

# Autonomous Hamiltonian flows, Hofer’s geometry and persistence modules

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**Abstract** We find robust obstructions to representing a Hamiltonian diffeomorphism as a full  $k$ -th power,  $k \geq 2$ , and in particular, to including it into a one-parameter subgroup. The robustness is understood in the sense of Hofer’s metric. Our approach is based on the theory of persistence modules applied in the context of filtered Floer homology. We present applications to geometry and dynamics of Hamiltonian diffeomorphisms.

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## Contents

1	Introduction and main results	228
1.1	Distance to autonomous diffeomorphisms and $k$ -th powers	228
1.2	“Vector fields generate few diffeomorphisms”	230
1.3	Further motivation	230
1.4	Constraints on autonomous diffeomorphisms	231
1.5	Constraints on full $p$ -th powers	234
1.6	Hamiltonian diffeomorphisms and persistence modules	235
1.7	A Hamiltonian egg-beater map	236
2	Floer homology in a non-contractible class of orbits	238

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3	Loop rotation operators	247
4	Invariants	252
4.1	The $\mathbb{Z}_k$ spectral spread	252
4.2	A primer on persistence modules	260
4.2.1	Persistence modules	261
4.2.2	Morphisms	261
4.2.3	The structure theorem	262
4.2.4	Interleaving distance	262
4.2.5	Bottleneck distance between barcodes	263
4.2.6	Isometry theorem	263
4.2.7	Multiplicity function	263
4.3	Persistence modules with a $\mathbb{Z}_k$ -action	264
4.4	A new invariant of Hamiltonian diffeomorphisms	267
5	Hamiltonian egg-beater and the proof of the main results	270
5.1	Proof of Theorems 1.2 and 1.3	270
5.2	Topological set up	271
5.3	Constructing the egg-beater map	273
5.4	Detecting periodic points	275
5.5	The end of the proof of Proposition 5.1	285
6	Alternative approach in dimension 2	286
7	Interaction with the Conley conjecture	289
8	Discussion	290
8.1	Extension to monotone manifolds	290
8.2	Spectral spread and persistence modules	293
	References	295

## 1 Introduction and main results

This paper deals with robust obstructions to representing a Hamiltonian diffeomorphism as a full  $k$ -th power,  $k \geq 2$ , and in particular, to including it into a one-parameter subgroup. The robustness is understood in the sense of Hofer's metric. These obstructions yield applications to geometry and dynamics of Hamiltonian diffeomorphisms. On the geometric side, we prove that for certain symplectic manifolds the complement of the set of Hamiltonian diffeomorphisms admitting a root of order  $k$  (and a fortiori, of autonomous Hamiltonian diffeomorphisms) contains an arbitrarily large ball. We also establish a result of a dynamical flavor providing a symplectic take on Palis' dictum "Vector fields generate few diffeomorphisms": for symplectically aspherical manifolds, the subset of non-autonomous Hamiltonian diffeomorphisms contains a  $C^\infty$ -dense Hofer-open subset.

Our approach is based on the theory of persistence modules applied in the context of filtered Floer homology enhanced with a special periodic automorphism. The latter is induced by the natural action (by conjugation) of a Hamiltonian diffeomorphism  $\phi$  on the Floer homology of its power  $\phi^k$ .

### 1.1 Distance to autonomous diffeomorphisms and $k$ -th powers

Let  $d$  be the Hofer metric on the group  $\text{Ham}(M, \omega)$  of  $C^\infty$ -smooth Hamiltonian diffeomorphisms of a closed symplectic manifold  $(M, \omega)$ . We recall that  $d(f, g)$  for  $f, g \in \text{Ham}$  is defined as the infimum over all Hamiltonian paths  $\{\phi_t\}_{t \in [0,1]}$  with  $\phi_0 = f$ ,  $\phi_1 = g$ , of the quantity

$$\int_0^1 \left( \max_M H_t - \min_M H_t \right) dt,$$

where  $H_t(x) = H(t, x)$  is the time-dependent Hamiltonian that generates the path  $\{\phi_t\}$ . Recall that a Hamiltonian diffeomorphism is called *autonomous* if it is generated by a time-independent Hamiltonian function, or, in other words, it can be included into a one-parameter subgroup of  $\text{Ham}$ . Denote by  $\text{Aut} \subset \text{Ham}$  the set of all autonomous Hamiltonian diffeomorphisms. Define the quantity

$$\text{aut}(M, \omega) := \sup_{\phi \in \text{Ham}} d(\phi, \text{Aut}).$$

**Conjecture 1.1**  $\text{aut}(M, \omega) = +\infty$  for all closed symplectic manifolds  $(M, \omega)$ .

In the present paper, we make a first step toward this conjecture. Recall that  $(M, \omega)$  is called symplectically aspherical if the class of the symplectic form and the first Chern class vanish on  $\pi_2(M)$ .

**Theorem 1.2** *Let  $\Sigma$  be a closed oriented surface of genus  $\geq 4$  equipped with an area form  $\sigma$ . Then for every closed symplectically aspherical manifold  $(M, \omega)$*

$$\text{aut}(\Sigma \times M, \sigma \oplus \omega) = +\infty.$$

In this theorem  $M$  is allowed to be the point.

Our main result is the following refinement of Theorem 1.2. Let  $k \geq 2$  be an integer. Write  $\text{Powers}_k = \{\phi = \psi^k \mid \psi \in \text{Ham}\}$  for the set of Hamiltonian diffeomorphisms admitting a root of order  $k$  and denote

$$\text{powers}_k := \sup_{\phi \in \text{Ham}} d(\phi, \text{Powers}_k).$$

**Theorem 1.3** *Let  $\Sigma$  be a closed oriented surface of genus  $\geq 4$  equipped with an area form  $\sigma$ , and  $k \geq 2$  an integer. Then for every closed symplectically aspherical manifold  $(M, \omega)$*

$$\text{powers}_k(\Sigma \times M, \sigma \oplus \omega) = +\infty.$$

Since for  $p$  dividing  $k$ , we have  $\text{Powers}_p \supset \text{Powers}_k$ , it suffices to prove Theorem 1.3 in the special case when  $k$  is a prime.

A few remarks are in order. Since every autonomous diffeomorphism admits a root of any order, Theorem 1.2 is an immediate consequence of Theorem 1.3. Nevertheless, we state it separately since it provides a symplectic take on an important phenomenon in dynamical systems, which will be discussed in the next section. Furthermore it admits a (somewhat simpler) independent proof in the course of which we introduce a new invariant of Hamiltonian diffeomorphisms. The split form  $\Sigma \times M$  of the symplectic manifold is crucial for producing specific examples of Hamiltonian diffeomorphisms which lie arbitrarily far from  $\text{Aut}$  and  $\text{Powers}_p$ . More comments on the proof of both theorems can be found in Sects. 1.4 and 1.5 of the Introduction.

## 1.2 “Vector fields generate few diffeomorphisms”

The phenomenon described in the title of this subsection, which is valid for various classes of dynamical systems, has been studied for more than four decades starting from seminal works by Brin [12] and Palis [42]. Nowadays it is known, for instance, that non-autonomous  $C^1$ -smooth symplectomorphisms (not necessarily Hamiltonian) form a  $C^1$ -open and dense set in the group of all  $C^1$ -symplectomorphisms, see Arnaud et al. [3] and Bonatti et al. [7, p. 929]. It is an easy consequence of a work by Ginzburg and Gürel [26] that a similar statement holds true for  $C^\infty$ -smooth Hamiltonian diffeomorphisms of certain symplectic manifolds, for instance of symplectically aspherical ones. The method developed in the present paper yields the following result.

**Theorem 1.4** *For a closed symplectically aspherical manifold, the set  $\text{Ham} \setminus \text{Aut}$  contains a  $C^\infty$ -dense subset which is open in the topology induced by Hofer’s metric.*

In this context, Conjecture 1.1 states that the set  $\text{Ham} \setminus \text{Aut}$  contains a Hofer ball of an arbitrary large radius.

*Remark 1.5* It sounds likely that, by a slight modification of the tools developed in this paper, one can show that  $\text{Ham} \setminus \text{Powers}_k$ ,  $k \geq 2$ , contains a  $C^\infty$ -dense Hofer-open subset.

## 1.3 Further motivation

There is a couple of additional circumstances which triggered our interest in Hofer’s geometry of the set  $\text{Ham} \setminus \text{Aut}$ . For certain closed symplectic manifolds  $(M, \omega)$ , one can show that a generic smooth function  $F \in C^\infty(M)$  generates a Hamiltonian flow  $\{f_t\}$ ,  $t \in \mathbb{R}$  with

$$c|s - t| < d(f_t, f_s) \leq c^{-1}|s - t|$$

for some  $c = c(F) > 0$ , that is a generic one-parameter subgroup of  $\text{Ham}$  is a quasi-geodesic. This holds true, for instance, for symplectically aspherical manifolds and for the sphere  $S^2$  (see e.g., Section 6.3.1 in [44]). For split manifolds of the form  $\Sigma \times M$ , where  $\Sigma$  is a surface of genus  $\geq 4$  and  $M$  is symplectically aspherical, Theorem 1.2 rules out existence of a constant  $r > 0$  such that the group  $\text{Ham}$  lies in the union of tubes of radius  $r$  around these quasi-geodesics. In contrast to this, in the case  $M = S^2$  (where Conjecture 1.1 is still open) we even cannot exclude existence of such a tube around a *specific* quasi-geodesic one-parameter subgroup constructed in [43].<sup>1</sup>

Autonomous Hamiltonian diffeomorphisms give rise to an interesting biinvariant metric on the group  $\text{Ham}(M, \omega)$  called the *autonomous metric*. It has been studied by Gambaudo and Ghys [24], Brandenbursky and Kedra [9] and Brandenbursky and Shelukhin [11]. Observe that the set  $\text{Aut} \subset \text{Ham}$  is conjugation invariant. Since the group  $\text{Ham}$  is simple [4], the normal subgroup generated by  $\text{Aut}$  coincides with

<sup>1</sup> This problem has been formulated by Misha Kapovich and L.P. at an Oberwolfach meeting in 2006.

Ham. In other words, every Hamiltonian diffeomorphism  $f$  can be decomposed into a product of  $k$  autonomous ones. The autonomous distance  $d_{\text{aut}}(\mathbf{1}, f)$  from the identity to  $f$  is defined as the minimal number  $k$  of terms in such a decomposition. The above-mentioned papers prove unboundedness of Ham with respect to  $d_{\text{aut}}$  on most surfaces. The comparison between the autonomous and the Hofer metrics is far from being understood (see [9] for a discussion). For instance, let us assume that  $M$  is a closed oriented surface of genus  $\geq 4$ . Let  $S(k) \subset \text{Ham}$  be the sphere of radius  $k \in \mathbb{N}$  with respect to the autonomous metric. Theorem 1.2 shows that  $d(f, S(1))$  can be made arbitrarily large, and in fact, as we shall see in the course of the proof, such  $f$ 's can be chosen from  $S(2)$ . However we cannot prove whether  $d(f, S(2))$  can be made arbitrarily large, and at the moment this problem sounds out of reach.

Let us mention finally that for most surfaces, an analogue of the quantity  $\text{aut}(\Sigma)$  for right-invariant ‘‘hydrodynamical’’ metrics on  $\text{Ham}(\Sigma)$  is infinite, see [11].

### 1.4 Constraints on autonomous diffeomorphisms

Even though Theorem 1.2 on autonomous diffeomorphisms is a formal consequence of Theorem 1.3 on full  $p$ -th powers, we present an independent proof. One of the reasons is that it uses less sophisticated tools. Here one can avoid the language of persistence modules even though it provides a useful intuition. Additionally, Floer homology with  $\mathbb{Z}_2$ -coefficients does the job and hence one can ignore orientation issues in Floer theory. Another reason for presenting an independent argument is that in the course of the proof we introduce a new invariant of Hamiltonian diffeomorphisms, the so-called spectral spread, which is useful in its own right. In contrast to this, the proof of Theorem 1.3 involves the theory of persistence modules in an essential way.

Let us outline our approach to Theorem 1.2. Let us mention that its proof simplifies significantly for the case of surfaces. It is instructive to discuss both the two- and the higher-dimensional cases.

We start with preliminaries on Hamiltonian flows and diffeomorphisms on a closed symplectic manifold  $(M, \omega)$ . Let  $F_t(x)$ ,  $t \in \mathbb{R}$  be a 1-periodic Hamiltonian function generating a Hamiltonian flow  $\{f_t\}$  with the time-one map  $\phi = f_1 \in \text{Ham}(M, \omega)$ . The 1-periodicity of the Hamiltonian yields

$$f_{t+1} = f_t f_1 \quad \forall t \in \mathbb{R}. \quad (1)$$

The 1-periodic orbits  $x(t) = f_t x$  of the flow correspond to fixed points  $x = x(0)$  of  $\phi$ .

A well-known (and nontrivial) fact is that the free homotopy class  $\alpha = \alpha(\phi, x)$  of the orbit  $x(t)$  depends only on the fixed point  $x$  of  $\phi$ , but not on the specific Hamiltonian flow  $\{f_t\}$  generating  $\phi$ . The class  $\alpha$  is called *primitive* if it cannot be represented by a multiply-covered loop, i.e., by a map

$$S^1 \rightarrow S^1 \rightarrow M,$$

where the left arrow is a nontrivial cover.

Let  $\alpha$  be a primitive free homotopy class on a closed orientable surface  $\Sigma$  of genus  $\geq 2$  equipped with an area form. We call  $\alpha$  *simple* if it can be represented by an embedded closed curve. Note that the non-constant periodic orbits of any autonomous flow on  $\Sigma$  are either multiply covered or have no self-intersections by the uniqueness theorem for ODEs.

*Constraint 1.6* If a Hamiltonian diffeomorphism  $\phi$  of a closed symplectic surface  $\Sigma$  possesses a fixed point  $x$  such that the free homotopy class  $\alpha(\phi, x)$  is primitive but non-simple, then  $\phi$  is not autonomous.

Let us mention that this constraint is purely two dimensional as all free homotopy classes of loops are simple in higher dimensions.

Let us return to the case of general closed symplectic manifolds. For an integer  $k \geq 2$  the Hamiltonian

$$F^{(k)} = kF_{kt} \tag{2}$$

generates the flow  $\{f_{kt}\}$  with the time-one map  $\phi^k$ . The 1-periodic orbits of this flow have the form  $x(t) = f_{kt}x$ , where  $\phi^k x = x$ . Next comes the following observation: (♣) For every  $i \in \mathbb{N}$ , the loop  $x(t + i/k)$  is again a closed orbit of  $\{f_{kt}\}$  corresponding to the fixed point  $\phi^i x$  of  $\phi^k$ .

The fixed point  $x$  of  $\phi^k$  is called *primitive*<sup>2</sup> if all  $\phi^i x$  are pair-wise distinct for  $i = 0, \dots, k - 1$ . For instance, if the free homotopy class  $\alpha(\phi^k, x)$  is primitive, the point  $x$  is primitive as well (but, in general, not vice versa!).

The fixed point  $x$  of  $\phi^k$  is called *isolated* if there are no other fixed points of  $\phi^k$  in a sufficiently small neighborhood of  $x$ . For instance, if  $x$  is *non-degenerate*, i.e., the differential of  $\phi^k$  at  $x$  does not have 1 as an eigenvalue, then  $x$  is isolated.

If the Hamiltonian  $F$  is time independent, primitive fixed points of  $\phi^k$  with  $k \geq 2$  are never isolated. They necessarily appear in  $S^1$ -families  $\{f_t x\}$ ,  $t \in S^1 = \mathbb{R}/(k\mathbb{Z})$ .

*Constraint 1.7* If a  $k$ -th power ( $k \geq 2$ )  $\phi^k$  of a Hamiltonian diffeomorphism  $\phi$  possesses an isolated primitive fixed point,  $\phi$  is not autonomous.

Our next task is to refine Constraints 1.6 and 1.7 so that they become robust with respect to (not necessarily small)  $C^0$ -perturbations of the Hamiltonian  $F_t$  generating  $\phi$ . To this end, we use the machine of filtered Floer homology.

For a free homotopy class  $\alpha$  on  $M$ , denote by  $\mathcal{L}_\alpha M$  the space of loops  $S^1 \rightarrow M$  representing  $\alpha$ . Under certain assumptions on  $M$  and  $\alpha$ , which will be stated precisely later, every Hamiltonian  $F$  as above determines an *action functional*  $\mathcal{A}_F : \mathcal{L}_\alpha M \rightarrow \mathbb{R}$ ,

$$z(t) \mapsto \int_{\bar{z}}^1 F_t(z(t))dt - \int_{\bar{z}} \omega.$$

Here  $\bar{z}$  is (any) annulus connecting  $z(t)$  with the loop playing the role of a base point in  $\mathcal{L}_\alpha M$ , chosen once and forever. The critical points of  $\mathcal{A}_F$  correspond to 1-periodic

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<sup>2</sup> We remark that elsewhere in the literature (cf. [25,26]) such fixed points are called *simple*, but this terminology would cause unnecessary confusion in the context of this paper.

orbits of  $f_t$  in the class  $\alpha$ . *Filtered Floer homology*  $HF^{(a,b)}(\phi)_\alpha$ ,  $b > a$ , is, roughly speaking, the Morse homology of the space

$$\{\mathcal{A}_F < b\} / \{\mathcal{A}_F < a\}.$$

In particular,  $HF^{(a,b)}(\phi)_\alpha \neq 0$  yields the existence of closed orbits of  $\{f_t\}$  in the class  $\alpha$ . An important feature of this existence mechanism for closed orbits is its robustness with respect to  $C^0$ -perturbations of the Hamiltonian  $F$ . In particular, if the map  $HF^{(a,b)}(\phi)_\alpha \rightarrow HF^{(a+c,b+c)}(\phi)_\alpha$  induced by inclusions of sublevel sets is nonzero, then every Hamiltonian flow  $g_t$  generated by a Hamiltonian  $G_t$  with

$$\max_{t \in [0,1], z \in M} |F_t(z) - G_t(z)| < c/2$$

possesses a 1-periodic trajectory in the class  $\alpha$ . This readily yields that every Hamiltonian flow generating a diffeomorphism  $\psi$  with  $d(\phi, \psi) < c/2$  must have a closed orbit in the class  $\alpha$ .

In the context of Constraint 1.6 above, we produce a sequence of Hamiltonian diffeomorphisms  $\phi_i$ ,  $i \rightarrow \infty$  of a surface  $\Sigma$  and primitive non-simple free homotopy classes  $\alpha_i$  on  $\Sigma$  so that the filtered Floer homology of  $\phi_i$  does not vanish in some window of width  $c_i \rightarrow \infty$ . In this way, we conclude that  $d(\phi_i, \text{Aut}) \rightarrow \infty$ , and hence  $\text{aut}(\Sigma) = +\infty$ .

In order to put Constraint 1.7 into the framework of filtered Floer homology, let us note that the main feature of the action functional associated with a time-independent Hamiltonian is that it is invariant under the canonical circle action

$$z(t) \mapsto z(t + s), \quad s \in S^1, \tag{3}$$

on the loop space  $\mathcal{L}_\alpha M$ . Nowadays a number of tools for tackling this action are available, such as equivariant Floer homology and the Batalin–Vilkovisky operator<sup>3</sup> (see e.g., [8]). The approach of the present paper is based on a trick<sup>4</sup> which can be described as follows. For any (not necessarily autonomous) Hamiltonian  $F$  create an artificial  $\mathbb{Z}_k$ -symmetry and then confront it with the  $S^1$ -symmetry inherent to autonomous Hamiltonians. More precisely, fix an integer  $k \geq 2$  and take the Hamiltonian  $F^{(k)}$  from Eq. (2) generating  $\phi^k$ . Look at the  $S^1$ -action (3) and define the *the loop rotation operator*

$$R_k : \mathcal{L}_\alpha M \rightarrow \mathcal{L}_\alpha M, \quad z(t) \rightarrow z(t + 1/k)$$

generating a cyclic subgroup  $\mathbb{Z}_k \subset S^1$ . A straightforward calculation shows that the action functional  $\mathcal{A}_{F^{(k)}}$  is invariant under  $R_k$ . In particular,  $R_k$  induces a filtration-preserving morphism  $[R_k]$  of the filtered Floer homology of  $F^{(k)}$ .

For an illustration, assume that  $\phi^k$  has a primitive isolated fixed point  $x$ . The closed orbits of the flow  $\{f_{kt}\}$  corresponding to the fixed points  $\phi^i x$ ,  $i = 0, \dots, k - 1$ ,

<sup>3</sup> The original unsuccessful attempt of the authors was to use the BV operator.

<sup>4</sup> It was communicated to us by Paul Seidel.

have the same action, say  $a$ , and represent the same free homotopy class, say,  $\alpha$ . Assume that some action window  $(a - c, a + c)$  does not contain any other critical values of  $\mathcal{A}_{F^{(k)}}$ . Then each  $\phi^i x$ ,  $i = 0, \dots, k - 1$ , defines an element  $\gamma_i := [\phi^i x]$  in  $HF^{(a-c, a+c)}(\phi^k)_\alpha$ . Furthermore these elements are pairwise distinct. Observation  $(\spadesuit)$  above readily yields that  $[R_k]$  cyclically permutes  $\gamma_i$ 's, i.e.,  $[R_k](\gamma_i) = \gamma_{i+1}$  (we put here  $\gamma_k = \gamma_0$ ).

On the other hand, if  $F$  is autonomous, the action functional  $\mathcal{A}_{F^{(k)}}$  is invariant under circle action (3), and hence it is invariant under the homotopy

$$\mathcal{L}_\alpha M \rightarrow \mathcal{L}_\alpha M, \quad z(t) \mapsto z(t + s/k), \quad s \in [0, 1]$$

joining  $R_k$  with the identity. This heuristic suggests that in the autonomous case  $[R_k] = \mathbf{1}$ , which we show later by Floer-theoretic tools. In the next sections, we shall extract a lower bound for the distance between  $\phi$  and  $\text{Aut}$  from, roughly speaking, the widths of the windows where  $[R_k] \neq \mathbf{1}$ . A precise realization of this strategy occupies Sects. 3–4.1 below.

### 1.5 Constraints on full $p$ -th powers

In the spirit of Milnor’s obstruction [38] for a diffeomorphism to have a square root that was applied in the Hamiltonian setting by Albers and Frauenfelder [1], we have the following constraint:

*Constraint 1.8* Assume that the  $k$ -th power (where  $k = p$  is a prime)  $\phi^k$  of a Hamiltonian diffeomorphism  $\phi$  possesses only isolated fixed points in a primitive class  $\alpha$ . Look at all non-parameterized (i.e., considered up to a cyclic shift)  $k$ -periodic orbits

$$S = \{x, \phi(x), \dots, \phi^{k-1}(x)\}$$

of  $\phi$ , where  $x$  is such a fixed point. If  $\phi$  is a  $p$ -th power, the number  $\#\{S\}$  of these orbits is divisible by  $p$ .

This statement has the following linear-algebraic proof that generalizes best to the setting of Floer homology.

*Proof* (Constraint 1.8) Let  $\mathcal{K} = \mathbb{Q}_p$  be the cyclotomic field obtained from  $\mathbb{Q}$  by adjoining a primitive root of unity  $\zeta_p$  of order  $p$ . For each set

$$S = \{x, \phi(x), \dots, \phi^{k-1}(x)\},$$

$x$  a fixed point of  $\phi^k$  in class  $\alpha$ , consider the vector space  $V_S$  over  $\mathcal{K}$  freely generated by  $S$ . In other words,  $V_S$  is dual to the  $\mathcal{K}$ -vector space of  $\mathcal{K}$ -valued functions on  $S$ . Let  $V = \bigoplus_S V_S$  be the  $\mathcal{K}$ -vector space freely generated by all fixed points of  $\phi^k$  in class  $\alpha$ . The map  $x \mapsto \phi(x)$  preserves each set  $S$ , and induces a linear map  $A : V \rightarrow V$  that satisfies  $A^k = 1$ , and moreover  $A = \bigoplus_S A_S$  with respect to the decomposition



$V = \bigoplus_S V_S$ , where  $A_S : V_S \rightarrow V_S$  is a linear map satisfying  $A_S^k = 1$ . It is easy to see that for each  $S$ ,  $A_S$  is diagonalizable, and the  $\zeta_p$ -eigenspace  $(V_S)_{\zeta_p}$  of  $A_S$  is one-dimensional. Therefore  $\dim V_{\zeta_p} = \sum_S \dim (V_S)_{\zeta_p} = \#\{S\}$ . If  $\phi$  admits a root  $\psi$  of order  $p$ , that is  $\phi = \psi^p$ , then  $\psi$  induces a linear map  $B : V \rightarrow V$  such that  $B^p = A$ . Note that  $B$  commutes with  $A$  and hence preserves  $V_{\zeta_p}$ . Moreover the restriction  $B'$  of  $B$  to  $V_{\zeta_p}$  satisfies  $(B')^p = \zeta_p \cdot \mathbf{1}$ . Therefore by algebraic Lemmas 4.14 and 4.16 below the number  $\#\{S\} = \dim V_{\zeta_p}$  is divisible by  $p$ . This finishes the proof.  $\square$

*Remark 1.9* Constraint 1.8 generalizes to the case when  $k$  is any integer divisible by  $p$ . Both the proof from [38] and the above proof generalize to this case. As a matter of curiosity, we present another equivalent short proof of this result. Consider the set  $Y$  of all fixed points of  $\phi^k$  in class  $\alpha$ . The map  $x \mapsto \phi(x)$  defines a free  $\mathbb{Z}_k$ -action on  $Y$ . Note that  $\{S\} \cong Y/\mathbb{Z}_k$ . Any  $p$ -th root  $\psi$  of  $\phi$  determines a  $G = \mathbb{Z}_{pk}$ -action whose restriction to  $\mathbb{Z}_k \cong p\mathbb{Z}_{pk} \subset \mathbb{Z}_{pk}$  agrees with the above  $\mathbb{Z}_k$ -action. Take  $x \in Y$ . Consider the stabilizer  $H = \text{Stab}_G(x) \subset G$  in  $G$  of  $x$ . Clearly  $H \cap p\mathbb{Z}_{pk} = \{0\}$ , as the  $\mathbb{Z}_k$ -action is free. Note that  $p \cdot H \subset H \cap p\mathbb{Z}_{pk} = \{0\}$ , whence  $H \subset \ker(\mathbb{Z}_{pk} \xrightarrow{p} p\mathbb{Z}_{pk}) = k\mathbb{Z}_{pk}$ . Whenever  $k$  is divisible by  $p$ , we have  $k\mathbb{Z}_{pk} \subset p\mathbb{Z}_{pk}$ , yielding  $H = 0$ . Hence the  $G$ -action is free. Therefore since  $\mathbb{Z}_k \cong p\mathbb{Z}_{pk} \subsetneq G$ , the induced  $G/p\mathbb{Z}_{pk} \cong \mathbb{Z}_p$ -action on  $Y/\mathbb{Z}_k$  is free, and hence  $\#\{S\} = \#(Y/\mathbb{Z}_k)$  is divisible by  $p$ . This concludes the proof.

### 1.6 Hamiltonian diffeomorphisms and persistence modules

The Floer homological version of Constraint 1.8 is based on the theory of one-parametric persistence modules (see [13, 28] for a survey). Let us sketch it very briefly leaving details for Sect. 4.2.

A barcode  $\mathcal{B} = \{(I_j, m_j)\}_{1 \leq j \leq N}$  is a finite collection of intervals (or bars)  $I_j = (a_j, b_j]$ ,  $a_j \in \mathbb{R}$ ,  $b_j \in \mathbb{R} \cup +\infty$  with multiplicities  $m_j \in \mathbb{N}$ . We say that two barcodes  $\mathcal{B}$  and  $\mathcal{C}$  are  $\delta$ -matched,  $\delta > 0$ , if after erasing some bars of lengths  $< 2\delta$  in  $\mathcal{B}$  and  $\mathcal{C}$ , the remaining ones can be matched bijectively so that the endpoints of the corresponding intervals lie at a distance  $< \delta$  from one another. The bottleneck distance  $d_{\text{bottle}}(\mathcal{B}, \mathcal{C})$  is defined as the infimum of such  $\delta$ . Note that if  $\mathcal{B}$  and  $\mathcal{C}$  have a different number of infinite rays, the bottleneck distance between them is infinite.

A persistence module  $V$  is a collection of finite-dimensional vector spaces  $V_t$ ,  $t \in \mathbb{R}$  over  $\mathcal{K}$  equipped with morphisms  $\pi_{st} : V_s \rightarrow V_t$ ,  $s < t$  which satisfy  $\pi_{sr} = \pi_{tr} \circ \pi_{st}$  for all  $s < t < r$ . It is assumed in addition that  $V_t = 0$  for  $t \ll 0$  and that the morphisms  $\pi_{st}$  satisfy certain regularity assumptions. According to the structure theorem, for every persistence module  $V$  there exists unique barcode  $\mathcal{B}(V) = \{(I_j, m_j)\}$  so that  $V \cong \bigoplus_{j=1}^N (Q(I_j))^{m_j}$ . Here the building block  $Q(I) = \{(Q(I)_t), \theta\}$  is given by  $(Q(I))_t = \mathcal{K}$  for  $t \in I$  and  $(Q(I))_t = 0$  otherwise, while the morphisms  $\theta$  are the identity maps within  $I$  and zeroes otherwise.

Given a (sufficiently non-degenerate) Hamiltonian diffeomorphism  $\phi$  of a closed aspherical (or, if required, atoroidal) symplectic manifold  $M$ , one can associate to it a number of canonical persistence modules  $V(\phi)$  coming from Floer theory. Our guiding principle is that the resulting mapping

$$(\text{Ham}(M), d) \rightarrow (\text{Barcodes}, d_{\text{bottle}}), \quad \phi \mapsto \mathcal{B}(V(\phi)) \tag{4}$$

is Lipschitz with respect to the Hofer and the bottleneck distances. Composing this map with a real-valued Lipschitz function  $\zeta$  on the space of barcodes, one gets a numerical invariant  $\zeta^\sharp(\phi) := \zeta(\mathcal{B}(V(\phi)))$  of Hamiltonian diffeomorphisms which is robust in Hofer’s metric. Varying persistence modules  $V(\phi)$  and functions  $\zeta$  yields a wealth of such invariants.<sup>5</sup>

As a warm up, take  $V(\phi) := HF^{(-\infty, a)}(\phi)_\alpha$ , where  $\alpha$  is the class of the point. For a barcode  $\mathcal{B}$  let  $c_1 \leq \dots \leq c_N$  be the endpoints of infinite rays sorted in the increasing order. One readily checks that for  $\zeta(\mathcal{B}) := c_k$ , the corresponding value  $\zeta^\sharp(\phi)$  is a *spectral invariant* of the Hamiltonian diffeomorphism  $\phi$  (see Schwarz [48]). Alternatively, taking  $\zeta$  to be the maximal length of a finite bar in  $\mathcal{B}$ , we recover *the boundary depth* of  $\phi$  as defined by Usher [53].<sup>6</sup>

In order to produce a Floer homological version of Constraint 1.8, we work (again) over the cyclotomic field  $\mathcal{K} = \mathbb{Q}(\zeta_p)$ , where  $p$  is a prime, and look at the  $\mathbb{Z}_p$ -action of the loop rotation operator  $A := [R_p]$  on  $HF^{(-\infty, a)}(\phi^p)_\alpha$ . Here  $\alpha$  is a primitive class of free loops. The persistence module  $L(\phi)$  of interest is the  $\zeta_p$ -eigenspace of this action. The Lipschitz function  $\zeta$  on the space of barcodes is defined, roughly speaking, as the length of the maximal interval in  $\mathcal{B}$  whose multiplicity is **not** divisible by  $p$  “in a stable way”. Define  $\mu_p$  as  $\zeta^\sharp$ .

Now, observe that for a full  $p$ -th power  $\tilde{\phi} = \psi^p$ , the loop rotation operator  $[R_{p^2}]$  associated to  $\psi$  induces a morphism  $B$  of  $HF^{(-\infty, a)}(\phi^p)_\alpha$  with  $B^p = A$ . Its restriction  $B'$  to  $L(\psi^p)$  satisfies  $(B')^p = \zeta_p \cdot \mathbf{1}$ . Arguing as in the proof of Constraint 1.8 above, we conclude that the dimension of  $L(\psi^p)_t$  is divisible by  $p$  for every  $t$ . Thus  $\zeta(\mathcal{B}(L(\psi^p))) = \mu_p(\psi^p) = 0$ . The vanishing of  $\mu_p(\psi^p)$  is the desired Floer theoretical version of Constraint 1.8.

Combining these facts together we get that for any  $\phi \in \text{Ham}(M, \omega)$

$$\mu_p(\phi) = |\mu_p(\phi) - \mu_p(\psi^p)| \leq \text{const} \cdot d(\phi, \psi^p).$$

This yields a lower bound on Hofer’s distance from  $\phi$  to Powers $_p$  which we use for the proof of Theorem 1.3.

The details and precise formulations are presented in Sect. 4.4. For reader’s convenience, we include a primer on persistence modules in Sect. 4.2.

### 1.7 A Hamiltonian egg-beater map

Our final task is to present specific examples of Hamiltonian diffeomorphisms  $\phi_\lambda$ ,  $\lambda > 0$  for which the distances  $d(\phi_\lambda, \text{Aut})$  and  $d(\phi_\lambda, \text{Powers}_p)$  become arbitrarily large

<sup>5</sup> In order to prove that the map (4) is Lipschitz, one uses a deep *isometry theorem* between the interleaving distance on persistence modules and the bottleneck distance on barcodes, see [6].

<sup>6</sup> Our understanding of this picture appeared in discussions with Michael Usher and Jun Zhang. For its extension to general symplectic manifolds, we refer the reader to the paper [54] by Usher and Zhang. See also the paper [22] by Fraser.

as  $\lambda \rightarrow +\infty$ . To this end, we use intuition coming from the transition to chaos in Hamiltonian dynamics. Observe that in dimension 2, autonomous Hamiltonian flows provide the simplest examples of integrable systems of classical mechanics. In particular, they exhibit deterministic dynamical behavior. This suggests that one should look for  $\phi_\lambda$  in the “opposite” class of chaotic Hamiltonian diffeomorphisms. With this in mind, we choose  $\phi_\lambda$  to be a (slightly modified) *egg-beater map* (see [21]), a cousin of well-studied linked twist maps [51]. We start with a pair of intersecting annuli (see Fig. 1 in Sect. 5 below) each of which carries a shear flow with the profile given by a tent-like function whose graph is sketched in Fig. 3. The map  $\phi_\lambda$  is the composition of time- $\lambda$  maps of these flows. It is known (at least at the numerical level) that this map exhibits chaotic behavior as  $\lambda \rightarrow +\infty$  [40] and possesses rich symbolic dynamics. Next, we embed the union of the annuli into the sphere  $S^2$ , insert handles into each connected component of the complement of the annuli and extend  $\phi_\lambda$  by the identity to the obtained surface  $\Sigma$  of genus  $\geq 4$ . Even though the egg-beater map has a wealth of periodic orbits, their number in a specially chosen free homotopy class of loops on  $\Sigma$  (here the handles enter the play) becomes independent of  $\lambda$ . This enables us to perform the Floer homological analysis of the egg-beater map based on persistence modules and loop rotation operators and eventually to end up with the desired lower bounds on  $d(\phi_\lambda, \text{Aut})$ . Incidentally, the same construction yields bounds for  $d(\phi_\lambda, \text{Powers}_p)$ . Moreover these bounds survive stabilization: they remain valid for  $\phi_\lambda \times \mathbf{1}$  on  $\Sigma \times M$ . In this way, we finish off the proof of Theorems 1.2 and 1.3, see Sect. 5 for details. Note that various versions of Hamiltonian egg-beater maps appeared in the context of algebra and geometry of Hamiltonian diffeomorphisms in the works by Kapovich [32], Brandenbursky and Kedra [10], Kim and Koberda [34] and Khanevsky [33].

We conclude the introduction by mentioning that some other aspects of symplectomorphisms admitting a square root have been recently studied by means of “hard” symplectic topology in [2,50].

*Organization of the paper* In Sect. 2, we set the stage and present the necessary background from Floer theory.

In Sect. 3, we introduce loop rotations operators coming from the natural circle action on the loop space of a symplectic manifold and relate it to the action by conjugation of a diffeomorphism  $\phi$  on its power  $\phi^k$ .

In Sects. 4.1 and 4.4, we define new invariants of Hamiltonian diffeomorphisms, the so-called spectral spread and its ramifications. Ultimately, our construction involves the theory of one-parametric persistence modules. A primer on this theory is presented in Sect. 4.2. In Sect. 4.3, we focus on persistence modules enhanced with a  $\mathbb{Z}_p$ -action and translate the geometric “distance to  $p$ -th powers” problem appearing in Theorem 1.3 into algebraic language.

In Sect. 5, we design a Hamiltonian egg-beater map and use the spectral spread and the results on persistence modules for proving Theorem 1.2 on autonomous diffeomorphisms and Theorem 1.3 on full  $p$ -th powers, respectively.

In Sect. 6, we present a simple Floer homological argument proving Theorem 1.2 for surfaces.

In Sect. 7, we prove Theorem 1.4 stating that for symplectically aspherical manifolds the subset of non-autonomous Hamiltonian diffeomorphisms contains a  $C^\infty$ -dense Hofer-open subset.

Finally, in Sect. 8, we outline a generalization of our results to monotone symplectic manifolds and discuss open problems.

## 2 Floer homology in a non-contractible class of orbits

We start with a description of the basic setup of this paper. Consider a symplectically aspherical manifold  $(M, \omega)$ , such that the class  $\alpha \in \pi_0(\mathcal{L}M)$  is symplectically atoroidal. Namely, put  $\mathcal{L}_\alpha M = p_{\pi_0}^{-1}(\alpha)$  for the preimage of  $\alpha$  under the natural projection  $p_{\pi_0} : \mathcal{L}M \rightarrow \pi_0(\mathcal{L}M)$ . We require that for a loop  $\rho$  in  $\mathcal{L}_\alpha M$ , considered as a map  $\rho : T^2 \rightarrow M$  from the two-torus,

$$\int_{T^2} \rho^* \omega = 0,$$

and

$$\int_{T^2} \rho^* c_1 = 0,$$

where  $c_1 = c_1(TM, \omega)$  denotes the first Chern class of  $(M, \omega)$ , and similar conditions hold for loops in the class  $pt_M \in \pi_0(\mathcal{L}M)$  of contractible loops.

In such manifolds, as far as Floer theory is concerned (see below), a capped periodic orbit  $(z, \bar{z})$  of the Hamiltonian flow in class  $\alpha$  can be identified with its starting point  $x = z(0)$ . Indeed, by the two vanishing conditions, one sees that neither its action nor its index depend on the choice of capping  $\bar{z}$ .

Denote  $\mathcal{E}^+(H) := \int_0^1 \max_M H(t, \cdot) dt$ ,  $\mathcal{E}^-(H) := \int_0^1 \min_M H(t, \cdot) dt$ , and  $\mathcal{E}(H) = \mathcal{E}^+(H) - \mathcal{E}^-(H)$ . For a segment  $I = (a, b)$  and  $d \in \mathbb{R}$  denote  $I + d := (a + d, b + d)$ .

We present the following general definition that organizes certain properties of Floer homology in action windows. For convenience, we shall use the language of (two-parametric) persistence modules. Let us emphasize that the genuine applications of persistence modules to our story (cf. the title of this paper) appear later on in Sect. 4, where we deal with a much more developed theory of one-parametric persistence modules.

Consider the partially ordered set  $\mathcal{I}$  of open intervals  $(a, b)$  where  $a \in \{-\infty\} \cup \mathbb{R}$ ,  $b \in \mathbb{R} \cup \{+\infty\}$ ,  $a < b$ , with the partial order  $I_1 = (a_1, b_1) \leq I_2 = (a_2, b_2)$  if  $a_1 \leq a_2$  and  $b_1 \leq b_2$ . Turn this partially ordered set into a category in the natural way, wherein  $\text{Hom}_{\mathcal{I}}(I_1, I_2)$  has 1 element if  $I_1 \leq I_2$  and is empty otherwise. For a subset  $S \subset \mathbb{R}$  we denote by  $\mathcal{I}^S$  the full subcategory of  $\mathcal{I}$  whose objects are intervals  $(a, b)$  with  $a, b \notin S$ .

Given a base field  $\mathcal{K}$ , denote by  $\text{Vect}_{\mathcal{K}}$  the category of finite-dimensional graded vector spaces over  $\mathcal{K}$ .

**Definition 2.1** We call a *restricted two-parametric (r2p) persistence module* of graded vector spaces over  $\mathcal{K}$  a pair  $(S, V)$  consisting of a compact subset  $S \subset \mathbb{R}$  with empty interior, *the spectrum* of the module, and a functor  $V : \mathcal{I}^S \rightarrow \text{Vect}_{\mathcal{K}}$  : a

collection of vector spaces  $V^{(a,b)}$  for each open interval  $(a, b)$ ,  $a, b \notin S$  and linear comparison maps

$$j_{(a_1,b_1),(a_2,b_2)} : V^{(a_1,b_1)} \rightarrow V^{(a_2,b_2)},$$

for each two intervals  $(a_1, b_1), (a_2, b_2)$  such that  $a_1 \leq a_2, b_1 \leq b_2$ , that satisfy

$$j_{I_2,I_3} \circ j_{I_1,I_2} = j_{I_1,I_3},$$

for  $I_1 \leq I_2 \leq I_3$ . Moreover for each  $a < b < c$  outside  $S$ , the data include a prescribed long exact sequence

$$V^{(a,b)} \xrightarrow{j_{(a,b),(a,c)}} V^{(a,c)} \xrightarrow{j_{(a,c),(b,c)}} V^{(b,c)} \xrightarrow{\delta_{a,b,c}} V^{(a,b)}[1],$$

where  $V^{(a,b)}[1]$  stands for the shift of the grading of  $V^{(a,b)}$  by 1.

We further require  $V$  to satisfy the following property:

- if  $[a, b] \subset \mathbb{R} \setminus S$ , then  $V^{(a,b)} = 0$ .

These data and properties imply that

- $j_{(a_1,b_1),(a_2,b_2)} = 0$  whenever  $(a_1, b_1) \leq (a_2, b_2)$  are disjoint.

Restricted two-parametric persistence modules form a particular case of two-dimensional<sup>7</sup> persistence modules as defined in [15].

*Remark 2.2* Note that  $r2p$  persistence modules with a given spectrum  $S$  form a category  $\mathcal{D}^S$  where a morphism  $\Phi$  between any two  $r2p$  persistence modules  $V, W \in \mathcal{D}^S$  is a collection of maps  $\Phi^{(a,b)} : V^{(a,b)} \rightarrow W^{(a,b)}$  for each interval  $(a, b)$ ,  $a, b \notin S$  that commutes with the comparison maps, i.e.,  $\Phi$  is a natural transformation of the corresponding functors.

*Remark 2.3* For a number  $\sigma \in \mathbb{R}$ , and a  $r2p$  persistence module  $(S, V)$  we can form its *shift*  $(S - \sigma, V^{\bullet+\sigma})$  by  $\sigma$ , which is a  $r2p$  persistence module with spectrum  $S - \sigma$  and defined on objects as  $(a, b) \mapsto V^{(a,b)+\sigma}$ , and on morphisms as  $j_{(a_1,b_1),(a_2,b_2)}^{V^{\bullet+\sigma}} = j_{(a_1,b_1)+\sigma,(a_2,b_2)+\sigma}^V$ .

*Example 2.4* We give a sketch of two examples from Morse homology, and continue to discuss a similar situation for Floer homology.

1. A basic example of such an  $r2p$  persistence module is associated to a Morse function  $f$  on a closed manifold  $X$ . Indeed put  $V^{(a,b)} = H(\{f < b\}, \{f \leq a\})$ , a relative homology group of sublevel sets of  $f$ . In fact this group is isomorphic to  $HM^{(a,b)}(f)$ , the homology of the Morse complex of  $f$ , generated by critical points of  $f$  with critical values in  $(a, b)$ , and  $S = \text{Spec}(f) = f(\text{Crit}(f))$  the set of all critical values of  $f$ . The comparison maps are given by natural morphisms between the corresponding complexes induced by inclusions, or alternatively by certain Morse continuation maps.

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<sup>7</sup> The term “two-dimensional” would cause confusion in our setting.

2. A more subtle example can be constructed from *any* smooth function  $f$  on a closed manifold. Put  $S = \text{Spec}(f) = f(\text{Crit}(f))$ , and let  $(a, b)$ ,  $a, b \notin S$  be an interval. Consider the set  $\mathcal{F}(f)$  of Morse functions  $f'$  sufficiently close to  $f$  so that  $a, b \notin \text{Spec}(f')$ , and all continuation maps  $C(f', f'') : HM^{(a,b)}(f') \rightarrow HM^{(a,b)}(f'')$  for  $f', f'' \in \mathcal{F}(f)$  are isomorphisms. Then define  $HM^{(a,b)}(f)$  as<sup>8</sup> the vector space of collections  $\{x_{f'} \in HM^{(a,b)}(f')\}_{f' \in \mathcal{F}(f)}$  satisfying  $C(f', f'')(x_{f'}) = x_{f''}$ . One shows that  $(S, \{HM^{(a,b)}(f)\}_{(a,b) \in \mathcal{I}^S})$  forms a  $r2p$  persistence module.

For example the constant function  $f \equiv 0$  on a closed manifold  $X$  gives the  $r2p$  persistence module with spectrum  $S = \{0\}$  and for  $a, b \notin S$ ,  $HM^{(a,b)}(f) = 0$  if  $0 \notin (a, b)$  and  $HM^{(a,b)}(f) \cong H(X)$ , the homology of  $X$ , if  $0 \in (a, b)$ , with the obvious comparison maps.

Let  $(M, \omega)$  be a closed connected symplectic manifold, and let  $\alpha \in \pi_0(\mathcal{L}M)$  be a free homotopy class of loops. Choose a reference path  $\eta_\alpha \in \mathcal{L}_\alpha M$ . Assuming that  $(M, \omega)$  is symplectically  $\alpha$ -atoroidal (and hence symplectically aspherical), we describe a construction of a Hamiltonian Floer homology for  $(M, \omega)$  in the class  $\alpha$ .

A time-periodic Hamiltonian  $H \in C^\infty(S^1 \times M, \mathbb{R})$  is called *mean-normalized* if  $\int_M H_t \omega^n = 0$  for all  $t$ , where  $\dim M = 2n$ . We write  $\mathcal{H}$  for the space of all mean-normalized Hamiltonians in  $C^\infty(S^1 \times M, \mathbb{R})$ .

For a Hamiltonian  $H \in \mathcal{H}$  the time-dependent Hamiltonian vector field  $X_H = X_H(t)$  is defined as

$$\iota_{X_H(t)}\omega = -dH_t,$$

where  $H_t(-) = H(t, -)$ . Denote by  $\mathcal{P}^\alpha(H)$  the set of 1-periodic orbits  $x(t)$  in the class  $\alpha$  of the Hamiltonian flow of  $X_H$ , namely solutions to

$$\dot{x}(t) = X_H(t, x(t)) \quad \forall t \in S^1, \tag{5}$$

$$[x(t)] = \alpha \in \pi_0(\mathcal{L}M). \tag{6}$$

Fix a base field  $\mathcal{K}$  (we pick  $\mathcal{K} = \mathbb{Z}/(2)$  for the proof of Theorems 1.2 and 1.4, and  $\mathcal{K} = \mathbb{Q}_p$ , the splitting field of  $x^p - 1 \in \mathbb{Q}[x]$  over  $\mathbb{Q}$ , for the proof of Theorem 1.3). Consider the vector space  $CF(H)_\alpha$  over  $\mathcal{K}$  freely generated by the set  $\mathcal{P}^\alpha(H)$ . Put  $\{\phi_H^t\}_{t \in [0,1]}$  for the flow generated by  $H$ . The elements of  $\mathcal{P}^\alpha(H)$  correspond to fixed points  $x$  of  $\phi_H := \phi_H^1$ , such that the loop  $\{\phi_H^t \cdot x\}_{t \in S^1 = [0,1]/\{0,1\}}$  lies in the class  $\alpha$ .

For an  $\alpha$ -non-degenerate Hamiltonian  $H$ , namely one for which the linearization  $D\phi_H(x)$  of  $\phi_H$  at every fixed point  $x$  corresponding to an element in  $\mathcal{P}^\alpha(H)$  has no eigenvectors with eigenvalue 1, the set  $\mathcal{P}^\alpha(H)$  is finite.

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<sup>8</sup> This definition is a specific representative of the isomorphism class of *limits of the indiscrete groupoid*, namely a category with exactly one morphism between any two objects, formed by  $\{HM^{(a,b)}(f')\}_{f' \in \mathcal{F}(f)}$  and the continuation maps, rendering each two of these vector spaces canonically isomorphic. Note that this representative of the limit of this diagram is canonically isomorphic by a unique isomorphism to a similar representative of the limit of any of its full subdiagrams, namely subdiagrams with the same morphism sets as in the diagram between any two of their objects (since all the continuation maps are isomorphisms). This observation is useful in showing that this definition satisfies the properties of a  $r2p$  persistence module.

We define the action functional  $\mathcal{A}_H : \mathcal{L}_\alpha M \rightarrow \mathbb{R}$  by choosing a path in  $\mathcal{L}_\alpha$  between  $\eta_\alpha$  and a point  $x \in \mathcal{L}_\alpha M$ , considering it as a cylinder  $\bar{x} : S^1 \times [0, 1] \rightarrow M$ , and computing

$$\mathcal{A}_H(x, \bar{x}) = \int_0^1 H(t, x(t))dt - \int_{\bar{x}} \omega.$$

Since our manifold is symplectically  $\alpha$ -atoroidal, the action functional  $\mathcal{A}_H$  does not depend on the choice of “capping”  $\bar{x}$ , and is hence well defined as a functional  $\mathcal{A}_H : \mathcal{L}_\alpha M \rightarrow \mathbb{R}$ . Its critical points are exactly the periodic orbits  $\mathcal{P}^\alpha(H)$  of the Hamiltonian flow of  $H$  in class  $\alpha$ . Put  $\text{Spec}_\alpha(H) := \mathcal{A}_H(\mathcal{P}^\alpha(H))$  for the spectrum of  $\mathcal{A}_H$  in the class  $\alpha$ . By [41]  $\text{Spec}_\alpha(H) \subset \mathbb{R}$  is a measure-zero subset, and hence has empty interior.

Since  $H$  is  $\alpha$ -non-degenerate,  $\mathcal{A}_H$  has isolated critical points in  $\mathcal{L}_\alpha$ —indeed there is a bijective correspondence between a 1-periodic orbit of a given flow and its initial point. As the manifold  $M$  is compact, we conclude that  $CF(H)_\alpha$  is a finite-dimensional  $\mathcal{K}$ -vector space. We grade it as follows by the Conley–Zehnder index [46], with the normalization that for a  $C^2$ -small autonomous Morse Hamiltonian, the Conley–Zehnder index of each of its critical points as a contractible periodic orbit of the Hamiltonian flow is equal to the Morse index of this critical point. We choose non-canonically a trivialization  $\Phi_\alpha$  of the symplectic vector bundle  $\eta_\alpha^* TM$  over  $S^1$ . Then any choice of a homotopy  $\bar{x}$  from  $\eta_\alpha$  to a 1-periodic orbit  $x$  of the flow  $\{\phi_H^t\}$  in the class  $\alpha$ , defines a homotopically canonical trivialization of  $x^* TM$ . We then compute the index of the path of symplectic matrices obtained from  $\{D\phi_H^t(x(0))\}_{t \in S^1}$  by the trivialization. Since our manifold is symplectically  $\alpha$ -atoroidal, this number does not depend on the choice of homotopy  $\bar{x}$ .

Choosing a generic  $\omega$ -compatible almost complex structure

$$J \in \mathcal{J} := C^\infty(S^1, \mathcal{J}(M, \omega))$$

depending on  $t \in S^1$ , so that transversality is achieved (cf. [20]), we define the matrix coefficients of the Floer differential  $\partial_{J,H} : CF(H)_\alpha \rightarrow CF(H)_\alpha[-1]$  by counting the number of (the dimension zero component of isolated) solutions  $u$  of the Floer equation

$$\partial_s u + J(t, u) (\partial_t u - X_H(t, u)) = 0.$$

By standard arguments (see e.g., [19,23,30,37])  $\partial_{J,H} \circ \partial_{J,H} = 0$ , and hence  $(CF(H)_\alpha, \partial_{J,H})$  is a chain complex. Moreover  $\mathcal{A}_H$  defines a function on the generators of  $CF(H)_\alpha$ , which extends to  $CF(H)_\alpha$  as a valuation, that is  $\mathcal{A}_H(0) := -\infty$ , and for a chain  $0 \neq c = \sum_j a_j x_j$ , where  $x_j \in \mathcal{P}^\alpha(H)$ ,

$$\mathcal{A}_H(c) := \max \{ \mathcal{A}_H(x_j) | a_j \neq 0 \}.$$

In particular  $\mathcal{A}_H(c_1 + c_2) \leq \mathcal{A}_H(c_1) + \mathcal{A}_H(c_2)$ , for all  $c_1, c_2 \in CF(H)_\alpha$ . By a standard action-energy estimate  $\mathcal{A}_H(\partial(c)) < \mathcal{A}_H(c)$ . Hence  $CF^{(-\infty, a)}(H)_\alpha := \{c \in$

$CF(H)_\alpha | \mathcal{A}_H(c) < a$  is a subcomplex of  $(CF(H)_\alpha, \partial_{J,H})$ , and as  $a$  runs through  $\mathbb{R} \setminus \text{Spec}_\alpha(H)$ , defines a filtration on  $CF^{(-\infty, a)}(H)_\alpha$ . We obtain a filtered complex which we denote by  $(CF(H)_\alpha, \partial_{J,H}, \mathcal{A}_H)$ .

For  $a \notin \text{Spec}_\alpha(H)$ , put  $CF^{(-\infty, a)}(H)_\alpha := \{c \in CF(H)_\alpha | \mathcal{A}_H(c) < a\}$ . This is a subcomplex of  $(CF(H)_\alpha, \partial_{J,H})$ . For a window  $(a, b)$  with  $a, b \notin \text{Spec}_\alpha(H)$ , define  $CF^{(a,b)}(H)_\alpha$  as the quotient complex

$$CF^{(a,b)}(H)_\alpha := CF^{(-\infty, b)}(H)_\alpha / CF^{(-\infty, a)}(H)_\alpha,$$

with the induced differential, which we denote  $\partial_{J,H}$ , by a slight abuse of notation. Put

$$HF^{(a,b)}(H, J)_\alpha := H \left( CF^{(a,b)}(H)_\alpha, \partial_{J,H} \right),$$

for the homology of this quotient complex. We have the following invariance statement.

**Proposition 2.5** *The assignment  $(a, b) \mapsto HF^{(a,b)}(H, J)_\alpha$  defines a  $r2p$  persistence module with spectrum  $S = \text{Spec}_\alpha(H)$ . Moreover there is a canonical isomorphism between this  $r2p$  persistence module and each other one obtained from a different choice of  $J$  and  $H$ , as long as the path  $\{\phi_H^t\}_{t \in [0,1]}$  remains in a fixed class in the universal cover  $\tilde{\mathcal{G}}$  of the group of Hamiltonian diffeomorphisms.*

This leads us to the following definition.

**Definition 2.6** Given a non-degenerate element  $\tilde{\phi}$  in  $\tilde{\mathcal{G}}$ , we define its  $r2p$  persistence module

$$(a, b) \mapsto HF^{(a,b)}(\tilde{\phi})$$

as<sup>9</sup> the  $r2p$  persistence module with spectrum  $S = \text{Spec}_\alpha(H)$ , which to an open interval  $(a, b)$  associates the vector space of collections

$$\{x_{(a,b),H,J} \in HF^{(a,b)}(H, J)_\alpha\}_{(H,J)}$$

indexed by all  $\alpha$ -regular pairs  $(H, J)$  such that the path  $\{\phi_H^t\}_{t \in [0,1]}$  lies in class  $\tilde{\phi}$ , satisfying the condition

$$\Psi_{(H,J),(H',J')}^{(a,b)}(x_{(a,b),H,J}) = x_{(a,b),H',J'},$$

for the canonical isomorphism

$$\Psi_{(H,J),(H',J')}^{(a,b)} : HF^{(a,b)}(H, J)_\alpha \rightarrow HF^{(a,b)}(H', J')_\alpha.$$

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<sup>9</sup> This definition is a canonical representative of the isomorphism class in  $\mathcal{D}^S$  of limits in  $\mathcal{D}^S$  of the  $\mathcal{D}^S$ -valued diagram defined by Proposition 2.5.



Proposition 2.5 follows for example from [53, Proposition 5.2]. Roughly speaking, and this can be made rigorous, the isomorphism between the  $r2p$  persistence modules for different choices of  $J$  is constructed by counting solutions to a Floer continuation map, and the isomorphism between the  $r2p$  persistence modules for two different choices  $\{\phi_H^t\}_{t \in [0,1]}$ ,  $\{\phi_{H'}^t\}_{t \in [0,1]}$  of the Hamiltonian path in a fixed class in the universal cover is obtained by a diffeomorphism of  $\mathcal{L}_\alpha$  given by the action of the contractible loop  $\{\gamma_t = \phi_{H'}^t(\phi_H^t)^{-1}\}_{t \in [0,1]}$  in  $\mathcal{G}$ , that is the difference between these paths.<sup>10</sup> This is the diffeomorphism

$$\begin{aligned} P(\gamma) : \mathcal{L}_\alpha &\rightarrow \mathcal{L}_\alpha, \\ z(t) &\mapsto \gamma_t(z(t)). \end{aligned}$$

Moreover (cf. [45, Section 13.1]) we have the diagram

$$(\mathcal{L}_\alpha M, \mathcal{A}_H) \xrightarrow{P(\gamma)} (\mathcal{L}_\alpha M, \mathcal{A}_{H'})$$

of spaces with functionals. That is

$$\mathcal{A}_H = P(\gamma)^* \mathcal{A}_{H'}. \tag{7}$$

*Remark 2.7* By the construction of the limit,  $HF^{(a,b)}(\tilde{\phi})_\alpha$  comes with a projection

$$\pi_{(H,J)} : HF^{(a,b)}(\tilde{\phi})_\alpha \rightarrow HF^{(a,b)}(H, J)_\alpha$$

for each  $\alpha$ -regular pair  $(H, J)$  with  $H$  generating a path in  $\tilde{\phi}$ . Each such projection  $\pi_{(H,J)}$  is in fact an isomorphism of  $r2p$  persistence modules.

In fact it is useful to have a definition of Floer homology for degenerate elements  $\tilde{\phi} \in \tilde{\mathcal{G}}$  as well, and many of the arguments that follow are based, at least intuitively, on the following extended definition.

**Definition 2.8** Given any  $\tilde{\phi} \in \tilde{\mathcal{G}}$ , represent it by a path  $\{\phi_H^t\}$  with Hamiltonian  $H \in C^\infty(S^1 \times M, \mathbb{R})$ . Let  $(a, b)$ ,  $a < b$ ,  $a, b \notin \text{Spec}_\alpha(H)$  be a fixed window. Then for any non-degenerate  $C^2$ -perturbation  $H'$  of  $H$  that is sufficiently small,<sup>11</sup>  $a, b \notin \text{Spec}_\alpha(H')$  still. Moreover decreasing if necessary the threshold of smallness, interpolation continuation establishes functorial isomorphisms between each pair of Floer homology groups<sup>12</sup> in  $\{HF^{(a,b)}(H', J')_\alpha\}$  where  $H'$  runs over the set  $\mathcal{H}^{\text{reg}}(H)$  of such  $C^2$ -small perturbations, and  $J' \in \mathcal{J}^{\text{reg}}(H')$  is an almost complex structure

<sup>10</sup> Here and below, we deal with certain transformations of loop spaces which naturally act on action functionals and on the Riemannian metrics on  $\mathcal{L}_\alpha M$  coming from loops of almost complex structures on  $M$ , thus inducing morphisms in Floer homology which are useful for our purposes. We call them diffeomorphisms since this way of thinking provides a right intuition for manipulating these Floer homological constructions. Incidentally, these transformations are genuine diffeomorphisms if understood in the sense of diffeology [31].

<sup>11</sup> We say that a perturbation  $H'$  of  $H$  is  $C^2$ -small if  $H' - H$  is a  $C^2$ -small function.

<sup>12</sup> In other words, these vector spaces and isomorphism maps form an *indiscrete groupoid* in  $\text{Vect}_{\mathcal{K}}$ .

such that the pair  $(H', J')$  is regular. We define  $HF^{(a,b)}(H)_\alpha$  as the vector space of collections<sup>13</sup>

$$\left\{ x_{(a,b),H',J'} \in HF^{(a,b)}(H', J')_\alpha \right\}_{H' \in \mathcal{H}^{\text{reg}}(H), J' \in \mathcal{J}^{\text{reg}}(H')}$$

such that

$$\Psi_{(H'_1, J'_1), (H'_2, J'_2)}^{(a,b)}(x_{(a,b),H'_1, J'_1}) = x_{(a,b),H'_2, J'_2}$$

for the canonical isomorphism

$$\Psi_{(H'_1, J'_1), (H'_2, J'_2)}^{(a,b)} : HF^{(a,b)}(H'_1, J'_1)_\alpha \rightarrow HF^{(a,b)}(H'_2, J'_2)_\alpha.$$

We note that the same construction with the indexing set given by any non-empty subset of the above indexing set yields a canonically isomorphic vector space, and that this implies that  $HF^{(a,b)}(H)_\alpha$  defines a  $r2p$  persistence module and that as in Proposition 2.5 this  $r2p$  persistence module does not depend on the representative  $\{\phi_H^t\}$  of  $\tilde{\phi}$ . This leads as in Definition 2.6 above to a definition of  $HF^{(a,b)}(\tilde{\phi})_\alpha$  for any class  $\tilde{\phi} \in \tilde{\mathcal{G}}$ .

*Remark 2.9* It is straightforward to check that Definition 2.8 agrees with Definition 2.6. Namely, that the resulting  $r2p$  persistence modules are canonically isomorphic.

*Remark 2.10* In fact, Proposition 2.5, and hence Definition 2.6, and consequently Definition 2.8 can be upgraded in our setting of a closed symplectically  $\alpha$ -atoroidal symplectic manifold to define a  $r2p$  persistence module

$$(a, b) \mapsto HF^{(a,b)}(\phi)$$

of a Hamiltonian diffeomorphism itself. Indeed, given two paths  $\{\phi_H^t\}$  and  $\{\phi_{H'}^t\}$  with endpoint  $\phi$  we consider the action  $P(\gamma)$  of the difference loop  $\{\gamma_t = \phi_H^t(\phi_{H'}^t)^{-1}\}_{t \in [0,1]}$  on the loop space  $\mathcal{L}_\alpha M$ .

It is easy to see that for  $x \in \mathcal{P}^\alpha(H)$ , its image  $x'(t) = P(\gamma)(x)(t) = \gamma_t(x(t)) \in \mathcal{P}^\alpha(H')$  satisfies

$$\mathcal{A}_{H'}(x') = \mathcal{A}_H(x) + A(\gamma, \eta_\alpha),$$

where  $A(\gamma, \eta_\alpha) = \mathcal{A}_G(P(\gamma)(\eta_\alpha))$ , where the action is normalized by the reference loop  $\eta_\alpha$ , and  $G \in \mathcal{H}$  is the Hamiltonian generating  $\gamma$ . Namely, for  $\eta \in \mathcal{L}_\alpha M$  we put

$$A(\gamma, \eta) = \int_0^1 G(t, \gamma_t(\eta(t)))dt - \int_{\overline{P(\gamma)(\eta)}} \omega,$$

---

<sup>13</sup> That is—we are considering a specific representative of the limit of the corresponding indiscrete groupoid.

where  $\overline{P(\gamma)}(\eta)$  is a cylinder in  $M$  defined by any path in  $\mathcal{L}_\alpha M$  between  $\eta_\alpha$  and  $P(\gamma)(\eta)$ . A little computation shows that  $A(\gamma, \eta)$  depends only on the class  $[\gamma]$  of  $\gamma$  in  $\pi_1(\text{Ham}(M))$  and the free homotopy class  $\alpha$  of  $\eta$ . In particular  $A(\gamma, \eta_\alpha)$  is invariant under reparametrizations of  $\gamma$  and  $\eta_\alpha$ . Reparametrizing  $\gamma$  and  $\eta_\alpha$  to be non-constant in disjoint time subintervals, we see that  $A(\gamma, \eta_\alpha) = A(\gamma, pt_M)$  and  $A(\gamma, pt_M) = 0$  by [48, Theorem 1.1, Corollary 4.15].

Similarly, it is easy to see that the Conley–Zehnder index of  $x \in \mathcal{P}^\alpha(H)$  and of its image  $x'(t) = P(\gamma)(x)(t) \in \mathcal{P}^\alpha(H')$  satisfy

$$CZ_{H'}(x') = CZ_H(x) + I(\gamma, \eta_\alpha),$$

where for  $\eta \in \mathcal{L}_\alpha$  we define  $I(\gamma, \eta)$  as the Maslov index of the following loop of symplectic matrices. Make a non-canonical choice of a cylinder  $\bar{\eta}$  from  $\eta_\alpha$  to  $\eta$ . This cylinder and the trivialization  $\Phi_\alpha$  of  $\eta_\alpha^*TM$  define homotopically-canonically a trivialization  $\Phi = \bar{\eta}_* \Phi_\alpha$  of  $\eta^*TM$ , that is an isomorphism  $\Phi : \eta^*TM \rightarrow S^1 \times \mathbb{R}^{2n}$  of symplectic vector bundles. Similarly, choosing a cylinder  $w$  from  $\eta$  to  $P(\gamma)\eta$  defines a trivialization  $w_* \Phi$  of  $P(\gamma)\eta^*TM$ . Since the differential  $D\gamma \circ \eta : \eta^*TM \rightarrow P(\gamma)\eta^*TM$  is an isomorphism of symplectic vector bundles, we obtain another trivialization  $P(\gamma)^*w_* \Phi$  of  $\eta^*TM$ . The loop of symplectic matrices we consider is the difference loop  $P(\gamma)^*w_* \Phi \circ \Phi^{-1}$  of the two trivializations. Since our manifold is symplectically  $\alpha$ -atoroidal, the Maslov index of this loop does not depend on the choices of cylinders made. Moreover  $I(\gamma, \eta)$  depends only on  $[\gamma] \in \pi_1(\text{Ham}(M))$  and the free homotopy class  $\alpha$  of  $\eta$ . Thus as above we conclude that  $I(\gamma, \eta) = I(\gamma, pt_M)$  and  $I(\gamma, pt_M) = 0$  by [49, Proposition 10.1].

Hence  $P(\gamma)$  enters the diagram  $(\mathcal{L}_\alpha M, \mathcal{A}_H) \xrightarrow{P(\gamma)} (\mathcal{L}_\alpha M, \mathcal{A}_{H'})$  of spaces with functionals, and hence determines an isomorphism of filtered complexes that preserves grading, and hence of the corresponding graded  $r2p$  persistence modules. Compare [53, Proposition 5.3].

A general observation is that given a Hamiltonian diffeomorphism  $g$ , and a regular class  $\tilde{\phi} \in \tilde{\mathcal{G}}$ , there is a natural morphism of  $r2p$  persistence modules

$$[P(g)] : HF^{(a,b)}(\tilde{\phi})_\alpha \rightarrow HF^{(a,b)}(g \circ \tilde{\phi} \circ g^{-1})_\alpha,$$

which we call the *push-forward map*. This morphism is built by acting by  $g$  on all the objects involved in the construction. Such morphisms in the context of fixed-point Floer homology of symplectomorphisms were recently introduced and used by Tonkonog [52]. The basic such action is the diffeomorphism

$$P(g) : \mathcal{L}_\alpha M \rightarrow \mathcal{L}_\alpha M, \\ z(t) \mapsto g(z(t)).$$

Given a Hamiltonian  $H \in \mathcal{H}$  that generates a representative of  $\tilde{\phi}$ , the Hamiltonian  $H \circ g^{-1}$  generates a representative of  $g \circ \tilde{\phi} \circ g^{-1}$  and the restriction  $P(g) : \mathcal{P}^\alpha(H) \rightarrow \mathcal{P}^\alpha(H \circ g^{-1}), x(t) \mapsto g(x(t))$  is a bijection. Moreover since  $g \in \text{Ham}(M)$ , the

symplectic area of the cylinder between the reference loop  $\eta_\alpha \in \mathcal{L}_\alpha$  and the loop  $g \circ \eta_\alpha \in \mathcal{L}_\alpha$  (which is well-defined since  $M$  is symplectically  $\alpha$ -atoroidal) is in fact zero.

Hence we have the diagram

$$(\mathcal{L}_\alpha M, \mathcal{A}_H) \xrightarrow{P(g)} (\mathcal{L}_\alpha M, \mathcal{A}_{H \circ g^{-1}})$$

of spaces with functionals, that is

$$P(g)^* \mathcal{A}_{H \circ g^{-1}} = \mathcal{A}_H, \tag{8}$$

the actions being computed with respect to the same reference loop  $\eta_\alpha$ .

What remains is to observe that the restricted map on generators, given a choice of almost complex structure  $J \in \mathcal{J}$ , extends naturally to an isomorphism of filtered Floer complexes

$$P(g) : (CF(H, J)_\alpha, \mathcal{A}_H) \rightarrow \left( (CF(H \circ g^{-1}, g_* J)_\alpha), \mathcal{A}_{H \circ g^{-1}} \right),$$

where  $(g_* J)_x = Dg(g^{-1}x)J_{g^{-1}x}D(g^{-1})(x)$  is the push-forward of the almost complex structure  $J$  by the diffeomorphism  $g$ .

**Definition 2.11** The isomorphism  $P(g)$  of filtered Floer complexes gives the map

$$[P(g)] : HF^{(a,b)}(\tilde{\phi})_\alpha \rightarrow HF^{(a,b)}(g \circ \tilde{\phi} \circ g^{-1})_\alpha$$

of  $r2p$  persistence modules, which we call the push-forward map.

The push-forward map is an example of an operator on Floer homology coming from actions of Hamiltonian loops on the loop space of  $M$ . Consider a contractible loop  $\gamma$  based at  $g \in \text{Ham}(M)$ . It acts by

$$\begin{aligned} P(\gamma) : \mathcal{L}_\alpha M &\rightarrow \mathcal{L}_\alpha M, \\ z(t) &\mapsto \gamma_t(z(t)) \end{aligned}$$

on the loop space of  $M$  in component  $\alpha$ . Given a Hamiltonian  $H \in \mathcal{H}$  there exists a natural Hamiltonian  $H' = P(\gamma)_* H$  such that

$$P(\gamma)^* \mathcal{A}_{H'} = \mathcal{A}_H.$$

It is given by

$$H'(t, x) = H_t \circ \gamma_t^{-1} + G_t,$$

where  $G_t \in \mathcal{H}$  is the Hamiltonian generating  $\gamma$ . Note that if  $H$  generates the Hamiltonian isotopy  $\{\phi_t\}$  then  $H'$  generates the Hamiltonian isotopy  $\{\gamma_t \phi_t g^{-1}\}$ . It is therefore

clear that  $H'$  is  $\alpha$ -non-degenerate if and only if  $H$  is, and that  $P(\gamma)$  establishes an action-preserving bijection  $\mathcal{P}^\alpha(H) \rightarrow \mathcal{P}^\alpha(H')$ , which moreover extends to an isomorphism of filtered Floer complexes

$$P(\gamma) : (CF(H, J)_\alpha, \mathcal{A}_H) \rightarrow ((CF(H', P(\gamma)_*J)_\alpha), \mathcal{A}_{H'}),$$

where  $(P(\gamma)_*J)_{t,x} = D\gamma_t(\gamma_t^{-1}x)J_{t,\gamma_t^{-1}x}D(\gamma_t^{-1})(x)$ .

*Remark 2.12* If  $\gamma$  is not contractible then

$$P(\gamma)^* \mathcal{A}_{H'} = \mathcal{A}_H - A(\gamma \cdot g^{-1}, \eta_\alpha),$$

where  $A$  is the value discussed in Remark 2.10 and was shown to vanish in our specific setting. Hence in our setting

$$P(\gamma)^* \mathcal{A}_{H'} = \mathcal{A}_H,$$

and  $P(\gamma)$  gives an automorphism of Floer homology in action windows precisely as discussed for the case of contractible loops in  $\text{Ham}(M)$ .

We need the following simple observation on the map  $P(\gamma)$  and continuation maps of almost complex structures.

**Lemma 2.13** *Given regular Floer continuation data  $\vec{J} = \{(H, J_s)\}_{s \in \mathbb{R}}$  with  $J_s \equiv J_0$  for  $s \ll 0$  and  $J_s \equiv J_1$  for  $s \gg 0$ , there is a commutative diagram of filtered Floer complexes*

$$\begin{CD} CF(H, J_0)_\alpha @>P(\gamma)>> CF(H', P(\gamma)_*J_0)_\alpha \\ @V C(\vec{J}) \downarrow VV @VV C(P(\gamma)_*\vec{J}) \downarrow V \\ CF(H, J_1)_\alpha @>P(\gamma)>> CF(H', P(\gamma)_*J_1)_\alpha \end{CD} \tag{9}$$

where  $P(\gamma)_*\vec{J}$  is the Floer continuation data  $\{(H, P(\gamma)_*(J_s))\}_{s \in \mathbb{R}}$ , and  $C(\vec{J}), C(P(\gamma)_*\vec{J})$  are Floer continuation maps.

This lemma is immediate once we change variables in the Floer continuation equation using the diffeomorphism  $P(\gamma)$ , namely there is a bijection between solutions  $\{u(s, t)\}$  of the continuation equation for the operator  $C(\vec{J})$  and solutions  $\{v(s, t)\}$  of the continuation equation for  $C(P(\gamma)_*\vec{J})$  given by  $u(s, t) \mapsto v(s, t) = \gamma_t(u(s, t))$ .

### 3 Loop rotation operators

Consider  $H \in \mathcal{H}$  generating the Hamiltonian diffeomorphism  $\phi_H$ . Take the new Hamiltonian function  $H^{(k)}(t, x) := kH(tk, x)$ . It generates  $\phi_{H^{(k)}} = \phi_H^k$ .

We note that  $\phi_H \phi_{H^{(k)}} \phi_H^{-1} = \phi_{H^{(k)}}$ , hence  $\phi_H$  acts on the Floer homology of  $\phi_{H^{(k)}}$ . Assuming that  $H$  is such that  $H^{(k)}$  is  $\alpha$ -non-degenerate, and denoting  $\tilde{\phi}_H^k$  the class in  $\tilde{\mathcal{G}}$  that  $H^{(k)}$  generates we have the morphism

$$[P_k] := [P(\phi_H)] : HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha \rightarrow HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha \tag{10}$$

of filtered Floer homology understood in the sense of the limit (see Definition 2.6 above).

On the other hand, since  $H^{(k)}(t, x) \equiv H^{(k)}(t + \frac{1}{k}, x)$ , it is easy to see that the loop rotation diffeomorphism

$$R_k : \mathcal{L}_\alpha M \rightarrow \mathcal{L}_\alpha M, \quad z(t) \mapsto z\left(t + \frac{1}{k}\right),$$

satisfies

$$(R_k)^* \mathcal{A}_{H^{(k)}} = \mathcal{A}_{H^{(k)}}. \tag{11}$$

Hence  $R_k$  restricts to an action-preserving bijection  $\mathcal{P}^\alpha(H^{(k)}) \rightarrow \mathcal{P}^\alpha(H^{(k)})$  and therefore for a generic almost complex structure  $J \in \mathcal{J}$ , defines an isomorphism of filtered Floer complexes

$$\left(CF\left(H^{(k)}, J\right)_\alpha, \mathcal{A}_{H^{(k)}}\right) \rightarrow \left(CF\left(H^{(k)}, (R_k)_* J\right)_\alpha, \mathcal{A}_{H^{(k)}}\right), \tag{12}$$

where  $((R_k)_* J)_t = J_{t+\frac{1}{k}}$ . Consider the induced morphism in filtered Floer homology, which is again understood in the sense of the limit:

$$[R_k] : HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha \rightarrow HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha. \tag{13}$$

*Remark 3.1* It is easy to verify that under the isomorphism  $\pi_{(H^{(k)}, J)} : HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha \rightarrow HF^{(a,b)}(H^{(k)}, J)_\alpha$ , the operator  $[R_k]$  takes the form of the composition

$$HF^{(a,b)}(H^{(k)}, J)_\alpha \rightarrow HF^{(a,b)}(H^{(k)}, (R_k)_* J)_\alpha \rightarrow HF^{(a,b)}(H^{(k)}, J)_\alpha$$

where the first map is induced by the chain-level isomorphism  $R_k$  and the second map is a continuation map along continuation datum  $\{(H^{(k)}, J_s)\}_{s \in \mathbb{R}}$  with  $J_s \equiv (R_k)_* J$  for  $s \ll 0$ , and  $J_s \equiv J$  for  $s \gg 0$ . Hence  $\pi_{(H^{(k)}, J)} \circ [R_k] \circ \pi_{(H^{(k)}, J)}^{-1}$  has an explicit chain-level description.

While it is not strictly necessary for the arguments below, it is useful for one’s intuition to know that the maps  $[P_k]$  and  $[R_k]$  on Floer homology in fact coincide. In particular it is easier to see in the  $[R_k]$  description that  $[R_k]^k = \mathbf{1}$ .

**Lemma 3.2** *The maps  $[R_k]$  and  $[P_k]$  coincide, and hence are simply different descriptions of the same map*

$$T_k : HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha \rightarrow HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha$$

of  $r2p$  persistence modules.

*Remark 3.3* It is evident from the definition of  $[\mathbf{R}_k]$  that  $(T_k)^k = \mathbf{1}$ , and hence  $T_k$  defines a  $\mathbb{Z}_k = \mathbb{Z}/(k)$ -representation on  $HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha$ .

*Morse theoretical digression* The proof of Lemma 3.2 rests on the following picture in Morse theory. Consider a Morse function  $f$  on a closed manifold  $X$  and an isotopy  $\{\psi_r\}_{r \in [0,1]}$  such that  $\psi_r^* f = f$  for all  $r \in [0, 1]$ . Then  $\psi = \psi_1$  acts on the Morse homology  $HM^{(a,b)}(f)$  in any window  $(a, b)$ , and moreover it acts by  $\mathbf{1}$ . This can be seen immediately by considering the (relative) singular homology of sublevel sets  $(\{f < b\}, \{f < a\})$  of  $f$ , and constructing a chain homotopy of the map induced by  $\psi$  to  $\mathbf{1}$  by considering the cylinders of the singular cycles traced by the isotopy  $\{\psi_r\}$ .

Let us sketch the Morse homological argument proving the above-mentioned statement. It will be important in the sequel as it readily extends to the Floer theoretical context. Fix a generic Riemannian metric  $\rho$  on  $X$ , and denote  $\rho_r := (\psi_r)_* \rho$ . Observe that the map  $\psi_r$  canonically identifies the filtered Morse complexes  $(CM_*(f, \rho_r), d_r)$  for all  $r$  with  $(W_*, \partial) := (CM_*(f, \rho), d)$ .

Consider the family of metrics  $\rho_{r,s}$ ,  $r \in [0, 1]$ ,  $s \in \mathbb{R}$  such that for  $s \ll -1$   $\rho_{r,s} = \rho_r = (\psi_r)_*(\rho)$  and for  $s \gg 1$   $\rho_{r,s} = \rho$ . Look at the gradient flow equation

$$\frac{du}{ds} = -\nabla_{\rho_{r,s}} f(u(s)), \tag{14}$$

where both  $r$  and  $u$  are considered as variables. Look at the isolated solutions  $(r, u)$  of this equation. In light of the identification above, they define a map  $\mathcal{S} : W_* \rightarrow W_{*+1}$  which does not increase the filtration induced by  $f$ .

With this identification, the action of the diffeomorphism  $\psi$  on the homology of  $W$  is given (on the chain level) by the continuation map  $C$  induced by the path of metrics  $\rho_{1,s}$ . We claim that  $C$  is chain homotopic to the identity, i.e.,

$$C - \mathbf{1} = \partial \mathcal{S} - \mathcal{S} \partial. \tag{15}$$

To see this, let us analyze the space of solutions of (14) connecting critical points with equal Morse indices. Given regularity, it can be compactified to a manifold with boundary of dimension 1. Boundary contributions appear when either  $r = 0$  or  $r = 1$ , or when there is breaking of trajectories. For  $r = 0$  they correspond to  $\mathbf{1}$ , for  $r = 1$ , the solutions satisfy the continuation equation for  $C$ , while the breaking of trajectories gives us  $\partial \mathcal{S} - \mathcal{S} \partial$ . This proves (15) and completes our digression.

*Proof (of Lemma 3.2) Step 1* Take  $r \in [0, 1]$ . Consider two Hamiltonians depending on the parameter  $r$ ,  $E_r = H^{(k)} \circ (\phi_H^r)^{-1}$  and  $F_r = H_{t+r/k}^{(k)}$ . The former generates the Hamiltonian path

$$\alpha_r(t) := \phi_H^r \phi_{H^{(k)}}^t (\phi_H^r)^{-1} = \phi_H^r \phi_H^{kt} (\phi_H^r)^{-1}.$$

The latter Hamiltonian generates the path

$$\beta_r(t) := \phi_{H^{(k)}}^{t+r/k} (\phi_{H^{(k)}}^{r/k})^{-1} = \phi_H^{tk+r} (\phi_H^r)^{-1}.$$

Observe that both paths have the same endpoints,

$$\alpha_r(0) = \beta_r(0) = \mathbf{1}, \quad \alpha_r(1) = \beta_r(1) = \phi_H^r \phi_H^k (\phi_H^r)^{-1},$$

and, as one readily checks, they are homotopic with fixed endpoints. An explicit homotopy can be given by  $\delta_s(t) = \phi_H^r (\phi_H^{sr})^{-1} \phi_H^{kt+sr} (\phi_H^r)^{-1}$ ,  $s \in [0, 1]$ , so that  $\delta_0 = \alpha_r$ , and  $\delta_1 = \beta_r$ . In other words,  $\beta_r = \gamma_r \alpha_r$ , where  $\gamma_r$  is a contractible loop. It follows that  $P(\gamma_r)^* \mathcal{A}_{F_r} = \mathcal{A}_{E_r}$ . Furthermore since the conjugation by  $\phi_H^r$  takes the path  $\phi_{H^{(k)}}^t$  to  $\phi_{E_r}^t$  we have  $P(\phi_H^r)^* \mathcal{A}_{E_r} = \mathcal{A}_{H^{(k)}}$ . We conclude that

$$P(\phi_H^r)^* P(\gamma_r)^* \mathcal{A}_{F_r} = \mathcal{A}_{H^{(k)}}. \tag{16}$$

*Step 2* Next, for  $r \in [0, 1]$ , let  $R_{r,k}$  be the diffeomorphism

$$R_{r,k} : \mathcal{L}_\alpha M \rightarrow \mathcal{L}_\alpha M, \quad z(t) \mapsto z(t + r/k).$$

It satisfies

$$R_{r,k}^* \mathcal{A}_{F_r} = \mathcal{A}_{H^{(k)}}. \tag{17}$$

Put  $Q_r := R_{r,k}^{-1} P(\gamma_r) P(\phi_H^r)$ . Combining (17) with (16) we get that

$$Q_r^* \mathcal{A}_{H^{(k)}} = \mathcal{A}_{H^{(k)}}. \tag{18}$$

It follows that  $Q_r$  is a path of diffeomorphisms of loop spaces preserving  $\mathcal{A}_{H_k}$ . Thus for every  $r$ ,  $Q_r$  induces the same morphism of filtered Floer homologies of  $H^{(k)}$ . The proof of this literally imitates the Morse theoretical argument presented above. The action functional  $\mathcal{A}_{H_k}$  on the loop space  $\mathcal{L}_\alpha M$  plays the role of the Morse function  $f$  on  $X$ , the diffeomorphisms  $Q_r$  correspond to  $\psi_r$  and loops of almost complex structures on  $M$  (which remained behind the scenes in our exposition) determine Riemannian metrics on  $\mathcal{L}_\alpha M$ .

Therefore diffeomorphisms  $Q_r$  induce the same morphism  $V \rightarrow V$ , where we abbreviate  $V := HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha$  and understand this space in the sense of the limit according to Definition 2.6 above. Since  $[Q_0] = \mathbf{1}$ , we have that

$$[Q_1] = [R_{1,k}^{-1}] \circ [P(\gamma_1)] \circ [P(\phi_H)] = \mathbf{1}.$$

It remains to notice that each of the factors  $[R_{1,k}^{-1}]$ ,  $[P(\gamma_1)]$  and  $[P(\phi_H)]$  in the previous equation is an automorphism of  $V$ , and moreover  $[R_{1,k}] = [R_k]$ ,  $[P(\gamma_1)] = \mathbf{1}$  since  $\gamma_1$  is a contractible loop, and  $[P(\phi_H)] = [P_k]$ . Therefore  $[R_k] = [P_k]$  as required.  $\square$

The fact that  $T_k$  is a morphism of  $r2p$  persistence modules in particular implies the following. Consider the natural comparison map

$$j_d := j_{(a,b),(a+d,b+d)} : HF^{(a,b)}(H^{(k)})_\alpha \rightarrow HF^{(a+d,b+d)}(H^{(k)})_\alpha$$



between Floer homology groups in two windows  $(a, b)$  and  $(a + d, b + d)$  for  $d \geq 0$ . Then the following diagram commutes.

$$\begin{array}{ccc}
 HF^{(a,b)}(H^{(k)})_\alpha & \xrightarrow{j_d} & HF^{(a+d,b+d)}(H^{(k)})_\alpha \\
 \text{T}_k \downarrow & & \text{T}_k \downarrow \\
 HF^{(a,b)}(H^{(k)})_\alpha & \xrightarrow{j_d} & HF^{(a+d,b+d)}(H^{(k)})_\alpha
 \end{array} \tag{19}$$

Similarly, the map  $\text{T}_k$  commutes with Floer continuation maps  $HF(F^{(k)})_\alpha \rightarrow HF(G^{(k)})_\alpha$  induced from continuation maps  $HF(F)_\alpha$  to  $HF(G)_\alpha$  as follows: a family  $\{H_r\}_{r \in \mathbb{R}}$  of Hamiltonians with  $H_r \equiv F$  for  $r \ll 0$ , and  $H_r \equiv G$  for  $r \gg 0$  induces the family  $\{H_r^{(k)}\}$  between  $F^{(k)}$ ,  $G^{(k)}$ . We note that an interpolation homotopy between  $F$  and  $G$  induces in this way an interpolation homotopy between  $F^{(k)}$  and  $G^{(k)}$ .

**Lemma 3.4** *Consider an interpolation continuation map*

$$C(F^{(k)}, G^{(k)}) : HF^{(a,b)}(F^{(k)})_\alpha \rightarrow HF^{(a,b)+k \mathcal{E}^+(G-F)}(G^{(k)})_\alpha.$$

Then the following diagram commutes.

$$\begin{array}{ccc}
 HF^{(a,b)}(F^{(k)})_\alpha & \xrightarrow{C(F^{(k)}, G^{(k)})} & HF^{(a,b)+k \mathcal{E}^+(G-F)}(G^{(k)})_\alpha \\
 \text{T}_k \downarrow & & \text{T}_k \downarrow \\
 HF^{(a,b)}(F^{(k)})_\alpha & \xrightarrow{C(F^{(k)}, G^{(k)})} & HF^{(a,b)+k \mathcal{E}^+(G-F)}(G^{(k)})_\alpha
 \end{array} \tag{20}$$

*Proof* Consider an interpolation

$$C_s(t, x) = C(s, t, x) = (1 - \rho(s))F^{(k)}(x, t) + \rho(s)G^{(k)}(x, t)$$

between  $F^{(k)}$  and  $G^{(k)}$ . This interpolation, upon the choice of a generic  $(s, t)$ -dependent compatible almost complex structure with  $J(s, t) \equiv J_-(t)$ , for  $s \ll 0$  and  $J(s, t) \equiv J_+(t)$  for  $s \gg 0$  regular for  $F^{(k)}$  and  $G^{(k)}$ , respectively, by standard action estimates, gives rise to the Floer continuation map

$$HF^{(a,b)}(F^{(k)})_\alpha \xrightarrow{C(F^{(k)}, G^{(k)})} HF^{(a,b)+k \mathcal{E}^+(G-F)}(G^{(k)})_\alpha$$

the matrix coefficient between  $x \in \mathcal{P}^\alpha(F^{(k)})$  and  $y \in \mathcal{P}^\alpha(G^{(k)})$  with  $i(x) = i(y)$  of whose chain-level operator is given by counting solutions to the equation

$$\partial_s u + J(s, t, u)(\partial_t u - X_{C_s}(t, u)) = 0,$$

with asymptotic boundary conditions  $u(s, t) \xrightarrow{s \rightarrow -\infty} x$ ,  $u(s, t) \xrightarrow{s \rightarrow \infty} y$ . Note that the operator on the level of homology does not depend on the choice of  $J(s, t)$ . Similarly to Lemma 2.13, as  $C(s, t + \frac{1}{k}) \equiv C(s, t)$ , the transformation  $u(s, t) \mapsto v(s, t) :=$

$R_k u(s, t) = u(s, t + \frac{1}{k})$  establishes a one-to-one correspondence between the solutions  $u(s, t)$  of the above continuation equation and the solutions  $v(s, t)$  of the equation

$$\partial_s v + J \left( s, t + \frac{1}{k} s, v \right) (\partial_t v - X_{C_s}(t, v)) = 0$$

with asymptotic boundary conditions  $v(s, t) \xrightarrow{s \rightarrow -\infty} R_k(x)$ ,  $v(s, t) \xrightarrow{s \rightarrow \infty} R_k(y)$ . Since the continuation map on the homology level does not depend on the choice of almost complex structures, the lemma now follows immediately.  $\square$

## 4 Invariants

### 4.1 The $\mathbb{Z}_k$ spectral spread

In the spirit of persistent homology (see [13, 28, 55] for surveys and Sect. 4.2 below), a rapidly developing area lying on the borderline of algebraic topology and topological data analysis, we shall use windows where  $T_k$  does not act trivially to separate  $\phi_H$  from autonomous Hamiltonian diffeomorphisms.

For  $k \in \mathbb{Z}$ ,  $k \geq 2$ , and a Hamiltonian  $H \in \mathcal{H}$  we make the following definition. Put

$$S_k := T_k - \mathbf{1}.$$

Composing  $S_k$  with the comparison map  $j_d : HF^{(a,b)}(H^{(k)})_\alpha \rightarrow HF^{(a+d,b+d)}(H^{(k)})_\alpha$ , we obtain a map

$$j_d \circ S_k = S_k \circ j_d : HF^{(a,b)}(H^{(k)})_\alpha \rightarrow HF^{(a+d,b+d)}(H^{(k)})_\alpha.$$

This brings us to the definition of one of the main invariants of this paper.

**Definition 4.1** (The  $\mathbb{Z}_k$  spectral spread)

$$w_{k,\alpha}(H) := \sup \{d \geq 0 : j_d \circ S_k \neq 0 \text{ for some window } (a, b)\}$$

The following proposition shows that this indeed defines an invariant and lists its properties relevant to subsequent discussion.

**Proposition 4.2** *The assignment  $H \mapsto w_{k,\alpha}(H)$  satisfies the following properties:*

- (i) *The invariant  $w_{k,\alpha}(H) \geq 0$  is a finite number. In fact*

$$w_{k,\alpha}(H) \leq k \cdot (\mathcal{E}^+(H) - \mathcal{E}^-(H)).$$

- (ii)  *$w_{k,\alpha}(H)$  depends only on the diffeomorphism  $\phi_H$ , hence we write  $w_{k,\alpha}(\phi_H)$ .*

(iii)  $w_{k,\alpha} : \mathcal{G} \rightarrow [0, \infty)$  is Lipschitz in Hofer's metric on  $\mathcal{G}$ . In terms of Hamiltonians

$$|w_{k,\alpha}(F) - w_{k,\alpha}(G)| \leq k \cdot (\mathcal{E}^+(F - G) - \mathcal{E}^-(F - G)).$$

In particular  $w_{k,\alpha}$  extends to arbitrary Hamiltonian diffeomorphisms.

- (iv)  $w_{k,\alpha}(\phi) = 0$  for every autonomous Hamiltonian diffeomorphism  $\phi$ .
- (v) The invariant  $w_{k,\alpha}$  is equivariant with respect to the action of  $\text{Symp}(M)$  on  $\text{Ham}(M)$  by conjugation and the natural action on  $\pi_0(\mathcal{L})$ . That is given  $\psi \in \text{Symp}(M)$ , and  $\phi \in \text{Ham}(M)$ , we have

$$w_{k,\psi \cdot \alpha}(\psi \phi \psi^{-1}) = w_{k,\alpha}(\phi).$$

- (vi) The invariant  $w_{k,\alpha}$  does not change under stabilization. Given a closed connected symplectically aspherical manifold  $(N, \omega_N)$ , and denoting by  $pt_N \in \pi_0(\mathcal{L}M)$  the class of contractible loops and  $\mathbf{1}_N \in \text{Ham}(N)$  the identity transformation, we have for all  $\phi \in \text{Ham}(M)$

$$w_{k,\alpha \times pt_N}(\phi \times \mathbf{1}_N) = w_{k,\alpha}(\phi),$$

where  $\alpha \times pt_N \in \pi_0(\mathcal{L}(M \times N))$  comes from the natural set isomorphism

$$\pi_0(\mathcal{L}(M \times N)) \cong \pi_0(\mathcal{L}M) \times \pi_0(\mathcal{L}N).$$

The following two statements deal with lower bounds on the invariant  $w_{k,\alpha}$  in certain specific situations.

**Proposition 4.3** *Assume that  $H^{(k)}$  is non-degenerate. If a fixed point  $x$  of  $\phi_{H^{(k)}} = \phi_H^k$  of index  $i$  is a primitive  $k$ -periodic point of  $\phi_H$  (for example if its free homotopy class  $\alpha = \alpha(\phi_H^k, x)$  is primitive), and the actions of all other generators of  $CF(H^{(k)})_\alpha$  of index  $i + 1$  and  $i - 1$  either coincide with the action of  $x$  or differ from it by at least  $D > 0$  (in absolute value), then  $w_{k,\alpha}(\phi_H) \geq D$ .*

Putting  $D$  to be the minimal action gap in the spectrum (which is finite), we immediately obtain the following statement, which holds for an arbitrary symplectically aspherical symplectic manifold  $(M, \omega)$ .

**Corollary 4.4** *If  $\phi^k$  is non-degenerate, and  $\phi$  has a primitive contractible  $k$ -periodic orbit, then  $w_{k,pt_M}(\phi) > 0$ .*

In the remainder of this section, we prove Propositions 4.2 and 4.3.

*Proof* (Proposition 4.2)

Property (i) is an immediate consequence of properties (iii) and (iv).

For (ii) we note that if  $\{\phi_H^t\}$  and  $\{\phi_{H'}^t\}$  have the same endpoint  $\phi$ , then the action  $P(\gamma^{(k)})$  of the  $k$ -iterated loop  $\gamma_t^{(k)} := \gamma_{kt}$  of the difference loop  $\{\gamma_t =$

$\{\phi_{H^t}^t(\phi_H^t)^{-1}\}_{t \in [0,1]}$  on the loop space  $\mathcal{L}_\alpha M$  by Remark 2.10 provides an identification the  $r2p$  persistence modules

$$\left[ P(\gamma^{(k)}) \right] : HF^{(a,b)}(\{\{\phi_{H^{(k)}}^t\}\})_\alpha \rightarrow HF^{(a,b)}(\{\{\phi_{H^{r(k)}}^t\}\})_\alpha.$$

What remains to show is that this identification commutes with  $T_k$ . To this end, note that the diffeomorphism  $R_k : \mathcal{L}_\alpha M \rightarrow \mathcal{L}_\alpha M$  which induces  $T_k$  on the level of  $r2p$  persistence modules commutes with the diffeomorphism  $P(\gamma^{(k)})$ . The proof is now immediate.

*Remark 4.5* Note that in fact, since we only consider shifts of intervals in the definition of  $w_{k,\alpha}$ , for the purposes of this proof, it is not actually necessary to show that the shift  $I(\gamma, \eta)$  (cf. Remark 2.10) of action functionals vanishes.

For (iii) we proceed to show the statement for Hamiltonians as follows. First assume that  $k \mathcal{E}(F - G) < w_{k,\alpha}(G)$ . We will prove

$$|w_{k,\alpha}(G) - w_{k,\alpha}(F)| \leq k \cdot \mathcal{E}(F - G).$$

Put

$$\begin{aligned} u(t) &= \max_M(F(t, \cdot) - G(t, \cdot)), \quad v(t) = \min_M(F(t, \cdot) - G(t, \cdot)), \\ u &= \mathcal{E}^+(F - G) = \int_0^1 u(t) dt, \quad v = \mathcal{E}^-(F - G) = \int_0^1 v(t) dt, \\ \Delta &= w_{k,\alpha}(G) - k \cdot \mathcal{E}(F - G). \end{aligned}$$

Note that  $\Delta > 0$  by assumption. Below we'll use  $u - v = \mathcal{E}(F - G)$  and

$$G(t, \cdot) + u(t) \geq F(t, \cdot) \geq G(t, \cdot) + v(t).$$

By Diagram (19) and Lemma 3.4 we obtain the following commutative diagram, where  $j_\Delta = j_{I+ku, I+ku+\Delta}$ ,  $C_+ = C(G^{(k)}, F^{(k)})$ ,  $C_- = C(F^{(k)}, G^{(k)})$ ,  $w = w_{k,\alpha}(G)$ :

$$\begin{array}{ccccccc} HF^I(G^{(k)})_\alpha & \xrightarrow{C_+} & HF^{I+ku}(F^{(k)})_\alpha & \xrightarrow{j_\Delta} & HF^{I+ku+\Delta}(F^{(k)})_\alpha & \xrightarrow{C_-} & HF^{I+w}(G^{(k)})_\alpha \\ S_k \downarrow & & S_k \downarrow & & S_k \downarrow & & S_k \downarrow \\ HF^I(G^{(k)})_\alpha & \xrightarrow{C_+} & HF^{I+ku}(F^{(k)})_\alpha & \xrightarrow{j_\Delta} & HF^{I+ku+\Delta}(F^{(k)})_\alpha & \xrightarrow{C_-} & HF^{I+w}(G^{(k)})_\alpha \end{array}$$

By functoriality of continuation maps, the composition of the top row equals  $j_w : HF^I(G^{(k)})_\alpha \rightarrow HF^{I+w}(G^{(k)})_\alpha$ . Hence by definition of  $w_{k,\alpha}(G)$ , decreasing the translation  $\Delta > 0$  to  $\Delta - \delta > 0$  by a small  $\delta > 0$  if necessary, the composition  $S_k \circ C(F^{(k)}, G^{(k)}) \circ j_\Delta \circ C(G^{(k)}, F^{(k)})$  of the top row with the rightmost vertical arrow does not vanish. By the commutativity of the diagram, this implies that  $C(F^{(k)}, G^{(k)}) \circ j_\Delta \circ S_k \circ C(G^{(k)}, F^{(k)}) \neq 0$ , and hence

$$0 \neq j_\Delta \circ S_k : HF^{I+ku}(F^{(k)}) \rightarrow HF^{I+ku+\Delta}(F^{(k)}).$$

Consequently  $w_{k,\alpha}(F) \geq \Delta$ . Hence by definition of  $\Delta$ ,  $w_{k,\alpha}(G) - w_{k,\alpha}(F) \leq k \cdot \mathcal{E}(F - G)$ .

If  $w_{k,\alpha}(G) \geq w_{k,\alpha}(F)$ , this proves the statement. If not, then  $w_{k,\alpha}(F) \geq w_{k,\alpha}(G) > k \mathcal{E}(F - G)$ . Hence by the above argument having  $F$  and  $G$  switch places, we get  $w_{k,\alpha}(F) - w_{k,\alpha}(G) \leq k \cdot \mathcal{E}(F - G)$ .

Hence under the assumption that  $k \mathcal{E}(F - G) \leq w_{k,\alpha}(G)$ , we have

$$|w_{k,\alpha}(G) - w_{k,\alpha}(F)| \leq k \cdot \mathcal{E}(F - G).$$

It remains to remove the assumption from the statement. We argue as follows. By the above statement, if either  $w_{k,\alpha}(G) > k \cdot \mathcal{E}(F - G)$  or  $w_{k,\alpha}(F) > k \cdot \mathcal{E}(F - G)$  then  $|w_{k,\alpha}(F) - w_{k,\alpha}(G)| \leq k \cdot \mathcal{E}(F - G)$ .

In the remaining case, we have both  $w_{k,\alpha}(F) \leq k \cdot \mathcal{E}(F - G)$  and  $w_{k,\alpha}(G) \leq k \cdot \mathcal{E}(F - G)$ . Assume without loss of generality that  $w_{k,\alpha}(F) \geq w_{k,\alpha}(G)$ . Then since  $w_{k,\alpha}(G) \geq 0$ ,

$$0 \leq w_{k,\alpha}(F) - w_{k,\alpha}(G) \leq w_{k,\alpha}(F) \leq k \cdot \mathcal{E}(F - G).$$

So the required inequality holds.

For (iv) consider an autonomous Hamiltonian diffeomorphism  $\phi$  generated by an autonomous Hamiltonian  $h \in C^\infty(M, \mathbb{R})_0$ . Consider a non-degenerate Hofer  $\epsilon$ -small perturbation of  $\phi$  generated by time-periodic Hamiltonian  $H \in \mathcal{H}$  with  $\mathcal{E}(H - h) < \epsilon$ . We use formula (13) for the loop rotation operator. Note that the family of Hamiltonians  $\{H_r^{(k)}\}_{r \in [0,1]}$  satisfies  $\mathcal{E}(H_r^{(k)} - kh) < k \cdot \epsilon$ , for all  $r \in [0, 1]$ . Since  $(\mathbb{R}_{r,k}^{-1})^* \mathcal{A}_{H^{(k)}} = \mathcal{A}_{H_r^{(k)}}$ , this means that for all  $x \in \mathcal{L}_\alpha M$ ,

$$\left(\mathbb{R}_{r,k}^{-1}\right)^* \mathcal{A}_{H^{(k)}} - \mathcal{A}_{H^{(k)}} < 2k \cdot \epsilon.$$

*Remark 4.6* At this point, a short intuitive remark is in order. Given a Morse function on a closed manifold  $X$ , and an isotopy  $\{\psi_r\}_{r \in [0,1]}$  with endpoint  $\psi_1 = \psi$  of  $X$ , such that  $(\psi_r)^* f - f < \delta$  for all  $r \in [0, 1]$  and some  $\delta > 0$  we can see that for a relative singular cycle  $c$  in the singular chain complex of the pair  $(\{f < b\}, \{f \leq a\})$  the image of the cycle  $\psi_*(c) - c$  in the singular chain complex of the pair  $(\{f < b + \delta\}, \{f \leq a + \delta\})$  is a boundary, by considering the trace of  $c$  under the isotopy  $\{\psi_r\}$ .

We add that while in this proof we perturb to achieve regularity, we may have considered the  $r2p$  persistence module of the degenerate Hamiltonian  $h$  itself, and then relied on the above intuitive picture with  $\delta = 0$ .

We construct a null-homotopy  $\mathcal{N} : CF(H^{(k)}, J)_\alpha \rightarrow CF(H^{(k)}, J)[1]_\alpha$  of

$$S'_k = C_1 \circ R_k - C_0$$

on  $CF(H^{(k)}, J)_\alpha$ , for a generic almost complex structure  $J \in \mathcal{J}$ , where

$$C_0 : CF(H^{(k)}, J)_\alpha \rightarrow CF(H^{(k)}, J)_\alpha$$

is a self-continuation map and

$$C_1 : CF(H^{(k)}, (R_k)_* J)_\alpha \rightarrow CF(H^{(k)}, J)_\alpha$$

is a continuation map. Note that  $S'_k$  induces  $S_k$  on homology, understood via the isomorphism  $\pi_{(H^{(k)}, J)} : HF^{(a,b)}(\tilde{\phi}_H^k)_\alpha \rightarrow HF^{(a,b)}(H^{(k)}, J)_\alpha$  (see Remark 3.1). The matrix element  $\langle \mathcal{N}(x), y \rangle$  between two generators  $x, y$  of indices  $i, i + 1$  is defined by counting solutions to the following parametrized Floer equation, with the boundary conditions  $\{y_r = y\}_{r \in [0,1]}$  and  $\{x_r = R_{r,k}(x) \in \mathcal{P}^\alpha(H_r^{(k)})\}_{r \in [0,1]}$  corresponding to  $x$  and  $y$ .

For  $r \in [0, 1]$ , choose an interpolation  $\widehat{H}_s^{(k)}, s \in \mathbb{R}$  between the function  $\widehat{H}_0^{(k)}(r, t, x) = H^{(k)}(t, x)$  and the function  $\widehat{H}_1^{(k)}(s, t, x) = H_r^{(k)}(t, x)$ . For a fixed  $r, s$  we obtain a Hamiltonian  $\widehat{H}_{r,s}^{(k)} \in \mathcal{H}$ .

Choose a generic family of almost complex structures

$$\{J(r, s) \in \mathcal{J}\}_{(r,s) \in [0,1] \times \mathbb{R}}$$

such that  $J(r, s, t) \equiv (R_{r,k})_* J(t)$  for  $s \ll 0$  and  $J(r, s, t) \equiv J(t)$  for  $s \gg 0$ . Then we count solutions  $(r, u_r)$  of the parametric Floer equation

$$\partial_s u_r + J(r, s, t, u_r) \left( \partial_t u_r - X_{\widehat{H}_{r,s}^{(k)}}(t, u_r) \right) = 0,$$

with boundary conditions  $u_r(s, t) \xrightarrow{s \rightarrow -\infty} x_r$  and  $u_r(s, t) \xrightarrow{s \rightarrow +\infty} y_r$ .

Considering the breaking of solutions in one-parameter families, we see that  $\mathcal{N}$  is a null-homotopy of  $S'_k$ , that is

$$S'_k = C_1 \circ R_k - C_0 = \partial \mathcal{N} - \mathcal{N} \partial.$$

We analyze the effect of this operator on the action filtration. The usual action estimates in Floer theory show that this operator does not raise the action filtration more than

$$\max_{r \in [0,1]} \mathcal{E} \left( H^{(k)} - H_r^{(k)} \right) \leq 2k \cdot \epsilon,$$

since  $0 \leq \mathcal{E}(H^{(k)} - H_r^{(k)}) = \mathcal{E}(H^{(k)} - k \cdot h + k \cdot h - H_r^{(k)}) \leq \mathcal{E}(H^{(k)} - k \cdot h) + \mathcal{E}(k \cdot h - H_r^{(k)}) \leq k \cdot \epsilon + k \cdot \epsilon$ .

Hence for the perturbation  $H \in \mathcal{H}$  of  $h$  we have  $w_{k,\alpha}(H) \leq 2k \cdot \epsilon$ . By property (iii), taking  $\epsilon \rightarrow 0$ , we obtain  $w_{k,\alpha}(\phi) = 0$ .

For (v) consider the diffeomorphism of loop spaces

$$\begin{aligned} P(\psi) : \mathcal{L}_\alpha M &\rightarrow \mathcal{L}_{\psi \cdot \alpha} M, \\ z(t) &\mapsto \psi(z(t)). \end{aligned}$$

This diffeomorphism satisfies

$$P(\psi)^* \mathcal{A}_{H^{(k)} \circ \psi^{-1}} = \mathcal{A}_{H^{(k)}} - I(\psi, \alpha),$$

for a constant  $I(\psi, \alpha)$  equal to the (well-defined) symplectic area of a cylinder between  $\eta_{\psi \cdot \alpha}$  and  $\psi \circ \eta_\alpha$  (recall that  $\eta_\alpha \in \mathcal{L}_\alpha M$  is a fixed reference loop). This means that  $P(\psi)$  defines an isomorphism

$$[P(\psi)] : HF^{(a,b)}(\phi^k)_\alpha \rightarrow HF^{(a,b)+I(\psi,\alpha)}(\psi \phi^k \psi^{-1})_{\psi \cdot \alpha}$$

between the  $r2p$  persistence module of  $\phi^k$  in class  $\alpha$ , and the  $r2p$  persistence module of  $\psi \phi^k \psi^{-1}$  in class  $\psi \cdot \alpha$  shifted by  $I(\psi, \alpha)$ . Since the invariant is independent on shifts of  $r2p$  persistence modules, it remains to see that  $[P(\psi)]$  commutes with  $T_k$ , which immediately follows from the fact that the diffeomorphisms  $P(\psi) : \mathcal{L}M \rightarrow \mathcal{L}M$ , covering  $\alpha \rightarrow \psi \cdot \alpha$  on  $\pi_0$  and  $R_k : \mathcal{L}M \rightarrow \mathcal{L}M$  covering  $\mathbf{1}$  on  $\pi_0$ , commute.

For (vi) we use the following easy lemma.

**Lemma 4.7** *Let  $I_1, I_2, I_3, I_4$ , with  $I_q = (a_q, b_q)$ ,  $1 \leq q \leq 4$  be intervals such that  $a_1 \leq a_2 \leq a_3 \leq a_4$  and  $b_1 \leq b_2 \leq b_3 \leq b_4$ . If  $j_{(a_1, b_1), (a_4, b_4)} \circ S_k \neq 0$  then  $j_{(a_2, b_2), (a_3, b_3)} \circ S_k \neq 0$ .*

*Proof* The lemma follows immediately from the identity

$$j_{I_1, I_4} \circ S_k = j_{I_3, I_4} \circ j_{I_2, I_3} \circ j_{I_1, I_2} \circ S_k = j_{I_3, I_4} \circ (j_{I_2, I_3} \circ S_k) \circ j_{I_1, I_2}.$$

□

Put  $w = w_{k,\alpha}(\phi)$ , and  $w' = w_{k,\alpha \times pt_N}(\phi \times \mathbf{1}_N)$ . We note that it is enough to show that

1. if  $w > 0$  then  $w' \geq w$ , and
2. if  $w' > 0$  then  $w \geq w'$ .

Indeed, this would imply equality even in the case when one of the values  $w, w'$  vanishes.

In the proof of both directions, we note that by property (iii) we can assume that  $\phi^k$  is generated by a non-degenerate Hamiltonian  $H^{(k)} \in \mathcal{H}$  for the class  $\alpha$ , with minimal gap  $\epsilon > 0$  in the spectrum  $\text{Spec}_{k,\alpha}(H) := \mathcal{A}_{H^{(k)}}(\mathcal{P}^\alpha(H^{(k)}))$ .

Then we replace  $\mathbf{1}_N$  by the time-one map  $\phi_f^1$  of a Morse function  $f$ , considered as an autonomous Hamiltonian, such that  $kf$  is  $C^2$ -small for the purposes of Floer theory, and satisfies the estimate  $|kf|_{C^0} < \delta < \frac{1}{10}\epsilon$ . Denote by  $\text{Spec}_{k,\alpha}(H) + (-\delta, \delta)$  the  $\delta$ -neighborhood of  $\text{Spec}_{k,\alpha}(H)$  in  $\mathbb{R}$ .

By the Kunneth theorem in Hamiltonian Floer homology combined with the calculation of Hamiltonian Floer homology for  $C^2$ -small Morse Hamiltonians, and the observation that it is clearly natural with respect to the operator  $T_k$ , we see that for each window  $(a, b)$  with

$$a, b \notin \text{Spec}_{k,\alpha}(H) + (-\delta, \delta)$$

we have the isomorphism

$$HF^{(a,b)}(H^{(k)} + kf)_{\alpha \times p t_N} \cong HF^{(a,b)}\left(H^{(k)}\right)_{\alpha} \otimes HM(kf), \tag{21}$$

where  $HM(kf)$  denotes the Morse homology of  $kf$ , which is isomorphic to the singular homology of  $N$ , and the operator  $T_k$  acts as

$$T_k^{H^{(k)}+kf} \cong T_k^{H^{(k)}} \otimes \mathbf{1}_{HM(kf)},$$

with respect to the isomorphism (21) above, where for a normalized 1-periodic Hamiltonian  $G$ ,

$$T_k^{G^{(k)}} : HF^{(a,b)}\left(G^{(k)}\right)_{\alpha} \rightarrow HF^{(a,b)}\left(G^{(k)}\right)_{\alpha}$$

is the loop rotation operator.

To prove direction 1 assume in addition that  $\delta < \frac{1}{10}w$ , and choose, by definition of  $w$ , a number  $d > w - \delta$  and a window  $(a, b)$  such that

$$J_{(a,b),(a+d,b+d)}^{H^{(k)}} \circ S_k^{H^{(k)}} \neq 0.$$

By having chosen  $\delta$  sufficiently small, there exist windows  $(a_1, b_1)$  and  $(a_4, b_4)$  such that

$$\begin{aligned} a &\leq a_1 < a + 2\delta, \\ b &\leq b_1 < b + 2\delta, \\ a + d - 2\delta &< a_4 \leq a + d, \\ b + d - 2\delta &< b_4 \leq b + d, \end{aligned}$$

and

$$a_1, b_1, a_4, b_4 \notin \text{Spec}_{k,\alpha}(H) + (-\delta, \delta).$$

In particular we have

$$J_{(a_1,b_1),(a_4,b_4)}^{H^{(k)}} \circ S_k^{H^{(k)}} \neq 0$$



by Lemma 4.7. Moreover since  $a_1, b_1, a_4, b_4 \notin \text{Spec}_{k,\alpha}(H) + (-\delta, \delta)$ , we conclude that

$$j_{(a_1, b_1), (a_4, b_4)}^{H^{(k)}+kf} \circ S_k^{H^{(k)}+kf} = \left( j_{(a_1, b_1), (a_4, b_4)}^{H^{(k)}} \circ S_k^{H^{(k)}} \right) \otimes \mathbf{1}_{HM(kf)} \neq 0.$$

It is easy to see that

$$\begin{aligned} a_3 &:= a_1 + d - 4\delta, \\ b_3 &:= b_1 + d - 4\delta \end{aligned}$$

satisfy

$$\begin{aligned} a_1 &< a_3 < a_4 \\ b_1 &< b_3 < b_4, \end{aligned}$$

and hence applying Lemma 4.7 with

$$(a_2, b_2) = (a_1, b_1)$$

we have

$$j_{(a_2, b_2), (a_3, b_3)}^{H^{(k)}+kf} \circ S_k^{H^{(k)}+kf} \neq 0$$

and hence

$$w_{k,\alpha \times pt_N}(\phi \times \phi_f^1) \geq w_{k,\alpha}(\phi) - 4\delta.$$

Since by property (iii)

$$w_{k,\alpha \times pt_N}(\phi \times \mathbf{1}_N) \geq w_{k,\alpha \times pt_N}(\phi \times \phi_f^1) - 2\delta,$$

we obtain

$$w_{k,\alpha \times pt_N}(\phi \times \mathbf{1}_N) \geq w_{k,\alpha}(\phi) - 6\delta.$$

Since this inequality holds for all sufficiently small  $\delta$ , this finishes the proof of direction 1. The proof of direction 2 is very similar to that of direction 1 and hence we omit its details.

This finishes the proof of the proposition. □

*Proof* (Proposition 4.3)

Consider the generator  $x$  of index  $i$  of the Floer complex  $CF(H^{(k)})_\alpha$ . Put  $\mathcal{A}$  for its critical value. By assumption on the action difference, we see that  $x$  defines a non-trivial cohomology class  $[x]$  in  $HF^{(\mathcal{A}-Q, \mathcal{A}+R)}(H^{(k)}, J)_\alpha$ , where  $0 < Q, R < D$ . So

do the generators  $\phi_H^j(x)$  for  $0 \leq j \leq k-1$ , which are all different since  $x$  is primitive. Moreover these classes persist under comparison maps  $HF^{(\mathcal{A}-Q, \mathcal{A}+R)}(H^{(k)}, J)_\alpha \rightarrow HF^{(\mathcal{A}-Q', \mathcal{A}+R')}(H^{(k)}, J)_\alpha$ ,  $0 < Q, Q', R, R' < D$ ,  $Q \geq Q', R \leq R'$  between Floer homology groups of different windows of this type. Note that for all  $\epsilon > 0$  sufficiently small, the action of all orbits of index  $i$  in the interval  $(\mathcal{A} - \epsilon, \mathcal{A} + \epsilon)$  is precisely  $\mathcal{A}$ . Recall the isomorphism  $\pi_{(H^{(k)}, J)} : HF^{(a,b)}(\phi^k)_\alpha = HF^{(a,b)}(\tilde{\phi}^k)_\alpha \rightarrow HF^{(a,b)}(H^{(k)}, J)_\alpha$  of  $r2p$  persistence modules from Remark 2.7. It is sufficient to prove that  $\pi_{(H^{(k)}, J)} \circ S_k \circ \pi_{(H^{(k)}, J)}^{-1}$  is non-zero on the image  $W$  of  $j_{D-2\epsilon} : HF^{(\mathcal{A}-D+\epsilon, \mathcal{A}+\epsilon)}(H^{(k)}, J)_\alpha \rightarrow HF^{(\mathcal{A}-\epsilon, \mathcal{A}+D-\epsilon)}(H^{(k)}, J)_\alpha$ . Indeed, in this case  $w_{k,\alpha}(H) \geq D - 2\epsilon$ , for each  $\epsilon > 0$  sufficiently small, and therefore  $w_{k,\alpha}(H) \geq D$ .

Look at the continuation map

$$C : HF^{(\mathcal{A}-\epsilon, \mathcal{A}+D-\epsilon)}(H^{(k)}, J')_\alpha \rightarrow HF^{(\mathcal{A}-\epsilon, \mathcal{A}+D-\epsilon)}(H^{(k)}, J)_\alpha$$

along regular Floer continuation data  $\{(H^{(k)}, J_s)\}_{s \in \mathbb{R}}$ , where  $J_s \equiv J'$  for  $s \ll 0$ , and  $J_s \equiv J$  for  $s \gg 0$ . This map takes the class  $[\phi_H^j(x)]$  to  $[\phi_H^j(x)] + y$ , where  $y$  is a linear combination of the fixed points of  $\phi_H^j$  in the class  $\alpha$  whose indices coincide with the index of  $x$  and whose actions lie in the interval  $(\mathcal{A} - \epsilon, \mathcal{A})$ . Observe that since  $\epsilon$  is small enough, no such fixed points exist. Therefore  $y = 0$  and it follows that

$$C \left( \left[ \phi_H^j(x) \right] \right) = \left[ \phi_H^j(x) \right]. \tag{22}$$

Relation (22) combined with Remark 3.1 shows that

$$\pi_{(H^{(k)}, J)} \circ T_k \circ \pi_{(H^{(k)}, J)}^{-1}$$

considered as an endomorphism of  $HF^{(\mathcal{A}-\epsilon, \mathcal{A}+D-\epsilon)}(H^{(k)}, J)_\alpha$  sends  $[\phi_H^j(x)]$  to  $[\phi_H^{j+1}(x)]$  (note that  $\phi_H^k(x) = x$ ). Consider the  $k$ -dimensional subspace  $V$  of  $HF^{(\mathcal{A}-\epsilon, \mathcal{A}+D-\epsilon)}(H^{(k)}, J)_\alpha$  spanned by the classes

$$[x], [\phi_H(x)], \dots, [\phi_H^{k-1}(x)].$$

It follows that  $\pi_{(H^{(k)}, J)} \circ S_k \circ \pi_{(H^{(k)}, J)}^{-1}|_V$  is non-zero. This finishes the proof.  $\square$

### 4.2 A primer on persistence modules

We start by noting that there is a weaker version  $\widehat{w}_{k,\alpha} \leq w_{k,\alpha}$  of the  $\mathbb{Z}_k$  spectral spread where all intervals are unbounded from below—that is when  $a = -\infty$  everywhere in the definition. This version still satisfies Properties (i)–(vi) of Proposition 4.2. Moreover it can be reformulated in the language of *barcodes of one-parametric persistence modules* (see Sect. 8.2) which captures additional information about the  $\mathbb{Z}_k$ -action of  $T_k$  on filtered Floer homology, which we subsequently use to prove Theorem 1.3. In

this section, we collect preliminaries on persistence modules, see [6, 13, 14, 16, 18, 28]. Let us mention that by default, a persistence module is assumed to be one-parametric, and that we work in the simplest setting suitable for our purposes.

### 4.2.1 Persistence modules

Let  $\mathcal{K}$  be a field. A (one-parametric) *persistence module* is given by a pair  $(V, \pi)$ , where

- (i)  $V_t, t \in \mathbb{R}$  are finite-dimensional  $\mathcal{K}$ -vector spaces;
- (ii) (*compact support*)  $V_s = 0$  for all  $|s|$  sufficiently large;
- (iii) (*persistence*)  $\pi_{st} : V_s \rightarrow V_t, s < t$  are morphisms satisfying  $\pi_{sr} = \pi_{tr} \circ \pi_{st}$  for all  $s, t, r \in \mathbb{R}$ .
- (iv) (*semicontinuity*) For every  $r \in \mathbb{R}$  there exists  $\epsilon > 0$  such that  $\pi_{st}$  is an isomorphism for all  $r - \epsilon < s < t \leq r$ ;
- (v) (*finite spectrum*) For all but a finite number of points  $r \in \mathbb{R}$ , there exists a neighborhood  $U$  of  $r$  such that  $\pi_{st}$  is an isomorphism for all  $s < t$  with  $s, t \in U$ . The exceptional points form *the spectrum* of the persistence module.

For the sake of brevity, we often denote the persistence module  $(V, \pi)$  or simply  $V$  and write  $\mathcal{S}(V)$  for its spectrum.

It is instructive to mention that for every pair  $a < b$  of consecutive points of the spectrum, the morphism  $\pi_{st}$  is an isomorphism for all  $s, t$  with  $a < s < t \leq b$ . This readily follows from axioms (iii) and (iv) and the compactness of  $[s, t]$ .

*Remark 4.8* While the above is the one we use to prove Theorem 1.3, it in fact makes sense to extend the compact support axiom by requiring only that  $V^s = 0$  for  $s \ll 0$  and that  $\pi_{s,t}$  are isomorphisms for all  $s, t$  sufficiently large.

*Example 4.9* Filtered Floer homology of a symplectically  $\alpha$ -atoroidal symplectic manifold for a non-degenerate Hamiltonian diffeomorphism  $\phi$  as defined in Sect. 2 gives an example of a persistence module. We refer to it as the *Floer persistence module*. More specifically put  $V_*(\phi)^a = HF_*^{(-\infty, a)}(\phi)_\alpha$  and  $\pi_{s,t} = j_{t-s}$  the level comparison maps. When  $\alpha$  is not the contractible class, this is a persistence module as defined above. If  $\alpha = pt_M$ , then it is a persistence module in the sense of Remark 4.8.

### 4.2.2 Morphisms

A morphism  $A$  between persistence modules  $(V, \pi)$  and  $(V', \pi')$  is a family of linear maps  $A_t : V_t \rightarrow V'_t$  which respect the persistence morphisms:

$$A_t \pi_{st} = \pi'_{st} A_s$$

for all  $s < t$ .

Persistence modules and their morphisms form a category. Thus we can speak about isomorphic persistence modules. One readily checks that isomorphic modules have the same spectra.

*Example 4.10* The continuation map  $HF^{(-\infty, a)}(F)_\alpha \rightarrow HF^{(-\infty, a + \mathcal{E}^+(G - F))}(G)_\alpha$  between Floer homologies of two non-degenerate Hamiltonians  $F, G$  gives a morphism between the corresponding persistence modules. Note that the second persistence module is shifted by  $\mathcal{E}^+(G - F)$ .

### 4.2.3 The structure theorem

Let us formulate the main structure theorem for (compactly supported semi-continuous) persistence modules, see e.g., [18].<sup>14</sup>

Given two persistence modules  $(V, \pi)$  and  $(V', \pi')$ , we define their direct sum as

$$(\{V_t \oplus V'_t\}, \{\pi_{st} \oplus \pi'_{st}\}).$$

Let  $I = (a, b]$  be an interval, where  $a, b \in \mathbb{R}$ . Introduce a persistence module  $Q(I) = (\{Q(I)_t\}, \theta)$  given by  $(Q(I))_t = \mathcal{K}$  for  $t \in I$  and  $(Q(I))_t = 0$  otherwise, while the morphisms  $\theta$  are the identity maps within  $I$  and zeroes otherwise.

**Theorem 4.11** (The structure theorem for persistence modules) *For every persistence module  $V$  there exists a unique collection of pair-wise distinct intervals  $I_j = (a_j, b_j]$ ,  $a_j, b_j \in \mathcal{S}(V)$ ,  $j = 1, \dots, N$  and multiplicities  $m_1, \dots, m_N \in \mathbb{N}$  so that*

$$V \cong \bigoplus_{j=1}^N (Q(I_j))^{m_j}, \tag{23}$$

where

$$(Q(I_j))^{m_j} = Q(I_j) \oplus \dots \oplus Q(I_j)$$

$m_j$  times.

The collection of intervals  $\{I_j, m_j\}$  with multiplicities appearing in the normal form of the persistence module  $V$  is called the barcode of  $V$  and is denoted by  $\mathcal{B}(V)$ .

### 4.2.4 Interleaving distance

For a persistence module  $V = (\{V_t\}, \{\pi_{st}\})$  and  $\delta \in \mathbb{R}$  denote by  $V^\delta$  the shifted module

$$V^\delta = (\{V_{t+\delta}\}, \{\pi_{s+\delta, t+\delta}\}).$$

Note that  $\mathcal{S}(V^\delta) = \mathcal{S}(V) - \delta$ . Observe that for  $\delta \in \mathbb{R}_{>0}$ , the persistence modules  $V$  and  $V^\delta$  are related by a canonical shift morphism  $\phi_V(\delta)$  given by  $\pi_{t, t+\delta} : V_t \rightarrow V_{t+\delta}$ .

For a morphism  $f : V \rightarrow W$  write  $f(\delta)$  for the induced morphism  $V^\delta \rightarrow W^\delta$ .

We say that two persistence modules  $V$  and  $W$  are  $\delta$ -interleaved ( $\delta > 0$ ) if there exists morphisms  $f : V \rightarrow W^\delta$  and  $g : W \rightarrow V^\delta$  so that

$$f(\delta)g = \phi_W(2\delta), \quad g(\delta)f = \phi_V(2\delta).$$

---

<sup>14</sup> It would be interesting to compare this result with the work of Barannikov [5,35,36].

The interleaving distance  $d_{\text{inter}}$  between two isomorphism classes of persistence modules equals the infimum of  $\delta$  such that these modules are  $\delta$ -interleaved.

Observe that since any  $V$  is compactly supported,  $\phi_V(\delta) = 0$  for all sufficiently large  $\delta$ , so the interleaving distance between any two modules is finite: take such a  $\delta$  and  $f = g = 0$ .

#### 4.2.5 Bottleneck distance between barcodes

Recall that a *matching* between finite sets  $X$  and  $Y$  is a bijection  $\mu : X' \rightarrow Y'$ , where  $X' \subset X$  and  $Y' \subset Y$ . We denote  $X' = \text{coim}(\mu)$  and  $Y' = \text{im}(\mu)$ .

Given a barcode  $\mathcal{B} = \{I_i, m_i\}$ ,  $i = 1, \dots, M$ , consider a set

$$\langle \mathcal{B} \rangle := \left\{ I_i^{(k_i)}, k_i = 1, \dots, m_i, i = 1, \dots, M \right\}.$$

Let  $\langle \mathcal{B} \rangle_\epsilon \subset \langle \mathcal{B} \rangle$  be a subset consisting of all intervals of length  $> \epsilon$ .

For an interval  $I = (a, b]$  put  $I^{-\delta} = (a - \delta, b + \delta]$ .

For  $\delta > 0$ , define a  $\delta$ -matching between barcodes  $\mathcal{B}$  and  $\mathcal{C}$  as a matching  $\mu$  between  $\langle \mathcal{B} \rangle$  and  $\langle \mathcal{C} \rangle$  such that

- $\langle \mathcal{B} \rangle_{2\delta} \subset \text{coim}(\mu)$ ;
- $\langle \mathcal{C} \rangle_{2\delta} \subset \text{im}(\mu)$ ;
- $\mu(I) = J \implies I \subset J^{-\delta}, J \subset I^{-\delta}$ .

The bottleneck distance  $d_{\text{bottle}}(\mathcal{B}, \mathcal{C})$  is defined as the infimum of  $\delta$  such that the barcodes  $\mathcal{B}$  and  $\mathcal{C}$  admit a  $\delta$ -matching.

#### 4.2.6 Isometry theorem

The interleaving distance between two persistence modules and the bottleneck distance between their barcodes are related by the Isometry Theorem—cf. Bauer and Lesnick [6] and references therein.

**Theorem 4.12** (The Isometry Theorem for persistence modules and barcodes)  
 $d_{\text{inter}}(V, W) = d_{\text{bottle}}(\mathcal{B}(V), \mathcal{B}(W))$ .

#### 4.2.7 Multiplicity function

Let  $\mathcal{B}$  be a barcode (recall that in this section all barcodes consist of finite intervals only since they correspond to compactly supported persistence modules).

For an interval  $I \subset \mathbb{R}$  denote the by  $m(\mathcal{B}, I)$  the number of bars in  $\mathcal{B}$  (with multiplicities!) containing  $I$ . For an interval  $I = (a, b]$  of length  $> 2c$  put  $I^c = (a + c, b - c]$ .

**Proposition 4.13** Assume that  $d_{\text{bottle}}(\mathcal{B}, \mathcal{C}) < c$  and for an interval  $I$  of length  $> 4c$

$$m(\mathcal{B}, I) = m(\mathcal{B}, I^{2c}) = l. \tag{24}$$

Then  $m(\mathcal{C}, I^c) = l$ .

*Proof* First, (24) yields that if a bar  $E$  of  $\mathcal{B}$  contains  $I^{2c}$ , it necessarily contains  $I$ .

Next, by definition of the bottleneck distance, there exists a  $\delta$ -matching  $\nu$  between  $\mathcal{B}$  and  $\mathcal{C}$  with  $\delta < c$ . Note that the set of bars of  $\mathcal{B}$  containing  $I$  lies in  $\text{coim}(\nu)$  and the set of bars of  $\mathcal{C}$  containing  $I^c$  lies in  $\text{im}(\nu)$ . We claim that  $\nu$  establishes a bijection between these two sets.

Let  $E \in \langle \mathcal{B} \rangle$  be a bar containing  $I$ . Then  $E \subset \nu(E)^{-\delta}$ , implying

$$I^c \subset E^\delta \subset \nu(E).$$

In the other direction, let  $J \in \langle \mathcal{C} \rangle$  be a bar containing  $I^c$ . Then  $J = \nu(E)$  for some  $E \in \langle \mathcal{B} \rangle$  and by the same argument as before

$$I^{2c} \subset J^\delta \subset E.$$

Hence by our first observation  $E$  contains  $I$ . This finishes the proof. □

### 4.3 Persistence modules with a $\mathbb{Z}_k$ -action

The main result of the present section, Theorem 4.22 below, is an algebraic counterpart of the geometric “distance to  $k$ -th powers” problem appearing in Theorem 1.3.

Let  $(V, \pi)$  be a  $\mathbb{Z}_k$  persistence module, which is a persistence module equipped with a  $\mathbb{Z}_k$ -action, given by an automorphism  $A : V \rightarrow V$  of persistence modules (this means that  $A$  is a morphism  $V \rightarrow V$  with  $A^k = \mathbf{1}$ ). We denote this data by  $(V, \pi; A)$ . For simplicity of exposition we assume that  $k = p$  is a prime, and fix it.

**Assumption** We shall assume that the ground field  $\mathcal{K}$  has characteristic  $\neq p$ , contains all  $p$ -th roots of unity,<sup>15</sup> and fixing a primitive  $p$ -th root of unity  $\zeta_p$ , the equation  $x^p - (\zeta_p)^q = 0$ , for any integer  $q$  not divisible by  $p$ , has no solutions in  $\mathcal{K}$ . The following lemma gives an example of a field satisfying this condition.

**Lemma 4.14** *Let  $p \geq 2$  be a prime number, and let  $\mathbb{Q}_p$  be the cyclotomic field obtained from  $\mathbb{Q}$  by adjoining a primitive  $p$ -th root of unity  $\zeta_p$ . Then the equation  $x^p = \zeta_p^q$ , where  $q \in \mathbb{Z}$  with  $\text{gcd}(p, q) = 1$ , has no solution  $x$  in  $\mathbb{Q}_p$ .*

*Proof* Assume on the contrary that  $x$  is a solution. Choose a primitive  $p^2$ -th root of unity  $\zeta_{p^2}$  such that  $\zeta_{p^2}^p = \zeta_p$ . In the algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$ , for some  $\tau \in \mathbb{N}$ ,  $x = \zeta_{p^2}^q \cdot (\zeta_p)^\tau \in \mathbb{Q}_p$ , thus  $\zeta_{p^2}^q$  lies in  $\mathbb{Q}_p$ . Also,  $\zeta_{p^2}^p = \zeta_p$  lies in  $\mathbb{Q}_p$ . Taking  $u, v \in \mathbb{Z}$  with  $uq + vp = 1$ , we get that  $\zeta_{p^2} = (\zeta_{p^2}^q)^u \cdot (\zeta_{p^2}^p)^v \in \mathbb{Q}_p$ . Thus  $\mathbb{Q}_{p^2} = \mathbb{Q}_p$ , contradicting basic theory of cyclotomic fields (cf. [39, Corollary 7.8]). □

*Remark 4.15* For  $p = 2$  a  $\mathbb{Z}_p$  persistence module is a persistence module  $V$  with involution  $A$ , the primitive root of unity is  $\zeta_2 = -1$ , the condition on  $\mathcal{K}$  is that  $\text{char}(\mathcal{K}) \neq 2$  and  $x^2 + 1 = 0$  has no solution in  $\mathcal{K}$ , and an example of such a field  $\mathcal{K}$  is simply  $\mathbb{Q}_2 = \mathbb{Q}$ . Another example is  $\mathcal{K} = \mathbb{Z}_3$ .

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<sup>15</sup> That is the polynomial  $x^p - 1 \in \mathcal{K}[x]$ , which is separable by the assumption  $\text{char}(\mathcal{K}) \neq p$ , splits over  $\mathcal{K}$ .

The following lemma serves to relate the assumption on the ground field  $\mathcal{K}$  to the multiplicities of barcodes of persistence modules.

**Lemma 4.16** *Let  $p$  be a prime number,  $V$ —vector space over  $\mathcal{K}$  with an operator  $B$  satisfying  $B^p = \zeta_p \cdot \mathbf{1}$ . Then  $p$  divides  $\dim V$ .*

*Proof* As  $(\det B)^p = (\zeta_p)^{\dim V}$ , the lemma follows immediately from the property of  $\mathcal{K}$ . □

An equivariant interleaving between  $\mathbb{Z}_p$  persistence modules  $V$  and  $W$  is defined as an interleaving respecting the  $\mathbb{Z}_p$ -actions. Given two  $\mathbb{Z}_p$  persistence modules, define the distance  $\widehat{d}_{\text{inter}}$  between them as the infimum of  $\delta$  such that they admit an equivariant  $\delta$ -interleaving. Clearly,  $\widehat{d}_{\text{inter}} \geq d_{\text{inter}}$ .

A  $\mathbb{Z}_p$  persistence module  $(V, \pi; A)$  is called a full  $p$ -th power if  $A = B^p$  for some morphism  $B : V \rightarrow V$ . Note that  $B^{p^2} = \mathbf{1}$  and  $B$  automatically commutes with  $A$ .

**Definition 4.17** For a  $\mathbb{Z}_p$  persistence module  $V := (V, \pi; A)$  set

$$\kappa(V) = \inf_W \widehat{d}_{\text{inter}}(V, W),$$

where the infimum is taken over all full  $p$ -th powers  $W$ .

Our objective is to give a lower bound for  $\kappa(V)$  through a barcode of an auxiliary persistence module  $(L_\zeta)_t = \ker(A_t - \zeta \cdot \mathbf{1})$ , for  $\zeta \neq 1$  a  $p$ -th root of unity. Clearly  $A_t$  descends to the morphism  $A_t = \zeta \cdot \mathbf{1} : (L_\zeta)_t \rightarrow (L_\zeta)_t$ . We call  $L_\zeta$  the  $\zeta$ -eigenspace of the  $\mathbb{Z}_p$  persistence module  $V$ .

*Remark 4.18* We remark that the persistence module  $(L)_t = V_t/\text{Fix}(A_t)$ , which appears in Sect. 8.2 below, in this situation is isomorphic to the direct sum of eigenspaces with eigenvalue different from 1 :

$$L \cong \bigoplus_{\zeta \neq 1, \zeta^p=1} L_\zeta.$$

Indeed it is an easy exercise to check that, fixing a primitive  $p$ -th root of unity  $\zeta$ , the operators  $\{\pi_k : V \rightarrow V\}_{0 \leq k \leq p-1}$  given by  $\pi_k(v) = \frac{1}{p} \sum \zeta^{-jk} A^j(v)$  give a splitting

$$V \cong \bigoplus_{0 \leq k \leq p-1} L_{\zeta^k}.$$

Indeed  $\text{Image}(\pi_k) \subset L_{\zeta^k}$  and  $v = \sum_{k=0}^{p-1} \pi_k(v)$  for all  $v \in V$ .

Note that every equivariant interleaving between the  $\mathbb{Z}_p$  persistence modules  $V$  and  $W$  descends to an interleaving between the  $\zeta$ -eigenspaces, and hence, by the isometry theorem,

$$\widehat{d}_{\text{inter}}(V, W) \geq d_{\text{inter}}(L_\zeta, K_\zeta) = d_{\text{bottle}}(\mathcal{B}(L_\zeta), \mathcal{B}(K_\zeta)). \tag{25}$$

**Proposition 4.19** *Let  $L_\zeta$  be the  $\zeta$ -eigenspace of a full  $p$ -th power  $\mathbb{Z}_p$  persistence module for a primitive  $p$ -th root of unity  $\zeta$ . Then for every interval  $I \subset \mathbb{R}$  the multiplicity  $m(\mathcal{B}(L_\zeta), I)$  is divisible by  $p$ .*

*Proof* Assume that  $A_t = B_t^p$ . Since  $B_t$  commutes with  $A_t$  it preserves  $(L_\zeta)_t$ . Let  $B'_t : (L_\zeta)_t \rightarrow (L_\zeta)_t$  denote the restriction  $B'_t = (B_t)|_{(L_\zeta)_t}$  to  $(L_\zeta)_t$ . Note that  $B'_t$  satisfies  $(B'_t)^p = \zeta$ . Furthermore denoting by  $\theta_{st} : (L_\zeta)_s \rightarrow (L_\zeta)_t$  the persistence morphisms and using that  $B'_t \theta_{st} = \theta_{st} B'_s$ , we get that  $\text{Image}(\theta_{st})$  is invariant under  $B'_t$ .

By Lemma 4.16 one concludes that the dimension of  $\text{Image}(\theta_{st})$  is **divisible by  $p$**  for all  $s < t$ . By looking at the normal form of  $\{(L_\zeta)_t\}$ , we readily conclude the desired statement, since for an interval  $I = (a, b]$ , every bar containing  $I$  contributes 1 to  $\dim \text{Image}(\theta_{a+, b-})$ , while all other bars contribute 0. □

**Definition 4.20** For a primitive  $p$ -th root of unity  $\zeta$  define  $\mu_{p,\zeta}(V, A)$  as the supremum of those  $c \geq 0$  for which there exists an interval  $I$  of length  $> 4c$  and such that  $m(\mathcal{B}(L_\zeta), I) = m(\mathcal{B}(L_\zeta), I^{2c}) = l$  with  $l \not\equiv 0 \pmod p$ , where  $L_\zeta$  is the  $\zeta$ -eigenspace of  $V$ . The *multiplicity-sensitive spread*  $\mu_p(V, A)$  of a  $\mathbb{Z}_p$  persistence module  $V$  is defined as

$$\mu_p(V, A) = \max_{\zeta \neq 1: \zeta^p=1} \mu_{p,\zeta}(V, A).$$

**Proposition 4.21**  $|\mu_p(V) - \mu_p(W)| \leq \widehat{d}_{\text{inter}}(V, W)$  for every pair of  $\mathbb{Z}_p$  persistence modules  $V$  and  $W$ .

*Proof* We first claim that it is enough to show that

$$|\mu_{p,\zeta}(V) - \mu_{p,\zeta}(W)| \leq \widehat{d}_{\text{inter}}(V, W)$$

for all  $p$ -th roots of unity  $\zeta \neq 1$ . By the symmetry between  $V$  and  $W$  we can assume that  $\mu_p(V) \geq \mu_p(W) \geq 0$ . Let  $\zeta_0$  be such that  $\mu_p(V) = \max_{p,\zeta_0}(V)$ , that is the maximum in Definition 4.20 is attained at  $\zeta_0$ . Then

$$|\mu_p(V) - \mu_p(W)| = \mu_{p,\zeta_0}(V) - \mu_p(W) \leq \mu_{p,\zeta_0}(V) - \mu_{p,\zeta_0}(W),$$

whence the claim is immediate.

Now fix a  $p$ -th root of unity  $\zeta \neq 1$ . Let  $L_\zeta$  and  $K_\zeta$  be the  $\zeta$ -eigenspaces of  $V$  and  $W$ , respectively. By (25) it is enough to show

$$|\mu_{p,\zeta}(V) - \mu_{p,\zeta}(W)| \leq d_{\text{bottle}}(\mathcal{B}(L_\zeta), \mathcal{B}(K_\zeta)). \tag{26}$$

Let  $d_{\text{bottle}}(\mathcal{B}(L_\zeta), \mathcal{B}(K_\zeta)) = \epsilon$ . By symmetry between  $V$  and  $W$  it suffices to show that

$$\mu_{p,\zeta}(V) - \mu_{p,\zeta}(W) \leq \epsilon. \tag{27}$$

Let  $\mu_{p,\zeta}(V) = c > 0$ . Assume without loss of generality that  $\epsilon < c$ . Take an interval  $I$  such that  $m(\mathcal{B}(L_\zeta), I) = m(\mathcal{B}(L_\zeta), I^{2c}) = l$  with  $l \not\equiv 0 \pmod p$ . This yields

$$m(\mathcal{B}(L_\zeta), I^{2c}) = m(\mathcal{B}(L_\zeta), I^{2c-2\epsilon}) = l, \quad m(\mathcal{B}(L_\zeta), I) = m(\mathcal{B}(L_\zeta), I^{2\epsilon}) = l,$$



and hence by Proposition 4.13

$$m(\mathcal{B}(K_\zeta), I^{2c-\epsilon}) = m(\mathcal{B}(K_\zeta), I^\epsilon) = l.$$

It follows that  $\mu_{p,\zeta}(W) \geq c - \epsilon$  which yields (27). □

**Theorem 4.22**  $\kappa(V) \geq \mu_p(V)$  for every  $\mathbb{Z}_p$  persistence module  $V$ .

*Proof* By Proposition 4.19  $\mu_p(W) = 0$  for every full  $p$ -th power  $\mathbb{Z}_p$  persistence module  $W$ . Thus by Proposition 4.21

$$\widehat{d}_{\text{inter}}(V, W) \geq \mu_p(V),$$

and hence  $\kappa(V) \geq \mu_p(V)$ . □

### 4.4 A new invariant of Hamiltonian diffeomorphisms

Fix a prime number  $p$ . Write  $\mathcal{G}_p$  for the set of Hamiltonian diffeomorphisms whose primitive  $p$ -periodic orbits are non-degenerate. Take any  $\phi \in \mathcal{G}_p$  and fix a primitive free homotopy class  $\alpha$  of loops. Then for each degree  $r \in \mathbb{Z}$

$$V_r(\phi) := \left( HF_r^{(-\infty, \cdot)}(\phi^p)_\alpha, [\mathbb{R}_p] \right)$$

is a  $\mathbb{Z}_p$  persistence module. Here  $\mathbb{R}_p$  is the loop rotation operator induced by  $t \mapsto t + 1/p$ , or, equivalently, induced by the action of  $\phi$  on the filtered Floer homology of  $\phi^p$ . Clearly, it induces a  $\mathbb{Z}_p$ -action on the persistence module  $HF_r^{(-\infty, \cdot)}(\phi^p)_\alpha$ . Fixing a primitive  $p$ -th root of unity  $\zeta \neq 1$ , we write  $L_r(\phi) = L_r(\phi)_\zeta$  for the  $\zeta$ -eigenspace of  $V_r(\phi)$ , and  $\mathcal{B}_r(\phi) = \mathcal{B}_r(\phi)_\zeta$  for the barcode of  $L_r(\phi)$ .

By Lemma 3.4, for every  $\phi, \psi \in \mathcal{G}_p$  the  $\mathbb{Z}_p$  persistence modules  $V_r(\phi)$  and  $V_r(\psi)$  are equivariantly  $\delta$ -interleaved with  $\delta = p \cdot d(\phi, \psi)$ , where  $d$  stands for Hofer's metric. Therefore (25) implies that

$$d_{\text{bottle}}(\mathcal{B}_r(\phi), \mathcal{B}_r(\psi)) \leq \widehat{d}_{\text{inter}}(V_r(\phi), V_r(\psi)) \leq p \cdot d(\phi, \psi) \quad \forall \phi, \psi \in \mathcal{G}_p. \quad (28)$$

In particular, the barcode mapping

$$\phi \mapsto \mathcal{B}_r(\phi)$$

is Lipschitz with respect to Hofer's metric  $d$  on  $\text{Ham}(M)$  and the bottleneck distance on the space of barcodes.

Put  $\mu_{p,\zeta}(r, \phi) = \mu_{p,\zeta}(V_r(\phi), A_r(\phi))$ , and

$$\mu_p(r, \phi) = \max_{\zeta \neq 1: \zeta^p=1} \mu_{p,\zeta}(r, \phi) = \mu_p(V_r(\phi)).$$

Define the *multiplicity-sensitive spread*  $\mu_p(\phi)$  of a Hamiltonian diffeomorphism as

$$\mu_p(\phi) = \max_{r \in \mathbb{Z}} \mu_p(r, \phi).$$

By (28) and (26) we get that

$$|\mu_p(\phi) - \mu_p(\psi)| \leq p \cdot d(\phi, \psi). \tag{29}$$

Since  $\mathcal{G}_p$  is Hofer-dense in  $\text{Ham}(M)$ , it follows that the invariant  $\mu_p$  can be extended by continuity to the whole  $\text{Ham}(M)$ , and the extension is still  $p$ -Lipschitz in Hofer’s norm.

The multiplicity-sensitive spread yields the following estimate for the distance to Powers $_p$ :

**Theorem 4.23**  $d(\phi, \text{Powers}_p) \geq \frac{1}{p} \cdot \mu_p(\phi)$  for all  $\phi \in \text{Ham}(M)$ .

*Proof* Let  $\theta = \psi^p$  be a  $p$ -th power of a Hamiltonian diffeomorphism. Then  $V_r(\theta)$  is a full  $p$ -th power  $\mathbb{Z}_p$  persistence module for each  $r \in \mathbb{Z}$ . Indeed, given a 1-periodic Hamiltonian  $F(t, x)$  generating  $\psi$ , we can choose the 1-periodic Hamiltonian  $H(t, x)$  generating  $\theta$  as  $H(t, x) = F^{(p)}(t, x) = pF(pt, x)$ . Then  $H^{(p)}(t, x) = p^2F(p^2t, x) = F^{(p^2)}(t, x)$ . Therefore the operator

$$[\mathbb{R}_{p^2}(F)] : HF_r^{(-\infty, a)}(H^{(p)})_\alpha \rightarrow HF_r^{(-\infty, a)}(H^{(p)})_\alpha$$

is well defined, and moreover the identity

$$t + \underbrace{\frac{1}{p^2} + \dots + \frac{1}{p^2}}_{p \text{ times}} = t + \frac{1}{p}$$

in  $S^1$  shows that

$$[\mathbb{R}_{p^2}(F)]^p = [\mathbb{R}_p(H)].$$

By (28) and Theorem 4.22

$$p \cdot d(\phi, \text{Powers}_p) \geq \max_{r, \zeta} \kappa(L_r(\phi)_\zeta) \geq \mu_p(\phi),$$

as required. □

Finally we observe that the multiplicity-sensitive spread is invariant under stabilization.

**Theorem 4.24** For  $\phi \in \text{Ham}(M)$ ,  $\alpha \in \pi_0(\mathcal{LM})$ , and any closed connected symplectically aspherical manifold  $N$ , consider the stabilization  $\phi \times \mathbf{1} \in \text{Ham}(M \times N)$  of  $\phi$ . Then we have

$$\mu_p(\phi) = \mu_p(\phi \times \mathbf{1}_N),$$

the latter value being computed in the class  $\alpha \times pt_N$  in  $\pi_0(\mathcal{L}(M \times N))$ .

*Proof* Clearly it is enough to prove that

$$\mu_{p,\zeta}(\phi) = \mu_{p,\zeta}(\phi \times \mathbf{1}_N),$$

for all  $p$ -th roots of unity  $\zeta \neq 1$ .

Put  $c = \mu_{p,\zeta}(\phi) = \max_r \mu_{p,\zeta}(r, \phi)$ , and  $c' = \mu_{p,\zeta}(\phi \times \mathbf{1}_N)$ . As carried out in the proof of Proposition 4.2, Item (vi), we perturb  $\mathbf{1}_N$  to the time-one map of the Hamiltonian flow of a  $C^2$ -small Morse function on  $N$ , and argue up to a small  $\delta > 0$ . We omit these considerations herein.

We show first that  $c \leq c'$ . Take the minimal  $r$  with  $\mu_{p,\zeta}(r, \phi) = c$ . Let  $I \subset \mathbb{R}$  be an interval such that  $m(\mathcal{B}_r(\phi)_\zeta, I) = m(\mathcal{B}_r(\phi)_\zeta, I^{2c}) = l$ , where  $l \not\equiv 0 \pmod p$ . Consider  $\mu_{p,\zeta}(\phi \times \mathbf{1})$ . By the Kunnet theorem in Floer homology, the barcode  $\mathcal{B}_r(\phi \times \mathbf{1})_\zeta$  consists of  $b_i = \dim H_i(N)$  copies of  $\mathcal{B}_{r-i}(\phi)_\zeta$ ,  $i = 0, 1, \dots, \dim N$ .

Note that for  $i > 0$  none of  $\mathcal{B}_{r-i}(\phi)_\zeta$  contains both  $I$  and  $I^{2c}$  with equal multiplicities not divisible by  $p$ , since otherwise we get a contradiction to  $r$  being the minimal degree where the maximum  $c$  is attained.

Thus we are left with  $i = 0$ , and since  $b_0 = 1$ , we have  $l$  copies of  $I$  and  $I^{2c}$  in the corresponding piece of the barcode  $\mathcal{B}_r(\phi \times \mathbf{1})_\zeta$ . It follows that

$$\mu_{p,\zeta}(\phi \times \mathbf{1}) \geq \mu_{p,\zeta}(r, \phi \times \mathbf{1}) \geq c = \mu_{p,\zeta}(\phi).$$

In the other direction, we show that  $c \geq c'$ . Assume that  $m(\mathcal{B}_r(\phi \times \mathbf{1})_\zeta, I) = m(\mathcal{B}_r(\phi \times \mathbf{1})_\zeta, I^{2c'}) = l$ , where  $l \not\equiv 0 \pmod p$ . Define  $S = \{i \mid b_i \neq 0\}$ . For each  $i \in S$  put  $m_i, m'_i$  for the multiplicities of  $I$  and  $I^{2c'}$  in  $\mathcal{B}_{r-i}(\phi)_\zeta$ .

We claim that  $m_i = m'_i$  for all  $i \in S$ . Indeed, by definition of the multiplicity function,  $m_i \leq m'_i$ . Hence the claim follows by the identities

$$l = m(\mathcal{B}_r(\phi \times \mathbf{1})_\zeta, I) = \sum_{i \in S} m_i b_i,$$

$$l = m(\mathcal{B}_r(\phi \times \mathbf{1})_\zeta, I^{2c'}) = \sum_{i \in S} m'_i b_i.$$

To conclude the proof, we note that  $l = \sum m_i b_i \not\equiv 0 \pmod p$ , and hence there exists  $i_0$  with  $m_{i_0} \not\equiv 0 \pmod p$ . Hence

$$\mu_{p,\zeta}(\phi) \geq \mu_{p,\zeta}(r - i_0, \phi) \geq c' = \mu_{p,\zeta}(\phi \times \mathbf{1}).$$

□

### 5 Hamiltonian egg-beater and the proof of the main results

Here we prove the following statement and show how to deduce Theorems 1.2 and 1.3 from it.

**Proposition 5.1** *On a surface  $\Sigma$  of genus  $g \geq 4$ , there exists a family  $\{\phi_\lambda\}$ , for  $\lambda$  in an unbounded increasing sequence in  $\mathbb{R}$ , of Hamiltonian diffeomorphisms of  $\Sigma$ , and a family of primitive classes  $\alpha_\lambda \in \pi_0(\mathcal{L}\Sigma)$  such that for a positive integer  $p$  and  $\lambda$  large,  $(\phi_\lambda)^p$  has exactly  $2^{2p}$   $p$ -tuples  $\{z, \phi_\lambda(z), \dots, (\phi_\lambda)^{p-1}(z)\}$  of (primitive) non-degenerate fixed points in class  $\alpha_\lambda$ , with action differences*

$$|\mathcal{A}(z) - \mathcal{A}(z')| \geq c \cdot \lambda + O(1),$$

for each  $z, z'$  belonging to different  $p$ -tuples, as  $\lambda \rightarrow \infty$  for a certain constant  $c > 0$ .

*Remark 5.2* It sounds likely that by slightly modifying our construction one can prove an analogue of Proposition 5.1 and hence of Theorems 1.2 and 1.3, in the case when the genus of the surface  $\Sigma$  is  $g \geq 2$ .

#### 5.1 Proof of Theorems 1.2 and 1.3

By Proposition 4.3, Proposition 5.1 implies that

$$w_{p, \alpha_\lambda}(\phi_\lambda) \geq c \cdot \lambda + O(1),$$

as  $\lambda \rightarrow \infty$ , for a constant  $c > 0$ . Consequently, Proposition 4.2: (iii), (iv), and (vi) yield Theorem 1.2.

Further, among the  $2^{2p}$   $p$ -tuples of fixed points of  $\phi_\lambda^p$  in the class  $\alpha_\lambda$  choose the  $p$ -tuple, say  $\{z, \phi_\lambda(z), \dots, (\phi_\lambda)^{p-1}(z)\}$  with the minimal action. Let  $r$  be the index of  $z$ . Fix a small  $\varepsilon > 0$  and choose the interval

$$I_{\varepsilon, \lambda} = \left( \mathcal{A}(z) + \frac{1}{2}\varepsilon \cdot \lambda, \mathcal{A}(z) + \left( c - \frac{1}{2}\varepsilon \right) \cdot \lambda \right]$$

of length

$$|I_{\varepsilon, \lambda}| = (c - \varepsilon)\lambda > 4 \cdot \frac{\lambda}{4}(c - 2\varepsilon).$$

For a primitive  $p$ -th root of unity  $\zeta$ , we claim that the multiplicity of the interval  $I_{\varepsilon, \lambda}$  in the barcode of the  $\zeta$ -eigenspace  $L_r(\phi_\lambda)_\zeta$  of the  $\mathbb{Z}_p$  persistence module  $V_r(\phi)$  equals 1:

$$m(\mathcal{B}_r(\phi_\lambda), I_{\varepsilon, \lambda}) = 1$$

Indeed, when  $\lambda$  is large, the  $\zeta$ -eigenspace  $(L_r(\phi_\lambda)_\zeta)_t$  for all

$$\mathcal{A}(z) < t < \mathcal{A}(z) + (c - \varepsilon) \cdot \lambda$$

is one-dimensional, and is in fact spanned by  $\sum_{j=0}^{p-1} \zeta^{-j} [(\phi_\lambda)^j(z)]$ . Moreover also

$$m \left( \mathcal{B}_r(\phi_\lambda), I_{\varepsilon, \lambda}^{\frac{\lambda}{2}(c-2\varepsilon)} \right) = 1.$$

By the definition of the multiplicity-sensitive spread, we conclude that  $\mu_p(\phi_\lambda) \geq \lambda(c - 2\varepsilon)/4$ , and hence by Theorem 4.23,  $d(\phi_\lambda, \text{Powers}_p) \geq \lambda(c - 2\varepsilon)/4p$ . This, together with Theorem 4.24, proves Theorem 1.3.  $\square$

The rest of this section is dedicated to the proof of Proposition 5.1.

### 5.2 Topological set up

We begin by describing the topological setting of the construction. To this end, consider the union of two annuli  $C_V$  and  $C_H$  in the standard  $\mathbb{R}^2$ , each symplectomorphic to

$$C_* = [-1, 1] \times \mathbb{R}/L\mathbb{Z},$$

for a number  $L \geq 4$ , intersecting at two squares  $A, B$ . Here the subscripts  $V$  and  $H$  stand for “vertical” and “horizontal”, as seen from the square  $A$ —compare Fig. 1. Then consider this configuration of annuli as being symplectically embedded in  $S^2$  and glue at least one handle into each connected component. This gives the surface  $\Sigma$  of genus  $\geq 4$ .

More precisely, consider the cylinder  $C_*$  with coordinates  $x, y$  and standard symplectic form  $dx \wedge dy$ . Take two copies of this cylinder, denoted by  $C_V$  and  $C_H$ . Consider the squares  $S_0 = [-1, 1] \times [-1, 1]/L\mathbb{Z}$  and  $S_1 = [-1, 1] \times [L/2 - 1, L/2 + 1]/L\mathbb{Z}$ . This gives us four squares  $S_{V,0}, S_{V,1} \subset C_V$  and  $S_{H,0}, S_{H,1} \subset C_H$ . Consider the symplectomorphism  $VH_{0,1} : S_{V,0} \sqcup S_{V,1} \rightarrow S_{H,0} \sqcup S_{H,1}$ , given by  $VH \sqcup VH'$ , where  $VH : S_{V,0} \rightarrow S_{H,0}$  is given by  $VH(x, [y]) = (-y, [x])$ , and  $VH' : S_{V,1} \rightarrow S_{H,1}$  is given by  $VH'(x, [y]) = (y - L/2, [-x + L/2])$ . Glue the two cylinders along  $VH_{0,1}$  to obtain the following surface with boundary

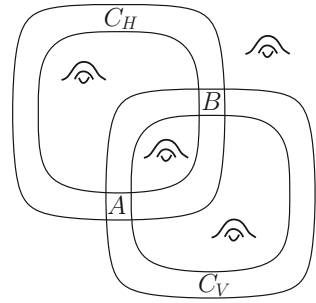
$$C := C_V \cup_{VH_{0,1}} C_H.$$

Next we consider a symplectic embedding  $\iota_0 : C \hookrightarrow S^2$  of  $C$  into  $S^2$  (this can be seen by first embedding it symplectically into  $\mathbb{R}^2$  with the standard symplectic structure inside a large ball and then composing with an embedding of that ball into  $S^2$ , or alternatively embed  $C$  as the union of tubular neighborhoods of two orthogonal great circles in  $S^2$ ), and identify  $C$  with the image of the embedding.

Note that  $S^2 \setminus C$  has 4 connected components. We glue at least one handle into each of these components, to obtain a surface  $\Sigma$  of genus  $g \geq 4$ , with a symplectic embedding (Fig. 1)

$$\iota : C \hookrightarrow \Sigma.$$

**Fig. 1**  $C$  embedded in  $\Sigma$



This embedding is *incompressible*, that is it induces an injective map on  $\pi_1$ . Moreover by considering geodesic representatives of free homotopy classes of loops in  $\Sigma$  with respect to a well-chosen hyperbolic metric, which renders the components of the boundary of  $\iota(C)$  geodesic, one sees that this embedding also induces an injection  $\pi_0(\mathcal{L}C) \rightarrow \pi_0(\mathcal{L}\Sigma)$ .

We continue to describe the general form of the class  $\alpha_\lambda$  in the image of the injection  $\pi_0(\mathcal{L}C) \rightarrow \pi_0(\mathcal{L}\Sigma)$ . It is sufficient to describe its preimage in  $\pi_0(\mathcal{L}C) \cong \pi_1(C)/\text{conj}$ .

Note that  $\pi_1(C) \cong \pi_1(\Gamma)$ , for a graph  $\Gamma$  underlying the oriented graph consisting of two vertices  $A, B$  corresponding to the squares along which the images of  $C_V$  and  $C_H$  in  $C$  intersect (we denote these squares by  $A, B$  as well), and four edges: there are two edges  $q_1, q_3$  from  $A$  to  $B$ , and two edges  $q_2, q_4$  from  $B$  to  $A$ —see Fig. 2. We find it useful to orient these edges so that the concatenation  $a = q_1\#q_2$  corresponds to a generator of  $\pi_1(C_V)$  based at a point in  $A$ , and  $b = q_3\#q_4$  corresponds to a generator of  $\pi_1(C_H)$  based at a point in  $A$ . Specifically, we represent these two generators by the loops

$$\gamma(a) = \{(0, [t]) \in C_V\}_{t \in \mathbb{R}/\mathbb{Z}}$$

and

$$\gamma(b) = \{(0, [t]) \in C_H\}_{t \in \mathbb{R}/\mathbb{Z}}$$

based at the center

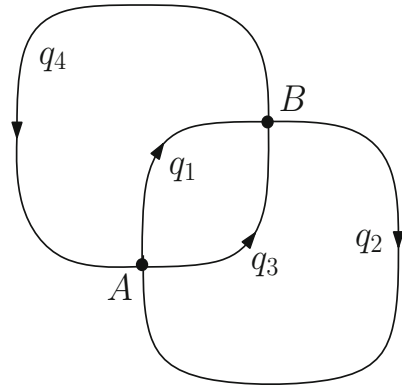
$$0_A := (0, [0]) \in A$$

of  $A$ . Put also  $c = q_3\#q_2$ . Since contracting the edge  $q_2$  establishes a homotopy equivalence of  $\Gamma$  with a bouquet  $V_3$  of 3 circles, with  $a, b, c$  mapping to three generators of  $\pi_1(V_3, [d])$  under the quotient map, we see that

$$\pi_1(\Gamma, A) \cong \text{Free}\langle a, b, c \rangle,$$

the free group on 3 generators.

Fig. 2 The graph  $\Gamma$



The general form of the class we consider is

$$\alpha = a^{m_1} b^{n_1} a^{m_2} b^{n_2} \dots \cdot a^{m_p} b^{n_p} / \text{conj},$$

for  $m_1, n_1, m_2, n_2, \dots, m_p, n_p \in \mathbb{Z}_{>0}$ , to be specified later, such that for each  $i < j$ ,  $(m_i, n_i) \neq (m_j, n_j)$ , as elements in  $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ . Put  $\tilde{\alpha} = a^{m_1} b^{n_1} a^{m_2} b^{n_2} \dots \cdot a^{m_p} b^{n_p}$ .

### 5.3 Constructing the egg-beater map

We proceed to describe the dynamical system of this example. Fix a large parameter  $\lambda \gg 1$ . We later constrain it to an unbounded increasing subsequence of  $\mathbb{R}$ . We first construct the diffeomorphism  $\phi_\lambda \in \text{Ham}(\Sigma, \sigma)$ , and based on its properties we choose an appropriate class  $\alpha_\lambda$  in the image of the injection  $\pi_0(\mathcal{L}C) \rightarrow \pi_0(\mathcal{L}\Sigma)$ , and the constraints on  $\lambda$ .

**Notation** For brevity we denote by  $\varepsilon_s := \text{sign}(s) \in \{\pm 1\}$  the sign of  $s \neq 0$ . For  $s = 0$  we define  $\varepsilon_0 := 0$ .

We construct the Hamiltonian diffeomorphisms  $\phi_\lambda$  as a small smoothing of a piecewise smooth homeomorphism. Consider the function  $u_0 : [-1, 1] \rightarrow \mathbb{R}$  (Fig. 3), given by

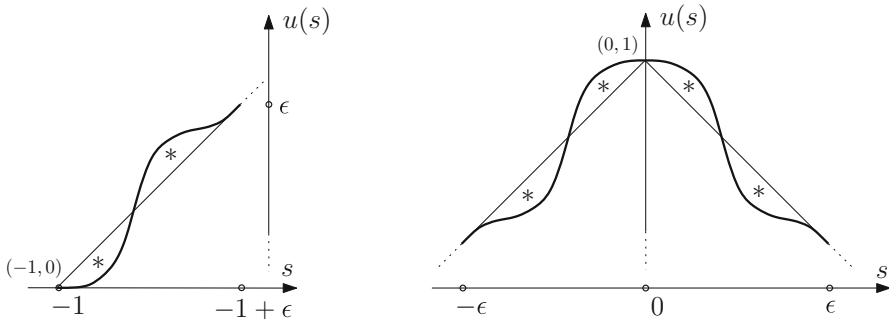
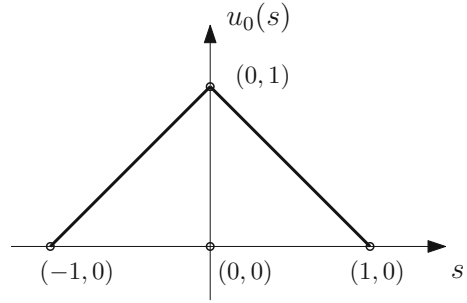
$$u_0(s) = 1 - |s| = 1 - \varepsilon_s \cdot s.$$

On the cylinder  $C_* = [-1, 1] \times \mathbb{R}/L\mathbb{Z}$ , consider the homeomorphism

$$\begin{aligned} f_0 &= f_{0,\lambda} : C_* \rightarrow C_*, \\ f_0(x, [y]) &= (x, [y + \lambda u_0(x)]). \end{aligned}$$

For a smoothing  $u \in C^\infty([-1, 1], \mathbb{R})$  of  $u_0$  with  $\text{supp}(u) \subset (-1, 1)$ , that is sufficiently  $C^0$ -close to  $u_0$  and coincides with  $u_0$  outside a sufficiently small neighborhood of  $U$  of  $\{-1, 0, 1\}$ , define the Hamiltonian diffeomorphism

**Fig. 3** Graph of  $u_0$



**Fig. 4** Graph of  $u$  near  $-1$  (left) and  $0$  (right). The areas marked  $*$  are equal

$$f = f_\lambda : C_* \rightarrow C_*,$$

$$f(x, [y]) = (x, [y + \lambda u(x)]).$$

*Remark 5.3* Note for a sequence  $u_j$  converging to  $u$  in the  $C^0$  topology, the flow  $\{f_j^t(x, [y]) = (x, [y + t\lambda u_j(x)])\}_{0 \leq t \leq 1}$  converges uniformly to  $\{f_0^t(x, [y]) = (x, [y + t\lambda u_0(x)])\}_{0 \leq t \leq 1}$ .

It is easy to achieve, in addition, that  $u$  is non-negative, even, and moreover such that

$$\int_{-1}^s (u(\sigma) - u_0(\sigma)) d\sigma$$

is supported in the small neighborhood  $U$  of  $\{-1, 0, 1\}$  outside which  $u$  coincides with  $u_0$ —see e.g., Fig. 4.

Note that  $f^{-1} : C_* \rightarrow C_*$  is given by  $f^{-1}(x, [y]) = (x, [y - \lambda u(x)])$ , and that  $f$  is supported away from the boundary of  $C_*$ . Moreover  $f$  is a time-one map of an autonomous Hamiltonian flow  $\{f^t\}_{t \in [0, 1]}$  with Hamiltonian given by

$$h(x, [y]) = h(x) = -\frac{1}{2} + \int_{-1}^x u(\sigma) d\sigma.$$



We note that by our choice of smoothing  $u$ , the function  $h : [-1, 1] \rightarrow \mathbb{R}$  is odd, and for all  $s \in [-1, 1] \setminus U$ , we have

$$h(s) = h_0(s),$$

where

$$h_0(s) = -\frac{1}{2} + \int_{-1}^s u_0(\sigma) d\sigma = s - \varepsilon_s \frac{s^2}{2}$$

would be the Hamiltonian were we to consider the piecewise smooth homeomorphism  $f_0$  as a Hamiltonian transformation. Note that  $h_0$  is an odd function, and hence  $h_0(s) = \varepsilon_s h_0(|s|)$ .

The Hamiltonian symplectomorphism  $f$  defines two Hamiltonian symplectomorphisms of  $C$ , one supported on  $C_V$  and one supported on  $C_H$ , and both supported away from the boundary of  $C$ , and hence by extension by  $\mathbf{1}$ —two Hamiltonian symplectomorphisms of  $\Sigma$  supported on  $C$ ,  $f_V$  and  $f_H$ . Let us emphasize also that the Hamiltonian  $h$  takes different values on the boundary components of the annulus  $C$ . Nevertheless, it extends from  $C_V$  and from  $C_H$  to a smooth Hamiltonian on  $\Sigma$  since these annuli separate  $\Sigma$ . In particular,  $f_V$  and  $f_H$  are genuine Hamiltonian diffeomorphisms of  $\Sigma$ .

For our fixed  $\lambda \in \mathbb{R}$ , we define

$$\phi_\lambda := f_{\lambda,H} \circ f_{\lambda,V}.$$

Let  $\{f_V^t\}, \{f_H^t\}$  denote the Hamiltonian isotopies of  $\Sigma$  (or interchangeably of  $C$ ) with endpoints  $f_V, f_H$  generated by the corresponding normalized autonomous Hamiltonians. Hence these are simply two copies of  $\{f^t\}$ . We consider the Hamiltonian isotopy

$$\{f_V^t\} \# \{f_H^t f_V\}$$

generating  $\phi_\lambda$  and the Hamiltonian isotopy

$$\{f_V^t\} \# \{f_H^t f_V\} \# \dots \# \left\{ f_V^t (f_H f_V)^{p-1}(z) \right\} \# \left\{ f_H^t f_V (f_H f_V)^{p-1} \right\}$$

generating

$$(\phi_\lambda)^p = (f_H f_V)^p.$$

### 5.4 Detecting periodic points

We study the fixed points  $z$  of  $(\phi_\lambda)^p$  in the class  $\alpha \in \pi_0(\mathcal{L}\Sigma)$ . In particular this means that

$$(\phi_\lambda)^p z = z$$

and the free homotopy class of the orbit

$$\gamma(z) := \{f_V^t(z)\} \# \{f_H^t f_V(z)\} \# \dots \# \{f_V^t (f_H f_V)^{p-1}(z)\} \# \{f_H^t f_V (f_H f_V)^{p-1}(z)\}$$

satisfies

$$[\gamma(z)] = \alpha.$$

*Terminology* In what follows we refer to

$$\{f_V^t(z)\}, \{f_H^t f_V(z)\}, \dots, \{f_V^t (f_H f_V)^{p-1}(z)\}, \{f_H^t f_V (f_H f_V)^{p-1}(z)\}$$

as the *intermediate paths* of the orbit  $\gamma(z)$ , and to their endpoints as the *intermediate points* of this orbit.

We start solving the system of equations

$$(\phi_\lambda)^p z = z, \tag{30}$$

$$[\gamma(z)]_{\pi_0(\mathcal{L}\Sigma)} = \alpha \tag{31}$$

by a topological analysis. Note that as all our Hamiltonian diffeomorphisms and flows are supported in  $C$  and the embedding  $\iota : C \rightarrow \Sigma$  is incompressible on both  $\pi_1$  and free homotopy classes of loops, it is sufficient to restrict our consideration to  $C$ . As  $\pi_0(\mathcal{L}C) = \pi_1(C)/\text{conj}$  and as explained above  $\pi_1(C) \cong \text{Free}\langle a, b, c \rangle$  is a free group on three generators, we can reduce the second equation from  $\pi_0(\mathcal{L}C)$  to  $\pi_1(C)$  using the following notion from combinatorial group theory.

A word in a free group on a set  $X$  is *cyclically reduced* if all its cyclic conjugations are reduced. Equivalently, this word is reduced, and moreover if it is written cyclically, then the resulting cyclic word is also reduced.

It is a well-known result in combinatorial group theory that conjugacy classes of words in a free group  $\text{Free}\langle a, b, c \rangle$  are classified by cyclically reduced words in  $a, b, c$  up to cyclic permutations (see e.g., [17]).

The word  $\tilde{\alpha} = a^{m_1} b^{n_1} a^{m_2} b^{n_2} \dots a^{m_p} b^{n_p}$  with  $m_i, n_i > 0$  is clearly cyclically reduced, and any other word  $v$  of the form  $v = a^{k_1} c^{e_1} b^{l_1} c^{e_2} \dots a^{k_p} c^{e_{2p-1}} b^{l_p} c^{e_{2p}}$  with  $k_1, l_1, \dots, k_p, l_p \in \mathbb{Z}_{>0}$  can be obtained from  $\tilde{\alpha}$  by a cyclic permutation if and only if

$$v = \tilde{\alpha}^{(j)} = a^{m_{p-j+1}} b^{n_{p-j+1}} \dots a^{m_p} b^{n_p} a^{m_1} b^{n_1} \dots a^{m_{p-j}} b^{n_{p-j}}$$

for some  $1 \leq j \leq p$ . Note that  $\tilde{\alpha}^{(p)} = \tilde{\alpha}$ . Moreover the word  $v$  is cyclically reduced, and is hence conjugate to  $\tilde{\alpha}$  in exactly these  $p$  cases. Below we prove the following.

**Lemma 5.4** *All fixed points  $z$  of  $(\phi_\lambda)^p$  in class  $\alpha$  satisfy*

$$(\phi_\lambda)^j z \in A$$

for all  $0 \leq j \leq p$ . Moreover all the intermediate points of  $\gamma(z)$  lie in  $A$ .

Therefore the system of Eqs. (30), (31) reduces to the  $p$  systems of equations indexed by  $1 \leq j \leq p$ .

$$(\phi_\lambda)^p z = z, \tag{32}$$

$$[\gamma(z)]_{\pi_1(C,0_A)} = \tilde{\alpha}^{(j)}, \tag{33}$$

The proof of Lemma 5.4 also shows the following statement.

**Lemma 5.5** *The map  $z \mapsto \phi_\lambda z$  establishes a bijection from the set of solutions of the system of Eqs. (32), (33) for  $j$  to the set of solutions of the system of Eqs. (32), (33) for  $j + 1$ . The inverse bijection is given by the map  $z \mapsto (\phi_\lambda)^{p-1} z = (\phi_\lambda)^{-1} z$ .*

Therefore the set of solutions of the system of Eqs. (30), (31) consists of  $p$ -tuples

$$\{z, \phi_\lambda z, \dots, (\phi_\lambda)^{p-1} z\}$$

where  $z$  is a fixed point of  $(\phi_\lambda)$  with  $[\gamma(z)]_{\pi_1(C,0_A)} = \tilde{\alpha}$ . We remark that the fixed points  $\{(\phi_\lambda)^j z\}_{1 \leq j \leq p}$  have the same action and index values.

*Proof of Lemma 5.4* We note that if  $[\gamma(z)] = \alpha$  then  $z \in A \cup B$ . Indeed, if  $z \in C_H \setminus A \cup B$ , then the path  $\beta_1 = \{f_V^t(z)\}$  is constant, in contradiction to the form of  $[\gamma(z)]$ .

Indeed, assume it were constant. Consider the cyclically reduced form (up to cyclic conjugation) of the word  $[\gamma(z)]$  in  $\text{Free}\langle a, b, c \rangle / \text{conj}$ . It must be a cyclic conjugation of  $\tilde{\alpha}$ , and hence contains no  $c$  contribution, and its word norm in the alphabet  $\{a^m, b^n, c^l\}_{l,m,n \in \mathbb{Z}}$  is  $2p$ . However, since each intermediate path contributes at most 1 to the word norm (those along  $C_V$  contribute at most 1 to the  $\{a^m\}$  count, and those along  $C_H$ —to the  $\{b^n\}$  count), in case  $\beta_1$  is constant, the word norm of the resulting cyclically reduced form would be strictly smaller than  $2p$ .

Similarly, if  $z \in C_V \setminus A \cup B$ , then the path

$$\{f_H^t f_V (f_H f_V)^{p-1}(z) = f_H^{t-1} (f_H f_V)^p(z) = f_H^{t-1}(z)\}$$

is constant, in contradiction to the form of  $[\gamma(z)]$ . Similarly, the intermediate points

$$f_V(z), f_H f_V(z), f_V f_H f_V(z), \dots, (f_H f_V)^{p-1}(z), f_V (f_H f_V)^{p-1}(z)$$

of the loop  $\gamma(z)$  lie in  $A \cup B$ .

The following table summarizes the possible types of trajectories of  $f_V^t$  and  $f_H^t$ , that join  $A \rightarrow A, A \rightarrow B, B \rightarrow A, B \rightarrow B$ . These correspond to morphisms in the fundamental groupoid of the graph  $\Gamma$ . Note that we adopt a non-standard notation for composition of morphisms in the fundamental groupoid—for example we write  $a = [q_1 \# q_2] = q_1 q_2$  instead of  $a = q_2 q_1$ —to make the relation to geometry more intuitive. Let  $m \in \mathbb{Z}_{\geq 0}$  be a generic notation for a non-negative integer. The condition  $m \geq 0$  in this table is a consequence of the fact that  $u$  is a non-negative function.

	$A \rightarrow A$	$A \rightarrow B$	$B \rightarrow A$	$B \rightarrow B$
$f_V$	$a^m$	$a^m q_1$	$q_2 a^m$	$q_2 a^{m-1} q_1$
$f_H$	$b^m$	$b^m q_3$	$q_4 b^m$	$q_4 b^{m-1} q_3$

The trajectory  $\gamma(z)$  is represented by a composable path of morphisms in the fundamental groupoid of the graph  $\Gamma$ . The table above together with the relations  $q_3 q_2 = c$  and  $q_1 q_4 = ac^{-1}b$  readily yields the following constraint on this path: if either  $z$  or any of the intermediate points

$$f_V(z), f_H f_V(z), f_V f_H f_V(z), \dots, f_V(f_H f_V)^{p-1}(z)$$

lie in  $B$ , the (cyclically reduced) word  $[\gamma(z)]$  necessarily contains either  $c$  or  $c^{-1}$ . But this contradicts the form of  $\alpha$ . Thus  $z$  and all the intermediate points lie in  $A$ , as required.  $\square$

It remains to study the solutions to the system of Eqs. (32), (33) for  $j = p$  with the help of Lemma 5.4. Lemma 5.4 and its proof imply that for any such solution  $z$  the classes of the intermediate paths of  $\gamma(z)$  in  $\pi_1(C, A)$  are well defined and in fact satisfy

$$\begin{aligned} [\{f_V^t(z)\}] &= a^{m_1}, [\{f_H^t f_V(z)\}] = b^{n_1}, \\ [\{f_V^t f_H f_V(z)\}] &= a^{m_2}, [\{f_H^t f_V f_H f_V(z)\}] = b^{n_2}, \\ &\dots \\ [\{f_V^t (f_H f_V)^{p-1}(z)\}] &= a^{m_p}, [\{f_H^t f_V (f_H f_V)^{p-1}(z)\}] = b^{m_p} \end{aligned} \tag{34}$$

Knowing this, we pass to the universal cover of  $C_*$  and translate the sequence

$$z \mapsto f_V(z) \mapsto f_H f_V(z) \mapsto f_V f_H f_V(z) \mapsto \dots \mapsto f_V(f_H f_V)^{p-1}(z) \mapsto z$$

of intermediate points in terms of explicit points in  $(-1, 1) \times (-1, 1)$ .

To this end, for  $m \in \mathbb{Z}$ , denote by

$$\begin{aligned} r = r_m : (-1, 1) \times \mathbb{R} &\rightarrow (-1, 1) \times \mathbb{R}, \\ r(x, y) &= (x, y - m \cdot L). \end{aligned}$$

the *reduction map*. Note that the images of  $(x, y)$  and  $r(x, y)$  in  $C_* = [-1, 1] \times \mathbb{R}/L\mathbb{Z}$  are equal, and that  $r$  maps  $(-1, 1) \times (m \cdot L - 1, m \cdot L + 1)$  isomorphically to  $(-1, 1) \times (-1, 1)$ . Call  $VH : S_{0,V} \rightarrow S_{0,H}, (x, [y]) \mapsto (-y, [x])$  and  $HV : S_{0,H} \rightarrow S_{0,V}, (x, [y]) \mapsto (y, [-x])$  the *flip maps*. Using these maps, we decode the equation  $(\phi_\lambda)^p z = z$  for  $z = (x_0, y_0)$ , in the class  $\tilde{\alpha}$ , as

$$(x_{2p}, y_{2p}) = (x_0, y_0),$$

where

$$\begin{aligned} (x_1, y_1) &= VH \circ r_{m_1} \circ \tilde{f}(x_0, y_0), \\ (x_2, y_2) &= HV \circ r_{n_1} \circ \tilde{f}(x_1, y_1) \\ &= HV \circ r_{n_1} \circ \tilde{f} \circ VH \circ r_{m_1} \circ \tilde{f}(x_0, y_0), \end{aligned}$$

and in general for  $0 \leq j \leq p - 1$

$$\begin{aligned} (x_{2j+2}, y_{2j+2}) &= HV \circ r_{n_{j+1}} \circ \tilde{f}(x_{2j+1}, y_{2j+1}) \\ &= HV \circ r_{n_{j+1}} \circ \tilde{f} \circ VH \circ r_{m_{j+1}} \circ \tilde{f}(x_{2j}, y_{2j}). \end{aligned}$$

See the diagram in Fig. 5 and an illustration in Fig. 6. Here  $\tilde{f}$  is the lift of  $f$  to the universal cover  $\tilde{C}_* \cong [-1, 1] \times \mathbb{R}$ , given by the Hamiltonian flow  $\{f^t\}$  of the Hamiltonian  $h$ .

The sequence of points

$$(x_0, y_0) \mapsto (x_1, y_1) \mapsto \cdots \mapsto (x_{2p-1}, y_{2p-1}) \mapsto (x_{2p}, y_{2p}) = (x_0, y_0) \quad (35)$$

corresponds the sequence of intermediate points (34).

The equation  $(x_{2p}, y_{2p}) = (x_0, y_0)$  gives the following system of Eqs. (38), (39).

We will choose certain  $\{v_j, \mu_j \in (0, 1)\}$ , with  $\frac{\kappa}{\kappa'} \in \mathbb{Q}$  for every  $\kappa, \kappa' \in \{v_j, \mu_j\}$  and fix them. We consider  $\lambda$  with the condition that

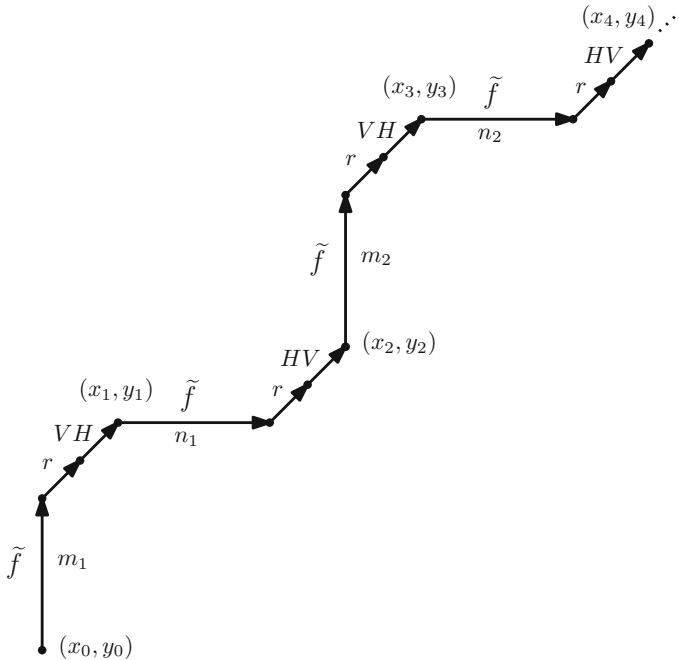


Fig. 5 Diagram for the equation  $(\phi_\lambda)^P(x_0, y_0) = (x_0, y_0)$

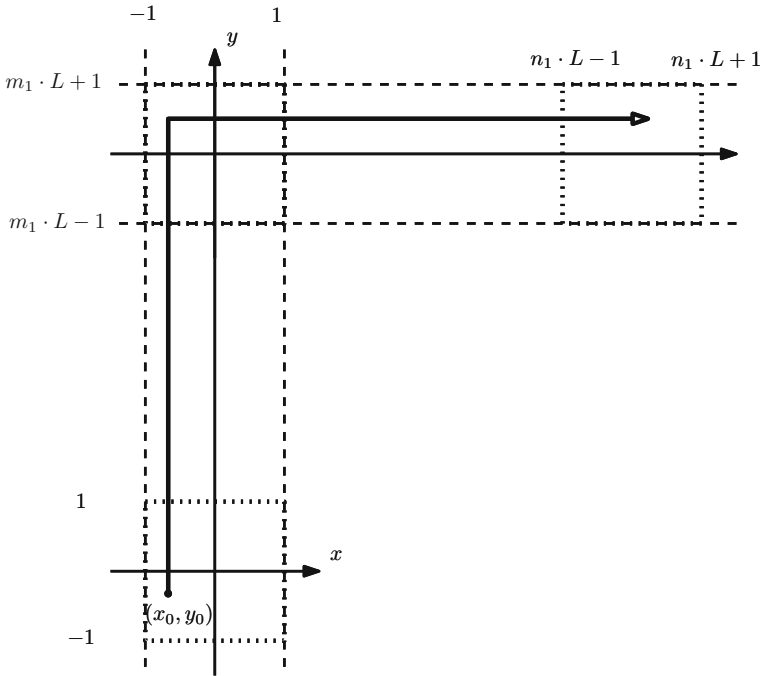


Fig. 6 Decoding the equation via diagram

$$n_j = \frac{\nu_j}{L} \lambda, \quad m_j = \frac{\mu_j}{L} \lambda$$

are positive integers for all  $1 \leq j \leq p$ . By construction there are arbitrarily large  $\lambda$  with this property, and each such choice of  $\lambda$  determines a primitive free homotopy class  $\alpha_\lambda$ .

For purposes of calculation let  $\mu, \nu \in (0, 1)$  be such that  $\mu\lambda/L, \nu\lambda/L \in \mathbb{Z}$ . Consider the composition

$$HV \circ r_{\nu\lambda/L} \circ \tilde{f} \circ VH \circ r_{\mu\lambda/L} \circ \tilde{f}.$$

Whenever  $(x, y)$  is such that

$$\tilde{f}(x, y) \in (-1, 1) \times (\mu\lambda - 1, \mu\lambda + 1), \tag{36}$$

$$\tilde{f}(VH \circ r_{\mu\lambda/L} \circ \tilde{f}(x, y)) \in (-1, 1) \times (\nu\lambda - 1, \nu\lambda + 1), \tag{37}$$

the image  $HV \circ r_{\nu\lambda/L} \circ \tilde{f} \circ VH \circ r_{\mu\lambda/L} \circ \tilde{f}(x, y) \in (-1, 1)^2$  of  $(x, y)$  under this composition is given by the formula

$$\Phi_\lambda^{\mu, \nu}(x, y) = (x + \lambda u(y + \lambda u(x) - \mu\lambda) - \nu\lambda, y + \lambda u(x) - \mu\lambda).$$

Moreover

$$pr_{C_V} (HV \circ r_{v\lambda/L} \circ \tilde{f} \circ VH \circ r_{\mu\lambda/L} \circ \tilde{f}(x, y)) = \phi_\lambda(x, y)$$

where  $pr_{C_V} : \tilde{C}_V \rightarrow C_V$  is the natural projection. Clearly  $pr_{C_V}$  maps  $(-1, 1)^2$  isomorphically onto  $A$ .

Note that by Eq. (34) the intermediate points of the trajectory of a solution in class  $\alpha$  satisfy these conditions. Hence the even intermediate points satisfy the system of equations:

$$(x_{2j+2}, y_{2j+2}) = \Phi_\lambda^{\mu_{j+1}, \nu_{j+1}}(x_{2j}, y_{2j}), \quad 0 \leq j \leq p-1 \tag{38}$$

$$(x_{2p}, y_{2p}) = (x_0, y_0). \tag{39}$$

In fact  $j$  can be considered modulo  $p$  in these equations, emphasizing their cyclic nature. Then the equation for  $(x_0, y_0)$  can be written as

$$\Phi_\lambda^{\mu_p, \nu_p} \circ \dots \circ \Phi_\lambda^{\mu_1, \nu_1}(x_0, y_0) = (x_0, y_0), \tag{40}$$

and for  $(x_{2j_0}, y_{2j_0})$  in general, as

$$\Phi_\lambda^{\mu_{j_0+p}, \nu_{j_0+p}} \circ \dots \circ \Phi_\lambda^{\mu_{j_0}, \nu_{j_0}}(x_{2j_0}, y_{2j_0}) = (x_{2j_0}, y_{2j_0}). \tag{41}$$

We remark that considering only the even intermediate points we do not lose data. Indeed any odd intermediate point  $(x_{2j+1}, y_{2j+1})$ ,  $0 \leq j \leq p-1$  satisfies

$$(x_{2j+1}, y_{2j+1}) = (-y_{2j+2}, x_{2j}). \tag{42}$$

We claim that for  $\lambda \gg 1$ , given any  $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2p}) \in \{\pm 1\}^{2p}$ , there exists a unique solution

$$\{(x_{2j}, y_{2j})\}_{j=0}^{p-1} = (z_{\vec{\varepsilon}}, (\phi_\lambda)z_{\vec{\varepsilon}}, \dots, (\phi_\lambda)^{p-1}z_{\vec{\varepsilon}})$$

of the system of Eqs. (38),(39) such that

$$\{\varepsilon_{x_{2j}}, \varepsilon_{y_{2j}}\}_{j=0}^{p-1} = \vec{\varepsilon},$$

that is for each  $0 \leq j \leq p-1$ ,

$$(\varepsilon_{x_{2j}}, \varepsilon_{y_{2j}}) = (\varepsilon_{2j+1}, \varepsilon_{2j+2}).$$

Observe that for fixed  $\{\mu_i\}, \{\nu_i\}$  and  $\lambda \gg 1$ , the  $x$ -coordinates of all the solutions of Eq. (40) for  $u_0$  are separated from  $-1, 0, 1$ . Thus there exists a smoothing  $u$  of  $u_0$  in a sufficiently small neighborhood of  $-1, 0, 1$  such that the solution sets of (40) for  $u$  and  $u_0$  coincide. This readily follows from Remark 5.3 above and a basic compactness argument. In light of this, we proceed with the analysis of the equations for  $u_0$ .

Using the definition  $u_0(s) = 1 - \varepsilon_s s$ , we simplify Eq. (40) to a linear equation in two variables  $(x_0, y_0)$  as follows. First note that

$$\Phi_\lambda^{\mu_{j+1}, \nu_{j+1}}(x_{2j}, y_{2j}) = A_{\lambda, j+1}(x_{2j}, y_{2j}) + b_{\lambda, j+1},$$

where for all  $0 \leq j \leq p - 1$ ,

$$A_{\lambda, j+1} = \begin{pmatrix} 1 + \varepsilon_{2j+4}\varepsilon_{2j+1}\lambda^2 & -\varepsilon_{2j+4}\lambda \\ -\varepsilon_{2j+1}\lambda & 1 \end{pmatrix} \tag{43}$$

and

$$b_{\lambda, j+1} = (-\varepsilon_{2j+4}(1 - \mu_{j+1})\lambda^2 + (1 - \nu_{j+1})\lambda, (1 - \mu_{j+1})\lambda). \tag{44}$$

Note that  $\det(A_{\lambda, j+1}) = 1$ . For future use in Sect. 8, we remark that in fact  $A_{\lambda, j+1}$  is a product of 2 parabolic matrices as follows:

$$A_{\lambda, j+1} = \begin{pmatrix} 1 & -\varepsilon_{2j+4}\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varepsilon_{2j+1}\lambda & 1 \end{pmatrix}. \tag{45}$$

We also record that

$$(A_{\lambda, j+1})^{-1}b_{\lambda, j+1} = \left( (1 - \nu_{j+1})\lambda, (1 - \mu_{j+1})\lambda + \varepsilon_{2j+1}(1 - \nu_{j+1})\lambda^2 \right), \tag{46}$$

and in particular its coordinates are polynomials of degree at most 2 in  $\lambda$ .

Equation (40) simplifies to:

$$(A_{\lambda, p} \circ \dots \circ A_{\lambda, 1} - \mathbf{1})(x_0, y_0) = -v_{\lambda, 0}, \tag{47}$$

$$v_{\lambda, 0} = \sum_{j=0}^{p-2} A_{\lambda, p} \circ \dots \circ A_{\lambda, j+2}(b_{\lambda, j+1}) + b_{\lambda, p}. \tag{48}$$

We use the convention that the last summand  $b_{\lambda, p}$  corresponds to the index  $j = p - 1$ . We claim that for  $\lambda \gg 1$ , this equation is non-singular and has a unique solution  $z = z_{\vec{\varepsilon}} = (x_0, y_0)$ . We require the following observation on product matrices similar to the ones involved in (47), (48). Denote for  $j \geq i$ ,

$$\bar{A}_{\lambda, j, i} = A_{\lambda, j} \circ \dots \circ A_{\lambda, i},$$

and

$$\bar{A}_\lambda = \bar{A}_{\lambda, p, 1} = A_{\lambda, p} \circ \dots \circ A_{\lambda, 1}.$$

**Lemma 5.6** Put  $\bar{\varepsilon}_{j, i} = \prod_{k=i-1}^{j-1} \varepsilon_{2k+1}\varepsilon_{2k+4}$ , where  $\varepsilon_l$  is considered with  $l$  modulo  $2p$ . Note that  $\bar{\varepsilon} := \bar{\varepsilon}_{p, 1} = \prod_{k=1}^{2p} \varepsilon_k$ . Then for all  $1 \leq i \leq j \leq p$ ,

$$\bar{A}_{\lambda, j, i} = \begin{pmatrix} \bar{\varepsilon}_{j, i}\lambda^{2(j-i+1)} + \dots & -\bar{\varepsilon}_{j, i}\varepsilon_{2i-1}\lambda^{2(j-i+1)-1} + \dots \\ -\bar{\varepsilon}_{j, i}\varepsilon_{2j+2}\lambda^{2(j-i+1)-1} + \dots & \bar{\varepsilon}_{j, i}\varepsilon_{2i-1}\varepsilon_{2j+2}\lambda^{2(j-i+1)-2} + \dots \end{pmatrix},$$



where the dots denote lower order terms in  $\lambda$ . In particular

$$\bar{A}_\lambda = \begin{pmatrix} \bar{\varepsilon}\lambda^{2p} + \dots & -\bar{\varepsilon}\varepsilon_1\lambda^{2p-1} + \dots \\ -\bar{\varepsilon}\varepsilon_2\lambda^{2p-1} + \dots & \bar{\varepsilon}\varepsilon_1\varepsilon_2\lambda^{2p-2} + \dots \end{pmatrix}.$$

The proof of this lemma is a straightforward induction. Now, since for every  $2 \times 2$  matrix  $A$  with  $\det A = 1$ , one has  $\det(A - \mathbf{1}) = 2 - \text{trace}(A)$ , we compute

$$\det(\bar{A}_\lambda - \mathbf{1}) = 2 - \text{trace}(\bar{A}_\lambda) = -\bar{\varepsilon}\lambda^{2p} + \dots,$$

where the dots denote lower order terms in  $\lambda$ . In particular for  $\lambda \gg 1$ ,

$$\det(\bar{A}_\lambda - \mathbf{1}) \neq 0,$$

concluding the proof of the claim.

We now claim that the combined solution  $\{(x_{2j}, y_{2j})\}_{j=0}^{p-1}$  can moreover be shown directly to satisfy the following asymptotics as  $\lambda \rightarrow \infty$ . For each  $0 \leq j \leq p - 1$ ,

$$(x_{2j}, y_{2j}) = (\varepsilon_{2j+1}(1 - \mu_j), \varepsilon_{2j+2}(1 - \nu_{j-1})) + O\left(\frac{1}{\lambda}\right). \tag{49}$$

In particular,  $z = (x_0, y_0)$  lies in  $(-1, 1)^2$ . Moreover by these asymptotics and Eq. (42), for each  $0 \leq j \leq p - 1$ ,  $(x_{2j}, y_{2j})$  satisfies Conditions (36), (37) with respect to  $(\mu_{j+1}, \nu_{j+1})$ . Therefore  $z$  is a solution of the system (32), (33) (for  $j = p$ ) and its intermediate points are given by  $\{(x_k, y_k)\}_{k=0}^{2p-1}$  which indeed lie in  $(-1, 1)^2$ . This together with Lemma 5.5 finishes the first, existence and uniqueness, part of Proposition 5.1.

We proceed with the proof of these asymptotics. Our convention is that  $\mu_j, \nu_j$  are considered with  $j$  modulo  $p$ , and  $\varepsilon_{2j+1}, \varepsilon_{2j+2}$  are considered modulo  $2p$ . Hence for  $j = 0$  the required asymptotics are:

$$(x_0, y_0) = (\varepsilon_1(1 - \mu_1), \varepsilon_2(1 - \nu_p)) + O\left(\frac{1}{\lambda}\right).$$

Note that for a  $2 \times 2$  matrix  $A$  with  $\det(A) = 1$  and with  $(A - 1)$  invertible, one has the identity  $(A - 1)^{-1} = \frac{1}{\det(A-1)}(A^{-1} - 1)$ . Hence by Eq. (40)

$$-z = \frac{1}{\det(\bar{A}_\lambda - 1)} \left( (A_{\lambda,1})^{-1} \circ \dots \circ (A_{\lambda,p})^{-1} - 1 \right) v_{\lambda,0}.$$

For  $0 < j < p - 1$ , the summand

$$\bar{A}_{\lambda,p,j+2}(b_{\lambda,j+1})$$

in  $v_{\lambda,0}$  contributes the following summand in the expression for  $(-z)$  :

$$\frac{1}{\det(\bar{A}_\lambda - 1)} \left( (\bar{A}_{\lambda,j,1})^{-1} \circ (A_{\lambda,j+1})^{-1} (b_{\lambda,j+1}) - \bar{A}_{\lambda,p,j+2} (b_{\lambda,j+1}) \right).$$

Since  $\det(\bar{A}_\lambda - 1)$  is a polynomial of degree  $2p$  in  $\lambda$  with non-trivial leading coefficient, the coordinates of  $(\bar{A}_{\lambda,j,1})^{-1} \circ (A_{\lambda,j+1})^{-1} (b_{\lambda,j+1})$  are, by (46), polynomials in  $\lambda$  of degree at most  $2j + 2 \leq 2(p - 2) + 2 = 2p - 2$ , and the coordinates of  $\bar{A}_{\lambda,p,j+2} (b_{\lambda,j+1})$  are polynomials in  $\lambda$  of degree at most  $2(p - (j + 2) + 1) + 2 = 2p - 2j \leq 2p - 2$ , this summand contributes  $O(\frac{1}{\lambda^2})$  to the expression for  $(-z)$ .

We are left with two terms in the sum, corresponding to  $j = 0$  and  $j = p - 1$ . First consider  $j = 0$ . The corresponding summand in the expression for  $(-z)$  is

$$\begin{aligned} & \frac{1}{\det(\bar{A}_\lambda - 1)} \left( (A_{\lambda,1})^{-1} (b_{\lambda,1}) - \bar{A}_{\lambda,p,2} (b_{\lambda,1}) \right) \\ &= -\frac{1}{\det(\bar{A}_\lambda - 1)} \bar{A}_{\lambda,p,2} (b_{\lambda,1}) + O\left(\frac{1}{\lambda^2}\right) \end{aligned}$$

by Lemma 5.6

$$\begin{aligned} &= -\frac{1}{-\bar{\varepsilon}\lambda^{2p} + O(\lambda^{2p-1})} \begin{pmatrix} \bar{\varepsilon}_{p,2}\lambda^{2p-2} + \dots & -\bar{\varepsilon}_{p,2}\varepsilon_3\lambda^{2p-3} + \dots \\ -\bar{\varepsilon}_{p,2}\varepsilon_2\lambda^{2p-3} + \dots & \bar{\varepsilon}_{p,2}\varepsilon_3\varepsilon_2\lambda^{2p-4} + \dots \end{pmatrix} (b_{\lambda,1}) \\ &+ O\left(\frac{1}{\lambda^2}\right) \end{aligned}$$

by Eq. (44)

$$\begin{aligned} &= \frac{1}{\bar{\varepsilon}\lambda^{2p} + O(\lambda^{2p-1})} \left( -\bar{\varepsilon}_{p,2}\varepsilon_4(1 - \mu_1)\lambda^{2p} + O(\lambda^{2p-1}), O(\lambda^{2p-1}) \right) \\ &= -(\varepsilon_1(1 - \mu_1), 0) + O\left(\frac{1}{\lambda}\right). \end{aligned}$$

Now consider  $j = p - 1$ . The corresponding summand in the expression for  $(-z)$  is

$$\begin{aligned} & \frac{1}{\det(\bar{A}_\lambda - 1)} \left( (A_{\lambda,p-1,1})^{-1} \circ (A_{\lambda,p})^{-1} (b_{\lambda,p}) - b_{\lambda,p} \right) \\ &= \frac{1}{\det(\bar{A}_\lambda - 1)} (A_{\lambda,p-1,1})^{-1} \circ (A_{\lambda,p})^{-1} (b_{\lambda,p}) + O\left(\frac{1}{\lambda^2}\right). \end{aligned}$$

Note that by Lemma 5.6

$$(\bar{A}_{\lambda,p-1,1})^{-1} = \begin{pmatrix} \bar{\varepsilon}_{p-1,1}\varepsilon_1\varepsilon_2p\lambda^{2p-4} + \dots & \bar{\varepsilon}_{p-1,1}\varepsilon_1\lambda^{2p-3} + \dots \\ \bar{\varepsilon}_{p-1,1}\varepsilon_2p\lambda^{2p-3} + \dots & \bar{\varepsilon}_{p-1,1}\lambda^{2p-2} + \dots \end{pmatrix},$$

and by (46)

$$(A_{\lambda,p})^{-1}b_{\lambda,p} = \left( (1 - \nu_p)\lambda, (1 - \mu_p)\lambda + \varepsilon_{2p-1}(1 - \nu_p)\lambda^2 \right).$$

Hence

$$\begin{aligned} & \frac{1}{\det(\overline{A}_\lambda - 1)} (A_{\lambda,p-1,1})^{-1} \circ (A_{\lambda,p})^{-1}(b_{\lambda,p}) + O\left(\frac{1}{\lambda^2}\right) \\ &= \frac{1}{-\overline{\varepsilon}\lambda^{2p} + O(\lambda^{2p-1})} (O(\lambda^{2p-1}), \overline{\varepsilon}_{p-1,1}\varepsilon_{2p-1}(1 - \nu_p)\lambda^{2p} + O(\lambda^{2p-1})) \\ &= - (0, \varepsilon_2(1 - \nu_p)) + O\left(\frac{1}{\lambda}\right). \end{aligned}$$

In conclusion, we have obtained the asymptotics  $z = (\varepsilon_1(1 - \mu_1), \varepsilon_2(1 - \nu_p)) + O(\frac{1}{\lambda})$ , as required. The asymptotics for  $(\phi_\lambda)^{j_0}z$  are obtained similarly, starting with Eq. (41) with index  $j_0$ . This finishes the proof of the claim.

It is easy to see that these fixed points of  $(\phi_\lambda)^p$  are non-degenerate. Indeed, given a solution  $z_{\vec{\varepsilon}} = (x_0, y_0)$ , we compute

$$(D(\phi_\lambda)^p)_{(x_0,y_0)} - \mathbf{1} = \overline{A}_\lambda - \mathbf{1},$$

which we have previously shown to be non-singular for  $\lambda \gg 1$ .

### 5.5 The end of the proof of Proposition 5.1

It remains to calculate the action gaps for the periodic orbits found above. Choose

$$\eta_{\alpha_\lambda} := \gamma(a)^{\#m_1} \# \gamma(b)^{\#n_1} \# \dots \# \gamma(a)^{\#m_p} \# \gamma(b)^{\#n_p}$$

as the reference loop in the class  $\alpha_\lambda$ . Then for  $\vec{\varepsilon} \in \{\pm 1\}^{2p}$  we calculate that the action of the orbit corresponding to  $z_{\vec{\varepsilon}}$  is

$$A(z_{\vec{\varepsilon}}) = \frac{\lambda}{2} \sum_{j=0}^{p-1} (\varepsilon_{2j+1}(1 - \mu_{j+1})^2 - \varepsilon_{2j+4}(1 - \nu_{j+1})^2) + O(1), \tag{50}$$

as  $\lambda \rightarrow \infty$ .

Indeed each of the  $2p$  terms corresponds to an intermediate path, and for example, using the asymptotics (49) and the definition of  $h_0(s)$ , the first term is

$$\begin{aligned} A_1 &= \int_0^1 \lambda h(f_V^t(x_0, y_0)) dt - \int_{\{f_V^t(x_0, y_0)\}_{t \in [0,1]}} x dy \\ &= \lambda h(x_0) - \lambda \mu_1 x_0 + O(1) = \varepsilon_{x_0} (\lambda h(|x_0|) - \lambda \mu_1 |x_0|) + O(1) \end{aligned}$$

$$\begin{aligned}
 &= \lambda \varepsilon_{x_0} \left( (1 - \mu_1) - \frac{(1 - \mu_1)^2}{2} - \mu_1(1 - \mu_1) \right) + O(1) \\
 &= \frac{\lambda}{2} \varepsilon_{x_0} (1 - \mu_1)^2 + O(1).
 \end{aligned}$$

Recall that in our conventions  $\varepsilon_{x_0} = \varepsilon_1$ , hence the term we just computed corresponds to the first summand for  $j = 0$ . There exist  $\{\mu_j, \nu_j \in (0, 1)\}_{1 \leq j \leq p}$  with all pairwise ratios in  $\mathbb{Q}$ , such that the coefficients

$$\left\{ \sum_{j=0}^{p-1} \left( \varepsilon_{2j+1} (1 - \mu_{j+1})^2 - \varepsilon_{2j+4} (1 - \nu_{j+1})^2 \right) \right\}_{\vec{\varepsilon} \in \{\pm 1\}^{2p}}$$

of  $\frac{\lambda}{2}$  in Eq. (50) are all different, and hence the action differences between the different  $p$ -tuples of fixed points of  $(\phi_\lambda)^p$  in the free homotopy class  $\alpha_\lambda$  grow linearly. This finishes the proof of Proposition 5.1. □

### 6 Alternative approach in dimension 2

For Theorem 1.2 above, there is an alternative proof in dimension 2, that is when  $M$  is a point. We present it here and note that it relies very strongly on two-dimensional methods.

*Proof* (A 2d proof of Theorem 1.2 in 2d)

As in the proof of Proposition 5.1 we fix a large  $\lambda \gg 1$ , and choose an appropriate small smoothing  $u$  of  $u_0(s) = 1 - |s|$ , and construct  $\phi_\lambda$  as the composition  $f_H f_V$  of diffeomorphisms of two annuli embedded in  $\Sigma$  as in Fig. 1, where in each annulus (isomorphic to  $C_* = [-1, 1] \times \mathbb{R}/L\mathbb{R}$ , for  $L \geq 4$ ) the diffeomorphism is given by  $f(x, [y]) = (x, y + \lambda u(x))$ . Note that  $f$  is the time-one map of the autonomous Hamiltonian isotopy  $f^t(x, [y]) = (x, [y + t \cdot \lambda u(x)])$ .

For a free homotopy class  $\beta$  of loops in a surface  $\Sigma$ , denote by

$$\text{si}(\beta) = \min \# \{ \text{double points of } b \},$$

where the minimum is taken over all immersed loops  $b : S^1 \rightarrow \Sigma$  in general position with  $[b] = \beta$ . This is the geometric self-intersection number of  $\beta$ . In particular,  $\beta$  is represented by a simple closed curve if and only if  $\text{si}(\beta) = 0$ . Recall that by Constraint 1.6 from Sect. 1.4 a Hamiltonian diffeomorphism  $\phi$  is not autonomous if it has a non-constant orbit in a primitive class  $\alpha$  that is not simple, i.e.,  $\text{si}(\alpha(\phi, z)) > 0$  for a fixed point  $z$ . Now consider the diffeomorphism

$$\phi_\lambda \in \text{Ham}(\Sigma)$$

constructed earlier. Fix coefficients  $0 < \mu, \nu < 1$ , with  $\frac{\mu}{\nu} \in \mathbb{Q}_{>0}$  to be determined later. Consider all  $\lambda \in \mathbb{R}_{>0}$ , such that  $m = \frac{\mu}{L} \lambda, n = \frac{\nu}{L} \lambda$  satisfy  $m, n \in \mathbb{Z}_{>0}$ . Clearly there are arbitrarily large  $\lambda$  with this property.

Consider the free homotopy class  $\beta_\lambda = a^m b^n / \text{conj}$ . By [29]

$$\text{si}(\beta_\lambda) = m \cdot n + (m - 1) \cdot (n - 1) > 0,$$

and hence  $\beta_\lambda$  is not simple.

**Claim 6.1** *There exist  $\mu, \nu \in (0, 1)$  with  $\frac{\mu}{\nu} \in \mathbb{Q}_{>0}$ , such that for  $\lambda \gg 1$  the diffeomorphism  $\phi_\lambda$  has exactly 4 non-degenerate critical orbits with distinct action values in the class  $\beta_\lambda$  and minimal action gap  $D_\lambda = c \cdot \lambda + O(1)$ , as  $\lambda \rightarrow \infty$ .*

Claim 6.1 implies that for each  $0 < c' < c$ ,  $0 < \epsilon$  and  $\lambda$  sufficiently large, there exists a window  $(\mathcal{A} - c' \cdot \lambda - \epsilon, \mathcal{A} + c' \cdot \lambda + \epsilon)$ , with  $\mathcal{A}$  the action value of a critical orbit of  $\phi_\lambda$  in class  $\beta_\lambda$ , that does not contain the critical value of any other orbit in this class. Hence the comparison map

$$HF^{(\mathcal{A} - c' \cdot \lambda - \epsilon, \mathcal{A} + \epsilon)}(\phi_\lambda)_{\beta_\lambda} \rightarrow HF^{(\mathcal{A} - \epsilon, \mathcal{A} + c' \cdot \lambda + \epsilon)}(\phi_\lambda)_{\beta_\lambda},$$

is an isomorphism. We claim that this implies that

$$d(\phi_\lambda, \theta) \geq \frac{c'}{2} \cdot \lambda$$

for any autonomous  $\theta \in \text{Ham}(M)$ . Indeed assume that  $d(\phi_\lambda, \theta) < \frac{c'}{2} \cdot \lambda$ . Consider the diagram of continuation maps

$$HF^{(\mathcal{A} - c' \cdot \lambda - \epsilon, \mathcal{A} + \epsilon)}(\phi_\lambda)_{\beta_\lambda} \rightarrow HF^{(\mathcal{A} - \frac{c'}{2} \cdot \lambda - \epsilon, \mathcal{A} + \frac{c'}{2} \cdot \lambda + \epsilon)}(\theta)_{\beta_\lambda} \rightarrow HF^{(\mathcal{A} - \epsilon, \mathcal{A} + c' \cdot \lambda + \epsilon)}(\phi_\lambda)_{\beta_\lambda}.$$

Since by naturality of continuation maps this composition is a comparison map that is by construction an isomorphism, we see that  $HF^{(\mathcal{A} - \frac{c'}{2} \cdot \lambda - \epsilon, \mathcal{A} + \frac{c'}{2} \cdot \lambda + \epsilon)}(\theta)_{\beta_\lambda}$  does not vanish, and hence, there exists a critical orbit of  $\theta$  in the primitive non-simple homotopy class  $\beta_\lambda$ , in contradiction to Constraint 1.6. Note that here we are working with Floer homology of a degenerate Hamiltonian diffeomorphism, but the implication still holds by continuous dependence on parameters of solutions of ODE's in a fixed time interval. □

*Proof* (Claim 6.1) We calculate the fixed points of  $\phi_\lambda$  in class  $\beta_\lambda$  and their actions. We will prove later that all fixed points  $(x_0, y_0)$  of  $\phi_\lambda$  in class  $\beta_\lambda$  lie in the square  $A$ , and in fact their class in  $\pi_1(C, 0_A)$  is exactly  $a^m b^n$ . For now, consider all such fixed points. It turns out that the fixed point equation is

$$u(x_0) = \mu \tag{51}$$

$$u(-y_0) = \nu. \tag{52}$$

Indeed since, as we show below, the intermediate paths  $\{f_V^t(z)\}$  and  $\{f_H^t f_V(z)\}$  have classes

$$[\{f_V^t(z)\}] = a^m, \quad [\{f_H^t f_V(z)\}] = b^n$$

in  $\pi_1(C, A)$ . Hence lifting to the universal cover of  $C_*$  the fixed point equation in this class translates to

$$(x_0, y_0) = HV \circ r_n \circ \tilde{f} \circ VH \circ r_m \circ \tilde{f}(x_0, y_0),$$

where for  $p \in \mathbb{Z}$ ,

$$\begin{aligned} r_p &: (-1, 1) \times \mathbb{R} \rightarrow (-1, 1) \times \mathbb{R}, \\ r_p(x, y) &= (x, y - p \cdot L) \end{aligned}$$

is the *reduction map* that maps  $(-1, 1) \times (p \cdot L - 1, p \cdot L + 1)$  isomorphically onto  $(-1, 1) \times (-1, 1)$ , and  $\tilde{f}$  is the lift of  $f$  to  $\tilde{C}_* = [-1, 1] \times \mathbb{R}$  given by the isotopy  $\{f^t\}$ . That is  $\tilde{f}(x, y) = (x, y + \lambda u(x))$ . Let us derive in detail the fixed point equation. We compute

$$\begin{aligned} (x_0, y_0) &= HV \circ r_n \circ \tilde{f} \circ VH \circ r_m \circ \tilde{f}(x_0, y_0) \\ &= HV \circ r_n \circ \tilde{f} \circ VH(x_0, y_0 + \lambda u(x_0) - \lambda \mu) \\ &= HV \circ r_n \circ \tilde{f}(-y_0 - \lambda u(x_0) + \lambda \mu, x_0) \\ &= HV \circ r_n(-y_0 - \lambda u(x_0) + \lambda \mu, x_0 + \lambda u(-y_0 - \lambda u(x_0) + \lambda \mu)) \\ &= HV(-y_0 - \lambda u(x_0) + \lambda \mu, x_0 + \lambda u(-y_0 - \lambda u(x_0) + \lambda \mu) - \lambda v) \\ &= (x_0 + \lambda u(-y_0 - \lambda u(x_0) + \lambda \mu) - \lambda v, y_0 + \lambda u(x_0) - \lambda \mu). \end{aligned}$$

Hence the fixed point equation reduces to the system of equations

$$\lambda u(x_0) - \lambda \mu = 0, \tag{53}$$

$$\lambda u(-y_0 - \lambda u(x_0) + \lambda \mu) - \lambda v = 0, \tag{54}$$

which is clearly equivalent to the system (51), (52).

A calculation shows that for a sufficiently small smoothing  $u$  of  $u_0$ , as described in Sect. 5, the system (51), (52) has four solutions

$$(x_0, y_0) = z_{\vec{\varepsilon}} = (\varepsilon_1(1 - \mu), \varepsilon_2(1 - v)),$$

where  $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \{\pm 1\}^2$ . Note that these solutions lie in the region where  $u = u_0$ .

Choose  $\eta_{\beta_\lambda} := \gamma(a)^{\#m} \# \gamma(b)^{\#n}$  as the reference loop in the class  $\beta_\lambda$ . An easy calculation shows that the action of the fixed point  $z_{\vec{\varepsilon}}$  corresponding to  $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \{\pm 1\}^2$  is

$$\mathcal{A}_H(z_{\vec{\varepsilon}}) = \frac{\lambda}{2} \left( \varepsilon_1(1 - \mu)^2 - \varepsilon_2(1 - v)^2 \right).$$

Hence for a proof of Claim 6.1, it is sufficient to choose  $\mu, v$  outside the line  $\{\mu - v = 0\}$ .

We leave the rather easy proof of non-degeneracy to the interested reader.

It remains to show that any fixed point  $(x_0, y_0)$  in class  $\beta_\lambda$  lies in  $A$ . First of all, clearly  $(x_0, y_0) \in A \cup B$ , since otherwise, by consideration of the supports of  $f_H, f_V$ , one would have had to have  $m = 0$  or  $n = 0$ . There are 4 types of intermediate paths  $\{f_V^l(x_0, y_0)\} \# \{f_H^k(x_0, y_0)\}$ ,  $A \rightarrow A \rightarrow A, A \rightarrow B \rightarrow A, B \rightarrow A \rightarrow B, B \rightarrow B \rightarrow B$ . These orbits have representatives of type  $a^k b^l, a^k c b^l, c^{-1} a^k b^l, c a^k c^{-1} b^l$  in  $\pi_1(\Sigma)$ , respectively (here  $k, l$  are generic notation for a pair integers). We see that since we can consider the problem in  $\mathcal{L}(C)$  instead of  $\mathcal{L}\Sigma$ , by a hyperbolic geometry argument,  $a^m b^n$ , being a cyclically reduced word in the free group  $Free\langle a, b, c \rangle$ , is conjugate to one of these representatives if and only if it is of type  $A \rightarrow A \rightarrow A$  and  $k = m, l = n$ . Hence  $(x_0, y_0) \in A$ .

This proves Claim 6.1 and hence finishes the alternative proof in dimension 2. □

### 7 Interaction with the Conley conjecture

The goal of this section is to prove Theorem 1.4.

*Proof* (Theorem 1.4) Put  $\mathcal{G} = \text{Ham}$ , and for  $\phi \in \mathcal{G}$  denote by  $\mathcal{P}^k(\phi)$  the set of all its primitive contractible  $k$ -periodic orbits. Define

$$\mathcal{G}_k := \left\{ \phi \in \mathcal{G} \mid \text{all } x \in \mathcal{P}^k(\phi) \text{ are non-degenerate} \right\},$$

$$\mathcal{G}'_k := \left\{ \phi \in \mathcal{G}_k \mid \mathcal{P}^k(\phi) \text{ is non-empty} \right\}.$$

Note that  $\mathcal{G}_k$  is an open dense subset of  $\mathcal{G}$  in  $C^\infty$ -topology, and hence

$$\mathcal{G}_\infty := \bigcap_{k \geq 1} \mathcal{G}_k$$

is a residual subset of  $\mathcal{G}$ . Moreover  $\mathcal{G}'_k$  is an open subset of  $\mathcal{G}$  and by Constraint 1.7 for  $k \geq 2$ ,

$$\mathcal{G}'_k \subset \text{Ham} \setminus \text{Aut}. \tag{55}$$

The non-degenerate case of the Conley conjecture [47] implies that if  $\phi \in \mathcal{G}_1$  then  $\mathcal{P}^k(\phi)$  is non-empty for some  $k \geq 2$ . Therefore if  $\phi \in \mathcal{G}_\infty$  then, being in particular in  $\mathcal{G}_1$ , it lies in  $\mathcal{G}'_k$  for some  $k \geq 2$ . Thus

$$\mathcal{G}_\infty \subset \bigcup_{k \geq 2} \mathcal{G}'_k,$$

whence, as  $\mathcal{G}_\infty$ , being residual, is dense in the  $C^\infty$ -topology,  $\bigcup_{k \geq 2} \mathcal{G}'_k$  is dense in the  $C^\infty$ -topology. Hence by (55), we conclude that  $\text{Ham} \setminus \text{Aut}$  contains an open dense set in the  $C^\infty$ -topology.

Corollary 4.4 implies that this  $C^\infty$  open and dense subset is contained in a Hofer-open subset  $\bigcup_{k \geq 2} \{\phi \in \mathcal{G} \mid w_{k, pt_M}(\phi) > 0\}$ , which by Proposition 4.2 Item (iv) is in turn contained in  $\text{Ham} \setminus \text{Aut}$ , which completes the proof of Theorem 1.4. □

## 8 Discussion

### 8.1 Extension to monotone manifolds

Consider a spherically monotone symplectic manifold  $(M, \omega)$  - that is

$$\langle [\omega], B \rangle = \kappa \cdot \langle c_1(TM), B \rangle$$

for all

$$B \in \text{Image}(\pi_2(M) \rightarrow H_2(M, \mathbb{Z}))$$

for a constant  $\kappa > 0$ . Assume that  $(M, \omega)$  is moreover  $\alpha$ -toroidally-monotone for a class  $\alpha \in \pi_0(\mathcal{L}M)$ . This means that

$$\langle [\omega], B \rangle = \kappa \cdot \langle c_1(TM), B \rangle$$

(the same  $\kappa!$ ) for all

$$B \in \text{Image}(\pi_1(\mathcal{L}_\alpha M) \rightarrow H_2(M, \mathbb{Z})),$$

where the map sends a loop in  $\mathcal{L}_\alpha M$  to the fundamental class of the associated  $T^2$ -cycle in  $M$ . We make a choice of a base point  $\eta_\alpha \in \mathcal{L}_\alpha M$  and of a trivialization of  $\eta_\alpha^*(TM, S^1)$  as a symplectic vector bundle over  $S^1$ .

Let  $k \geq 2$  be an integer. The Hamiltonian Floer homology  $HF_*(\tilde{\phi}^k)_\alpha$  in class  $\alpha$  with coefficients in a base field  $\mathcal{K}$  can be defined as a graded module over the Novikov ring (i.e., the ring of semi-infinite Laurent series)  $\mathcal{K}[[q^{-1}, q]$ , where  $\deg(q) = 2N_M$ , twice the minimal Chern number of  $(M, \omega)$ . The filtered Floer homology  $HF_*^{(-\infty, b)}(\tilde{\phi}^k)_\alpha$  will be a module over the ring  $\mathcal{K}[[q^{-1}]]$  of formal power series in  $q^{-1}$ . Noting that since multiplication by  $q$  shifts the degree by  $2N_M$  the homology groups  $HF_r^{(-\infty, b)}(\tilde{\phi}^k)_\alpha$  in a given degree  $r$  will be finite-dimensional vector spaces over the base field  $\mathcal{K}$ . A similar remark holds for degree  $r$  Floer homology groups  $HF_r^{(a, b)}(\tilde{\phi}^k)_\alpha$  in finite action windows.

Thus it is straightforward to see that the considerations of Sect. 4 are fully applicable in this case and give us an invariant  $w_{k, \alpha, r}(\tilde{\phi})$  and a  $\mathbb{Z}_k$  persistence module (in the sense of Remark 4.8)

$$(V_r(k, \alpha, \tilde{\phi}), A_r(k, \alpha, \tilde{\phi})) = \left( HF_r^{(-\infty, \cdot)}(\tilde{\phi}^k)_\alpha, [\mathbf{R}_k(\tilde{\phi})] \right)$$

with  $A_r(k, \alpha, \tilde{\phi})^k = \mathbf{1}$ . Here the  $\mathbb{Z}_k$  action  $A_r(k, \alpha, \tilde{\phi}) = [\mathbf{R}_k]$  is induced by the loop rotation operator.

Put  $\text{Area}(q) = \Omega := \kappa \cdot N_M$ , for the minimal positive symplectic area of a sphere in  $M$ . Now we note that multiplication by  $q$  on the chain level gives isomorphisms

$$(q \cdot) : V_r(k, \alpha, \tilde{\phi}) \rightarrow V_{r+2N_M}(k, \alpha, \tilde{\phi})^\Omega$$



and

$$(q \cdot) : HF_r^{(a,b)}(\tilde{\phi})_\alpha \rightarrow HF_{r+2N_M}^{(a,b)+\Omega}(\tilde{\phi})_\alpha.$$

This means, since the definitions of our invariants depend only on action differences, that  $\max_{r \in \mathbb{Z}}(w_{k,\alpha,r}) = \max_{r \in \mathbb{Z}/(2N_M)}(w_{k,\alpha,r})$ . Considering the actions of loops in Ham on Floer homology, which result in isomorphisms up to shifts in grading and in the action filtration, we see that

$$w_{k,\alpha}(\tilde{\phi}) := \max_{r \in \mathbb{Z}/(2N_M)} w_{k,\alpha,r}(\tilde{\phi})$$

depends only on the projection  $\phi$  of  $\tilde{\phi}$  to Ham. This invariant  $w_{k,\alpha}(\phi)$  still satisfies all the necessary properties.

Similarly, considering the actions of loops in Ham on Floer homology, we see that if  $\phi \in \mathcal{G}_p$  has a root  $\psi$  of order  $p$ , then for any degree  $r$  and  $k = p$ , prime, the  $\mathbb{Z}_p$  persistence module  $(V_r(p, \tilde{\phi}, \alpha), A_r(p, \tilde{\phi}, \alpha))$  given by the filtered Floer homology of  $\tilde{\phi}^p$  in class  $\alpha$  for any lift  $\tilde{\phi}$  of  $\phi$  to  $\widetilde{\text{Ham}}$ , and the loop rotation operator on it, is a full  $p$ -th power  $\mathbb{Z}_p$  persistence module. Indeed, taking a lift  $\tilde{\psi}$  of  $\psi$  to  $\widetilde{\text{Ham}}$ , we obtain a special lift  $\tilde{\psi}^p$  of  $\phi$ , whose persistence module  $(V_r(p, \tilde{\psi}^p, \alpha), A_r(p, \tilde{\psi}^p, \alpha))$  is a full  $p$ -th power  $\mathbb{Z}_p$  persistence module (with full root of order  $p$  furnished by  $A_r(p^2, \tilde{\psi}, \alpha)$ ). Moreover  $(V_r(p, \tilde{\phi}, \alpha), A_r(p, \tilde{\phi}, \alpha))$  is isomorphic, by the action of a corresponding loop in Ham to  $(V_{r'}(p, \tilde{\psi}^p, \alpha)^c, A_{r'}(p, \tilde{\psi}^p, \alpha)^c)$  for some  $r'$  with a certain shift  $c \in \mathbb{R}$ , and is therefore a full  $p$ -th power  $\mathbb{Z}_p$  persistence module.

In fact, this property is invariant under shifts of the persistence module and up to shifts in degree, as explained above, and hence, we can consider only a finite number  $(2N_M)$  of degrees  $r$ . Moreover we see that the multiplicity-sensitive spread  $\mu_p(V, A)$  of a  $\mathbb{Z}_p$  persistence module  $(V, A)$  from Definition 4.20 is clearly independent of shifts. Hence we conclude that for Hamiltonian diffeomorphism  $\phi \in \text{Ham}$  the multiplicity-sensitive spread  $\mu_p(\phi)$  is well defined and satisfies Theorem 4.23, giving a lower bound on the Hofer distance between  $\phi$  and any  $\theta \in \text{Powers}_p$ .

It seems likely that this construction applies to negatively monotone symplectic manifolds, for which by a result of Ginzburg and Gürel [27] the Conley conjecture holds. Hence one expects to generalize Theorem 1.4 to this case.

Theorem 1.4 does generalize to the case of  $\mathbb{C}P^n$  or more generally symplectic manifolds  $(M, \omega)$  with minimal Maslov number  $N_M \geq n + 1$ , where  $2n = \dim M$ , by Ginzburg and Gürel [26]. We leave the details to the interested reader.

As for generalization of Theorems 1.2 and 1.3, we consider the following adaptations of our example.

*Example 8.1* Consider the surface  $\Sigma$  of genus at least 4, and a symplectically aspherical manifold  $N$  of  $\dim N = 2n - 2$ . Consider the diffeomorphism  $\phi_\lambda$  of  $\Sigma$  and its stabilization  $\phi_\lambda \times \mathbf{1}_N$ . Take the one-point size  $\Omega$  blow-up

$$M = (\text{Bl}_{(x,y)} X, \omega_M)$$

of

$$X = \Sigma \times N$$

at a point  $(x, y)$  with  $x$  outside the union of annuli where  $\phi_\lambda$  is supported. Denote by  $E \subset M$  the exceptional divisor. Let  $\bar{\alpha}_\lambda \in \pi_0(\mathcal{L}M)$  be the lift of the class  $\alpha \times pt_N \in \pi_0(\mathcal{L}(\Sigma \times N))$ . As  $\dim M = 2n$ , the evaluation of  $[\omega_M]$  on  $\text{Image}(\pi_2(M) \rightarrow H_2(M, \mathbb{Z}))$  and on  $\text{Image}(\pi_1(\mathcal{L}_{\bar{\alpha}_\lambda} M) \rightarrow H_2(M, \mathbb{Z}))$  is generated by its values on  $\mathbb{C}P^1 \subset E$ , and moreover  $\langle [\omega_M], [\mathbb{C}P^1] \rangle = \Omega$ ,  $\langle c_1(TM), [\mathbb{C}P^1] \rangle = n - 1$ .

Look at the fixed points of  $\phi_\lambda^p$  in the class  $\alpha_\lambda$ . Let their actions, in increasing order, be  $a_1, \dots, a_l$  so that  $a_i - a_j \sim \lambda$ , and the indices, calculated with respect to a well-chosen trivialization of  $T\Sigma$  along the basic cycles  $a, b$  on  $\Sigma$ , are  $I_1, \dots, I_l$ .

The trivialization we use: on one of the two annuli (identified with  $T^*S^1$ ) the trivialization of  $T\Sigma$  is given by the vertical Lagrangian distribution, and on the other by the Lagrangian distribution tangent to the zero section (clearly, these two trivializations are homotopic)—it is easy to see that these two trivializations glue well together to a trivialization along  $\alpha_\lambda$ .

Since by the proof of Proposition 5.1, Eq. (45), the linearization of  $\phi_\lambda^p$  at each fixed point is the product of  $2p$  parabolic  $2 \times 2$  matrices, and since multiplication with  $\mathbf{1}_N$  contributes a finite correction to Conley–Zehnder indices, there exists  $C > 0$  such that  $|I_j| \leq C$  for all  $j$ . The crucial point here is that  $C$  is independent of  $\lambda$ .

Recall that  $\Omega$  stands for the area of  $\mathbb{C}P^1$  in the exceptional divisor.

Now fix degree  $r = I_1$ , for instance. Observe that the actions with this index are  $a_1$ , as well as  $a_k + \Omega \cdot q_k$ , where  $q_k$  is the solution of  $I_1 = I_k + 2(n - 1)q_k$ ,  $k \neq 1$ . Observe that  $q_k$  are bounded and thus the differences  $|a_1 - (a_k + \Omega \cdot q_k)|$  are of order  $\lambda$ . A similar analysis of actions goes through for  $r = I_1 + 1$  and  $r = I_1 - 1$ . It follows by Proposition 5.1 that in the degree  $r := I_1$  we have that  $w_{p,\alpha,r}(\tilde{\phi}) \sim \lambda$ , and similarly,  $\mu_p(\phi) \sim \lambda$ , as required.

*Example 8.2* Using the same argument as in Example 8.1, we can show analogues of Theorems 1.2, 1.3 for the manifold  $\Sigma \times N$ , where now  $N$  is any spherically monotone symplectic manifold. The diffeomorphisms we take are stabilizations  $\phi_\lambda \times \mathbf{1}_N$  of  $\phi_\lambda$  from Sect. 5, and the class is  $\alpha_\lambda \times pt_N$ .

It would be interesting to see how the methods of this paper extend further to non-monotone symplectic manifolds.

*Remark 8.3* (M. Khanevsky) The following short argument of Michael Khanevsky shows that our construction of  $\phi_\lambda$  when  $C$  is embedded in  $S^2$  gives a family of Hamiltonian diffeomorphisms that is at a bounded distance to  $\text{Aut}$  in  $\text{Ham}(S^2)$ . Indeed on  $S^2$ , the diffeomorphisms  $f_\lambda = f_{\lambda,V}$  and  $f_{\lambda,H}$  are conjugate (for example when the annuli are considered as tubular neighborhoods of two orthogonal great circles on  $S^2$ , the two maps are conjugate by a rotation). Denote the conjugating Hamiltonian diffeomorphism by  $\psi$ . Then

$$d(\phi_\lambda, f_{2\lambda}) = d(f_{\lambda,H} f_\lambda, f_{2\lambda}) = d(\psi f_\lambda \psi^{-1} f_\lambda, f_{2\lambda})$$

(by the bi-invariance of Hofer’s metric)

$$= d(\psi f_\lambda \psi^{-1} f_\lambda^{-1}, 1) \leq d(\psi, 1) + d(f_\lambda \psi^{-1} f_\lambda^{-1}) = 2d(\psi, 1).$$

Hence  $d(\phi_\lambda, \text{Aut}) \leq d(\phi_\lambda, \{f_\lambda\}_{\lambda \in \mathbb{R}}) \leq 2d(\psi, 1)$  for all  $\lambda$ .

This remark highlights the role of the handles which were attached to  $S^2$  in order to get the egg-beater map on the surface  $\Sigma$  of higher genus. In the course of the proof, we used the handles in order to distinguish periodic orbits of the map according to the free homotopy classes. Choosing carefully a homotopy class, we have been left with a “small” collection of orbits having large action gaps, which enabled us to apply the machinery of the persistence modules. Khanevsky’s example shows that the handles are not just a technical amenity. They serve as an obstruction to the existence of the conjugating Hamiltonian diffeomorphism  $\psi \in \text{Ham}(\Sigma)$  between  $f_H$  and  $f_V$ .

### 8.2 Spectral spread and persistence modules

We start with a modified and simplified version of the spectral spread  $w_{k,\alpha}$ . The modification affects the choice of the windows in Definition 4.1: instead of arbitrary windows  $(a, b)$ , we work with semi-infinite windows  $(-\infty, b)$ . Let us spell this out.

For  $k \in \mathbb{Z}$ ,  $k \geq 1$ , and a Hamiltonian  $H \in \mathcal{H}$  we make the following definition. Put

$$S_k := T_k - \mathbf{1},$$

where  $T_k$  is the loop rotation operator. Composing  $S_k$  with the comparison map  $j_d : HF^{(-\infty, b)}(H^{(k)})_\alpha \rightarrow HF^{(-\infty, b+d)}(H^{(k)})_\alpha$ , we obtain a map

$$j_d \circ S_k = S_k \circ j_d : HF^{(-\infty, b)}(H^{(k)})_\alpha \rightarrow HF^{(-\infty, b+d)}(H^{(k)})_\alpha.$$

**Definition 8.4** (*Modified spread*)

$$\hat{w}_{k,\alpha}(H) := \sup \{d \geq 0 : j_d \circ S_k \neq 0 \text{ for some window } (-\infty, b)\}$$

Note that  $\hat{w}_{k,\alpha} \leq w_{k,\alpha}$ . Nevertheless our proof of Theorem 1.2 remains unchanged if one uses  $\hat{w}$  instead of  $w$ .

The main point of this section is that the modified spectral spread  $\hat{w}_{k,\alpha}$  admits a concise description in terms of persistent homology (see Sect. 4.2). In what follows, we use Floer homology with coefficients in any field  $\mathcal{K}$ .

In the context of Floer homology and loop rotation operators, we have to deal with persistence modules equipped with extra data, the family of morphisms  $A_t : V_t \rightarrow V_t$ . Furthermore we assume that

$$A_t \circ \theta_{st} = \theta_{st} \circ A_s \quad \forall s < t. \tag{56}$$

We denote the enriched persistence module by  $(V, \theta; A)$ .

**Definition 8.5** The spread  $\hat{w}(V, \theta; A)$  is defined as the supremum of  $d > 0$  such that for some  $s \in \mathbb{R}$  the map  $\theta_{s,s+d}(A_s - \mathbf{1}) \neq 0$ .

It turns out that the spread can be expressed in terms of the barcode of an auxiliary persistence module which is defined as follows. Put  $W_t = \{v \in V_t : A_t v_t = v_t\}$  and  $L_t = V_t/W_t$ . We write  $p_t : V_t \rightarrow L_t$  for the natural projection. By (56)  $\theta_{st}(W_s) \subset W_t$ , and hence the morphism  $\theta_{st}$  descends to a morphism  $\pi_{st} : L_s \rightarrow L_t$ . We get a new persistence module  $(L, \pi)$ .

Put  $\beta(L, \pi)$  for the longest finite bar in the barcode of the persistence module  $(L, \pi)$ . This notion is related to Usher’s notion of boundary depth (cf. [53]).

**Theorem 8.6**  $\hat{w}(V, \theta; A) = \beta(L, \pi)$ .

*Proof* Note that for  $s < t$  and  $v \in V_s$

$$\begin{aligned} \theta_{st}(A_s - \mathbf{1})v &= (A_t - \mathbf{1})\theta_{st}v \neq 0 \Leftrightarrow \\ A_t\theta_{st}v &\neq \theta_{st}v \Leftrightarrow \theta_{st}v \notin W_t \Leftrightarrow p_t(\theta_{st}(v)) = \pi_{st}(p_s(v)) \neq 0. \end{aligned}$$

This yields

$$\theta_{st}(A_s - \mathbf{1}) \neq 0 \Leftrightarrow (A_t - \mathbf{1})\theta_{st} \neq 0 \Leftrightarrow \pi_{st} \neq 0.$$

Note that the reverse implication holds since  $p_s$  is onto. Hence  $\hat{w}(V, \theta; A) = \beta(L, \pi)$ . □

**Problem 8.7** It would be interesting to extract the invariant  $w_{k,\alpha}$  in a similar way from two-parametric persistence modules.

It is easy to see that for two persistence modules  $K, L$ ,

$$|\beta(K) - \beta(L)| \leq 2d_{\text{bottle}}(\mathcal{B}(K), \mathcal{B}(L)),$$

which shows [cf. Eq. (28)] that for an integer  $k \geq 2$ ,  $\alpha \in \pi_0(\mathcal{LM})$ , and  $\phi \in \mathcal{G}_k$ , the map  $\phi \mapsto \hat{w}_{k,\alpha}(\phi) = \hat{w}(V(k, \alpha, \phi), A(k, \alpha, \phi))$  is Lipschitz with constant  $2k$  with respect to the Hofer metric. However, by considering continuation maps directly and separating the positive and the negative parts of the Hofer metric, we show in Proposition 4.2, Item (iii) that in fact it is Lipschitz with constant  $k$ . It would be interesting to modify the bottleneck distance on the space of barcodes to account for this discrepancy.

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