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Witt vectors as a polynomial functor

D. Kaledin[1,2,3]

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Abstract For every commutative ring A, one has a functorial commutative ring $W(A)$ of p-typical Witt vectors of A, an iterated extension of A by itself. If A is not commutative, it has been known since the pioneering work of L. Hesselholt that $W(A)$ is only an abelian group, not a ring, and it is an iterated extension of the Hochschild homology group $HH_0(A)$ by itself. It is natural to expect that this construction generalizes to higher degrees and arbitrary coefficients, so that one can define "Hochschild–Witt homology" $WHH_*(A, M)$ for any bimodule M over an associative algebra A over a field k. Moreover, if one want the resulting theory to be a trace theory, then it suffices to define it for $A = k$. This is what we do in this paper, for a perfect field k of positive characteristic p. Namely, we construct a sequence of polynomial functors W_m, $m \geq 1$ from k-vector spaces to abelian groups, related by restriction maps, we prove their basic properties such as the existence of Frobenius and Verschiebung maps, and we show that W_m are trace functors. The construction is very simple, and it only depends on elementary properties of finite cyclic groups.

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To Sasha Beilinson, on his birthday.

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✉ D. Kaledin
kaledin@mi.ras.ru

1 Algebraic Geometry Section, Steklov Mathematics Institute, Moscow, Russia

2 Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow, Russia

3 Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang, Korea

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Introduction

Recall that to any commutative ring A, one canonically associates the ring $W(A)$ of *p-typical Witt vectors* of A (a reader not familiar with the subject can find a great overview for example in [12, Section 0.1]). Witt vectors are functorial in A, and $W(A)$ is the inverse limit of rings $W_m(A)$ of *m-truncated p-typical Witt vectors* numbered by integers $m \geq 1$. We have $W_1(A) \cong A$, and for any m, $W_{m+1}(A)$ is an extension of A by $W_m(A)$.

If A is annihilated by a prime p and perfect—that is, the Frobenius endomorphism $F : A \to A$ is bijective—then one has $W_m(A) \cong W(A)/p^m$, and in particular, $A \cong W(A)/p$. If A is not perfect, this is usually not true. However, if A is sufficiently nice—for example, if it is the algebra of functions on a smooth affine algebraic variety—then $W(A)$ has no p-torsion. Thus roughly speaking, the Witt vectors construction provides a functorial way to associate a ring of characteristic 0 to a ring of characteristic p.

Historically, this motivated a lot of interest in the construction. In particular, one of the earliest attempts to construct a Weil cohomology theory, due to Serre [23], was to consider $H^{\bullet}(X, W(\mathcal{O}_X))$, where X is an algebraic variety over a finite field k of positive characteristic p, and $W(\mathcal{O}_X)$ is the sheaf obtained by taking the Witt vectors of its structure sheaf $\mathcal{O}_X$.

This attempt did not quite work, and the focus of attention switched to other cohomology theories discovered by A. Grothendieck: étale cohomology first of all, but also cristalline cohomology introduced slightly later. Much later, Illusie [12] discovered what could be thought of as a vindication of Serre's original approach. They proved that any smooth algebraic variety X over a perfect field k of positive characteristic can be equipped with a functorial *de Rham–Witt complex* $W\Omega^{\bullet}_X$, an extension of the usual de Rham complex $\Omega^{\bullet}_X$. In degree 0, one has $W\Omega^0_X \cong W(\mathcal{O}_X)$, but in higher degrees, one needs a new construction. The resulting complex computes cristalline cohomology $H^{\bullet}_{cris}(X)$ of X, in the sense that for proper X, one has a canonical isomorphism $H^{\bullet}_{cris}(X) \cong H^{\bullet}(X, W\Omega^{\bullet}_X)$, and cristalline cohomology is known to be a Weil cohomology theory.

Yet another breakthrough in our understanding of Witt vectors happenned in 1995, and it was due to Hesselholt [9]. What he did was to construct Witt vectors $W(A)$ for an arbitrary associative ring A. Hesselholt's $W(A)$ is also the inverse limit of its truncated version $W_m(A)$, and if A is commutative and unital, then it coincides with the classical Witt vectors ring. But if A is not commutative, $W(A)$ is not even a ring—it is only an abelian group. We have $W_1(A) = A/[A, A]$, the quotient of the algebra A by the subgroup spanned by commutators of its elements, and for any m, $W_{m+1}(A)$ is an extension of $A/[A, A]$ by a quotient of $W_m(A)$.

In the context of non-commutative algebra and non-commutative algebraic geometry, one common theme of the two constructions is immediately obvious: Hochschild homology. On one hand, for any associative ring, $A/[A, A]$ is the 0-th Hochschild homology group $HH_0(A)$. On the other hand, for a smooth affine algebraic variety $X = \operatorname{Spec} A$, the spaces $H^0(X, \Omega^i_X)$ of differential forms on X are identified with the Hochshchild homology groups $HH_i(A)$ by the famous theorem of Hochschild, Kostant and Rosenberg. Thus one is lead to expect that a unifying theory would use an associative unital k-algebra A as an input, and product what one could call "Hochschild–Witt homology groups" $WHH_{\bullet}(A)$ such that in degree 0, $WHH_0(A)$ coincides with Hesselholt's Witt vectors, while for a commutative A with smooth spectrum $X = \operatorname{Spec} A$, we would have natural identifications $WHH_i(A) \cong H^0(X, W\Omega^i_X)$.

But "expect" is perhaps a wrong word here, since this relation to Hochschild homology is already made abundantly clear in Hesselholt's work. In fact, while the main construction in [9] is purely algebraic, its origins are in algebraic topology—specificaly, the theory of Topological Cyclic Homology and cyclotomic trace of Bokstedt et al. [3,11], itself a development of Topological Hochschild Homology of Bokstedt [2]. Among other things, [3] associates a certain spectrum $TR(A, p)$ to any ring spectrum A. If A is a k-algebra, $\operatorname{char} k = p$, then $TR(A, p)$ is an Eilenberg-Mac Lane spectrum, thus essentially a chain complex. The Witt vectors group $W(A)$ constructed explicitly by Hesselholt is the homology group of this complex in degree 0. He also proved [8,10] that for a commutative k-algebra A with smooth spectrum X, all the homotopy groups of the spectrum $TR(A, p)$ coincide with the de Rham–Witt forms $H^0(X, W\Omega^{\bullet}_X)$. Thus the homotopy groups of $TR(A, p)$ are already perfectly good candidates for hypothetical Hochschild–Witt homology groups. What we lack is an algebraic construction of these groups in degrees > 0.

However, while the relevance of Hochschild homology for Witt vectors has been well-understood by topologists from the very beginning, one of its features has been somewhat overlooked. Namely, Hochschild homology is in fact a theory with two variables—an algebra A and an A-bimodule M (that is, a module over the product $A^o \otimes A$ of A with its opposite algebra A^o). To obtain $HH_{\bullet}(A)$, one takes as M the diagonal bimodule A, but the groups $HH_{\bullet}(A, M)$ are well-defined for any bimodule. Moreover, Hochschld homology has the following trace-like property: for any two algebras A, B, a left module M over $A^o \otimes B$, and a left module N over $B^o \otimes A$, we have a canonical isomorphism

$$HH_\bullet(A, M \overset{\mathsf{L}}{\otimes}_B N) \cong HH_\bullet(B, N \overset{\mathsf{L}}{\otimes}_A M),$$

subject to some natural compatibility conditions. This was axiomatized and studied in the recent work of Ponto [22], with references to even earlier work. It has also been observed and axiomatized under the name of "trace theory" and "trace functor" in [15]. The essential point is this: if one wants to have a generalization of Hochschild homology that is a functor of two variables A, M and has trace isomorphisms, then it suffices to define it for $A = k$—there is general machine that automatically produces the rest. Thus one can trade the first variable for the second one: instead of constructing $WHH_\bullet(A)$ for an arbitrary A, one can construct $WHH_\bullet(k, M)$ for an arbitrary k-vector space M. This is hopefully simpler. In particular, it is reasonable to expect that $WHH_i(k, M) = 0$ for $i \geq 1$, so that the problem reduces to constructing a single functor from k-vector spaces to abelian groups.

One possible approach to this is to go back to the original paper [1] of Bloch that motivated [8, 12], and use algebraic K-theory. It seems that this approach indeed works; in fact, a definition along these lines has been sketched by the author a couple of years ago and presented in several talks. The main technical ingredient of the definition is a certain completed version of K_1 of free non-commutative algebras in several variables. The construction works over an arbitrary commutative ring, not just over a perfect field, and provides a version of "universal", or "big" Witt vectors; the p-typical Witt vectors that we discuss here are then extracted by a separate procedure. Unfortunately, the complete construction turns out to be longer than one would wish, and the rather elementary nature of the resulting Witt vectors functor is somewhat obscured by the machinery needed to define it. This is one of the reasons why despite many promises, the story has not been yet written down.

The goal of the present paper is to alleviate the situation by presenting a very simple and direct alternative construction motivated by recent work of V. Vologodsky. In a nutshell, here is the basic idea: instead of trying to associate an abelian group to a k-vector space M directly, one should lift M to a free $W(k)$-module in some way, use it for the construction, and then prove that the result does not depend on the lifting. The resulting definition only works over a perfect field k of characteristic p and assumes that we already know the classical Witt vectors ring $W(k)$. However, it produces directly an inverse system of p-typical Witt vectors functors W_m, and it only uses elementary properties of cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$, $n \geq 0$. The functors W_m are polynomial, thus the "polynomial functor" of the title.

The paper is organized as follows. The longish Sect. 1 contains the necessary preliminaries. Most of the material is standard, and we include it mostly to set up notation. We also give a short recollection on the theory of Mackey functors, the only piece of relatively high technology required in the body of the paper. In principle, even this could have been avoided. However, Mackey functors do help—in particular, they provide for free some very useful canonical filtrations on our Witt vectors, and explain the origin of the projection formula relating the Frobenius and the Verschiebung maps.

Sections 1.3 and 1.4 give an overview of Mackey functors, Sect. 1.5 contains some less standard material specifically adapted to our needs, and Sect. 1.6 shows how things work for cyclic groups. We should mention that an intimate relation between $\mathbb{Z}/p^m\mathbb{Z}$-Mackey functors and m-truncated p-typical Witt vectors is very well known and documented in the literature.

Having finished with the preliminaries, we give our two main definitions, those of Witt vectors and of restriction maps between them. This, together with the proof of correctness, is the subject of Sect. 2. Section 3 explores the basic structure of the Witt vectors functors W_m constructed in Sect. 2. We also prove that W_m are pseudotensor functors (this is Sect. 3.3), and give a slightly more explicit inductive description of W_m that is closer to [9] and to the original construction of Witt (this is Sect. 3.4). Then in the last Sect. 4, we recall the definition of a trace functor from [15], and prove that our Witt vectors W_m are trace functors in a natural way. We finish the paper with some results on compatibility between all the structures we have on W_m, namely, the pseudotensor structure of Sect. 3.3, the trace functor structure of Sect. 4.2, and some remnants of the Mackey functors structure that they carry by definition.

As mentioned above, after we define Witt vectors functors on k-vector spaces and equip them with trace isomorphisms, constructing a Hochschild–Witt complex for any k-algebra A with coefficients in an A-bimodule M becomes automatic. However, in the present paper, we do not do this. We feel that exploring the resulting Hochschild–Witt complex deserves a separate treatment, and we relegate it to a companion paper [18]. The same goes for the comparison results with Hesselholt's Witt vectors and with the de Rham–Witt complex. Any possible comparison results with $TR(A, p)$ would require much more technology than we presently have, so will return to it elsewhere. Another very interesting subject for comparison is a version of non-commutative Witt vectors given recently by Cuntz and Deninger [4]. At the moment, we do not know what is the relation between the two constructions, and we feel that it deserves further research.

1 Preliminaries

1.1 Small categories

For any category $\mathcal{C}$, we will denote by $\mathcal{C}^o$ the opposite category. We will denote by pt the point category (one object, one morphism). For any group G, we will denote by pt_G the groupoid with one object with automorphism group G. For any integer $l \geq 1$, we will simplify notation by letting $\mathsf{pt}_l = \mathsf{pt}_{\mathbb{Z}/l\mathbb{Z}}$. We note that for any functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between categories $\mathcal{C}_1$, $\mathcal{C}_2$, giving a G-action on F is equivalent to extending it to a functor

$$\widetilde{F} : \mathsf{pt}_G \times \mathcal{C}_1 \to \mathsf{pt}_G \times \mathcal{C}_2 \tag{1.1}$$

that commutes with projections to pt_G. If the functor F admits a left resp. right-adjoint functor F', then G acts on F' by adjunction, and $\widetilde{F}$ is left resp. right-adjoint to $\widetilde{F}'$.

For any small category I and any category $\mathcal{C}$, we will denote by $\operatorname{Fun}(I, \mathcal{C})$ the category of functors from I to $\mathcal{C}$. For any ring A, we denote by A-mod the category of left A-modules, and for any small category I, we will simplify notation by letting $\mathrm{Fun}(I, A) = \mathrm{Fun}(I, A\text{-mod})$. This is an abelian category; we will denote by $\mathcal{D}(I, A)$ its derived category. In particular, for any group G, $\mathrm{Fun}(\mathsf{pt}_G, A)$ is the category $A[G]$-mod of left modules over the group algebra $A[G]$. For any functor $\gamma : I \to I'$ between small categories, we denote by $\gamma^* : \mathrm{Fun}(I', A) \to \mathrm{Fun}(I, A)$ the pullback functor, and we denote by $\gamma_!, \gamma_* : \mathrm{Fun}(I, A) \to \mathrm{Fun}(I', A)$ its left and right adjoint (the left and right Kan extensions along γ). The functor γ^* is exact, hence descends to derived categories, and the derived functors $L^\bullet\gamma_!, R^\bullet\gamma_* : \mathcal{D}(I, A) \to \mathcal{D}(I', A)$ are left and right adjoint to $\gamma^* : \mathcal{D}(I', A) \to \mathcal{D}(I, A)$.

If the ring A is commutative, then the category $\mathrm{Fun}(I, A)$ is a symmetric unital tensor category with respect to the pointwise tensor product. For any functor $\gamma : I' \to I$, the pullback functor γ^* is a tensor functor. Recall that a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between unital symmetric monoidal categories is *pseudotensor* if it is equipped with functorial maps

$$\varepsilon : 1 \to F(1), \quad \mu : F(M) \otimes F(N) \to F(M \otimes N), \quad M, N \in \mathcal{C}_1, \tag{1.2}$$

where 1 stands for the unit object, and these maps are associative and unital in the obvious sense. A pseudotensor functor is *symmetric* if the maps μ are also commutative. Then a right-adjoint to a symmetric tensor functor is automatically symmetric pseudotensor by adjunction. In particular, for any functor $\gamma : I' \to I$, the functor $\gamma_* : \mathrm{Fun}(I', A) \to \mathrm{Fun}(I, A)$ is symmetric pseudotensor.

We will assume known the notions of a fibration, a cofibration and a bifibration of small categories originally introduced in [7]. We also assume known the following useful base change lemma: if we are given a cartesian square

$$\begin{array}{ccc} I'_1 & \xrightarrow{f_1} & I_1 \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ I' & \xrightarrow{f} & I \end{array}$$

of small categories, and π is a cofibration, then π' is a cofibration, and the base change map $L^\bullet\pi'_! \circ f_1^* \to f^* \circ L^\bullet\pi_!$ is an isomorphism. Dually, if π is a fibration, then π' is a fibration, and $f^* \circ R^\bullet\pi_* \cong R^\bullet\pi'_* \circ f_1^*$. For a proof, see e.g. [13, Lemma 1.7].

1.2 Trace maps

One specific example of a base change situation is a bifibration $\pi : I' \to I$ whose fiber is equivalent to a groupoid pt_G. In this case, for any $E \in \mathrm{Fun}(I', A)$ and any object $i' \in I'$ with image $i = \pi(i') \in I$, the A-module $E(i')$ carries a natural action of the group G, and we have base change isomorphisms

$$\pi_! E(i) \cong E(i')_G, \qquad \pi_* E(i) \cong E(i')^G,$$

where in the right-hand side, we have coinvariants and invariants with respect to G. If the group G is finite, then for any $A[G]$-module V, we have a natural trace map

$$\mathsf{tr}_G = \sum_{g \in G} g : V_G \to V^G \tag{1.3}$$

whose cokernel is the Tate cohomology group $\check{H}^0(G, V)$. Taken together, these maps then define a natural trace map

$$\mathsf{tr}_\pi : \pi_! E \to \pi_* E. \tag{1.4}$$

This map is functorial in E and compatible with the base change. If $E = \pi^* E'$ for some $E' \in \mathrm{Fun}(I, A)$, then

$$\mathsf{tr}_\pi = |G| \, \mathsf{id} : \pi_! \pi^* E' \to \pi_* \pi^* E', \tag{1.5}$$

where $|G|$ is the order of the finite group G. In the general case, we denote by $\mathsf{tr}_\pi^\dagger : E \to E$ the composition

$$E \xrightarrow{\ l\ } \pi^* \pi_! E \xrightarrow{\pi^*(\mathsf{tr}_\pi)} \pi^* \pi_* E \xrightarrow{\ r\ } E, \tag{1.6}$$

where l and r are the adjunction maps, and we denote by

$$\check{\pi}_* : \mathrm{Fun}(I', A) \to \mathrm{Fun}(I, A) \tag{1.7}$$

the functor sending E to the cokernel of the map (1.4). For any $i' \in I'$ with $i = \pi(i')$, we have a natural identification $\check{\pi}_*(E)(i) \cong \check{H}^0(G, E(i'))$. We note that even if the categories I, I' are not small, the map $\mathsf{tr}_\pi^\dagger : E \to E$ of (1.6) is perfectly well-defined for every functor $E : I' \to k$-mod.

Lemma 1.1 *Assume that the ring A is commutative. Then the functor $\check{\pi}_*$ of (1.7) is pseudotensor.*

Proof Since by definition, $\check{\pi}_*$ is a quotient of a pseudotensor functor π_*, it suffices to check that the maps μ of (1.2) for the functor π_* descend to maps $\check{\pi}_*(M) \otimes_A \check{\pi}_*(N) \to \check{\pi}_*(M \otimes_A N)$, $M, N \in \mathrm{Fun}(I', A)$. This can be checked pointwise on I. Thus we may assume that $I = \mathsf{pt}$ is a point, M and N are $A[G]$-modules, and $\check{\pi}_*$ is the Tate cohomogy functor $\check{H}^0(G, -)$. Then the claim is well-known (and easily follows from the obvious equality $\mathsf{tr}_G(m \otimes n) = \mathsf{tr}_G(m) \otimes n$, $m \in M$, $n \in N^G \subset N$). □

Finally, assume that we have a normal subgroup $N \subset G$ with the quotient $W = G/N$, and a bifibration $\pi : I'' \to I$ with fiber pt_G factors as $\pi = \pi' \circ \pi''$, where $\pi'' : I'' \to I'$ is a bifibration with fiber pt_N, and $\pi' : I' \to I$ is a bifibration with fiber W. Then we have a commutative diagram

$$\begin{array}{ccccc}
\pi'_! \circ \pi''_! & \xrightarrow{\ \sim\ } & \pi_! & \xrightarrow{\mathsf{tr}_{\pi'}} & \pi'_* \circ \pi''_! \\
{\scriptstyle \pi'_!(\mathsf{tr}_{\pi''})}\downarrow & & \downarrow{\scriptstyle \mathsf{tr}_\pi} & & \downarrow{\scriptstyle \pi'_*(\mathsf{tr}_{\pi''})} \\
\pi'_! \circ \pi''_* & \xrightarrow{\mathsf{tr}_{\pi'}} & \pi_* & \xrightarrow{\ \sim\ } & \pi'_* \circ \pi''_*.
\end{array} \tag{1.8}$$

In particular, we have

$$\mathsf{tr}_\pi = \pi'_*(\mathsf{tr}_{\pi''}) \circ \mathsf{tr}_{\pi'} = \mathsf{tr}_{\pi'} \circ \pi'_!(\mathsf{tr}_{\pi''}). \tag{1.9}$$

Moreover, taking cokernels of the vertical maps in (1.8), we obtain natural maps

$$\begin{aligned} l_\pi :& \pi'_! \circ \check{\pi}''_* = \pi'_!(\mathrm{Coker}(\mathsf{tr}_{\pi''})) \cong \mathrm{Coker}(\pi'_!(\mathsf{tr}_{\pi''})) \to \check{\pi}_*, \\ r_\pi :& \check{\pi}_* \to \mathrm{Coker}(\pi'_*(\mathsf{tr}_{\pi''})) \to \pi'_*(\mathrm{Coker}(\mathsf{tr}_{\pi''})) = \pi'_* \circ \check{\pi}''_*, \end{aligned}$$

such that $r_\pi \circ l_\pi = \mathsf{tr}_{\pi'}$. By adjunction, these maps induce maps

$$l^\dagger_\pi : \check{\pi}''_* \to \pi'^* \circ \check{\pi}_*, \qquad r^\dagger_\pi : \pi'^* \circ \check{\pi}_* \to \check{\pi}''_*, \tag{1.10}$$

and we have

$$r^\dagger_\pi \circ l^\dagger_\pi = \mathsf{tr}^\dagger_{\pi'} : \check{\pi}''_* \to \check{\pi}_*. \tag{1.11}$$

1.3 Mackey functors

Mackey functors were introduced by Dress [5], with current definition due to Lindner [20]. Good expositions of the theory can be found e.g. [21,24]. We only give a brief overview following [16, Section 2].

Assume given a group G and a ring A. Denote by Γ_G the category of finite G-sets—that is, finite sets equipped with a left G-action. Denote by $Q\Gamma_G$ the category with the same objects, and with morphisms from S_1 to S_2 given by isomorphism classes of diagrams

$$S_1 \xleftarrow{p_1} S \xrightarrow{p_2} S_2 \tag{1.12}$$

in Γ_G, with composition given by pullbacks. Any map $f : S_1 \to S_2$ between G-set defines two maps $f_* : S_1 \to S_2$, $f^* : S_2 \to S_1$ in the category $Q\Gamma_G$, one by letting $p_1 = \mathsf{id}$, $p_2 = f$ in (1.12), and the other by letting $p_1 = f$, $p_2 = \mathsf{id}$. Sending f to f_* resp. f^* gives inclusions $\Gamma_G, \Gamma_G^o \subset Q\Gamma_G$. Say that a functor $E \in \mathrm{Fun}(\Gamma_G^o, A)$ is *additive* if for any $S, S' \in \Gamma_G$ with disjoint union $S\coprod S'$, the natural map

$$E(S\textstyle\coprod S') \to E(S) \oplus E(S')$$

is an isomorphism. An *A-valued G-Mackey functor* is a functor M from $Q\Gamma_G$ to left A-modules whose restriction to Γ_G^o is additive. Mackey functors form a full abelian subcategory $\mathcal{M}(G, A) \subset \mathrm{Fun}(Q\Gamma_G, A)$. The embedding functor $\mathcal{M}(G, A) \to \mathrm{Fun}(Q\Gamma_G, A)$ admits a left-adjoint additivization functor $\mathsf{Add} : \mathrm{Fun}(Q\Gamma_G, A) \to \mathcal{M}(G, A)$.

If $G = \{e\}$ is the trivial group consisting only of its unity element e, so that $\Gamma_G = \Gamma$ is the category of finite sets, then the category $Q\Gamma$ is in fact equivalent to the category of free finitely generated commutative monoids, and $\mathcal{M}(G, A)$ is naturally equivalent to A-mod. The functor $\widetilde{M} : Q\Gamma \to A$-mod corresponding to an A-module M under the equivalence sends a finite set S to $M[S]$, the sum of copies of the module M numbered by elements of the set S.

In general, any finite G-set S decomposes into a disjoint union of G-orbits: we have

$$S = \coprod_{s \in G\backslash S} [G/H_s], \tag{1.13}$$

where $H_s \subset G$ are cofinite subgroups, and $[G/H]$, $H \subset G$ a cofinite subgroup denotes the quotient G/H considered as a G-set via the action by left shifts. Then by additivity, the values of a G-Mackey functor M at all finite G-sets are completely determined by its values $M([G/H])$, $H \subset G$ a cofinite subgroup. To determine M itself, one needs to specify also the maps $f_* : M([G/H_1]) \to M([G/H_2])$, $f^* : M([G/H_2]) \to M([G/H_1])$ for any G-equivariant map $f : [G/H_1] \to [G/H_2]$. For any composable pair of maps f, g, we must have $f_* \circ g_* = (f \circ g)_*$, $g^* \circ f^* = (f \circ g)^*$. In addition, given two maps $f : [G/H_1] \to [G/H]$, $g : [G/H_2] \to [G/H]$, we must have

$$g^* \circ f_* = \sum_s f_{s*} \circ g_s^*, \tag{1.14}$$

where the sum is over all the terms $[G/H_s]$ in the orbit decomposition (1.13) of the fibered product $S = [G/H_1] \times_{[G/H]} [G/H_2]$, and $f_s : [G/H_s] \to [G/H_2]$, $g_s : [G/H_s] \to [G/H_1]$ are the natural projections. We note that the existences of the maps f, g implies that $H_1, H_2 \subset G$ can be conjugated to a subgroup in $H \subset G$. If we do the conjugation, then we have a natural identification

$$G\backslash S = G\backslash([G/H_1] \times_{[G/H]} [G/H_2]) \cong H\backslash([H/H_1] \times [H/H_2]) \cong H_1\backslash H/H_2.$$

Because of this, (1.14) is known as the *double coset formula*.

1.4 Fixed points and products

For any subgroup $H \subset G$, a finite G-set S is also an H-set by restriction, so that we have a natural functor $\psi^H : \Gamma_G \to \Gamma_H$. This functor preserves pullbacks, thus induces a functor $Q\psi^H : Q\Gamma_G \to Q\Gamma_H$. The Kan extension $Q\psi^H_!$ then sends additive functors to additive functors, so that we obtain a functor

$$\Psi^H = Q\psi^H_! : \mathcal{M}(G, A) \to \mathcal{M}(H, A).$$

This is known as the *categorical fixed points functor*. Note that the centralizer $Z_H \subset G$ of the group $H \subset G$ acts on ψ^H, thus on Ψ^H, so that Ψ^H can be promoted to a functor

$$\widetilde{\Psi}^H : \mathcal{M}(G, A) \to \mathcal{M}(H, A[Z_H]). \tag{1.15}$$

If $H \subset G$ is cofinite, then ψ^H admits a left-adjoint functor $\gamma^H : \Gamma_H \to \Gamma_G$ that sends a finite H-set S to $(G \times S)/H$, where H acts on G by right shifts. This functor also preserves pullbacks, and moreover, the induced functor $Q\gamma^H Q\Gamma_H \to Q\Gamma_G$: is also adjoint to $Q\psi^H$ (both on the left and on the right). Then by adjunction, we have

$$\Psi^H \cong Q\gamma^{H*}. \tag{1.16}$$

On the other hand, assume given a normal subgroup $N \subset G$, and let $W = G/N$ be the quotient group. Then every finite W-set is by restriction a finite G-set, so that we obtain a functor $\Gamma_W \to \Gamma_G$. This has a right-adjoint $\varphi^N : \Gamma_G \to \Gamma_W$ sending a G-set S to the fixed points subset S^N. The functor φ^N preserves pullbacks, and the induced functor $Q\varphi^N_!$ preserves additivity, so that we obtain the *geometric fixed points functor*

$$\Phi^N = Q\varphi^N_! : \mathcal{M}(G, A) \to \mathcal{M}(W, A).$$

The functor Φ^N has a right-adjoint *inflation functor*

$$\mathsf{Infl}^N = \varphi^{N*} : \mathcal{M}(W, A) \to \mathcal{M}(G, A).$$

The inflation functor is fully faithful. Its essential image consists of G-Mackey functors that are *supported at* N in the sense that $M([G/H]) = 0$ unless H contains N.

Assume now that the ring A is commutative. The Cartesian product preserves pullbacks in each variable, this gives a functor

$$m : Q\Gamma_G \times Q\Gamma_G \to Q\Gamma_G. \tag{1.17}$$

The Kan extension $m_!$ does *not* in any sense preserve additivity; however, one can still define a product on the category $\mathcal{M}(G, A)$ by setting

$$M_1 \circ M_2 = \mathsf{Add}(m_!(M_1 \boxtimes_A M_2)). \tag{1.18}$$

This is a symmetric unital tensor product. It is right-exact in each variable. If $G = \{e\}$ is the trivial group, so that $\mathcal{M}(G, A) \cong A$-mod, then $M_1 \circ M_2 \cong M_1 \otimes_A M_2$, and the unit object is the free module A. In general, the unit object is the so-called *Burnside Mackey functor* A given by

$$\mathcal{A} = \mathsf{Add}(Qp_! A), \tag{1.19}$$

where $p : \Gamma \to \Gamma_G$ is the tautological embedding sending a finite set to itself with the trivial G-action.

Since $\psi^H \circ m \cong m \circ (\psi^H \times \psi^H)$ and $\varphi^H \circ m \cong m \circ (\varphi^H \times \varphi^H)$, both fixed points functors Ψ^H, Φ^H are tensor functors with respect to the product (1.18). Since the isomorphism $\psi^H \circ m \cong m \circ (\psi^H \times \psi^H)$ is Z_H-equivariant, the extended fixed points functor $\widetilde{\Psi}^H$ of (1.15) is also a tensor functor.

While the product $M_1 \circ M_2$ of two G-Mackey functors M_1, M_2 does not admit a simple description, such a description is possible for the spaces of maps from $M_1 \circ M_2 \to M_3$ to a third G-Mackey functor M_3. Namely, we have the following useful result.

Lemma 1.2 *Assume given a commutative ring A, a group G, and three G-Mackey functors $M_1, M_2, M_3 \in \mathcal{M}(G, A)$. Then giving a map*

$$\mu : M_1 \circ M_2 \to M_3$$

is equivalent to giving a map $\mu : M_1([G/H]) \otimes_A M_2([G/H]) \to M_3([G/H])$ for any cofinite subgroup $H \subset G$ such that for any map $f : [G/H'] \to [G/H]$ of G-sets, we have

$$\begin{aligned} \mu \circ (f_1^* \times f_2^*) &= f_3^* \circ \mu, \\ \mu \circ (f_*^1 \times \mathsf{id}) &= f_*^3 \circ \mu \circ (\mathsf{id} \times f_2^*), \\ \mu \circ (\mathsf{id} \times f_*^2) &= f_*^3 \circ \mu \circ (f_1^* \times \mathsf{id}), \end{aligned} \tag{1.20}$$

where f_1^*, f_2^*, f_3^*, resp. f_*^1, f_*^2, f_*^3 are the maps f^* resp. f_* for the Mackey functors M_1, M_2, M_3.

Proof This is a reformulation of [16, Lemma 2.6]. □

Informally, the first equation in (1.20) means that the maps f^* are multiplicative, and the other two equations say that f_* satisfies a version of the projection formula with respect to f^*. Note that among other things, Lemma 1.2 implies that for any cofinite $H \subset G$, the evaluation functor $M \mapsto M([G/H])$ has a natural pseudotensor structure. This can also been seens explicitly as follows. For $H = G$, we have a functorial identification

$$M([G/G]) \cong Qp^*(M),$$

where $p : \Gamma \to \Gamma_G$ is the tautological embedding of (1.19), so that evaluation at $[G/G]$ is right-adjoint to the functor $Qp_!$. Since Qp commutes with the product functor (1.17), $Qp_!$ is tensor, and Qp^* is pseudotensor by adjunction. For a general cofinite H, precompose with the tensor functor Ψ^H.

1.5 The functor Q

If the group G is finite, then the trivial subgroup $\{e\} \subset G$ is cofinite, and the centralizer $\mathbb{Z}_{\{e\}} \subset G$ is the whole G. Denote by

$$U = \widetilde{\Psi}^{\{e\}} : \mathcal{M}(G, A) \to \mathcal{M}(\{e\}, A[G]) \cong A[G]\text{-mod} \tag{1.21}$$

the corresponding extended fixed points functor (1.15). By (1.16), U sends a Mackey functor $M \in \mathcal{M}(G, A)$ to its value $M([G/\{e\}])$ at the biggest G-orbit $[G/\{e\}]$, and G acts on $M[G/\{e\}]$ via its action on $G = [G/\{e\}]$ by right shifts. The functor U has a left and a right-adjoint $L, R : A[G]\text{-mod} \cong \mathcal{M}(\{e\}, A[G]) \to \mathcal{M}(G, A)$ given by

$$L(E) = (\psi^*(E))_G, \quad R(E) = (\psi^*(E))^G, \qquad E \in A[G]\text{-mod}, \tag{1.22}$$

where we simplify notation by writing $\psi = \psi^{\{e\}}$, and G acts both on E and on ψ. Explicitly, for any $A[G]$-module E and any subgroup $H \subset G$, we have

$$\begin{aligned} L(E)([G/H]) &\cong (E[G/H])_G \cong E_H, \\ R(E)([G/H]) &\cong (E[G/H])^G \cong E^H. \end{aligned} \tag{1.23}$$

We note that E^H, E_H only depend on H and not on G. In fact, for any subgroup $H \subset G$, the isomorphisms (1.16) and (1.22) provide canonical identifications

$$\Psi^H L(E) \cong L'(E'), \quad \Psi^H R(E) \cong R'(E'), \qquad E \in A[G]\text{-mod}, \tag{1.24}$$

where $L', R' : R[H]\text{-mod} \to \mathcal{M}(H, A)$ are the functors L, R for the group H, and E' is E treated as an $A[H]$-module by restriction. In particular, taking $H = \{e\}$, we see that $U \circ L \cong U \circ R \cong \mathsf{Id}$, so that both L and R are fully faithful. By adjunction, we then have a natural trace map

$$\mathsf{tr} : L \to R. \tag{1.25}$$

For any subgroup $H \subset G$, it is compatible with the identification (1.24).

Definition 1.3 For any $A[G]$-module E, the G-Mackey functor $Q(E)$ is the cokernel of the trace map (1.25).

It is clear that $Q(E)$ is functorial in E, so that we obtain a functor

$$Q : A[G]\text{-mod} \to \mathcal{M}(G, A).$$

Explicitly, for any subgroup $H \subset G$, the map $L(E)([G/H]) \to R(E)([G/H])$ induced by the trace map (1.25) is the trace map tr_H of (1.3), and we have the identification

$$Q(E)([G/H]) \cong \check{H}^0(G, E[G/H]) \cong \check{H}^0(H, E), \tag{1.26}$$

where as in Sect. 1.1, $\check{H}^\bullet(-,-)$ stands for Tate cohomology. The maps f_*, f^* associated to a G-equivariant map $f : [G/H'] \to [G/H]$ can aslo be described explicitly, but we will need it only in one case: $H = G$, $H' = N$ is a normal subgroup with the quotient $W = G/N$, $f : [G/H] \to [G/G]$ is the only map. Then

$$f_* = l_\pi^\dagger, \qquad f^* = r_\pi^\dagger, \tag{1.27}$$

where $l_\pi^\dagger$, $r_\pi^\dagger$ are the natural maps (1.10) for the projection $\pi : \mathsf{pt}_G \to \mathsf{pt}$. We also note that for any subgroup $H \subset G$, (1.24) provides an identification

$$\Psi^H Q(E) \cong Q'(E'), \qquad E \in A[G]\text{-mod}, \tag{1.28}$$

where E' is as in (1.24), and Q' is the functor Q for the group H.

We will also need the following slightly more elaborate description of the functor Q. Let $Q\widetilde{\psi}$ be the functor (1.1) associated to the action of G on $Q\psi$, and consider the full embedding

$$A[G]\text{-mod} \cong \mathcal{M}(\{e\}, A[G]) \subset \operatorname{Fun}(Q\Gamma, A[G]) \cong \operatorname{Fun}(\mathsf{pt}_G \times Q\Gamma_G, A).$$

Then for any $E \in A[G]\text{-mod} \subset \operatorname{Fun}(\mathsf{pt}_G \times Q\Gamma, A)$, we have

$$L(E) = \pi_! Q\widetilde{\psi}^* E, \qquad R(E) = \pi_* Q\widetilde{\psi}^* E, \tag{1.29}$$

where $\pi : \mathsf{pt}_G \times Q\gamma_G \to Q\Gamma_G$ is the projection, and the trace map (1.25) coincides with the trace map tr_π of (1.4). Therefore $Q(E) \cong \check{\pi}_* Q\widetilde{\psi}^* E$.

Lemma 1.4 *For any finite group G and commutative ring A, the functor Q : $A[G]$-mod $\to \mathcal{M}(G, A)$ of Definition 1.3 is symmetric pseudotensor, and for any subgroup $H \subset G$, the functorial isomorphism (1.26) is compatible with the pseudotensor structures.*

Proof Since the functor U is tensor, its right-adjoint R is symmetric pseudotensor by adjunction. Then as in Lemma 1.1, since Q is a quotient of R, the map ε of (1.2) for the functor R induces a map ε for Q, and to prove that Q is pseudotensor, it suffices to show that the maps μ descend to the corresponding maps for Q—the associativity, commutivity and unitality are then automatic. Moreover, compatibility of (1.26) with the pseudotensor structures is also automatic, since the second of the isomorphisms (1.23) is compatible with the pseudotensor structures by adjunction.

Since the product (1.18) is right-exact, $Q(M) \circ Q(N)$ is the cokernel of the map

$$(L(M) \circ R(N)) \oplus (R(M) \circ L(N)) \xrightarrow{(\mathsf{tr} \circ \mathsf{id}) \oplus (\mathsf{id} \circ \mathsf{tr})} R(M) \circ R(N)$$

for any $M, N \in A[G]$-mod, and since R is symmetric, it suffices to show that there exists a functorial map

$$L(M) \circ R(N) \to L(M \otimes N) \tag{1.30}$$

that fits into a commutative diagram

$$\begin{array}{ccc} L(M) \circ L(N) & \xrightarrow{\mathsf{tr} \circ \mathsf{id}} & R(M) \circ R(N) \\ \downarrow & & \downarrow \mu \\ L(M \otimes N) & \xrightarrow{\mathsf{tr}} & R(M \otimes N). \end{array}$$

To see this, use (1.29). Denote $\gamma = \gamma^{\{e\}}$, and let $Q\widetilde{\gamma}$ be the functor (1.1) associated to the G-action on $Q\gamma$. Since the product of a free G-set and an arbitrary G-set is free, we have a commutative diagram

$$\begin{array}{ccccc} Q\Gamma \times \mathsf{pt}_G \times Q\Gamma_G & \xrightarrow{(Q\widetilde{\gamma}\times\mathsf{id})\circ(t\times\mathsf{id})} & \mathsf{pt}_G \times Q\Gamma_G \times Q\Gamma_G & \xrightarrow{\mathsf{id}\times m} & \mathsf{pt}_G \times Q\Gamma_G \\ {\scriptstyle \mathsf{id}\times Q\widetilde{\psi}}\downarrow & & & & \| \\ Q\Gamma \times \mathsf{pt}_G \times Q\Gamma & \xrightarrow{(\mathsf{id}\times m)\circ(t\times\mathsf{id})} & \mathsf{pt}_G \times Q\Gamma & \xrightarrow{Q\widetilde{\gamma}} & \mathsf{pt}_G \times Q\Gamma_G, \end{array}$$

where $t : Q\Gamma \times \mathsf{pt}_G \to \mathsf{pt}_G \times Q\Gamma$ is the transposition, and since $Q\widetilde{\psi}^* \cong Q\widetilde{\gamma}_!$, this diagram together with (1.29) induces a functorial projection formula isomorphism

$$L(E \otimes_A U(M)) \cong L(E) \circ M, \qquad E \in A[G]\text{-mod}, M \in \mathcal{M}(G, A).$$

By adjunction, this isomorphism is compatible with the trace maps, and the inverse isomorphism yields the map (1.30). □

1.6 Cyclic groups

Now fix a prime p, and let $G = \mathbb{Z}_p$ be the group of p-adic integers. Then the lattice of cofinite subgroups $H \subset G$ is very simple—they are all of the form $p^m G \subset G, m \geq 0$. In particular, finite G-orbits are numbered by non-negative integers, and it turns out that the category $\mathcal{M}(G, A)$ admits a simple explicit description.

Namely, denote by I the groupoid of G-orbits and their isomorphisms, with $[p^m] \in I$ being the orbit $[G/p^m G]$. Explicitly, $\operatorname{Aut}([p^m]) = \mathbb{Z}/p^m\mathbb{Z}$, so that I is the disjoint union of groupoids $\mathsf{pt}_{p^m}, m \geq 0$. Let $I_p \subset I$ be the full subcategory spanned by orbits other than the point (in other words, I_p is the union of groupoids pt_{p^m} with $m \geq 1$). Denote by $i : I_p \to I$ the natural embedding. On the other hand, for any $m \geq 1$, we have a natural quotient map $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^{m-1}\mathbb{Z}$ and the corresponding functor $\mathsf{pt}_{p^m} \to \mathsf{pt}_{p^{m-1}}$. Taking all these functors together, we obtain a functor

$$\pi : I_p \to I.$$

The functor π is a bifibration with fiber pt_p, so that in particular, we have the trace map tr_π of (1.4).

Lemma 1.5 (i) *For any ring A, the category $\mathcal{M}(\mathbb{Z}_p, A)$ is equivalent to the category of triples $\langle E, V, F\rangle$ of a object $E \in \operatorname{Fun}(I, A)$ and two maps*

$$i^*E \xrightarrow{V} \pi^*E \xrightarrow{F} i^*E$$

such that $F \circ V : i^*E \to i^*E$ is equal to the natural map $\mathsf{tr}_\pi^\dagger$ of (1.6).

(ii) *Assume that the ring A is commutative, and assume given G-Mackey functors $M_1, M_2, M_3 \in \mathcal{M}(G, A)$ corresponding to triples $\langle E_1, V_1, F_1\rangle$, $\langle E_2, V_2, F_2\rangle$, $\langle E_3, V_3, F_3\rangle$. Then maps $M_1 \circ M_2 \to M_3$ are in a natural one-to-one correspondence with maps $\mu : E_1 \otimes_A E_2 \to E_3$ such that $\mu \circ (F_1 \times F_2) = F_3 \circ \mu$ and*

$$\mu \circ (V_1 \times \mathsf{id}) = V_3 \circ \mu \circ (\mathsf{id} \times F_2), \quad \mu \circ (\mathsf{id} \times V_2) = V_3 \circ \mu \circ (F_1 \times \mathsf{id}).$$

Proof By definition, we have a natural embedding $e : I \to \Gamma_G$ that extends further to an embedding $\widetilde{e} : I \to Q\Gamma_G$. Fix a map $\rho : [G/p^{m+1}G] \to [G/p^m G]$ for every $m \geq 0$. Then any morphism $f : [G/p^n G] \to [G/p^m G]$ in Γ_G uniquely decomposes as

$$f = \overline{f} \circ \rho^{n-m}, \qquad \overline{f} \in \mathrm{Aut}([G/p^m G]), \tag{1.31}$$

and all the maps ρ together define a map of functors $\rho : e \circ i \to e \circ \pi$. In one direction, the equivalence of (i) sends a G-Mackey functor $M \in \mathcal{M}(G, A)$ to $\widetilde{e}^* M \in \mathrm{Fun}(I, A)$, with $V = \rho_*$, $F = \rho^*$; the equality $F \circ V = \mathsf{tr}^\dagger_\pi$ follows from the double coset formula (1.14). In the other direction, $\langle E, V, F\rangle$ gives a G-Mackey functor M such that $M([G/p^m G]) = E([p^m])$, $m \geq 0$, and for any map $f : [G/p^n G] \to [G/p^m G]$, we have

$$f_* = E(\overline{f})^{-1} \circ V^{n-m}, \qquad f^* = F^{n-m} \circ E(\overline{f}),$$

where $\overline{f} \in \mathrm{Aut}([p^m])$ is given by (1.31). The double coset formula (1.14) then follows from the equality $F \circ V = \mathsf{tr}^\dagger_\pi$. Finally, (ii) immediately follows from Lemma 1.2. □

We note that if for any integer $m \geq 0$ we denote $G_m = G/p^m G \cong \mathbb{Z}/p^m\mathbb{Z}$, then for any $n \geq m$, we have a fully faithful inflation functor

$$\mathsf{Infl}^n_m : \mathcal{M}(G_m, A) \to \mathcal{M}(G_n, A), \tag{1.32}$$

and for any $m \geq 0$, we have the fully faithful inflation functor

$$\mathsf{Infl}_m : \mathcal{M}(G_m, A) \to \mathcal{M}(G, A). \tag{1.33}$$

Therefore Lemma 1.5 also describes the category $\mathcal{M}(G_m, A)$ for any integer $m \geq 0$—this is the full subcategory spanned by triples $\langle E, V, F\rangle$ such that $E([p^n]) = 0$ for $n > m$.

Apart from Lemma 1.5, one can also study G_m-Mackey functors by induction on m. Namely, for any $m \geq 1$ and $M \in \mathcal{M}(G_m, A)$, we have adjunction maps

$$l : L(U(M)) \to M, \qquad r : M \to R(U(M)), \tag{1.34}$$

and we note that since $U(l)$ and $U(r)$ are isomorphisms, both the kernel $\mathrm{Ker}\, r$ and the cokernel $\mathrm{Coker}\, l$ are supported at $p^{m-1}G_1 \subset G_m$, so that both are effectively G_{m-1}-Mackey functors. We then introduce the following inductive definition.

Definition 1.6 1. A G_m-Mackey functor $M \in \mathcal{M}(G_m, A)$ is *perfect* if either $m = 0$, or the map l of (1.34) is injective, the map r is surjective, and both $\mathrm{Ker}\, r$ and $\mathrm{Coker}\, l$ are perfect G_{m-1}-Mackey functors.

2. Assume given a perfect G_m-Mackey functor M. Then the *co-standard filtration* $F_\bullet$ on M is the increasing filtration such that $F_m M = M$ and $F_i M = F_i(\mathrm{Ker}\, r)$, $0 \leq i \leq m-1$, and the *standard filtration* $F^\bullet$ on M is the decreasing filtration such that $F^m M = \mathrm{Im}\, l$, and $F^i M = q^{-1}(F^i\, \mathrm{Coker}\, l)$, $0 \leq i \leq m-1$, where $q : M \to \mathrm{Coker}\, l$ is the quotient map.

Example 1.7 Already for $m = 1$, Definition 1.6 (i) is not vacuous. Indeed, let k be a field of characteristic $p = \operatorname{char} k$, and take $E \in \operatorname{Fun}(I, k)$ with $E([p^m]) = 0$, $m \neq 1$, and $E([p]) = k$ with the trivial action of $G_1 = \mathbb{Z}/p\mathbb{Z}$. Then the trace map $\operatorname{tr}_{\mathbb{Z}/p\mathbb{Z}} : k \to k$ vanishes, so that $V = F = 0$ satisfies the condition of Lemma 1.5 (i). The resulting G_1-Mackey functor is not perfect.

As we see from Example 1.7, not all G_m-Mackey functors are perfect. However, those of interest to us in the rest of the paper will be, and the standard and co-standart filtrations will prove useful.

2 Witt vectors

2.1 Preliminaries on cyclic groups

Fix a perfect field k of positive characteristic $p = \operatorname{char} k$. Let $W(k)$ be the ring of p-typical Witt vectors of the field k. Since k is perfect, we have $W(k)/p \cong k$, and for any integer $n \geq 1$, we have the truncated Witt vectors ring $W_n(k) = W(k)/p^n$. We also have the Frobenius automorphism $F : W(k) \to W(k)$ lifting the absolute Frobenius automorphism of the field k.

As in Sect. 1.6, let $G = \mathbb{Z}_p$ be the group of p-adic integers, and for any integer $m \geq 0$, let $G_m = \mathbb{Z}/p^m\mathbb{Z} = G/p^m G$. Any finite G-set S decomposes as

$$S = \coprod_{i \geq 0} S_{[i]}, \tag{2.1}$$

where $S_{[i]} \subset S$ is the union of elements $s \in S$ whose stabilizer is exactly $p^i G \subset G$. Any finite G_m-set S is canonically a G-set via the quotient map $G \to G_m$; its decomposition (2.1) only contains terms $X_{[i]}$ with $i \leq m$.

Lemma 2.1 *Assume given integers $m \geq 0$, $n \geq 1$ and a finite G_m-set S, and consider the free $W_n(k)$-module $E = W_n(k)[S]$ spanned by S, with the G_m-action induced from S.*

(i) *We have a natural identification $E^{G_m} = W_n(k)[S/G_m]$.*
(ii) *Moreover, assume that $n \geq m$. Then we have a natural identification*

$$\check{H}^0(G_m, E) = \bigoplus_{0 \leq i < m} W_{m-i}(k)[S_{[i]}/G_m],$$

where $S_{[i]} \subset S$ are the components of the decomposition (2.1).

Proof (i) is obvious. For (ii), note that (2.1) induces a G_m-invariant direct sum decomposition of E, so that it suffices to consider the case $S = S_{[i]}$, $0 \leq i \leq m$. In this case, S is actually a G_i-set, and the G_i-action on S is free. Therefore the trace map tr_{G_m} is equal to $p^{m-i} \operatorname{tr}_{G_i}$ by (1.5), and the trace map tr_{G_i} is a bijection. □

Note that for any integers $m \geq 0$, $n \geq 1$, the quotient map $W_{n+1}(k) \to W_n(k)$ induces a natural map

$$\check{H}^0(G_m, W_{n+1}(k)[S]) \to \check{H}^0(G_m, W_n(k)[S]).$$

Lemma 2.1 (ii) then implies that this map is an isomorphism if $n \geq m$.

Now assume given a free finitely generated $W_n(k)$-module E, and denote by

$$E_{(m)} = E^{\otimes_{W_n(k)} p^m} \tag{2.2}$$

its p^m-th tensor power with the $W_n(k)$-module structure given by

$$a \cdot e = F^m(a)e, \qquad a \in W_n(k), e \in E^{\otimes_{W_n(k)} p^m}. \tag{2.3}$$

Let the group G_m act on $E_{(m)}$ by permutations. Moreover, let $E'_{(m)}$ be the same $W_n(k)$-module considered as a representation of G_{m+1} via the quotient map $G_{m+1} \to G_m$, and denote

$$Q_m(E) = \check{H}^0(G_m, E_{(m)}), \qquad Q'_m(E) = \check{H}^0(G_{m+1}, E'_{(m)}). \tag{2.4}$$

Then we have

$$E_{(m)}^{G_m} \cong (E'_{(m)})^{G_{m+1}},$$

and $\mathsf{tr}_{G_{m+1}} = p\,\mathsf{tr}_{G_m}$ on $E_{(m)} \cong E'_{(m)}$ by (1.9) and (1.5), so that we obtain a natural map

$$r : Q'_m(E) \to Q_m(E). \tag{2.5}$$

Every element $e \in E$ gives an element $e^{\otimes p^m} \in E'_{(m)}$; this element is G_{m+1}-invariant and descends to elements

$$e'_{(m)} \in Q'_m(E), \quad e_{(m)} = r(e'_{(m)}) \in Q_m(E). \tag{2.6}$$

Lemma 2.2 *Assume that $n \geq m$, and assume given two elements $a, b \in E$ such that $a = b \mod p$. Then $a'_{(m)} = b'_{(m)}$ and $a_{(m)} = b_{(m)}$.*

Proof Since $e_{(m)} = r(e'_{(m)})$, it suffices to prove the first claim. By functoriality, it suffices to consider the universal situation: $E = W_n(k)[S]$ is the free module generated by a set S with two elements, s_0 and s_1, and we have $a = s_0$, $b = s_0 + ps_1$. Then elements $\widetilde{s}$ of the set S^{p^m} are of the form

$$\widetilde{s} = s_{f(1)} \times s_{f(2)} \times \cdots \times s_{f(p^m)}, \tag{2.7}$$

where f is a function that assigns 0 or 1 to any integer $1 \leq j \leq p^m$. For such an element, denote

$$|\widetilde{s}| = \sum_{1 \leq j \leq p^m} f(j),$$

so that $|\widetilde{s}|$ is the number of integers j with $f(j) = 1$. Then we have

$$b^{\otimes p^m} - a^{\otimes p^m} = \sum_{\widetilde{s}} p^{|\widetilde{s}|}\widetilde{s} \in E^{(m)} = W_n(k)[S^{p^m}]^{G_m}, \tag{2.8}$$

where the sum is over all elements $\widetilde{s} \in S^{p^m}$ not equal to $s_0^{p^m}$—that is, with $|\widetilde{s}| \geq 1$. Moreover, if $\widetilde{s}$ lies in $S_{(i)}^{p^m}$ for some integer i, then $|\widetilde{s}|$ must be divisible by p^{m-i}, and since $|\widetilde{s}| \geq 1$, we actually have $|\widetilde{s}| \geq p^{m-i}$. Since $p^{m-i} \geq m - i + 1$ for any $i \leq m$, Lemma 2.1 (ii) implies that every term in the right-hand side of (2.8) vanishes after projecting to $\check{H}^0(G_{m+1}, E'_{(m)})$. □

2.2 Polynomial Witt vectors

Now fix integers $n \geq m \geq 1$, let E be a free $W_n(k)$-module, and consider the corresponding $W_n(k)[G_m]$-modules $E_{(m)}$, $E'_{(m-1)}$ of (2.2). Denote

$$\widetilde{Q}_m(E) = Q(E_{(m)}), \widetilde{Q}'_{m-1}(E) = Q(E'_{(m-1)}) \in \mathcal{M}(G_m, W_n(k)),$$

where $Q(-)$ is as in Definition 1.3. Note that by (1.26), this notation is consistent with (2.4)—the Mackey functor $\widetilde{Q}_m(E)$ gives $Q_m(E)$ after evaluation at the trivial G_m-orbit $[G_m/G_m]$, and similarly for $\widetilde{Q}'_m(E)$. Note also that since $n \geq m$, every $W_m(k)$-module is automatically a $W_n(k)$-module by virtue of the map $W_n(k) \to W_m(k)$, so that we have a full embedding $W_m(k)\text{-mod} \subset W_n(k)\text{-mod}$.

Proposition 2.3 *For any integer $m \geq 1$, there exists functors*

$$\widetilde{W}_m, \widetilde{W}'_{m-1} : k\text{-mod} \to \mathcal{M}(G_{m-1}, W_m(k))$$

such that for any free $W_n(k)$-module E, we have functorial isomorphisms

$$\widetilde{Q}_m(E) \cong \mathsf{Infl}^m_{m-1} \widetilde{W}_m(E/p), \qquad \widetilde{Q}'_{m-1}(E) \cong \mathsf{Infl}^m_{m-1} \widetilde{W}'_{m-1}(E/p),$$

where Infl *are the inflation functors* (1.32).

Proof The proofs of both claims are the same, so let us start with $\widetilde{W}_m$. By (1.26), we know that $Q(E_{(m)})([G_m/\{e\}]) = 0$, so that $\widetilde{Q}_m(E)$ is supported at $p^{m-1}G_1 \subset G_m$, thus lies in the image of the fully faithful inlfation functor Infl^m_{m-1}. Denote by q the functor from free $W_n(k)$-modules to k-vector spaces that sends E to E/p. We have to show that $\widetilde{Q}_m$ canonically factors through q. Since q is essentially surjective, the issue is the morphisms: we have to show that for two free $W_n(k)$-modules M, N, and two maps $a, b : M \to N$ with $q(a) = q(b)$, we have $\widetilde{Q}_m(a) = \widetilde{Q}_m(b)$. Moreover, since the functor $\widetilde{Q}_m$ obviously commutes with filtered colimits, it suffices to consider finitely generated free $W_n(k)$-modules.

Let $E = \mathrm{Hom}_{W_n(k)}(M, N)$. This is also a free finitely generated $W_n(k)$-module, we have the action map

$$\alpha : M \otimes_{W_n(k)} E \to N,$$

and the map $a : M \to N$ decomposes as

$$M = M \otimes_{W_n(k)} W_n(k) \xrightarrow{\mathrm{id} \otimes \widetilde{a}} N \otimes_{W_n(k)} E \xrightarrow{\alpha} N,$$

where $\widetilde{a} : W_n(k) \to E$ is the map sending 1 to a.

Now, the p^m-th tensor power functor is tensor, and the functor Q is pseudotensor by Lemma 1.4. Therefore $\widetilde{Q}_m$ is pseudotensor, and the map $\widetilde{Q}_m(a)$ can be decomposed as

$$\widetilde{Q}_m(M) \cong \widetilde{Q}_m(M) \circ \mathcal{A} \xrightarrow{\mathrm{id} \circ \widetilde{Q}_m(\widetilde{a})} \widetilde{Q}_m(M) \circ \widetilde{Q}_m(E) \xrightarrow{\mu}$$
$$\xrightarrow{\mu} \widetilde{Q}_m(M \otimes_{W_n(k)} E) \xrightarrow{\widetilde{Q}_m(\alpha)} \widetilde{Q}_m(N),$$

where $\mathcal{A} \in \mathcal{M}(G_m, W_n(k))$ is the Burnside Mackey functor, and μ comes from the pseudotensor structure on $\widetilde{Q}_m$. We also have an analogous decomposition for b, so that in the end, it suffices to prove that

$$\widetilde{Q}_m(\widetilde{a}) = \widetilde{Q}_m(\widetilde{b}) : \mathcal{A} \to \widetilde{Q}_m(E).$$

But by (1.19) and (1.26), we have $\mathrm{Hom}(\mathcal{A}, \widetilde{Q}_m(E)) \cong \check{H}^0(G_m, E_{(m)})$, and in terms of this identification, we obviously have $\widetilde{Q}_m(\widetilde{a}) = a_{(m)}$, $\widetilde{Q}_m(\widetilde{b}) = b_{(m)}$. Then we are done by the second claim of Lemma 2.2.

For $\widetilde{W}'_{m-1}$, the argument is exactly the same, except that we need to invoke the first claim of Lemma 2.2. □

Definition 2.4 For any k-vector space E and integer $m \geq 1$, the *m-truncated extended polynomial Witt vectors Mackey functor* $\widetilde{W}_m(E)$ is the $W_m(k)$-valued G_{m-1}-Mackey functor provided by Proposition 2.3, and the $W_m(k)$-module of *m-truncated polynomial Witt vectors*

$$W_m(E) = \widetilde{W}_m(E/k)[G_m/G_m]$$

is its value at the trivial G_m-orbit $[G_m/G_m]$.

We note that by (1.26), m-truncated polynomial Witt vectors $W_m(E)$ can be also described as follows: we have

$$W_m(E) \cong Q_m(\widetilde{E}) = \check{H}^0(G_m, \widetilde{E}_{(m)}), \tag{2.9}$$

where $\widetilde{E}$ is any flat $W_m(k)$-module equipped with an isomorphism $\widetilde{E}/p \cong E$, and Q_m is the functor (2.4).

2.3 Restriction maps

The reader will notice immediately that Definition 2.4 only uses the functor $\widetilde{W}_m$ of Proposition 2.3. The role of the functor $\widetilde{W}'_m$ is that it allows one to relate m-truncated Witt vectors for different m.

Namely, we have natural maps r of (2.5), and taken together, they provide a canonical map

$$r : \widetilde{W}'_m(E) \to \mathsf{Infl}^m_{m-1} \widetilde{W}_m(E) \tag{2.10}$$

for any $m \geq 1$ and any k-vector space E.

On the other hand, if we take $m = 0$, so that $G_{m+1} = G_1 = \mathbb{Z}/p\mathbb{Z}$, then we tautologically have $\check{H}^0(G_1, \widetilde{E}) \cong \widetilde{E}/p$ for any n and any $W_n(k)$-module $\widetilde{E}$ with the trivial G_1-action, so that we have a canonical identification $\widetilde{W}'_0(E) \cong E$ (since $G_0 = \{e\}$, we have $\mathcal{M}(G_0, W_1(k)) \cong k$-mod, so that $\widetilde{W}'(E)$ is simply a k-vector space). But then, there also exists a canonical Frobenius-semilinear identification

$$\check{H}^0(\mathbb{Z}/p\mathbb{Z}, E^{\otimes p}) \cong E$$

constructed e.g. in [13, Lemma 2.3]. Thus for $m = 0$, we have an isomorphism of functors

$$\widetilde{W}_{m+1} \cong \widetilde{W}'_m. \tag{2.11}$$

It turns out that the same is true for $m \geq 1$, and we will now prove it.

Definition 2.5 For $n \geq 1$ and any free $W_n(k)$-module E, a $\mathbb{Z}/p\mathbb{Z}$-equivariant Frobenius-semilinear map $c : E \to E^{\otimes_{W_n(k)} p}$ is *admissible* if the induced map

$$Q(c) : \widetilde{W}'_0(E/p) \to \widetilde{W}_1(E/p)$$

is the standard isomorphism (2.11).

At this point, we do not need to know the precise form of the identification (2.11). It suffices to say that if $E = W_n(k)[S]$ is the free module spanned by a set S, then the diagonal map $\delta : S \to S^p$ induces a Frobenius-semilinear map

$$c_S : E \to E^{\otimes_{W_n(k)} p}, \qquad \sum_s a_s \cdot s \mapsto \sum_s a_s^p \cdot \delta(s), \tag{2.12}$$

and this map is admissible.

Proposition 2.6 *Assume given integers $n \geq m \geq 1$, a free $W_n(k)$-module E, and two admissible maps $c_1, c_2 : E \to E^{\otimes_{W_n(k)} p}$. Then the corresponding maps*

$$Q(c_1^{\otimes p^m}), Q(c_2^{\otimes p^m}) : \widetilde{W}'_m(E/p) \to \widetilde{W}_{m+1}(E/p)$$

coincide, and both are isomorphisms.

Proof As in the proof of Proposition 2.3, it suffices to consider finitely generated modules. Moreover, it clearly suffices to consider the case when $E = W_n(k)[S]$ for some set S, and c_1 is the standard map c_S of (2.12). In this case, at least $Q(c_1^{\otimes p^m})$ is certainly an isomorphism by Lemma 2.1. We have the decomposition (2.1) of the product S^p, and it induces the decomposition

$$S^{p^{m+1}} = \left(S^p\right)^{p^m} = \coprod_f S^p_{[f(1)]} \times \cdots \times S^p_{[f(p^m)]}, \tag{2.13}$$

where f is as in (2.7). However, if an element $\widetilde{s} \in S^{p^{m+1}}$ is fixed by some non-trivial subgroup in G_{m+1}, it must also be fixed by the smallest non-trivial subgroup $\mathbb{Z}/p\mathbb{Z} \subset G_{m+1}$, the kernel of the quotient map $G_{m+1} \to G_m$. Therefore in the decomposition (2.1) for the G_{m+1}-set $S^{p^{m+1}}$, all the terms except for $S^{p^{m+1}}_{[m+1]}$ lie in the component of (2.13) corresponding to the constant function $f = 0$. By Lemma 2.1, this means that if we decompose

$$E_{(m+1)} = W_n(k)[S^{p^{m+1}}] \cong E'_{(m)} \oplus E_0, \tag{2.14}$$

where E_0 is spanned by all the other components in (2.13), then any element $e \in E_0$ invariant under G_{m+1} vanishes after projection to the quotient $W_{m+1}(E/p) \cong \check{H}^0(G_{m+1}, E_{(m+1)})$. The same is also true if we project to $\check{H}^0(H, E_{(m+1)})$ for some subgroup $H \subset G_{m+1}$, so that $Q(E_0) = 0$ by (1.26).

Now decompose

$$c_2 = b_0 + b_1,$$

with b_i taking values in $W_n(k)[S^p_{(i)}]$, $i = 0, 1$, and consider the binomial decomposition

$$c_2^{\otimes p^m} = \sum_f b_{f(1)} \otimes \cdots \otimes b_{f(p^m)}, \tag{2.15}$$

with the same meaning of f as in (2.13). Then all the terms except for $b_0^{\otimes p^m}$ take values in E_0 of (2.14), thus vanish after we apply the functor Q. Therefore we may assume right away that $b_1 = 0$, so that $c_2 = a \circ c_S$ for some endomorphism $a : E \to E$ of the module E. Since c_2 is admissible, we must have $a = \mathsf{id}$ modulo p, and then we are done by Proposition 2.3. □

Corollary 2.7 *For any integer $m \geq 1$, the standard isomorphism of Proposition rm 2.6 provides a functorial isomorphism* (2.11)*, so that the morphisms* (2.10) *provide functorial maps*

$$R : \widetilde{W}_{m+1} \to \mathsf{Infl}^m_{m-1} \widetilde{W}_m, \qquad R : W_{m+1} \to W_m. \tag{2.16}$$

Proof For the first claim, it suffices to prove that the standard isomorphism of Proposition 2.6 commutes with $W_\bullet(f)$ for any morphism $f : E_1 \to E_2$ of free

$W_n(k)$-modules. If f is a split injection, so that $E_2 \cong E_1 \oplus E_1'$, then the claim is obvious—we can choose bases in E_1 and E_1', consider the corresponding base in E_2, and notice that the already the maps (2.12) commute with f. If f is surjective, the same argument works. A general map is a composition of a split injection and a surjection (say, via its graph decomposition). □

By virtue of Corollary 2.7, for any k-vector space E, we can consider the inverse limit

$$\widetilde{W}(E) = \lim_{\overleftarrow{R}} \mathsf{Infl}_{m-1} \widetilde{W}_m(E) \in \mathcal{M}(G, W(k)), \tag{2.17}$$

where $\mathsf{Infl}_{\bullet}$ are the inflation functors (1.33). We call it the *extended polynomial Witt vectors* G-Mackey functor of the vector space E. Evaluating at the trivial orbit $[G/G]$, we obtain the $W(k)$-module

$$W(E) = \widetilde{W}(E/k)([G/G]) = \lim_{\overleftarrow{R}} W_m(E). \tag{2.18}$$

We call it the *polynomial Witt vectors module* of the vector space E.

3 Basic properties

3.1 Exact sequences and filtrations

Let us now prove some elementary properties of the truncated Witt vectors functors of Definition 2.4. We start with the following.

Lemma 3.1 *For any integer $m \geq 1$, the map $\widetilde{W}_{m+1} \to \widetilde{W}_{m+1}$ given by multiplication by p factors as*

$$\widetilde{W}_{m+1} \xrightarrow{R} \mathsf{Infl}_{m-1}^m \widetilde{W}_m \xrightarrow{C} \widetilde{W}_{m+1},$$

where R is the restriction map (2.16), and C is a certain functorial map. Moreover, R is surjective, and C is injective.

Proof Identify $\widetilde{W}_m' \cong \widetilde{W}_{m+1}$ as in Corollary 2.7, and let E be an arbitrary free $W_m(k)$-module. Then by (1.9), (1.5) and (1.29), we have a commutative diagram

$$\begin{array}{ccccc} R(E'_{(m)}) & \xrightarrow{\mathsf{id}} & R(E_{(m)}) & \xrightarrow{p\,\mathsf{id}} & R(E'_{(m)}) \\ \uparrow{\scriptstyle \mathsf{tr}} & & \uparrow{\scriptstyle \mathsf{tr}} & & \uparrow{\scriptstyle \mathsf{tr}} \\ L(E'_{(m)}) & \xrightarrow{p\,\mathsf{id}} & L(E_{(m)}) & \xrightarrow{\mathsf{id}} & L(E'_{(m)}), \end{array}$$

where tr are the trace maps (1.25). Taking the cokernels of these maps, we obtain the desired factorization. The fact that R is surjective and C is injective then immediately follows from Lemma 2.1. □

Now for any k-vector space E and integer $l \geq 1$, denote

$$C_l(E) = \left(E^{\otimes l}\right)_{\mathbb{Z}/l\mathbb{Z}}, \qquad C^l(E) = \left(E^{\otimes l}\right)^{\mathbb{Z}/l\mathbb{Z}}, \tag{3.1}$$

where the cyclic group $\mathbb{Z}/l\mathbb{Z}$ acts by permutations, and for any $m \geq 0$, let $C_{(m)}(E) = C_{p^m}(E), C^{(m)}(E) = C^{p^m}(E)$, with the k-vector space structure twisted by the absolute Frobenius of k as in (2.3). Note that $C_{(m)}(E)$ and $C^{(m)}(E)$ canonically extend to G_m-Mackey functors

$$\widetilde{C}_{(m)}(E) = L(E_{(m)}), \qquad \widetilde{C}^{(m)}(E) = R(E_{(m)}),$$

in the sense that we have natural identifications

$$\widetilde{C}_{(m)}(E)([G_m/G_m]) \cong C_{(m)}(E), \quad \widetilde{C}^{(m)}(E)([G_m/G_m]) \cong C^{(m)}(E) \tag{3.2}$$

for any k-vector space E.

Lemma 3.2 *For any integer $m \geq 1$, the natural maps R, C of Lemma 3.1 fit into functorial short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{C}_{(m)} & \stackrel{l}{\longrightarrow} & \widetilde{W}_{m+1} & \stackrel{R}{\longrightarrow} & \mathsf{Infl}^m_{m-1}\widetilde{W}_m & \longrightarrow & 0, \\ 0 & \longrightarrow & \mathsf{Infl}^m_{m-1}\widetilde{W}_m & \stackrel{C}{\longrightarrow} & \widetilde{W}_{m+1} & \stackrel{r}{\longrightarrow} & \widetilde{C}^{(m)} & \longrightarrow & 0. \end{array} \tag{3.3}$$

Proof As in Lemma 3.1, identify $\widetilde{W}_{m+1} \cong \widetilde{W}'_m$. Then by definition and (1.26), we have $U(\widetilde{W}'_m(E)) \cong \check{H}^0(\mathbb{Z}/p\mathbb{Z}, \widetilde{E}'_{(m)}) \cong E_{(m)}$ for any free $W_m(k)$-module $\widetilde{E}$ with reduction $E = \widetilde{E}/p$. Therefore we can take the adjunctions maps (1.34) as l and r in (3.3). Then since $U \circ \mathsf{Infl}^m_{m-1} = 0$, $R \circ l = r \circ C = 0$ by adjunction, and it suffices to prove that the sequences are exact after evaluation at an arbitrary k-vector space E. This immediately follows from Lemma 2.1 (choose a basis S in E, and use the explicit decompositions of Lemma 2.1 (ii) to compute $\widetilde{W}_\bullet$). ◻

Corollary 3.3 *For any integer $m \geq 1$ and k-vector space E, the G_{m-1}-Mackey functor $\widetilde{W}_m(E)$ is perfect in the sense of Definition 1.6 (i), and its assosiated graded quotients gr^i, gr_i with respect to the standard and co-standard filtrations of Definition 1.6 (ii) are given by*

$$\mathsf{gr}^i\,\widetilde{W}_m(E) \cong \mathsf{Infl}^{m-1}_i\,\widetilde{C}_{(i)}(E), \quad \mathsf{gr}_i\,\widetilde{W}_m(E) \cong \mathsf{Infl}^{m-1}_i\,\widetilde{C}^{(i)}(E)$$

for any $0 \leq i \leq m-1$.

Proof Clear. ◻

Let us now evaluate our G-Mackey functors at the trivial G-orbit $[G/G]$. Then (3.2) and (3.3) yield functorial short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{(m)}(E) & \xrightarrow{l} & W_{m+1}(E) & \xrightarrow{R} & W_m(E) & \longrightarrow & 0, \\ 0 & \longrightarrow & W_m(E) & \xrightarrow{C} & W_{m+1}(E) & \xrightarrow{r} & C^{(m)}(E) & \longrightarrow & 0. \end{array}$$

In fact, we can say more. Namely, for any $m \geq 0$ and k-vector space E, denote by $\Phi_m(E)$ the image of the trace map

$$\mathsf{tr}_{G_m} : C_{(m)}(E) \to C^{(m)}(E).$$

Then for any $m \geq 0$, sending $e \in E_{(m)}$ to $e^{\otimes p} \in E_{(m+1)}$ gives a functorial k-linear map

$$C : C_{(m)}(E) \to C_{(m+1)}(E).$$

If the vector space E is finite-dimensional, then we can dualize this map to obtain a functorial map

$$R : C^{(m+1)}(E) \to C^{(m)}(E),$$

and since both sides commute with filtered colimits in E, we can extend this map to arbitrary k-vector spaces.

Lemma 3.4 *For any $m \geq 0$ and any k-vector space E, we have functorial short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{(m)}(E) & \xrightarrow{C} & C_{(m+1)}(E) & \longrightarrow & \Phi_{m+1}(E) & \longrightarrow & 0, \\ 0 & \longrightarrow & \Phi_{(m+1)}(E) & \longrightarrow & C^{(m+1)} & \xrightarrow{R} & C^{(m)}(E) & \longrightarrow & 0, \end{array} \tag{3.4}$$

and commutative diagrams

$$\begin{array}{ccc} W_{m+1}(E) & \xrightarrow{r} & C^{(m+1)}(E) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ W_m(E) & \xrightarrow{r} & C^{(m)}(E) \end{array} \qquad \begin{array}{ccc} C_{(m)} & \xrightarrow{l} & W_m(E) \\ {\scriptstyle C}\downarrow & & \downarrow{\scriptstyle C} \\ C_{(m+1)} & \xrightarrow{l} & W_{m+1}(E). \end{array} \tag{3.5}$$

Proof Note that we have a functorial four-term sequence

$$0 \longrightarrow C_{(m)} \xrightarrow{C} C_{(m+1)} \xrightarrow{\mathsf{tr}_{G_{m+1}}} C^{(m+1)} \xrightarrow{R} C^{(m)} \longrightarrow 0.$$

and Lemma 2.1 immediately shows that the sequence is exact (indeed, for any vector space $E = k[S]$ with a basis S, the cokernel of the map $\mathsf{tr}_{G_{m+1}} : C_{(m+1)}(E) \to$

$C^{(m+1)}(E)$ is $\check{H}^o(G_{m+1}, E^{\otimes p^{m+1}})$, this coincides with $C^{(m+1)}(E)$ by Lemma 2.1 (ii), and dually for the kernel $\operatorname{Ker} \mathsf{tr}_{G_{m+1}}$). Together with the definition of the functor $\Phi_{\bullet}$, this yields the exact sequences (3.4). As for (3.5), then since all the maps are functorial, it suffices to check commutativity after choosing a basis S in E. Then the first claim immediately follows from Lemma 2.1 and the definition of the restriction map $R : W_{m+1} \to W_m$, and the second then follows by Lemma 3.1. □

By induction, Lemma 3.4 shows that the functor $C_{(m)}$ has a natural increasing "co-standard" $F_{\bullet}$ filtration with $\mathsf{gr}_i\, C_{(m)} \cong \Phi_i$, $0 \leq i \leq m-1$, and the functor $C^{(m)}$ has a natural descreasing "standard" filtration $F^{\bullet}$ with $\mathsf{gr}^j\, C^{(m)} \cong \Phi_j$, $0 \leq j \leq m-1$. The Witt vectors functor W_m has both filtrations, and they are transversal. Here is a picture of the associated graded quotient $\mathsf{gr}^{\bullet}_{\bullet}\, W_m = \mathsf{gr}^{\bullet}\, \mathsf{gr}_{\bullet}\, W_m = \mathsf{gr}_{\bullet}\, \mathsf{gr}^{\bullet}\, W_m$:

$$\begin{array}{cccccc} & & & & & \Phi_0 \\ & & & & \Phi_0 & \Phi_1 \\ & & & \Phi_0 & \Phi_1 & \Phi_2 \\ & & \cdots & \cdots & \cdots & \cdots \\ & & \Phi_0 & \cdots & \cdots & \Phi_{m-3} \\ & \Phi_0 & \Phi_1 & \cdots & \Phi_{m-3} & \Phi_{m-2} \\ \Phi_0 & \Phi_1 & \Phi_2 \ \ldots & \Phi_{m-3} & \Phi_{m-2} & \Phi_{m-1} \end{array} \tag{3.6}$$

The indices i, j correspond to rows and columns of the table that we number starting from 0, and we have $\mathsf{gr}_i^j\, W_m \cong \Phi_{i+j+1-m}$, or 0 if $i + j + 1 < m$. Multplication by p acts diagonally and induces an isomorphism $\mathsf{gr}^i_j \cong \mathsf{gr}^{i-1}_{j+1}$, or vanishes if $j = m-1$. The bottom row is the subfunctor $C_{(m-1)} \subset W_m$, and the rightmost column is the quotient functor $C^{(m-1)}$.

When we pass to the inverse limit (2.18), only the standard filtration $F^{\bullet}$ survives—we have $\mathsf{gr}^i\, W(E) \cong C_{(i)}$, $i \geq 0$, and $W(E)/F^i W(E) \cong W_i(E)$ for $i \geq 1$. We can also consider the inverse limit

$$C^{(\infty)}(E) = \lim_{\overset{R}{\leftarrow}} C^{(m)}(E),$$

and it also carries the standard filtration. The second exact sequence of (3.3) then provides a functorial isomorphism

$$W(E)/p \cong C^{(\infty)}(E), \tag{3.7}$$

and Lemma 3.1 shows that the $W(k)$-module $W(E)$ is torsion-free.

3.2 Frobenius and Verschiebung

All the cofinite subgroups $p^n G \subset G$, $n \geq 0$ in the group $G = \mathbb{Z}_p$ are abstractly isomorphic to G itself. For any integer $n \geq 0$ and coefficient ring A, denote by

$$\Psi^n = \Psi^{p^n G} : \mathcal{M}(G, A) \to \mathcal{M}(G, A)$$

the categorical fixed points functor with respect to the subgroup $p^nG \subset G$, and moreover, for any $m \geq n$, denote

$$\Psi^n = \Psi^{p^nG_{m-n}} : \mathcal{M}(G_m, A) \to \mathcal{M}(G_{m-n}, A).$$

Since by (1.16) we obviously have $\Psi^n \circ \mathsf{Infl}_m \cong \mathsf{Infl}_{m-n} \circ \Psi^n$, the notation is consistent. In fact, it is even consistent in the sense that $\Psi^{n_1} \circ \Psi^{n_2} \cong \Psi^{n_1+n_2}$ for any $n_1, n_2 \geq 0$.

Lemma 3.5 *For any k-vector space E and integers $m \geq n \geq 1$, we have a functorial isomorphism*

$$\Psi^n \widetilde{W}_{m+1}(E) \cong \widetilde{W}_{m-n+1}(E_{(n)}),$$

where $E_{(n)}$ is the k-vector space (2.2). These isomorphisms commute with restriction maps R and induce a functorial isomorphism

$$\Psi^n \widetilde{W}(E) \cong \widetilde{W}(E_{(n)}).$$

Proof For $\widetilde{W}_m$, the statement immediately follows from its definition and (1.28), and compatibility with the restriction maps is clear from their construction. For $\widetilde{W}$, pass to the limit. □

Lemma 3.5 allows us to evaluate G-Mackey functors $\mathsf{Infl}_m \widetilde{W}_{m+1}(E)$, $m \geq 0$ and their inverse limit $\widetilde{W}(E)$ at non-trivial G-orbits $[G/p^nG]$, $n \geq 1$. Namely, for any integer $n \geq 1$, denote

$$\begin{aligned} W^n(E) &= \widetilde{W}(E)([G/p^nG]),\\ W^n_m(E) &= \widetilde{W}_m(E)([G_{m-1}/p^nG_{m-n-1}]), \qquad m > n, \end{aligned}$$

with $W^0(E) = W(E)$ and $W^0_m(E) = W_m(E)$. By (1.26), we have a natural isomorphism

$$W^n_m(E) \cong \check{H}^0(G_{m-n}, \widetilde{E}_{(m)}), \tag{3.8}$$

with the same meaning of $\widetilde{E}$ as in (2.9). On the other hand, Lemma 3.5 provides natural isomorphisms

$$W^n(E) \cong W(E_{(n)}), \quad W^n_m(E) \cong W_{m-n}(E_{(n)}), \quad m > n. \tag{3.9}$$

By definition, $W^n(E)$ and $W^n_m(E), m > n$ carry actions of the group G_n. If $m = n+1$, then (3.9) reduces to an isomorphism

$$W^n_{n+1}(E) \cong W_1(E_{(n)} \cong E_{(n)}. \tag{3.10}$$

Effectively, $W^n_{n+1}(E)$ is simply $U(\widetilde{W}_{n+1}(E/k))$, so that in particular, the isomorphism (3.10) is G_n-equivariant. To see what happens for $m \geq m+2$, equip $W^n_m(E)$ with the standard and co-standard filtrations induced by those of Definition 1.6 (ii). Both filtrations are indexed by integers i, $n \leq i \leq m-1$, and preserved by G_n. Then (1.24) provides isomorphisms

$$\Psi^n \widetilde{C}_{(i)}(E) \cong \widetilde{C}_{(i-n)}(E_{(n)}), \quad \Psi^n \widetilde{C}^{(i)}(E) \cong \widetilde{C}^{(i-n)}(E_{(n)}), \qquad i \geq n,$$

and Corollary 3.3 then shows that we have

$$\mathsf{gr}^i\, W^n_m(E) \cong C_{(i-n)}(E_{(n)}), \quad \mathsf{gr}^i\, W^n_m(E) \cong C_{(i-n)}(E_{(n)}) \tag{3.11}$$

for any i, $n \leq i < m$. The G_n-action in both cases is the residual action of $G_n = G_i/p^n G_{i-n}$ on $C_{(i-n)}(E_{(n)}) = (E_{(i)})_{G_{i-n}}$ resp. $C_{(i-n)}(E_{(n)}) = (E_{(i)})_{G_{i-n}}$. In particular, we have

$$\begin{aligned} \mathsf{gr}^i\, W^n_m(E)_{G_n} &\cong C_{(i)}(E) \cong \mathsf{gr}^i\, W_m(E), \\ \mathsf{gr}_i\, W^n_m(E)^{G_n} &\cong C^{(i)}(E) \cong \mathsf{gr}_i\, W_m(E) \end{aligned} \tag{3.12}$$

for any $m > i \geq n \geq 0$.

Remark 3.6 As soon as $i > n$, the residual G_n-action on $C_{(i-n)}(E_{(n)})$, $C^{(i-n)}(E_{(n)})$ does *not* coincide with action induced by the natural action on $E_{(n)}$. Explicitly, the action of the generator $1 \in \mathbb{Z}/p^n\mathbb{Z} = G_n$ is induced by the map

$$\sigma_i = \sigma_n^{\otimes p^{i-n}} \circ (\sigma_{i-n} \otimes \mathsf{id}^{\otimes p^n - 1}) : E_{(i)} \to E_{(i)},$$

where $\sigma_n : E_{(n)} \to E_{(n)}$ resp. $\sigma_{i-n} : E_{(i-n)} \to E_{(i-n)}$ are the actions of the generators of the groups G_n resp. G_{i-n}, and we use the identifications $E_{(i)} \cong (E_{(n)})_{(i-n)} \cong (E_{(i-n)})_{(n)}$. In fact, σ_i is not even conjugate to $\sigma_n^{\otimes p^{i-n}}$. To see this, it suffices to consider the case when E is finite-dimensional, choose a basis, and compute the dimensions of the space of invariants and coinvariants with respect to the two G_n-action, e.g. by Lemma 2.1.

Consider now the maps V and F provided by Lemma 1.5. Explicitly, these can be described in terms of the isomorphisms (2.9) and (3.8)—by (1.27), we have

$$V = l^\dagger_\pi : W^1_m(E) \to W_m(E), \qquad F = r^\dagger_\pi : W_m(E) \to W^1_m(E), \tag{3.13}$$

where $l^\dagger_\pi$, $r^\dagger_\pi$ are the maps (1.10), $\pi : \mathsf{pt}_m \to \mathsf{pt}$ is the natural projection, and we factorize it as

$$\mathsf{pt}_m \xrightarrow{\pi''} \mathsf{pt}_1 \xrightarrow{\pi'} \mathsf{pt}.$$

Iterating the maps V and F, we obtain natural G_n-invariant maps

$$V^n : W^n(E) \to W(E), \qquad F^n : W(E) \to W^n(E),$$

and similarly for $W^n_m(E), m > n$. Being G_n-invariant, these maps factor through maps

$$\overline{V}^n : W^n(E)_{G_n} \to W(E), \qquad \overline{F}^n : W(E) \to W^n(E)^{G_n}, \tag{3.14}$$

and again similarly for $W^n_m(E)$, $m > n$.

Lemma 3.7 *For any k-vector space E and positive integers $m > n \geq 1$, the maps (3.14) fit into short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^n_m(E)_{G_n} & \xrightarrow{\overline{V}^n} & W_m(E) & \xrightarrow{R^{m-n}} & W_{m-n}(E) & \longrightarrow & 0, \\ 0 & \longrightarrow & W_{m-n}(E) & \xrightarrow{C^{m-n}} & W_m(E) & \xrightarrow{\overline{F}^n} & W^n_m(E)^{G_n} & \longrightarrow & 0, \end{array}$$

where R and C are the maps of Lemma 3.1.

Proof Since R and C are maps of Mackey functors, they commute with V and F, and this immediately implies that $R^{n-m} \circ V^n = V^n \circ R^{n-m} = 0$ on W^n_m and $F^n \circ C^{m-n} = C^{m-n} \circ F^n = 0$ on W_{m-n}. To prove that the sequences are exact, use induction on $m-n$. If $m = n$, we let $W^m_m(E) = 0$, and the statement is trivially true. For $m > n$, the lowest non-trivial term in the standard filtration on $W^n_m(E)$ is $F^m W^n_m(E) = \operatorname{gr}^m W^n_m(E)$, and by (3.11), we have $W^n_m(E)/\operatorname{gr}^m W^n_m(E) \cong W^n_{m-1}(E)$. Then since taking G_n-coinvariants is a right-exact functor, the sequence

$$\operatorname{gr}^m W^n_m(E)_{G_n} \longrightarrow W^n_m(E)_{G_n} \longrightarrow W^n_{m-1}(E)_{G_n} \longrightarrow 0$$

is exact on the right. However, by (3.12), the map $\operatorname{gr}^m W^n_m(E)_{G_n} \to W_m(E)$ induced by $\overline{V}^n$ is injective, so that the sequence is also exact on the left. Then $W^n_m(E)/\operatorname{gr}^m W^n_m(E)_{G_n} \cong W^n_{m-1}(E)_{G_n}$ and $W_m(E)/\operatorname{gr}^m W^n_m(E)_{G_n} \cong W_m/\operatorname{gr}^m W_m(E) \cong W_{m-1}(E)$, so that proving that the first one of our two sequences is exact for W_m is equivalent to proving it for W_{m-1}. By a dual argument, exactly the same holds for the second sequence; this gives the induction step. □

Example 3.8 If $E = k$ is one-dimensional, then it is immediately obvious from (2.9) that $W_m(k)$ is the free $W_m(k)$-module of rank 1, so that our notation is consistent. Then $V : W_{m-1}(k) \to W_m(k)$, $F : W_m(k) \to W_{m-1}(k)$ are the standard Verschiebung and Frobenius maps of the Witt vectors ring $W_m(k)$. Lemma 3.7 simply states that for any $n < m$, $V^n : W_{m-n}(k) \to W_m(k)$ is injective, with image $p^{m-n}W_m(k) \subset W_m(k)$, and $F^n : W_m(k) \to W_{m-n}(k)$ is surjective, with kernel $p^n W_m(k) \subset W_m(k)$.

Corollary 3.9 *On $W^n(E)$, the map $\overline{F}^n$ is an isomorphism, while the map $\overline{V}^n$ fits into a short exact sequence*

$$0 \longrightarrow (W^n(E))_{G_n} \xrightarrow{\overline{V}^n} W(E) \xrightarrow{R} W_n(E) \longrightarrow 0.$$

Moreover, for any integer i, we have $\check{H}^{2i+1}(G_n, W^n(E)) = 0$, while the group $\check{H}^{2i}(G_n, W^n(E))$ is canonically isomorphic to $W_n(E)$.

Proof The first statement follows from Lemma 3.7 by taking the inverse limit. For the second statement, compute Tate cohomology $\check{H}^\bullet(G_n, -)$ by the standard periodic complex, and note that by Lemma 1.5, $F^n \circ V^n$ is the trace map tr_{G_n}. □

3.3 Multiplication

Since both k and the Witt vectors rings $W_m(k)$, $m \geq 1$ are commutative, the categories k-mod and $\mathcal{M}(G_{m-1}, W_m(k))$, $m \geq 1$ are symmetric tensor categories. We then have the following result.

Proposition 3.10 *For every integer $m \geq 1$, the extended Witt vectors functor $\widetilde{W}_m$ of Definition 2.4 is symmetric pseudotensor, and the restriction map (2.16) of Corollary 2.7 is compatible with the pseudotensor structures.*

Proof Since $\widetilde{W}_m$ commutes with filtered colimits, it suffices to prove both claims after restricting it to finite-dimensional vector spaces. As in the proof of Proposition 2.3, let q be the essentially surjective reduction functor from the category $W_m(k)\text{-mod}^{ff} \subset W_m(k)$-mod of free finitely generated $W_m(k)$-modules to the category $k\text{-mod}^f \subset k$-mod of finite dimensional k-vector spaces. Then for any target category $\mathcal{C}$, the pullback functor $q^* : \mathrm{Fun}(k\text{-mod}^f, \mathcal{C}) \to \mathrm{Fun}(W_m(k)\text{-mod}^{ff}, \mathcal{C})$ is fully faithful, and the same is true for the pullback functor

$$\begin{aligned}(q \times q)^* : &\mathrm{Fun}(k\text{-mod}^f \times k\text{-mod}^f, \mathcal{C})\\ &\to \mathrm{Fun}(W_m(k)\text{-mod}^{ff} \times W_m(k)\text{-mod}^{ff}, \mathcal{C}).\end{aligned}$$

Therefore to construct the maps (1.2) for the functor $\widetilde{W}_m$, it suffices to construct them for the functor $\widetilde{W}_m \circ q \cong \widetilde{Q}_m$. But we have $\widetilde{Q}_m(E) = Q(E_{(m)})$, the p^m-th tensor power functor is symmetric and tensor, and the functor Q is symmetric pseudotensor by Lemma 1.4.

As for the restriction maps (2.16), what we need to prove is that they commute with the structure maps ε, μ of (1.2), and this can be checked pointwise, that is, after evaluating at arbitrary $M, N \in k\text{-mod}^f$. Then it suffices to choose bases in M and N, and notice that the standard map c_S of (2.12) is obviously multiplicative. □

As an immediate corollary of Proposition 3.10, we can pass to the inverse limit with respect to the restriction maps (2.16) and obtain a natural symmeric pseudotensor functors structure on the extended Witt vectors functor $\widetilde{W}$ of (2.17). Another immediate corollary is a symmetric pseudotensor structure on the polynomial Witt vectors functors W_m, $m \geq 1$ and their inverse limit W obtained by evaluating at the trivial G-orbit $[G/G]$. Since by Lemma 1.4, the pseudotensor structure on the functor Q is compatible with the isomorphism (1.26), the pseudotensor structures on W_m, $m \geq 1$ can also be characterized directly—these are the only pseudotensor structures compatible with the isomorphism (2.9) and the obvious pseudotensor structure on the functor Q_m.

We observe that the unit object in k-mod is the one-dimensional vector space k, and the unit object in $W_m(k)$-mod is the free module $W_m(k)$ of rank 1. Then by Example 3.8, the map ε of the pseudotensor structure on the functor W_m is an isomorphism. For any k-vector space E, applying the map μ to the tautological isomorphism $k \otimes E \cong E$ gives a map $W_m(k) \otimes_{W_m(k)} W_m(E) \to W_m(E)$, and this map is also tautologically an isomorphisms. As it happens, these tautologies already give one thing essentially for free.

Lemma 3.11 *For any k-vector space E and integer $m \geq 1$, we have*

$$V \circ F = p\,\mathsf{id} : W_m(E) \to W_m(E).$$

Proof Applying Lemma 1.5 (ii) to the tautological isomorphism $k \otimes E \cong E$, we see that it suffices to prove the claim for $E = k$. But then F and V obviously commute, so that $VF = FV = \mathsf{tr}_{\mathbb{Z}/p\mathbb{Z}}$, and the $\mathbb{Z}/p\mathbb{Z}$-action on $W_m(k)$ is trivial. ◻

Another immediate corollary of the tautological isomoprhism $W_m(k) \cong W_m(k)$ is the existence of a functorial map

$$T : E = \mathrm{Hom}_k(k, E) \to W_m(E) = \mathrm{Hom}_{W_m(k)}(W_m(k), W_m(E)) \tag{3.15}$$

induced by the functor W_m. The map T is not additive—it is only a map of sets, a version of the Teichmüller representative map for our polynomial Witt vectors W_m. However, it is obviously multiplicative, in the the sense that

$$T(e \otimes e') = \mu(T(e) \otimes T(e')), \qquad e, e' \in E, \tag{3.16}$$

and compatible with restriction maps (2.16), in that $R \circ T = T$. Passing to the limit, we obtain a functorial map of sets

$$T : E \to W(E), \qquad E \in k\text{-mod}. \tag{3.17}$$

Spelling out the definition of the functor W_m, we see that one can describe the Teichmüller map (3.15) as follows. For any $e \in E$, choose a free $W_m(k)$-module $\widetilde{E}$ so that $E \cong \widetilde{E}/p$, lift e to an element $\widetilde{e} \in E$, and consider the elements $\widetilde{e}_{(m)} \in Q_m(\widetilde{E}) = W_m(E)$, $\widetilde{e}'_{(m-1)} \in Q'_{m-1}(\widetilde{E}) = W'_{m-1}(E)$ of (2.6). Then both these elements only depend on e by Lemma 2.2, the canonical isomorphism of Proposition 2.6 sends $\widetilde{e}'_{(m-1)}$ to $\widetilde{e}_{(m)}$, and we have

$$T(e) = \widetilde{e}_{(m)} = \widetilde{e}'_{(m-1)} \in W_m(E) \cong W'_{m-1}(E). \tag{3.18}$$

For a more non-trivial application of Proposition 3.10, note that for any finite-dimensional k-vector space E with the dual vector space E^*, we have the natural pairing map $E \otimes E^* \to k$. Together with the pseudotensor structure on W_m, it induces a natural pairing

$$W_m(E) \otimes_{W_m(k)} W_m(E^*) \xrightarrow{\ \mu\ } W_m(E \otimes E^*) \longrightarrow W_m(k). \tag{3.19}$$

We observe that by Lemma 1.5 (ii), this pairing must be compatible with the maps F and V in the following sense: we have

$$\langle V(a), b\rangle = V(\langle a, F(b)\rangle) \in W_m(k) \tag{3.20}$$

for any $a \in W_{m-1}(E_{(1)})$, $b \in W_m(E^*)$.

Lemma 3.12 *For any $m \geq 1$ and finite-dimensional k-vector space E, (3.19) is a perfect pairing, so that $W_m(E)$ and $W_m(E^*)$ are dual modules over the Gorenstein ring $W_m(k)$.*

Proof Let $W_m(E)^* = \mathrm{Hom}_{W_m(k)}(W_m(E), W_m(k))$ be the $W_m(k)$-module dual to $W_m(E)$. Then the pairing (3.19) induces a map

$$\rho : W_m(E^*) \to W_m(E)^*,$$

and we have to prove that this map is an isomorphism. If $m = 1$, we have $W_1(E^*) \cong E^*$, $W_1(E) \cong E$, so the claim is clear. In the general case, we note that by Lemma 2.1 and (2.9), the source and the target of the map ρ are $W_m(k)$-modules of the same finite length, so that it suffices to prove that ρ is injective. Assume by induction that this is proved for W_{m-1}. Then since the Verschiebung map $V : W_{m-1}(k) \to W_m(k)$ is injective, (3.20) implies that for any $a \in \mathrm{Ker}\,\rho$, we have $F(a) = 0$. By Lemma 3.7, this means that a lies in the image of the map C^{m-1}, and by induction, we have $a = 0$. □

We note by Lemma 3.7 and (3.20), the perfect pairing (3.19) interchanges the standard and costandard filtrations on W_m. In terms of the diagram (3.6), one notes that the functors Φ_i are obviously self-dual, that is, we have $\Phi_i(E)^* \cong \Phi_i(E^*)$, and the pairing (3.19) then corresponds to flipping the table along the main diagonal.

Assume now given an algebra A over k. Then since W_m is symmetric pseudotensor, $W_m(A)$, $m \geq 1$ and $W(A)$ are rings, and if the algebra A is commutative, these rings are also commutative.

Unfortunately, at this point our notation stops being consistent: for any commutative k-algebra A different from k itself, our ring $W_m(A)$ is definitely *not* isomorphic to the standard m-truncated p-typical Witt vectors ring $W_m^{st}(A)$ of A, and $W(A)$ is not isomorphic to the standard Witt vectors ring $W^{st}(A)$. Indeed, already the associated graded quotients with respect to the standard filtration are different: we have $\mathrm{gr}^i W(A) \cong C_{(i)}(A)$ and $\mathrm{gr}^i W^{st}(A) \cong A$. In fact, our construction is by its nature relative over the field k, so that $W(E/k)$ would have been perhaps better notation for our polynomial Witt vectors. In this sense, the standard Witt vectors ring ought to correspond to $W(A/A)$. Unfortunately, under our present definition of polynomial Witt vectors this makes no sense, since the constrution crucially depends on the identification (2.11), and it only holds for perfect k. As we have mentioned in the Introduction, it is possible to give a competely different definition that does not require this assumption, but this is much more difficult technically, and we will return to it elsewhere.

3.4 Explicit descriptions

Let us now give some more explicit descriptions of the functors W_m. First of all, for any set S and integer $m \geq 0$, denote

$$C_{(m)}(S) = S^{p^m}/G_m, \qquad \overline{C}_{(m)} = S^{p^m}_{[m]}/G_m = C_{(m)}(S) \setminus C_{(m-1)}(S),$$

where the embedding $C_{(m-1)}(S) \subset C_{(m)}(S)$ is induced by the diagonal embedding $S^{p^{m-1}} \subset S^{p^m}$. Note that this is consistent with our earlier notation, in that we have $C_{(m)}(k[S]) \cong k[C_{(m)}(S)]$. Then the Witt vectors $W_m(E)$ of the k-vector space $E = k[S]$ spanned by S are immediately given by Lemma 2.1—we have a natural identification

$$W_m(E) \cong \bigoplus_{0 \leq i < m} W_{m-i}(k)[\overline{C}_{(i)}(S)].$$

Now more generally, assume given a k-vector space E graded by S, that is, assume that we have

$$E = \bigoplus_{s \in S} E_s$$

for some k-vector spaces E_s, $s \in S$.

Lemma 3.13 *For any S-graded vector space E, and for any choice of splittings $\tau : \mathcal{C}_{(i)}(S) \to S^{p^i}_{[i]}$ of the quotient maps $S^{p^i}_{[i]} \to \mathcal{C}_{(i)}(S)$, $0 \leq i < m$, we have a natural identification*

$$W_m(E) \cong \bigoplus_{0 \leq i < m} \bigoplus_{f \in \overline{C}_{(i)}(S)} W_{m-i}(E_{\tau(f)(1)} \otimes \cdots \otimes E_{\tau(f)(p^i)}). \tag{3.21}$$

Proof For any $i \geq 0$, we have a natural decomposition

$$E_{(i)} = E^{\otimes p^i} \cong \bigoplus_{f \in S^{p^i}} E_{f(1)} \otimes \cdots \otimes E_{f(p^i)},$$

so that the map $V^i : W_{m-i}(E_{(i)}) \to W_m(E)$ induces a map

$$V^i : W_{m-i}(E_{\tau(f)(1)} \otimes \cdots \otimes E_{\tau(f)(p^i)}) \to W_m(E)$$

for any $f \in \overline{C}_{(i)}(S)$. To prove that the direct sum of these maps is an isomorphism, it suffices to prove that it becomes an isomorphism after we pass to the associated graded quotients with respect to the standard filtration. This immediately follows from (3.12) and an obvious computation for the cyclic power functors $C_{(i)}(-)$, $0 \leq i < m$. □

Remark 3.14 The independence of the choice of τ in Lemma 3.13 is not surprising. In fact, as we will see in Sect. 4, W_m has a natural structure of a trace functor in the sense of [15]. In particular, this provides a natural identification

$$W_m(E_1 \otimes \cdots \otimes E_n) \cong W_m(E_{\sigma(1)} \otimes \cdots \otimes E_{\sigma(n)})$$

for any k-vector spaces $E_1, \dots, E_n$ and any cyclic permutation σ of the set of indices $\{1, \dots, n\}$.

If a k-vector space is graded by integers, then we can lift it to a $\mathbb{Z}$-graded flat $W_m(k)$-module $\widetilde{E}$, and then by (2.9), $W_m(E)$ inherits a natural grading. However, the restriction maps $R : W_{m+1}(E) \to W_m(E)$ do not preserve this grading—conversely, they multiply it by p. To make the graded consistent with the restriction maps, we need to rescale it by p^m. Thus a natural grading on $W_m(E)$, $m \geq 1$ and on the limit $W(E)$ is indexed not by integers but by elements $a \in \mathbb{Z}[p^{-1}]$ in the localization at p of the ring $\mathbb{Z}$. To see this grading in terms of the decomposition (3.21), denote

$$|f| = \sum_{j=1}^{p^i} \tau(f)(j)$$

for any $i \geq 0$ and $f \in \overline{C}_{(i)}(\mathbb{Z})$. Then the component $W_m(E)_a \subset W_m(E)$ of degree a is given by

$$W_m(E)_a = \bigoplus_{0 \leq i < m} \bigoplus_{|f| = p^i a} W_{m-i}(E_{\tau(f)(1)} \otimes \cdots \otimes E_{\tau(f)(p^i)}). \tag{3.22}$$

Passing to the limit, one obtains an obvious version of (3.21) and (3.22) for the limit Witt vectors functor W.

We now observe that even if a k-vector space E has no distinguished basis or grading, we still have the Teichmüller representative maps (3.17), (3.15). We recall that these maps are not additive. Nevertheless, combining them with the powers V^m of the Verschiebung map V, we can still obtain a functorial surjective map of sets

$$\widetilde{T} : \prod_{m \geq 0} E_{(m)} \to W(E) \tag{3.23}$$

given by

$$\widetilde{T}(\langle e_0, e_1, \dots \rangle) = \sum_{m \geq 0} V^m(T(e_m)),$$

where the sum is convergent with respect to the limit topology on $W(E)$.

In principle, one can use the map (3.23) to obtain an alternative description of the functor $W(E)$ that is closer to the classical construction of Witt vectors. We will not do it completely. However, let us show how one can describe the first non-trivial extension

$$0 \longrightarrow W^1(E) \xrightarrow{\overline{V}} W(E) \xrightarrow{R} E \longrightarrow 0$$

in the standard filtration on $W(E)$. The Teichmüller map (3.17) gives a set-theoretic section of the restriction map R, so that we obtain a functorial isomorphism of sets

$$W(E) \cong W^1(E) \times E,$$

and in terms of this isomorphism, the abelian group structure on $W(E)$ is given by

$$(e_0' \times e_0) + (e_1' \times e_1) = (e_0' + e_1' - c(e_0, e_1)) \times (e_0 \times e_1), \tag{3.24}$$

where $c(-,-)$ is a certain functorial symmetric 2-cocycle of the group E with coefficients in $W^1(E) \subset W(E)$. For any $m \geq 2$, we can project the cocycle c to $W_m(E)$ and obtain a cocycle

$$c : E \times E \to W_m^1(E) \tag{3.25}$$

that gives the group $W_m(E)$.

Lemma 3.15 (i) *Let $M = \mathbb{Z}[S]$ be the free abelian group generated by the set $S = \{s_0, s_1\}$ with two elements, and for any integer $n \geq 1$, let $\sigma : S^n \to S^n$ be the cyclic permutation of order n. Then there exist elements $c_i \in M^{\otimes p^i}$, $i \geq 1$ such that for any $n \geq 1$, we have*

$$(s_0 + s_1)^{\otimes p^n} = s_0^{\otimes p^n} + s_1^{\otimes p^n} + \sum_{i=1}^{n} \sum_{j=0}^{p^i-1} \sigma^j(c_i^{\otimes p^{n-i}}) \in M^{\otimes p^n}. \tag{3.26}$$

(ii) *Moreover, for any such set of elements c_i, $1 \leq i < m$ satisfying (3.26) for $n < m$, the cocycle (3.25) is given by*

$$c(e_0, e_1) = \widetilde{T}(0 \times c_1(e_0, e_1) \times c_2(e_0, e_1) \times c_{m-1}(e_0, e_1), \qquad e_0, e_1 \in E,$$

where $\widetilde{T}$ is the map (3.23), and $c_i(e_0, e_1)$ stands for the image of c_i under the map $M \to E$ sending s_0 to e_1 and s_1 to e_1.

Proof For (ii), it suffices to use the interpretation (3.18) of the Teichmüller map in terms of the element $\widetilde{e}'_{(m-1)}$, and recall that for any i such that $0 < i < m$, the map $V^i : Q'_{m-1-i}(\widetilde{E}_{(i)}) \to Q'_{m-1}(\widetilde{E})$ is induced by the trace map

$$\mathsf{tr}_{G_i} : H^0(G_{m-i}, \widetilde{E}'_{(m-1)}) \to H^0(G_m, \widetilde{E}'_{(m-1)}).$$

For (i), assume by induction that we already have elements c_i, $1 \leq i < m$ satisfying (3.26) for $n < m$. Then the difference

$$\overline{c}_m = (s_0 + s_1)^{\otimes p^m} - s_0^{\otimes p^{m-1}} - s_1^{\otimes p^{m-1}} - \sum_{i=1}^{m-1} \sum_{j=0}^{p^i-1} \sigma^j(c_i^{\otimes p^{m-i}})$$

is invariant under σ, thus gives an element in $H^0(G_m, M^{\otimes p^m})$. An element c_m satisfies (3.26) for $n = m$ if and only if

$$\overline{c}_m = \sum_{j=0}^{p^m-1} \sigma^j(c_m),$$

thus to show that it exists, it suffices to check that $\overline{c}_m$ projects to 0 in the Tate cohomology group $\check{H}^0(G_m, M^{\otimes p^m})$. This cohomology group is certainly annihilated by $p^m = |G_m|$, thus it does not change if we replace M with $M \otimes \mathbb{Z}_p$, and then (2.9) provides an identification

$$\check{H}^0(G_m, M^{\otimes p^m}) \cong W_m(E),$$

where $E = M/p$ is treated as a vector space over the prime field $\mathbb{Z}/p\mathbb{Z}$. To see that $\overline{c}_m$ projects to 0 in this group, it suffices to use the interpretation (3.18) of the map T in terms of the element $\widetilde{e}_{(m)}$, and apply (ii). □

Remark 3.16 Lemma 3.15 (i) has a priori nothing to do with the functors W_m, and might admit an explicit combinatorial proof [for example, certain explicit polynomials δ_i are introduced in [9], and the proof of [9, Proposition 1.2.3] seems to also prove that they satisfy (3.26)]. Having obtained the universal polynomials c_i in whatever fashion, one might try to reconstruct the functors W_m by induction on m via (3.24). However, one would then also need a $\mathbb{Z}/p\mathbb{Z}$-action on $W_{m-1}(E^{\otimes p})$ that produces the quotient $W_m^1(E) = W_{m-1}(E^{\otimes p})_{\mathbb{Z}/p\mathbb{Z}}$, and this requires a structure of a trace functor that we explore in Sect. 4. One might be able to reconstruct this structure explicitly as well, but we did not pursue this.

4 Trace functors

4.1 Definitions and examples

A *trace functor* from a unital monoidal category $\mathcal{E}$ to a category $\mathcal{C}$ is a functor $P : \mathcal{E} \to \mathcal{C}$ equipped with functorial isomorphisms

$$\tau_{M,N} : P(M \otimes N) \to P(N \otimes M), \qquad M, N \in \mathcal{E}$$

such that for any object $M \in \mathcal{E}$, we have $\tau_{1,M} = \tau_{M,1} = \mathsf{id}$, and for any three objects $M, N, L \in \mathcal{E}$, we have

$$\tau_{L,M,N} \circ \tau_{N,L,M} \circ \tau_{M,N,L} = \mathsf{id}, \tag{4.1}$$

where $1 \in \mathcal{E}$ denotes the unit object, we identify $1 \otimes M \cong M \cong M \otimes 1$ by means of the unitality isomorphism of the category $\mathcal{E}$, and $\tau_{A,B,C}$ for any $A, B, C \in \mathcal{E}$ is the composition of the map $\tau_{A,B\otimes C}$ and the map induced by the associativity isomorphism $(B \otimes C) \otimes A \cong B \otimes (C \otimes A)$.

It seems that the notion of a trace functor has been around in some form at least since 1960-ies. This particular definition is taken from [15], and it admits a convenient repackaging using A. Connes' cyclic category Λ. We recall (see e.g. [19] for [6]) that objects in Λ correspond to cellular decompositions of a circle S^1. For any positive integer $n \geq 1$, we have a natural object $[n] \in \Lambda$ corresponding to the unique decomposition with n 0-cells, called *vertices*, and n 1-cells. We denote the set of vertices by $V([n])$. Morphisms in Λ are homotopy classes of cellular maps satisfying certain condition. In particular, any morphism $f : [n'] \to [n]$ induces a map $f : V([n']) \to V([n])$; moreover, it is known that for any $v \in V([n])$, the preimage $f^{-1}(v) \subset V([n'])$ carries a natural total order.

Now, for any monoidal category $\mathcal{E}$, one constructs a category $\mathcal{E}^\natural$ as follows:

(i) Objects in $\mathcal{E}^\natural$ are pairs $\langle [l], E_\bullet \rangle$ of an object $[l] \in \Lambda$ and a collection $E_v \in \mathcal{E}$, $v \in V([l])$ of objects in $\mathcal{E}$ numbered by vertices of $[l]$.

(ii) Morphisms from $\langle [l'], E'_\bullet \rangle$ to $\langle [l], E_\bullet \rangle$ are given by a map $f : [l'] \to [l]$ in Λ and a collection of morphisms

$$f_v : \bigotimes_{v' \in f^{-1}(v)} E'_{v'} \to E_v, \qquad v \in V([n]),$$

where the product is taken in the order prescribed by the total order on $f^{-1}(v)$.

We have a natural projection $\rho : \mathcal{E}^\natural \to \Lambda$ sending $\langle [l], E_\bullet \rangle$ to $[l]$. This projection is a cofibration whose fiber over $[l] \in \Lambda$ is the product $\mathcal{E}^{V([l])}$ of copies of $\mathcal{E}$ numbered by vertices $v \in V([l])$. A map in $\mathcal{E}^\natural$ is cocartesian with respect to ρ if and only if all its components f_v are invertible.

To avoid size issues, assume that the monoidal category $\mathcal{E}$ is small. Then trace functors from $\mathcal{E}$ to $\mathcal{C}$ form a category $\mathrm{Tr}(\mathcal{E}, \mathcal{C})$ in an obvious way, and we have the following result.

Lemma 4.1 *The category* $\mathrm{Tr}(\mathcal{E}, \mathcal{C})$ *is equivalent to the category of functors* $P^\natural : \mathcal{E}^\natural \to \mathcal{E}$ *such that* $P^\natural(f)$ *is invertible for any map* f *in* $\mathcal{E}^\natural$ *cocartesian with respect to the projection* ρ.

The proof is in [15, Lemma 2.5]; let us just say that the correspondence $P^\natural \mapsto P$ simply restricts $P^\natural$ to the fiber $\mathcal{E}^\natural_{[1]} \cong \mathcal{E}$ of the projection $\rho : \mathcal{E}^\natural \to \Lambda$ over the object $[1] \in \Lambda$.

Remark 4.2 Lemma 4.1 has one immediate corollary. Note that the forgetful functor $\mathrm{Tr}(\mathcal{E}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{C})$ is faithful. Thus if we have two functors $P_1^\natural, P_2^\natural : \mathcal{E}^\natural \to \mathcal{C}$ corresponding to trace functors $P_1, P_2 \in \mathrm{Tr}(\mathcal{E}, \mathcal{C})$, and two maps $a, a' : P_1^\natural \to P_2^\natural$ such that $a = a'$ on $\mathcal{E}^\natural_{[1]} \subset \mathcal{E}^\natural$, then Lemma 4.1 shows that $a = a'$ on the whole $\mathcal{E}^\natural$.

If a monoidal category $\mathcal{E}$ is symmetric, for example if $\mathcal{E}$ is the category of vector spaces over a field k, then any functor $P : \mathcal{E} \to \mathcal{C}$ is tautologically a trace functor, with the maps $\tau_{\bullet,\bullet}$ provided by the commutativity isomorphism of the category $\mathcal{E}$. However, even in this case, there could be some non-trivial trace functor structures. The basic example of such considered in detail in [15] concern the cyclic power functor C_l, $l \geq 1$ of (3.1), and an explanation of how it works in terms of the eqivalence of Lemma 4.1 has been given in [17, Subsection 4.1]. Let us reproduce it.

Recall (e.g. from [6, Appendix]) that for any integer $l \geq 1$, the category Λ has a cousin Λ_l described as follows. For any $[n] \in \Lambda$, the automorphism group $\operatorname{Aut}([n])$ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ generated by the clockwise rotation σ of the circle. If $n = ml$ is divisible by l, then $\tau = \sigma^m$ generates the cyclic subgroup $\mathbb{Z}/l\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z}$. Then objects in $[m] \in \Lambda_l$ are numbered by positive integers $m \geq 1$, and morphisms from $[m']$ to $[m]$ are τ-equivariant morphisms from $[m'l]$ to $[ml]$ in the category Λ. Sending $[m]$ to $[ml]$ gives a functor $i_l : \Lambda_l \to \Lambda$. On the other hand, taking the quotient of a circle S^1 by the automorphism τ and equipping it with the induced cellular decomposition gives a functor $\pi_l : \Lambda_l \to \Lambda$, $\pi_l([m]) = [m]$. The functor π_l is a bifibration with fiber pt_l. For any m, it also induces the natural quotient map $\pi_l : V(i_l([m])) \to V([m])$.

Now, it was observed in [17] that for any monoidal category $\mathcal{E}$ and integer $l \geq 1$, we can construct a canonical commutative diagram

$$\begin{array}{ccccc} \mathcal{E}^\natural & \xleftarrow{i_l^{\mathcal{E}}} & \mathcal{E}_l^\natural & \xrightarrow{\pi_l^{\mathcal{E}}} & \mathcal{E}^\natural \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ \Lambda & \xleftarrow{i_l} & \Lambda_l & \xrightarrow{\pi_l} & \mathcal{E}^\natural, \end{array} \tag{4.2}$$

where the square on the right-hand side is cartesian, and the functor $i_l^{\mathcal{E}}$ sends $\langle [m], c_\bullet \rangle$ to $\langle i_l([m]), c_\bullet^l \rangle$, with the collection $c_\bullet^l$ given by $c_v^l = c_{\pi_l(v)}$, $v \in V(i_l([m]))$. Both $\pi_l^{\mathcal{E}}$ and $i_l^{\mathcal{E}}$ are cocartesian functors with respect to the cofibrations ρ.

Assume now that $\mathcal{E}$ is the category of finite-dimensional k-vector spaces, so that the category $\operatorname{Fun}(\mathcal{E}^\natural, k)$ is well-defined, and consider the object

$$C_l^\natural = \pi_{l!}^{\mathcal{E}} i_l^{\mathcal{E}*} T \in \operatorname{Fun}(\mathcal{E}^\natural, k),$$

where $T \in \operatorname{Fun}(\mathcal{E}^\natural, k)$ is the object corresponding to the tautological trace functor $\mathcal{E} \to k\text{-mod}$, $V \mapsto V$. Then for any $V \in \mathcal{E} \cong \mathcal{E}_{[1]}^\natural \subset \mathcal{E}^\natural$, we have $C_l^\natural(V) \cong C_l(V)$ by base change. Also by base change, $C_l^\natural : \mathcal{E}^\natural \to k$-mod satisfies the conditions of Lemma 4.1, and corresponds to a non-trivial trace functor structure on C_l.

4.2 Constructions

We now observe that exactly the same construction as for the cyclic power functor can be used to make the polynomial Witt vectors W into a trace functor. Namely, let k be a perfect field of characteristic $p > 0$, fix integers $n \geq m \geq 1$, and denote by

$\mathcal{E}_{(n)} = W_n(k)\text{-mod}^{ff} \subset W_n(k)\text{-mod}$ the category of finitely generated free modules over the Witt vectors ring $W_n(k)$. We have the quotient functor $\mathcal{E}_{(n)} \to \mathcal{E}$, $V \mapsto V/p$, where $\mathcal{E} = k\text{-mod}^f$ is the category of finite-dimensional k-vector spaces. Since the quotient functor is monoidal, it induces a functor $q : \mathcal{E}^\natural_{(n)} \to \mathcal{E}^\natural$. Consider the diagram (4.2) with $l = p^m$, and to simplify notation, let

$$i^{(n)}_{p^m} = i^{\mathcal{E}_{(n)}}_{p^m}, \qquad \pi^{(n)}_{p^m} = \pi^{\mathcal{E}_{(n)}}_{p^m}.$$

Then $\pi^{(n)}_{p^m}$ is a bifibration with fiber pt_{p^m}, so that we have the functor $\check{\pi}^{(n)}_{p^m *}$ of (1.7). Denote

$$Q^\natural_m = \check{\pi}^{(n)}_{p^m *} i^{(n)*}_{p^m} T^{(m)} \in \operatorname{Fun}(\mathcal{E}^\natural_{(n)}, W_n(k)), \tag{4.3}$$

where $T \in \operatorname{Fun}(\mathcal{E}^\natural_{(n)}, W_n(k))$ is the tautological functor corresponding to the embedding $\mathcal{E}_{(n)} \subset W_n(k)\text{-mod}$, and $T^{(m)}$ is T with the $W_n(k)$-module structure twisted by F^m as in (2.3). Note that by base change, $Q^\natural_m$ satisfies the conditions of Lemma 4.1. Moreover, for any $E \in \mathcal{E}_{(n)} = \mathcal{E}^\natural_{(n)[1]} \subset \mathcal{E}^\natural_{(n)}$, we have $Q^\natural_m(E) \cong \check{H}^0(G_m, E_{(m)})$, where $E_{(m)}$ is as in Proposition 2.3.

Proposition 4.3 (i) *For any integer $m \geq 1$, there exists an object $W^\natural_m \in \operatorname{Fun}(\mathcal{E}^\natural, W_m(k))$ such that $Q^\natural_m \cong q^* W^\natural_m$, independently of the choice of an integer $n \geq m$, and we have $W^\natural_m(E) \cong W_m(E)$ for any $E \in \mathcal{E} \cong \mathcal{E}^\natural_{[1]} \subset \mathcal{E}^\natural$.*

(ii) *Moreover, for any $m \geq 1$, the restriction map R of (2.16) extends to a map $W^\natural_{m+1} \to W^\natural_m$, and the Teichmüller representative map (3.15) extends to a map $T : W^\natural_1 \to W^\natural_m$.*

Proof For (i), note that q is essentially surjective, so that as in Proposition 2.3, the issue is the morphisms: we have to check that for two morphisms a, b in $\mathcal{E}^\natural_{(n)}$ with $q(a) = q(b)$, we have $Q^\natural_m(a) = Q^\natural_m(b)$. Every morphism f in $\mathcal{E}^\natural_{(n)}$ decomposes as $f_v \circ f_c$, where f_c is cocartesian with respect to the projection $\rho : \mathcal{E}^\natural_{(n)} \to \Lambda$, and f_v is contained in a fiber of this projection. If $q(a) = q(b)$, then in particular $\rho(a) = \rho(b)$, so that we may assume that $a_c = b_c$, and it suffices to check that $Q^\natural_m(a_v) = Q^\natural_m(b_m)$. Moreover, every object $[n] \in \Lambda$ admits a morphism $f : [n] \to [1]$, so that for any object $\widetilde{E} \in \mathcal{E}^\natural$, we have a cocartesian map $\widetilde{E} \to \widetilde{E}_0$ with $\rho(\widetilde{E}_0) = [1]$. Therefore we may further assume that a_v and b_v lie in the fiber $\mathcal{E} \cong \mathcal{E}^\natural_{[1]}) \subset \mathcal{E}^\natural$ of the cofibration ρ. Then the claim immediately follows from Proposition 2.3.

For (ii), the claim about the Teichmüller map T is clear from its explicit description given in the proof of Lemma 3.15, so what we need to construct is the restriction map R. To do this, we imitate the construction of the restriction map R of Corollary 2.7. Note that the functors $i_{p^{m+1}}$, resp. $\pi_{p^{m+1}}$ factor through i_{p^m}, resp. π_{p^m}—we have natural functors

$$\widetilde{i}, \widetilde{\pi} : \Lambda_{p^{m+1}} \to \Lambda_{p^m}$$

such that $i_{p^{m+1}} \cong i_{p^m} \circ \widetilde{i}$ and $\pi_{p^{m+1}} \cong \pi_{p^m} \circ \widetilde{\pi}$. The functor $\widetilde{\pi}$ is a bifibration with fiber pt_p; over an object of Λ, it is the fibration $\mathsf{pt}_{p^{m+1}} \to \mathsf{pt}_{p^m}$ corresponding to the quotient map $G_{m+1} \to G_m$. Moreover, both $\widetilde{\pi}$ and $\widetilde{i}$ lift to functors

$$\widetilde{i}^{(n)}, \widetilde{\pi}^{(n)} : \mathcal{E}^{\natural}_{(n)p^{m+1}} \to \mathcal{E}^{\natural}_{(n)p^m}$$

so that we have the same factorization, and $\widetilde{\pi}^{(n)}$ is also a bifibration with fiber pt_p. Denote

$$Q'^{\natural}_m = \check{\pi}^{(n)}_{p^{m+1}*} \widetilde{\pi}^{(n)*} i^{(n)*}_{p^m} T^{(m)}.$$

Then for any $E \in \mathcal{E}_{(n)} = \mathcal{E}^{\natural}_{(n)[1]} \subset \mathcal{E}^{\natural}_{(n)}$, we have $Q'^{\natural}_m(E) \cong \check{H}^0(G_m, E'_{(m)})$, where $E'_{(m)}$ is as in Proposition 2.3, so that by the same argument as in (i), we have

$$Q'^{\natural}_m = q^* W'^{\natural}_m$$

for some $W'^{\natural}_m \in \operatorname{Fun}(\mathcal{E}^{\natural}, W_n(k))$. Moreover, as in (2.5), we have a natural map

$$r : W'^{\natural}_m \to W^{\natural}_m,$$

so that it suffices to construct an isomorphism $W'^{\natural}_m \cong W^{\natural}_{m+1}$. By Lemma 4.1, this is equivalent to proving that the isomorphism (2.11) of Corollary 2.7 is compatible with the trace functor structures—that is, commutes with the maps $\tau_{\bullet,\bullet}$.

However, the category Γ of finite sets is also a monoidal category, with the monoidal structure given by cartesian product, and we have a natural monoidal functor $\nu : \Gamma \to \mathcal{E}$ sending a finite set S to the vector space $k[S]$. Then $\nu^* W^{\natural}_{m+1}$, $\nu^* W'^{\natural}_m$ correspond to trace functors from Γ to $W(k)$-mod, and since ν is essentially surjective, it suffices to check that (2.11) commutes with $\tau_{\bullet,\bullet}$ after restricting to Γ. In other words, it suffices to prove that it extends to an isomorphism $\nu^* W'^{\natural}_m \cong \nu^* W^{\natural}_{m+1}$. Moreover, the pullback functor q^* is fully faithful, and ν factors through q by means of a monoidal functor $\widetilde{\nu} : \Gamma \to \mathcal{E}_{(n)}$. Therefore it suffices to extend (2.11) to an isomorphism $\varepsilon : \widetilde{\nu}^* Q'^{\natural}_m \to \widetilde{\nu}^* Q^{\natural}_{m+1}$. But by base change, $\widetilde{\nu}^* Q'^{\natural}_m$ and $\widetilde{\nu}^* Q^{\natural}_{m+1}$ are given by

$$\widetilde{\nu}^* Q'^{\natural}_m = \check{\pi}^{\Gamma}_{p^{m+1}!} \widetilde{\pi}^{\Gamma*} \widetilde{T}^{(m)}, \qquad \widetilde{\nu}^* Q^{\natural}_{m+1} = \check{\pi}^{\Gamma}_{p^{m+1}!} \widetilde{i}^{\Gamma*} \widetilde{T}^{(m)},$$

where we denote $\widetilde{T}^{(m)} = i^{\Gamma*}_{p^n} \widetilde{\nu}^* T^{(m)}$. The canonical maps c_S, $S \in \Gamma$ of (2.12) together give a map

$$\widetilde{\pi}^{\Gamma*} \widetilde{T}^{(m)} \to \widetilde{i}^{\Gamma*} \widetilde{T}^{(m)}. \tag{4.4}$$

This map induces a map $\varepsilon : \widetilde{\nu}^* Q'^{\natural}_m \to \widetilde{\nu}^* Q^{\natural}_{m+1}$ whose restriction to $\Gamma = \Gamma^{\natural}_{[1]} \subset \Gamma^{\natural}$ is exactly the isomorphism (2.11). Since both $Q'^{\natural}_m$ and $Q^{\natural}_{m+1}$ send cocartesian maps in $\mathcal{E}^{\natural}$ to invertible maps, and every object in $\Gamma^{\natural}$ admits a cocartesian map to an object in $\Gamma^{\natural}_{[1]}$, the map ε must be an isomorphism everywhere. □

As a corollary of Proposition 4.3, we see that the m-truncated polynomial Witt vectors functors $W_m, m \geq 1$ have natural trace functors structures at least if we restrict them to finite-dimensional k-vector spaces, and the restriction maps $R : W_{m+1} \to W_m$ together with the Teichmüller maps $T : W_1 \to W_m$ are trace functor maps. Since there functors commute with filtered colimits, both statements immediately extend to all k-vector spacs. The inverse limit $W = \lim_{\overleftarrow{R}} W_m$ then also has a structure of a trace functor, and the Teichmüller map (3.17) is a trace functor map.

4.3 Extensions

Let us now prove that the trace functor structure on W is compatible with the two other structures it has—namely, the structure of a G-Mackey functor, and the pseudotensor structure of Proposition 3.10. As it happens, the proofs are quite straightforward, and the main issue is formulating the exact meaning of compatibility.

For the pseudotensor structure, this is also straightforward. For any unital monoidal category $\mathcal{E}$, the category $\mathcal{E}^2 = \mathcal{E} \times \mathcal{E}$ is also unital monoidal with respect to the coordinatewise monoidal structure. For any trace functor P from $\mathcal{E}$ to a category $\mathcal{C}$, we have a natural trace functor $P^2 : \mathcal{E}^2 \to \mathcal{C}^2$. If $\mathcal{E}$ is symmetric, then the product functor $m_{\mathcal{E}} : \mathcal{E}^2 \to \mathcal{E}$ is a monoidal functor, so that it defines a natural functor

$$m_{\mathcal{E}}^{\natural} : \mathcal{E}^{2\natural} \to \mathcal{E}^{\natural}$$

cocartesian over Λ. Moreover, the unit object $1 \in \mathcal{E}$ extends to a natural cocartesian section $1_{\Lambda} : \Lambda \to \mathcal{E}^{\natural}$ of the projection $\rho^{\mathcal{E}} : \mathcal{E}^{\natural} \to \Lambda$ such that $1_{\Lambda}([n]) = 1^{V([n])} \in \mathcal{E}^{V([n])}$, $[n] \in \Lambda$.

Definition 4.4 A *pseudotensor structure* on a trace functor P from a symmetric monoidal category $\mathcal{E}$ to a monoidal category $\mathcal{C}$ with the product functor $m_{\mathcal{C}} : \mathcal{C}^2 \to \mathcal{C}$ is given by functorial maps

$$\varepsilon : 1 \to P^{\natural}(1_{\Lambda}), \qquad \mu : m_{\mathcal{C}} \circ P^{2\natural} \to P^{\natural} \circ m_{\mathcal{E}}^{\natural} \tag{4.5}$$

that are associative, commutative and unital in the obvious sense.

As for the G-Mackey structure, one immediate observation is that finite G-orbits and objects $[n] \in \Lambda$ have one thing in common: their automorphisms form finite cyclic groups. We emphasize this similarity by using the notation $[p^m]$, $m \geq 0$ for finite G-orbits in Sect. 1.6. In fact, we have a natural functor $\delta : I \to \Lambda$, $\delta([p^m]) = [p^m]$, where as in Sect. 1.6, I is the groupoid of finite G-orbits and their isomorphisms. Moreover, this functor fits into a commutative diagram

$$\begin{array}{ccccc}
I & \xleftarrow{\pi} & I_p & \xrightarrow{i} & I \\
{\scriptstyle\delta}\downarrow & & \downarrow & & \downarrow{\scriptstyle\delta} \\
\Lambda & \xleftarrow{\pi_p} & \Lambda_p & \xrightarrow{i_p} & \Lambda,
\end{array} \tag{4.6}$$

where the squares are cartesian, and I_p, i, π are as in Sect. 1.6. One is tempted to think that one can add morphisms of G-orbits to morphisms already in Λ to obtain a bigger category $\Lambda R \supset \Lambda$, and then try to imitate the definition of Mackey functors to obtain some sort of a category $Q\Lambda R$. This is indeed possible; the resulting category is the *cyclotomic category* that appeared e.g. in [14], and when the theory is fully developed, it provides a notion of a "cyclotomic Mackey functor". However, doing all this properly requires a lot of work and a lot of space, and we will return to it elsewhere. For the purposes of the present paper, we simply use an explicit description of G-Mackey functors given in Lemma 1.5, and introduce the following somewhat *ad hoc* definition.

Definition 4.5 An FV*-structure* on a trace functor P from a monoidal category $\mathcal{E}$ to the category A-mod of modules over a ring A is a pair of maps

$$P^\natural \circ i_p^{\mathcal{E}} \xrightarrow{V} P^\natural \circ \pi_p^{\mathcal{E}} \xrightarrow{F} P^\natural \circ i_p^{\mathcal{E}}$$

such that $F \circ V : P^\natural \circ i_p^E \to P^\natural \circ i_p^{\mathcal{E}}$ is equal to the natural map $\mathsf{tr}^\dagger_{\pi_p^{\mathcal{E}}}$ of (1.6).

Here $i_p^{\mathcal{E}}, \pi_p^{\mathcal{E}} : \mathcal{E}_p^\natural \to E^\natural$ are the functors of (4.2), and we recall that the trace map $\mathsf{tr}^\dagger_{\pi_p^E}$ is well-defined even if the category $\mathcal{E}$ is not small. Note that for any object $E \in \mathcal{E} = \mathcal{E}^\natural_{[1]} \subset \mathcal{E}^\natural$, we have a natural functor $E^\natural : I \to \mathcal{E}^\natural$ given by

$$E^\natural([p^m]) = i^{\mathcal{E}}_{p^m}(\widetilde{E}), \tag{4.7}$$

and we have $\rho^{\mathcal{E}} \circ E^\natural \cong \delta$. Therefore by (4.6), for any trace functor $P^\natural : \mathcal{E} \to A$-mod equipped with an FV-structure, and for any object $E \in \mathcal{E}$, the composition $P^\natural \circ E^\natural$ with the induced maps V, F satisfies the conditions of Lemma 1.5 (i) and defines an A-valued G-Mackey functor.

Finally, assume that the monoidal category $\mathcal{E}$ is symmetric, the ring A is commutative, and we are given a trace functor P from $\mathcal{E}$ to A-mod that has both a pseudotensor structure $\langle \varepsilon, \mu \rangle$ in the sense of Definition 4.4 and an FV-structure $\langle V, F \rangle$ in the sense of Definition 4.5.

Definition 4.6 The structures $\langle \varepsilon, \mu \rangle$ and $\langle V, F \rangle$ on a trace functor P from $\mathcal{E}$ to A-mod are *compatible* if we have

$$\mu \circ (F \times F) = F \circ \mu, \tag{4.8}$$

$$\mu \circ (V \times \mathsf{id}) = V \circ \mu \circ (\mathsf{id} \times F), \tag{4.9}$$

$$\mu \circ (\mathsf{id} \times V) = V \circ \mu \circ (F \times \mathsf{id}). \tag{4.10}$$

We can now formulate our result about Witt vectors. Note that a pseudotensor structure on a trace functor P from $\mathcal{E}$ to $\mathcal{C}$ gives a pseudotensor structure on the underlying functor $P : \mathcal{E} \to \mathcal{C}$ by restriction to the fiber $\mathcal{E} = \mathcal{E}^\natural_{[1]} \subset \mathcal{E}$. Let $\mathcal{E} =$ k-mod, the category of vector spaces over our perfect ring k of characteristic p, and let $\mathcal{C} = W(k)$-mod, the category of modules over the Witt vectors ring $W(k)$.

Proposition 4.7 *The Witt vectors trace functor W from $\mathcal{E} = k$-mod to abelian groups of Sect. 4.2 has a natural $W(k)$-linear pseudotensor structure $\langle \varepsilon, \mu \rangle$ and an FV-structure $\langle V, F \rangle$ such that*

(i) *the two structures are compatible in the sense of Definition 4.6,*
(ii) *$\langle \varepsilon, \mu \rangle$ restricts to the pseudotensor structure of Proposition 3.10 on the fiber k-mod $= k$-mod$^\natural_{[1]} \subset k$-mod$^\natural$, and*
(iii) *for any object $E \in \mathcal{E}$, the functor $W^\natural \circ E^\natural : I \to W(k)$-mod with the maps V, F corresponds to the extended Witt vectors Mackey functor $\widetilde{W}(E)$ of (2.17) under the equivalence of Lemma 1.5.*

Moreover, for any integer $n \geq 1$, the maps V, F of the FV-structure induce $W_{n+1}(k)$-linear functorial maps

$$W_n^{(1)\natural} \circ i_p^{k\text{-mod}} \xrightarrow{\ V\ } W_{n+1}^\natural \circ \pi_p^{k\text{-mod}} \xrightarrow{\ F\ } W_n^{(1)\natural} \circ i_p^{k\text{-mod}}, \tag{4.11}$$

and the pseudotensor structure $\langle \varepsilon, \mu \rangle$ induces a pseudotensor structure on the quotient W_n of the trace functor W.

Proof As in the proofs of Propositions 3.10 and 4.3, it suffices to prove everything after replacing $\mathcal{E}$ with the category k-mod^f of finite-dimensional k-vector spaces, and then passing to the categories $\mathcal{E}_{(n)} = W_n(k)$-mod^{ff}, $n \geq 1$ by means of the fully faithful pullback functor q^*. Thus to construct a pseudotensor structure on W, it suffices to construct a system of pseudotensor structures on trace functors $Q_n = q^* W_n$, $n \geq 1$ compatible with the restriction maps R. Fix such an integer n, and to simplify notation, let $m^\natural = m^\natural_{\mathcal{E}_{(n)}}$, $i_{p^n} = i_{p^n}^{\mathcal{E}_{(n)}}$, $\pi_{p^n} = \pi_{p^n}^{\mathcal{E}_{(n)}}$. Then since the tautological embedding $\mathcal{E}_{(n)} \subset W_n(k)$-mod is a tensor functor, we have

$$m^{\natural *} T^{(n)} \cong T^{(n)} \boxtimes T^{(n)},$$

and since $i_{p^n} \circ m^\natural \cong m^\natural \circ i_{p^n}$, this gives an isomorphism

$$m^{\natural *} i_{p^n}^* T^{(n)} \cong i_{p^n}^* T^{(n)} \boxtimes i_{p^n}^* T^{(n)}.$$

Therefore to obtain a pseudotensor structure on $Q_n^\natural = \check{\pi}_{p^n *} i_{p^n}^* T^{(n)}$, it suffices to recall that the functor $\check{\pi}_{p^n *}$ is pseudotensor by Lemma 1.1. On the fiber $\mathcal{E} = E^\natural_{[1]} \subset \mathcal{E}^\natural$, $Q_n^\natural$ restricts to the functor Q_m, so that the compatibility statement (ii) immediately follows from the characterisation of the pseudotensor structure on W_n given in Sect. 3.3. Then Proposition 3.10 implies that our pseudotensor structures agree with the restriction maps on $\mathcal{E}^\natural_{[1]} \subset \mathcal{E}^\natural$, and as explained in Remark 4.2, Lemma 4.1 shows that they agree everywhere.

To obtain the FV-structure, note that for every $n \geq 1$, we have

$$i_p^* Q_n^\natural \cong \check{\pi}_{p^n *} i_{p^{n+1}}^* T^{(n)}$$

by base change. Simplify notation further by writing $\widetilde{T} = i^*_{p^{n+1}} T^{(n+1)}$, $\pi = \pi_{p^{n+1}}$, and let $\pi' = \pi_p$, $\pi'' = \pi_{p^n}$. Then (1.10) provides natural maps $l^\dagger_\pi$, $r^\dagger_\pi$, and evaluating them at $\widetilde{T}$, we obtain morphisms

$$V = l^\dagger_\pi(\widetilde{T}) : i^*_p Q^{(1)\natural}_n \to \pi^*_p Q^\natural_{n+1}, \qquad F = r^\dagger_\pi(\widetilde{T}) : \pi^*_p Q^\natural_{n+1} \to i^*_p Q^{(1)\natural}_n$$

whose composition $F \circ V$ is exactly as required in Definition 4.5 by virtue of (1.11). Moreover, since q^* is fully faithful, the maps F and V descend to Witt vectors trace functors $W^\natural_\bullet$ and give the diagram (4.11).

Now observe that if choose an object $E \in \mathcal{E}$ and restrict $W^\natural_n$ and $W^\natural_{n+1}$ to the category I via the functor (4.7), then by (3.13), our maps V and F restrict exactly to the maps V and F corresponding to the Mackey functor $\widetilde{W}_{n+1}(E)$ under the equivalence of Lemma 1.5. This implies in particular that the maps V and F for different integers n agree with the restriction maps R on $\mathcal{E} = \mathcal{E}^\natural_{[1]} \subset \mathcal{E}^\natural$, and then by Lemma 4.1, they must agree everywhere. Therefore we can pass to the inverse limit and obtain an FV-structure on the trace functor $W^\natural$. Moreover, this FV-structure satisfies the compatibility condition (iii).

To finish the proof, it remains to notice that the remaining condition (i) amounts to checking (4.8), and this can be done pointwise, that is, after evaluation at an arbitrary object $\widetilde{E} \in \mathcal{E}^\natural$. Moreover, by Lemma 4.1, it suffices to consider objects $\widetilde{E} \in \mathcal{E}^\natural_{[1]} \subset \mathcal{E}^\natural$, and by (iii), the statement then immediately follows from Lemma 1.5 (ii). □

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