Instanton Floer homology and contact structures

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Abstract We define an invariant of contact 3-manifolds with convex boundary using Kronheimer and Mrowka's sutured instanton Floer homology theory. This is the first invariant of contact manifolds—with or without boundary—defined in the instanton Floer setting. We prove that our invariant vanishes for overtwisted contact structures and is nonzero for contact manifolds with boundary which embed into Stein fillable contact manifolds. Moreover, we propose a strategy by which our contact invariant might be used to relate the fundamental group of a closed contact 3-manifold to properties of its Stein fillings. Our construction is inspired by a reformulation of a similar invariant in the monopole Floer setting defined by Baldwin and Sivek (arXiv:1403.1930, 2014).

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1 Introduction

Floer-theoretic invariants of contact manifolds have been responsible for many important results in low-dimensional topology. Notable examples of such invariants include

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the invariants of closed contact 3-manifolds defined by Kronheimer and Mrowka [13] and by Ozsváth and Szabó [25] in monopole and Heegaard Floer homology, respectively. Among other applications, these have been used to show that their respective homology theories detect the Thurston norm and that knot Floer homology detects the Seifert genus [20,24], to prove Dehn surgery characterizations of the unknot [15] and the trefoil and figure eight [7,26], and to prove many results about contact 3-manifolds, such as a complete solution to the existence problem for tight contact structures on Seifert fiber spaces [22].

Also important is the work in [11], where Honda, Kazez, and Matić extend Ozsváth and Szabó's construction, using sutured Heegaard Floer homology to define an invariant of *sutured contact manifolds*, which are triples of the form (M, Γ, ξ) where (M, ξ) is a contact 3-manifold with convex boundary and $\Gamma \subset \partial M$ is a multicurve dividing the characteristic foliation of ξ on ∂M. Applications of this invariant include the vanishing of the Ozsváth–Szabó contact class in the presence of Giroux torsion [8], which implies that manifolds with Giroux torsion cannot be strongly symplectically fillable; and a proof that in any contact 3-manifold, the complement of the binding of a supporting open book decomposition is torsion-free [6]. Recently, we defined an analogous invariant of sutured contact manifolds in Kronheimer and Mrowka's sutured monopole Floer homology theory [1].

The goal of this paper is to define an invariant of sutured contact manifolds in Kronheimer and Mrowka's sutured instanton Floer homology (SHI). To the best of our knowledge, this is the first invariant of contact manifolds—with or without boundary—defined in the instanton Floer setting. Like the Heegaard Floer invariants but in contrast with the monopole invariants, our instanton Floer contact invariant is defined using the full relative Giroux correspondence. Its construction is inspired by a reformulation of the monopole Floer invariant in [1] which was used there to prove that the monopole invariant is well defined.

A unique feature of the instanton Floer viewpoint is the central role played by the fundamental group. Along these lines, we conjecture a means by which our contact invariant in SHI might be used to relate the fundamental group of a closed contact 3-manifold to properties of its Stein fillings, a relationship which has been largely unexplored to this point.

Below, we sketch the construction of our contact invariant, describe some of its most important properties, state some conjectures, and discuss plans for future work which include using the constructions in this paper to define invariants of bordered manifolds in the instanton Floer setting.

1.1 A contact invariant in SHI

Suppose (M, Γ) is a balanced sutured manifold. Roughly speaking, a *closure* of (M, Γ) is formed by gluing on some auxiliary piece and "closing up" by identifying the remaining boundary components. In [21], Kronheimer and Mrowka defined an invariant of balanced sutured manifolds in terms of the instanton Floer homology groups of these related closed 3-manifolds. They proved that the groups associated to different closures of a given sutured manifold are all isomorphic. In this way, their invariant assigns to (M, Γ) an isomorphism class of $\mathbb{C}$-modules, denoted by $\underline{SHI}(M, \Gamma)$.

In [2], we introduced a refinement of their construction which assigns to (M, Γ) a *projectively transitive system of* $\mathbb{C}$-*modules*, denoted by $\underline{\mathbf{SHI}}(M, \Gamma)$. This system records the collection of $\mathbb{C}$-modules—all isomorphic to $\underline{SHI}(M, \Gamma)$—associated to different closures of (M, Γ) together with *canonical* isomorphisms relating these modules, where these isomorphisms are well defined up to multiplication in $\mathbb{C}^\times$. We refer to this system as the *sutured instanton homology* of (M, Γ).

A key step in constructing our contact invariant is to first define maps on sutured instanton homology associated to contact handle attachments. That is, suppose (M_i, Γ_i) is a balanced sutured manifold obtained by attaching a contact i-handle to (M, Γ) for some $i \in \{0, 1, 2, 3\}$. We define a map

$$\mathscr{H}_i : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M_i, -\Gamma_i)$$

which depends only on the smooth data involved in this handle attachment. Our construction of these maps is almost identical to that of the analogous maps in sutured monopole homology [1]; in particular, these maps are defined in terms of the maps on instanton Floer homology induced by natural cobordisms between closures.

Suppose now that (M, Γ, ξ) is a sutured contact manifold. According to the relative Giroux correspondence, this contact manifold admits a partial open book decomposition. This implies that (M, Γ, ξ) can be obtained by attaching contact 2-handles to a sutured contact manifold $H(S)$ formed from rounding the corners of a tight, vertically invariant contact structure on $S \times I$, where S is a compact surface with boundary (the surface S and the contact 2-handle attachments are specified by the partial open book decomposition). Let

$$\mathscr{H} : \underline{\mathbf{SHI}}(-H(S)) \to \underline{\mathbf{SHI}}(-M, -\Gamma)$$

be the composition of the maps associated to the contact 2-handle attachments above. Since $H(S)$ is a product sutured manifold, its sutured instanton homology has rank one with generator $\mathbf{1} \in \underline{\mathbf{SHI}}(-H(S)) \cong \mathbb{C}$, and we define the contact invariant of (M, Γ, ξ) to be

$$\theta(M, \Gamma, \xi) := \mathscr{H}(\mathbf{1}) \in \underline{\mathbf{SHI}}(-M, -\Gamma).$$

Our main theorem (stated later as Theorem 4.3) is the following.

Theorem 1.1 *The element* $\theta(M, \Gamma, \xi)$ *is well defined.*

That is to say, this element does not depend on the chosen partial open book decomposition (by the Giroux correspondence, it suffices to prove that this element is preserved under positive stabilization of the open book).[1]

We also show that this contact invariant behaves naturally with respect to the contact handle attachment maps, per the following (stated later as Theorem 4.8).

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[1] The analogous contact invariant in sutured monopole homology is defined without reference to the relative Giroux correspondence, but admits the same formulation as above.

Theorem 1.2 *Suppose (M_i, Γ_i, ξ_i) is obtained from (M, Γ, ξ) by attaching a contact i-handle and $\mathscr{H}_i$ is the associated map for $i = 0, 1,$ or 2. Then, $\mathscr{H}_i(\theta(M, \Gamma, \xi)) = \theta(M_i, \Gamma_i, \xi_i)$.*[2]

The invariant θ shares several important features with Honda, Kazez, and Matić's invariant and with our contact invariant in sutured monopole homology (besides the one above). Among these are the following two results (stated later as Theorems 4.10 and 4.12). Interestingly, the proofs of both theorems below are substantially different from those of their counterparts in [1] in the sutured monopole homology setting.

Theorem 1.3 *If (M, Γ, ξ) is overtwisted, then $\theta(M, \Gamma, \xi) = 0$.*

For the next theorem, suppose (Y, ξ) is a closed contact 3-manifold and let $Y(n)$ denote the sutured manifold obtained by removing n disjoint Darboux balls for any $n \geq 1$.

Theorem 1.4 *If (Y, ξ) is Stein fillable, then $\theta(Y(n), \xi|_{Y(n)}) \neq 0$.*

As we shall see, the corollary below (stated later as Corollary 4.13) follows from Theorems 1.4 and 1.2.

Corollary 1.5 *If (M, Γ, ξ) embeds into a Stein fillable manifold, then $\theta(M, \Gamma, \xi) \neq 0$.*

In a related direction, we conjecture the following, which is an instanton Floer analogue of a theorem of Plamenevskaya regarding the contact invariant in Heegaard Floer homology [27].

Conjecture 1.6 *Suppose J_1, J_2 are Stein structures on a smooth 4-manifold X such that $c_1(J_1) - c_1(J_2)$ is nontorsion. Let ξ_1, ξ_2 be the induced contact structures on $Y = \partial X$. Then, the contact invariants*

$$\theta(Y(1), \xi_1|_{Y(1)}) \quad \text{and} \quad \theta(Y(1), \xi_2|_{Y(1)})$$

are linearly independent in $\underline{\textbf{SHI}}(-Y(1))$.

Note that one needs some kind of naturality for a statement like that in Conjecture 1.6 since "linear independence" has little meaning if elements are only well defined up to isomorphism. This is precisely the sort of consideration that motivated our work in [2].

As we explain in Sect. 5, a positive answer to Conjecture 1.6 would imply the following link between the fundamental group of a contact 3-manifold and properties of its Stein fillings.

Conjecture 1.7 *Suppose Y is an integer homology 3-sphere which bounds a Stein 4-manifold (X, J) with $c_1(J) \neq 0$. Then, there exists a nontrivial homomorphism $\rho : \pi_1(Y) \to SU(2)$.*

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[2] We believe the contact 3-handle maps also preserve the contact invariant but do not prove this here.

It was pointed out to us by Tom Mrowka that the conclusion of Conjecture 1.7 holds by arguments similar to those used in the proof of the Property P conjecture [14] if the Stein filling has $b_2^+ > 0$. However, this leaves a lot of Stein fillable contact structures behind. For instance, Etnyre [5] shows that if a contact structure is supported by a planar open book, then all of its Stein fillings are negative definite.

In light of Conjecture 1.7, it is natural to ask whether there exist *any* integer homology spheres other than S^3 whose fundamental group admits no nontrivial $SU(2)$ representations? The main result of [17] implies that the answer is "no" among integer homology spheres arising from surgery on knots in S^3. In general, however, the question seems to be wide open.

Finally, it is worth mentioning that Conjecture 1.7 would also follow from Plamenevskaya's work in [27], combined with the conjectural isomorphism between $\underline{SHI}(Y(1))$ and $\widehat{HF}(Y) \otimes \mathbb{C}$ proposed in [21], but the latter seems more difficult to establish than Conjecture 1.6.

1.2 Future directions

Two of our future projects involve defining sutured cobordism maps and bordered invariants in the instanton Floer setting as mentioned briefly below.

Suppose (M, Γ) is a sutured submanifold of (M', Γ') and ξ is a contact structure on $M' \smallsetminus \text{int}(M)$ with dividing set $\Gamma \cup \Gamma'$. Note that $(M' \smallsetminus \text{int}(M), \Gamma \cup \Gamma', \xi)$ can be obtained from a vertically invariant contact structure on $\partial M \times I$ by attaching contact handles. Given such a handle decomposition H, we may then define

$$\Phi_{\xi,H} : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M', -\Gamma')$$

to be the corresponding composition of contact handle attachment maps. A similar map was defined by Honda et al. in [10] in the setting of sutured Heegaard Floer homology (see also [1] in the setting of sutured monopole homology). Their map depends only on ξ, and we conjecture that the same is true for the map above.

Conjecture 1.8 *The map $\Phi_{\xi,H}$ is independent of H.*

A positive answer to this conjecture would allow us assign well-defined maps to cobordisms between sutured manifolds in the instanton Floer setting—in the language of [2], to extend $\underline{\mathbf{SHI}}$ to a functor from **CobSut** to $\mathbb{C}$-**PSys**—following Juhasz's strategy [12], as explained in the analogous context of sutured monopole in [1, Subsection 1.3]. And this, in turn, would allow us to define invariants of bordered 3-manifolds using instanton Floer homology, following a strategy of Zarev [28]; again, see [1, Subsection 1.3] for the analogous discussion in the sutured monopole Floer setting. Namely, Zarev [28] constructed the homology of the bordered Heegaard Floer invariants as a direct sum of sutured Heegaard Floer homology groups. The invariant associated to a parametrized surface is a differential graded algebra, and he also described the product on its homology in terms of Juhász's sutured cobordism maps [12]. A complete bordered instanton theory would require these constructions at the chain level rather

than just on homology, but being able to implement Zarev's construction would be an important first step.

1.3 Organization

In Sect. 2, we provide the necessary background on projectively transitive systems, sutured instanton homology, and the relative Giroux correspondence. In Sect. 3, we define the contact handle attachment maps mentioned above. In Sect. 4, we define the contact invariant θ and establish some basic properties of this invariant, proving Theorems 1.2, 1.3, 1.4, and Corollary 1.5. Finally, in Sect. 5, we explain how a positive answer to Conjecture 1.6 would imply a positive answer to Conjecture 1.7.

2 Preliminaries

In this section, we review the notion of a projectively transitive system, the construction of sutured instanton homology, and the relative Giroux correspondence.

2.1 Projectively transitive systems of $\mathbb{C}$-modules

In [2], we introduced *projectively transitive systems* to make precise the idea of a collection of modules being canonically isomorphic up to multiplication by a unit. We recount their definition and related notions below, focusing on modules over $\mathbb{C}$.

Definition 2.1 Suppose M_α and M_β are $\mathbb{C}$-modules. We say that elements $x, y \in M_\alpha$ are *equivalent* if $x = u \cdot y$ for some $u \in \mathbb{C}^\times$. Likewise, homomorphisms

$$f, g : M_\alpha \to M_\beta$$

are *equivalent* if $f = u \cdot g$ for some $u \in \mathbb{C}^\times$.

Remark 2.2 We will write $x \doteq y$ or $f \doteq g$ to indicate that two elements or homomorphisms are equivalent, and will denote their equivalence classes by $[x]$ or $[f]$.

Note that *composition* of equivalence classes of homomorphisms is well defined, as is the *image* of the equivalence class of an element under an equivalence class of homomorphisms.

Definition 2.3 A *projectively transitive system of $\mathbb{C}$-modules* consists of a set A and:

(1) a collection of $\mathbb{C}$-modules $\{M_\alpha\}_{\alpha \in A}$ and
(2) a collection of equivalence classes of homomorphisms $\{g_\beta^\alpha\}_{\alpha, \beta \in A}$ such that:
    (a) elements of the equivalence class g_β^α are isomorphisms from M_α to M_β,
    (b) $g_\alpha^\alpha = [id_{M_\alpha}]$,
    (c) $g_\gamma^\alpha = g_\gamma^\beta \circ g_\beta^\alpha$.

Remark 2.4 The equivalence classes of homomorphisms in a projectively transitive system of $\mathbb{C}$-modules can be thought of as specifying canonical isomorphisms between the modules in the system that are well defined up to multiplication by units in $\mathbb{C}$.

The class of projectively transitive systems of $\mathbb{C}$-modules forms a category $\mathbb{C}$-**PSys** with the following notion of morphism.

Definition 2.5 A *morphism* of projectively transitive systems of $\mathbb{C}$-modules

$$F : (A, \{M_\alpha\}, \{g_\beta^\alpha\}) \to (B, \{N_\gamma\}, \{h_\delta^\gamma\})$$

is a collection of equivalence classes of homomorphisms $F = \{F_\gamma^\alpha\}_{\alpha \in A, \, \gamma \in B}$ such that:

(1) elements of the equivalence class F_γ^α are homomorphisms from M_α to N_γ,

(2) $F_\delta^\beta \circ g_\beta^\alpha = h_\delta^\gamma \circ F_\gamma^\alpha$.

Note that F is an *isomorphism* iff the elements in each equivalence class F_γ^α are isomorphisms.

Remark 2.6 A collection of equivalence classes of homomorphisms $\{F_\gamma^\alpha\}$ with indices ranging over any nonempty subset of $A \times B$ can be uniquely completed to a morphism as long as this collection satisfies the compatibility in (2) where it makes sense.

Remark 2.7 Suppose $\{\mathcal{S}_\alpha\}_{\alpha \in A}$ is a collection of projectively transitive systems of $\mathbb{C}$-modules and

$$\{f_{\alpha,\beta} : \mathcal{S}_\alpha \to \mathcal{S}_\beta\}_{\alpha, \beta \in A}$$

is a collection of isomorphisms of projectively transitive systems of $\mathbb{C}$-modules which satisfy the transitivity $f_{\alpha,\gamma} = f_{\beta,\gamma} \circ f_{\alpha,\beta}$ for all $\alpha, \beta, \gamma \in A$. Then, this *transitive system of systems* defines an even larger projectively transitive system of $\mathbb{C}$-modules in a natural way, whose set of constituent $\mathbb{C}$-modules is the union over all $\alpha \in A$ of the sets of $\mathbb{C}$-modules making up the systems $\mathcal{S}_\alpha$.

Definition 2.8 An *element* of a projectively transitive system of $\mathbb{C}$-modules

$$x \in \mathcal{M} = (A, \{M_\alpha\}, \{g_\beta^\alpha\})$$

is a collection of equivalence classes of elements $x = \{x_\alpha\}_{\alpha \in A}$ such that:

(1) elements of the equivalence class x_α are elements of M_α,

(2) $x_\beta = g_\beta^\alpha(x_\alpha)$.

Remark 2.9 As in Remark 2.6, a collection of equivalence classes of elements $\{x_\alpha\}$ with indices ranging over any nonempty subset of A can be uniquely completed to an element of $\mathcal{M}$ as long as this collection satisfies the compatibility in (2) where it makes sense.

We say that x is a *generator* in $\mathcal{M}$ if each M_α is isomorphic to $\mathbb{C}$ and each x_α is the equivalence class of a generator—i.e., nonzero. The *zero* element $0 \in \mathcal{M}$ is the collection of equivalence classes of the elements $0 \in M_\alpha$. Finally, it is clear how to define the image $F(x)$ of an element $x \in \mathcal{M}$ under a morphism $F : \mathcal{M} \to \mathcal{N}$ of projectively transitive systems of $\mathbb{C}$-modules.

Remark 2.10 Given a $\mathbb{C}$-module M, we can also think of M as the projectively transitive system of $\mathbb{C}$-modules given (in an abuse of notation) by

$$M = (\{\star\}, \{M\}, \{[id_M]\})$$

consisting of the single $\mathbb{C}$-module M together with the equivalence class of the identity map, so that it makes sense to write $\mathcal{S} \cong M$, for any other object $\mathcal{S} \in \mathbb{C}$-**PSys**.

2.2 Sutured instanton homology

In this subsection, we describe our refinement in [2] of Kronheimer and Mrowka's sutured instanton homology, as defined in [21].

2.2.1 Closures of balanced sutured manifolds

Definition 2.11 A *balanced sutured manifold* (M, Γ) is a compact, oriented, smooth 3-manifold M with a collection Γ of disjoint, oriented, smooth curves in ∂M called *sutures*. Let $R(\Gamma) = \partial M \smallsetminus \Gamma$, oriented as a subsurface of ∂M. We require that:

(1) neither M nor $R(\Gamma)$ has closed components,
(2) $R(\Gamma) = R_+(\Gamma) \sqcup R_-(\Gamma)$ with $\partial R_+(\Gamma) = -\partial R_-(\Gamma) = \Gamma$,
(3) $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$.

An *auxiliary surface* for (M, Γ) is a compact, connected, oriented surface F with $g(F) > 0$ and $\pi_0(\partial F) \cong \pi_0(\Gamma)$. Suppose F is an auxiliary surface for (M, Γ), $A(\Gamma)$ is a closed tubular neighborhood of Γ in ∂M, and

$$h : \partial F \times [-1, 1] \to A(\Gamma)$$

is an orientation-reversing diffeomorphism which sends $\partial F \times \{\pm 1\}$ to $\partial(R_\pm(\Gamma) \smallsetminus A(\Gamma))$. One forms a *preclosure* of M

$$M' = M \cup_h F \times [-1, 1]$$

by gluing $F \times [-1, 1]$ to M according to h and rounding corners. This preclosure has two diffeomorphic boundary components, $\partial_+ M'$ and $\partial_- M'$. We may therefore glue $\partial_+ M'$ to $\partial_- M'$ by some diffeomorphism to form a closed manifold Y containing a distinguished surface

$$R := \partial_+ M' = -\partial_- M' \subset Y.$$

In [21], Kronheimer and Mrowka define a *closure* of (M, Γ) to be any pair (Y, R) obtained in this way. Our definition of closure, as needed for naturality, is slightly more involved.

Definition 2.12 ([2]) A *marked odd closure* of (M, Γ) is a tuple $\mathscr{D} = (Y, R, r, m, \eta, \alpha)$ consisting of:

(1) a closed, oriented, 3-manifold Y,
(2) a closed, oriented, surface R with $g(R) \geq 2$,
(3) an oriented, nonseparating, embedded curve $\eta \subset R$,
(4) a smooth, orientation-preserving embedding $r : R \times [-1, 1] \hookrightarrow Y$,
(5) a smooth, orientation-preserving embedding $m : M \hookrightarrow Y \smallsetminus \text{int}(\text{Im}(r))$ such that:
    (a) m extends to a diffeomorphism

$$M \cup_h F \times [-1, 1] \to Y \smallsetminus \text{int}(\text{Im}(r))$$

    for some $A(\Gamma)$, F, h, as above,
    (b) m restricts to an orientation-preserving embedding

$$R_+(\Gamma) \smallsetminus A(\Gamma) \hookrightarrow r(R \times \{-1\}).$$

(6) an oriented, embedded curve $\alpha \subset Y$ such that:
    (a) α is disjoint from $\text{Im}(m)$,
    (b) α intersects $r(R \times [-1, 1])$ in an arc of the form $r(\{p\} \times [-1, 1])$ for some $p \in R$.

The *genus* $g(\mathscr{D})$ refers to the genus of R.

Remark 2.13 Suppose $\mathscr{D} = (Y, R, r, m, \eta, \alpha)$ is a marked odd closure of (M, Γ). Then, the tuple

$$-\mathscr{D} := (-Y, -R, r, m, -\eta, -\alpha),$$

obtained by reversing the orientations of Y, R, η, and α is a marked odd closure of $-(M, \Gamma) := (-M, -\Gamma)$, where r and m are the induced embeddings of $-R \times [-1, 1]$ and $-M$ into $-Y$.

2.2.2 Instanton Floer homology

Before defining sutured instanton homology, we recall the basic set up of instanton Floer homology from [21].

    Suppose Y is a closed, oriented, smooth 3-manifold and $w \to Y$ is a Hermitian line bundle such that $c_1(w)$ has odd pairing with some class in $H_2(Y; \mathbb{Z})$. Let $E \to Y$ be a $U(2)$ bundle with an isomorphism $\theta : \Lambda^2 E \to w$. Let $\mathcal{C}$ be the space of $SO(3)$ connections on $\text{ad}(E)$, and let $\mathcal{G}$ be the group of determinant-1 gauge transformations of E (the automorphisms of E that respect θ). The associated instanton Floer homology

group, which Kronheimer and Mrowka denote by $I_*(Y)_w$, is the $\mathbb{Z}/8\mathbb{Z}$-graded $\mathbb{C}$-module arising from the Morse homology of the Chern–Simons functional on $\mathcal{C}/\mathcal{G}$ (cf. [3]). Given any closed, embedded surface $R \subset Y$ there is a natural operator

$$\mu(R) : I_*(Y)_w \to I_*(Y)_w$$

of degree -2. When R has genus at least 2, Kronheimer and Mrowka define the submodule

$$I_*(Y|R)_w \subset I_*(Y)_w$$

to be the eigenspace of $\mu(R)$ with eigenvalue $2g(R) - 2$. In this paper, in contrast with [2,21], we will instead use the $(2 - 2g(R))$-eigenspace for the sake of consistency with [1], but it makes no difference as far as the proofs of invariance or naturality (or for that matter anything in this paper) are concerned.

Suppose α is an oriented, smooth 1-cycle in Y which intersects a closed, embedded surface in an odd number of points. One can associate to (Y, α) an instanton Floer group after first choosing bundles w, E, and an isomorphism θ as above, where the first Chern class is Poincaré dual to α. This Floer group is itself not an invariant of (Y, α) as it depends on these auxiliary choices. However, given a pair (Y, α), the Floer groups associated to any two sets of auxiliary choices are related by a canonical isomorphism which is well defined up to sign (cf. [18, Section 4]). In particular, the pair (Y, α) defines a projectively transitive system of $\mathbb{C}$-modules, which we will denote by $I_*(Y)_\alpha$. The canonical isomorphisms respect the eigenspace decompositions and, so, for a closed embedded surface $R \subset Y$, we may also define the projectively transitive system of $\mathbb{C}$-modules $I_*(Y|R)_\alpha$.

Suppose R_1 and R_2 are embedded surfaces in Y_1 and Y_2 as above. A cobordism (W, ν) from (Y_1, α_1) to (Y_1, α_2) together with an embedded surface $R_W \subset W$ containing R_1 and R_2 as components gives rise to a map of projectively transitive systems

$$I_*(W|R_W)_\nu : I_*(Y_1|R_1)_{\alpha_1} \to I_*(Y_2|R_2)_{\alpha_2}.$$

This map depends only on the homology class $[\nu] \subset H_2(W, \partial W; \mathbb{Z})$ and the isomorphism class of (W, ν), where two pairs are isomorphic if they are diffeomorphic by a map which intertwines the boundary identifications.

2.2.3 Sutured instanton homology

Following Kronheimer and Mrowka [21], we made the following definition in [2].

Definition 2.14 Given a marked odd closure $\mathscr{D} = (Y, R, r, m, \eta, \alpha)$ of (M, γ), the *twisted sutured instanton homology of* $\mathscr{D}$ is the projectively transitive system of $\mathbb{C}$-modules

$$\underline{SHI}(\mathscr{D}) := I_*(Y|r(R \times \{0\}))_{\alpha \sqcup \eta}.$$

Remark 2.15 If w and u are line bundles over Y with first Chern classes represented by α and η, then the line bundle $w \otimes u$ has first Chern class represented by $\alpha \sqcup \eta$.

In [2], we constructed canonical isomorphisms

$$\underline{\Psi}_{\mathscr{D},\mathscr{D}'} : \underline{SHI}(\mathscr{D}) \to \underline{SHI}(\mathscr{D}')$$

for any two marked odd closures $\mathscr{D}, \mathscr{D}'$ of (M, Γ), so that

$$\underline{\Psi}_{\mathscr{D},\mathscr{D}''} = \underline{\Psi}_{\mathscr{D}',\mathscr{D}''} \circ \underline{\Psi}_{\mathscr{D},\mathscr{D}'}$$

for all $\mathscr{D}, \mathscr{D}', \mathscr{D}''$. In other words, the systems in $\{\underline{SHI}(\mathscr{D})\}_{\mathscr{D}}$ and the maps in $\{\underline{\Psi}_{\mathscr{D},\mathscr{D}'}\}_{\mathscr{D},\mathscr{D}'}$ form a transitive system of systems and, therefore, a larger projectively transitive system of $\mathbb{C}$-modules as explained in Remark 2.7. These isomorphisms are defined almost exactly as in the monopole setting—in terms of 2-handle or splicing cobordisms depending on whether the genera of $\mathscr{D}$ and $\mathscr{D}'$ are the same or different.

Definition 2.16 The *sutured instanton homology of* (M, Γ) is the projectively transitive system of $\mathbb{C}$-modules $\underline{\textbf{SHI}}(M, \Gamma)$ defined by the transitive system of systems above.

Sutured instanton homology is functorial in the following sense. Suppose

$$f : (M, \Gamma) \to (M', \Gamma')$$

is a diffeomorphism of sutured manifolds and $\mathscr{D}' = (Y', R', r', m', \eta', \alpha')$ is a marked odd closure of (M', Γ'). Then,

$$\mathscr{D}'_f := (Y', R', r', m' \circ f, \eta', \alpha') \tag{1}$$

is a marked odd closure of (M, Γ). Let

$$id_{\mathscr{D}'_f,\mathscr{D}'} : \underline{SHI}(\mathscr{D}'_f) \to \underline{SHI}(\mathscr{D}')$$

be the identity map on $\underline{SHI}(\mathscr{D}'_f) = \underline{SHI}(\mathscr{D}')$. The equivalence classes underlying these identity maps can be completed to a morphism (as in Remark 2.6)

$$\underline{\textbf{SHI}}(f) : \underline{\textbf{SHI}}(M, \Gamma) \to \underline{\textbf{SHI}}(M', \Gamma'),$$

which is an invariant of the isotopy class of f. We proved in [2] that these morphisms behave as expected under composition of diffeomorphisms, so that $\underline{\textbf{SHI}}$ defines a functor from **DiffSut** to $\mathbb{C}$-**PSys**, where **DiffSut** is the category of balanced sutured manifolds and isotopy classes of diffeomorphisms between them.

The following will be important in our definition of the instanton Floer contact invariant.

Proposition 2.17 *If* (M, Γ) *is a product sutured manifold, then* $\underline{\textbf{SHI}}(M, \Gamma) \cong \mathbb{C}$.

Proof Let F be an auxiliary surface for (M, Γ) with $g(F) \geq 2$. Thinking of (M, Γ) as obtained from $(S \times [-1, 1], \partial S \times \{0\})$ by rounding corners, we can form a preclosure of (M, Γ) by gluing $F \times [-1, 1]$ to $S \times [-1, 1]$ according to a map

$$h : \partial F \times [-1, 1] \rightarrow \partial S \times [-1, 1]$$

of the form $f \times id$ for some diffeomorphism $f : \partial F \rightarrow \partial S$. This preclosure is then a product $M' = (S \cup F) \times [-1, 1]$. To form a marked odd closure, we take $R = S \cup F$ and glue $R \times [-1, 1]$ to M' by the "identity" maps

$$R \times \{\pm 1\} \rightarrow S \times \{\mp 1\}.$$

An oriented, nonseparating curve $\eta \subset S \cup F$ and a curve $\alpha = \{p\} \times S^1$ for any point $p \in F$ gives a marked odd closure

$$\mathscr{D} = ((S \cup F) \times S^1, (S \cup F), r, m, \eta, \alpha).$$

Here, we are thinking of S^1 as the union of two copies of $[-1, 1]$, and r and m as the obvious embeddings. The system $\underline{SHI}(\mathscr{D})$ is then given by

$$I_*((S \cup F) \times S^1 | (S \cup F) \times \{0\})_{\alpha \sqcup \eta} \cong \mathbb{C},$$

where this isomorphism follows from [21, Proposition 7.8]. $\qquad\qquad\qquad\square$

2.3 The relative Giroux correspondence

Below, we review the relative Giroux correspondence between partial open books and sutured contact manifolds. Our discussion of this correspondence differs slightly in style but not in substance from the discussions in [4,11].

Definition 2.18 A *partial open book* is a quadruple $(S, P, h, \mathbf{c})$, where:

(1) S is a surface with nonempty boundary,
(2) P is a subsurface of S,
(3) $h : P \rightarrow S$ is an embedding which restricts to the identity on $\partial P \cap \partial S$,
(4) $\mathbf{c} = \{c_1, \ldots, c_n\}$ is a set of disjoint, properly embedded arcs in P such that $S \smallsetminus \mathbf{c}$ deformation retracts onto $S \smallsetminus P$.

Remark 2.19 The collection $\mathbf{c}$ of *basis arcs* for P is not typically recorded in the data of a partial open book. Usually, it is just required that S be obtained from $\overline{S \smallsetminus P}$ by successive 1-handle attachments. The basis arcs specify a 1-handle decomposition of P. Given that we are specifying basis arcs, we do not technically need to record the subsurface P. We do so anyhow to emphasize the equivalence between Definition 2.18 and the more commonplace definition of partial open book found in [4,11].

Suppose $(S, P, h, \mathbf{c})$ is a partial open book. Consider the $[-1, 1]$-invariant contact structure ξ_S on $S \times [-1, 1]$ for which each $S \times \{t\}$ is convex with collared Legendrian
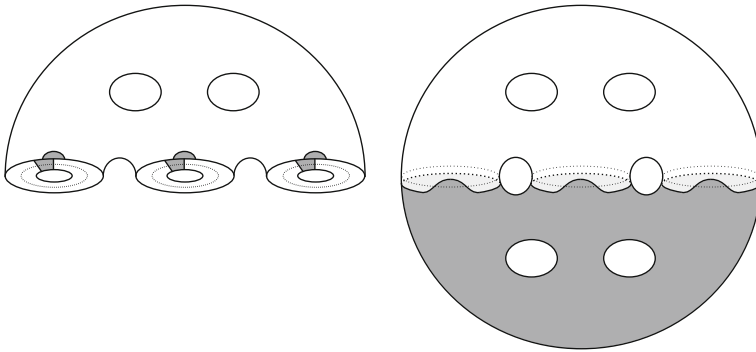
Fig. 1 Left, $(S \times [-1, 1], \xi_S)$, with negative region shaded, for a genus 2 surface with 3 boundary components. *Right*, the convex boundary of the sutured contact handlebody $H(S)$ obtained by rounding corners

boundary and the dividing set on $S \times \{1\}$ consists of k boundary parallel arcs, one for each component of ∂S, oriented in the same direction as the boundary, as shown in Fig. 1. Let $H(S)$ be the *product sutured contact manifold* obtained from $(S \times [-1, 1], \xi_S)$ by rounding corners.

Remark 2.20 Note that $H(S)$ is precisely the sort of contact handlebody that appears in the Heegaard splitting associated to an open book with page S.

Let γ_i be the curve on $\partial H(S)$ corresponding to

$$(c_i \times \{1\}) \cup (\partial c_i \times [-1, 1]) \cup (h(c_i) \times \{-1\}) \subset \partial(S \times [-1, 1]). \tag{2}$$

Let $M(S, P, h, \mathbf{c})$ be the sutured contact manifold obtained from $H(S)$ by attaching contact 2-handles along the curves in

$$\boldsymbol{\gamma}(h, \mathbf{c}) := \{\gamma_1, \ldots, \gamma_n\}. \tag{3}$$

We will use $H(S)$ and $M(S, P, h, \mathbf{c})$ to refer both to these sutured contact manifolds and to the sutured manifolds underlying them.

Definition 2.21 A *partial open book decomposition* of (M, Γ, ξ) is a partial open book $(S, P, h, \mathbf{c})$ together with a contactomorphism

$$f : M(S, P, h, \mathbf{c}) \to (M, \Gamma, \xi).$$

The "existence" part of the relative Giroux correspondence between partial open books and sutured contact manifolds, proven by Honda et al. in [11], says the following.

Theorem 2.22 *Every sutured contact manifold admits a partial open book decomposition.*

Below, we describe how different partial open book decompositions of (M, Γ, ξ) are related. Suppose $(S, P, h, \mathbf{c})$ and $(S', P', h', \mathbf{c}')$ are partial open books. Note that a diffeomorphism

$$g : (S, P, \mathbf{c}) \to (S', P', \mathbf{c}') \tag{4}$$

which intertwines h and h' gives rise to a canonical isotopy class of contactomorphisms

$$\tilde{g} : H(S) \to H(S') \tag{5}$$

and therefore to a canonical isotopy class of contactomorphisms

$$\tilde{\tilde{g}} : M(S, P, h, \mathbf{c}) \to M(S', P', h', \mathbf{c}'). \tag{6}$$

Definition 2.23 We say that $(S, P, h, \mathbf{c}, f)$ and $(S', P', h', \mathbf{c}', f')$ are *isomorphic* partial open book decompositions if there exists a diffeomorphism g as in (4) such that $f = f' \circ \tilde{\tilde{g}}$.

Definition 2.24 A *positive stabilization* of the partial open book $(S, P, h, \mathbf{c})$ is a partial open book $(S', P', h', \mathbf{c}')$ such that:

(1) S' is obtained by attaching a 1-handle H_0 to S,
(2) $P' = P \cup H_0$,
(3) $h' = D_\beta \circ h$, where β is a curve on S' meeting a cocore c_0 of H_0 exactly once, and D_β denotes a positive Dehn twist along β,
(4) $\mathbf{c}' = \mathbf{c} \cup \{c_0\}$.

Suppose $(S', P', h', \mathbf{c}')$ is a positive stabilization of $(S, P, h, \mathbf{c})$ as in the definition above. Let $M(S', P', h', c_0)$ be the sutured contact manifold obtained from $H(S')$ by attaching a contact 2-handle along the curve $\gamma_0' \subset \partial H(S')$ obtained from c_0 as in (2). Note that $M(S', P', h', c_0)$ is obtained from $H(S)$ by attaching a Darboux ball in the form of canceling contact 1- and 2-handles. In particular, there is a canonical isotopy class of contactomorphisms

$$q : M(S', P', h', c_0) \to H(S) \tag{7}$$

which restricts to the identity away from this Darboux ball and sends the curves $\gamma_1', \dots, \gamma_n' \subset \partial M(S', P', h', c_0)$ to $\gamma_1, \dots, \gamma_n \subset \partial H(S)$. Such a map gives rise to a canonical isotopy class of contactomorphisms

$$\bar{q} : M(S', P', h', \mathbf{c}') \to M(S, P, h, \mathbf{c}). \tag{8}$$

Definition 2.25 A *positive stabilization* of the partial open book decomposition $(S, P, h, \mathbf{c}, f)$ is a partial open book decomposition $(S', P', h', \mathbf{c}', f' = f \circ \bar{q})$, where $(S', P', h', \mathbf{c}')$ is a positive stabilization of $(S, P, h, \mathbf{c})$ and $\bar{q}$ is the contactomorphism in (8).

The "uniqueness" part of the relative Giroux correspondence says the following.

Theorem 2.26 *Given two partial open book decompositions of the same sutured contact manifold, it is possible to positively stabilize each some number of times so that the resulting partial open book decompositions are isomorphic.*

Remark 2.27 As stated, Theorem 2.26 is a combination of the results in [4,11]. Namely, Etgü and Özbağcı's work in [4] implies that a partial open book decomposition of (M, Γ, ξ) as defined above determines a contact cell decomposition of (M, Γ, ξ). In [11], Honda, Kazez, and Matić prove that two contact cell decompositions of (M, Γ, ξ) admit a common subdivision, and subdividing in their sense corresponds to positive stabilization as defined above.

3 Contact handle attachment maps

In this section, we define the contact handle attachment maps in sutured instanton homology mentioned in the introduction. Our construction of these maps is nearly identical to that of the corresponding maps in sutured monopole homology [1], except that we make comparatively little reference to contact geometry here.

3.1 0-handles

Attaching a contact 0-handle to (M, Γ) is equivalent to taking the disjoint union of (M, Γ) with the Darboux ball (B^3, S^1, ξ_{std}). Let (M_0, Γ_0) be this disjoint union. It is not hard to construct a marked odd closure of (M_0, Γ_0) which is also a marked odd closure of (M, Γ). We may therefore define the 0-handle attachment map to be the "identity" map.

Indeed, suppose M_0' is a preclosure of (M_0, Γ_0) formed from an auxiliary surface F_0. Then, there are natural identifications

$$\partial_\pm M_0' = R_\pm(\Gamma) \cup F_0 \cup R_\pm(S^1).$$

Let R be a copy of $\partial_+ M_0'$. Let Y_0 be the closed 3-manifold obtained by gluing $R \times [-1, 1]$ to M_0' by the "identity" map from $R \times \{-1\}$ to $\partial_+ M_0'$ and by a map from $R \times \{+1\}$ to $\partial_- M_0'$ which sends a point

$$p \in F_0 \subset R \quad \text{to} \quad p \in F_0 \subset \partial_- M_0'.$$

Let $\eta \subset R$ be an oriented, nonseparating curve contained in $F_0 \subset R$, and let $\alpha \subset Y_0$ be the union of the oriented arcs

$$\{p\} \times [-1, 1] \subset F_0 \times [-1, 1] \subset M_0' \quad \text{and}$$
$$\{p\} \times [-1, 1] \subset F_0 \times [-1, 1] \subset R \times [-1, 1].$$
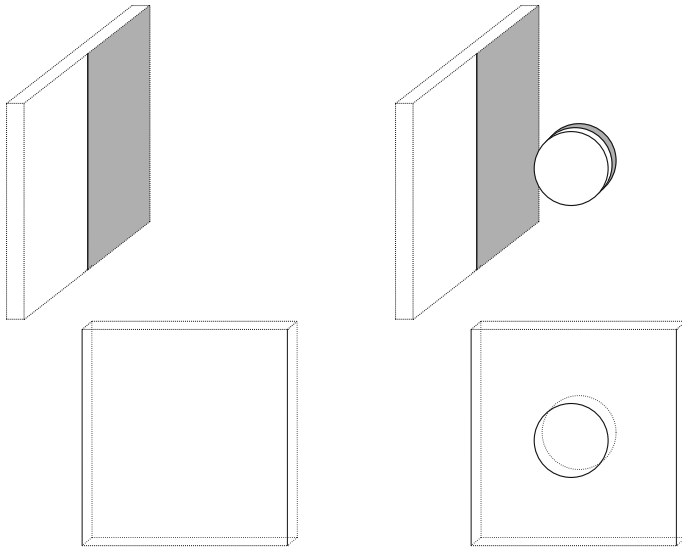
Then,

$$\mathscr{D}_0 = (Y_0, R, r, m_0, \eta, \alpha)$$

Fig. 2 *Left*, a neighborhood of part of ∂M (*top*), and the thickened surface $F \times [-1, 1]$ which is glued to it along Γ to produce a preclosure M'_0 of (M, Γ) (*bottom*). *Right*, the same neighborhood together with a 0-handle, viewed as part of (M_0, Γ_0) (*top*), and the thickened surface $F_0 \times [-1, 1]$ which is glued to it along Γ_0 to realize M'_0 as a preclosure of (M_0, Γ_0) (*bottom*)

is a marked odd closure of (M_0, Γ_0), where r and m_0 are the obvious embeddings of $R \times [-1, 1]$ and M_0 into Y_0.

Note that M'_0 is also a preclosure of (M, Γ) in a natural way, formed using the auxiliary surface $F = F_0 \cup R_+(S^1)$, as illustrated in Fig. 2. It is then clear that

$$\mathscr{D} = (Y_0, R, r, m, \eta, \alpha)$$

is a marked odd closure of (M, Γ), where m is the restriction of m_0 to $M \subset M_0$. In particular, $\underline{SHI}(-\mathscr{D}) = \underline{SHI}(-\mathscr{D}_0)$. This leads to the following definition.

Definition 3.1 We define the 0-handle attachment map

$$\mathscr{H}_0 : \mathbf{\underline{SHI}}(-M, -\Gamma) \to \mathbf{\underline{SHI}}(-M_0, -\Gamma_0)$$

to be the morphism determined by the identity map

$$id_{-\mathscr{D}, -\mathscr{D}_0} : \underline{SHI}(-\mathscr{D}) \to \underline{SHI}(-\mathscr{D}_0).$$

To prove that $\mathscr{H}_0$ is independent of the choices made in its construction, we need to show that if $\mathscr{D}_0, \mathscr{D}'_0$ are marked odd closures of (M_0, Γ_0) constructed as above, and $\mathscr{D}, \mathscr{D}'$ are the corresponding marked odd closures of (M, Γ), then the diagram

$$\begin{array}{ccc}
\underline{SHI}(-\mathscr{D}) & \xrightarrow{id_{-\mathscr{D},-\mathscr{D}_0}} & \underline{SHI}(-\mathscr{D}_0) \\
\Big\downarrow{\scriptstyle\underline{\Psi}_{-\mathscr{D},-\mathscr{D}'}} & & \Big\downarrow{\scriptstyle\underline{\Psi}_{-\mathscr{D}_0,-\mathscr{D}'_0}} \\
\underline{SHI}(-\mathscr{D}') & \xrightarrow[id_{-\mathscr{D}',-\mathscr{D}'_0}]{} & \underline{SHI}(-\mathscr{D}'_0)
\end{array}$$

commutes, where $\underline{\Psi}_{-\mathscr{D},-\mathscr{D}'}$ and $\underline{\Psi}_{-\mathscr{D}_0,-\mathscr{D}'_0}$ are the canonical isomorphisms relating the systems associated to different closures. But this follows from the fact that $\underline{\Psi}_{-\mathscr{D}_0,-\mathscr{D}'_0}$ is a composition of maps associated to 2-handle and *splicing* cobordisms, and $\underline{\Psi}_{-\mathscr{D},-\mathscr{D}'}$ can be defined via the exact same composition (refer to [2] for the definition of these maps and [1, Subsection 4.2] for the same argument in the sutured monopole Floer context).

3.2 1-handles

Suppose D_- and D_+ are disjoint embedded disks in ∂M which each intersect Γ in a single properly embedded arc. To attach a contact 1-handle to (M, Γ) along these disks, we glue the contact manifold $(D^2 \times [-1, 1], \xi_{D^2})$ to (M, Γ) by diffeomorphisms

$$D^2 \times \{-1\} \to D_- \quad \text{and} \quad D^2 \times \{+1\} \to D_+,$$

which preserve and reverse orientations, respectively, and identify the dividing sets with the sutures, and then we round corners, as illustrated in Fig. 3. Let (M_1, Γ_1) be the resulting sutured manifold. As in the 0-handle case, it is not hard to construct a marked odd closure of (M_1, Γ_1) which is also a marked odd closure of (M, Γ), as shown in Fig. 4, so that we may define the contact 1-handle attachment map to be the "identity" map in this case as well.

Indeed, suppose M'_1 is a preclosure of (M_1, Γ_1) formed from an auxiliary surface F_1. Then, there are natural identifications

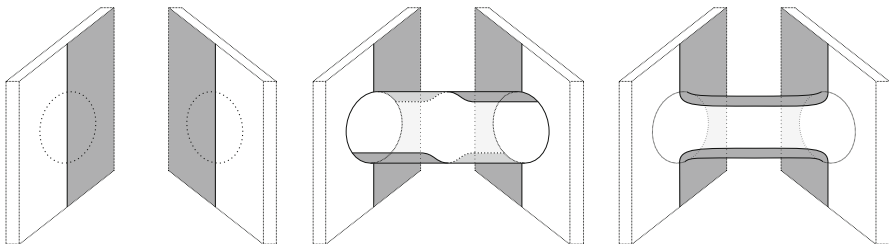$$\partial_\pm M'_1 = R_\pm(\Gamma_1) \cup F_0.$$

Fig. 3 *Left*, a collar neighborhood of a subsurface of ∂M containing the disks $D_-, D_+ \subset \partial M$, whose boundaries are *dotted*. *Middle*, attaching the contact 1-handle. *Right*, the 1-handle attachment after rounding corners
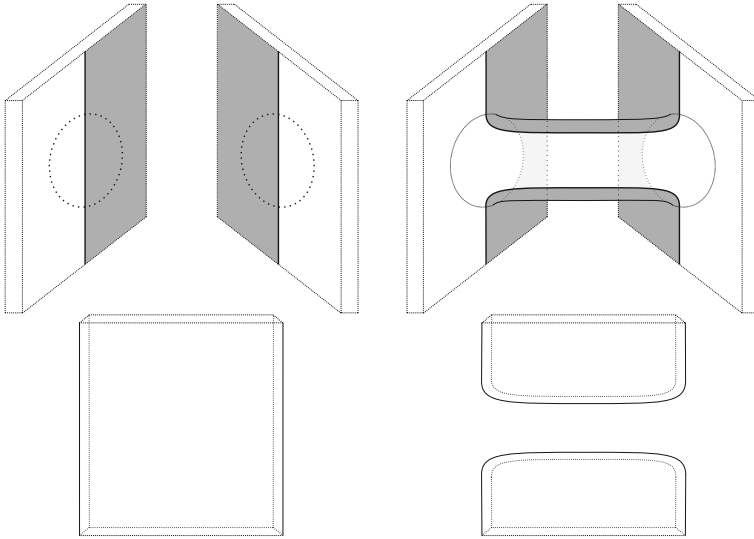
Fig. 4 *Left*, a neighborhood of a subsurface of ∂M where a 1-handle is about to be attached (*top*), and a piece of the thickened surface $F \times [-1, 1]$ which is glued along Γ to form M_1' as a preclosure of (M, Γ) (*bottom*). *Right*, the corresponding region of M_1, with 1-handle in place (*top*), and the piece of $F_1 \times [-1, 1]$ glued along Γ_1 which is used to realize M_1' as a preclosure of (M_1, Γ_1) (*bottom*)

Let R be a copy of $\partial_+ M_1'$. Let Y_1 be the closed 3-manifold obtained by gluing $R \times [-1, 1]$ to M_1' by the "identity" map from $R \times \{-1\}$ to $\partial_+ M_1'$ and by a map from $R \times \{+1\}$ to $\partial_- M_1'$ which sends a point

$$p \in F_1 \subset R \quad \text{to} \quad p \in F_1 \subset \partial_- M_1'.$$

Let $\eta \subset R$ be an oriented, nonseparating curve contained in $F_1 \subset R$, and let $\alpha \subset Y_1$ be the union of the oriented arcs

$$\{p\} \times [-1, 1] \subset F_1 \times [-1, 1] \subset M_1' \quad \text{and}$$
$$\{p\} \times [-1, 1] \subset F_1 \times [-1, 1] \subset R \times [-1, 1].$$

Then,

$$\mathscr{D}_1 = (Y_1, R, r, m_1, \eta, \alpha)$$

is a marked odd closure of (M_1, Γ_1), where r and m_1 are the obvious embeddings of $R \times [-1, 1]$ and M_1 into Y_1.

In complete analogy with the 0-handle case, we note that M_1' is also a preclosure of (M, Γ), the point being that the union of $F_1 \times [-1, 1]$ with the contact 1-handle is a product $F \times [-1, 1]$, where F is an auxiliary surface for (M, Γ). It follows that

$$\mathscr{D} = (Y_1, R, r, m, \eta, \alpha)$$

is a marked odd closure of (M, Γ), where m is the restriction of m_1 to $M \subset M_1$. In particular, $\underline{SHI}(-\mathscr{D}) = \underline{SHI}(-\mathscr{D}_1)$. This leads to the following definition.

Definition 3.2 We define the 1-handle attachment map

$$\mathscr{H}_1 : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M_1, -\Gamma_1)$$

to be the morphism determined by the identity map

$$id_{-\mathscr{D}, -\mathscr{D}_1} : \underline{SHI}(-\mathscr{D}) \to \underline{SHI}(-\mathscr{D}_1).$$

The same reasoning as in the 0-handle case shows that the map $\mathscr{H}_1$ is independent of the choices made in its construction.

3.3 2-handles

In this subsection, we define the map associated to contact 2-handle attachment. Along the way, we define a map associated to surgery on a framed knot in a sutured manifold.

Suppose γ is an embedded curve in ∂M which intersects Γ in two points. Let $A(\gamma)$ be an annular neighborhood of γ intersecting Γ in two cocores. To attach a contact 2-handle to (M, Γ, ξ) along γ, we glue $(D^2 \times [-1, 1], \xi_{D^2})$ to (M, Γ, ξ) by an orientation-reversing diffeomorphism

$$\partial D^2 \times [-1, 1] \to A(\gamma)$$

which identifies positive regions with negative regions, and then round corners, as illustrated in Fig. 5. Let (M_2, Γ_2) be the resulting sutured manifold. We will show that there exists a marked odd closure of (M_2, Γ_2) which is obtained from a marked odd closure of (M, Γ) via integer surgery and will accordingly define the 2-handle attachment map to be the map induced by the four-dimensional 2-handle cobordism corresponding to this surgery, roughly speaking.

We construct the aforementioned closure of (M_2, Γ_2) in a slightly roundabout way. Let us first consider the sutured manifold (M_1, Γ_1) obtained from (M_2, Γ_2) by
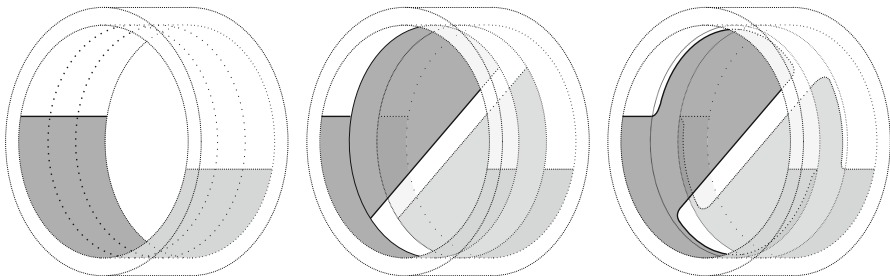

Fig. 5 *Left*, a collar neighborhood N of a subsurface of ∂M containing $A(\gamma) \subset \partial M$, whose boundary is *dotted*. *Middle*, attaching the contact 2-handle. *Right*, the 2-handle attachment after rounding corners
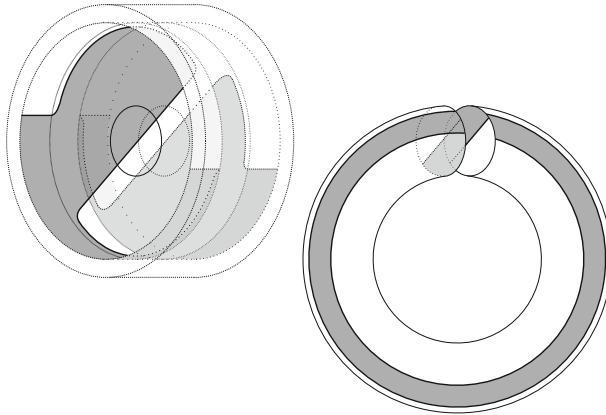
Fig. 6 Attaching a contact 1-handle to form M_1. The *circles* on $D^2 \times \{\pm 1\}$ indicate where the feet of this handle are to be attached. The union of the 1-handle below with the portion of M_1 shown above is the *solid torus N_1*

attaching a contact 1-handle along disks in the interiors of the $D^2 \times \{\pm 1\}$ boundary components of the contact 2-handle, as indicated in Fig. 6. Let

$$\mathscr{H}_1 : \underline{\textbf{SHM}}(-M_2, -\Gamma_2) \to \underline{\textbf{SHM}}(-M_1, -\Gamma_1)$$

be the corresponding 1-handle attachment map, as defined in Sect. 3.2. It is not hard to see that (M_1, Γ_1) is diffeomorphic to the sutured manifold obtained from (M, Γ) by performing ∂M-framed surgery on a parallel copy γ' of γ in the interior of M.

To be precise, let us suppose that γ' is contained in the solid torus neighborhood $N \subset M$ shown in Fig. 5. Let $N_1 \subset M_1$ be the *solid torus* obtained from N by attaching the 1- and 2-handles as indicated in Figs. 5 and 6. Note that

$$(M \smallsetminus N, \Gamma|_{M \smallsetminus N}) = (M_1 \smallsetminus N_1, \Gamma_1|_{M_1 \smallsetminus N_1}). \tag{9}$$

Furthermore, the restriction of the identity map on these complements to $\partial M \smallsetminus N = \partial M_1 \smallsetminus N_1$ extends uniquely, up to isotopy, to a diffeomorphism of pairs

$$(\partial M, \Gamma) \to (\partial M_1, \Gamma_1).$$

The identity map on the complement in (9) therefore extends naturally to a diffeomorphism

$$(M \smallsetminus N', \Gamma) \to (M_1 \smallsetminus N_1', \Gamma_1),$$

where $N' \subset \text{int}(N)$ and $N_1' \subset \text{int}(N_1)$ are slightly smaller solid tori. This provides a canonical, up to isotopy, diffeomorphism

$$f : (M', \Gamma') \to (M_1, \Gamma_1), \tag{10}$$

where (M', Γ') is the sutured manifold obtained from (M, Γ) via ∂M-framed surgery on γ'.

In order to define the contact 2-handle map, we first define a morphism associated to this surgery. In fact, we take this opportunity to define a map associated to surgery on any framed knot in the interior of a sutured manifold.

Suppose $\mathscr{D} = (Y, R, r, m, \eta, \alpha)$ is a marked odd closure of (M, Γ). Suppose K is a framed knot in the interior of M, and let (M', Γ') be the sutured manifold obtained via surgery on K with respect to this framing. Let Y' be the 3-manifold obtained from Y by performing surgery on $m(K)$ with respect to the induced framing. Then, $\mathscr{D}' = (Y', R, r', m', \eta, \alpha)$ is a marked odd closure of (M', Γ'), where r' is the map induced by r and m' is the embedding of M' into Y' induced by m. Let W be the 2-handle cobordism from Y to Y' obtained from $Y \times [0, 1]$ by attaching the 2-handle corresponding to the above surgery, and let $\nu \subset W$ be the obvious cylindrical cobordism from

$$(r(\eta \times \{0\}) \sqcup \alpha) \subset Y \quad \text{to} \quad (r'(\eta \times \{0\}) \sqcup \alpha) \subset Y'.$$

We define

$$F_K : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M', -\Gamma')$$

to be the morphism induced by the map

$$I_*(-W|-R)_{-\nu} : \underline{SHI}(-\mathscr{D}) \to \underline{SHI}(-\mathscr{D}').$$

To prove that F_K is well defined, we must show that the diagram

$$
\begin{array}{ccc}
\underline{SHI}(-\mathscr{D}_1) & \xrightarrow{I_*(-W_1|-R_1)_{-\nu_1}} & \underline{SHI}(-\mathscr{D}_1') \\
\underline{\Psi}_{-\mathscr{D}_1,-\mathscr{D}_2} \downarrow & & \downarrow \underline{\Psi}_{-\mathscr{D}_1',-\mathscr{D}_2'} \\
\underline{SHI}(-\mathscr{D}_2) & \xrightarrow{I_*(-W_2|-R_2)_{-\nu_2}} & \underline{SHI}(-\mathscr{D}_2')
\end{array}
$$

commutes, for any two marked odd closures $\mathscr{D}_1$, $\mathscr{D}_2$ of (M, Γ), where $\mathscr{D}_1'$, $\mathscr{D}_2'$ are the induced marked odd closures of (M', Γ'). As explained in [1, Subsection 4.2] in the context of sutured monopole homology, this diagram commutes because the cobordisms used to define these maps commute: W_1 and W_2 are built by attaching 2-handles along curves in the regions $m_1(M)$ and $m_2(M)$, while the vertical isomorphisms are defined from cobordisms built by attaching 2-handles or splicing along tori outside of these regions.

Let us now return to the situation at hand, where (M_2, Γ_2) is obtained from (M, Γ) by attaching a contact 2-handle along γ, and f is the diffeomorphism in (10).

Definition 3.3 We define the 2-handle attachment map

$$\mathscr{H}_2 : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M_2, -\Gamma_2)$$

to be the composition $\mathscr{H}_2 = \mathscr{H}_1^{-1} \circ \mathbf{SHI}(f) \circ F_{\gamma'}$.

That $\mathscr{H}_2$ is independent of γ' follows from the fact that any two such parallel copies of γ are related by an ambient isotopy of M supported in N.

Remark 3.4 Unpacking the composition above, we see that $\mathscr{H}_2$ may also be formulated as follows. Suppose $\mathscr{D} = (Y, R, r, m, \eta, \alpha)$ is a marked odd closure of (M, Γ), and let $\mathscr{D}'$ be the induced marked odd closure of the surgered manifold (M', Γ') as above. Then,

$$\mathscr{D}_2 = (Y', R, r', m_2, \eta, \alpha)$$

is a marked odd closure of (M_2, Γ_2), where m_2 is the restriction of $m' \circ f^{-1}$ to $M_2 \subset M_1$. Let

$$id_{-\mathscr{D}', -\mathscr{D}_2} : \underline{SHI}(-\mathscr{D}') \to \underline{SHI}(-\mathscr{D}_2)$$

be the identity map on $\underline{SHI}(-\mathscr{D}') = \underline{SHI}(-\mathscr{D}_2)$. Then, $\mathscr{H}_2$ is the morphism induced by the map

$$id_{-\mathscr{D}', -\mathscr{D}_2} \circ I_*(-W|-R)_{-\nu} : \underline{SHI}(-\mathscr{D}) \to \underline{SHI}(-\mathscr{D}_2).$$

In other words, the 2-handle map is really just the map of systems induced by the cobordism map corresponding to surgery along the curve of attachment.

3.4 3-handles

Attaching a contact 3-handle to (M, Γ) amounts to gluing the Darboux ball (B^3, S^1, ξ_{std}) to (M, Γ) along an S^2 boundary component of M with one suture, identifying positive regions with negative regions and vice versa. Let (M_3, Γ_3) be the result of this gluing. We will assume that ∂M is disconnected, so that M_3 has boundary. Let p be a point in M_3 in the interior of the Darboux ball we glued in. Then, there is a canonical isotopy class of diffeomorphisms

$$f : (M, \Gamma) \to (M', \Gamma'),$$

where (M', Γ') is the sutured manifold obtained by taking the connected sum of (M_3, Γ_3) with (B^3, S^1) at the point p. Let (M_0, Γ_0) be the disjoint union of (M_3, Γ_3) with (B^3, S^1), and let

$$\mathscr{H}_0 : \mathbf{SHI}(-M_3, -\Gamma_3) \to \mathbf{SHI}(-M_0, -\Gamma_0)$$

be the corresponding 0-handle attachment map, as defined in Sect. 3.1. Suppose

$$\mathscr{D}_0 = (Y_0, R, r, m, \eta, \alpha)$$

is a marked odd closure of (M_0, Γ_0). Then,

$$\mathcal{D}' = (Y', R, r, m', \eta, \alpha)$$

is a marked odd closure of (M', Γ'), where Y' is the self connected sum obtained from Y_0 by removing Darboux balls around $m(p)$ and some point in $m(B^3) \subset Y_0$ and gluing in $S^2 \times I$, and m' is the embedding of M' into Y' induced by m. In particular, Y' is a connected sum of Y_0 with $S^1 \times S^2$. Let W be the natural 1-handle cobordism from Y_0 to Y', and let $\nu \subset W$ be the natural cylindrical cobordism from

$$(r(\eta \times \{0\}) \sqcup \alpha) \subset Y_0 \quad \text{to} \quad (r(\eta \times \{0\}) \sqcup \alpha) \subset Y'.$$

Let

$$F_\# : \underline{\mathbf{SHI}}(-M', -\Gamma') \to \underline{\mathbf{SHI}}(-M_0, -\Gamma_0)$$

be the morphism determined by the map

$$I_*(W|-R)_\nu : \underline{SHI}(-\mathcal{D}') \to \underline{SHI}(-\mathcal{D}_0).$$

Definition 3.5 We define the 3-handle attachment map

$$\mathscr{H}_3 : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M_3, -\Gamma_3)$$

to be the composition $\mathscr{H}_3 = \mathscr{H}_0^{-1} \circ F_\# \circ \underline{\mathbf{SHI}}(f)$.

To show that this map is well defined, we only need to argue that $F_\#$ is well defined. But this follows from same sort of reasoning as was used to argue that F_K is well defined: Namely, the 1-handle cobordism used to define $F_\#$ is formed via 1-handle attachment along balls in the interiors of Y' and Y_0 and therefore commutes with the 2-handle and splicing cobordisms used to define the canonical isomorphisms in the systems $\underline{\mathbf{SHI}}(-M', -\Gamma')$ and $\underline{\mathbf{SHI}}(-M_0, -\Gamma_0)$.

3.5 A further property

Below, we prove a lemma which will be useful for defining the contact invariant in Sect. 4. Suppose

$$f : (M, \Gamma) \to (M', \Gamma')$$

is a diffeomorphism and (M_i, Γ_i) is obtained from (M, Γ) by attaching a contact i-handle along an attaching region $S \subset \partial M$. Note that f extends uniquely, up to isotopy, to a sutured diffeomorphism

$$\bar{f} : (M_i, \Gamma_i) \to (M'_i, \Gamma'_i),$$

where (M'_i, Γ'_i) is obtained from (M', Γ') by attaching a contact i-handle along the attaching region $f(S) \subset \partial M'$. Then, we have the following.

Lemma 3.6 *The diagram*

$$
\begin{array}{ccc}
\underline{\textbf{SHI}}(-M, -\Gamma) & \xrightarrow{\mathscr{H}_i} & \underline{\textbf{SHI}}(-M_i, -\Gamma_i) \\
\downarrow{\scriptstyle \underline{\textbf{SHI}}(f)} & & \downarrow{\scriptstyle \underline{\textbf{SHI}}(\bar{f})} \\
\underline{\textbf{SHI}}(-M', -\Gamma') & \xrightarrow[\mathscr{H}'_i]{} & \underline{\textbf{SHI}}(-M'_i, -\Gamma'_i)
\end{array}
$$

commutes, where $\mathscr{H}_i$ and $\mathscr{H}'_i$ are the appropriate contact i-handle attachment maps.

Proof The composition $\underline{\textbf{SHI}}(\bar{f}) \circ \mathscr{H}_i$ is ultimately defined in terms of the map on instanton Floer homology induced by a natural cobordism (the identity cobordism, a 1-handle cobordism, or a 2-handle cobordism) from a closure of $(-M, -\Gamma)$ to a closure of $(-M'_i, -\Gamma'_i)$. Unraveling definitions, it is clear that the composition $\mathscr{H}'_i \circ \underline{\textbf{SHI}}(f)$ is determined by the same cobordism map. □

4 A contact invariant in sutured instanton homology

In this section, we use the relative Giroux correspondence to define the contact invariant

$$
\theta(M, \Gamma, \xi) \in \underline{\textbf{SHI}}(-M, -\Gamma)
$$

outlined in the introduction. We then establish some basic properties of this invariant, such as the fact that it vanishes for overtwisted contact structures and is nonzero for the complement of a Darboux ball in a Stein fillable contact manifold.

4.1 The contact invariant

Suppose (M, Γ, ξ) is a sutured contact manifold with partial open book decomposition $(S, P, h, \textbf{c}, f)$. Recall that $M(S, P, h, \textbf{c})$ is obtained from $H(S)$ by attaching contact 2-handles along the curves in the set $\boldsymbol{\gamma}(h, \textbf{c})$ defined in (3). Let

$$
\mathscr{H} : \underline{\textbf{SHI}}(-H(S)) \to \underline{\textbf{SHI}}(-M(S, P, h, \textbf{c}))
$$

be corresponding composition of contact 2-handle attachment morphisms.

Definition 4.1 We define

$$
\theta(S, P, h, \textbf{c}, f) := \underline{\textbf{SHI}}(f)(\mathscr{H}(\textbf{1})) \in \underline{\textbf{SHI}}(-M, -\Gamma),
$$

where $\textbf{1}$ is the generator of $\underline{\textbf{SHI}}(-H(S)) \cong \mathbb{C}$.

Definition 4.2 We define

$$\theta(M, \Gamma, \xi) := \theta(S, P, h, \mathbf{c}, f) \in \underline{\mathbf{SHI}}(-M, -\Gamma)$$

for any partial open book decomposition $(S, P, h, \mathbf{c}, f)$ of (M, Γ, ξ).

That the element $\theta(M, \Gamma, \xi)$ is well defined is the content of the following theorem.

Theorem 4.3 *The element $\theta(S, P, h, \mathbf{c}, f)$ is independent of the partial open book decomposition $(S, P, h, \mathbf{c}, f)$ of (M, Γ, ξ).*

The rest of this subsection is devoted to the proof of Theorem 4.3. As a first step, we have the following lemma.

Lemma 4.4 *If $(S, P, h, \mathbf{c}, f)$ and $(S', P', h', \mathbf{c}', f')$ are isomorphic partial open book decompositions, then $\theta(S, P, h, \mathbf{c}, f) = \theta(S', P', h', \mathbf{c}', f')$.*

Proof We must show that

$$\underline{\mathbf{SHI}}(f)(\mathscr{H}(\mathbf{1})) = \underline{\mathbf{SHI}}(f')(\mathscr{H}'(\mathbf{1}')), \tag{11}$$

where $\mathscr{H}$ and $\mathscr{H}'$ are the compositions of contact 2-handle maps used to define $\theta(S, P, h, \mathbf{c}, f)$ and $\theta(S', P', h', \mathbf{c}', f')$ and $\mathbf{1}$ and $\mathbf{1}'$ are the generators of $\underline{\mathbf{SHI}}(-H(S))$ and $\underline{\mathbf{SHI}}(-H(S'))$.

Since these open book decompositions are isomorphic, there exist maps $\tilde{g}$ and $\bar{\tilde{g}}$ as in (5) and (6) such that $f = f' \circ \bar{\tilde{g}}$. Note that we have a commutative diagram

$$
\begin{array}{ccccc}
\underline{\mathbf{SHI}}(-H(S)) & \xrightarrow{\mathscr{H}} & \underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c})) & \xrightarrow{\underline{\mathbf{SHI}}(f)} & \underline{\mathbf{SHI}}(-M, -\Gamma) \\
\downarrow{\scriptstyle \underline{\mathbf{SHI}}(\tilde{g})} & & \downarrow{\scriptstyle \underline{\mathbf{SHI}}(\bar{\tilde{g}})} & & \downarrow{\scriptstyle id} \\
\underline{\mathbf{SHI}}(-H(S')) & \xrightarrow{\mathscr{H}'} & \underline{\mathbf{SHI}}(-M(S', P', h', \mathbf{c}')) & \xrightarrow{\underline{\mathbf{SHI}}(f')} & \underline{\mathbf{SHI}}(-M, -\Gamma).
\end{array}
$$

The leftmost square commutes by Lemma 3.6 and the rightmost square commutes since

$$\underline{\mathbf{SHI}}(f) = \underline{\mathbf{SHI}}(f' \circ \bar{\tilde{g}}) = \underline{\mathbf{SHI}}(f') \circ \underline{\mathbf{SHI}}(\bar{\tilde{g}}).$$

The equality in (11) then follows as long as $\underline{\mathbf{SHI}}(\tilde{g})$ sends $\mathbf{1}$ to $\mathbf{1}'$, but it does since this map is an isomorphism and $\mathbf{1}$ and $\mathbf{1}'$ are the generators. □

Since isomorphic partial open book decompositions give rise to the same contact element, by Lemma 4.4, it suffices, for the proof of Theorem 4.3, to establish the following.

Proposition 4.5 *If the partial open book decomposition $(S', P', h', \mathbf{c}', f')$ is a positive stabilization of $(S, P, h, \mathbf{c}, f)$, then $\theta(S', P', h', \mathbf{c}', f') = \theta(S, P, h, \mathbf{c}, f)$.*

Proof Suppose the partial open book decomposition $(S', P', h', \mathbf{c}', f')$ of (M, Γ, ξ) is a positive stabilization of $(S, P, h, \mathbf{c}, f)$. Recall from Definition 2.24 that $\mathbf{c}$ denotes a collection of basis arcs $\{c_1, \ldots, c_n\} \subset P$ and that $\mathbf{c}' = \mathbf{c} \cup \{c_0\}$, where the new basis arc $c_0 \subset P'$ is the cocore of the handle attached to P to produce P'. Let

$$\mathscr{H} : \underline{\mathbf{SHI}}(-H(S)) \to \underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c}))$$
$$\mathscr{H}' : \underline{\mathbf{SHI}}(-H(S')) \to \underline{\mathbf{SHI}}(-M(S', P', h', \mathbf{c}'))$$

be the compositions of contact 2-handle maps used to define the elements $\theta(S, P, h, \mathbf{c}, f)$ and $\theta(S', P', h', \mathbf{c}', f')$. To prove Proposition 4.5, we must show that

$$\underline{\mathbf{SHI}}(f)(\mathscr{H}(\mathbf{1})) = \underline{\mathbf{SHI}}(f')(\mathscr{H}'(\mathbf{1}')), \tag{12}$$

where $\mathbf{1}$ and $\mathbf{1}'$ are the generators of $\underline{\mathbf{SHI}}(-H(S))$ and $\underline{\mathbf{SHI}}(-H(S'))$. Let

$$\mathscr{H}^{c_0} : \underline{\mathbf{SHI}}(-H(S')) \to \underline{\mathbf{SHI}}(-M(S', P', h', c_0))$$

be the morphism associated to the 2-handle attachment along the curve

$$\gamma_0 = (c_0 \times \{1\}) \cup (\partial c_0 \times [-1, 1]) \cup (h'(c_0) \times \{-1\})$$

in $\partial H(S') = \partial(S' \times [-1, 1])$, which is defined as in (2), and let

$$\mathscr{H}^{c>0} : \underline{\mathbf{SHI}}(-M(S', P', h', c_0)) \to \underline{\mathbf{SHI}}(-M(S', P', h', \mathbf{c}'))$$

be the morphism associated to the composition of 2-handle attachments along the analogously defined curves $\gamma_1', \ldots, \gamma_n'$, so that

$$\mathscr{H}' = \mathscr{H}^{c>0} \circ \mathscr{H}^{c_0}.$$

Finally, let q and $\bar{q}$ be the contactomorphisms in (7) and (8), so that $f' = f \circ \bar{q}$. Then, we have

$$\underline{\mathbf{SHI}}(f') \circ \mathscr{H}' = \underline{\mathbf{SHI}}(f) \circ \underline{\mathbf{SHI}}(\bar{q}) \circ \mathscr{H}^{c>0} \circ \mathscr{H} = \underline{\mathbf{SHI}}(f) \circ \mathscr{H} \circ \underline{\mathbf{SHI}}(q) \circ \mathscr{H}^{c_0},$$

where the second equality is an application of Lemma 3.6. Thus, for (12), it suffices to show that

$$\underline{\mathbf{SHI}}(q)(\mathscr{H}^{c_0}(\mathbf{1}')) = \mathbf{1},$$

which is equivalent to proving that $\mathscr{H}^{c_0}$ is nonzero.

By definition, the curve $\gamma_0 \subset \partial H(S')$ is obtained from the curve

$$(c_0 \times \{1\}) \cup (\partial c_0 \times [-1, 1]) \cup (D_\beta(c_0) \times \{-1\}) \subset \partial(S' \times [-1, 1]),$$

shown in Fig. 7, by rounding corners. Suppose $\mathscr{D} = (Y, R, r, m, \eta, \alpha)$ is a marked odd closure of $H(S')$, let γ_0' be a parallel copy of γ_0 in the interior of Y, and let Y'
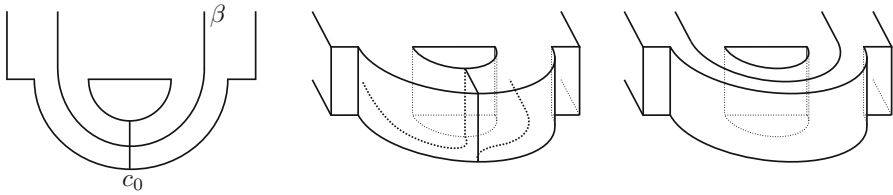
Fig. 7 *Left*, the surface S' with the cocore c_0 of the 1-handle H_0 and the curve β. *Middle*, the curve γ_0 in $H(S')$ drawn as $(c_0 \times \{1\}) \cup (\partial c_0 \times [-1, 1]) \cup (D_\beta(c_0) \times \{-1\})$ in $S' \times [-1, 1]$. *Right*, the curve β' drawn as $\beta \times \{1\} \subset S' \times \{1\}$

be the result of 0-surgery on $m(\gamma'_0)$ with respect to the framing induced by $\partial H(S')$. By the construction of the contact 2-handle map in the previous section, we know that there is an embedding

$$m' : M(S', P', h', c_0) \to Y'$$

such that $\mathscr{D}' = (Y', R, r, m', \eta, \alpha)$ is a marked odd closure of $M(S', P', h', c_0)$. Let W be the 2-handle cobordism from Y to Y' obtained from $Y \times [0, 1]$ by attaching a 2-handle corresponding to this surgery. Then, $\mathscr{H}^{c_0}$ is the morphism determined by the induced map

$$I_*(-W|-R)_{-\nu} : \underline{SHI}(-\mathscr{D}) \to \underline{SHI}(-\mathscr{D}'), \tag{13}$$

where $\nu \subset W$ is the obvious cylindrical cobordism from

$$(r(\eta \times \{0\}) \sqcup \alpha) \subset Y \quad \text{to} \quad (r'(\eta \times \{0\}) \sqcup \alpha) \subset Y'.$$

Note that γ_0 is isotopic to the curve $\beta' \subset \partial H(S')$ corresponding to $\beta \times \{1\} \subset S' \times \{1\}$, by an isotopy which sends the $\partial H(S')$-framing on γ_0 to the $(\partial H(S') + 1)$-framing on β'. Since β' is contained in the positive region of $\partial H(S')$, the image $m(\beta')$ is isotopic to $r(b \times \{t\})$ for some embedded curve $b \subset R$ and any $t \in [0, 1]$, by an isotopy which sends the $\partial H(S')$-framing on $m(\beta')$ to the $r(R \times \{t\})$-framing on $r(b \times \{t\})$. We may therefore think of W as the cobordism associated to $(+1)$-surgery on $r(b \times \{t\})$. But this is exactly the sort of cobordism used to define the canonical isomorphisms relating the sutured instanton homologies associated to different closures of a sutured manifold, as described in [2, Section 9]. In particular, the map in (13) is an isomorphism, proving that $\mathscr{H}^{c_0}$ is nonzero. $\qquad\square$

This completes the proof of Theorem 4.3.

4.2 Properties

Below, we establish some properties of the invariant $\theta(M, \Gamma, \xi)$. The first result below says that the invariant θ behaves functorially with respect to contact $(+1)$-surgery.

Proposition 4.6 *Suppose K is a Legendrian knot in the interior of (M, Γ, ξ) and that (M', Γ', ξ') is the result of contact $(+1)$-surgery on K. Then, the map*

$$F_K : \underline{SHI}(-M, -\Gamma) \to \underline{SHI}(-M', -\Gamma')$$

corresponding to this surgery, as defined in Sect. 3.3, sends $\theta(M, \Gamma, \xi)$ to $\theta(M', \Gamma', \xi')$.

Proof Let $(S, P, h, \mathbf{c}, f)$ be a partial open book decomposition of (M, Γ, ξ) such that $K = f(\Lambda)$, where Λ is a pushoff of

$$\lambda \times \{-1\} \subset S \times [-1, 1] \subset M(S, P, h, \mathbf{c})$$

into the interior of $M(S, P, h, \mathbf{c})$, where $\lambda \subset P$ is a curve which intersects $c_1 \in \mathbf{c}$ in a single point, is disjoint from all other c_i, and is not homotopic to ∂P. We further require that the contact framing on K agrees with the contact framing on Λ (which is induced by the S-framing on λ). One can construct an $(S, P, h, \mathbf{c}, f)$ with these properties by including K in the Legendrian graph used to define the partial open book, as described in [11].

Let $H'(S)$ and $M'(S, P, h, \mathbf{c})$ be the contact manifolds obtained from $H(S)$ and $M(S, P, h, \mathbf{c})$, respectively, by performing contact $(+1)$-surgery on Λ. The contactomorphism f naturally induces a contactomorphism

$$\bar{f} : M'(S, P, h, \mathbf{c}) \to (M', \Gamma', \xi')$$

such that the diagram

$$
\begin{array}{ccc}
\underline{SHI}(-M(S, P, h, \mathbf{c})) & \xrightarrow{\ \ \underline{SHI}(f)\ \ } & \underline{SHM}(-M, -\Gamma) \\
\Big\downarrow{\scriptstyle F_\Lambda} & & \Big\downarrow{\scriptstyle F_K} \\
\underline{SHI}(-M'(S, P, h, \mathbf{c})) & \xrightarrow[\ \underline{SHI}(\bar{f})\ ]{} & \underline{SHM}(-M', -\Gamma')
\end{array}
$$

commutes, by the same sort of argument as was used in the proof of Lemma 3.6. Note that there is a canonical isotopy class of contactomorphism

$$g : H'(S) \to H(S)$$

which sends the attaching set

$$\boldsymbol{\gamma}(h, \mathbf{c}) \subset \partial H'(S) \quad \text{to} \quad \boldsymbol{\gamma}(h \circ D_\lambda^{-1}, \mathbf{c}) \subset \partial H(S).$$

This map naturally induces a contactomorphism

$$\bar{g} : M'(S, P, h, \mathbf{c}) \to M(S, P, h \circ D_\lambda^{-1}, \mathbf{c}).$$

Thus,

$$(S, P, h \circ D_\lambda^{-1}, \mathbf{c}, f_\lambda := \bar{f} \circ (\bar{g})^{-1})$$

is a partial open book decomposition for (M', Γ', ξ').

Let

$$\mathcal{H} : \underline{\mathbf{SHI}}(-H(S)) \to \underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c}))$$
$$\mathcal{H}' : \underline{\mathbf{SHI}}(-H(S')) \to \underline{\mathbf{SHI}}(-M'(S, P, h, \mathbf{c}))$$
$$\mathcal{H}_\lambda : \underline{\mathbf{SHI}}(-H(S)) \to \underline{\mathbf{SHI}}(-M(S, P, h \circ D_\lambda^{-1}, \mathbf{c}))$$

be the compositions of contact 2-handle maps associated to the attaching sets

$$\boldsymbol{\gamma}(h, \mathbf{c}) \subset \partial H(S) \quad \text{and} \quad \boldsymbol{\gamma}(h, \mathbf{c}) \subset \partial H'(S) \quad \text{and} \quad \boldsymbol{\gamma}(h \circ D_\lambda^{-1}, \mathbf{c}) \subset \partial H(S),$$

respectively. The commutativity of the diagram

$$
\begin{array}{ccc}
\underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c})) & \xrightarrow{\underline{\mathbf{SHI}}(f)} & \underline{\mathbf{SHI}}(-M, -\Gamma) \\
\downarrow{\scriptstyle \underline{\mathbf{SHI}}(\bar{g}) \circ F_\Lambda} & & \downarrow{\scriptstyle F_K} \\
\underline{\mathbf{SHI}}(-M(S, P, h \circ D_\lambda^{-1}, \mathbf{c})) & \xrightarrow{\underline{\mathbf{SHI}}(f_\lambda)} & \underline{\mathbf{SHI}}(-M', -\Gamma')
\end{array}
$$

follows immediately from that of the previous diagram combined with the fact that

$$\underline{\mathbf{SHI}}(\bar{f}) = \underline{\mathbf{SHI}}(f_\lambda) \circ \underline{\mathbf{SHI}}(\bar{g}).$$

Since

$$\underline{\mathbf{SHI}}(f)(\mathcal{H}(\mathbf{1})) = \theta(M, \Gamma, \xi),$$
$$\underline{\mathbf{SHI}}(f_\lambda)(\mathcal{H}_\lambda(\mathbf{1})) = \theta(M', \Gamma', \xi'),$$

by definition, it suffices for the proof of the proposition to show that

$$(\underline{\mathbf{SHI}}(\bar{g}) \circ F_\Lambda)(\mathcal{H}(\mathbf{1})) = \mathcal{H}_\lambda(\mathbf{1}). \tag{14}$$

For this, consider the diagram

$$
\begin{array}{ccc}
\underline{\mathbf{SHI}}(-H(S)) & \xrightarrow{\mathscr{H}} & \underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c})) \\
\Big\downarrow{\scriptstyle F_\Lambda} & & \Big\downarrow{\scriptstyle F_\Lambda} \\
\underline{\mathbf{SHI}}(-H'(S)) & \xrightarrow{\mathscr{H}'} & \underline{\mathbf{SHI}}(-M'(S, P, h, \mathbf{c})) \\
\Big\downarrow{\scriptstyle \underline{\mathbf{SHI}}(g)} & & \Big\downarrow{\scriptstyle \underline{\mathbf{SHI}}(\bar{g})} \\
\underline{\mathbf{SHI}}(-H(S)) & \xrightarrow[\mathscr{H}_\lambda]{} & \underline{\mathbf{SHI}}(-M(S, P, h \circ D_\lambda^{-1}, \mathbf{c})).
\end{array}
$$

The top square commutes since the 2-handle cobordisms between closures used to define the maps obviously commute. Moreover, the leftmost map

$$
F_\Lambda : \underline{\mathbf{SHI}}(-H(S)) \to \underline{\mathbf{SHI}}(-H'(S))
$$

is induced by the same sort of 2-handle cobordism that defines the canonical isomorphisms between different closures of the same genus, since Λ is isotopic to a curve contained in the negative region of $\partial H(S)$ (see [2, Section 9]). In particular, it is an isomorphism and therefore sends $\mathbf{1}$ to $\mathbf{1}$. It follows that the rightmost map

$$
F_\Lambda : \underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c})) \to \underline{\mathbf{SHI}}(-M'(S, P, h, \mathbf{c}))
$$

satisfies

$$
F_\Lambda(\mathscr{H}(\mathbf{1})) = \mathscr{H}'(\mathbf{1}). \tag{15}
$$

The bottom square in the diagram commutes by Lemma 3.6, and since $\underline{\mathbf{SHI}}(g)$ is an isomorphism, it sends $\mathbf{1}$ to $\mathbf{1}$. Hence,

$$
\underline{\mathbf{SHI}}(\bar{g})(\mathscr{H}'(\mathbf{1})) = \mathscr{H}_\lambda(\mathbf{1}). \tag{16}
$$

Putting (15) and (16) together, we obtain (14), completing the proof of Proposition 4.6. $\square$

Next, we show that the invariant θ behaves as one would expect with respect to contactomorphism.

Proposition 4.7 *Suppose*

$$
g : (M, \Gamma, \xi) \to (M', \Gamma', \xi')
$$

is a contactomorphism. Then, the map

$$
\underline{\mathbf{SHI}}(g) : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M', -\Gamma')
$$

sends $\theta(M, \Gamma, \xi)$ *to* $\theta(M', \Gamma', \xi')$.

Proof Suppose $(S, P, h, \mathbf{c}, f)$ is a partial open book decomposition for (M, Γ, ξ). Then, clearly $(S, P, h, \mathbf{c}, g \circ f)$ is a partial open book decomposition for (M', Γ', ξ'). Letting

$$\mathscr{H} : \underline{\mathbf{SHI}}(-H(S)) \to \underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c}))$$

be the corresponding composition of contact 2-handle maps, we have that

$$\begin{aligned}
\theta(M', \Gamma', \xi') &:= \underline{\mathbf{SHI}}(g \circ f)(\mathscr{H}(\mathbf{1})) = \underline{\mathbf{SHI}}(g)(\underline{\mathbf{SHI}}(f)(\mathscr{H}(\mathbf{1}))) \\
&= \underline{\mathbf{SHI}}(g)(\theta(M, \Gamma, \xi)),
\end{aligned}$$

as desired. $\qquad\square$

As explained in the introduction, the contact invariant θ behaves naturally with respect to the maps induced by handle attachments.

Theorem 4.8 *Suppose* (M_i, Γ_i, ξ_i) *is obtained from* (M, Γ, ξ) *by attaching a contact* i-*handle and* $\mathscr{H}_i$ *is the associated contact handle attachment map for* $i = 0, 1,$ *or* 2. *Then,*

$$\mathscr{H}_i : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M_i, -\Gamma_i)$$

sends $\theta(M, \Gamma, \xi)$ *to* $\theta(M_i, \Gamma_i, \xi_i)$.

Proof Let us first assume that $i = 0$, and let us adopt all the notation from Sect. 3.1. Suppose $(S, P, h, \mathbf{c}, f)$ is a partial open book decomposition for (M, Γ, ξ). Then, $(S', P, h, \mathbf{c}, f')$ is a partial open book decomposition of (M_0, Γ_0, ξ_0), where S' is the disjoint union of S with D^2 and f' is the disjoint union of f with a contactomorphism

$$H(D^2) \to (B^3, S^1, \xi_{std}).$$

Consider the diagram

$$
\begin{array}{ccc}
\underline{\mathbf{SHI}}(-H(S)) & \xrightarrow{\ \mathscr{H}_0''\ } & \underline{\mathbf{SHI}}(-H(S')) \\
{\scriptstyle\mathscr{H}}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathscr{H}'} \\
\underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c}, f)) & \xrightarrow{\ \mathscr{H}_0'\ } & \underline{\mathbf{SHI}}(-M(S', P, h, \mathbf{c}, f')) \\
{\scriptstyle\underline{\mathbf{SHI}}(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle\underline{\mathbf{SHI}}(f')} \\
\underline{\mathbf{SHI}}(-M, -\Gamma) & \xrightarrow[\ \mathscr{H}_0\ ]{} & \underline{\mathbf{SHI}}(-M_0, -\Gamma_0).
\end{array}
$$

where $\mathscr{H}, \mathscr{H}'$ are the compositions of contact 2-handle maps of the sort used to define θ, and $\mathscr{H}_0, \mathscr{H}_0', \mathscr{H}_0''$ are the obvious contact 0-handle maps. Since the map $\mathscr{H}_0''$ is

an isomorphism (and therefore sends **1** to **1**), we need only check that this diagram commutes. But this is straightforward from the definitions of these maps—on the level of closures, the identity and 2-handle cobordisms defining these maps commute. The map $\mathscr{H}_0$ thus preserves the contact invariant as desired.

Let us now assume that $i = 1$ and adopt all the notation from Sect. 3.2. The proof in this case is similar. We can find a partial open book decomposition $(S, P, h, \mathbf{c}, f)$ for (M, Γ, ξ) such that $(S', P, h, \mathbf{c}, f')$ is a partial open book decomposition for (M_1, Γ_1, ξ_1), where S' is the surface obtained by attaching a 1-handle to S away from P, and

$$f' : M(S', P, h, \mathbf{c}) \to (M_0, \Gamma_0, \xi_0)$$

is a contactomorphism which restricts to f on $M(S, P, h, \mathbf{c}) \subset M(S', P, h, \mathbf{c})$. (To find open book decompositions with this property, we first construct a partial open book decomposition for (M_0, Γ_0, ξ_0) from a contact cell decomposition whose Legendrian graph contains the core of the contact 1-handle. We can then arrange that the resulting partial open book decomposition is precisely of the form $(S', P, h, \mathbf{c}, f')$, where $(S, P, h, \mathbf{c}, f)$ is a partial open book decomposition for (M, Γ, ξ), as described above.) As in the previous case, it suffices to check that the diagram

$$
\begin{array}{ccc}
\underline{\mathbf{SHI}}(-H(S)) & \xrightarrow{\ \mathscr{H}_1''\ } & \underline{\mathbf{SHI}}(-H(S')) \\
\ \ \downarrow{\scriptstyle\mathscr{H}} & & \ \ \downarrow{\scriptstyle\mathscr{H}'} \\
\underline{\mathbf{SHI}}(-M(S, P, h, \mathbf{c}, f)) & \xrightarrow{\ \mathscr{H}_1'\ } & \underline{\mathbf{SHI}}(-M(S', P, h, \mathbf{c}, f')) \\
\ \ \downarrow{\scriptstyle\underline{\mathbf{SHI}}(f)} & & \ \ \downarrow{\scriptstyle\underline{\mathbf{SHI}}(f')} \\
\underline{\mathbf{SHI}}(-M, -\Gamma) & \xrightarrow{\ \ \mathscr{H}_1\ \ } & \underline{\mathbf{SHI}}(-M_1, -\Gamma_1).
\end{array}
$$

commutes, where $\mathscr{H}, \mathscr{H}'$ are the compositions of contact 2-handle maps of the sort used to define θ, and $\mathscr{H}_1, \mathscr{H}_1', \mathscr{H}_1''$ are the obvious contact 1-handle maps. Again, this commutativity is straightforward from the definitions of these maps. The map $\mathscr{H}_1$ thus preserves the contact invariant as desired.

Let us now assume that $i = 2$ and adopt all the notation from Sect. 3.3. The contact 2-handle attachment map

$$\mathscr{H}_2 : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M_2, -\Gamma_2)$$

is defined by $\mathscr{H}_2 = \mathscr{H}_1^{-1} \circ \underline{\mathbf{SHI}}(f) \circ F_{\gamma'}$. We have shown that $\mathscr{H}_1$ preserves the contact invariant; we can assume that γ' is Legendrian so that (M', Γ', ξ') is obtained from (M, Γ, ξ) by contact $(+1)$-surgery on γ', which means that $F_{\gamma'}$ preserves the contact invariant, by Proposition 4.6; finally, we can assume that f is a contactomorphism

$$f : (M', \Gamma', \xi') \to (M_1, \Gamma_1, \xi_1)$$

(see the discussion in [1, Subsubsection 4.2.3]) and therefore preserves the contact invariant by Proposition 4.7. The map $\mathscr{H}_2$ thus preserves the contact invariant as desired. □

Remark 4.9 Suppose (M, Γ) is a *sutured submanifold* of (M', Γ'), as defined in [10]. Let ξ be a contact structure on $M' \smallsetminus \mathrm{int}(M)$ with convex boundary and dividing set Γ on ∂M and Γ' on $\partial M'$. As explained in Sect. 1.2, the sutured contact manifold $(M' \smallsetminus \mathrm{int}(M), \Gamma \cup \Gamma', \xi')$ can be obtained from a vertically invariant contact structure on $\partial M \times I$ by attaching contact handles. Given a contact handle decomposition H of this sort, we define

$$\Phi_{\xi, H} : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M', -\Gamma')$$

to be the corresponding composition of contact handle attachment maps, as in the introduction. Note that if the contact handles in H are 0-, 1-, and 2-handles only and if ξ_M is a contact structure on M which agrees with ξ near ∂M, then

$$\Phi_{\xi, H}(\theta(M, \Gamma, \xi_M)) = \theta(M', \Gamma', \xi_M \cup \xi)$$

by Theorem 4.8.

We remark that Theorem 4.8 does not necessarily apply to the contact 3-handle map

$$\mathscr{H}_3 : \underline{\mathbf{SHI}}(-M, -\Gamma) \to \underline{\mathbf{SHI}}(-M_3, -\Gamma_3)$$

of Sect. 3.4. The problem lies with the morphism $F_\#$ of Definition 3.5, which is defined from a map on instanton Floer homology corresponding to a four-dimensional 1-handle cobordism; it is not clear that this morphism should take one contact class to another. In sutured monopole homology, this is not an issue (see [1, Subsubsection 4.2.4]), because the sutured contact class is defined in terms of an invariant of closed contact 3-manifolds, and since the above cobordism is Stein it is known to take one such contact invariant to another. However, in this setting we cannot hope to apply the same argument, because $\theta(M, \Gamma, \xi)$ does not come from an instanton Floer invariant of closed contact manifolds; indeed, no such invariant currently exists. Fortunately, we will not need a version of Theorem 4.8 for contact 3-handles in this paper.

Next, we show that θ vanishes for overtwisted contact structures.

Theorem 4.10 *If (M, Γ, ξ) is overtwisted, then $\theta(M, \Gamma, \xi) = 0$.*

Proof Let $N \subset M$ be a neighborhood of an overtwisted disk D. Take a Darboux ball in $N \smallsetminus D$ and let K be a Legendrian right-handed trefoil in this ball with $tb(K) = 1$ and $rot(K) = 0$. Then, the connected sum $K' = K \# \partial D$ is a Legendrian trefoil with $tb(K') = 2$, and it has a connected Seifert surface $\Sigma \subset N \subset M$ of genus 1.

Let (M_-, Γ_-, ξ_-) be the result of contact (-1)-surgery on K'. Suppose $\mathscr{D} = (Y, R, r, m, \eta, \alpha)$ is a marked odd closure of (M, Γ) and let $\mathscr{D}_- = (Y_-, R, r, m_-, \eta, \alpha)$ be the induced closure of (M_-, Γ_-, ξ_-), where Y_- is obtained from Y via contact (-1)-surgery on $m(K')$. Let X be the associated 2-handle cobordism from Y to Y_-. Now, (M, Γ, ξ) can be thought of as being obtained from (M_-, Γ_-, ξ_-) via contact $(+1)$-surgery on a Legendrian pushoff $K'' \subset M_-$ of K'. The associated 2-handle cobordism from Y_- to Y is isomorphic to $-X$. The morphism

$$F_{K''} : \underline{\mathbf{SHI}}(-M_-, -\Gamma_-) \to \underline{\mathbf{SHI}}(-M, -\Gamma)$$

is therefore the equivalence class of the map associated to X, viewed as a cobordism from $-Y_-$ to $-Y$.

We can cap off Σ to a closed surface $\Sigma' \subset X$ of genus 1 with self-intersection

$$\Sigma' \cdot \Sigma' = tb(K') - 1 = 1.$$

This surface violates the adjunction inequality $\Sigma' \cdot \Sigma' \leq 2g(\Sigma') - 2$, which implies that the map induced by the cobordism X is zero [16]. It follows that $F_{K''} \equiv 0$. But this map sends $\theta(M_-, \Gamma_-, \xi_-)$ to $\theta(M, \Gamma, \xi)$, by Proposition 4.6. Thus, $\theta(M, \Gamma, \xi) = 0$. $\square$

Remark 4.11 The idea above of using the right-handed trefoil was suggested to us by Peter Kronheimer and has been used to prove similar results; see [23], for example.

Given a closed 3-manifold Y, we denote by $Y(n)$ the sutured manifold obtained by removing n disjoint 3-balls from Y, where the suture on each component of $\partial Y(n)$ consists of a single curve. The following is perhaps the most important result of this subsection.

Theorem 4.12 *Suppose (Y, ξ) is a closed contact manifold which is Stein fillable. Then, the invariant $\theta(Y(n), \xi|_{Y(n)})$ of the sutured contact manifold obtained from (Y, ξ) by removing n Darboux balls is nonzero.*

As promised in the introduction, we have the following corollary.

Corollary 4.13 *If (M, Γ, ξ) embeds as a sutured contact submanifold of a Stein fillable contact manifold, then $\theta(M, \Gamma, \xi) \neq 0$.*

Proof Suppose (M, Γ, ξ) embeds in the Stein fillable contact manifold (Y, ξ). Then, (M, Γ, ξ) also embeds into the complement $(Y(n), \xi|_{Y(n)})$ of some n Darboux balls for any $n \geq 1$. By choosing these Darboux balls appropriately, we can arrange that $Y(n) \smallsetminus \text{int}(M)$ has a contact handle decomposition consisting of 0-, 1-, and 2-handles only. This corollary then follows from Theorem 4.12 and the discussion in Remark 4.9. $\square$

In order to prove Theorem 4.12, we first establish the following.

Lemma 4.14 *For any $k \geq 0$ and any $n \geq 1$, $\underline{\mathbf{SHI}}((\#^k(S^1 \times S^2))(n)) \cong \mathbb{C}^{2^{k+n-1}}$.*

Proof Note that $(\#^k(S^1 \times S^2))(n)$ can be obtained from the disjoint union of k copies of $(S^1 \times S^2)(1)$ with one copy of $S^3(n)$ via k contact 1-handle attachments. Note that each $(S^1 \times S^2)(1)$ is obtained from $S^3(2)$ by attaching a single contact 1-handle and that $S^3(n)$ is obtained from the disjoint union of $n-1$ copies of $S^3(2)$ by attaching $n-2$ contact 1-handles. Since contact 1-handle attachment has no effect on the rank of sutured instanton homology, it follows that $\underline{\textbf{SHI}}((\#^k(S^1 \times S^2))(n))$ is isomorphic to the sutured instanton homology of the disjoint union of $k+n-1$ copies of $S^3(2)$. In particular,

$$\underline{\textbf{SHI}}((\#^k(S^1 \times S^2))(n)) \cong \bigotimes_{i=1}^{k+n-1} \underline{\textbf{SHI}}(S^3(2)).$$

So, it suffices for the proof of this lemma to show that

$$\underline{\textbf{SHI}}(S^3(2)) \cong \mathbb{C}^2. \tag{17}$$

Let L_k denote the k-component unlink. Then, $S^3(L_k)$ refers to the sutured manifold given as the complement of a regular neighborhood of L_k, with 2 meridional sutures on each boundary component. Note that $S^3(L_k)$ can be obtained from $S^3(k)$ by attaching k contact 1-handles. Thus,

$$\underline{\textbf{SHI}}(S^3(k)) \cong \underline{\textbf{SHI}}(S^3(L_k)).$$

The isomorphism class of the modules which make up the system $\underline{\textbf{SHI}}(S^3(L_k))$ is what Kronheimer and Mrowka call the instanton knot homology of L_k, denoted by $KHI(L_k)$, so it suffices for (17) to show that

$$KHI(L_2) \cong \mathbb{C}^2.$$

In [19], Kronheimer and Mrowka show that KHI satisfies an oriented skein exact triangle. Applying this to a diagram of L_1 with a single crossing, as in Fig. 8, we have

$$KHI(L_1) \xrightarrow{\ f\ } KHI(L_1)$$
$$\searrow \qquad \swarrow$$
$$KHI(L_2).$$

Since $S^3(1)$ is a product sutured manifold, we have that $\underline{\textbf{SHI}}(S^3(1)) \cong \mathbb{C}$, which implies that $KHI(L_1) \cong \mathbb{C}$. The map f is therefore either zero or an isomorphism. If the latter, then $KHI(L_2) \cong 0$, which would imply that $\underline{\textbf{SHI}}(S^3(2)) \cong 0$. But this is impossible since $S^3(2)$ is taut [21, Theorem 7.12]. Thus, $f \equiv 0$, which implies that $KHI(L_2) \cong \mathbb{C}^2$. □

We may now prove Theorem 4.12.

Fig. 8 Diagrams in an oriented skein triangle

$$L_1 \qquad\qquad L_1 \qquad\qquad L_2$$
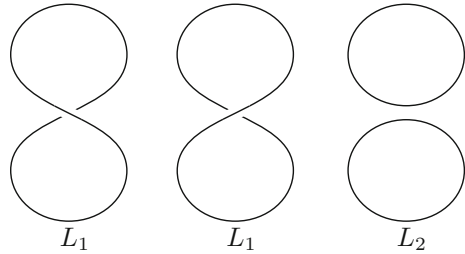
Proof of Theorem 4.12 Since (Y, ξ) is Stein fillable, it is the result of contact (-1)-surgery on some link in the standard tight $(\#^k(S^1 \times S^2), \xi_k)$. Let $\#^k(S^1 \times S^2)(n)$ be the sutured contact manifold obtained by removing n Darboux balls away from this link and let $Y(n)$ be the corresponding sutured contact manifold obtained via surgery (we are suppressing the contact structures from the notation). Then, repeated application of Proposition 4.6 gives rise to a map

$$\underline{\mathbf{SHI}}(-Y(n)) \to \underline{\mathbf{SHI}}(-(\#^k(S^1 \times S^2)(n)))$$

which sends $\theta(Y(n))$ to $\theta(\#^k(S^1 \times S^2)(n))$. So, it suffices to show that

$$\theta(\#^k(S^1 \times S^2)(n)) \neq 0. \tag{18}$$

The sutured contact manifold $\#^k(S^1 \times S^2)(n)$ has a partial open book decomposition given by $(S, P, id, \mathbf{c}, f)$, where S is obtained from the disk D^2 by attaching k unlinked 1-handles $h_1, \ldots, h_k$; $\mathbf{c} = \{c_1, \ldots, c_{k+n-1}\}$, where $c_1, \ldots, c_{k-1}$ are cocores of the 1-handles $h_1, \ldots, h_{k-1}$ and $c_k, \ldots, c_{k+n-1}$ are parallel cocores of the 1-handle h_k; P is a regular neighborhood of these cocores, as shown in Fig. 9. Define $M_0 = H(S)$ and let M_i be the sutured contact manifold obtained by attaching contact 2-handles to $H(S)$ along the curves $\gamma_1, \ldots, \gamma_i \subset \boldsymbol{\gamma}(h, \mathbf{c})$ for $i \geq 1$. In particular, M_i is obtained from M_{i-1} by attaching a contact 2-handle along $\gamma_i \subset \partial M_{i-1}$. Let

$$\mathscr{H}_{\gamma_i} : \underline{\mathbf{SHI}}(-M_{i-1}) \to \underline{\mathbf{SHI}}(-M_i)$$

denote the corresponding morphism. Note that $M_{k+n-1} = M(S, P, id, \mathbf{c})$ and the contact invariant $\theta(\#^k(S^1 \times S^2)(n))$ is the image of

$$(\mathscr{H}_{\gamma_{k+n-1}} \circ \cdots \circ \mathscr{H}_{\gamma_1})(\mathbf{1}) \tag{19}$$

under the map $\underline{\mathbf{SHI}}(f)$. So, to prove (18), it suffices to show that the class in (19) is nonzero. For this, it suffices to show that each $\mathscr{H}_{\gamma_i}$ is injective.

Let $\mathscr{D}_{i-1} = (Y_{i-1}, R, r, m_{i-1}, \eta, \alpha)$ be a marked odd closure of M_{i-1} and let $\mathscr{D}_i = (Y_i, R, r, m_i, \eta, \alpha)$ be the induced closure of M_i, where Y_i is obtained from Y_{i-1} by performing (∂M_{i-1})-framed surgery on $m(\gamma_i')$, where γ_i' is a pushoff of γ_i into the interior of M_{i-1}. The 2-handle cobordism associated to this surgery gives rise to a map

Fig. 9 Page S obtained from the disk by attaching k unlinked 1-handles showing the cocores $c_1, \ldots, c_{k-1}$ on the first $k-1$ handles and the cocores $c_k, \ldots, c_r$ on the last handle, where $r = k + n - 1$
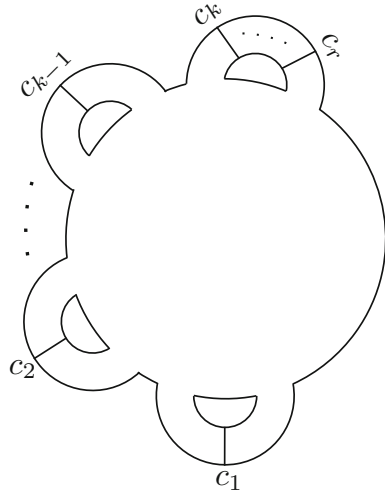
$$g : \underline{SHI}(-\mathscr{D}_{i-1}) \to \underline{SHI}(-\mathscr{D}_i)$$

whose equivalence class agrees with the map $\mathscr{H}_{\gamma_i}$. Note that γ_i is a unknot in M_{i-1} such that the framing induced by the bounding disk agrees with the (∂M_{i-1})-framing. In other words, Y_i is obtained from Y_{i-1} via 0-surgery on the unknot $m(\gamma_i')$. Let $\mathscr{D}_{i-1}' = (Y_{i-1}', R, r, m_{i-1}', \eta, \alpha)$ be the closure of M_{i-1} in which Y_{i-1}' is obtained from Y_{i-1} by (-1)-surgery on this unknot. Then, g fits into the surgery exact triangle

$$\underline{SHI}(-\mathscr{D}_{i-1}) \xrightarrow{g} \underline{SHI}(-\mathscr{D}_i) \tag{20}$$

$$\underline{SHI}(-\mathscr{D}_{i-1}').$$

For $i = 1, \ldots, k$, let $(S_i, P_i, id, \mathbf{c}_i = \{c_1, \ldots, c_i\})$ be the partial open book in which S_i is the surface obtained from the disk by attaching the first i 1-handles $h_1, \ldots, h_i$. Then, $M(S_i, P_i, id, \mathbf{c}_i)$ is diffeomorphic to $(\#^i(S^1 \times S^2))(1)$. Note that M_i is obtained from $M(S_i, P_i, id, \mathbf{c}_i)$ by attaching contact 1-handles. Therefore,

$$\underline{\mathbf{SHI}}(-M_i) \cong \underline{\mathbf{SHI}}(-(\#^i(S^1 \times S^2))(1)) \cong \mathbb{C}^{2^i},$$

where the latter isomorphism is by Lemma 4.14. It follows that

$$\underline{SHI}(-\mathscr{D}_{i-1}') \cong \mathbb{C}^{2^{i-1}} \quad \text{and} \quad \underline{SHI}(-\mathscr{D}_{i-1}) \cong \mathbb{C}^{2^{i-1}} \quad \text{and} \quad \underline{SHI}(-\mathscr{D}_i) \cong \mathbb{C}^{2^i}.$$

The exactness of the triangle in (20) then implies that g is injective for $i = 1, \ldots, k$.

For $i = k + 1, \ldots, k + n - 1$, M_i is diffeomorphic to $(\#^k(S^1 \times S^2))(1 + i - k)$. Therefore,

$$\underline{\mathbf{SHI}}(-M_i) \cong \underline{\mathbf{SHI}}(-(\#^k(S^1 \times S^2))(1 + i - k)) \cong \mathbb{C}^{2^i}$$

in this case as well, by Lemma 4.14. We therefore have again that

$$\underline{SHI}(-\mathscr{D}'_{i-1}) \cong \mathbb{C}^{2^{i-1}} \quad \text{and} \quad \underline{SHI}(-\mathscr{D}_{i-1}) \cong \mathbb{C}^{2^{i-1}} \quad \text{and} \quad \underline{SHI}(-\mathscr{D}_i) \cong \mathbb{C}^{2^i}.$$

The exactness of the triangle in (20) then implies that g is injective for $i = k + 1, \ldots, k + n - 1$.

Putting all of this together, we have shown that $\mathscr{H}_{\gamma_i}$ is injective for all $i = 1, \ldots, k + n - 1$, completing the proof. $\qquad\square$

5 Stein fillings and the fundamental group

Below, we demonstrate how Conjecture 1.7 follows from Conjecture 1.6. Suppose Y is an integer homology 3-sphere which bounds a Stein 4-manifold (X, J) with $c_1(J) \neq 0$. The long exact sequence of the pair (X, Y), combined with Poincaré duality, tells us that

$$H^2(X) \cong H_2(X, Y) \cong H_2(X).$$

Moreover, $H_2(X)$ is nontorsion since X can be built out of 1- and 2-handles. Thus, $H^2(X)$ is nontorsion. In particular, the difference between two unequal elements in $H^2(X)$ is nontorsion. Let $\bar{J}$ be the conjugate Stein structure on X, so that $c_1(\bar{J}) = -c_1(J)$. It then follows from the discussion above that $c_1(J) \neq c_1(\bar{J})$ and, hence, that $c_1(J) - c_1(\bar{J})$ is nontorsion. Assuming that Conjecture 1.6 is true, it follows that the rank of $\underline{\mathbf{SHI}}(-Y(1))$ is at least 2. But

$$rk(\underline{\mathbf{SHI}}(-Y(1))) = rk(\underline{\mathbf{SHI}}(-Y(U))),$$

where U is an unknot in Y. Therefore,

$$rk(\underline{KHI}(Y, U)) \geq 2. \tag{21}$$

We claim that there exists an irreducible homomorphism

$$\rho : \pi_1(Y \smallsetminus U) \to SU(2) \tag{22}$$

which sends a chosen meridian m of U to $\mathbf{i} \subset SU(2)$. The argument is similar to that used in the proof of [21, Proposition 7.17]. Suppose there are no irreducibles. Observe that there is only one reducible homomorphism. Indeed, reducibles have abelian image and so must factor through homomorphisms

$$H_1(Y \smallsetminus U; \mathbb{Z}) \to SU(2)$$

sending $[m]$ to **i**. But $H_1(Y \smallsetminus U; \mathbb{Z}) \cong \mathbb{Z}$ since Y is an integer homology 3-sphere, so there is exactly one such homomorphism. Since there are no irreducibles, this reducible homomorphism corresponds to the unique generator of a chain complex for the reduced singular instanton knot homology $I^\natural(Y, U)$, which implies that

$$I^\natural(Y, U) \cong \mathbb{Z}.$$

There are several ways to see this; it follows easily, for instance, from the work of Hedden et al. [9]. On the other hand, Kronheimer and Mrowka proved in [18] that

$$\underline{KHI}(Y, U) \cong I^\natural(Y, U) \otimes \mathbb{C},$$

so the inequality in (21) implies that $I^\natural(Y, U)$ has rank at least 2, a contradiction. It follows that there exists an irreducible homomorphism

$$\rho : \pi_1(Y \smallsetminus U) \cong \pi_1(Y) * \mathbb{Z} \to SU(2)$$

as claimed. Such a ρ then induces a homomorphism

$$\rho_Y : \pi_1(Y) \to SU(2)$$

which must be nontrivial (otherwise, ρ would be reducible), completing this discussion.

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