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Global solutions of nonlinear wave equations with large data

Shiwu Yang

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Abstract In this paper, we give a criterion on the Cauchy data for the semilinear wave equations satisfying the null condition in $\mathbb{R}^+ \times \mathbb{R}^3$ such that the data can be arbitrarily large, while the solution is still globally in time in the future.

Keywords Large data · Global solutions · Semilinear wave equations

Mathematics Subject Classification Primary 35L05

1 Introduction

In this paper, we study the Cauchy problem to the semilinear wave equations

$$\begin{cases} \Box\phi = (-\partial_t^2 + \Delta)\phi = F(\phi, \partial\phi), \\ \phi(0, x) = \phi_0(x), \partial_t\phi(0, x) = \phi_{1(x)} \end{cases} \tag{1}$$

on the Minkowski space of $\mathbb{R}^{3+1}$. The nonlinearity F is assumed to satisfy the null condition outside a large cylinder $\{(t, x)||x| \leq R\}$, that is,

$$F(\phi, \partial\phi) = A^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + O\left(|\phi|^3 + |\partial\phi|^3 + |\phi|^N + |\partial\phi|^N\right), \quad |x| \geq R, \tag{2}$$

where $A^{\alpha\beta}$ are constants such that $A^{\alpha\beta}\xi_\alpha\xi_\beta = 0$ whenever $\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ and N is a given integer larger than 3. Inside the cylinder, we assume F is at least quadratic

S. Yang (✉)
DPMMS, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK
e-mail: S.Yang@dpmms.cam.ac.uk

Birkhäuser

in terms of ϕ, $\partial\phi$. The data (ϕ_0, ϕ_1) are assumed to be smooth but can be arbitrarily large in the energy space.

The long time behavior of solutions of nonlinear wave equations has drawn considerable attention in the past decades. When the data are small, the classical result of Christodoulou [1] and Klainerman [8] shows that the solution of the nonlinear wave equation (1) is globally in time for all sufficiently small initial data. Generalizations and variants can be found in [5,11,15,18,19,23,24,29] and references therein. One of the most remarkable related results should be the global nonlinear stability of Minkowski space first proved by Christodoulou and Klainerman [3] and later an alternative proof contributed by Lindblad and Rodnianski [14].

For the general large data case which is concerned in the current study, global existence results for the related wave map problems in dimension $2+1$ have been established with initial energy below that of any nontrivial harmonic maps, see e.g., [12,25,26,28]. Failure of this energy constriction may lead to finite time blow up, see e.g., [13,20–22,27]. For the $3+1$ dimensional case of Eq. (1), the concrete example

$$\Box\phi = |\partial_t\phi|^2 - |\nabla\phi|^2$$

shows that the solution can blow up in finite time for general large data. For the details, we refer to Klainerman [6].

In a recent work [30] of Wang–Yu, they constructed an open set of Cauchy data for the semilinear wave equations satisfying the null condition such that the energy is arbitrarily large, while the solution exists globally in the future. The construction is indirect. They in fact impose the radiation data at the past null infinity and then solve the equation to some time $t_0 < 0$ to obtain the Cauchy data. Their work relied on the *short pulse* method of Christodoulou in his monumental work [2] on the formation of trapped surface. Extensions and refinements of Christodoulou's result are contributed by, e.g., Klainerman and Rodnianski [10], Luk and Rodnianski [16,17], Klainerman et al. [9], Yu [31,35,36].

However, the nonlinear terms considered in Wang–Yu's work are quite restrictive. In fact, only quadratic null forms are allowed and cubic or higher-order nonlinearities are excluded for consideration due to the short pulse method. In this paper, we use the new approach developed in [4,32–34] to treat the nonlinear wave equations (1) with large data. We are able to give a criterion on the initial data such that the solution exists globally in the future, while the energy can be arbitrarily large. In particular, our approach applies to equations with any higher-order nonlinearities. Combined with the techniques developed in [34], our result here can even be extended to quasilinear wave equations. Moreover, we no longer require the data to have compact support as in [32–34]. This in particular implies that those results also hold for data satisfying conditions in this paper.

Before we state the main result, we define the necessary notations. We use the coordinate system $(t, x) = (x^0, x^1, x^2, x^3)$ of the Minkowski space. We denote $\partial_0 = \partial_t$, $\partial_i = \partial_{x^i}$, $\partial = (\partial_t, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla)$. We may also use the standard polar coordinates (t, r, ω). Let $\nabla\!\!\!/$ denote the induced covariant derivative and $\Delta\!\!\!/$ the induced Laplacian on the spheres of constant r. We also define the null coordinates $u = \frac{t-r}{2}$, $v = \frac{t+r}{2}$ and denote the corresponding partial derivatives

$$\partial_u = \partial_t - \partial_r, \quad \partial_v = \partial_t + \partial_r, \quad \overline{\partial_v} = (\partial_v, \nabla\!\!\!/), \quad \overline{\partial_u} = (\partial_u, \nabla\!\!\!/)$$

for $r > 0$. The vector fields that will be used as commutators are

$$Z = \{\partial_t, \Omega_{ij} = x_i \partial_j - x_j \partial_i\}.$$

Let α be a positive constant. Without loss of generality, we may assume $\alpha < \frac{1}{4}$. Denote

$$\begin{aligned} E_0(R) &= \sum_{k \le 4} \int_{\{r \ge R\} \cap \mathbb{R}^3} r^{1+\alpha} \, |\overline{\partial_v}(r Z^k \phi)|^2 \mathrm{d}r \mathrm{d}\omega \Big|_{t=0} \\ &\quad + \int_{\{r \le R\} \cap \mathbb{R}^3} |\partial Z^k \phi|^2 + |\phi|^2 \mathrm{d}x \Big|_{t=0}, \\ E_1(R) &= \sum_{k \le 4} \int_{\{r \ge R\} \cap \mathbb{R}^3} |\partial_u (r Z^k \phi)|^2 + |r Z^k \phi|^2 \mathrm{d}r \mathrm{d}\omega \Big|_{t=0}. \end{aligned}$$

These quantities can be uniquely determined by the initial data (ϕ_0, ϕ_1) together with the Eq. (1). We have the following main result:

Theorem 1 *Consider the Cauchy problem for the semilinear wave equation* (1) *satisfying the null condition* (2) *with some integer* $N \ge 3$. *For all* $\alpha \in (0, 1)$, *there exists a constant* $R(\alpha)$, *depending only on* α, *and a constant* ϵ_0, *depending only on the highest order* N *of the nonlinearity, such that if the initial data satisfy the estimate*

$$E_0(R) \le R^{-2+\alpha}, \quad E_1(R) \le R^{\epsilon_0 \alpha} \tag{3}$$

for some $R \ge R(\alpha)$, *then the solution* ϕ *exists globally in the future and obeys the estimates:*

$$\begin{aligned} &|\overline{\partial_v}\phi| \le C_\delta (1+r)^{-\frac{3}{2}+\delta}, \quad \delta > 0; \\ &|\partial_u \phi| \le C_\delta (1+r)^{-1+\delta} (1+t-r+R)^{-\frac{1}{2}-\frac{1}{2}\alpha}, \quad \delta > 0, \quad t + R \ge r; \\ &|\phi| \le C(1+r)^{-1} R^{\frac{1}{2}\epsilon_0 \alpha}, \quad t + R < r, \end{aligned}$$

where the constant C_δ *depends on* δ, α *and the constant* C *depends only on* α.

Remark 1 Similar result holds for equations in higher dimensions without assuming the null condition.

The Theorem implies that the energy of the initial data can be as large as $R^{\epsilon_0 \alpha}$. Since R can be any constant larger than a fixed constant $R(\alpha)$, the energy together with the higher-order Sobolev norm can be arbitrarily large. Moreover, the amplitude of the solution, at least in a small region, can have size $R^{\frac{1}{2}\epsilon_0 \alpha}$. In Wang–Yu's work, the construction of the Cauchy data is indirect and only the size of the energy has a lower bound. The amplitude or the L^∞ estimates of the solution are unclear except the upper bound. From this point of view, the problem we consider here is a large data problem.

The existence of the initial data (ϕ_0, ϕ_1) satisfying the conditions in the Theorem can be seen as follows: For any fixed $\alpha \in (0, 1)$ and any $R \geq R(\alpha)$, let ϕ_0 be small in the ball with radius R in $\mathbb{R}^3$. Here, $R(\alpha)$ is a sufficiently large constant depending only on α. Outside the ball, the energy of ϕ_0 is allowed to be as large as $R^{\epsilon_0 \alpha}$. Then for ϕ_1, we require it to be small inside the ball with radius R. Outside the ball, it is close to $\partial_r \phi_0$. This will definitely give a large set of initial data (ϕ_0, ϕ_1) satisfying the conditions in the Theorem.

We will use the new approach developed in [4,32–34] to prove the main Theorem. A key ingredient of this new approach is the p-weighted energy inequality originally introduced by Dafermos and Rodnianski [4]. This inequality can be obtained by using the vector field $r^p \partial_v$ as multipliers in a neighborhood of the null infinity. It in particular implies that the p-weighted energy $E_0(R)$, see the definition before the main Theorem, keeps small if initially it is. This allows us to relax the size of the transversal derivative of the solution, which is $E_1(R)$ in the theorem.

We will first construct the solution of the nonlinear wave equation outside the light cone, that is the region $r \geq R + t$, and show that the energy flux through the outgoing null hypersurface $r = t + R$ is small. And then, we prove the solutions exist globally inside the light cone, for which we are not able to apply the results, e.g., in [34] directly. In the previous results, the smallness needed in order to close the bootstrap argument for nonlinear problem is guaranteed by assuming the data to be sufficiently small. Hence, it is not necessary to keep track of the dependence of the radius R of the constants in the argument. However, in this paper, the smallness comes from the radius R and thus we need an argument with all the dependence on R.

2 Preliminaries and energy identities

We briefly recall the energy identity for wave equations, for details, we refer to Yang [34]. Let m be the Minkowski metric. We make a convention that the Greek indices run from 0 to 3, while the Latin indices run from 1 to 3. We raise and lower indices of any tensor relative to the metric m, e.g., $\partial^\gamma = m^{\gamma\mu}\partial_\mu$. Recall the energy-momentum tensor

$$\mathbb{T}_{\mu\nu}[\phi] = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m_{\mu\nu} \partial^\gamma \phi \partial_\gamma \phi.$$

Given a vector field X, we define the currents

$$J^X_\mu[\phi] = \mathbb{T}_{\mu\nu}[\phi] X^\nu, \qquad K^X[\phi] = \mathbb{T}^{\mu\nu}[\phi] \pi^X_{\mu\nu},$$

where $\pi^X_{\mu\nu} = \frac{1}{2}\mathcal{L}_X g_{\mu\nu}$ is the deformation tensor of the vector field X. For any function χ, we define the vector field $\tilde{J}^X[\phi]$

$$\tilde{J}^X[\phi] = \tilde{J}^X_\mu[\phi]\partial^\mu = \left(J^X_\mu[\phi] - \frac{1}{2}\partial_\mu \chi \cdot \phi^2 + \frac{1}{2}\chi \partial_\mu \phi^2 \right) \partial^\mu. \tag{4}$$

For any bounded region $\mathcal{D}$ in $\mathbb{R}^{3+1}$, using Stokes' formula, we have the energy identity

$$\iint_{\mathcal{D}} \Box_g \phi(\chi\phi + X(\phi)) + K^X[\phi] + \chi \partial^\gamma \phi \partial_\gamma \phi - \frac{1}{2}\Box_g \chi \cdot \phi^2 \mathrm{dvol} = \int_{\partial\mathcal{D}} i_{\tilde{J}^X[\phi]} \mathrm{dvol}, \tag{5}$$

where $\partial\mathcal{D}$ denotes the boundary of the domain $\mathcal{D}$, and $i_Y \mathrm{dvol}$ denotes the contraction of the volume form dvol with the vector field Y which gives the surface measure of the boundary.

3 The solution on the region $\{r \geq t + R\}$

In this section, we construct the solution of the nonlinear wave equation (1) on the region $\{r \geq R + t\}$. First, we define some notations. For $R \leq r_1 \leq r_2$, we use S_{r_1,r_2} to denote the following outgoing null hypersurface emanating from the sphere with radius r_1

$$S_{r_1,r_2} := \left\{ u = -\frac{r_1}{2}, \quad r_1 \leq r \leq r_2 \right\}.$$

Similarly, define $\bar{C}_{r_1,r_2}$ to be the following incoming null hypersurface emanating from the sphere with radius r_2

$$\bar{C}_{r_1,r_2} := \left\{ v = \frac{r_2}{2}, \quad r_1 \leq r \leq r_2 \right\}.$$

On the initial hypersurface $\mathbb{R}^3$, the annulus with radii r_1, r_2 is

$$B_{r_1,r_2} := \{t = 0, \quad r_1 \leq r \leq r_2\}.$$

We use S_r to be short for $S_{r,\infty}$. Similarly, we have $\bar{C}_r$ and B_r.

We use $\mathcal{D}_{r_1,r_2}$ to denote the region bounded by S_{r_1,r_2}, B_{r_1,r_2}, $\bar{C}_{r_1,r_2}$. Let $E[\phi](\Sigma)$ be the energy flux for ϕ through the hypersurface Σ in the Minkowski space. In particular,

$$E[\phi](S_{r_1,r_2}) = \int_{S_{r_1,r_2}} |\overline{\partial_v}\phi|^2 r^2 \mathrm{d}v \mathrm{d}\omega, \quad E[\phi](\bar{C}_{r_1,r_2}) = \int_{\bar{C}_{r_1,r_2}} |\overline{\partial_u}\phi|^2 r^2 \mathrm{d}u \mathrm{d}\omega,$$

where $\overline{\partial_u} = (\partial_u, \nabla\!\!\!/)$. On the initial hypersurface

$$E[\phi](B_{r_1,r_2}) = \int_{B_{r_1,r_2}} |\partial\phi|^2 \mathrm{d}x.$$

3.1 Energy estimates

In the energy identity (5), take the region $\mathcal{D}$ to be $\mathcal{D}_{r_1,r_2}$, the vector field $X = \partial_t$ and the function $\chi = 0$. We obtain the classical energy estimate

$$2\iint_{\mathcal{D}_{r_1,r_2}} \Box\phi \cdot \partial_t\phi \mathrm{dvol} + E[\phi](S_{r_1,r_2}) + E[\phi](\bar{C}_{r_1,r_2}) = E[\phi](B_{r_1,r_2}). \quad (6)$$

We also need an integrated energy estimate adapted to the region $\mathcal{D}_{r_1,r_2}$. For some small positive constant ϵ, depending only on α, we construct the vector field X and choose the functions f, χ as follows

$$X = f(r)\partial_r, \quad f = 2\epsilon^{-1} - \frac{2\epsilon^{-1}}{(1+r)^{\epsilon}}, \quad \chi = r^{-1}f.$$

We then can derive from the energy identity (5) that

$$I^{\epsilon}[\phi]_{r_1}^{r_2} \leq C_{\epsilon}(E[\phi](B_{r_1}) + E[\phi](S_{r_1,r_2}) + E[\phi](\bar{C}_{r_1,r_2}) + D^{\epsilon}[\Box\phi]_{r_1}^{r_2}), \quad (7)$$

where we denote

$$I^{\epsilon}[\phi]_{r_1}^{r_2} := \iint_{\mathcal{D}_{r_1,r_2}} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} \mathrm{d}x\mathrm{d}t, \quad D^{\epsilon}[F]_{r_1}^{r_2} := \iint_{\mathcal{D}_{r_1,r_2}} (1+r)^{1+\epsilon}|F|^2 \mathrm{d}x\mathrm{d}t.$$

here, $\bar{\partial}\phi = (\partial\phi, \frac{\phi}{1+r})$. The constant C_{ϵ} depends only on ϵ and is independent of r_1, r_2. For the derivation of the above estimate (7), it is almost the same as Proposition 1 of Yang [34] or Proposition 2 of Yang [33]. The only point we have to point out here is that we use the fact that the solution ϕ goes to zero as $r \to \infty$ on the initial hypersurface. We thus can use a Hardy's inequality to control the integral of $\frac{|\phi|^2}{(1+r)^2}$. This is also the reason that we have $E[\phi](B_{r_1})$, which is $E[\phi](B_{r_1,\infty})$ according to our notations, instead of $E[\phi](B_{r_1,r_2})$ on the right-hand side of the above estimate (7).

Combine the above two estimates (6), (7). We derive the following integrated energy estimates.

Proposition 1 *We have*

$$E[\phi](S_{r_1,r_2}) + E[\phi](\bar{C}_{r_1,r_2}) + I^{\epsilon}[\phi]_{r_1}^{r_2} \leq C_{\epsilon}(E[\phi](B_{r_1}) + D^{\epsilon}[\Box\phi]_{r_1}^{r_2}) \quad (8)$$

for some constant C_{ϵ} depending only on ϵ.

Proof For the derivation of the integrated energy estimate (7), we refer to [33] or [34]. Then, from the energy identity (6), we can estimate

$$E[\phi](S_{r_1,r_2}) + E[\phi](\bar{C}_{r_1,r_2}) \leq E[\phi](B_{r_1,r_2}) + \frac{1}{2}C_{\epsilon}^{-1}I^{\epsilon}[\phi]_{r_2}^{r_2} + 2C_{\epsilon}D^{\epsilon}[\Box\phi]_{r_2}^{r_2},$$

where C_ϵ is the constant in the integrated energy estimate (7). Then, the integrated energy estimate (7) can be improved to be

$$I^\epsilon[\phi]_{r_1}^{r_2} \leq 4C_\epsilon(E[\phi](B_{r_1}) + D^\epsilon[\Box\phi]_{r_1}^{r_2}).$$

This together with the previous estimate proves the proposition. Here, according to our notation, C_ϵ is a constant depending only on the small constant ϵ. □

Next, we consider the p-weighted energy inequality. In the energy identity (5), we take

$$X = f\partial_v, \quad \chi = r^{p-1}, \quad f = r^p, \quad 0 \leq p \leq 2.$$

We can compute

$$\begin{aligned}
&\int_{B_{r_1,r_2}} i_{\tilde{J}^X[\phi]}\mathrm{dvol} = \frac{1}{2}\int_{B_{r_1,r_2}} f\left(|\partial_v\psi|^2 + |\nabla\!\!\!/\psi|^2\right) - \partial_r\left(fr\phi^2\right) + f'r\phi^2\mathrm{d}r\mathrm{d}\omega,\\
&\int_{S_{r_1,r_2}} i_{\tilde{J}^X[\phi]}\mathrm{dvol} = \int_{S_{r_1,r_2}} f|\partial_v\psi|^2 - \frac{1}{2}\partial_v\left(fr\phi^2\right)\mathrm{d}v\mathrm{d}\omega,\\
&\int_{\bar{C}_{r_1,r_2}} i_{\tilde{J}^X[\phi]}\mathrm{dvol} = -\int_{\bar{C}_{r_1,s_2}} f|\nabla\!\!\!/\psi|^2 + f'r\phi^2 + \frac{1}{2}\partial_u\left(fr\phi^2\right)\mathrm{d}u\mathrm{d}\omega,\\
&\iint_{\mathcal{D}_{r_1,r_2}} K^X[\phi] + \chi\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\Box\chi\cdot\phi^2\mathrm{dvol}\\
&= \iint_{\mathcal{D}_{r_1,r_2}} \frac{1}{2}f'|\partial_v\psi|^2 + \left(\chi - \frac{1}{2}f'\right)|\nabla\!\!\!/\psi|^2 - \frac{1}{2}\partial_v\left(f'r\phi^2\right)\mathrm{d}r\mathrm{d}t.
\end{aligned}$$

here, $\psi = r\phi$. We can do integration by parts on $\mathcal{D}_{r_1,r_2}$ to estimate the integral of $\partial_v(f'r\phi^2)$. Alternatively, we can modify the current vector field $\tilde{J}^X[\phi]$ defined in line (4) to be

$$\hat{J}^X[\phi] = \tilde{J}^X[\phi] + \frac{1}{2}f'r\phi^2\partial_v.$$

Notice that

$$-\int_{B_{r_1,r_2}}\partial_r(fr\phi^2)\mathrm{d}r\mathrm{d}\omega - \int_{\bar{C}_{r_1,r_2}}\partial_u(fr\phi^2)\mathrm{d}u\mathrm{d}\omega + \int_{S_{r_1,r_2}}\partial_v(fr\phi^2)\mathrm{d}v\mathrm{d}\omega = 0.$$

Then, from the energy identity (5) and the above calculations, we obtain

$$\begin{aligned}
&\iint_{\mathcal{D}_{r_1,r_2}} r^{p-1}(p|\partial_v\psi|^2 + (2-p)|\nabla\!\!\!/\psi|^2)\mathrm{d}r\mathrm{d}t\mathrm{d}\omega + \int_{\bar{C}_{r_1,r_2}} r^p|\nabla\!\!\!/\psi|^2\mathrm{d}u\mathrm{d}\omega\\
&\quad + \int_{S_{r_1,r_2}} r^p|\partial_v\psi|^2\mathrm{d}v\mathrm{d}\omega = \int_{B_{r_1,r_2}} r^p|\overline{\partial_v}\psi|^2\mathrm{d}r\mathrm{d}\omega - 2\iint_{\mathcal{D}_{r_1,r_2}} r^{p-1}\Box\phi\partial_v\psi\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{9}$$

3.2 Bootstrap argument

We assume initially

$$E_0(R) \leq R^{-\beta}, \quad E_1(R) \leq R^{\epsilon_1}.$$

for some positive constant β, ϵ_1, which will be determined later. The definition of $E_0(R)$ can be found in the introduction before the statement of the main Theorem. We impose the following bootstrap assumption on the nonlinearity $F(\partial\phi)$ in the Eq. (1)

$$\sum_{k\leq 4} \iint_{\mathcal{D}_R} |Z^k F|^2 r^{2+\alpha} \mathrm{d}x\mathrm{d}t \leq 2R^{-\beta}. \tag{10}$$

Then, the p-weighted energy inequality (9) obtained in the end of the previous subsection implies that

$$\begin{aligned}
&\iint_{\mathcal{D}_{r_1,r_2}} r^{p-1}\left(p|\partial_v Z^k\psi|^2 + (2-p)|\nabla\!\!\!/ Z^k\psi|^2\right)\mathrm{d}r\mathrm{d}t\mathrm{d}\omega \\
&\quad + \int_{S_{r_1,r_2}} r^p|\partial_v Z^k\psi|^2 \mathrm{d}v\mathrm{d}\omega + \int_{\bar{C}_{r_1,r_2}} r^p|\nabla\!\!\!/ Z^k\psi|^2\mathrm{d}u\mathrm{d}\omega \\
&\leq r_1^{p-1-\alpha}\int_{B_{r_1,r_2}} r^{1+\alpha}|\overline{\partial_v}\psi|^2\mathrm{d}r\mathrm{d}\omega + \iint_{\mathcal{D}_{r_1,r_2}} \frac{p}{2} r^{p-1}|\partial_v Z^k\psi|^2\mathrm{d}r\mathrm{d}t\mathrm{d}\omega \\
&\quad + \iint_{\mathcal{D}_{r_1,r_2}} \frac{2}{p} r^{p+1}|Z^k\Box\phi|^2\mathrm{d}x\mathrm{d}t \\
&\leq r_1^{p-1-\alpha}E_0(R) + \frac{p}{2}\iint_{\mathcal{D}_{r_1,r_2}} r^{p-1}|\partial_v Z^k\psi|^2\mathrm{d}r\mathrm{d}t\mathrm{d}\omega + r_1^{p-1-\alpha} \\
&\quad \times \iint_{\mathcal{D}_{r_1,r_2}} \frac{2}{p} r^{2+\alpha}|Z^k\Box\phi|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

The second term in the last line can be absorbed. Then, let r_2 goes to infinity, we can obtain the following p-weighted energy estimate

$$\begin{aligned}
&\iint_{\mathcal{D}_{r_1}} r^{p-1}|\overline{\partial_v} Z^k\psi|^2\mathrm{d}r\mathrm{d}t\mathrm{d}\omega + \int_{S_{r_1}} r^p|\partial_v Z^k\psi|^2\mathrm{d}v\mathrm{d}\omega + \int_{\bar{C}_{r_1,r_2}} r^p|\nabla\!\!\!/ Z^k\psi|^2\mathrm{d}u\mathrm{d}\omega \\
&\lesssim R^{-\beta} r_1^{p-1-\alpha}
\end{aligned} \tag{11}$$

for all $k \leq 4, r_2 \geq r_1 \geq R, 0 < p \leq 1+\alpha$. Here and in the following, we make a convention that $A \lesssim B$ means $A \leq CB$ for some constant C depending only on α and is independent of R, r_1.

Note that the assumption (3) in particular implies that

$$\int_\omega r^2|Z^k\phi(0,r,\omega)|^2\mathrm{d}\omega \lesssim R^{\epsilon_1}, \quad k \leq 4, \quad r \geq R. \tag{12}$$

Using the p-weighted energy inequality (11) when $p = 1+\alpha$, on S_{r_1}, we can estimate

$$\begin{aligned}\int_\omega |rZ^k\phi|^2(t,r,\omega)\mathrm{d}\omega &\leq \int_\omega |rZ^k\phi(0,r_1,\omega)|^2\mathrm{d}\omega \\ &\quad + \int_{S_{r_1}} r^{1+\alpha}|\partial_v(rZ^k\phi)|^2\mathrm{d}v\mathrm{d}\omega\cdot\alpha^{-1}r_1^{-\alpha} \\ &\lesssim \int_\omega |rZ^k\phi(0,r_1,\omega)|^2\mathrm{d}\omega + R^{-\beta}r_1^{-\alpha}, \quad k\leq 4. \qquad (13)\end{aligned}$$

In particular, we have

$$\int_\omega |rZ^k\phi|^2(t,r,\omega)\mathrm{d}\omega \lesssim R^{\epsilon_1}, \quad k \leq 4.$$

We also need an inequality to estimating $\partial_u\psi$, $\psi = r\phi$. From the energy inequality (8), we can show that

$$\begin{aligned}&\int_{\bar{C}_{R,r_2}} |\overline{\partial_u}Z^k\psi|^2\mathrm{d}u\mathrm{d}\omega \lesssim E[\phi](\bar{C}_{R,r_2}) + \int_\omega r|Z^k\phi|^2(u_{r_2},v_{r_2},\omega)\mathrm{d}\omega \\ &\lesssim R^{\epsilon_1}, \quad k\leq 4, \quad r_2 \geq R, \qquad (14)\end{aligned}$$

where $u_r = \frac{r-R}{2}$, $v_r = \frac{r+R}{2}$, $\overline{\partial_u} = (\partial_u, \nabla\!\!\!/)$.

We now improve the bootstrap assumption (10). The quadratic part of the nonlinearity F is a null form $Q(\phi,\phi)$. Note that

$$Z^kQ(\phi,\phi) = \sum_{k_1+k_2\leq k} Q\left(Z^{k_1}\phi, Z^{k_2}\phi\right).$$

Here, Q denotes a general null form. They may stand for different null forms with different constants $A^{\mu\nu}$. For details, we refer to, e.g., [7]. We denote

$$\phi_1 = Z^{k_1}\phi, \quad \phi_2 = Z^{k_2}\phi, \quad \psi_1 = r\phi_1, \quad \psi_2 = r\phi_2.$$

This ϕ_1 is only a notation and should not be confused with the initial data ϕ_1. Note that

$$\begin{aligned}|r^2Z^kQ(\phi,\phi)| \lesssim \sum_{k_1+k_2\leq k} &|\bar{\partial}\psi_1||\phi_2| + |\nabla\!\!\!/\psi_1|(|\partial_t\psi_2| + |\partial_v\psi_2|) + |\nabla\!\!\!/\psi_1||\nabla\!\!\!/\psi_2| \\ &+|\partial_u\psi_1||\partial_v\psi_2|.\end{aligned}$$

For the proof of the above inequality, we refer to, e.g., Lemma 7 in [33]. The key point of this estimate is that the null form does not allow the worst term $\partial_u\phi_1\cdot\partial_u\phi_2$ in the estimate. Then by using Sobolev embedding, for $k\leq 4$, we have the estimate

$$\begin{aligned}\int_\omega |r^2 Z^k Q(\phi,\phi)|^2 \mathrm{d}\omega \lesssim & \sum_{k_1\le 4, k_2\le 4} \int_\omega |\bar{\partial}\psi_1|^2 \mathrm{d}\omega \cdot \int_\omega |\phi_2|^2 \mathrm{d}\omega \\ &+ \sum_{k_1\le 4, k_2\le 2} \int_\omega |\overline{\partial_u}\psi_1|^2 \mathrm{d}\omega \cdot \int_\omega |\partial_v \psi_2|^2 \mathrm{d}\omega \\ &+ \sum_{k_1\le 2, k_2\le 4} \int_\omega |\partial_u\psi_1|^2 \mathrm{d}\omega \cdot \int_\omega |\partial_v \psi_2|^2 \mathrm{d}\omega \\ &+ \int_\omega |\nabla\!\!\!/ Z^4 \psi|^2 |\partial_t \psi|^2 \mathrm{d}\omega. \end{aligned} \tag{15}$$

The last term in the above estimates needs special consideration. We comment here that all the other terms (after using Sobolev embedding on the unit sphere) estimating the term $|\nabla\!\!\!/ \psi_1||\partial_t\psi_2|$ in the previous inequality are grouped to the first term in the above estimate.

We will use estimate (12) or (13) to bound $\|\phi_2\|_{L^2(\mathbb{S}^2)}, k_2 \le 4$. For the good term $\|\partial_v\psi_2\|_{L^2(\mathbb{S}^2)}, k_2 \le 3$, we will rely on the equation together with the p-weighted energy estimates. In fact, we have

$$\begin{aligned}\sup_{v=\frac{r}{2}, -\frac{r}{2}\le u\le -\frac{R}{2}} r^\alpha \|\partial_v\psi_2\|^2_{L^2(\mathbb{S}^2)} \lesssim & \; r^\alpha \int_\omega |\partial_v\psi_2|^2(0,r,\omega)\mathrm{d}\omega \\ &+ \left| \int_{\bar{C}_{R,r}} \partial_u (r^\alpha |\partial_v \psi_2|^2) \mathrm{d}u \mathrm{d}\omega \right| \\ \lesssim & \; r^\alpha \int_\omega |\partial_v\psi_2|^2(0,r,\omega)\mathrm{d}\omega \\ &+ \int_{\bar{C}_{R,r}} r^{\alpha-1}|\partial_v\psi_2|^2 + r^\alpha |\partial_u\partial_v\psi_2||\partial_v\psi_2| \mathrm{d}u\mathrm{d}\omega \\ \lesssim & \; r^\alpha \int_\omega |\partial_v\psi_2|^2(0,r,\omega)\mathrm{d}\omega + \int_{\bar{C}_{R,r}} r^{\alpha-1}|\partial_v\psi_2|^2 \mathrm{d}u\mathrm{d}\omega \\ &+ \int_{\bar{C}_{R,r}} r^{\alpha+1}(|\nabla\!\!\!/ \Omega\phi_2|^2 + |r Z^{k_2} F|^2)\mathrm{d}u\mathrm{d}\omega, \end{aligned}$$

where we have used the equation for ϕ_2 in null coordinates (u, v, ω). We have to note that the coordinate $(0, r, \omega)$ appeared in the above estimate is with respect to the polar coordinate (t, r, ω). Integrate the above estimate with respect to r from R to infinity. We obtain

$$\begin{aligned}&\int_R^\infty \sup_{v=\frac{r}{2}, -\frac{r}{2}\le u\le -\frac{R}{2}} r^\alpha \|\partial_v\psi_2\|^2_{L^2(\mathbb{S}^2)} \mathrm{d}r \\ &\lesssim \int_{B_R} r^\alpha |\partial_v\psi_2|^2 \mathrm{d}r\mathrm{d}\omega + \iint_{\mathcal{D}_R} r^{\alpha-1}|\partial_v\psi_2|^2 \mathrm{d}r\mathrm{d}t\mathrm{d}\omega + \iint_{\mathcal{D}_R} r^{\alpha-1}|\nabla\!\!\!/ \Omega\psi_2|^2 \\ &\quad + r^{\alpha+3}|Z^{k_2}F|^2 \mathrm{d}t\mathrm{d}\omega\mathrm{d}r \lesssim R^{-1-\beta}. \end{aligned} \tag{16}$$

here, we have used the assumption on the initial data that $E_0(R) \leq R^{-\beta}$. The bound for F follows from the bootstrap assumption (10). The estimates for $|\partial_v \psi_2|^2$, $|\nabla\!\!\!/ \Omega\psi_2|^2$ are due to the p-weighted energy inequality (11) and the fact that $k_2 \leq 3$.

Similarly, for $\|\partial_u \psi_1\|_{L^2(\mathbb{S}^2)}$, $k_1 \leq 3$, we have

$$\begin{aligned}
&\int_R^\infty \sup_{u=-\frac{r}{2}, v\geq\frac{r}{2}} r^{-1}\|\partial_u\psi_1\|^2_{L^2(\mathbb{S}^2)}\mathrm{d}r \\
&\lesssim R^{-1}\int_{B_R}|\partial_u\psi_1|^2\mathrm{d}r\mathrm{d}\omega + \iint_{\mathcal{D}_R} r^{-2}|\partial_u\psi_1|^2 dt\mathrm{d}r\mathrm{d}\omega \\
&\quad + \iint_{\mathcal{D}_R}|\nabla\!\!\!/ \Omega\phi_1|^2 + |rZ^{k_1}F|^2\mathrm{d}r\mathrm{d}t\mathrm{d}\omega \\
&\lesssim R^{-1+\epsilon_1} + R^{-1+\epsilon} I^\epsilon[\phi_1]_R^\infty + R^{-2-\alpha-\beta} \\
&\lesssim R^{-1+\epsilon_1+\epsilon}.
\end{aligned}$$

here, the estimate for the integrated energy estimate $I^\epsilon[\phi_1]_R^\infty$ follows from (8) in which the bounds for $D^\epsilon[Z^{k_1}F]_R^\infty$ are guaranteed by the bootstrap assumption (10).

On the right-hand side of the null form estimate (15), we are left to estimate the special term $|\nabla\!\!\!/ \psi_1||\partial_t\psi|$, $\psi_1 = Z^4\psi$. We can show that

$$\begin{aligned}
\iint_{\mathcal{D}_R}|\nabla\!\!\!/ \psi_1|^2|\partial_t\psi|^2 r^\alpha \mathrm{d}r\mathrm{d}t\mathrm{d}\omega &\lesssim \int_R^\infty\int_{\frac{r}{2}}^\infty\left(v+\frac{r}{2}\right)^\alpha\int_\omega|\nabla\!\!\!/ \psi_1|^2\cdot\int_\omega|\psi_2|^2\mathrm{d}\omega\mathrm{d}r\mathrm{d}v \\
&\lesssim \int_R^\infty\int_{\frac{r}{2}}^\infty\left(v+\frac{r}{2}\right)^\alpha\int_\omega|\nabla\!\!\!/ \psi_1|^2 \\
&\quad\times\left(\int_\omega|\psi_2|^2\left(r,\frac{r}{2},\omega\right)\mathrm{d}\omega + R^{-\beta}\left(v+\frac{r}{2}\right)^{-\alpha}\right)\mathrm{d}r\mathrm{d}v \\
&\lesssim R^{-2\beta-\alpha} + \int_R^\infty\int_{-\frac{r}{2}}^{-\frac{R}{2}}\int_\omega\left(\frac{r}{2}-u\right)^\alpha|\nabla\!\!\!/ \psi_1|^2\mathrm{d}\omega \\
&\quad\times\int_\omega|\psi_2|^2\left(r,-\frac{r}{2},\omega\right)\mathrm{d}\omega\mathrm{d}u\mathrm{d}r.
\end{aligned}$$

Here, we have used estimate (13) and $k_2 \leq 3$. We note that when u is fixed, the p-weighted energy inequality (11) implies that

$$\int_{-\frac{r}{2}}^{-\frac{R}{2}}\int_\omega\left(\frac{r}{2}-u\right)^{1+\alpha}|\nabla\!\!\!/ \psi_1|^2\mathrm{d}\omega\mathrm{d}u \leq \int_{\bar{C}_{R,r}} r^{1+\alpha}|\nabla\!\!\!/ \psi_1|^2\mathrm{d}u\mathrm{d}\omega \lesssim R^{-\beta}.$$

Thus, we can show that

$$\begin{aligned}
\iint_{\mathcal{D}_R}|\nabla\!\!\!/ \psi_1|^2|\partial_t\psi|^2 r^\alpha\mathrm{d}r\mathrm{d}t d\omega &\lesssim R^{-2\beta-\alpha} + \int_R^\infty R^{-\beta-1}\int_\omega|\psi_2|^2(0,r,\omega)\mathrm{d}\omega\mathrm{d}r \\
&\lesssim R^{-2\beta-\alpha} + R^{-1-\beta+\epsilon_1}.
\end{aligned}$$

Therefore from the null form estimate (15), we can derive that

$$\begin{aligned}
&\iint_{\mathcal{D}_R} r^{\alpha}|r^2 Z^k Q(\phi,\phi)|^2 \mathrm{d}r\mathrm{d}t\mathrm{d}\omega\\
&\lesssim R^{\epsilon_1}\iint_{\mathbb{D}_R} r^{-2+\alpha}|\partial\psi_1|^2\mathrm{d}t\mathrm{d}r\mathrm{d}\omega+\int_R^{\infty}\sup_u r^{\alpha}\|\partial_v\psi_2\|^2_{L^2(\mathbb{S}^2)}\int_{\bar{C}_{R,r}}|\overline{\partial_u}\psi_1|^2\mathrm{d}u\mathrm{d}\omega\mathrm{d}r\\
&\quad+\int_R^{\infty}\sup_v r^{-1}\|\partial_u\psi_1\|^2_{L^2(\mathbb{S}^2)}\int_{S_{r_1}} r^{1+\alpha}|\partial_v\psi_2|^2\mathrm{d}v\mathrm{d}\omega dr_1+R^{-2\beta-\alpha}+R^{-1-\beta+\epsilon_1}\\
&\lesssim R^{2\epsilon_1-1+\alpha+\epsilon}+R^{\epsilon_1-1-\beta}+R^{-1+\epsilon_1+\epsilon-\beta}+R^{-2\beta-\alpha}+R^{-1-\beta+\epsilon_1}.
\end{aligned}$$

For cubic or higher-order nonlinearities, we first conclude from estimate (16) that

$$\int_0^{r_1-R}\int_{\omega} r_1^{\alpha}\left(|\partial_v Z^k\psi|^2+|\partial_v\partial_t Z^k\psi|^2\right)\mathrm{d}\omega\mathrm{d}t\lesssim R^{-1-\beta},\quad k\le 2.$$

In particular, we have

$$\int_{\omega}|\partial_v Z^k\psi|^2\mathrm{d}\omega\lesssim R^{-1-\beta}r_1^{-\alpha},\quad k\le 2.$$

Since we have shown that

$$\int_{\omega}|Z^k\psi|^2\mathrm{d}\omega\lesssim R^{\epsilon_1},\quad k\le 4,$$

we then have

$$\int_{\omega}|\partial Z^k\psi|^2\mathrm{d}\omega\lesssim R^{\epsilon_1},\quad k\le 2.$$

Thus for cubic or higher-order nonlinearities, we can bound

$$\begin{aligned}
\iint_{\mathcal{D}_R}|Z^k(F-Q)|^2 r^{2+\alpha}\mathrm{d}x\mathrm{d}t&\lesssim\sum_{k\le 4}\iint_{\mathcal{D}_R}|\partial Z^k\phi|^2 r^{-4+2+\alpha}R^{2(N-2)\epsilon_1}\mathrm{d}x\mathrm{d}t\\
&\lesssim R^{(2N-3)\epsilon_1+\alpha+\epsilon-1}.
\end{aligned}$$

here, we recall that N is the order of the highest order nonlinearity. To summarize, we have shown that

$$\iint_{\mathcal{D}_R}|Z^k F|^2 r^{2+\alpha}\mathrm{d}x\mathrm{d}t\lesssim R^{(2N-3)\epsilon_1+\alpha+\epsilon-1}+R^{-1+\epsilon_1+\epsilon-\beta}+R^{-2\beta-\alpha}.$$

If we take

$$\beta=1-2\alpha,\quad \epsilon=\frac{\alpha}{20},\quad \epsilon_1=\frac{\alpha}{2N},\tag{17}$$

we then have

$$\iint_{\mathcal{D}_R} |Z^k F|^2 r^{2+\alpha} \mathrm{d}x dt \lesssim R^{-\beta-\frac{1}{5}\alpha}.$$

According to our notations, the implicit constant in the above estimate depends only on α. Hence, let the constant R be sufficiently large, depending only on α, we then can improve the bootstrap assumption (10). Once we have improved the bootstrap assumption (10), the proof for the existence of a unique solution of the Eq. (1) on the region $\{r \geq R+t\}$ is standard, see the end of Yang [33].

Remark 2 In particular, the small constant ϵ_0 in the main Theorem can be $\epsilon_0 = \frac{1}{2N}$.

4 The solution on $\{r \leq R+t\}$

We have constructed the solution of the Eq. (1) outside the light cone $\{r \geq R+t\}$. In this section, we will prove that the solution also exists globally in the future inside the light cone which is the region $\{r \leq R+t\}$. We use the foliation

$$S_\tau := \left\{ u = u_\tau = \frac{\tau - R}{2}, \quad \frac{\tau + R}{2} = v_\tau \leq v \right\}, \quad \Sigma_\tau := \{t = \tau, \quad r \leq R\} \cup S_\tau.$$

The energy flux through Σ_τ for the scalar field ϕ is $E[\phi](\tau)$. For $\tau_2 \geq \tau_1$, we define

$$I^\epsilon[\phi]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} \mathrm{d}x \mathrm{d}\tau, \quad D^\epsilon[F]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} (1+r)^{1+\epsilon} |F|^2 \mathrm{d}x \mathrm{d}\tau.$$

We have the integrated energy estimate and the energy estimate

$$E[\phi](\tau_2) + I^\epsilon[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\nabla\!\!\!/ \phi|^2}{1+r} \mathrm{d}x \mathrm{d}\tau \lesssim E[\phi](\tau_1) + D^\epsilon[F]_{\tau_1}^{\tau_2}, \qquad (18)$$

see Proposition 1 of Yang [34] or Proposition 2 of Yang [33]. As before, the implicit constant here depends only on ϵ.

4.1 The p-weighted energy inequality

As we have discussed in the introduction, the smallness needed to close the bootstrap argument for nonlinear problem in this paper comes from the radius R while in the previous work, e.g., [33] the smallness comes from the data. In particular, the previous argument cannot be applied directly to the settings in this paper. Instead, we need an argument with all the dependence of the constants on the radius R. To be more precise, we first consider one of the key ingredients the p-weighted energy inequality. We recall the p-weighted energy identity originally introduced by Dafermos and Rodnianski [4]

$$\begin{aligned}
&\int_{S_{\tau_2}^v} r^p(\partial_v\psi)^2 \mathrm{d}v\mathrm{d}\omega + \int_{\tau_1}^{\tau_2}\int_{S_\tau^v} 2r^{p+1}F\cdot\partial_v\psi \mathrm{d}v\mathrm{d}\tau\mathrm{d}\omega \\
&\quad + \int_{\tau_1}^{\tau_2}\int_{S_\tau^v} r^{p-1}\left(p(\partial_v\psi)^2 + (2-p)|\nabla\!\!\!/\psi|^2\right)\mathrm{d}v\mathrm{d}\tau\mathrm{d}\omega \\
&\quad + \int_{\bar{C}(\tau_1,\tau_2,v)} r^p|\nabla\!\!\!/\psi|^2 \mathrm{d}u\mathrm{d}\omega \\
&= \int_{S_{\tau_1}^v} r^p(\partial_v\psi)^2\mathrm{d}v\mathrm{d}\omega + \int_{\tau_1}^{\tau_2} r^p\left(|\nabla\!\!\!/\psi|^2 - (\partial_v\psi)^2\right)\mathrm{d}\omega\mathrm{d}\tau|_{r=R},
\end{aligned}$$

where $\psi = r\phi$, $F = \Box\phi$. Note that the boundary term on $\{r = R\}$ is proportional to R^p. Hence, we can simply take $p = 0$ to estimate it. First for any τ, we have

$$\int_{S_\tau^v} (\partial_v\psi)^2 \mathrm{d}v\mathrm{d}\omega \le 5E[\phi](\tau).$$

For the proof of this inequality, see e.g., Corollary 1 in [33]. For the inhomogeneous term $F\partial_v\psi$ when $p = 0$, we can estimate it as follows:

$$\left|\int_{\tau_1}^{\tau_2}\int_{S_\tau^v} rF\cdot\partial_v\psi \mathrm{d}v\mathrm{d}\tau\mathrm{d}\omega\right| \lesssim D^\epsilon[F]_{\tau_1}^{\tau_2} + E[\phi](\tau_1).$$

Therefore for general p, we have the estimate for the boundary term

$$\left|\int_{\tau_1}^{\tau_2} r^p\left(|\nabla\!\!\!/\psi|^2 - (\partial_v\psi)^2\right)\mathrm{d}\omega\mathrm{d}\tau|_{r=R}\right| \lesssim R^p(D^\epsilon[F]_{\tau_1}^{\tau_2} + E[\phi](\tau_1)).$$

Since the boundary term on the incoming null hypersurface $\bar{C}(\tau_1, \tau_2, v)$ has a good sign, to obtain a useful estimate from the p-weighted energy identity, it suffices to estimate the integral of the inhomogeneous term $r^{p+1}F\partial_v\psi$ in the above p-weighted energy identity. On S_τ, we control it as follows

$$2r^{p+1}|F\partial_v\psi| \le r^p|\partial_v\psi|^2\tau_+^{-1-\epsilon} + r^{p+2}|F|^2\tau_+^{1+\epsilon}, \quad \tau_+ = 1+\tau.$$

The integral of the first term $r^p|\partial_v\psi|^2\tau_+^{-1-\epsilon}$ will be bounded by using Gronwall's inequality. Thus, we derive

$$\int_{S_{\tau_2}} r^p(\partial_v\psi)^2 \mathrm{d}v\mathrm{d}\omega + \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{p-1}\left(p|\partial_v\psi|^2 + (2-p)|\nabla\!\!\!/\psi|^2\right)\mathrm{d}v\mathrm{d}\omega\mathrm{d}\tau$$
$$\lesssim R^p(E[\phi](\tau_1) + D^\epsilon[F]_{\tau_1}^{\tau_2}) + \int_{S_{\tau_1}} r^p|\partial_v\psi|^2\mathrm{d}v\mathrm{d}\omega$$
$$+ \int_{\tau_1}^{\tau_2} \tau_+^\epsilon D_+^{p-1}[F]_\tau^{\tau_2}\mathrm{d}\tau + (\tau_1)_+^{1+\epsilon} D_+^{p-1}[F]_{\tau_1}^{\tau_2}, \tag{19}$$

where

$$D_+^\alpha[F]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2}\int_{S_\tau}(1+r)^{1+\alpha}|F|^2\mathrm{d}x\mathrm{d}\tau.$$

4.2 The data

To study the equation in the interior region $\{r \le t+R\}$, we need the initial data on the boundary which consists of the outgoing null hypersurface S_0, that is $\{v \ge \frac{R}{2}, u = -\frac{R}{2}\}$ and the initial ball with radius R. However, we are not able to get the desired estimates of the solution on the particular cone $S_{R,\infty}$. The idea is that we instead find some one nearby. Since the data on the ball with radius R can be arbitrarily small according to our assumptions, the results in the Sect. 3 also hold if we replace R with $\frac{R}{2}$ (the data between are small). As long as R is sufficiently large, we still can construct the solutions satisfying the estimates in Sect. 3 on the larger region $\{r \ge t + \frac{R}{2}\}$. Therefore from the p-weighted energy inequality (11), we have

$$\int_{S_{r_1,\infty}} r^{1+\alpha}|\partial_v Z^k\psi|^2\mathrm{d}v\mathrm{d}\omega \lesssim R^{-\beta}, \quad \beta = 1-2\alpha, \quad k \le 4, \quad \forall r_1 \ge \frac{R}{2}, \tag{20}$$
$$\int_{\frac{1}{2}R}^\infty \int_{S_{r,\infty}} r^\alpha|\overline{\partial_v} Z^k\psi|^2\mathrm{d}v\mathrm{d}\omega\mathrm{d}r \lesssim R^{-\beta}. \tag{21}$$

Here, note that we have fixed β in line (17). From the integral version of the p-weighted energy estimate (21), we in particular can choose a slice $S_{r_0,\infty}$ for some $r_0 \in [\frac{1}{2}R, R]$ such that

$$\int_{S_{r_0,\infty}} r^\alpha|\overline{\partial_v} Z^k\psi|^2\mathrm{d}v\mathrm{d}\omega \lesssim R^{-1-\beta}.$$

Therefore for the energy flux through $S_{r_0,\infty}$, we can show that

$$\sum_{|k|\le 4} E[Z^k\phi](S_{r_0,\infty}) \le \sum_{|k|\le 4}\int_{S_{r_0,\infty}} |\overline{\partial_v} Z^k\psi|^2\mathrm{d}v\mathrm{d}\omega + \sum_{|k|\le 4} r_0\int_\omega |\phi|^2(0, r_0, \omega)\mathrm{d}\omega$$
$$\lesssim r_0^{-\alpha}R^{-\beta-1} + \sum_{|k|\le 4}\int_{|x|\le R}|\partial Z^k\phi|^2\mathrm{d}x \lesssim R^{-2+\alpha}. \tag{22}$$

Here, we used the Sobolev embedding on the ball with radius R on the initial hypersurface.

In summary, we have constructed the solution on the extended exterior region $\{r \geq t + \frac{R}{2}\}$ and we can find some cone $S_{r_0,\infty}$ for some $r_0 \in [\frac{1}{2}R, R]$ such that the solution on this cone satisfies the estimates (20), (22). The data on the ball with radius r_0 remain small. So, in the sequel, we will construct solutions on the interior region $\{r \leq t + r_0\}$. However, to avoid too many constants, we still use R to denote r_0.

4.3 Bootstrap argument

We now use the above initial data to establish the decay of the energy flux. We impose the following bootstrap assumptions on the nonlinearity F for all $k \leq 4$

$$D^{\epsilon}\left[Z^k F\right]_{\tau_1}^{\tau_2} \leq 2 \min\left\{R^{-\beta}(\tau_1)_+^{-1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{-\alpha}\right\},$$
$$D_+^{\alpha}\left[Z^k F\right]_{\tau_1}^{\tau_2} \leq 2\tau_+^{-1-\alpha} R^{-\beta}. \tag{23}$$

We show the decay of $E[Z^k\phi](\tau)$. Let $p = 1+\alpha$ in the p-weighted energy inequality (19). We have

$$\int_{S_{\tau_2}} r^{1+\alpha}|\partial_v \psi|^2 \mathrm{d}v\mathrm{d}\omega + \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{\alpha}|\partial_v\psi|^2 \mathrm{d}v\mathrm{d}\omega\mathrm{d}\tau \lesssim R^{1+\alpha-2+\alpha} + R^{-\beta} = R^{-\beta}.$$

Hence, we can choose a dyadic sequence $\{\tau_n\}$ such that

$$\int_{S_{\tau_n}} r^{\alpha}|\partial_v\psi|^2 \mathrm{d}v\mathrm{d}\omega \lesssim (\tau_n)_+^{-1} R^{-\beta}.$$

Interpolation leads to

$$\int_{S_{\tau_n}} r|\partial_v\psi|^2 \mathrm{d}v\mathrm{d}\omega \lesssim R^{-\beta}(\tau_n)_+^{-\alpha}.$$

Then, take $p = 1$ in the p-weighted energy inequality (19). We derive

$$\int_{\tau_n}^{\tau'} E[\phi](\tau)\mathrm{d}\tau \lesssim R^{-\beta}(\tau_n)_+^{-\alpha} + RE[\phi](\tau_n) + R^{1+\epsilon}\left(E[\phi](\tau_n) + (\tau_n)_+^{-\alpha} R^{-1-\beta-\epsilon}\right)$$
$$\lesssim R^{-\beta}(\tau_n)_+^{-\alpha} + R^{1+\epsilon}E[\phi](\tau_n), \quad \tau' \geq \tau_n.$$

In the energy estimate (18), set $\tau_1 = 0$. We have

$$E[\phi](\tau) \lesssim R^{-2+\alpha}.$$

For $\tau' \geq \tau$, we have

$$E[\phi](\tau') \lesssim E[\phi](\tau) + R^{-\beta}\tau_+^{-1-\alpha}.$$

We thus can conclude that

$$(\tau' - \tau_n)E[\phi](\tau') \lesssim R^{-\beta}(\tau_n)_+^{-\alpha} + R^{1+\epsilon}E[\phi](\tau_n).$$

In particular, we have

$$E[\phi](\tau) \lesssim \tau_+^{-1}\left(R^{-\beta} + R^{1+\epsilon}R^{-2+\alpha}\right) \lesssim \tau_+^{-1}R^{-\beta}.$$

This then implies that

$$E[\phi](\tau_{n+1}) \lesssim R^{-\beta}(\tau_n)_+^{-1-\alpha} + R^{1+\epsilon-\beta}(\tau_n)_+^{-2}.$$

As τ_n is dyadic, we then infer that

$$E[\phi](\tau) \lesssim R^{-\beta}\tau_+^{-1-\alpha} + R^{1+\epsilon-\beta}\tau_+^{-2}.$$

Summarizing, we have the following energy decay estimate

Proposition 2 *For any $k \leq 4$, we have*

$$I^{\epsilon}[Z^k\phi]_{\tau_1}^{\tau_2} + D^{\epsilon}[Z^kF]_{\tau_1}^{\tau_2} + E[Z^k\phi](\tau) \lesssim A(\tau),$$

where

$$A(\tau) := \min\left\{R^{-\beta}\tau_+^{-1-\alpha} + R^{1+\epsilon-\beta}\tau_+^{-2}, \quad R^{-2+\alpha}, \quad R^{-\beta}\tau_+^{-1}\right\}.$$

In particular, we have

$$E[Z^k\phi](\tau) \lesssim \min\left\{R^{-\gamma}\tau_+^{-1-\alpha}, R^{-2+\alpha}\right\}, \quad \gamma = \beta - (1+\epsilon)\alpha.$$

Proof The estimate for the energy flux $E[\phi](\tau)$ follows from the above argument. The estimate for the integrated energy $I^{\epsilon}[Z^k\phi]_{\tau_1}^{\tau_2}$ follows from (18) and the bound for the inhomogeneous term F is a restatement of the bootstrap assumption (23). □

The following lemma will be used to show the C^1 estimate of the solution.

Lemma 1

$$\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau \cap \{r \geq 1\}} r^{1-\epsilon}|\partial_u\partial_v Z^k\phi|^2 dx d\tau \lesssim A(\tau_1), \quad \forall k \leq 3.$$

Proof Using the equation for $Z^k\phi$ (commutation of the equation (1) with Z^k), we have

$$\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau\cap\{r\geq 1\}} r^{1-\epsilon}|\partial_u\partial_v Z^k\phi|^2 \mathrm{d}x\mathrm{d}\tau \lesssim \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau\cap\{r\geq 1\}} r^{1-\epsilon}(r^{-1}|\partial Z^k\phi| + |\not{\Delta} Z^k\phi| + |Z^k F|)^2 \mathrm{d}x\mathrm{d}\tau$$
$$\lesssim I^\epsilon[Z^k\phi]_{\tau_1}^{\tau_2} + I^\epsilon[\Omega Z^k\phi]_{\tau_1}^{\tau_2} + D^\epsilon[Z^k F]_{\tau_1}^{\tau_2}$$
$$\lesssim A(\tau_1)$$

for all $k \leq 3$. □

Next, we improve the bootstrap assumption (23). We mainly consider the quadratic nonlinearity $Q(\phi,\phi)$, which satisfies the null condition. We first estimate $D_+^\alpha[F]_{\tau_1}^{\tau_2}$. On S_τ, we can estimate

$$\int_\omega |rZ^k\phi|^2(\tau,r,\omega)\mathrm{d}\omega \leq \int_\omega |rZ^k\phi(\tau,R,\omega)|^2\mathrm{d}\omega + \int_{S_\tau} r^{1+\alpha}|\partial_v(rZ^k\phi)|^2 \mathrm{d}v\mathrm{d}\omega\cdot\alpha^{-1}R^{-\alpha}$$
$$\lesssim \int_\omega |rZ^k\phi(\tau,R,\omega)|^2\mathrm{d}\omega + R^{-\beta}R^{-\alpha} \lesssim R^{-1+\alpha}.$$

Let $\bar{C}_{\tau_1,\tau_2,v_1}$ be the incoming null hypersurface between Σ_{τ_1} and Σ_{τ_2}, defined as follows:

$$\bar{C}_{\tau_1,\tau_2,v_1} := \{v = v_1, \quad u_{\tau_1} \leq u \leq u_{\tau_2}\}.$$

The energy estimate on the region $\{v \geq v_1, \quad u_{\tau_1} \leq u \leq u_{\tau_2}\}$ then implies that

$$\int_{C_{\tau_1,\tau_2,v_1}} |\overline{\partial_u} Z^k\psi|^2 \mathrm{d}u\mathrm{d}\omega \lesssim A(\tau_1), \quad k \leq 4.$$

For the detailed proof of this estimate, we refer to, e.g., Lemma 8 in [33] or Lemma 11 in [32]. Then from estimate (15), we can show that

$$D_+^\alpha[Z^k Q]_{\tau_1}^{\tau_2} \lesssim R^{-1+\alpha}\int_{\tau_1}^{\tau_2}\int_{S_\tau} |\bar{\partial}\psi_1|^2 r^{-3+\alpha}\mathrm{d}r\mathrm{d}t\mathrm{d}\omega$$
$$+ \sum_{k_1\leq 2} R^{-\beta}\int_{\tau_1}^{\tau_2}\sup_v r^{-2}\int_\omega |\partial_u\psi_1|^2\mathrm{d}\omega\mathrm{d}\tau$$
$$+ A(\tau_1)\sum_{k_2\leq 2}\int_{v_{\tau_1}}^{\infty}\sup_u r^{\alpha-1}\int_\omega |\partial_v\psi_2|^2\mathrm{d}\omega\mathrm{d}v$$
$$+ \int_{\tau_1}^{\tau_2}\int_{S_\tau} |\nabla\!\!\!/ Z^4\psi|^2 r^\alpha E[Z^3\phi](\tau)\mathrm{d}r\mathrm{d}\omega\mathrm{d}\tau.$$

here, we still use the notation that $\phi_1 = Z^{k_1}\phi, \phi_2 = Z^{k_2}\phi, \psi_1 = r\phi_1$. Now on S_τ, $\tau_1 \leq \tau \leq \tau_2$, we can estimate

$$
\begin{aligned}
r^{-\alpha}\int_{\omega}(\partial_u\psi_1)^2 \mathrm{d}\omega &\lesssim r^{-\alpha}\int_{\omega}(\partial_u\psi_1)^2\mathrm{d}\omega\Big|_{v=v_{\tau_2}} + \int_{S_\tau} r^{-1-\alpha}|\partial_u\psi_1|^2\mathrm{d}v\mathrm{d}\omega \\
&\quad + \int_{S_\tau} r^{-1-\alpha}(\partial_u\psi_1)^2\mathrm{d}v\mathrm{d}\omega + \int_{S_\tau} r^{1-\alpha}(\partial_v\partial_u\psi_1)^2\mathrm{d}v\mathrm{d}\omega \\
&\lesssim \int_{\omega}(\partial_u\psi_1)^2\mathrm{d}\omega\Big|_{v=v_{\tau_2}} + \int_{S_\tau} r^{-1-\epsilon}(|\partial\psi_1|^2 + |\partial\Omega\psi_1|^2)\mathrm{d}v\mathrm{d}\omega \\
&\quad + \int_{S_\tau} r^{3-\alpha}|Z^{k_1}F|^2\mathrm{d}v\mathrm{d}\omega.
\end{aligned}
$$

Similarly, on $\bar{C}_{\tau_1,\tau_2,v}$, we have

$$
\begin{aligned}
r^{\alpha}\int_{\omega}(\partial_v\psi_2)^2\mathrm{d}\omega &\lesssim r^{\alpha}\int_{\omega}(\partial_v\psi_2)^2\mathrm{d}\omega\Big|_{u=u_{\tau_1}} + \int_{\bar{C}_{\tau_1,\tau_2,v}} r^{\alpha}(\partial_v\psi_2)^2\mathrm{d}u\mathrm{d}\omega \\
&\quad + \int_{\bar{C}_{\tau_1,\tau_2,v}} r^{\alpha}(\partial_u\partial_v\psi_2)^2\mathrm{d}u\mathrm{d}\omega + \int_{\bar{C}_{\tau_1,\tau_2,v}} r^{\alpha-1}(\partial_v\psi_2)^2\mathrm{d}u\mathrm{d}\omega \\
&\lesssim r^{\alpha}\int_{\omega}(\partial_v\psi_2)^2\mathrm{d}\omega\Big|_{u=u_{\tau_1}} + \int_{\bar{C}_{\tau_1,\tau_2,v}} r^{\alpha}(\partial_v\psi_2)^2 + r^{\alpha}(\not{\Delta}\psi_2)^2 \\
&\quad + r^{\alpha+2}|Z^{k_2}F|^2\mathrm{d}u\mathrm{d}\omega.
\end{aligned}
$$

Therefore we can show that

$$
\begin{aligned}
D_+^{\alpha}[Z^kQ]_{\tau_1}^{\tau_2} &\lesssim R^{-3+2\alpha+\epsilon}A(\tau_1) + R^{-\beta-2+\alpha}A(\tau_1) + A(\tau_1)R^{-1-\beta} + A(\tau_1)R^{-\beta} \\
&\lesssim A(\tau_1)R^{-\beta}.
\end{aligned}
$$

The estimate for cubic or higher-order nonlinearities is better and we can conclude that

$$
D_+^{\alpha}[Z^kF]_{\tau_1}^{\tau_2} \lesssim A(\tau_1)R^{-\beta}, \quad k \le 4. \tag{24}
$$

Next, we estimate the integral inside the cylinder with radius R. We have

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_{r\le R}(1+r)^{1+\epsilon}|Z^kF|^2\mathrm{d}x\mathrm{d}\tau &\lesssim \int_{\tau_1}^{\tau_2}\int_{r\le R}(1+r)^{1+\epsilon}|\partial\phi_1|^2|\partial\phi_2|^2\mathrm{d}x\mathrm{d}\tau \\
&\lesssim \int_{\tau_1}^{\tau_2}\int_{r\le 1}|\partial\phi_1|^2|\partial\phi_2|^2\mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_{\tau_1}^{\tau_2}\int_{1\le r\le R} r^{1+\epsilon}|\partial\phi_1|^2|\partial\phi_2|^2\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

Here, we omitted the summation sign for simplicity and the right-hand side should be interpreted as the sum for all $k_1 + k_2 \le k \le 4$. The integral on the cylinder with radius 1 can be estimated by using elliptic estimates together with the wave equation and Sobolev embedding. This will rely on the commutator ∂_t. More specifically, we have the elliptic equation for the solution on the disk with radius one at a fixed time.

Then, the estimate of $\partial_{tt}\phi$ will lead to the desired estimate of ϕ inside the disk. For much more detailed argument, we refer to, e.g., Proposition 6 in [33]. To estimate the second part, we claim that

$$\int_{\omega} r|\partial(Z^k\phi)|^2 \mathrm{d}\omega \lesssim A(\tau), \quad k \le 2, \quad 1 \le r \le R. \tag{25}$$

In fact from Lemma 1, we have

$$\int_{1\le r\le R} r^{1-\epsilon}|\partial_u\partial_v Z^k\phi|^2 \mathrm{d}x \lesssim A(\tau), \quad k \le 2,$$
$$\int_{1\le r\le R} |\partial_u\partial_t Z^k\phi|^2 \mathrm{d}x \lesssim E[\partial_t Z^k\phi](\tau) \lesssim A(\tau), \quad k \le 3.$$

This implies that

$$\int_{1\le r\le R} |\partial_u\partial_r Z^k\phi|^2 \mathrm{d}x \lesssim A(\tau), \quad k \le 2.$$

In particular, we can show that

$$r\int_{\omega} |\partial_u Z^k\phi|^2 \mathrm{d}\omega \le A(\tau), \quad 1 \le r \le R, \quad k \le 2.$$

This leads to the above claim (25). Hence, we can show that

$$\int_{\tau_1}^{\tau_2}\int_{1\le r\le R} r^{1+\epsilon}|\partial\phi_1|^2|\partial\phi_2|^2 \mathrm{d}x\mathrm{d}\tau \lesssim R^{\epsilon}\int_{\tau_1}^{\tau_2} A(\tau)^2 \mathrm{d}\tau \lesssim A(\tau_1)R^{\epsilon-\beta}.$$

Inside the cylinder with radius 1, by using elliptic theory, we can show that

$$|\partial Z^k\phi|^2 \lesssim A(\tau), \quad k \le 2.$$

For the details, we refer to, e.g., the end of the second last section of Yang [34]. Therefore, we can estimate

$$\int_{\tau_1}^{\tau_2}\int_{r\le 1} |\partial\phi_1|^2|\partial\phi_2|^2 \mathrm{d}x\mathrm{d}\tau \lesssim \int_{\tau_1}^{\tau_2} A(\tau)^2 \mathrm{d}\tau \lesssim A(\tau_1)R^{-\beta+\epsilon}.$$

Combined with the estimate (24), we then have shown that

$$D^{\epsilon}[Z^k F]_{\tau_1}^{\tau_2} \le D_{+}^{\alpha}[Z^k F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{r\le R} (1+r)^{1+\epsilon}|Z^k F|^2 \mathrm{d}x\mathrm{d}\tau$$
$$\lesssim A(\tau_1)R^{\epsilon-\beta}, \quad \forall k \le 4.$$

Simply considering the total decay in R and $(\tau_1)_+$, we see from the definition of $A(\tau)$ in Proposition 2 that

$$A(\tau) \leq 2\tau_+^{-1-\alpha} R^{\epsilon+\alpha-\beta}.$$

Therefore, we have the estimate for $D_+^\alpha[Z^k F]_{\tau_1}^{\tau_2}$

$$D_+^\alpha[Z^k F]_{\tau_1}^{\tau_2} \lesssim (\tau_1)_+^{-1-\alpha} R^{-\beta} R^{\epsilon+\alpha-\beta}, \quad \forall k \leq 4.$$

For $D^\epsilon[Z^k F]_{\tau_1}^{\tau_2}$, when $(\tau_1)_+ \leq R$, we have

$$A(\tau_1) \leq R^{-2+\alpha} \leq R^\epsilon \min\{R^{-\beta}(\tau_1)_+^{-1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{-\alpha}\}.$$

Here, recall that $\beta = 1 - 2\alpha$, $\epsilon = \frac{\alpha}{20}$. When $(\tau_1)_+ \geq R$, we can show that

$$\begin{aligned} A(\tau_1) &\leq R^{-\beta}(\tau_1)_+^{-1-\alpha} + R^{1+\epsilon-\beta}(\tau_1)_+^{-2} \leq 2R^{1+\epsilon-\beta}(\tau_1)_+^{-2} \\ &\leq 2R^{2\epsilon+\alpha} \min\left\{R^{-\beta}(\tau_1)_+^{-1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{-\alpha}\right\} \end{aligned}$$

In any case, we have

$$A(\tau_1) \leq 2R^{2\epsilon+\alpha} \min\left\{R^{-\beta}(\tau_1)_+^{-1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{-\alpha}\right\}$$

Therefore, we have

$$D^\epsilon[Z^k F]_{\tau_1}^{\tau_2} \lesssim A(\tau_1)R^{\epsilon-\beta} \lesssim R^{3\epsilon+\alpha-\beta} \min\left\{R^{-\beta}(\tau_1)_+^{-1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{-\alpha}\right\}$$

for all $k \leq 4$. Recall that $\epsilon = \frac{\alpha}{20}$, $\beta = 1 - 2\alpha$ and $\alpha < \frac{1}{4}$. We conclude that for sufficiently large R, depending only on α, we can improve the bootstrap assumption (23). Then, the construction of the solution on the region $\{r \leq t + R\}$ will be the same as that in, e.g., [33] (the last section).

Finally for the pointwise bound of the solution, the case inside the cone $t + R \geq r$ can be obtained in the same way as that in, e.g., [34]. The pointwise estimate for the solution in the region $t + R < r$ is derived from the estimate (12) after using Sobolev embedding on the unit sphere. We are not able to get the pointwise estimate for the full derivative of the solution in the region $t + R < r$ as initially we are lack of this pointwise estimate.

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