when $i = (n-1)(h-1)$ and vanishes otherwise. Moreover, for $\chi, \chi' \in \mathcal{T}$, we have $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi] \cong H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi']$ if and only if $\chi = \chi'$.*

6.1 Proof of Proposition 6.1

Note that from Sect. 4, the representation

$$W_\chi := \operatorname{Ind}_{H'(\mathbb{F}_{q^n})}^{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}(\chi^\sharp)$$

is irreducible and isomorphic to ρ_χ in Case 1, and is a direct sum of $q^{n/2}$ copies of ρ_χ in Case 2. Thus the statement of the proposition is equivalent to:

$$\dim \operatorname{Hom}_{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}\left(W_\chi, H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right) = \begin{cases} \delta_{i,(n-1)(h-1)} & \text{if } (n,h) \text{ is in Case 1,} \\ q^{n/2} \cdot \delta_{i,(n-1)(h-1)} & \text{if } (n,h) \text{ is in Case 2.} \end{cases}$$

We use Proposition 3.3 to reduce the computation of the space of homomorphisms $\operatorname{Hom}_{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}\left(W_\chi, H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right)$ to a computation of the cohomology of a certain scheme S with coefficients in a certain constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}$. Then, to compute $H_c^i(S, \mathscr{F})$, we inductively apply Proposition 3.4. This will allow us to reduce the computation to a computation involving a 0-dimensional scheme in Case 1 and a 1-dimensional scheme in Case 2. We will treat these cases simultaneously until the final step.

Step 0 We first establish some notation. Note the resemblance to the notation in Lemma 5.9.

- Let

$$I := \{i : n(h-1)/2 < i \le n(h-1),\ n \nmid i\}$$
$$J := \{i : 1 \le i \le n(h-1)/2,\ n \nmid i\}$$

  and set $d := \#I = \lfloor (n-1)(h-1)/2 \rfloor$. Note that $\mathcal{A}' \cup J = \{1, 2, \ldots, n(h-1)\}$.
- Set $I_0 := I$ and $J_0 := J$. For $1 \le i \le d$, let

$$r_i := \max I_{i-1}, \quad I_i := I_{i-1} \smallsetminus \{r_i\}, \quad J_i := J_{i-1} \smallsetminus \{(h-1)n - r_i\}.$$

  Note that $I_d = \varnothing$. In Case 1, $J_d = \varnothing$, and in Case 2, $J_d = \{n(h-1)/2\}$.
- For a finite set A, we will write $\mathbb{A}[A]$ to denote the affine space $\mathbb{A}^{\#A}$ with coordinates labelled by A.
- For $m \in \mathbb{N}$, we will denote by $[m]$ the unique integer in $\{1, \ldots, n\}$ with $m \equiv [m]$ modulo n, and denote by $\bar{m}$ the unique integer in $\{0, \ldots, n-1\}$ with $m \equiv \bar{m}$ modulo n.
- For any finite-type scheme S over $\mathbb{F}_{q^n}$, we consider $H_c^\bullet(S, \overline{\mathbb{Q}}_\ell) := \bigoplus_{i \in \mathbb{Z}} H_c^i(S, \overline{\mathbb{Q}}_\ell)$ as a finite-dimensional graded vector space over $\overline{\mathbb{Q}}_\ell$ equipped with an action of Fr_{q^n}. We write $H_c^i(S, \overline{\mathbb{Q}}_\ell)[-1] := H_c^{i-1}(S, \overline{\mathbb{Q}}_\ell)$ and we write $H_c^i(S, \overline{\mathbb{Q}}_\ell)(-1)$, to denote that the action of Fr_{q^n} on $H_c^i(S, \overline{\mathbb{Q}}_\ell)$ is multiplied by q^n.

Step 1 We apply Proposition 3.3 to the following set-up:

- $U_{h,k}^{n,q}$ together with the connected subgroup H', both of which are defined over $\mathbb{F}_{q^n}$
- a morphism $s\colon U_{h,k}^{n,q}/H' \to U_{h,k}^{n,q}$ defined by identifying $U_{h,k}^{n,q}/H'$ with affine space $\mathbb{A}[J]$ and setting $s\colon (x_i)_{i \in J} \mapsto 1 + \sum_{i \in J} x_i \tau^i$
- the algebraic group morphism $f\colon H' \to H$ given by $\sum_{i \in \mathcal{A}'} x_i \tau^i \mapsto \sum_{n|i} x_i \tau^i$
- a character $\chi\colon H(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$
- $Y_h := L_{q^n}(X_h)$, a locally closed subvariety of $U_{h,k}^{n,q}$ satisfying $X_h = L_{q^n}^{-1}(Y_h)$

Since X_h has a right-multiplication action of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$, the cohomology groups $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ inherit a $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$-action. By Proposition 3.3, we have graded vector space isomorphisms

$$\boxed{\operatorname{Hom}_{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}\left(W_\chi, H_c^\bullet\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right) \cong H_c^\bullet(\beta^{-1}(Y_h), P^*\mathcal{L}_\chi)}$$

compatible with the action of Fr_{q^n}. Here, $\mathcal{L}_\chi$ is the local system on H corresponding to χ, the morphism $\beta\colon (U_{h,k}^{n,q}/H') \times H' \to U_{h,k}^{n,q}$ is given by $\beta(x, g) = s(\mathrm{Fr}_{q^n}(x)) \cdot g \cdot s(x)^{-1}$, and the morphism $P\colon \beta^{-1}(Y_h) \to H$ is the composition $\beta^{-1}(Y_h) \hookrightarrow (U_{h,k}^{n,q}/H') \times H' \xrightarrow{\mathrm{pr}} H' \xrightarrow{f} H$.

We now work out an explicit description of $\beta^{-1}(Y_h) \subset \mathbb{A}[J] \times H'$. For $1 \leq m \leq h-1$, let g_{mn} be the polynomial defined in Lemma 5.4. Write $x = (x_i)_{i \in J} \in \mathbb{A}[J]$ and $g = 1 + \sum_{i \in \mathcal{A}'} x_i \tau^i \in H'(\overline{\mathbb{F}}_q)$. For any $y = 1 + \sum_{i \in \mathcal{A}'} y_i \tau^i \in H'(\overline{\mathbb{F}}_q)$ such that $L_{q^n}(y) = g$, we have

$$\beta(x, g) = \mathrm{Fr}_{q^n}(s(x)) \cdot L_{q^n}(y) \cdot s(x)^{-1} = L_{q^n}(s(x) \cdot y).$$

We see that $\beta(x, g) \in Y_h$ if and only if $s(x) \cdot y \in X_h$. Let $s(x) \cdot y = 1 + \sum a_i \tau^i$. By Lemma 5.4, we know that $s(x) \cdot y \in X_h$ if and only if $g_{mn}(a) = 0$ for $m = 1, \dots, h-1$. Recall from Lemma 5.8 that using the identity $L_{q^n}(y) = 1 + \sum_{i \in I} x_i \tau^i$, each polynomial $g_{mn}(a)$, which *a priori* is a polynomial in x_j for $j \in J$ and y_i for $i \in \mathcal{A}'$, is in fact a polynomial in x_i for $1 \leq i \leq n(h-1)$.

Step 2 By Lemma 5.9(a), for each $m = 1, \dots, h-1$, the polynomial $g_{mn}(s(x) \cdot y)$ is of the form $x_{mn} +$ (stuff with x_i for $i < mn$). Thus the coordinates x_{mn} of $\beta^{-1}(Y_h) \subset \mathbb{A}[\mathcal{A}' \cup J_0]$ are uniquely determined by the other coordinates. Equivalently, the morphism $(x_i)_{i \in \mathcal{A}' \cup J_0} \mapsto (x_i)_{i \in I_0 \cup J_0}$ gives an isomorphism $\beta^{-1}(Y_h) \cong \mathbb{A}[I_0 \cup J_0] =: S^{(0)}$. Then

$$\boxed{H_c^\bullet(\beta^{-1}(Y_h), P^*\mathcal{L}_\chi) = H_c^\bullet(S^{(0)}, (P^{(0)})^*\mathcal{L}_\chi),}$$

where $P^{(0)}\colon S^{(0)} \to H$ is the morphism determined by P and the isomorphism $\beta^{-1}(Y_h) \cong S^{(0)}$; it is the map determined by $(x_i)_{i \in I_0 \cup J_0} \mapsto (x_n, x_{2n}, \dots, x_{(h-1)n})$, where for $m = 1, \dots, h-1$, we view x_{mn} as a polynomial in x_i for $i \in I_0 \cup J_0$.

Step 3: Base case We now apply Proposition 3.4 to the following set-up:

- Let $S^{(0)} = \mathbb{A}[I_0 \cup J_0]$.
- Let $S_2^{(0)} = \mathbb{A}[I_1 \cup J_0]$.
- Note that $S^{(0)} = S_2^{(0)} \times \mathbb{A}[\{r_1\}]$.
- Let $f\colon S_2^{(0)} \to \mathbb{G}_a$ be the morphism $(x_i)_{i \in I_1 \cup J_0} \mapsto x_{n(h-1)-r_1}$.
- Set $v \in S_2^{(0)}$ and $w = x_{r_1}$. By Lemma 5.9, we may write

$$P^{(0)}(v, w) = g\left(f(v)^{q^{[lr_1]}} w - f(v)^{q^n} w^{q^{n-[lr_1]}} - (f(v)t_1(w))^{q^n} + f(v)t_1(w)\right) \cdot P_2^{(0)}(v),$$

  where $g\colon \mathbb{G}_a \to H$ is the morphism $z \mapsto 1 + z\tau^{n(h-1)}$. Observe that this is the negative of the expression appearing in Lemma 5.9 since we solved for $x_{(h-1)n}$ in the equation $g_{(h-1)n}(s(x) \cdot y) = 0$.
- Let $S_3^{(0)} = \mathbb{A}[I_1 \cup J_1]$ so that this is the subscheme of $S_2^{(0)} = \mathbb{A}[I_1 \cup J_0]$ defined by $f = 0$, and let $P_3^{(0)} := P_2^{(0)}|_{S_3^{(0)}}\colon S_3^{(0)} \to H$.

Then by Proposition 3.4, as graded vector spaces with an action of Fr_{q^n}, we have

$$\boxed{H_c^\bullet\left(S^{(0)}, (P^{(0)})^*\mathcal{L}_\chi\right) \cong H_c^\bullet\left(S_3^{(0)}, (P_3^{(0)})^*\mathcal{L}_\chi\right)(-1)[-2].}$$

Step 3: Inductive step We now describe the inductive step for $d_0 \le d$. We apply Proposition 3.5 to the following set-up:

- Let $S^{(d_0)} := S_3^{(d_0-1)} = \mathbb{A}[I_{d_0} \cup J_{d_0}]$.
- Let $S_2^{(d_0)} = \mathbb{A}[I_{d_0+1} \cup J_{d_0}]$.
- Note that $S^{(d_0)} = S_2^{(d_0)} \times \mathbb{A}[\{r_{d_0}\}]$.
- Let $f\colon S_2^{(d_0)} \to \mathbb{G}_a$ be the morphism $(x_i)_{i \in I_{d_0+1} \cup J_{d_0}} \mapsto x_{n(h-1)-r_{d_0}}$.
- Set $v \in S_2^{(d_0)}$ and $w = x_{r_{d_0}}$. Let $t_{d_0}(x)$ be as in Lemma 5.9 so that, by the same lemma, the morphism $P^{(d_0)} := P_3^{(d_0-1)}\colon S^{(d_0)} \to H$ has the following form:

$$P^{(d_0)}(v, w) = g\left(f(v)^{q^{[lr_{d_0}]}} w - f(v)^{q^n} w^{q^{n-[lr_{d_0}]}} - (f(v)t_{d_0}(w))^{q^n} + f(v)t_{d_0}(w)\right) \cdot P_2^{(d_0)}(v),$$

  where as in Step 3: Base case, the morphism $g\colon \mathbb{G}_a \to H$ is $z \mapsto 1 + z\tau^{n(h-1)}$.
- Let $S_3^{(d_0)} = \mathbb{A}[I_{d_0+1} \cup J_{d_0+1}]$ so that this is the subscheme of $S_2^{(d_0)} = \mathbb{A}[I_{d_0+1} \cup J_{d_0}]$ defined by $f = 0$, and let $P_3^{(d_0)} := P_2^{(d_0)}|_{S_3^{(d_0)}}\colon S_3^{(d_0)} \to H$.

Then by Proposition 3.4, as graded vector spaces with an action of Fr_{q^n}, we have

$$\boxed{H_c^\bullet\left(S^{(d_0)}, (P^{(d_0)})^*\mathcal{L}_\chi\right) \cong H_c^\bullet\left(S_3^{(d_0)}, (P_3^{(d_0)})^*\mathcal{L}_\chi\right)(-1)[-2].}$$

Step 4: Case 1 Step 3 allows us to reduce the computation about the cohomology of $S^{(0)}$ to a computation about the cohomology of $S^{(d)} := S_3^{(d-1)}$, which is a point. Thus Fr_{q^n} acts trivially on the cohomology of $S^{(d)}$ and for all $i \in \mathbb{Z}$,

$$\boxed{\dim H_c^i\left(S^{(d)}, (P^{(d)})^*\mathcal{L}_\chi\right) = \delta_{0,i}.}$$

Step 4: Case 2 Step 3 allows us to reduce the computation about the cohomology of $S^{(0)}$ to a computation about the cohomology of $S^{(d)} := S_3^{(d-1)} = \mathbb{A}[\{n(h-1)/2\}]$. The morphism $P^{(d)}$ is

$$P^{(d)} \colon S^{(d)} \to H, \quad a_{n(h-1)/2} \mapsto 1 + a_{n(h-1)/2}^{q^{n/2}}\left(a_{n(h-1)/2}^{q^n} - a_{n(h-1)/2}\right)\tau^{n(h-1)}.$$

Then we claim that for all $i \in \mathbb{Z}$

$$H_c^i\left(\mathbb{G}_a, (P^{(d)})^*\mathcal{L}_\chi\right) = H_c^i(\mathbb{G}_a, P^*\mathcal{L}_\psi),$$

where ψ is the restriction of χ to $\mathbb{F}_{q^n} \to \overline{\mathbb{Q}}_\ell^\times$ and P_0 is the morphism

$$P_0 \colon \mathbb{G}_a \to \mathbb{G}_a, \quad x \mapsto x^{q^{n/2}}(x^{q^n} - x).$$

We now compute the cohomology groups $H_c^i(\mathbb{G}_a, P^*\mathcal{L}_\psi)$ in the same way as in [4, Section 6.5.6 and Proposition 6.6.1]. We may write $P = f_1 \circ f_2$ where $f_1(x) = x^{q^{n/2}} - x$ and $f_2(x) = x^{q^{n/2}+1}$. Since f_1 is a group homomorphism, then $f_1^*\mathcal{L}_\psi \cong \mathcal{L}_{\psi \circ f_1}$. By assumption ψ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer, so $\psi \circ f_1$ is nontrivial. Furthermore, $\psi \circ f_1$ is trivial on $\mathbb{F}_{q^{n/2}}$. Thus the character $\psi \circ f_1 : \mathbb{F}_{q^n} \to \overline{\mathbb{Q}}_\ell^\times$ satisfies the hypotheses of [4, Proposition 6.6.1], and thus Fr_{q^n} acts on $H_c^1(\mathbb{G}_a, P_0^*\mathcal{L}_\psi)$ via multiplication by $-q^{n/2}$ and

$$\dim H_c^i\left(\mathbb{G}_a, P_0^*\mathcal{L}_\psi\right) = q^{n/2} \cdot \delta_{1,i}.$$

Thus for all $i \in \mathbb{Z}$,

$$\boxed{\dim H_c^i\left(S^{(d)}, (P^{(d)})^*\mathcal{L}_\chi\right) = q^{n/2} \cdot \delta_{1,i}.}$$

Step 5 We now put together all of the boxed equations. For all $i \in \mathbb{Z}$,

$$\begin{aligned}
\mathrm{Hom}_{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}\left(W_\chi, H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right) &\cong H_c^i\left(\beta^{-1}(Y_h), P^*\mathcal{L}_\chi\right) \\
&= H_c^i\left(S^{(0)}, (P^{(0)})^*\mathcal{L}_\chi\right) \\
&\cong H_c^{i-2}\left(S_3^{(0)}, (P_3^{(0)})^*\mathcal{L}_\chi\right)(-1) \\
&= H_c^{i-2}\left(S^{(1)}, (P^{(1)})^*\mathcal{L}_\chi\right)(-1) \\
&\cong H_c^{i-2d}\left(S^{(d)}, (P^{(d)})^*\mathcal{L}_\chi\right)(-d).
\end{aligned}$$

Therefore if we are in Case 1, then

$$\dim \operatorname{Hom}_{H(\mathbb{F}_{q^n})}\left(W_\chi, H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right) = \delta_{(n-1)(h-1),i}.$$

Moreover, the Frobenius Fr_{q^n} acts on $\operatorname{Hom}_{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}\left(W_\chi, H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)\right)$ via multiplication by the scalar $q^{n(n-1)(h-1)/2}$.

If we are in Case 2, then

$$\dim \operatorname{Hom}_{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}\left(W_\chi, H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right) = q^{n/2} \cdot \delta_{(n-1)(h-1),i}.$$

Moreover, the Frobenius Fr_{q^n} acts on $\operatorname{Hom}_{U_{h,k}^{n,q}(\mathbb{F}_{q^n})}\left(W_\chi, H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)\right)$ via multiplication by the scalar $-q^{n(n-1)(h-1)/2}$.

Finally, observe that if we are in Case 1, then $(n-1)(h-1)$ is even and if we are in Case 2, then $(n-1)(h-1)$ is odd, and therefore Fr_{q^n} acts on $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ by multiplication by $(-q^n)^{(n-1)(h-1)}$.

6.2 Proof of Proposition 6.2

By Corollary 5.5, X_h is a separated, finite-type scheme over $\mathbb{F}_{q^n}$ and the action of $(\zeta, h, g) \in \mathbb{F}_{q^n}^\times \times H(\mathbb{F}_{q^n}) \times U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ on X_h defines a finite-order automorphism. Moreover, $(\zeta, h, g) = (1, h, g) \cdot (\zeta, 1, 1)$, where $(1, h, g)$ is a p-power-order automorphism and $(\zeta, 1, 1)$ is an automorphism with prime-to-p order. By the Deligne–Lusztig fixed point formula [8, Theorem 3.2],

$$\sum_i (-1)^i \operatorname{Tr}\left((\zeta, h, g)^*; H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right) = \sum_i (-1)^i \operatorname{Tr}\left((1, h, g)^*; H_c^i\left(X_h^\zeta, \overline{\mathbb{Q}}_\ell\right)\right).$$

It is easy to calculate X_h^ζ. Indeed, it can be identified with the subvariety of all elements of $U_{h,k}^{n,q}$ of the form $1 + \sum_{1 \leq i \leq h-1} a_{ni} \tau^{ni}$. Then the determinant condition on X_h implies that $a_{ni} \in \mathbb{F}_{q^n}$ and hence X_h^ζ is just a discrete set naturally identified with $H(\mathbb{F}_{q^n})$ and the left and right actions of $H(\mathbb{F}_{q^n})$ are given by left and right multiplication. Therefore $H_c^i(X_h^\zeta, \overline{\mathbb{Q}}_\ell) = 0$ for $i > 0$ so

$$\sum_i (-1)^i \operatorname{Tr}\left((1, h, g)^*; H_c^i\left(X_h^\zeta, \overline{\mathbb{Q}}_\ell\right)\right) = \operatorname{Tr}\left((1, h, g)^*; H_c^0\left(X_h^\zeta, \overline{\mathbb{Q}}_\ell\right)\right).$$

Furthermore, as a $(H(\mathbb{F}_{q^n}) \times H(\mathbb{F}_{q^n}))$-representation, $H_c^0(X_h^\zeta, \overline{\mathbb{Q}}_\ell)$ is the pullback of the regular representation of $H(\mathbb{F}_{q^n})$ along the multiplication map $H(\mathbb{F}_{q^n}) \times H(\mathbb{F}_{q^n}) \to H(\mathbb{F}_{q^n})$. Thus

$$H_c^0\left(X_h^\zeta, \overline{\mathbb{Q}}_\ell\right) = \bigoplus_{\chi_0 \colon H(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times} \chi_0 \otimes \chi_0$$

as representations of $H(\mathbb{F}_{q^n}) \times H(\mathbb{F}_{q^n})$. Therefore

$$\sum_{h \in H(\mathbb{F}_{q^n})} \chi(h)^{-1} \sum_i (-1)^i \operatorname{Tr}\left((\zeta, h, g)^*; H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right) = \chi(g) \cdot \# H(\mathbb{F}_{q^n}).$$

This is equivalent to

$$\sum_i (-1)^i \operatorname{Tr}\left((\zeta, 1, g)^*; H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)[\chi]\right) = \chi(g),$$

and since $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi] = 0$ for $i \neq (n-1)(h-1)$ by Proposition 6.1, the desired result follows.

6.3 Proof of Proposition 6.3 and Theorem 6.4

By Proposition 6.1, any $\rho \in \mathcal{G}$ occurs exactly once in $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)$. Recalling that $\mathcal{G}$ is the set of irreducible representations of $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ whose central character has trivial $\operatorname{Gal}(L/K)$-stabilizer, observe that ρ must occur in $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ for some $\chi \in \mathcal{T}$. Conversely, each irreducible constituent of $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ must be in $\mathcal{G}$, and therefore

$$\bigoplus_{\rho \in \mathcal{G}} \rho = \bigoplus_{\chi \in \mathcal{T}} H_c^{(n-1)(h-1)}\left(X_h, \overline{\mathbb{Q}}_\ell\right)[\chi].$$

By Theorem 4.3, the left-hand side has at most $\#\mathcal{T}$ irreducible constituents. By Proposition 6.2, each $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ for $\chi \in \mathcal{T}$ is nonzero, and therefore the right-hand side has at least $\#\mathcal{T}$ irreducible constituents. Therefore both sides must have exactly $\#\mathcal{T}$ irreducible constituents, $\#\mathcal{G} = \#\mathcal{T}$, and the $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$-representations $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ for $\chi \in \mathcal{T}$ are irreducible and mutually nonisomorphic. This proves Proposition 6.3 and Theorem 6.4.

7 Division algebras and Jacquet–Langlands transfers

Our goal in this final section is to understand two connections. The first, explained in Sect. 7.1, is to unravel the relationship between Theorem 6.4 and the representations arising from Deligne–Lusztig constructions of division algebras. Because Theorem 6.4 proves a conjecture of Boyarchenko (see [2, Conjecture 5.18]) for primitive characters χ, we can use [2, Proposition 5.19] to explicitly describe this relationship.

The second connection, explained in Sect. 7.2, is to unravel the relationship between the representations described in Sect. 7.1 with respect to the local Langlands and Jacquet–Langlands correspondences. We prove that the correspondence $\theta \mapsto H_\bullet(\widetilde{X})[\theta]$ is consistent with the correspondence given by the composition of the local Langlands and Jacquet–Langlands correspondences, and therefore the homology of Deligne–Lusztig constructions gives a geometric realization of the Jacquet–Langlands correspondence between division algebras of different invariants.

7.1 Deligne–Lusztig constructions for division algebras

We temporarily drop the assumption on the characteristic of K as the following discussion is not restricted to the positive characteristic setting. Let $\widehat{K}^{\mathrm{nr}}$ be the completion of the maximal unramified extension of K and let φ denote the Frobenius automorphism of $\widehat{K}^{\mathrm{nr}}$ (inducing $x \mapsto x^q$ on the residue field).

Consider the following automorphisms of $\mathrm{GL}_n(\widehat{K}^{\mathrm{nr}})$:

$$F_1(g) = \varpi_k^{-1}\varphi(g)\varpi_k, \quad \varpi_k = \begin{pmatrix} 0 & 1_{n-1} \\ \pi^k & 0 \end{pmatrix},$$
$$F_2(g) = \varpi^{-k}\varphi(g)\varpi^k, \quad \varpi = \begin{pmatrix} 0 & 1_{n-1} \\ \pi & 0 \end{pmatrix}.$$

Here, we write $\varphi(g)$ to mean the matrix obtained by applying φ to each entry of g. For $i = 1, 2$, let $\mathbb{G}_i$ be the algebraic group over K with Frobenius F_i. Let $\mathbb{T}_i \subset \mathbb{G}_i$ be the algebraic group corresponding to the diagonal matrices over $\widehat{K}^{\mathrm{nr}}$. Then we have

$$\mathbb{G}_1(K) \xrightarrow{\cong} \mathbb{G}_2(K), \quad \mathbb{T}_1(K) \xrightarrow{\cong} \mathbb{T}_2(K),$$

where the isomorphism is given by $f : g \mapsto \gamma^{-1} \cdot g \cdot \gamma$, where $\gamma = \gamma_0 \cdot \mathrm{diag}(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n})$ for a permutation matrix γ_0 and for some $\lambda_i \in \mathbb{Z}$. Since the image of ϖ in the Weyl group has order n, we may choose γ_0 so that $e_1 \cdot \gamma_0 = e_1$, where e_1 is the first elementary row vector.

Let $\widetilde{G} := \mathbb{G}_i(\widehat{K}^{\mathrm{nr}}) = \mathrm{GL}_n(\widehat{K}^{\mathrm{nr}})$ and $\widetilde{T} := \mathbb{T}_i(\widehat{K}^{\mathrm{nr}})$. Let $\mathbb{B} \subset \mathbb{G}_i \otimes_K \widehat{K}^{\mathrm{nr}}$ be the Borel subgroup consisting of upper triangular matrices and let $\mathbb{U}$ be its unipotent radical. Note that $\widetilde{T}$ consists of all diagonal matrices and $\widetilde{U} := \mathbb{U}(\widehat{K}^{\mathrm{nr}})$ consists of unipotent upper triangular matrices. Let $\widetilde{U}^- \subset \mathrm{GL}_n(\widehat{K}^{\mathrm{nr}})$ denote the subgroup consisting of unipotent lower triangular matrices.

The Deligne–Lusztig construction X associated to the pair $(\mathrm{GL}_n(\widehat{K}^{\mathrm{nr}}), F_1)$ described in [12] is the quotient

$$X := (\widetilde{U} \cap F_1^{-1}(\widetilde{U}))\backslash\{A \in \widetilde{G} : F_1(A)A^{-1} \in \widetilde{U}\}.$$

The quotient X carries an action of $\mathbb{T}_1(K) \times \mathbb{G}_1(K) \cong L^\times \times D^\times$ induced by the action

$$(t, g) * x := t^{-1}xg, \quad \text{for } t \in \mathbb{T}_1(K),\, g \in \mathbb{G}_1(K), \text{ and } x \in \widetilde{G}.$$

By [2, Corollary 4.3], X can be identified with the set

$$\widetilde{X} := \left\{A \in \widetilde{G} : F_1(A)A^{-1} \in \widetilde{U} \cap F_1(\widetilde{U}^-)\right\},$$

and this choice of section $X \to \widetilde{G}$ respects the $(\mathbb{T}_1(K) \times \mathbb{G}_1(K))$-action. By [2, Lemma 4.4], a matrix $A \in \widetilde{G}$ belongs to $\widetilde{X}$ if and only if it has the form

$$\begin{aligned} A &= x(A_0, \dots, A_{n-1}) \\ &:= \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} \\ \pi^k \varphi(A_{n-1}) & \varphi(A_0) & \varphi(A_1) & \cdots & \varphi(A_{n-2}) \\ \pi^k \varphi^2(A_{n-2}) & \pi^k \varphi^2(A_{n-1}) & \varphi^2(A_0) & \cdots & \varphi^2(A_{n-3}) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \pi^k \varphi^{n-1}(A_1) & \pi^k \varphi^{n-1}(A_2) & \pi^k \varphi^{n-1}(A_3) & \cdots & \varphi^{n-1}(A_0) \end{pmatrix}, \end{aligned} \tag{7.1}$$

where $A_i \in \widehat{K}^{\text{nr}}$ for $0 \leq i \leq n-1$ and $\det(A) \in K^\times$. We remark here that in [2], k is assumed to be 1, but the proofs of [2, Corollary 4.3, Lemma 4.4] work for arbitrary k by simply replacing π with π^k. (In fact, the identification of X with $\widetilde{X}$ and the explicit description in (7.1) hold without our running hypothesis that $(k, n) = 1$.) We may therefore write

$$\widetilde{X} = \bigsqcup_{m \in \mathbb{Z}} \widetilde{X}^{(m)},$$

where $\widetilde{X}^{(m)}$ consists of all $A \in \widetilde{X}$ with $\det(A) \in \pi^m \mathcal{O}_K^\times$. Note that the action of ϖ_k takes each $\widetilde{X}^{(m)}$ isomorphically onto $\widetilde{X}^{(m+k)}$, and the action of π takes each $\widetilde{X}^{(m)}$ isomorphically onto $\widetilde{X}^{(m+n)}$. By assumption, $(k, n) = 1$ and so the $\widetilde{X}^{(m)}$ are all isomorphic. It is therefore sufficient to show that $\widetilde{X}^{(0)}$ can be realized as the $\overline{\mathbb{F}}_q$-points of a scheme. To do this, we use Lemma 7.1, whose proof is very similar to that of [2, Lemma 4.5].

Lemma 7.1 [2, Lemma 4.5] *If a matrix A of the form 7.1 satisfies* $\det(A) \in \mathcal{O}_K^\times$, *then* $A_j \in \pi^{-\lfloor jk/n \rfloor} \widehat{\mathcal{O}}_K^{nr}$ *for* $0 \leq j \leq n-1$ *and* $A_0 \in (\widehat{\mathcal{O}}_K^{nr})^\times$.

Proof Write $A = (a_{ij})$ and let $v_j = \nu(A_j)$ for $0 \leq j \leq n-1$. By definition,

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}.$$

Let $\tau \in S_n$ be the n-cycle given by $(123 \cdots n)$. Note that $\tau^j(i) = [i + j]$ and hence the summand of $\det(A)$ corresponding to τ^j only involves A_j. It is easy to see that

$$\nu\left(\prod a_{i,\tau^j(i)}\right) = n \cdot v_j + j \cdot k, \quad \text{for } 0 \leq j \leq n-1.$$

We now calculate the valuation of the summand corresponding to a fixed $\sigma \in S_n$, where σ is not a power of τ. Set

$$\alpha(i) := \begin{cases} \sigma(i) - i, & \text{if } i \leq \sigma(i), \\ \sigma(i) - i + n, & \text{if } i > \sigma(i), \end{cases} \qquad \beta(i) := \begin{cases} 0, & \text{if } i \leq \sigma(i), \\ k, & \text{if } i > \sigma(i). \end{cases}$$

Then the valuation of the σ-summand is $\sum(v_{\alpha(i)} + \beta(i))$. Since $\sum(\sigma(i) - i) = 0$, we have $\frac{k}{n}\sum\alpha(i) = \sum\beta(i)$, and therefore

$$\nu\left(\prod_{i=1}^n a_{i,\sigma(i)}\right) = \sum_{i=1}^n (v_{\alpha(i)} + \beta(i)) = \frac{1}{n}\sum_{i=1}^n (nv_{\alpha(i)} + k\alpha(i)).$$

Since $(k, n) = 1$, the set $\{nv_j + jk : 0 \leq j \leq n-1\}$ consists of n distinct numbers, and hence

$$\nu(\det(A)) = \min_{0\leq j\leq n-1}\{nv_j + jk\}.$$

By assumption $\nu(\det(A)) = 0$, and this implies that

$$nv_j + jk \geq 0, \quad \text{for } 0 \leq j \leq n-1. \tag{7.2}$$

Conversely, since $(k, n) = 1$, if (7.2) is satisfied, then $\nu(\det(A)) = 0$ only if $v_0 = \nu(A_0) = 0$. □

We have now shown that a matrix of the form (7.1) with determinant in $\mathcal{O}_K^\times$ is of the form

$$A(A_0, A_1, \ldots, A_{n-1}) := x(A_0', A_1', \ldots, A_{n-1}'),$$

for some $A_0 \in (\widehat{\mathcal{O}}_K^{\mathrm{nr}})^\times$ and $A_j \in \widehat{\mathcal{O}}_K^{\mathrm{nr}}$ for $1 \leq j \leq n-1$, where we write

$$A_j' := \pi^{-\lfloor jk/n\rfloor} A_j, \quad \text{for } 0 \leq j \leq n-1.$$

For any integer $h \geq 1$, the set

$$\Big\{A(A_0, A_1, \ldots, A_{n-1}) : A_0 \in (\widehat{\mathcal{O}}_K^{\mathrm{nr}}/\pi^h\widehat{\mathcal{O}}_K^{\mathrm{nr}})^\times, \\ A_j \in \widehat{\mathcal{O}}_K^{\mathrm{nr}}/\pi^{h-1}\widehat{\mathcal{O}}_K^{\mathrm{nr}} \text{ for } 1 \leq j \leq n-1, \\ \det(A(A_0, \ldots, A_{n-1})) \in (\mathcal{O}_K/\pi^h\mathcal{O}_K)^\times\Big\}$$

can be naturally viewed as the set of $\overline{\mathbb{F}}_q$-points of a finite-type scheme $\widetilde{X}_h^{(0)}$ over $\mathbb{F}_q$. If R is an $\mathbb{F}_q$-algebra, then for $h \geq 1$, let $\mathbb{W}_h(R) = R[\![\pi]\!]/(\pi^h)$ if K has positive characteristic and let $\mathbb{W}_h(R)$ be the R-points of the truncated ramified Witt vectors of K if K has characteristic zero. Then determinant of a matrix $A(A_0, A_1 \ldots, A_{n-1})$ for $A_0 \in \mathbb{W}_h(R)^\times$ and $A_1, \ldots, A_{n-1} \in \mathbb{W}_{h-1}(R)$ can be viewed as an element of $\mathbb{W}_h(R)^\times$, and $\widetilde{X}_h^{(0)}$ is then the closed $\mathbb{F}_q$-subscheme of $\mathbb{W}_h^\times \times \mathbb{W}_{h-1}^{n-1}$ defined as the fiber of

$$\mathbb{W}_h^\times \times \mathbb{W}_{h-1}^{n-1} \to \mathbb{W}_h^\times, \quad (A_0, A_1, \ldots, A_{n-1}) \mapsto \frac{\varphi(\det(A(A_0, A_1, \ldots, A_{n-1})))}{\det(A(A_0, A_1, \ldots, A_{n-1}))}$$

over the identity element of $\mathbb{W}_h^\times$. By Lemma 7.1, we have $\widetilde{X}^{(0)} = \varprojlim_h \widetilde{X}_h^{(0)}(\overline{\mathbb{F}}_q)$ and we may define $\widetilde{X}_h^{(m+1)} := \varpi \cdot \widetilde{X}_h^{(m)}$ for all $m \in \mathbb{Z}$ so that $\widetilde{X}^{(m)} = \varprojlim_h \widetilde{X}_h^{(m)}(\overline{\mathbb{F}}_q)$. Thus $\widetilde{X}^{(m)}$ is the set of $\overline{\mathbb{F}}_q$-points of a (pro-)scheme.

Note that $\widetilde{X}_h^{(0)}$ has a left-multiplication action of $\mathcal{O}_L^\times/U_L^h$ and a right-multiplication action of $\mathcal{O}_D^\times/U_D^{n(h-1)+1}$, and these actions are defined over $\mathbb{F}_{q^n}$ and by the following subgroups of $\mathbb{G}_1(K)$:

$$\mathcal{O}_L^\times/U_L^h \cong \left\{A(A_0, 0, \ldots, 0) : A_0 \in (\mathcal{O}_L/\pi^h\mathcal{O}_L)^\times\right\}$$

$$\mathcal{O}_D^\times/U_D^{n(h-1)+1} \cong \left\{A(A_0, A_1, \ldots, A_{n-1}) : \begin{array}{c} A_0 \in (\mathcal{O}_L/\pi^h\mathcal{O}_L)^\times, \\ A_j \in \mathcal{O}_L/\pi^{h-1}\mathcal{O}_L \text{ for } 1 \leq j \leq n-1 \end{array}\right\}.$$

We now define ℓ-adic homology groups of $\widetilde{X}^{(0)}$.

Lemma 7.2 (Boyarchenko [2, Lemma 4.7]) *Set* $W_h := \ker(\mathbb{W}_h(\mathbb{F}_{q^n})^\times \to \mathbb{W}_{h-1}(\mathbb{F}_{q^n})^\times)$ *for* $h \geq 2$. *The action of* W_h *on* $\widetilde{X}_h^{(m)}$ *preserves every fiber of the natural map* $\widetilde{X}_h^{(m)} \to \widetilde{X}_{h-1}^{(m)}$, *the induced morphism* $W_h\backslash\widetilde{X}_h^{(m)} \to \widetilde{X}_{h-1}^{(m)}$ *is smooth, and each of its fibers is isomorphic to the affine space* $\mathbb{A}^{n-1}$ *over* $\overline{\mathbb{F}}_q$.

Proof The proof of [2, Lemma 4.7] is independent of the invariant k/n of the division algebra D once replace the matrix $A_h(a_0, a_1, \ldots, a_{n-1})$ by the matrix $A(a_0, a_1, \ldots, a_{n-1})$ of Eq. (7.1). Note that there is a minor typo in the proof: In 6.11.2, the isomorphism of schemes

$$\mathbb{O}_{K,h-1}^\times \times \mathbb{O}_{K,h-2}^{n-1} \times \mathbb{G}_a^n \to \mathcal{O}_{K,h}^\times \times \mathcal{O}_{K,h-1}^{n-1}$$

should be given by

$$(a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}) \mapsto (\hat{a}_0 + b_0\pi^h, \hat{a}_1 + b_1\pi^{h-1}, \ldots, \hat{a}_{n-1} + b_{n-1}\pi^{h-1}).$$

□

For a smooth scheme S of pure dimension d, set $H_i(S, \overline{\mathbb{Q}}_\ell) := H_c^{2d-i}(S, \overline{\mathbb{Q}}_\ell(d))$. By Lemma 7.2, we have an isomorphism

$$H_i(\widetilde{X}_h^{(m)}, \overline{\mathbb{Q}}_\ell) \to H_i(\widetilde{X}_h^{(m)}, \overline{\mathbb{Q}}_\ell)^{W_h}.$$

In particular, we have a natural embedding $H_i(\widetilde{X}_{h-1}^{(m)}, \overline{\mathbb{Q}}_\ell) \hookrightarrow H_i(\widetilde{X}_h^{(m)}, \overline{\mathbb{Q}}_\ell)$. We define

$$H_i(\widetilde{X}^{(m)}, \overline{\mathbb{Q}}_\ell) := \varinjlim_h H_i(\widetilde{X}_h^{(m)}, \overline{\mathbb{Q}}_\ell), \quad H_i\left(\widetilde{X}, \overline{\mathbb{Q}}_\ell\right) := \bigoplus_{m \in \mathbb{Z}} H_i\left(\widetilde{X}^{(m)}, \overline{\mathbb{Q}}_\ell\right).$$

The vector space $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ inherits commuting smooth actions of $L^\times$ and $D^\times$. Therefore, given a smooth character $\theta\colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$, we may consider the subspace $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \subset H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ wherein $L^\times$ acts by θ. If θ has level h, then $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ is a subspace of $H_i(\widetilde{X}_h, \overline{\mathbb{Q}}_\ell)$, where $\widetilde{X}_h := \bigsqcup_{m \in \mathbb{Z}} \widetilde{X}_h^{(m)}$. One can show that $\widetilde{X}_h$ is equal to the translates of $\widetilde{X}_h^{(0)}$ under the action of $(L^\times/U_L^h) \times (D^\times/U_D^{n(h-1)+1})$. It therefore follows that if $\widetilde{\Gamma}_h$ is the stabilizer of $\widetilde{X}_h^{(0)}$, then

$$H_i(\widetilde{X}_h, \overline{\mathbb{Q}}_\ell) \cong \operatorname{Ind}_{\widetilde{\Gamma}_h}^{(L^\times/U_L^h)\times\left(D^\times/U_D^{n(h-1)+1}\right)} \left(H_i(\widetilde{X}_h^{(0)}, \overline{\mathbb{Q}}_\ell)\right).$$

This type of argument is crucial in the proof of Theorem 7.8.

7.1.1 Boyarchenko's conjectures

Strictly speaking, [2, Conjectures 5.16 and 5.18] require D to be a division algebra of invariant $1/n$ over a non-Archimedean local field K of positive characteristic. In this section, we formulate extensions of Boyarchenko's conjectures for any division algebra D of dimension n^2 over any non-Archimedean local field K with residue field $\mathbb{F}_q$.

The morphism $f : \mathbb{G}_1 \to \mathbb{G}_2$ given by $g \mapsto \gamma^{-1} \cdot g \cdot \gamma$ is injective. Set

$$\widetilde{X}_h'^{(0)} := f\left(\widetilde{X}_h^{(0)}\right)$$

so that if we write $A'(A_0, \ldots, A_{n-1}) := \gamma^{-1} \cdot A(A_0, \ldots, A_{n-1}) \cdot \gamma$, then

$$\widetilde{X}_h'^{(0)}(\overline{\mathbb{F}}_q) = \Big\{A'(A_0, \ldots, A_{n-1}) : A_0 \in \left(\widehat{\mathcal{O}}_K^{\mathrm{nr}}/\pi^h \widehat{\mathcal{O}}_K^{\mathrm{nr}}\right)^\times, \\ A_j \in \widehat{\mathcal{O}}_K^{\mathrm{nr}}/\pi^{h-1}\widehat{\mathcal{O}}_K^{\mathrm{nr}} \text{ for } 1 \le j \le n-1, \\ \det(A'(A_0, \ldots, A_{n-1})) \in \left(\mathcal{O}_K/\pi^h\mathcal{O}_K\right)^\times\Big\}.$$

The group $(\mathcal{O}_L^\times/U_L^h) \times (\mathcal{O}_D^\times/U_D^{n(h-1)+1})$ acts on $\widetilde{X}_h'^{(0)}$ via f. Hence we obtain the lemma:

Lemma 7.3 *For all $i \ge 0$, as representations of $\mathcal{O}_L^\times/U_L^h \times \mathcal{O}_D^\times/U_D^{n(h-1)+1}$,*

$$H_c^i\left(\widetilde{X}_h^{(0)}, \overline{\mathbb{Q}}_\ell\right) \cong H_c^i\left(\widetilde{X}_h'^{(0)}, \overline{\mathbb{Q}}_\ell\right).$$

For any $\mathbb{F}_{q^n}$-algebra R, define

$$X_h(R) := \left\{A'(A_0, \ldots, A_{n-1}) \in \widetilde{X}_h^{(0)} : A_0 \in \mathbb{W}_h^{(1)}(R)\right\}, \tag{7.3}$$

where if $V : \mathbb{W}_{h-1} \to \mathbb{W}_{h-1}$ is the Verschiebung morphism, then $\mathbb{W}_h^{(1)} := 1 + V\mathbb{W}_{h-1} \subset \mathbb{W}_h^\times$. We remark that we have abused notation here in the sense that when K has positive characteristic, the X_h defined in Eq. (7.3) is the image of the X_h defined in Definition 2.5 under $\iota_{h,k}$. Since the definition of $U_{h,k}^{n,q}$ is not available when K has characteristic 0 and Boyarchenko's conjectures can be formulated without $U_{h,k}^{n,q}$, we choose to proceed as in Eq. (7.3).

Let Γ_h denote the stabilizer of X_h in $\mathcal{O}_L^\times/U_L^h \times \mathcal{O}_D^\times/U_D^{n(h-1)+1}$. Then by Lemma 7.3,

$$H_c^i(\widetilde{X}_h^{(0)}, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Ind}_{\Gamma_h}^{\mathcal{O}_L^\times/U_L^h \times \mathcal{O}_D^\times/U_D^{n(h-1)+1}} \left(H_c^i\left(X_h, \overline{\mathbb{Q}}_\ell\right)\right).$$

Boyarchenko's conjectures concern the cohomology groups $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ as representations of $U_L^1/U_L^h \times U_D^1/U_D^{n(h-1)+1} \subset \Gamma_h$.

Conjecture 7.4 (Boyarchenko [2, Conjecture 5.16]) *For* $i \geq 0$*, we have* $H_c^i(X_h, \overline{\mathbb{Q}}_\ell) = 0$ *unless* i *or* n *is even, and* Fr_{q^n} *acts on* $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ *by the scalar* $(-1)^i q^{ni/2}$.

Conjecture 7.5 (Boyarchenko [2, Conjecture 5.18]) *Given a character* $\chi : U_L^1/U_L^h \to \overline{\mathbb{Q}}_\ell$*, there exists* $r \geq 0$ *such that* $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi] = 0$ *for all* $i \neq r$*. Moreover,* $H_c^r(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ *is an irreducible representation of* $U_D^1/U_D^{n(h-1)+1}$.

Remark 7.6 It is useful to have an explicit formula for $A'(A_0, \ldots, A_{n-1})$. First observe that

$$A(A_0, \ldots, A_{n-1}) = D(A_0') + D(A_1')\varpi_k + \cdots + D(A_{n-1}')\varpi_k^{n-1},$$

where we write $D(x) = \mathrm{diag}(x, \varphi(x), \ldots, \varphi^{n-1}(x))$. Let γ_0 be the permutation matrix corresponding to the permutation $i \mapsto [(i + l - 1)k]$. Then

$$\gamma_0^{-1} \cdot c \cdot \gamma_0 = c^k, \quad \text{where } c = \begin{pmatrix} 0 & 1_{n-1} \\ 1 & 0 \end{pmatrix}.$$

This implies that

$$\gamma_0^{-1} \cdot D(x) \cdot \gamma_0 = \mathrm{diag}\left(x, \varphi^{[l]}(x), \ldots, \varphi^{[(n-1)l]}(x)\right) =: D'(x).$$

Therefore

$$\begin{aligned} A'(A_0, \ldots, A_{n-1}) &= D'(A_0') + D'(A_1')\varpi^k + \cdots + D'(A_{n-1}')\varpi^{(n-1)k} \\ &= D'(A_0) + D'(A_1)\varpi^{[k]} + \cdots + D'(A_{n-1})\varpi^{[(n-1)k]}, \end{aligned}$$

which, when expanded, is

$$\begin{pmatrix} A_0 & A_{[l]} & A_{[2l]} & \cdots & A_{[(n-1)l]} \\ \pi\varphi^{[l]}(A_{[(n-1)l]}) & \varphi^{[l]}(A_0) & \varphi^{[l]}(A_{[l]}) & \cdots & \varphi^{[l]}(A_{[(n-2)l]}) \\ \pi\varphi^{[2l]}(A_{[(n-2)l]}) & \pi\varphi^{[2l]}(A_{[(n-1)l]}) & \varphi^{[2l]}(A_0) & \cdots & \varphi^{[2l]}(A_{[(n-3)l]}) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \pi\varphi^{[(n-1)l]}(A_{[l]}) & \pi\varphi^{[(n-1)l]}(A_{[2l]}) & \cdots & \pi\varphi^{[(n-1)l]}(A_{[(n-1)l]}) & \varphi^{[(n-1)l]}(A_0) \end{pmatrix}.$$

Observe that when K has positive characteristic, after appropriately permuting the a_i's, the point $x(\sum a_{ni}\pi^i, \sum a_{ni+1}\pi^i, \ldots, \sum a_{ni+(n-1)}\pi^i)$ is $\iota_{h,k}(\sum a_i\tau^i)$ as defined in Eq. (2.3). ♢

From now on, assume char $K > 0$. Then Proposition 6.1 gives evidence supporting Conjecture 7.4, and by Theorem 6.4, we have:

Theorem 7.7 *Let $\chi\colon U_L^1/U_L^h \to \overline{\mathbb{Q}}_\ell^\times$ be a primitive character. Then Conjecture 7.5 holds.*

By [2, Proposition 5.19], we have

Theorem 7.8 *Let $\theta\colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a primitive character of level h and let $\chi\colon U_L^1/U_L^h \to \overline{\mathbb{Q}}_\ell^\times$ denote the restriction of θ to U_L^1.*

(a) *Pick any $\zeta \in \mathcal{O}_L^\times/U_L^{h-1}$ with the property that its image in $\mathbb{F}_{q^n}^\times$ generates $\mathbb{F}_{q^n}^\times$. The representation $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ extends uniquely to a representation η_θ° of the semidirect product $\mathcal{R}_{h,k,n,q}^\times(\mathbb{F}_{q^n}) \cong \mathcal{O}_D^\times/U_D^{n(h-1)+1}$ with $\mathrm{Tr}(\eta_\theta^\circ(\zeta)) = (-1)^{(n-1)(h-1)}\theta(\zeta)$.*

(b) *The inflation $\widetilde{\eta}_\theta^\circ$ of η_θ° to $\mathcal{O}_D^\times$ extends to a representation η_θ' of $\pi^{\mathbb{Z}} \cdot \mathcal{O}_D^\times$ by setting $\eta_\theta'(\pi) = \theta(\pi)$. Then*

$$H_{(n-1)(h-1)}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \cong \eta_\theta := \mathrm{Ind}_{\pi^{\mathbb{Z}}\cdot\mathcal{O}_D^\times}^{D^\times}(\eta_\theta')$$

and $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] = 0$ for $i \neq (n-1)(h-1)$.

(c) *$H_{(n-1)(h-1)}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ is an irreducible representation of dimension $n \cdot q^{n(n-1)(h-1)/2}$.*

Proof We outline the proof given in [2, Section 6.15].

The uniqueness in (a) follows from the irreducibility of $H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$. The representation η_θ° is the tensor product $\theta^\circ \otimes H_c^{(n-1)(h-1)}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ where $\theta^\circ(z, g) = \theta(z)$ for $(z, g) \in \langle\zeta\rangle \ltimes U_{h,k}^{n,q}(\mathbb{F}_{q^n}) = \mathcal{R}_{h,k,n,q}^\times(\mathbb{F}_{q^n})$. Finally, the trace identity is a special case of Proposition 6.2.

Let $\widetilde{X}_h := \bigsqcup_{m\in\mathbb{Z}} \widetilde{X}_h^{(m)}$. The action of $L^\times \times D^\times$ on $\widetilde{X}$ induces an action of the quotient $G_h := (L^\times/U_L^h) \times (D^\times/U_D^{n(h-1)+1})$ on $\widetilde{X}_h$. Moreover, $H_*(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \subset H_*(\widetilde{X}_h, \overline{\mathbb{Q}}_\ell)$, so it is enough to understand the cohomology of $\widetilde{X}_h$. Since $\widetilde{X}_h^{(m+1)} = \varpi\widetilde{X}_h^{(m)}$ and $\varpi \in \mathbb{G}_1(K) \cong D^\times$, we see that $\widetilde{X}_h$ is equal to the G_h-translates of $f(\iota_{h,k}(X_h)) \subset \widetilde{X}_h^{(0)}$. One can define an action of

$$\Gamma_h = \langle(\pi, \pi^{-1})\rangle \cdot \langle(\zeta, \zeta^{-1})\rangle \cdot \left(U_L^1/U_L^h \times U_D^1/U_D^{n(h-1)+1}\right) \subset G_h$$

on X_h so that $f \circ \iota_{h,k}$ is Γ_h-equivariant. Moreover, the stabilizer of $f(\iota_{h,k}(X_h))$ in G_h is exactly equal to Γ_h. The claim in (b) then follows from an analysis of the θ-eigenspace of $\mathrm{Ind}_{\Gamma_h}^{G_h}\left(H_i(X_h, \overline{\mathbb{Q}}_\ell)\right)$.

Let $\psi := \theta|_{U_L^{h-1}}$. For any $x \in U_L^{h-1} \subset U_D^{n(h-1)}$, we have $\eta_\theta'(x) = \psi(x)$ and

$$\eta_\theta'(\Pi \cdot x \cdot \Pi^{-1}) = \eta_\theta'(\varphi(x)) = \psi(x^{q^l}).$$

Since θ is primitive and l is coprime to n, it follows that the normalizer of η_θ' in $D^\times$ is equal to $\pi^{\mathbb{Z}} \cdot \mathcal{O}_D^\times$. Irreducibility then follows by Mackey's criterion. The dimension of the η_θ is equal to the product of $[D^\times : \pi^{\mathbb{Z}} \cdot \mathcal{O}_D^\times] = n$ and the dimension of η'_θ, so the desired result holds by Theorem 4.3 and Proposition 6.1. □

7.2 Local Langlands correspondences

It is known that automorphic induction is not compatible with induction on Weil groups in the sense that one must often keep track of a rectifying character when constructing the Langlands parameter σ_θ of the automorphic induction of a character $\theta \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$. Instead, we recall Langlands–Shelstad's theory of χ-datum [11, Section 2.5] to give a canonical construction of $\sigma_\theta \colon \mathcal{W}_K \to \mathrm{GL}_n(\mathbb{C})$. We then recall the statements of the local Langlands and Jacquet–Langlands correspondences and prove in Theorem 7.12 that the homology of X realizes the composition of these two correspondences. As always in this paper, L is the degree-n unramified extension of K.

Fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Let $T = \mathrm{Res}_{L/K}\,\mathbb{G}_m$ and let $G = \mathrm{GL}_n$. Viewing L as an n-dimensional K-vector space induces an embedding $T \hookrightarrow G$. Let $\Phi = \Phi(G, T)$ be the root system of T in G and recall that there is a natural action of the absolute Galois group Γ_K on Φ. For each $\lambda \in \Phi$, let L_λ and $L_{\pm\lambda}$ be the extensions of K corresponding to the subgroups $\{g \in \Gamma_K : {}^g\lambda = \lambda\}$ and $\{g \in \Gamma_K : {}^g\lambda = \pm\lambda\}$. We note that since L/K is unramified, it is Galois and hence $L_\lambda = L$. (In general, one only has $L_\lambda \supseteq L$.) We say that $\lambda \in \Phi$ is *symmetric* if $L_\lambda \neq L_{\pm\lambda}$ and *asymmetric* otherwise. Observe that if λ is symmetric, then L_λ is a quadratic extension of $L_{\pm\lambda}$.

Definition 7.9 A χ-*datum* is a collection of characters $\{\chi_\lambda\}_{\lambda\in\Phi}$ satisfying:

(i) $\chi_\lambda \colon L_\lambda^\times \to \mathbb{C}^\times$ is a homomorphism.
(ii) For each $\lambda \in \Phi$, we have $\chi_{-\lambda} = \chi_\lambda^{-1}$ and $\chi_{{}^w\lambda} = {}^w\chi_\lambda$ for all $w \in \mathcal{W}_K$.
(iii) If λ is symmetric, then $\chi_\lambda|_{L_{\pm\lambda}^\times}$ equals the quadratic character of $L_\lambda/L_{\pm\lambda}$.

Consider the dual groups of G and T given by $\widehat{G} := \mathrm{GL}_n(\mathbb{C})$ and $\widehat{T} := (\mathbb{C}^\times)^n$. A χ-datum $\{\chi_\lambda\}_{\lambda\in\Phi}$ determines an embedding $\chi \colon {}^LT \to {}^LG$, where ${}^LT$ and ${}^LG$ are the L-groups ${}^LG := \widehat{G} \rtimes \mathcal{W}_K$ and ${}^LT := \widehat{T} \rtimes \mathcal{W}_K$ (see [11, Section 2.6]). The local Langlands correspondence for T gives a natural isomorphism

$$\mathrm{Hom}(L^\times, \mathbb{C}^\times) \cong H^1(\mathcal{W}_K, \widehat{T}).$$

Let $\widetilde{\theta} \colon \mathcal{W}_K \to {}^LT$ be a 1-cocycle representing the image of θ under the above isomorphism. Then by [13, Proposition 6.5], the representation given by the composition

$$\mathcal{W}_K \xrightarrow{\widetilde{\theta}} {}^LT \xrightarrow{\chi} {}^LG \xrightarrow{\mathrm{pr}} \mathrm{GL}_n(\mathbb{C}) \tag{7.4}$$

is isomorphic to the induced representation

$$\mathrm{Ind}_{\mathcal{W}_L}^{\mathcal{W}_K}(\theta \cdot \mu), \quad \text{where } \mu = \prod_{[\lambda]\in\mathcal{W}_K\backslash\Phi} \chi_\lambda, \tag{7.5}$$

and where we view $\theta \cdot \mu$ as a character of $\mathcal{W}_L$ via local class field theory. Since L/K is unramified, it is easy to write down a natural choice of χ-datum. It is clear from the definition that a χ-datum $\{\chi_\lambda\}_{\lambda \in \Phi}$ is determined by $\{\chi_\lambda\}_{\lambda \in A}$, where A is any choice of coset representatives of $\Gamma_K \backslash \Phi$. The Γ_K-orbits of Φ are in bijection with the nontrivial double cosets of Γ_L in Γ_K (see [13, Proposition 3.1]), and we may write

$$(\Gamma_L \backslash \Gamma_K / \Gamma_L)' = \{[\phi^i] := \Gamma_L \phi^i \Gamma_L : 1 \leq i \leq n-1\},$$

where ϕ is the q-power Frobenius. By [13, Proposition 3.3], $[\phi^i]$ is symmetric if and only if n is even and $i = n/2$. It is clear that the following specifies a χ-datum for $T \hookrightarrow G$:

(i) If $[\phi^i]$ is symmetric, we let $\chi_{[\phi^i]}$ be the unramified character with $\chi_{[\phi^i]}(\pi) = -1$.
(ii) If $[\phi^i]$ is asymmetric, we let $\chi_{[\phi^i]} \equiv 1$.

Define σ_θ to be the $\mathcal{W}_K$-representation in Eq. (7.4) corresponding to the above canonical choice of χ-datum. Then by Eq. (7.5),

$$\sigma_\theta \cong \operatorname{Ind}_{\mathcal{W}_L}^{\mathcal{W}_K}(\theta \cdot \mu), \tag{7.6}$$

where $\mu \colon L^\times \to \mathbb{C}^\times$ is the character determined by $\mu|_{\mathcal{O}_L^\times} \equiv 1$ and $\mu(\pi) = (-1)^{n-1}$.

Pick any division algebra D of dimension n^2 over K. We now describe the relevant correspondences between representations of $L^\times$, $\mathcal{W}_K$, $\mathrm{GL}_n(K)$, and $D^\times$. Fix a character ϵ of $K^\times$ whose kernel is equal to the image of the norm $\mathrm{N}_{L/K} \colon L^\times \to K^\times$. Let $\mathscr{X}$ denote the set of all characters of $L^\times$ that have trivial stabilizer in $\mathrm{Gal}(L/K)$ and let $\mathcal{G}_K^\epsilon(n)$ denote the set of (isomorphism classes of) smooth irreducible n-dimensional representations σ of $\mathcal{W}_K$ that satisfy $\sigma \cong \sigma \otimes (\epsilon \circ \mathrm{rec}_F)$. Then

$$\begin{array}{ccc} \mathscr{X}/\mathrm{Gal}(L/K) & \xrightarrow{\chi\text{-datum}} & \mathcal{G}_K^\epsilon(n) \\ \theta & \longmapsto & \sigma_\theta \end{array}$$

is a bijection.

Now let $\mathcal{A}_K^\epsilon(n)$ denote the set of (isomorphism classes of) irreducible supercuspidal representations π of $\mathrm{GL}_n(K)$ such that $\pi \cong \pi \otimes (\epsilon \circ \det)$. There exists a canonical bijection

$$\begin{array}{ccc} \mathcal{G}_K^\epsilon(n) & \xrightarrow{\text{LLC}} & \mathcal{A}_K^\epsilon(n) \\ \sigma_\theta & \longmapsto & \pi_\theta \end{array}$$

known as the local Langlands correspondence.

Finally, let $\mathcal{A}_K'^\epsilon(n)$ denote the set of (isomorphism classes of) irreducible representations ρ of $D^\times$ such that $\rho \cong \rho \otimes (\epsilon \circ \mathrm{Nrd}_{D/K})$. Then the Jacquet–Langlands correspondence gives a bijection

$$\begin{array}{ccc} \mathcal{A}_K^\epsilon(n) & \xrightarrow{\text{JLC}} & \mathcal{A}_K'^\epsilon(n) \\ \pi_\theta & \longmapsto & \rho_\theta \end{array}$$

Remark 7.10 Since L/K is unramified, the restriction of ϵ to $\mathcal{O}_K^\times$ is trivial, and thus the composition $\epsilon \circ \mathrm{Nrd}_{D/K}$ is trivial on $E^\times \cdot \mathcal{O}_D^\times \supset \pi^{\mathbb{Z}} \cdot \mathcal{O}_D^\times$. Thus by the construction of η_θ, we have that η_θ is invariant under twisting by $\epsilon \circ \mathrm{Nrd}_{D/K}$. ♢

Theorem 7.8 describes a correspondence between $L^\times$-characters and $D^\times$-representations given by

$$\begin{array}{ccc} \{\text{primitive characters of } L^\times\} & \xrightarrow{\text{DL construction}} & \{\text{irreducible representations of } D^\times\} \\ \theta & \longmapsto & \eta_\theta := H_\bullet(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \end{array}$$

By Remark 7.10, we see that $\eta_\theta \in \mathcal{A}_K'^{\epsilon}(n)$. In Theorem 7.12, we prove that this correspondence matches the composition of the previous three, therefore giving a geometric realization of the Jacquet–Langlands correspondence.

Remark 7.11 The construction of the local Langlands and Jacquet–Langlands correspondences was already known. See, for example, [9]. Recent work of Boyarchenko and Weinstein (see [3]) gives a partially geometric construction of these correspondences using the representations $H_c^{n-1}(X_2, \overline{\mathbb{Q}}_\ell)[\psi]$ of $U_2^{n,q}(\mathbb{F}_{q^n})$. Note that in [4] and [3], the scheme X_2 is denoted by X and the group $U_2^{n,q}(\mathbb{F}_{q^n})$ is denoted by $U^{n,q}$ $(\mathbb{F}_{q^n})$. ♢

Theorem 7.12 *Let $\theta : L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a primitive character of level h and let ρ_θ be the $D^\times$-representation corresponding to θ under the local Langlands and Jacquet–Langlands correspondences. Then $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] = 0$ if $i \neq (n-1)(h-1)$ and*

$$H_{(n-1)(h-1)}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \cong \rho_\theta.$$

Proof By Eq. (7.6) and [3, Proposition 1.5(b)], we just need to show that $\eta_\theta := H_{(n-1)(h-1)}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ satisfies the following two properties:

(i) For any character ϵ of $K^\times$ whose kernel is equal to the image of the norm map $\mathrm{N}_{L/K} : L^\times \to K^\times$, we have $\eta_\theta \cong \eta_\theta \otimes (\epsilon \circ \mathrm{Nrd}_{D/K})$.

(ii) There exists a constant c such that $\operatorname{tr} \eta_\theta(x) = c \cdot \sum_{\gamma \in \mathrm{Gal}(L/K)} \theta^\gamma(x)$ for each very regular element $x \in \mathcal{O}_L^\times$.

Since L/K is unramified, the restriction of ϵ to $\mathcal{O}_K^\times$ is trivial, and thus the composition $\epsilon \circ \mathrm{Nrd}_{D/K}$ is trivial on $L^\times \cdot \mathcal{O}_D^\times \supset \pi^{\mathbb{Z}} \cdot \mathcal{O}_D^\times$. Thus by construction, η_θ is invariant under twisting by $\eta \circ \mathrm{Nrd}_{D/K}$. This proves (i).

We now prove (ii). By the construction of η_θ, since $\pi^{\mathbb{Z}} \cdot \mathcal{O}_D = L^\times \cdot U_D^1$, we have

$$\operatorname{tr} \eta_\theta(x) = \sum_{\substack{g \in D^\times / L^\times \cdot U_D^1 \\ gxg^{-1} \in L^\times \cdot U_D^1}} \operatorname{tr} \eta'_\theta(gxg^{-1}).$$

Now let $x \in \mathcal{O}_L^\times$ be very regular. By Proposition 6.2, $\eta_\theta^\circ(x) = (-1)^{(n-1)(h-1)}\theta(x)$. By [3, Lemma 5.1(b)], if $g \in D^\times$ is such that $gxg^{-1} \in L^\times \cdot U_D^1$, then $g \in N_{D^\times}(L^\times) \cdot U_D^1$, where $N_D^\times(L^\times)$ is the normalizer of $L^\times$ in $D^\times$. Therefore

$$\begin{aligned} \operatorname{tr} \eta_\theta(x) &= \sum_{g \in N_{D^\times}(L^\times) \cdot U_D^1 / L^\times \cdot U_D^1} \operatorname{tr} \eta'_\theta(gxg^{-1}) = \sum_g \operatorname{tr}(\eta_\theta^\circ(gxg^{-1})) \\ &= \sum_g (-1)^{(n-1)(h-1)} \theta(gxg^{-1}) = (-1)^{(n-1)(h-1)} \cdot \sum_{\gamma \in \mathrm{Gal}(L/K)} \theta^\gamma(x). \end{aligned}$$

□

The following corollary shows that the homology of Deligne–Lusztig constructions for division algebras gives a geometric realization of the Jacquet–Langlands correspondence.

Theorem 7.13 *Let D and D' be division algebras of rank n and let X_D and $X_{D'}$ be their corresponding Deligne–Lusztig constructions. For any primitive character $\theta \colon L \to \overline{\mathbb{Q}}_\ell^\times$ of level h, the Jacquet–Langlands transfer of $H_{(n-1)(h-1)}(X_D, \overline{\mathbb{Q}}_\ell)[\theta]$ is isomorphic to $H_{(n-1)(h-1)}(X_{D'}, \overline{\mathbb{Q}}_\ell)[\theta]$.*

Acknowledgements I am deeply grateful to Mitya Boyarchenko for introducing me to this area of research. I'd also like to thank Stephen DeBacker, Tasho Kaletha, Jake Levinson, David Speyer, Kam-Fai Tam, and Jared Weinstein for helpful conversations. Finally, I'd like to thank the referees for numerous helpful comments on both the exposition and the mathematics. This work was partially supported by NSF Grants DMS-0943832 and DMS-1160720.

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