Sel. Math. New Ser. (2018) 24:591–608
https://doi.org/10.1007/s00029-017-0370-2

**Selecta Mathematica**
**New Series**

# On Beilinson's equivalence for $p$-adic cohomology

**Tomoyuki Abe**[1] · **Daniel Caro**[2]

Published online: 23 November 2017


**Abstract** In this short paper, we construct a unipotent nearby cycle functor and show a $p$-adic analogue of Beilinson's equivalence comparing two derived categories: the derived category of holonomic arithmetic $\mathcal{D}$-modules and the derived category of arithmetic $\mathcal{D}$-modules whose cohomologies are holonomic.

**Mathematics Subject Classification** 14F10 · 14F30

## Introduction

In the theory of $p$-adic cohomology, the lack of a nearby cycle functor has been a big technical obstruction for proving important results. For example, [4,16] are few of such examples. In this short paper, we establish the theory of unipotent nearby cycle functor, and as an application, we prove a $p$-adic analogue of Beilinson's equivalence: for a smooth variety $X$ over $\mathbb{C}$, we have an equivalence of categories (see [5])

$$D^{\mathrm{b}}(\mathrm{Hol}(X)) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{hol}}(X).$$

✉ Daniel Caro
daniel.caro@unicaen.fr

Tomoyuki Abe
tomoyuki.abe@ipmu.jp

1 Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan

2 Laboratoire de Mathématiques Nicolas Oresme (LMNO), Université de Caen, Campus 2, 14032 Caen Cedex, France

Birkhäuser

For the construction of the unipotent nearby cycle functor, we follow the idea of [6]. The original construction of Beilinson's unipotent nearby cycles in the context of algebraic $\mathcal{D}$-modules is based on a key lemma whose proof is a consequence of the existence of $b$-functions. However, in our $p$-adic context, the definition of $b$-functions is problematic. To remedy this, we can use successfully another powerful tool, namely, Kedlaya's semistable reduction theorem, applied to overconvergent isocrystals with Frobenius structure.

Now, even though the proof of Beilinson's equivalence is written in a way that it can be adopted for many cohomology theories, we still need to figure out what the suitable definition of "holonomic modules" is in the $p$-adic context. A naive answer might be to consider overholonomic complexes (without Frobenius structure), namely $\mathrm{Ovhol}(X/K)$ in 1.4, introduced by the second author. However, we do not know if this category is closed under taking tensor products when modules do not admit Frobenius structure. Thus, the category does not seem appropriate for the equivalence because Beilinson's original proof uses the stability under Grothendieck six operations. Moreover, the full subcategory of overholonomic modules whose objects are endowed with some Frobenius structure is not thick. To resolve these issues, in this paper, we construct some kind of smallest triangulated subcategory of the category of overholonomic complexes which contains modules with Frobenius structure. Its construction allows us to come down by "devissage" to the case of modules with Frobenius structure.

Finally, we point out that techniques developed in this paper are crucial tools to construct the theory of arithmetic $\mathcal{D}$-modules for general schemes in [2], and we also expect more applications: unification of the rigid cohomology theory into arithmetic $\mathcal{D}$-modules (cf. [2, 1.3.11]), $p$-adic analogue of Fujiwara's trace formula, *etc.*.

The first section is devoted to construct the good triangulated category, and the unipotent nearby cycle functors is treated in the second section. Finally, as an application, in the third section we give a comparison of Euler characteristics as Laumon in $l$-adic cohomology in [25] with the remark that the use of unipotent nearby cycles theory is enough for the proof.

In this paper, we fix a complete discrete valuation ring $R$ of mixed characteristic. Its residue field is denoted by $k$, and we assume it to be perfect and of characteristic $p$. Let $q$ be a power of $p$, and we suppose that there exists a lifting $R \xrightarrow{\sim} R$ of the $q$-Frobenius automorphism of $k$, and fix one. We put $K := \mathrm{Frac}(R)$. If there is no ambiguity with $K$, we sometimes omit "$/K$" in the notation of some categories.

# 1 Overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-modules

## 1.1 On the stability under base change

Let $\mathcal{P}$ be a smooth formal scheme over $R$. Let $\mathcal{E}$ be an object of $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$, *i.e.* an overholonomic complex of $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-modules (see Definition [13, 3.1]). For the reader, we recall (see [18, Definition 3.2.1] or maybe also [20, 1.3.3]) the complex $\mathcal{E}$ is said to be *overholonomic after any base change* if for any morphism $R \to R'$ of complete discrete valuation rings of unequal characteristic with perfect residue fields, putting $\mathcal{S} := \mathrm{Spf}(R)$, $\mathcal{S}' := \mathrm{Spf}(R')$, $f : \mathcal{P}' := \mathcal{P} \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{P}$ the canonical morphism, then the object $f^*(\mathcal{E}) := \mathcal{D}^\dagger_{\mathcal{P}'/\mathcal{S}',\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathcal{P}/\mathcal{S},\mathbb{Q}}} f^{-1}\mathcal{E}$ remains to be an overholonomic complex of $\mathcal{D}^\dagger_{\mathcal{P}'/\mathcal{S}',\mathbb{Q}}$-modules. When $\mathcal{E}$ is a module, we say that $\mathcal{E}$ is an overholonomic after any base change $\mathcal{D}^\dagger_{\mathcal{P}'/\mathcal{S}',\mathbb{Q}}$-module.

We remark that the base change functor $f^*$ is exact, commutes with push-forwards, pull-backs, dual functors, local cohomological functors and preserves the coherence and the holonomicity (use Virrion's characterization of the holonomicity of [27, III.4]). For instance, if $Y$ is a subvariety of the special fiber of $\mathcal{P}$ and $Y' := f^{-1}(Y)$, for any overholonomic complex $\mathcal{E}$ of $\mathcal{D}^\dagger_{\mathcal{P}/\mathcal{S},\mathbb{Q}}$-modules, we get the isomorphism of coherent complexes $\mathbb{R}\underline{\Gamma}^\dagger_{Y'}(f^*\mathcal{E}) \xrightarrow{\sim} f^*\mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E})$ of $\mathcal{D}^\dagger_{\mathcal{P}'/\mathcal{S}',\mathbb{Q}}$-modules.

**Lemma 1.2** *Let $\mathcal{P}$ be a proper smooth formal scheme over $R$, and $\mathcal{E}$ be an object of $F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}})$, i.e. an overholonomic complex of $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}$-modules endowed with Frobenius structure. Then $\mathcal{E}$ is overholonomic after any base change.*

*Proof* Let $R \to R'$ be a morphism of complete discrete valuation rings of unequal characteristic with perfect residue fields, $\mathcal{S} := \mathrm{Spf}(R)$, $\mathcal{S}' := \mathrm{Spf}(R')$, $f : \mathcal{P}' := \mathcal{P} \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{P}$ be the canonical morphism. We have to prove that $f^*(\mathcal{E})$ is overholonomic. By devissage, we can suppose that there exists a quasi-projective subvariety $Y$ of the special fiber of $\mathcal{P}$ such that $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(Y, \mathcal{P})$, *i.e.* by definition of this category such that $\mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$. There exists an immersion of the form $Y \hookrightarrow \mathcal{Q}$, where $\mathcal{Q}$ is a projective formal scheme over $R$. We get an immersion $Y \hookrightarrow \mathcal{P} \times \mathcal{Q}$ and two projections $p_1 : \mathcal{P} \times \mathcal{Q} \to \mathcal{P}$, $p_2 : \mathcal{P} \times \mathcal{Q} \to \mathcal{Q}$. We recall that the categories $F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(Y, \mathcal{P})$ and $F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(Y, \mathcal{Q})$ are canonically equivalent. Let $\mathcal{F}$ be the object of $F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(Y, \mathcal{Q})$ corresponding to $\mathcal{E}$, *i.e.* $\mathcal{E} \xrightarrow{\sim} p_{1+}\mathbb{R}\underline{\Gamma}^\dagger_Y p_2^!(\mathcal{F})$. Let $Y'$, $\mathcal{Q}'$, $p_1'$, $p_2'$ be the base change of $Y$, $\mathcal{Q}$, $p_1$, $p_2$ by $f$. The complex $f^*(\mathcal{F})$ is endowed with a Frobenius structure by using [10, 2.1.6] and is holonomic because $f^*$ preserves the holonomicity. Therefore $f^*(\mathcal{F})$ is overholonomic by [15] since $\mathcal{Q}'$ is projective. Since $f^*(\mathcal{E}) \xrightarrow{\sim} p'_{1+}\mathbb{R}\underline{\Gamma}^\dagger_{Y'} p_2'^!(f^*\mathcal{F})$, the stability of overholonomicity implies that $f^*(\mathcal{E})$ is also overholonomic. □

**Lemma 1.3** *Let $\mathcal{P}$ be a smooth formal scheme over $R$. The category consisting of overholonomic after any base change $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}$-modules is a thick abelian subcategory of* $\mathrm{Mod}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}})$.

*Proof* Since the other properties are easy, we will only prove the stability under kernels and cokernels. Let $\phi$ be a homomorphism of overholonomic after base change $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}$-modules. Then these are holonomic by [14, 4.3]. Thus the kernel and cokernel of $\phi$ are holonomic by [*ibid.*, 2.14]. Since the functor $\mathbb{D}$ is exact on the category of holonomic modules, we get the overholonomicity of the kernel and cokernel of $\phi$, and then their overholonomicity after any base change. □

**1.4** A variety (*i.e.* a reduced scheme of finite type over $k$) $X$ is said to be *realizable* if there exists a smooth proper formal scheme $\mathcal{P}$ over $R$ such that $X$ can be embedded into $\mathcal{P}$. Since the cohomology theory does not change if we take the associated reduced scheme, in the following, we assume that schemes are always reduced. For any realizable variety $X$, choose $X \hookrightarrow \mathcal{P}$ an immersion with $\mathcal{P}$ a smooth proper formal scheme over $R$. By replacing "overholonomic" by "overholonomic after any base change" in [13, 4.16], the full subcategory of $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$ consisting of overholonomic after any base change $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-complexes $\mathcal{E}$ which are supported on the closure $\overline{X}$ of $X$ in $\mathcal{P}$ and which satisfy $\mathbb{R}\underline{\Gamma}^{\dagger}_{\overline{X}\setminus X}(\mathcal{E}) = 0$ does not depend on the choice of $\mathcal{P}$ and of the embedding of $X$ in $\mathcal{P}$. Hence, the objects of this category will be called "overholonomic after any base change complexes over $X/K$ (or simply $X$)". This category is denoted by $D^{\mathrm{b}}_{\mathrm{ovhol,bc}}(X/K)$

Now, for $\star \in \{\geq 0, \leq 0\}$, we define a full subcategory $D^{\star} \subset D^{\mathrm{b}}_{\mathrm{ovhol,bc}}(X/K)$ in a following way: We take an embedding $X \hookrightarrow \mathcal{P}$ as above. Let $\mathcal{U} \subset \mathcal{P}$ be an open subscheme which contains $X$ as a *closed* subscheme. Then $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovhol,bc}}(X/K) \subset D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$ is in $D^{\star}$ if and only if $\mathcal{E}|_{\mathcal{U}}$ is in $D^{\star}(\mathcal{D}^{\dagger}_{\mathcal{U},\mathbb{Q}})$, where we used the standard t-structure for the derived category of $\mathcal{D}^{\dagger}_{\mathcal{U},\mathbb{Q}}$-complexes. As in [3, 1.2.1], this construction does not depend on the auxiliary choices, and defines a t-structure on $D^{\mathrm{b}}_{\mathrm{ovhol,bc}}(X/K)$. The objects of its heart is called "overholonomic after any base change modules over $X/K$ (or simply $X$)", and denoted by $\mathrm{Ovhol}^{\mathrm{bc}}(X/K)$ or $\mathrm{Ovhol}^{\mathrm{bc}}(X)$.

Assume $X$ is smooth and realizable. Recall the category $\mathrm{Isoc}^{\dagger\dagger}(X)$ (see [3, 1.2.14] and references therein), which is a $\mathcal{D}^{\dagger}$-module theoretic interpretation of the category of overconvergent isocrystals on $X$.

*Remark* In the construction of the t-structure, if we can take a *divisor* $Z$ of $\mathcal{P}$ such that $\mathcal{U} = \mathcal{P}\setminus Z$, then the t-structure is nothing but the one induced by the standard t-structure on $D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$. However, objects of $\mathrm{Ovhol}^{\mathrm{bc}}(X)$ realized in $D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$ are complexes unless we can take such a divisor $Z$. See [3, 1.2.2].

**Lemma 1.5** *For a realizable variety $X$, any object of the abelian category* $\mathrm{Ovhol}^{\mathrm{bc}}(X)$ *satisfies the ascending and descending chain condition.*

*Proof* By base change, we can suppose that $k$ is uncountable. We prove the claim using the induction on the dimension of the support. From [14, 3.7][1] there exists an open dense subscheme $j\colon U \hookrightarrow X$ such that $X\setminus U$ is a divisor and $\mathcal{G} := \mathcal{E}|_U \in \mathrm{Isoc}^{\dagger\dagger}(U)$. By induction hypothesis, we are reduced to checking that $j_+(\mathcal{G})$ satisfies the ascending (resp. descending) chain condition. Take an irreducible submodule $\mathcal{G}' \subset \mathcal{G}$ in the category of overholonomic after any base change modules on $U$. Using [3, 1.4.7], we check that since $\mathcal{G}'$ is irreducible then so is $j_{!+}(\mathcal{G}')$ $(:= \mathrm{Im}(j_!(\mathcal{G}') \to j_+(\mathcal{G}')))$. Thus by induction hypothesis, $j_+(\mathcal{G}')$ satisfies the ascending (resp. descending) chain condition. Since $j_+$ is exact, if $\mathcal{G}$ is not irreducible then we conclude by using a second induction on the generic rank of $\mathcal{G}$. □

*Remark* For a smooth formal scheme $\mathcal{P}$ (which may not be proper), we may also show that any overholonomic module on $\mathcal{P}$ satisfies the ascending and descending chain conditions. The proof is similar.

[1] In the statement of [14, 3.7], we need to add that $k$ is uncountable or that the property to have finite fibers is stable under base change.

**Corollary 1.6** *Let $X$ be a realizable variety. Let $\mathcal{E}$ be an overholonomic after any base change module on $X$. Assume that $\mathcal{E}$ can be endowed with a $q^s$-Frobenius structure for an integer $s > 0$. Then any constituents of $\mathcal{E}$ in the category overholonomic after any base change module on $X$ can be endowed with a $q^{s'}$-Frobenius structure for some $s'$ a multiple of $s$.*

*Proof* The verification is similar to [21, 6.0–15]. Let us recall the argument. Let $\mathcal{A}$ be an abelian category which consists of objects whose lengths are finite, and $F$ be an endo-functor on $\mathcal{A}$. For an object $X \in \mathcal{A}$, assume given an isomorphism $\alpha\colon X \cong F(X)$. To show the corollary, it suffices to check that for any constituent $Y$ of $X$, there exists an integer $s > 0$ such that $Y \cong F^s(Y)$. Indeed, let $I$ be the multiset of isomorphism classes of irreducible constituents of $X$. The isomorphism $\alpha$ induces an automorphism $\alpha_*\colon I \xrightarrow{\sim} I$. Since $I$ is a finite multiset, for any $[Y] \in I$, there exists an integer $n > 0$ such that $\alpha_*^n([Y]) = [Y]$, which by definition implies that $Y \cong F^{s'}(Y)$ where $s' = ns$. □

**1.7** Let $X$ be a realizable variety. Let $\mathrm{Hol}_F(X/K)'$ be the subcategory of the category of overholonomic after any base change module on $X$ whose objects can be endowed with $q^s$-Frobenius structure for some integer $s > 0$, and let $\mathrm{Hol}_F(X/K)$ be the thick abelian subcategory generated by $\mathrm{Hol}_F(X/K)'$ in the category consisting of overholonomic after any base change modules on $X$. We denote by $D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$ the triangulated full subcategory of the category of overholonomic after any base change complexes on $X/K$ consisting of complexes whose cohomologies are in $\mathrm{Hol}_F(X/K)$. Recall that in this paper, if there is no ambiguity with $K$, we sometimes omit "$/K$" in the notation of some categories. By Lemma 1.3 and Corollary 1.6, we have:

**Corollary** *Any object of* $\mathrm{Hol}_F(X)$ *can be written as successive extensions of modules in* $\mathrm{Hol}_F(X)'$.

This corollary has the following consequences:

**Theorem 1.8** *Let $f\colon X \to Y$ be a morphism between realizable varieties.*

1. *The functor $f_+$ induces $D^{\mathrm{b}}_{\mathrm{hol},F}(X) \to D^{\mathrm{b}}_{\mathrm{hol},F}(Y)$.*
2. *The functor $f^!$ induces $D^{\mathrm{b}}_{\mathrm{hol},F}(Y) \to D^{\mathrm{b}}_{\mathrm{hol},F}(X)$.*
3. *The dual functor $\mathbb{D}$* ([3, 1.1.6 (i)]) *induces the functor $D^{\mathrm{b}}_{\mathrm{hol},F}(X)^{\circ} \to D^{\mathrm{b}}_{\mathrm{hol},F}(X)$ such that $\mathbb{D} \circ \mathbb{D} \cong \mathrm{id}$.*
4. *We have the bifunctor $\widetilde{\otimes}\colon D^{\mathrm{b}}_{\mathrm{hol},F}(X) \times D^{\mathrm{b}}_{\mathrm{hol},F}(X) \to D^{\mathrm{b}}_{\mathrm{hol},F}(X)$.*

*Moreover, these functors satisfy the properties listed in* [3, 1.3.14].

*Remark* Even if we replace $D^{\mathrm{b}}_{\mathrm{hol},F}$ by $D^{\mathrm{b}}_{\mathrm{ovhol,bc}}$, the theorem holds except for 4, which has been checked by the second author. For detailed references, we refer to [3, 1.1].

*Proof* Let us check the first three claims. As noted in the remark, the three functors are known to be defined if we replace $D^{\mathrm{b}}_{\mathrm{hol},F}$ by $D^{\mathrm{b}}_{\mathrm{ovhol,bc}}$. Since $D^{\mathrm{b}}_{\mathrm{hol},F}$ is a full subcategory of $D^{\mathrm{b}}_{\mathrm{ovhol,bc}}$, it suffices to check that these functors send $D^{\mathrm{b}}_{\mathrm{hol},F}$ to itself. Since Corollary 1.7 tells us that the latter category is generated by overholonomic modules with Frobenius structure, it suffices to verify the stability just for these objects. Since the functors commute with Frobenius pull-backs, the stability follows.

Let us check the last one. First, let us suppose that $X$ can be lifted to a proper smooth formal scheme $\mathcal{X}$. Then we have the functor

$$\boxtimes^{\dagger}_{\mathcal{O}_{\mathcal{X},\mathbb{Q}}}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}) \times D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{X}\times\mathcal{X},\mathbb{Q}}).$$

as in [17, (2.3.3.2)]. As in the case of the first three functors, $\boxtimes^{\dagger}_{\mathcal{O}_{\mathcal{X},\mathbb{Q}}}$ induces a functor between $D^{\mathrm{b}}_{\mathrm{hol},F}$. Indeed, similarly to the argument above using Corollary 1.7, the verification is reduced to the stability for overholonomic modules with Frobenius structure, and this case is verified in [17, Thm 4.2.3]. As in [17, Thm 4.2.7], we may check that the functor does not depend on the choice of lifting up to canonical equivalence, and defines a functor

$$\boxtimes^{\dagger}_{X}\colon D^{\mathrm{b}}_{\mathrm{hol},F}(X) \times D^{\mathrm{b}}_{\mathrm{hol},F}(X) \to D^{\mathrm{b}}_{\mathrm{hol},F}(X \times X).$$

when $X$ is proper smooth and liftable. Now, for a general realizable scheme $X$, we take an immersion $i\colon X \hookrightarrow P$ to a proper smooth liftable variety $P$, and define

$$\widetilde{\otimes} := \Delta^{!}_{X}(i \times i)^{!}\big(i_{+}(-) \boxtimes^{\dagger}_{P} i_{+}(-)\big)$$

where $\Delta_X\colon X \to X \times X$ is the diagonal immersion. We leave the reader to check that this is independent of the choice of auxiliary choices. Verification of properties in [3, 1.3.14] is similar, so we leave it to the reader. □

We recall that for a realizable variety $X$, we define

$$\otimes := \mathbb{D}\big(\mathbb{D}(-)\widetilde{\otimes}\mathbb{D}(-)\big)\colon D^{\mathrm{b}}_{\mathrm{hol},F}(X) \times D^{\mathrm{b}}_{\mathrm{hol},F}(X) \to D^{\mathrm{b}}_{\mathrm{hol},F}(X)$$

as in [3, 1.1.6 (iii)]. We point out that [17, (2.3.9.2)][2] shows that the definition of $\widetilde{\otimes}$ is compatible with that defined in [3, 1.1.6], in the sense that if we forget the Frobenius structure from [3], then the functor coincides with the one defined here. Thus the functors $\otimes$ is also compatible.

*Remark* Similarly to [3, 1.1.6(ii)], we can construct directly the bifunctor $\widetilde{\otimes}\colon D^{\mathrm{b}}_{\mathrm{hol},F}(X) \times D^{\mathrm{b}}_{\mathrm{hol},F}(X) \to D^{\mathrm{b}}_{\mathrm{hol},F}(X)$. In the proof of Theorem 1.8.4, we first construct the bifunctor $\boxtimes^{\dagger}_{X}$. By using the other functors, as showed in this proof, we notice that this is equivalent to the construction of $\widetilde{\otimes}$. The advantage to construct directly $\boxtimes^{\dagger}_{X}$ is that we can avoid speaking of Berthelot's categories of the form $\underrightarrow{LD}$ (see [11, 4.2]), even if these Berthelot's categories of the form $\underrightarrow{LD}$ are fundamental to check the transitivity of our functors.

**1.9** Using this category, we can state our main theorem as follows:

[2] Here a shift is missing, or in other words, $\delta^{!}$ should be replaced by $\delta^{*}$. The same for [*ibid.*, (2.3.9.1)].

**Theorem** *Let $X$ be a realizable variety. Then the canonical functor*

$$D^{\mathrm{b}}(\mathrm{Hol}_F(X/K)) \to D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$$

*is an equivalence of categories.*

*Proof* With the aid of the six formalism as we constructed in Theorem 1.8 and the next section, the proof of [5] can be adapted without any difficulties, so we only sketch the outline. We put $D(X) := D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$, which is endowed with t-structure whose heart is $M(X) := \mathrm{Hol}_F(X/K)$ by construction. For the derived category $D^{\mathrm{b}}(M(X))$, we consider the standard t-structure, so the heart is $M(X)$ as well. The first task is to define the functor $\mathrm{real}_X\colon D^{\mathrm{b}}(M(X)) \to D(X)$ which induces an identity on the hearts of the t-structures. This can be defined using the abstract non-sense presented in the appendix of [5].

1) In this first part, we prove Theorem 1.9 generically. For a generic point $\eta \in X$, we put $D(\eta) := 2\text{-}\varinjlim_{\eta\in U} D(U)$, and $M(\eta) := 2\text{-}\varinjlim_{\eta\in U} M(U)$. We prove that the functor

$$\mathrm{real}_\eta\colon D^{\mathrm{b}}(M(\eta)) \to D(\eta) \tag{1.9.1}$$

is an equivalence. Let $\eta \in U \subset X$ be an open subscheme, and $M_U, N_U$ are in $M(U)$. Since $D(\eta)$ has canonically a t-structure whose heart is $M(\eta)$ it suffices to show the existence of an open subscheme $\eta \in V \xrightarrow{j} U$, $O_V \in M(V)$ and $N_V := j^+N_U \hookrightarrow O_V$ such that the induced homomorphism $\mathrm{Ext}^i_{D(U)}(M_U, N_U) \to \mathrm{Ext}^i_{D(V)}(M_V, O_V)$ is zero for any $i > 0$. Now, we prove this latter property by induction on the dimension of $X$.

Since $k$ is perfect, by shrinking $U$, we may assume that $U$ is connected and smooth of dimension $d$, and $M_U$ and $N_U$ are contained in $\mathrm{Isoc}^{\dagger\dagger}(U)$ (use [14, 3.7]). Recall that by definition $\mathcal{H}om(M_U, N_U) := \mathbb{D}M_U\widetilde{\otimes}N_U$ (see [3, A.1]) and we have $\mathcal{H}om(M_U, N_U)[d] \in \mathrm{Isoc}^{\dagger\dagger}(U)$. By shrinking $U$ further, we may assume that there exists a smooth affine morphism $\P\colon U \to Z$ with 1-dimensional fibers such that $Z$ is smooth (notice that we can choose $Z$ as a dense open of $\mathbb{A}^{d-1}$ since, shrinking $U$ if necessary, $U$ is affine and étale over $\mathbb{A}^d$ and then use [14, 3.7]), and even assume that $L^q := \mathscr{H}^{q+(d-1)}\P_+\mathcal{H}om(M_U, N_U)$ are in $\mathrm{Isoc}^{\dagger\dagger}(Z)$ for any $q$ by shrinking $Z$. Note[3] that since $\P$ is affine then $L^q = 0$ for $q \neq 0, 1$ (use Proposition [3, 1.3.13.(i)] and Definition [3, 1.1.2]). We refer to [3, A.5] for the relation between $\mathcal{H}om$ and $\mathrm{Hom}_{D(X)}$. For an open subscheme $Y \subset Z$, let $U_Y := \P^{-1}(Y)$ and $\P_Y : U_Y \to Y$ is the one induced by $\P$. Since $\P$ is assumed to be affine and the dimension of each fiber is 1, we see that the Leray spectral sequence $E_2^{p,q} = \mathscr{H}^{p-(d-1)}p_{Y+}(L^q_Y) \Rightarrow \mathrm{Ext}^{p+q}_{D(U_Y)}(M_{U_Y}, N_{U_Y})$ degenerates at $E_3$, where $p_Y$ denotes the structural morphism of $Y$ and $L^q_Y$ denotes the restriction of $L^q$ to $Y$. For simplicity, we denote $\mathscr{H}^{p-(d-1)}p_{Z+}(-)$ by $H^p(Z, -)$. Using this degeneration, Beilinson splits the construction problem of $O_V$ into two: one

[3] One might wonder why we have the funny degree $\mathscr{H}^{q+(d-1)}$ in the definition of $L^q$. This is because our category Hol corresponds to the category of perverse sheaves in the $\ell$-adic situation. However, since the objects appearing in this argument are in $\mathrm{Isoc}^{\dagger\dagger}$, everything works as in Beilinson except that we need suitable shifts of degrees.

is to find an open subscheme $Y \subset Z$ and $N_{U_Y} \hookrightarrow P_{U_Y}$ such that $P_{U_Y} \in \mathrm{Isoc}^{\dagger\dagger}(U_Y)$ and $\mathscr{H}^d \P_{Y+}\mathcal{H}om(M_{U_Y}, N_{U_Y}) \to \mathscr{H}^d \P_{Y+}\mathcal{H}om(M_{U_Y}, P_{U_Y})$ is zero, and the other is to find an open subscheme $Y' \subset Z$ and $N_{U_{Y'}} \hookrightarrow Q_{U_{Y'}}$ such that $Q_{U_{Y'}} \in \mathrm{Isoc}^{\dagger\dagger}(U_{Y'})$ and $H^p(Z, \mathscr{H}^{d-1}\P_+\mathcal{H}om(M_U, N_U)) \to H^p(Y', \mathscr{H}^{d-1}\P_{Y'+}\mathcal{H}om(M_{U_{Y'}}, Q_{U_{Y'}}))$ is zero for all $p \geq 1$. The construction is written in [5, 2.1], but we recall briefly for the reader.

Let us construct $P_{U_Y}$. We put $\mathcal{H} := \mathcal{H}om(\P^+ L^1 \otimes M[1-d], N)$, where $\P^+ L^1 \otimes M[1-d] \in \mathrm{Isoc}^{\dagger\dagger}(U)$. Using [3, (A.1.1), A.8], we get $\P_+\mathcal{H} \xrightarrow{\sim} \mathcal{H}om(L^1[1-d], \P^+\mathcal{H}om(M, N))$. Since, the functor $\mathcal{H}om(L^1[1-d], -)$ is exact, we get $\mathscr{H}^i \P_+\mathcal{H} \xrightarrow{\sim} \mathcal{H}om(L^1[1-d], \mathscr{H}^i \P^+\mathcal{H}om(M, N))$. This yields the vertical isomorphisms of the following diagram:

$$\begin{array}{ccccc} \mathrm{Ext}^1(\P^+ L^1 \otimes M[1-d], N) & \longrightarrow & H^0(Z, \mathscr{H}^d \P_+\mathcal{H}) & \xrightarrow{\partial} & H^2(Z, \mathscr{H}^{d-1}\P_+\mathcal{H}) \\ & & \Big| \wr & & \Big| \wr \\ & & H^0(Z, \mathbb{D}L^1 \widetilde{\otimes} L^1[d-1]) & & H^2(Z, \mathbb{D}L^1 \widetilde{\otimes} L^0[d-1]), \end{array}$$

where the horizontal exact sequence comes from Leray spectral exact sequence (see also [3, A.4]). Let $\alpha \in H^0(Z, \mathbb{D}L^1 \widetilde{\otimes} L^1[d-1]) = \mathrm{Ext}^0_{D(Z)}(L^1, L^1)$ (see [3, A.5]) be the canonical element. Now, it is the time to use the induction hypothesis, to $\mathbb{D}L^0$ and $\mathbb{D}L^1$. Since $\mathbb{D}\mathbb{D}L^0 \xrightarrow{\sim} L^0$, this implies that there exist $Y \subset Z$, $K_Y \in \mathrm{Isoc}^{\dagger\dagger}(Y)$, and $\varphi \colon (\mathbb{D}L^1)_Y \hookrightarrow K_Y$ such that the induced arrow $\varphi_* \colon H^2(Z, \mathbb{D}L^1 \widetilde{\otimes} L^0[d-1]) \to H^2(Y, K_Y \widetilde{\otimes} L^0_Y[d-1])$ is 0. Thus, $\varphi_* \partial(\alpha) = 0$. By diagram chase, $\varphi_*(\alpha) \in H^0(Y, K_Y \widetilde{\otimes} L^1_Y[d-1])$ is the image of some $(\varphi(\alpha))^\sim \in \mathrm{Ext}^1(\P^+ K_Y \otimes M_{U_Y}[1-d], N_{U_Y})$. Now define $P_{U_Y}$ to be the object in the extension $0 \to N_{U_Y} \to P_{U_Y} \to \P^+ K_Y \otimes M_{U_Y}[1-d] \to 0$ corresponding to $(\varphi(\alpha))^\sim$, and one shows that this meets the demand.

Let us construct $Q_{U_{Y'}}$. By applying the induction hypothesis to the constant isocrystal on $Z$ and $L^0$ (remark that, using [3, A.5] we get $H^i(Z, L^0) = \mathrm{Ext}^i_{D(Z)}(\mathcal{O}_Z, L^0)$ where $\mathcal{O}_Z$ is the constant isocrystal on $Z$) we get $Y' \subset Z$, $R_{Y'}$ (the corresponding object in [5] is denoted by $Q_{Y'}$), and an injection $L^0_{Y'} \hookrightarrow R_{Y'}$ such that the induced map $H^i(Z, L^0) \to H^i(Z, R_{Y'})$ is 0 for $i > 0$. Define $Q_{U_{Y'}}$ (the corresponding object in [5] is denoted by $O_{U_{Y'}}$) by the cocartesian square:

$$\begin{array}{ccc} \P^+ R_{Y'} \otimes M_{U_{Y'}}[1-d] & \longrightarrow & Q_{U_{Y'}} \\ \uparrow & & \uparrow \\ \P^+ L^0_{Y'} \otimes M_{U_{Y'}}[1-d] & \longrightarrow & N_{U_{Y'}}, \end{array}$$

where the bottom horizontal arrow is the canonical arrow. Using [3, (A.1.1), A.5, A.8]), we get the first equality $\mathrm{Hom}(\P^+ L^0_{Y'} \otimes M_{U_{Y'}}[1-d], N_{U_{Y'}}) = \mathrm{Hom}(L^0_{Y'}, \P_+\mathcal{H}om(M_{U_{Y'}}, N_{U_{Y'}})[d-1]) = \mathrm{Hom}(L^0_{Y'}, \mathscr{H}^{d-1}\P_+\mathcal{H}om(M_{U_{Y'}}, N_{U_{Y'}}))$.

Hence, we get

$$\begin{array}{ccc} R_{Y'} & \longrightarrow & \mathscr{H}^{d-1}\P_{+}\mathcal{H}om(M_{U_{Y'}}, Q_{U_{Y'}}) \\ \uparrow & & \uparrow \\ L^0_{Y'} & \longrightarrow & \mathscr{H}^{d-1}\P_{+}\mathcal{H}om(M_{U_{Y'}}, N_{U_{Y'}}). \end{array}$$

(2) Using the generic case of part 1), let us prove Theorem 1.9 by induction on the dimension of $X$. Since $X$ is separated, any open immersion $j: U \to X$ with $U$ affine is affine, and in particular, $j_+$ sends $M(U)$ to $M(X)$ by [3, 1.3.13]. Thus, by standard argument, the claim is Zariski local, and we may assume $X$ to be affine. It suffices to show that for any $M$, $N$ in $M(X)$, and $k \geq 0$, the homomorphism $\mathrm{Ext}^k_{M(X)}(M, N) \to \mathrm{Ext}^k_{D(X)}(M, N)$, where $\mathrm{Ext}^k$ denotes the Yoneda's Ext functor, is an isomorphism. Using the equivalence of (1.9.1) and the formal properties of cohomological functors, it is a standard devissage argument to reduce to the case where the supports of $M$ and $N$ have dimension less than that of $X$ (cf. [5, 2.2.2–2.2.4]). Take a morphism $f: X \to \mathbb{A}^1$ such that $Y := f^{-1}(0)$ contains the support of $M$ and $N$. Let $i: Y \hookrightarrow X$ be the immersion. Using the induction hypothesis, we have

$$\mathrm{Ext}^k_{D(X)}(M, N) \underset{i^!}{\xrightarrow{\sim}} \mathrm{Ext}^k_{D(Y)}(M, N) \cong \mathrm{Ext}^i_{M(Y)}(M, N),$$

where the inverse of the first isomorphism is $i_+$. It remains to show that the canonical homomorphism induced by $i_+$

$$I: \mathrm{Ext}^k_{M(Y)}(M, N) \to \mathrm{Ext}^k_{M(X)}(M, N),$$

is a bijection for any $k$. For this we need the existence of the functors $\Phi_f$ and $\Xi_f$. These functors are defined and basic properties are shown in the next section (cf. Proposition 2.7). In fact, the inverse of $I$ can be constructed as

$$\Phi_{f*}: \mathrm{Ext}^k_{M(X)}(M, N) \xrightarrow{\Phi_f} \mathrm{Ext}^k_{M(Y)}(\Phi_f(M), \Phi_f(N)) \cong \mathrm{Ext}^k_{M(Y)}(M, N)$$

where we used the exactness of $\Phi_f$ in the first homomorphism, and the isomorphism holds since $M$ and $N$ are supported on $Y$. Since $\Phi_{f*} \circ I = \mathrm{id}$, it remains to show that $I \circ \Phi_{f*} = \mathrm{id}$. To check this, for an extension class $0 \to N \to C^1 \to \cdots \to C^i \to M \to 0$ in $M(X)$, we need to show that the class of $0 \to N \to \Phi_f(C^1) \to \cdots \to \Phi_f(C^i) \to M \to 0$ is the same. For this, Beilinson constructs an ingenious sequence of homomorphisms connecting the two using $\Xi_f$ as follows:

$$C^\bullet \to C^\bullet \oplus \Xi_f(C^\bullet_U) \to \big(C^\bullet \oplus \Xi_f(C^\bullet_U)\big)/j_!(C^\bullet_U) \leftarrow \Phi_f(C^\bullet),$$

where we refer to (2.5.1) for the last arrow. □

*Remark* This theorem is a generalization of [3, A.4]. In fact, Hom's in the category $D^{\mathrm{b}}(\mathrm{Hol}_F(X/K))$ can be computed by Yoneda extensions.

## 2 Unipotent nearby cycle functor

**2.1** For the convenience of the reader, we start by recalling some generalities on Beilinson's limit construction. Nothing is new in this paragraph, and the construction is explained in [6], even though it would not be easy to check the details. One can also refer to [26, 3.2] where Lichtenstein explains Beilinson's construction in more details.

Let $\Pi := \{(a,b) \in \mathbb{Z}^2\,;\, a \leq b\}$ be the partially ordered set[4] such that $(a,b) \leq (a',b') \Leftrightarrow a \geq a',\ b \geq b'$, which we consider as a category. For an abelian category $\mathfrak{A}$, we denoted by $\mathfrak{A}^{\Pi}$ the category of $\Pi$-shaped diagrams in $\mathfrak{A}$, in other words, the category of functors $\mathrm{Funct}(\Pi, \mathfrak{A})$. Concretely, objects of $\mathfrak{A}^{\Pi}$ are $\mathcal{E}^{\bullet,\bullet} = (\mathcal{E}^{a,b}, \alpha^{(a,b),(a',b')})$, where $(a,b)$, $(a',b')$ runs through elements of $\Pi$ so that $(a',b') \leq (a,b)$, $\mathcal{E}^{a,b}$ belong to $\mathfrak{A}$, and $\alpha^{(a,b),(a',b')} : \mathcal{E}^{a',b'} \to \mathcal{E}^{a,b}$ are morphisms of $\mathfrak{A}$, transitive with respect to the composition. We denoted by $\mathfrak{A}^{\Pi}_{\mathrm{a}}$ the full subcategory of $\mathfrak{A}^{\Pi}$ of objects $\mathcal{E}^{\bullet,\bullet} = (\mathcal{E}^{a,b}, \alpha^{(a,b),(a',b')})$ such that, for any $a \leq b \leq c$, the sequence $0 \to \mathcal{E}^{b,c} \to \mathcal{E}^{a,c} \to \mathcal{E}^{a,b} \to 0$ is exact. These objects are called *admissible*. Since this subcategory is closed under extension, this is an exact category so that the canonical functor $\mathfrak{A}^{\Pi}_{\mathrm{a}} \to \mathfrak{A}^{\Pi}$ is exact.

Let $M$ be the set of maps $\phi\colon \mathbb{Z} \to \mathbb{Z}$ which are order-preserving (*i.e.* $\phi(a) \leq \phi(b)$ for any $a \leq b$) and $\lim_{i \to \pm\infty} \phi(i) = \pm\infty$. For any $\phi \in M$, we put $\widetilde{\phi}(\mathcal{E}^{\bullet,\bullet}) := (\mathcal{E}^{\phi(a),\phi(b)})_{(a,b)\in\Pi}$. Let $S$ be the set of the canonical morphisms of the form $\widetilde{\phi}(\mathcal{E}^{\bullet,\bullet}) \to \widetilde{\psi}(\mathcal{E}^{\bullet,\bullet})$, where $\phi, \psi \in M$ satisfy $\phi \geq \psi$ and $\mathcal{E}^{\bullet,\bullet} \in \mathfrak{A}^{\Pi}$. We denote by $S_{\mathrm{a}}$ the elements of $S$ which are morphisms of $\mathfrak{A}^{\Pi}_{\mathrm{a}}$ as well. The sets $S$ and $S_{\mathrm{a}}$ are multiplicative[5].

1. Following [6, Appendix], we put $\underset{\longleftrightarrow}{\lim}\, \mathfrak{A} := S_{\mathrm{a}}^{-1}\mathfrak{A}^{\Pi}_{\mathrm{a}}$ and $\underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} \mathfrak{A} := S^{-1}\mathfrak{A}^{\Pi}$. For any $\mathcal{E}^{\bullet,\bullet}_{\mathrm{a}}, \mathcal{F}^{\bullet,\bullet}_{\mathrm{a}} \in \underset{\longleftrightarrow}{\lim}\, \mathfrak{A}$ and for any $\mathcal{E}^{\bullet,\bullet}, \mathcal{F}^{\bullet,\bullet} \in \underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} \mathfrak{A}$ we have the equalities
$$\mathrm{Hom}_{\underset{\longleftrightarrow}{\lim}\, \mathfrak{A}}(\mathcal{E}^{\bullet,\bullet}_{\mathrm{a}}, \mathcal{F}^{\bullet,\bullet}_{\mathrm{a}}) = \varinjlim_{\phi \in M} \mathrm{Hom}_{\mathfrak{A}^{\Pi}_{\mathrm{a}}}(\widetilde{\phi}\mathcal{E}^{\bullet,\bullet}_{\mathrm{a}}, \mathcal{F}^{\bullet,\bullet}_{\mathrm{a}}),$$
$$\mathrm{Hom}_{\underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} \mathfrak{A}}(\mathcal{E}^{\bullet,\bullet}, \mathcal{F}^{\bullet,\bullet}) = \varinjlim_{\phi \in M} \mathrm{Hom}_{\mathfrak{A}^{\Pi}}(\widetilde{\phi}\mathcal{E}^{\bullet,\bullet}, \mathcal{F}^{\bullet,\bullet}). \tag{2.1.1}$$
We get from (2.1.1) that the canonical functor $\underset{\longleftrightarrow}{\lim}\, \mathfrak{A} \to \underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} \mathfrak{A}$ is fully faithful. This enables us to denote by $\underset{\longleftrightarrow}{\lim}\colon \mathfrak{A}^{\Pi}_{\mathrm{a}} \to \underset{\longleftrightarrow}{\lim}\, \mathfrak{A}$ and $\underset{\longleftrightarrow}{\lim}\colon \mathfrak{A}^{\Pi} \to \underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} \mathfrak{A}$ the canonical functors. By definition of the exact structure, $\underset{\longleftrightarrow}{\lim}$ is exact (cf. [26, Prop A.3]).
2. Let $N(\mathfrak{A})$ be the full subcategory of $\mathfrak{A}^{\Pi}$ whose objects are null in $\underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} \mathfrak{A}$. Then, the category $N(\mathfrak{A})$ is a Serre subcategory of $\mathfrak{A}^{\Pi}$. Moreover, we have the equality $\mathfrak{A}^{\Pi}/N(\mathfrak{A}) = \underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} \mathfrak{A}$. In particular, $\underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} \mathfrak{A}$ is an abelian category. The proof is identical to [19, 1.2.4].

[4] The order is opposite to Beilinson's one in [6, A.3], and we followed that of Lichtenstein's in [26, 3.2.1].

[5] We remark that in [26], Lichtenstein calls a function $\phi$ to be order-preserving if $\phi(a) < \phi(b)$ for any $a < b$ unlike our convention. This prevents him to verify the multiplicativity without assuming further condition (see [*ibid.*, Footnote 5]). In our setting, it is straightforward to check this multiplicativity.

3. Let $\mathcal{E} \in \mathfrak{A}$. For any $c \in \mathbb{R}$, we pose $\mathcal{E}^c = \mathcal{E}$ if $c < 0$ and $\mathcal{E}^c = 0$ otherwise. For any $(a, b) \in \Pi$, we set $\mathcal{E}^{a,b} := \mathcal{E}^a/\mathcal{E}^b$. We get canonically the object $\mathcal{E}^{\bullet,\bullet} \in \mathfrak{A}_{\mathrm{a}}^{\Pi}$. By sending $\mathcal{E}$ to $\mathcal{E}^{\bullet,\bullet}$, we get the fully faithful exact functor $\mathfrak{A} \to \mathfrak{A}_{\mathrm{a}}^{\Pi}$.

**2.2** This paragraph will be useful in the proof of Lemma 2.4. Let $V \hookrightarrow U \hookrightarrow Y \hookrightarrow X$ be open immersions of realizable varieties. We have the abelian categories $F$-Ovhol$(U, X/K)$ and $F$-Ovhol$(V, Y/K)$ (see [3, 1.2.13]). Since the functor $|_{(V,Y)}$ (see notation [3, 1.2.9.(iii)]) is exact, it preserves admissible objects and yields the functor

$$|_{(V,Y)}\colon \underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} F\text{-Ovhol}(U, X/K) \to \underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} F\text{-Ovhol}(V, Y/K).$$

Let $\mathcal{E}^{\bullet,\bullet} \in \underset{\longleftrightarrow}{\lim}^{\mathrm{ab}} F\text{-Ovhol}(U, X/K)$. We remark that $\mathcal{E}^{\bullet,\bullet} = 0$ if and only if $\mathcal{E}^{\bullet,\bullet}|_{(U,Y)} = 0$. Let $\{U_i\}$ be an open covering of $U$. We notice that $\mathcal{E}^{\bullet,\bullet} = 0$ if and only if $\mathcal{E}^{\bullet,\bullet}|(U_i, U_i) = 0$ for any $i$.

**2.3** For a smooth formal scheme $\mathcal{X}$ and a divisor $Z$ of the special fiber of $\mathcal{X}$, recall that $\mathcal{O}_{\mathcal{X}}({}^{\dagger}Z)_{\mathbb{Q}}$ is the ring of overconvergent functions with poles at $Z$ on $\mathcal{X}$, and that $\mathcal{D}^{\dagger}_{\mathcal{X}}({}^{\dagger}Z)_{\mathbb{Q}}$ is the ring of differential operators with poles along $Z$ on $\mathcal{X}$ and with suitable convergence condition. See [9, 4.2.4, 4.2.5] for details. Set $\mathcal{O}_{\mathbb{G}_{m,k}} := \mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathcal{V}}}({}^{\dagger}\{0,\infty\})_{\mathbb{Q}}$, $\mathcal{D}_{\mathbb{G}_{m,k}} := \mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathcal{V}}}({}^{\dagger}\{0,\infty\})_{\mathbb{Q}} \otimes_{\mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathcal{V}}}} \mathcal{D}_{\widehat{\mathbb{P}}^1_{\mathcal{V}}}$ and $\mathcal{D}^{\dagger}_{\mathbb{G}_{m,k}} := \mathcal{D}^{\dagger}_{\widehat{\mathbb{P}}^1_{\mathcal{V}}}({}^{\dagger}\{0,\infty\})_{\mathbb{Q}}$ and let $t$ be the coordinate of $\widehat{\mathbb{P}}^1_{\mathcal{V}}$. We denote by $\mathcal{O}_{\mathbb{G}_{m,k}}[s, s^{-1}] \cdot t^s$ the free $\mathcal{O}_{\mathbb{G}_{m,k}}[s, s^{-1}]$-module of rank one generated by $t^s$. For any integer $a \in \mathbb{Z}$, the free $\mathcal{O}_{\mathbb{G}_{m,k}}[s]$-submodule of rank one generated by $s^a t^s$ is denoted by $s^a\mathcal{O}_{\mathbb{G}_{m,k}}[s] \cdot t^s$ or by $\mathcal{I}^a_{\mathbb{G}_{m,k}}$. Following Beilinson's notation, for integers $a \leq b$, we get a free $\mathcal{O}_{\mathbb{G}_{m,k}}$-module of finite type by putting

$$\mathcal{I}^{a,b}_{\mathbb{G}_{m,k}} := \mathcal{I}^a_{\mathbb{G}_{m,k}}/\mathcal{I}^b_{\mathbb{G}_{m,k}}.$$

We define a structure of $\mathcal{D}_{\mathbb{G}_{m,k}}$-module on $\mathcal{O}_{\mathbb{G}_{m,k}}[s, s^{-1}] \cdot t^s$ so that for $g \in \mathcal{O}_{\mathbb{G}_{m,k}}$ and $l \in \mathbb{Z}$, we have

$$\partial_t(s^l g \cdot t^s) = s^l \partial_t(g) \cdot t^s + s^{l+1} g/t \cdot t^s. \tag{2.3.1}$$

Hence, we get a canonical structure of $\mathcal{D}_{\mathbb{G}_{m,k}}$-module on $\mathcal{I}^{a,b}_{\mathbb{G}_{m,k}}$. Moreover, we have an isomorphism

$$\mathcal{I}^{a,b}_{\mathbb{G}_{m,k}} \xrightarrow{\sim} F^*\mathcal{I}^{a,b}_{\mathbb{G}_{m,k}}; \qquad s^l g \cdot t^s \mapsto q^l g \otimes (s^l \cdot t^s).$$

It is straightforward to check that this is an isomorphism of $\mathcal{D}_{\mathbb{G}_{m,k}}$-modules. Because of the existence of Frobenius structure, it follows by [9, 4.4.5] and [8, Thm 2.5.7] that the $\mathcal{D}_{\mathbb{G}_{m,k}}$-module structure naturally extends to a $\mathcal{D}^{\dagger}_{\mathbb{G}_{m,k}}$-module structure, and in particular $\mathcal{I}^{a,b}_{\mathbb{G}_{m,k}}$ is an object of $F$-Isoc$^{\dagger\dagger}(\mathbb{G}_{m,k}/K)$.

The multiplication by $s^n$ induces the isomorphism in $F$-Isoc$^{\dagger\dagger}(\mathbb{G}_{m,k}/K)$:

$$\sigma^n\colon \mathcal{I}^{a,b}_{\mathbb{G}_{m,k}} \xrightarrow{\sim} \mathcal{I}^{a+n,b+n}_{\mathbb{G}_{m,k}}(-n), \tag{2.3.2}$$

where $(-)$ denotes the Tate twist (cf. [1, 2.7]). Moreover, there is a non-degenerate pairing

$$\mathcal{I}^{a,b}_{\mathbb{G}_{m,k}} \otimes_{\mathcal{O}_{\mathbb{G}_{m,k}}} \mathcal{I}^{-b,-a}_{\mathbb{G}_{m,k}} \to \mathcal{O}_{\mathbb{G}_{m,k}}(-1); \qquad (x(s), g(s)) \mapsto \mathrm{Res}_{s=0} f(s) \cdot g(-s).$$

We can check easily that this pairing is compatible with Frobenius structure. By using [1, Prop 3.12], the pairing induces an isomorphism

$$\mathbb{D}(\mathcal{I}^{a,b}_{\mathbb{G}_{m,k}}) \xrightarrow{\sim} \mathcal{I}^{-b,-a}_{\mathbb{G}_{m,k}}. \tag{2.3.3}$$

Here, recall that $\mathbb{D}$ denotes the dual functor (cf. Theorem 1.8 (1.8)). As a variant, we put $\mathcal{I}^{a,b}_{\mathbb{G}_{m,k},\log} := s^a \mathcal{O}_{\widehat{\mathbb{A}}^1_{\mathcal{V}}}[s]t^s / s^b \mathcal{O}_{\widehat{\mathbb{A}}^1_{\mathcal{V}}}[s]t^s$. Then $\mathcal{I}^{a,b}_{\mathbb{G}_{m,k},\log}$ is a convergent isocrystal on the formal log-scheme $(\widehat{\mathbb{A}}^1_{\mathcal{V}}, \{0\})$.

**2.4** In the rest of this section, we will keep the following notation. Let $X$ be a realizable variety, $f \in \Gamma(X, \mathcal{O}_X)$ be a fixed function. Put $Z := f^{-1}(0) \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} Y := X \backslash Z$. Let $f|_Y : Y \to \mathbb{G}_{m,k}$ be the morphism induced by $f$ and put

$$\mathcal{I}^{a,b}_f := (f|_Y)^+(\mathcal{I}^{a,b}_{\mathbb{G}_{m,k}})[\dim Y - 1] \in F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(Y/K).$$

For $\mathcal{E} \in \mathrm{Hol}_F(Y/K)$, put $\mathcal{E}^{a,b} := \mathcal{E} \otimes \mathcal{I}^{a,b}_f[-\dim(Y)]$ (see the notation of $\otimes$ after Theorem 1.8). Since the functor $- \otimes \mathcal{I}^{a,b}_f[-\dim(Y)]$ is exact, we get that $\mathcal{E}^{a,b} \in \mathrm{Hol}_F(Y/K)$ and then the object $\mathcal{E}^{\bullet,\bullet} \in \mathrm{Hol}_F(Y/K)^{\Pi}_{\mathrm{a}}$. We note that since $j_!$, $j_+$ are exact functors by [3, 1.3.13], these functors preserve admissible objects.

**Lemma** *Let $\mathcal{E} \in \mathrm{Hol}_F(Y/K)$. The canonical morphism of* $\varprojlim\limits_{\longleftrightarrow} \mathrm{Hol}_F(X/K)$

$$\underset{\longleftrightarrow}{\lim}\, j_!(\mathcal{E}^{\bullet,\bullet}) \to \underset{\longleftrightarrow}{\lim}\, j_+(\mathcal{E}^{\bullet,\bullet})$$

*is an isomorphism.*

*Proof* We put $d := \dim(Y)$. Using the five lemma, we may assume that $\mathcal{E} \in F\text{-Ovhol}(Y/K)$. The proof is divided into several steps.

(**0**) By 2.2, it is sufficient to check that the canonical homomorphism is an isomorphism after applying the functor $|_{(X,X)}$, i.e. that the morphism $\underset{\longleftrightarrow}{\lim}\, (j, \mathrm{id})_!(\mathcal{E}^{\bullet,\bullet}|_{(Y,X)}) \to \underset{\longleftrightarrow}{\lim}\, (j, \mathrm{id})_+(\mathcal{E}^{\bullet,\bullet}|_{(Y,X)})$ is an isomorphism, where $(j, \mathrm{id}) \colon (Y, X) \to (X, X)$ is the morphism of couples induced by $j$ (see notation [3, 1.1.6]). By abuse of notation in this proof, we simply denote by $\mathcal{E} := \mathcal{E}|_{(Y,X)}$ (see notation [3, 1.2.9.(iii)]), $\mathcal{E}^{a,b} := \mathcal{E}|_{(Y,X)} \otimes_{(Y,X)} \mathcal{I}^{a,b}_f|_{(Y,X)}[-\dim Y] = \mathcal{E}^{a,b}|_{(Y,X)}$ (see notation [3, 1.1.6.(iii)]), and write $j \colon (Y, X) \to (X, X)$ instead of $(j, \mathrm{id})$.

(**1**) We prove the lemma under the following hypotheses: "Let $\mathcal{X}$ be a smooth formal $\mathcal{V}$-scheme with local coordinates denoted by $t_1, \ldots, t_d$ whose special fiber is $X$. For any $i = 1, \ldots, d$, we put $\mathcal{Z}_i = V(t_i)$. We suppose that there exist an open immersion $U \hookrightarrow Y$ such that $T := X \backslash U$ is a strict normal crossing divisor of $X$ and an overconvergent

$F$-isocrystal $\mathcal{G}$ on $(U, X)/K$ unipotent along $T$ so that $\mathcal{E} = \iota_!(\mathcal{G})$, where $\iota\colon (U, X) \to (Y, X)$ is the induced morphism of couples (see below in the proof for a concrete description of the notion of unipotence). We fix $0 \leq r' \leq r \leq d$. We suppose that the special fiber of $\mathcal{T} := \cup_{1 \leq n \leq r} \mathcal{Z}_n$ (resp. $\mathcal{Z} := \cup_{1 \leq n' \leq r'} \mathcal{Z}_{n'}$) is $T$ (resp. $Z$)."

We check the step 1) by induction on the integer $r'$ (in the induction, the scheme $X$ can vary and so can $f$, $Y$, $\mathcal{E}$, $\mathcal{X}$ etc.). Where $r' = 0$, this is obvious. Suppose $r' \geq 1$. Recall the functor $\mathcal{H}^{\dagger i}_{Z_1} := \mathbb{R}^i \underline{\Gamma}^{\dagger}_{Z_1}$ using the notation of [3, 1.1.8], and $({}^{\dagger} Z_1)$ is defined in the same place. Consider the following localization sequence of $F\text{-Ovhol}(X, X/K)^{\Pi}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}^{\dagger 0}_{Z_1} j_!(\mathcal{E}^{\bullet,\bullet}) & \longrightarrow & j_!(\mathcal{E}^{\bullet,\bullet}) & \longrightarrow & ({}^{\dagger} Z_1) j_!(\mathcal{E}^{\bullet,\bullet}) & \longrightarrow & \mathcal{H}^{\dagger 1}_{Z_1} j_!(\mathcal{E}^{\bullet,\bullet}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}^{\dagger 0}_{Z_1} j_+(\mathcal{E}^{\bullet,\bullet}) & \longrightarrow & j_+(\mathcal{E}^{\bullet,\bullet}) & \longrightarrow & ({}^{\dagger} Z_1) j_+(\mathcal{E}^{\bullet,\bullet}) & \longrightarrow & \mathcal{H}^{\dagger 1}_{Z_1} j_+(\mathcal{E}^{\bullet,\bullet}) \end{array}$$

whose horizontal sequences are exact. Put $X' := X \backslash Z_1$. By 2.2, the morphism $\underset{\longleftrightarrow}{\lim}\, ({}^{\dagger} Z_1) j_!(\mathcal{E}^{\bullet,\bullet}) \to \underset{\longleftrightarrow}{\lim}\, ({}^{\dagger} Z_1) j_+(\mathcal{E}^{\bullet,\bullet})$ of $\underset{\longleftrightarrow}{\lim}^{\mathrm{ab}}\, F\text{-Ovhol}(X', X/K)$ is an isomorphism if and only if so is after applying $|_{(X',X')}$. By using the induction hypothesis, this latter is an isomorphism and then the homomorphism $\underset{\longleftrightarrow}{\lim}\, ({}^{\dagger} Z_1) j_!(\mathcal{E}^{\bullet,\bullet}) \to \underset{\longleftrightarrow}{\lim}\, ({}^{\dagger} Z_1) j_+(\mathcal{E}^{\bullet,\bullet})$ is an isomorphism. Since $Z_1 \subset X \backslash Y$, we get $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z_1} j_+(\mathcal{E}^{\bullet,\bullet}) = 0$. Hence, it is sufficient to check that $\underset{\longleftrightarrow}{\lim}\, \mathcal{H}^{\dagger i}_{Z_1} j_!(\mathcal{E}^{\bullet,\bullet}) = 0$, for any $i = 0, 1$ by the exactness of $\underset{\longleftrightarrow}{\lim}$.

We have a strict normal crossing divisor of $\mathcal{Z}_1$ defined by $\mathcal{D}_1 := \bigcup_{i=2}^{r} \mathcal{Z}_1 \cap \mathcal{Z}_i$. We put $\mathcal{U} := \mathcal{X} \backslash \mathcal{T}$, and let $i_1 : \mathcal{Z}_1 \hookrightarrow \mathcal{X}$ be the canonical closed immersion. Since $\mathcal{G}$ is unipotent, following [23], this is equivalent to saying that there exists a convergent isocrystal $\mathcal{F}$ on the log scheme $(\mathcal{X}, M_{\mathcal{T}})$, where $M_{\mathcal{T}}$ means the log structure induced by $\mathcal{T}$ (we keep the same kind of notation below), so that $\mathcal{G} \xrightarrow{\sim} ({}^{\dagger} T)(\mathcal{F})$. By abuse of notation, we denote by $f$ (which can be written in the form of $u t_1^{a_1} \cdots t_{r'}^{a_{r'}} \in \mathcal{O}_{\mathcal{X}}$, with $a_i \in \mathbb{N}$ and $u \in \mathcal{O}^*_{\mathcal{X}}$), a lifting of $f$. We put

$$\mathcal{I}^{a,b}_{f,\log} := (f^{\sharp})^* (\mathcal{I}^{a,b}_{\mathbb{G}_{m,k,\log}}),$$

where $f^{\sharp}$ is the composition morphism of formal log-schemes $f^{\sharp} \colon (\mathcal{X}, M_{\mathcal{T}}) \to (\mathcal{X}, M_{\mathcal{Z}}) \to (\widehat{\mathbb{A}}^1_{\mathcal{V}}, M_{\{0\}})$ where the last morphism is induced by $f$. Since $\mathcal{I}^{a,b}_{f,\log}$ is a convergent isocrystal on the formal log scheme $(\mathcal{X}, M_{\mathcal{T}})$ with nilpotent residues, then so is

$$\mathcal{F}^{a,b} := \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X},\mathbb{Q}}} \mathcal{I}^{a,b}_{f,\log}.$$

Notice that we have the isomorphism in $F\text{-Isoc}^{\dagger\dagger}(Y, X/K)$ of the form $({}^{\dagger} Z)(\mathcal{I}^{a,b}_{f,\log}) \xrightarrow{\sim} \mathcal{I}^{a,b}_{f}|_{(Y,X)} = s^a\, \mathcal{O}_{\mathcal{X}}({}^{\dagger} Z)_{\mathbb{Q}}[s] \cdot f^s / s^b\, \mathcal{O}_{\mathcal{X}}({}^{\dagger} Z)_{\mathbb{Q}}[s] \cdot f^s$, which clarifies the notation.

We put $\mathcal{U}_1 := \mathcal{Z}_1 \backslash \mathcal{D}_1$, and let $\iota_1 := (\star, \mathrm{id}, \mathrm{id})\colon (U_1, Z_1, \mathcal{Z}_1) \to (Z_1, Z_1, \mathcal{Z}_1)$ be the canonical morphism of frames. Let $N_{1,\mathcal{F}^{a,b}}$ be the action induced by $t_1 \partial_1$ on $i_1^*(\mathcal{F}^{a,b})$ (following the terminology of [3, 3.2.11], this is the residue morphism). We put

$$\mathcal{G}^{a,b} := \mathcal{G} \otimes_{(U,X)} \mathcal{I}_f^{a,b}|_{(U,X)}[-d] \in F\text{-Isoc}^{\dagger\dagger}(U, X/K),$$

where $\otimes_{(U,X)}$ is the functor defined in [3, 1.1.6(iii)]. Since $({}^\dagger T)(\mathcal{I}_{f,\log}^{a,b}) \xrightarrow{\sim} \mathcal{I}_f^{a,b}|_{(U,X)}$, we have $({}^\dagger T)(\mathcal{F}^{a,b}) \xrightarrow{\sim} \mathcal{G}^{a,b}$. By Theorem [3, 3.4.19], we get the isomorphisms

$$\mathcal{H}^{\dagger 1}_{Z_1}(j_! \iota_!(\mathcal{G}^{a,b})) \xrightarrow{\sim} i_{1+} \circ \iota_{1!} \circ ({}^\dagger D_1) \left(\mathrm{coker}\, N_{1,\mathcal{F}^{a,b}}\right),$$
$$\mathcal{H}^{\dagger 0}_{Z_1}(j_! \iota_!(\mathcal{G}^{a,b})) \xrightarrow{\sim} i_{1+} \circ \iota_{1!} \circ ({}^\dagger D_1) \left(\ker N_{1,\mathcal{F}^{a,b}}\right).$$

Since $\mathcal{E}^{a,b} = \iota_!(\mathcal{G}) \otimes_{(Y,X)} \mathcal{I}_f^{a,b}|_{(Y,X)}[-d] \underset{[3,A6]}{\xrightarrow{\sim}} \iota_!\big(\mathcal{G} \otimes_{(U,X)} \mathcal{I}_f^{a,b}|_{(U,X)}\big)[-d] = \iota_!(\mathcal{G}^{a,b})$, then we get $\mathcal{H}^{\dagger i}_{Z_1} j_!(\mathcal{E}^{a,b}) \xrightarrow{\sim} \mathcal{H}^{\dagger i}_{Z_1}(\iota_!(\mathcal{G}^{a,b}))$. Hence, by functoriality and by exactness of the functor $i_{1+} \circ \iota_{1!} \circ ({}^\dagger D_1)$, we reduce to check that $\varinjlim\!\!\!\!\varprojlim N_{1,\mathcal{F}^{a,b}}$ is an isomorphism. Since $N_{1,\mathcal{F}^{a,b}} = N_{1,\mathcal{F}} \otimes \mathrm{id} + \mathrm{id} \otimes N_{1,\mathcal{I}^{a,b}_{f,\log}}$, and since there exists an integer $n$ (independent of $a, b$) such that $N^n_{1,\mathcal{F}} = 0$, then we reduce to checking that $\underset{\leftrightarrow}{\lim}\, N_{1,\mathcal{I}^{a,b}_{f,\log}}$ is an isomorphism, which is obvious since $N_{1,\mathcal{I}^{a,b}_{f,\log}}$ is the multiplication by $s$.

(**2**) Finally, let us reduce the lemma to 1). We proceed by induction on $\dim X$. We can suppose that $j$ is dominant. Recalling that $Y$ being reduced, there exists a dominant open immersion $U \to Y$ such that $U$ is smooth and $\mathcal{G} := \iota^+(\mathcal{E}) \in F\text{-Isoc}^{\dagger\dagger}(U, X/K)$, where $\iota\colon (U, X) \to (Y, X)$. By the induction hypothesis, we can suppose that $\mathcal{E} = \iota_!(\mathcal{G})$. Put $T := X \backslash U$. Then, we can suppose that $U, Y, X$ are integral and that $\iota$ is affine. Let $\alpha\colon \widetilde{X} \to X$ be a proper surjective generically finite and étale morphism, such that $\widetilde{X}$ is smooth and quasi-projective, $\widetilde{T} := \alpha^{-1}(T)$ is a strict normal crossing divisor of $\widetilde{X}$. We put $\alpha\colon (\widetilde{X}, \widetilde{X}) \to (X, X)$ (by abuse of notation), $\widetilde{Z} := \alpha^{-1}(Z)$, $\widetilde{Y} := \alpha^{-1}(Y)$, $\widetilde{U} := \alpha^{-1}(U)$, $\beta\colon (\widetilde{Y}, \widetilde{X}) \to (Y, X)$, $\gamma\colon (\widetilde{U}, \widetilde{X}) \to (U, X)$, $\widetilde{\iota}\colon (\widetilde{U}, \widetilde{X}) \hookrightarrow (\widetilde{Y}, \widetilde{X})$, $\widetilde{j}\colon (\widetilde{Y}, \widetilde{X}) \hookrightarrow (\widetilde{X}, \widetilde{X})$, $\widetilde{\mathcal{G}} := \gamma^!(\mathcal{G})$, $\widetilde{\mathcal{E}} := \widetilde{\iota}_!(\widetilde{\mathcal{G}})$. Notice that since $\widetilde{Z} := (f \circ \alpha^{-1}(0)$ is a divisor included in $\widetilde{T}$, then $\widetilde{Z}$ is also a strict normal crossing divisor. By Kedlaya's semistable reduction theorem [24], there exists such a morphism $\alpha$ satisfying moreover the following property: the object $\widetilde{\mathcal{G}} \in F\text{-Isoc}^{\dagger\dagger}(\widetilde{U}, \widetilde{X}/K)$ is unipotent. We know that $\mathcal{G}$ is a direct factor of $\mathscr{H}^0\gamma_+(\widetilde{\mathcal{G}})$ (see the proof of [12, 6.1.4] at the beginning of p.433). Then $\mathcal{E} = \iota_!(\mathcal{G})$ is a direct factor of $\iota_! \mathscr{H}^0 \gamma_!(\widetilde{\mathcal{G}}) \xrightarrow{\sim} \mathscr{H}^0 \beta_! \circ \widetilde{\iota}_!(\widetilde{\mathcal{G}})$. Thus we are reduced to checking the lemma for $\mathscr{H}^0 \beta_! \circ \widetilde{\iota}_!(\widetilde{\mathcal{G}})$. We have

$$\begin{aligned}\mathscr{H}^{-d} \beta_! \circ \widetilde{\iota}_!(\widetilde{\mathcal{G}}) \otimes_{(Y,X)} (\mathcal{I}_f^{a,b}|_{(Y,X)}) &\xrightarrow{\sim} \mathscr{H}^{-d} \beta_!\big(\widetilde{\iota}_!(\widetilde{\mathcal{G}}) \otimes_{(\widetilde{Y},\widetilde{X})} \beta^+(\mathcal{I}_f^{a,b}|_{(Y,X)})\big) \\ &= \mathscr{H}^{-d} \beta_!(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})}) \qquad (\star)\end{aligned}$$

where $\widetilde{f} = f \circ \alpha$ (the equality comes from $\widetilde{\iota}_!(\widetilde{\mathcal{G}}) = \widetilde{\mathcal{E}}$ and $\beta^+(\mathcal{I}_f^{a,b}|_{(Y,X)}) = \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})}$). By applying the exact functor $j_!$ (resp. $j_+$) to the composition isomorphism of ($\star$), we get the first isomorphisms of the following ones:

$$j_!(\mathscr{H}^{-d}\beta_! \circ \widetilde{\iota}_!(\widetilde{\mathcal{G}}) \otimes_{(Y,X)} (\mathcal{I}_f^{a,b}|_{(Y,X)})) \xrightarrow{\sim} j_! \circ \mathscr{H}^{-d}\beta_!(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})})$$
$$\xrightarrow{\sim} \mathscr{H}^{-d}\alpha_! \circ \widetilde{j}_!(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})})$$
$$j_+(\mathscr{H}^{-d}\beta_! \circ \widetilde{\iota}_!(\widetilde{\mathcal{G}}) \otimes_{(Y,X)} (\mathcal{I}_f^{a,b}|_{(Y,X)})) \xrightarrow{\sim} j_+ \circ \mathscr{H}^{-d}\beta_!(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})})$$
$$\xrightarrow{\sim} \mathscr{H}^{-d}\alpha_! \circ \widetilde{j}_+(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})}).$$

From 1), The fact that the canonical morphism $\varprojlim\!\!\!\!\!\!\longleftrightarrow \widetilde{j}_!(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})}[-d]) \to \underset{\longleftrightarrow}{\lim}\, \widetilde{j}_+(\widetilde{\mathcal{E}} \otimes \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})}[-d])$ is an isomorphism is local in $\widetilde{X}$. Hence, we reduce to the case where $\widetilde{X}$, $\widetilde{T}$, and $\widetilde{Z}$ satisfy the conditions of the part 1) of the proof in place of respectively $X$, $T$, and $Z$. Hence, from part 1), $\underset{\longleftrightarrow}{\lim}\, \widetilde{j}_!(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})}[-d]) \to \underset{\longleftrightarrow}{\lim}\, \widetilde{j}_+(\widetilde{\mathcal{E}} \otimes \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})}[-d])$ is an isomorphism. Then so is $\underset{\longleftrightarrow}{\lim}\, \mathscr{H}^{-d}\alpha_! \circ \widetilde{j}_!(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})}) \to \underset{\longleftrightarrow}{\lim}\, \mathscr{H}^{-d}\alpha_! \circ \widetilde{j}_+(\widetilde{\mathcal{E}} \otimes_{(\widetilde{Y},\widetilde{X})} \mathcal{I}_{\widetilde{f}}^{a,b}|_{(\widetilde{Y},\widetilde{X})})$. □

**2.5** Let $\mathcal{E} \in \mathrm{Hol}_F(Y/K)$. With the notation of 2.4, we put $\mathcal{E}_k^{a,b} := \mathcal{E}^{\max\{a,k\},\max\{b,k\}}$ for any integer $k \in \mathbb{Z}$. We get $\mathcal{E}_k^{\bullet,\bullet} \in \underset{\longleftrightarrow}{\lim}\, \mathrm{Hol}_F(Y/K)$. Now, for $\mathcal{E} \in \mathrm{Hol}_F(Y/K)$, we put

$$\Pi_{!+}^{a,b}(\mathcal{E}) := \underset{\longleftrightarrow}{\lim}\, j_+(\mathcal{E}_a^{\bullet,\bullet}) / \underset{\longleftrightarrow}{\lim}\, j_!(\mathcal{E}_b^{\bullet,\bullet})$$

in $\underset{\longleftrightarrow}{\lim}\, \mathrm{Hol}_F(X/K)$. By Lemma 2.4, this is in fact[6] in $\mathrm{Hol}_F(X/K)$, which yields a functor $\Pi_{!+}^{a,b} \colon \mathrm{Hol}_F(Y/K) \to \mathrm{Hol}_F(X/K)$. The following properties can be checked easily:

1. By (2.3.3), we have $\mathbb{D} \circ \Pi_{!+}^{a,b} \cong (\Pi_{!+}^{-b,-a} \circ \mathbb{D})(1)$.
2. The isomorphism $\sigma^n$ of 2.3.2 induces an isomorphism $\Pi_{!+}^{a,b} \xrightarrow{\sim} \Pi_{!+}^{a+n,b+n}(-n)$.

[6] Since this deduction is formal and not explained in [5], further explanations might be needless for experts, but we point out that the details are written down in Lichtenstein's thesis [26, Prop 3.21]. However, there is a small mistake in Lichtenstein's argument, as well as some obvious typos: he claims that there exists an isomorphism of diagrams $\widetilde{\varphi}\mathcal{F}_!^{a,b} \xrightarrow{\sim} \widetilde{\varphi}\mathcal{F}_*^{a,b}$ for some $\varphi \geq \mathbf{1}_{\mathbb{Z}}$ using the notation in *ibid.*, but this is wrong in general. This issue can be resolved as follows: Since $\alpha \colon \underset{\longleftrightarrow}{\lim}\mathcal{F}_!^{a,b} \to \underset{\longleftrightarrow}{\lim}\mathcal{F}_*^{a,b}$ is assumed to be an isomorphism, (2.1.1) tells us that there exist $\varphi \geq \mathbf{1}_{\mathbb{Z}}$ and a homomorphism of diagram $\beta \colon \widetilde{\varphi}\mathcal{F}_*^{a,b} \to \mathcal{F}_!^{a,b}$ which induces the inverse of $\alpha$ if we pass to the pro-ind category. Now, we consider the last big diagram in the proof of *ibid.*. Because of the mistake, we do not have the isomorphism #, but just the canonical homomorphism $\underset{\longleftrightarrow}{\lim}\mathcal{F}_{!,\varphi\ell}^{a,b} \to \underset{\longleftrightarrow}{\lim}\mathcal{F}_{*,\varphi\ell}^{a,b}$. However, we do have the homomorphism $\#' \colon \underset{\longleftrightarrow}{\lim}\mathcal{F}_{*,\varphi\ell}^{a,b} \to \underset{\longleftrightarrow}{\lim}\mathcal{F}_{!,\ell}^{a,b}$ (from the target of # to the target of (1)), induced by $\beta$, making the diagram commutative. Other homomorphisms or isomorphisms remain to be the same: since the isomorphism # is used only to show the existence of the isomorphism $\mathrm{coker}_{k,\ell} = \mathrm{coker}(\star)$, the existence of $\#'$ is enough to show the equality.

We put $\Psi_f^{(i)} := \Pi_{!+}^{i,i}$, $\Xi_f^{(i)} := \Pi_{!+}^{i,i+1}$, and put $\Psi_f := \Psi_f^{(0)}$, $\Xi_f := \Xi_f^{(0)}$. The functor $\Psi_f$ is called the *unipotent nearby cycle functor*. The isomorphisms

$$\varprojlim j_!(\mathcal{E}_i^{\bullet,\bullet})/\varprojlim j_!(\mathcal{E}_{i+1}^{\bullet,\bullet}) \cong j_!(\mathcal{E})(i), \qquad \varprojlim j_+(\mathcal{E}_i^{\bullet,\bullet})/\varprojlim j_+(\mathcal{E}_{i+1}^{\bullet,\bullet}) \cong j_+(\mathcal{E})(i)$$

induce exact sequences

$$0 \to j_!(\mathcal{E})(i) \xrightarrow{\alpha_-} \Xi_f^{(i)}(\mathcal{E}) \xrightarrow{\beta_-} \Psi_f^{(i)}(\mathcal{E}) \to 0,$$
$$0 \to \Psi_f^{(i+1)}(\mathcal{E}) \xrightarrow{\beta_+} \Xi_f^{(i)}(\mathcal{E}) \xrightarrow{\alpha_+} j_+(\mathcal{E})(i) \to 0.$$

We define a functor $\Phi_f\colon \mathrm{Hol}_F(X/K) \to \mathrm{Hol}_F(Z/K)$ as follows. Let $\mathcal{E} \in \mathrm{Hol}_F(X/K)$, and put $\mathcal{E}_Y := j^+(\mathcal{E})$. Let $\gamma_-\colon j_!(\mathcal{E}_Y) \to \mathcal{E}$ and $\gamma_+\colon \mathcal{E} \to j_+(\mathcal{E}_Y)$ be the adjunction homomorphisms. Consider the sequence

$$j_!\mathcal{E}_Y \xrightarrow{(\alpha_-,\gamma_-)} \Xi_f(\mathcal{E}_Y) \oplus \mathcal{E} \xrightarrow{(\alpha_+,-\gamma_+)} j_+(\mathcal{E}_Y). \tag{2.5.1}$$

The cohomology of this sequence is $\Phi_f(\mathcal{E})$, and the functor $\Phi_f$ is called the *unipotent vanishing cycle functor*.

*Remark 2.6* (i) In fact we have checked in the key lemma 2.4 that the canonical morphism $\alpha^{\bullet,\bullet}\colon j_!(\mathcal{E}^{\bullet,\bullet}) \to j_+(\mathcal{E}^{\bullet,\bullet})$ of $\mathrm{Hol}_F(X/K)_{\mathrm{a}}^{\Pi}$ becomes an isomorphism in $(S_{\mathrm{a}}^f)^{-1}\mathrm{Hol}_F(X/K)_{\mathrm{a}}^{\Pi}$. We remark that this is equivalent to saying that there exist an integer $N$ large enough and a morphism $\beta^{\bullet,\bullet}\colon j_+(\mathcal{E}^{\bullet+N,\bullet+N}) \to j_!(\mathcal{E}^{\bullet,\bullet})$ of $\mathrm{Hol}_F(X/K)_{\mathrm{a}}^{\Pi}$ so that the morphisms $\alpha^{\bullet,\bullet}\circ\beta^{\bullet,\bullet}$ and $\beta^{\bullet,\bullet}\circ\alpha^{\bullet+N,\bullet+N}$ are the canonical morphisms. Since the multiplication by $s^N$ factors through $j_+(\mathcal{E}^{\bullet,\bullet})(N) \xrightarrow{\sim} j_+(\mathcal{E}^{\bullet+N,\bullet+N}) \to j_+(\mathcal{E}^{\bullet,\bullet})$, we get that coker $\alpha^{a,b}$ and ker $\alpha^{a,b}$ are killed by $s^N$. For any integer $i \geq 0$, this implies that the projective system coker $(s^i\alpha^{a,b}(i))$ stabilizes for $b$ large enough (with $a$ and $i$ fixed). We remark that this limit is isomorphic to $\Pi_{!+}^{a,a+i}$, which is the analogue of the remark by Beilinson and Bernstein in [7, 4.2].

(ii) Crew constructed in [22] nearby and vanishing cycle functors in the local situation. These functors should be closely related to what we defined here. Let us take $X = \mathbb{A}^1$, and $f = 1$. Let $\mathscr{S}$ the formal disk around $0 \in X$, and let $\mathcal{M}$ be a $\mathcal{D}_{\mathscr{S}}^{\mathrm{an}}$-module with Frobenius structure using the notation of [*ibid.*, (4.1.10)]. Let $\mathcal{M}^{\mathrm{can}}$ be the canonical extension of $\mathcal{M}$ (cf. [*ibid.*, 8.2]), which is in $\mathrm{Hol}_F(X)$. Then we should have

$$\Psi_f(\mathcal{M}^{\mathrm{can}}) \otimes K^{\mathrm{ur}} \cong \mathbb{V}(\mathbb{D}(\mathcal{M}))^I, \qquad \Phi_f(\mathcal{M}^{\mathrm{can}}) \otimes K^{\mathrm{ur}} \cong \mathbb{W}(\mathbb{D}(\mathcal{M}))^I,$$

where $\mathbb{V}$ and $\mathbb{W}$ are nearby and vanishing cycle functor defined in [22, (6.1.7)], $K^{\mathrm{ur}}$ denotes the maximal unramified extension of $K$, and $I$ is the inertia subgroup of the Galois group of $k(\!(t)\!)$. We did not work out in detail to check this. The computation [3, 1.5.9 (iii)] might be used to show this.

**Proposition 2.7** *The functors* $\Pi_{!+}^{a,b}$ *and* $\Phi_f$ *are exact. When* $\mathcal{E}$ *is in* $\mathrm{Hol}_F(Z/K)$*, then* $\mathcal{E} \cong \Phi_f(\mathcal{E})$ *canonically.*

*Proof* The exactness of $\Pi^{a,b}_{!+}$ follows by that of $j_!$ and $j_+$. The exactness of $\Phi_f$ follows since $\alpha_-$ is injective and $\alpha_+$ is surjective. The last claim follows by definition. □

*Remark* Since we do not use it in the proof of the main theorem, we do not go into the details, but it is straightforward to get an analogue of [6, Prop 3.1], a gluing theorem of holonomic modules.

## 3 Comparison of Euler characteristics

Let $X$ be a realizable $k$-variety, $p_X\colon X \to \operatorname{Spec} k$ be the structural morphism and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$. We put $\chi(\mathcal{E}) := \sum_{i\in\mathbb{Z}}(-1)^i \dim_K H^i p_{X+}(\mathcal{E})$ and $\chi_c(\mathcal{E}) := \sum_{i\in\mathbb{Z}}(-1)^i \dim_K H^i p_{X!}(\mathcal{E})$. We denote by $K(D^{\mathrm{b}}_{\mathrm{hol},F}(X/K))$ the Grothendieck group of the triangulated category $D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$. We put $K(X) := K(D^{\mathrm{b}}_{\mathrm{hol},F}(X/K))$. We simply denote by $[\ ]\colon D^{\mathrm{b}}_{\mathrm{hol},F}(X/K) \to K(X)$ the additive universal function.

If $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to \mathcal{E}[1]$ is an exact triangle of $D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$ then we have the equalities $\chi(\mathcal{E}) = \chi(\mathcal{E}') + \chi(\mathcal{E}'')$ and $\chi_c(\mathcal{E}) = \chi_c(\mathcal{E}') + \chi_c(\mathcal{E}'')$. Hence, by the universal property, $\chi$ and $\chi_c$ factors respectively as homomorphisms of groups $\chi\colon K(X) \to \mathbb{Z}$ and $\chi_c\colon K(X) \to \mathbb{Z}$.

Let $f\colon X \to Y$ be a morphism of realizable $k$-varieties. Similarly, the push-forward $f_+$ and the extraordinary push-forward $f_!$ factors respectively as homomorphisms of groups $[f]_+\colon K(X) \to K(Y)$ and $[f]_!\colon K(X) \to K(Y)$.

**Lemma 3.1** *Let $f\colon X \to \mathbb{A}^1_k$ be a morphism of realizable $k$-varieties. Let $Y := f^{-1}(\mathbb{G}_{\mathrm{m,k}})$ $j\colon Y \subset X$ be the open immersion. We have the equality $[j]_+ = [j]_!$, i.e. $[j]_+$ commutes with dual homomorphisms.*

*Proof* From 2.5 for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$, we have the exact sequences

$$0 \to j_!(\mathcal{E}) \xrightarrow{\alpha_-} \Xi_f(\mathcal{E}) \xrightarrow{\beta_-} \Psi_f(\mathcal{E}) \to 0,\ 0 \to \Psi_f(\mathcal{E})(1) \xrightarrow{\beta_+} \Xi_f(\mathcal{E}) \xrightarrow{\alpha_+} j_+(\mathcal{E}) \to 0,$$

which yield the lemma. □

**Theorem 3.2** *Let $f\colon X \to Y$ be a morphism of realizable $k$-varieties. We have the equality $[f]_+ = [f]_!$.*

*Proof* Using 3.1, we can follow the (beginning of the) proof of [25, 1.1]. □

**Corollary 3.3** *Let $X$ be a realizable $k$-variety. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$. We have the equality $\chi(\mathcal{E}) = \chi_c(\mathcal{E})$. In other words, $\chi(\mathcal{E}) = \chi(\mathbb{D}_X(\mathcal{E}))$.*

**Acknowledgements** The first author (T.A.) was supported by Grant-in-Aid for Young Scientists (B) 25800004. The second author (D.C.) thanks Antoine Chambert-Loir for his suggestion to consider the comparison of Euler characteristics in the $p$-adic context. The second author (D.C) was supported by the I.U.F.

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