

Singular support of coherent sheaves and the geometric Langlands conjecture

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Abstract We define the notion of singular support of a coherent sheaf on a quasi-smooth-derived scheme or Artin stack, where “quasi-smooth” means that it is a locally complete intersection in the derived sense. This develops the idea of “cohomological” support of coherent sheaves on a locally complete intersection scheme introduced by D. Benson, S. B. Iyengar, and H. Krause. We study the behavior of singular support under the direct and inverse image functors for coherent sheaves. We use the theory of singular support of coherent sheaves to formulate the categorical geometric Langlands conjecture. We verify that it passes natural consistency tests: It is compatible with the geometric Satake equivalence and with the Eisenstein series functors. The latter compatibility is particularly important, as it fails in the original “naive” form of the conjecture.

Keywords Cohomological support · Geometric Langlands program · Derived algebraic geometry · Arthur parameters

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1 Introduction

1.1 What we are trying to do

1.1.1. Let G be a connected reductive group, and X a smooth, connected, and complete curve over a ground field k , assumed algebraically closed and of characteristic 0. Let $\mathrm{Bun}_G(X)$ be the moduli stack of G -bundles on X , and consider the (DG) category $\mathrm{D}\text{-mod}(\mathrm{Bun}_G(X))$.

The goal of the (classical, global, and unramified) geometric Langlands program is to express the category $\mathrm{D}\text{-mod}(\mathrm{Bun}_G(X))$ in terms of the Langlands dual group \check{G} ; more precisely, in terms of the (DG) category of quasi-coherent sheaves on the (DG) stack $\mathrm{LocSys}_{\check{G}}$ of local systems on X with respect to \check{G} .

The naive guess, referred to by A. Beilinson and V. Drinfeld for a number of years as “the best hope,” says that the category $\mathrm{D}\text{-mod}(\mathrm{Bun}_G(X))$ is simply equivalent to $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$. For example, this is indeed the case when G is torus, and the required equivalence is given by the Fourier transform of [18, 19] and [35, 36].

However, the “best hope” does not hold for groups other than the torus. For example, it fails in the simplest case of $G = SL_2$ and $X = \mathbb{P}^1$. An explicit calculation showing this can be found in [17].

1.1.2. There is a heuristic reason for the failure of the “best hope”:

In the classical theory of automorphic forms, one expects that automorphic representations are parametrized not just by Galois representations, but by Arthur parameters, i.e., in addition to a homomorphism from the Galois group to \check{G} , one needs to specify a nilpotent element in $\check{\mathfrak{g}}$ centralized by the image of the Galois group.

In addition, there has been a general understanding that the presence of the commuting nilpotent element must be “cohomological in nature.” As an incarnation of this, for automorphic representations realized in the cohomology of Shimura varieties, the nilpotent element in question acts as the Lefschetz operator of multiplication by the Chern class of the corresponding line bundle.

So, in the geometric theory one has been faced with the challenge of how to modify the Galois side, i.e., the category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, with the hint being that the solution should come from considering the stack of pairs (σ, A) , where σ is a \check{G} -local system, and A its endomorphism, i.e., a horizontal section of the associated local system $\check{\mathfrak{g}}_{\sigma}$.

The general feeling, shared by many people who have looked at this problem, was that the sought-for modification has to do with the fact that the DG stack $\mathrm{LocSys}_{\check{G}}$ is not smooth; i.e., we need to modify the category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ by taking into account the singularities of $\mathrm{LocSys}_{\check{G}}$.

1.1.3. The goal of the present paper is to provide such a modification and to formulate the appropriately modified version of the “best hope.”

In fact, one does not have to look very far for the possibilities to “tweak” the category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$. Recall that for any reasonable algebraic DG stack \mathcal{Z} , the

category $\mathrm{QCoh}(\mathcal{Z})$ is compactly generated by its subcategory $\mathrm{QCoh}(\mathcal{Z})^{\mathrm{perf}}$ of perfect complexes.

Now, if \mathcal{Z} is non-smooth, one can enlarge the category $\mathrm{QCoh}(\mathcal{Z})^{\mathrm{perf}}$ to that of coherent complexes, denoted $\mathrm{Coh}(\mathcal{Z})$. By passing to the ind-completion, one obtains the category $\mathrm{IndCoh}(\mathcal{Z})$, studied in [6] and [10].

(We emphasize that the difference between $\mathrm{QCoh}(\mathcal{Z})$ and $\mathrm{IndCoh}(\mathcal{Z})$ is not a “stacky” phenomenon: It is caused not by automorphisms of points of \mathcal{Z} , but rather by its singularities.)

Thus, one can try $\mathrm{IndCoh}(\mathrm{LocSys}_{\mathcal{G}})$ as a candidate for the Galois side of geometric Langlands. However, playing with the example of $X = \mathbb{P}^1$ shows that, whereas $\mathrm{QCoh}(\mathrm{LocSys}_{\mathcal{G}})$ was “too small” to be equivalent to $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, the category $\mathrm{IndCoh}(\mathrm{LocSys}_{\mathcal{G}})$ is “too large.”

So, a natural guess for the category on the Galois side is that it should be a (full) subcategory of $\mathrm{IndCoh}(\mathrm{LocSys}_{\mathcal{G}})$ that contains $\mathrm{QCoh}(\mathrm{LocSys}_{\mathcal{G}})$. This is indeed the shape of the answer that we will propose. Our goal is to describe the corresponding subcategory.

1.1.4. Let us go back to the situation of a nice (=QCA, in the terminology of [6]) algebraic DG stack \mathcal{Z} . In general, it is not so clear how to describe categories that lie between $\mathrm{QCoh}(\mathcal{Z})$ and $\mathrm{IndCoh}(\mathcal{Z})$. However, the situation is more manageable when \mathcal{Z} is *quasi-smooth*, that is, if its singularities are modeled, locally in the smooth topology, by a complete intersection.

For every point $z \in \mathcal{Z}$, we can consider the derived cotangent space $T_z^*(\mathcal{Z})$, which is a complex of vector spaces lying in cohomological degrees ≤ 1 . The assumption that \mathcal{Z} is quasi-smooth is equivalent to that the cohomologies vanish in degrees < -1 (smoothness is equivalent to the vanishing of H^{-1} as well).

It is easy to see that the assignment $z \mapsto H^{-1}(T_z^*(\mathcal{Z}))$ forms a well-defined *classical*¹ stack, whose projection to \mathcal{Z} is affine, which carries a canonical action of \mathbb{G}_m by dilations. We will denote this stack by $\mathrm{Sing}(\mathcal{Z})$.

We are going to show (see Sect. 4) that to every $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Z})$, one can assign its singular support, denoted $\mathrm{SingSupp}(\mathcal{F})$, which is a conical Zariski-closed subset in $\mathrm{Sing}(\mathcal{Z})$. It is easy to see that for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Z})$, its singular support is contained in the zero-section of $\mathrm{Sing}(\mathcal{Z})$. It is less obvious, but still true (see Theorem 4.2.6) that if \mathcal{F} is such that its singular support is the zero-section, then \mathcal{F} belongs to $\mathrm{QCoh}(\mathcal{Z}) \subset \mathrm{IndCoh}(\mathcal{Z})$. Thus, the singular support of an object of $\mathrm{IndCoh}(\mathcal{Z})$ exactly measures the degree to which this object does not belong to $\mathrm{QCoh}(\mathcal{Z})$.

For a fixed conical Zariski-closed subset $Y \subset \mathrm{Sing}(\mathcal{Z})$, we can consider the full subcategory $\mathrm{IndCoh}_Y(\mathcal{Z}) \subset \mathrm{IndCoh}(\mathcal{Z})$ consisting of those objects, whose singular support lies in Y (The paper [37] implies that the assignment $Y \mapsto \mathrm{IndCoh}_Y(\mathcal{Z})$ establishes a bijection between subsets Y as above and full subcategories of $\mathrm{IndCoh}(\mathcal{Z})$ satisfying certain natural conditions.)

We should mention that the procedure of assigning the singular support to an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Z})$ is cohomological in nature: We read it off (locally) from the action

¹ “Classical” as opposed to “DG.”

of the algebra of Hochschild cochains (on smooth affine charts of \mathcal{Z}) on our object. This loosely corresponds to the cohomological nature of the Arthur parameter. To the best of our knowledge, the general idea of using cohomological operators to define support is due to Benson et al. [4].

The singular support for (ind)coherent complexes on a quasi-smooth DG scheme is similar to the singular support (also known as the “characteristic variety”) of a coherent D -module on a smooth variety, and also, perhaps in a more remote way, to the singular support of constructible sheaves on a manifold. Because of this analogy, we use the name “singular support” (rather than, say, “cohomological support”) for the support of ind-coherent sheaves.

1.1.5. Returning to the Galois side of geometric Langlands, we note that the DG stack $\mathrm{LocSys}_{\check{G}}$ is indeed quasi-smooth. Moreover, we note that the corresponding stack $\mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$ classifies pairs (σ, A) , i.e., Arthur parameters. Thus, we rename

$$\mathrm{Arth}_{\check{G}} := \mathrm{Sing}(\mathrm{LocSys}_{\check{G}}).$$

Our candidate for a category lying between $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ and $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ corresponds to a particular closed subset of $\mathrm{Arth}_{\check{G}}$. Namely, let

$$\mathrm{Nilp}_{\mathrm{glob}} \subset \mathrm{Arth}_{\check{G}},$$

be the subset of pairs (σ, A) with nilpotent A .

So, we propose the following modified version of the “best hope”:

Conjecture 1.1.6 *There exists an equivalence of categories*

$$\mathrm{D}\text{-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$$

Thus, given $\mathcal{M} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, one cannot really speak of “the Arthur parameter” corresponding to \mathcal{M} . However, one can specify a conical closed subset of $\mathrm{Nilp}_{\mathrm{glob}}$ over which \mathcal{M} is supported.

1.1.7. Geometric Langlands correspondence is more than simply an equivalence of categories as in Conjecture 1.1.6. Rather, the sought-for equivalence must satisfy a number of compatibility conditions.

Two of these conditions are discussed in the present paper: Compatibility with the geometric Satake equivalence and compatibility with the Eisenstein series functors.

Two other conditions have to do with the description of the Whittaker D -module on the automorphic side, and of the construction of automorphic D -modules by localization from Kac-Moody representations.

On the Galois side, the Whittaker D -module is supposed to correspond to the structure sheaf on $\mathrm{LocSys}_{\check{G}}$. The localization functor should correspond to the direct image functor with respect to the map to $\mathrm{LocSys}_{\check{G}}$ from the scheme of opers.

Both of the latter procedures are insensitive to the singular aspects of $\mathrm{LocSys}_{\check{G}}$, which is why we do not discuss them in this paper. However, we plan to revisit these objects in a subsequent publication.

1.1.8. Conjecture 1.1.6 contains the following statement (which can actually be proved unconditionally):

Conjecture 1.1.9 *The monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ acts on $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$.*

The above corollary allows one to take fibers of the category $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ at points of $\mathrm{LocSys}_{\check{G}}$, or more generally S -points for any test scheme S . Namely, for $\sigma : S \rightarrow \mathrm{LocSys}_{\check{G}}$ we set

$$\mathrm{D}\text{-mod}(\mathrm{Bun}_G)_\sigma := \mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})} \mathrm{D}\text{-mod}(\mathrm{Bun}_G).$$

In particular, for a k -point σ of $\mathrm{LocSys}_{\check{G}}$, the corresponding category $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)_\sigma$ is that of Hecke eigensheaves with eigenvalue σ .

However, we do not know (and have no reasons to believe) that this procedure can be refined to $\mathrm{Arth}_{\check{G}}$. In other words, we do not expect that the category $\mathrm{QCoh}(\mathrm{Arth}_{\check{G}})$ should act on $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ and that it should be possible to take the fiber of $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ at a specified Arthur parameter.

1.1.10. Let \mathcal{Z} be a quasi-smooth DG stack and let $Y \subset \mathrm{Sing}(\mathcal{Z})$ be a conical Zariski-closed subset. If Y contains the zero-section of $\mathrm{Sing}(\mathcal{Z})$, then the category $\mathrm{IndCoh}_Y(\mathcal{Z})$ contains $\mathrm{QCoh}(\mathcal{Z})$ as a full subcategory.

In particular, $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ contains $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ as a full subcategory.

Accepting Conjecture 1.1.6, we obtain that $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ corresponds to a certain full subcategory of $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, consisting of objects whose support in $\mathrm{Arth}_{\check{G}}$ is contained in the zero-section. In other words, its support only contains Arthur parameters (σ, A) with $A = 0$.

We denote this category by $\mathrm{D}\text{-mod}_{\mathrm{temp}}(\mathrm{Bun}_G)$. In Corollary 12.8.7, we give an intrinsic characterization of this subcategory in terms of the action of the Hecke functors.

1.2 Results concerning Langlands correspondence

The main theorems of this paper fall into two classes. On the one hand, we prove some general results about the behavior of the categories $\mathrm{IndCoh}_Y(\mathcal{Z})$. On the other hand, we run some consistency checks on Conjecture 1.1.6.

We will begin with the review of the latter.

1.2.1. Recall that the Hecke category $\mathrm{Sph}(G, x) \simeq \mathrm{D}\text{-mod}(\mathrm{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)}$ is a monoidal category acting on $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ by the Hecke functors.

The (derived) geometric Satake equivalence identifies $\mathrm{Sph}(G, x)$ with a certain subcategory of the category of ind-coherent sheaves on the DG stack

$$\mathrm{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \mathrm{pt}/\check{G},$$

which is a monoidal category under convolution.

We prove (Theorem 12.5.3) that the resulting subcategory of $\text{IndCoh}(\text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G})$ is determined by a singular support condition.

Namely, there is a natural isomorphism

$$\text{Sing}(\text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G}) \simeq \check{\mathfrak{g}}^*/\check{G},$$

which allows us to view

$$\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G}$$

as a conical subset of $\text{Sing}(\text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G})$. Here, $\text{Nilp}(\check{\mathfrak{g}}^*) \subset \check{\mathfrak{g}}^*$ is the cone of nilpotent elements. We show that the geometric Satake equivalence identifies $\text{Sph}(G, x)$ with the full subcategory

$$\text{IndCoh}_{\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G}) \subset \text{IndCoh}(\text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G})$$

corresponding to this conical subset.

We also show (see Proposition 12.7.3) that the conjectural equivalence of “modified best hope” (Conjecture 1.1.6) is consistent with the geometric Satake equivalence. Namely, we construct a natural action of the monoidal category

$$\text{IndCoh}_{\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G})$$

on the category $\text{IndCoh}_{\text{Nilp}_{glob}}(\text{LocSys}_{\check{G}})$. This action should correspond to the action of the monoidal category $\text{Sph}(G, x)$ on $\text{D-mod}(\text{Bun}_G)$ under the equivalence of Conjecture 1.1.6.

1.2.2. The following fact was observed in [17], and independently, by R. Bezrukavnikov.

Take $X = \mathbb{P}^1$, and let $\delta_{\mathbf{1}} \in \text{D-mod}(\text{Bun}_G)$ be the D-module of δ -functions at the trivial bundle $\mathbf{1} \in \text{Bun}_G$. Then, the Hecke action of $\text{Sph}(G, x)$ on $\delta_{\mathbf{1}}$ defines an equivalence

$$\text{Sph}(G, x) \rightarrow \text{D-mod}(\text{Bun}_G).$$

It is easy to see that the action of $\text{IndCoh}(\text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G})$ on the sky-scraper of the unique k -point of $\text{LocSys}_{\check{G}}$ defines an equivalence

$$\text{IndCoh}(\text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G}) \rightarrow \text{IndCoh}(\text{LocSys}_{\check{G}}).$$

Thus, Theorem 12.5.3 implies the existence of an equivalence as stated in Conjecture 1.1.6 for $X = \mathbb{P}^1$.

To show that this equivalence is *the* equivalence, one needs to verify the additional properties that one expects the equivalence of Conjecture 1.1.6 to satisfy, see Sect. 1.1.7. Only some of these properties have been verified so far. So, an interested reader is welcome to tackle them.

1.2.3. A fundamental construction in the classical theory of automorphic functions is that of Eisenstein series. In the geometric theory, this takes the form of a functor

$$\mathrm{Eis}_!^P : \mathrm{D}\text{-mod}(\mathrm{Bun}_M) \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_G),$$

defined for every parabolic subgroup P with Levi quotient M .

The Eisenstein series functor $\mathrm{Eis}_!^P$ is defined as

$$(\mathfrak{p}^P)_! \circ (\mathfrak{q}^P)^*,$$

where \mathfrak{p}^P and \mathfrak{q}^P are the maps in the diagram

$$\begin{array}{ccc} & \mathrm{Bun}_P & \\ \mathfrak{p}^P \swarrow & & \searrow \mathfrak{q}^P \\ \mathrm{Bun}_G & & \mathrm{Bun}_M. \end{array} \quad (1.1)$$

It is not obvious that the functor $\mathrm{Eis}_!^P$ makes sense, because the functors $(\mathfrak{q}^P)^*$ and $(\mathfrak{p}^P)_!$ are being applied to D-modules that need not be holonomic. For some non-trivial reasons, the functor $\mathrm{Eis}_!^P$ is defined on the category $\mathrm{D}\text{-mod}(\mathrm{Bun}_M)$; See Sect. 13.1.1 for a quick summary and [8, Proposition 1.2] for a proof.

On the spectral side, one has an analogous functor

$$\mathrm{Eis}_{spec}^P : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}}) \rightarrow \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}),$$

defined as

$$(\mathfrak{p}_{spec}^P)_*^{\mathrm{IndCoh}} \circ (\mathfrak{q}_{spec}^P)^!$$

using the diagram

$$\begin{array}{ccc} & \mathrm{LocSys}_{\check{P}} & \\ \mathfrak{p}_{spec}^P \swarrow & & \searrow \mathfrak{q}_{spec}^P \\ \mathrm{LocSys}_{\check{G}} & & \mathrm{LocSys}_{\check{M}}. \end{array} \quad (1.2)$$

The Langlands correspondence for groups G and M is supposed to intertwine the functors $\mathrm{Eis}_!^P$ and Eis_{spec}^P (up to tensoring by a line bundle).

Thus, a consistency check for Conjecture 1.1.6 should imply:

Theorem 1.2.4 *The functor $\mathrm{Eis}_{\mathrm{Spec}}^P$ maps*

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \rightarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$$

We prove this theorem in Sect. 13 (see Proposition 13.2.6).

1.2.5. The main result of this paper is Theorem 13.3.6, which is a refinement of Theorem 1.2.4.

Essentially, Theorem 13.3.6 says that the choice of $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ as a subcategory of $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ containing $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ is the minimal one, if we want to have an equivalence with $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ compatible with the Eisenstein series functors.

More precisely, Theorem 13.3.6 says that the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ is generated by the essential images of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})$ under the functors $\mathrm{Eis}_{\mathrm{Spec}}^P$ for all parabolics P (including $P = G$).

1.3 Results concerning the theory of singular support

1.3.1. As was already mentioned, the idea of support based on cohomological operations was pioneered by Benson et al. in [4].

Namely, let \mathbf{T} be a triangulated category (containing arbitrary direct sums), and let A be an algebra graded by non-negative even integers that acts on \mathbf{T} . By this, we mean that every homogeneous element $a \in A_{2n}$ defines a natural transformation from the identity functor to the shift functor $\mathcal{F} \mapsto \mathcal{F}[2n]$. Given a homogeneous element $a \in A$, one can attach to it the full subcategory $\mathbf{T}_{\mathrm{Spec}(A)-Y_a} \subset \mathbf{T}$ consisting of a -local objects, and its left orthogonal, denoted \mathbf{T}_{Y_a} , to be thought of as consisting of objects “set-theoretically supported on the set of zeroes of a .” More generally, one can attach the corresponding subcategories

$$\mathbf{T}_Y \subset \mathbf{T} \supset \mathbf{T}_{\mathrm{Spec}(A)-Y}$$

to any conical Zariski-closed subset $Y \subset \mathrm{Spec}(A)$.

It is shown in *loc.cit.* that the categories $\mathbf{T}_Y \subset \mathbf{T}$ are very well-behaved. Namely, they satisfy essentially the same properties as when $\mathbf{T} = A\text{-mod}$, and we are talking about the usual notion of support in commutative algebra.

1.3.2. Let us now take \mathbf{T} to be the homotopy category of the DG category $\mathrm{IndCoh}(Z)$, where Z is an affine DG scheme. There is a universal choice of a graded algebra acting on \mathbf{T} , namely $\mathrm{HH}(Z)$, the Hochschild cohomology of Z .

We note that when Z is quasi-smooth, there is a canonical map of graded algebras

$$\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)}) \rightarrow \mathrm{HH}(Z),$$

where the grading on $\mathcal{O}_{\mathrm{Sing}(Z)}$ is obtained by scaling by 2 the action of \mathbb{G}_m along the fibers of $\mathrm{Sing}(Z) \rightarrow Z$.

Thus, by [4], we obtain the desired assignment

$$Y \subset \text{Sing}(Z) \rightsquigarrow \text{IndCoh}_Y(Z) \subset \text{IndCoh}(Z).$$

For a given $\mathcal{F} \in \text{IndCoh}(Z)$, its singular support is by definition the smallest Y such that

$$\mathcal{F} \subset \text{IndCoh}_Y(Z).$$

Remark 1.3.3 We chose the terminology ‘‘singular support’’ by loose analogy with the theory of D -modules. In the latter case, the singular support of a D -module is a conical subset of the (usual) cotangent bundle, which measures the degree to which the D -module is not lisse.

Remark 1.3.4 We do not presume to make a thorough review of the existing literature on the subject. However, in Appendix H we will indicate how the notion of singular support developed in this paper is related to several other approaches, due to D. Orlov, L. Positselski, G. Stevenson, and M. Umut Isik, respectively.

1.3.5. If the above definition of singular support sounds a little too abstract, here is how it can be rewritten more explicitly.

First, we consider the most basic example of a quasi-smooth (DG !) scheme. Namely, let \mathcal{V} be a smooth scheme, and let $\text{pt} \rightarrow \mathcal{V}$ be a k -point. We consider the DG scheme

$$\mathcal{G}_{\text{pt}/\mathcal{V}} := \text{pt} \times_{\mathcal{V}} \text{pt}.$$

Explicitly, let V denote the tangent space to \mathcal{V} at pt . Then for every *parallelization* of \mathcal{V} at pt , i.e., for an identification of the formal completion of $\mathcal{O}_{\mathcal{V}}$ at pt with $\widehat{\text{Sym}}(V)$, we obtain an isomorphism

$$\mathcal{G}_{\text{pt}/\mathcal{V}} \simeq \text{Spec}(\text{Sym}(V^*[1])).$$

Now, Koszul duality defines an equivalence of DG categories

$$\text{KD}_{\text{pt}/\mathcal{V}} : \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \simeq \text{Sym}(V[-2])\text{-mod}$$

(this equivalence does not depend on the choice of a parallelization).

It is easy to see that $\text{Sing}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \simeq V^*$. In terms of this equivalence, the singular support of an object in $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})$ becomes the usual support of the corresponding object in $\text{Sym}(V[-2])\text{-mod}$ (By definition, the support of a $\text{Sym}(V[-2])$ -module M is the support of its cohomology $H^\bullet(M)$ viewed as a graded $\text{Sym}(V)$ -module.).

For example, for $\mathcal{F} = \mathcal{O}_{\mathcal{G}_{\text{pt}/\mathcal{V}}}$, its singular support is $\{0\}$. By contrast, for \mathcal{F} being the skyscraper at the unique k -point of $\mathcal{G}_{\text{pt}/\mathcal{V}}$, its singular support is all of V^* .

1.3.6. Suppose now that a quasi-smooth DG scheme Z is given as a global complete intersection. By this, we mean that

$$Z = \text{pt} \times_{\mathcal{V}} \mathcal{U},$$

where \mathcal{U} and \mathcal{V} are smooth (Any quasi-smooth DG scheme can be locally written in this form, see Corollary 2.1.6.).

Then, it is easy to see that there exists a canonical closed embedding

$$\text{Sing}(Z) \hookrightarrow V^* \times Z, \quad (1.3)$$

where V^* is as above.

Now, we have a canonical isomorphism of DG schemes

$$Z \times_{\mathcal{U}} Z \simeq \mathcal{G}_{\text{pt}/\mathcal{V}} \times Z$$

(see Sect. 5.3.1), so we have a map, denoted

$$\text{act}_{\text{pt}/\mathcal{V}, Z} : \mathcal{G}_{\text{pt}/\mathcal{V}} \times Z \rightarrow Z.$$

We show in Corollary 5.6.7(a) that an object $\mathcal{F} \in \text{IndCoh}(Z)$ has its singular support inside $Y \subset \text{Sing}(Z) \subset V^* \times Z$ if and only if the object

$$(\text{KD}_{\text{pt}/\mathcal{V}} \otimes \text{Id}) \circ \text{act}_{\text{pt}/\mathcal{V}, Z}^!(\mathcal{F}) \in \text{Sym}(V[-2])\text{-mod} \otimes \text{IndCoh}(Z)$$

is supported on Y in the sense of commutative algebra.

1.3.7. Here is yet another, even more explicit, characterization of singular support of objects of $\text{Coh}(Z)$, suggested to us by V. Drinfeld.

Let (z, ξ) be an element of $\text{Sing}(Z)$, where z is a point of Z and $0 \neq \xi \in H^{-1}(T_z^*(Z))$. We wish to know when this point belongs to $\text{SingSupp}(\mathcal{F})$ for a given $\mathcal{F} \in \text{Coh}(Z)$.

Suppose that Z is written as in Sect. 1.3.6. Then by (1.3), ξ corresponds to a cotangent vector to \mathcal{V} at pt . Let f be a function on \mathcal{V} that vanishes at pt , and whose differential equals ξ . Let $Z' \subset \mathcal{U}$ be the hypersurface cut out by the pullback of f to \mathcal{U} . Note that Z' is singular at pt . Let i denote the closed embedding $Z \hookrightarrow Z'$.

We have:

Theorem 1.3.8 *The element (z, ξ) does not belong to $\text{SingSupp}(\mathcal{F})$ if and only if $i_*(\mathcal{F}) \in \text{Coh}(Z')$ is perfect on a Zariski neighborhood of z .*

This theorem will be proved as Corollary 7.3.6

1.3.9. Here are some basic properties of the assignment

$$Y \mapsto \text{IndCoh}_Y(Z):$$

(a) As was mentioned before,

$$\mathcal{F} \in \text{QCoh}(\mathcal{F}) \Leftrightarrow \text{SingSupp}(\mathcal{F}) = \{0\};$$

this is Theorem 4.2.6.

- (b) The assignment $Y \rightsquigarrow \text{IndCoh}_Y(Z)$ is Zariski-local (see Corollary 4.5.7). In particular, this allows us to define singular support on non-affine DG schemes.
- (c) For Z quasi-compact, the category $\text{IndCoh}_Y(Z)$ is compactly generated. This is easy for Z affine (see Corollary 4.3.2) and is a variant of the theorem of [38] in general (see Appendix C).
- (d) The category $\text{IndCoh}_Y(Z)$ has a t-structure whose eventually coconnective part identifies with that of $\text{QCoh}(Z)$. Informally, the difference between $\text{IndCoh}_Y(Z)$ and $\text{QCoh}(Z)$ is “at $-\infty$.” This is the content of Sect. 4.4.
- (e) For $\mathcal{F} \in \text{Coh}(Z)$,

$$\text{SingSupp}(\mathcal{F}) = \text{SingSupp}(\mathbb{D}_Z^{\text{Serre}}(\mathcal{F})),$$

where $\mathbb{D}_Z^{\text{Serre}}$ is the Serre duality anti-involution of the category $\text{Coh}(Z)$. This is Proposition 4.7.2.

- (f) Singular support can be computed point-wise. Namely, for $\mathcal{F} \in \text{IndCoh}(Z)$ and a geometric point $\text{Spec}(k') \xrightarrow{i_z} Z$, the graded vector space $H^\bullet(i_z^!(\mathcal{F}))$ is acted on by the algebra $\text{Sym}(H^1(T_z(Z)))$, and

$$\text{SingSupp}(\mathcal{F}) \subset Y \Leftrightarrow \forall z, \text{supp}_{\text{Sym}(H^1(T_z(Z)))} \left(H^\bullet(i_z^!(\mathcal{F})) \right) \subset Y \cap H^{-1}(T_z^*(Z)).$$

This is Proposition 6.2.2.

- (g) For Z_1 and Z_2 quasi-compact, and $Y_i \subset \text{Sing}(Z_i)$, we have

$$\text{IndCoh}_{Y_1}(Z_1) \otimes \text{IndCoh}_{Y_2}(Z_2) = \text{IndCoh}_{Y_1 \times Y_2}(Z_1 \times Z_2)$$

as subcategories of

$$\text{IndCoh}(Z_1) \otimes \text{IndCoh}(Z_2) \otimes \text{IndCoh}(Z_1 \times Z_2).$$

This is Lemma 4.6.4.

- (h) An estimate on singular support ensures preservation of coherence. For example, if $\mathcal{F}', \mathcal{F}'' \in \text{Coh}(Z)$ are such that the set-theoretic intersection of their respective singular supports is contained in the zero-section of $\text{Sing}(Z)$, then the tensor product $\mathcal{F}' \otimes \mathcal{F}''$ lives in finitely many cohomological degrees. This is proved in Proposition 7.2.2.

1.3.10 Functoriality Let $f : Z_1 \rightarrow Z_2$ be a map between quasi-smooth (and quasi-compact) DG schemes. We have the functors

$$f_*^{\text{IndCoh}} : \text{IndCoh}(Z_1) \rightarrow \text{IndCoh}(Z_2) \text{ and } f^! : \text{IndCoh}(Z_2) \rightarrow \text{IndCoh}(Z_1)$$

(they are adjoint if f is proper).

We wish to understand how they behave in relation to the categories $\text{IndCoh}_Y(Z)$. First, we note that there exists a canonical map

$$Z_1 \times_{Z_2} \text{Sing}(Z_2) \rightarrow \text{Sing}(Z_1);$$

we call this map “the singular codifferential of f ,” and denote it by $\text{Sing}(f)$.

The singular support behaves naturally under direct and inverse images:

Theorem 1.3.11 *Let $Y_i \subset \text{Sing}(Z_i)$ be conical Zariski-closed subsets.*

- (a) *If $\text{Sing}(f)^{-1}(Y_1) \subset Y_2 \times_{Z_2} Z_1$, then the functor f_*^{IndCoh} maps $\text{IndCoh}_{Y_1}(Z_1)$ to $\text{IndCoh}_{Y_2}(Z_2)$.*
- (b) *If $Y_2 \times_{Z_2} Z_1 \subset \text{Sing}(f)^{-1}(Y_1)$, then $f^!$ maps $\text{IndCoh}_{Y_2}(Z_2)$ to $\text{IndCoh}_{Y_1}(Z_1)$.*

This is proved in Proposition 7.1.3.

Suppose now that the map f is itself quasi-smooth. According to Lemma 2.4.3, this is equivalent to the condition that the singular codifferential map $\text{Sing}(f)$ be a closed embedding. For $Y_2 \subset \text{Sing}(Z_2)$, let

$$Y_1 := \text{Sing}(f)(Y_2 \times_{Z_2} Z_1)$$

be the corresponding subset in $\text{Sing}(Z_1)$.

In Corollary 7.6.2, we will show:

Theorem 1.3.12 *The functor $f^!$ defines an equivalence*

$$\text{QCoh}(Z_1) \otimes_{\text{QCoh}(Z_2)} \text{IndCoh}_{Y_2}(Z_2) \rightarrow \text{IndCoh}_{Y_1}(Z_1).$$

Finally, we have the following crucial result (see Theorem 7.8.2):

Theorem 1.3.13 *Assume that f is proper, and let $Y_i \subset \text{Sing}(Z_i)$ be such that the composed map*

$$\text{Sing}(f)^{-1}(Y_1) \hookrightarrow Z_1 \times_{Z_2} \text{Sing}(Z_2) \rightarrow \text{Sing}(Z_2)$$

is surjective onto Y_2 . Then, the essential image of $\text{IndCoh}_{Y_1}(Z_1)$ under f_^{IndCoh} generates $\text{IndCoh}_{Y_2}(Z_2)$.*

Remark 1.3.14 Theorems 1.3.12 and 1.3.13 are deeper than Theorem 1.3.11. Indeed, Theorem 1.3.11 provides upper bounds on the essential images $f_*^{\text{IndCoh}}(\text{IndCoh}_{Y_2}(Z_2))$ and $f^!(\text{IndCoh}_{Y_1}(Z_1))$; to a large extent, these bounds formally follow from definitions. On the other hand, Theorems 1.3.12 and 1.3.13 give a precise description of this image under some additional assumptions on the morphism f ; they are more “geometric” in nature.

1.3.15. Finally, we remark that Theorem 1.3.12 ensures that the assignment $Y \mapsto \text{IndCoh}_Y(Z)$ is local also in the smooth topology. This allows us to develop the theory of singular support on DG Artin stacks.

For a quasi-smooth DG Artin stack \mathcal{Z} , we introduce the classical Artin stack $\text{Sing}(\mathcal{Z})$ by descending $\text{Sing}(Z)$ over affine DG schemes Z mapping smoothly to \mathcal{Z} (any such Z is automatically quasi-smooth).

Given $Y \subset \text{Sing}(\mathcal{Z})$, we define the category $\text{IndCoh}_Y(\mathcal{Z})$ as the limit of the categories

$$\text{IndCoh}_{Z \times_{\mathcal{Z}} Y}(Z)$$

over $Z \rightarrow \mathcal{Z}$ as above.

One easily establishes the corresponding properties of the category $\text{IndCoh}_Y(\mathcal{Z})$ by reducing to the case of schemes. The one exception is the question of compact generation.

At the moment, we cannot show that $\text{IndCoh}_Y(\mathcal{Z})$ is compactly generated for a general nice (=QCA) algebraic DG stack. However, we can do it when \mathcal{Z} can be presented as a global complete intersection (see Corollary 9.2.7). Fortunately, this is the case for $\mathcal{Z} = \text{LocSys}_G$ for any affine algebraic group G .

However, we do prove that the category $\text{IndCoh}_Y(\mathcal{Z})$ is dualizable for a general \mathcal{Z} which is QCA (see Corollary 8.2.12).

1.4 How this paper is organized

This paper is divided into three parts. Part I contains miscellaneous preliminaries, Part II develops the theory of singular support for IndCoh , and Part III discusses the applications to geometric Langlands.

1.4.1. In Sect. 2, we recall the notion of quasi-smooth DG scheme and morphism. We show that this condition is equivalent to that of locally complete intersection. We also introduce the classical scheme $\text{Sing}(Z)$ attached to a quasi-smooth DG scheme Z .

In Sect. 3, we review the theory of support in a triangulated category acted on by a graded algebra. Most results of this section are contained in [4] (Note, however, that what we call support is the closure of the support in the terminology of *loc.cit.*).

1.4.2. In Sect. 4, we introduce the notion of singular support of objects of $\mathrm{IndCoh}(Z)$, where Z is a quasi-smooth DG scheme, and establish the basic properties listed in Sect. 1.3.9.

In Sect. 5, we study the case of a global complete intersection and prove the description of singular support via Koszul duality, mentioned in Sect. 1.3.6.

In Sect. 6, we show how singular support of an object $\mathcal{F} \in \mathrm{IndCoh}(Z)$ can be read off the behavior of the fibers of \mathcal{F} at geometric points of Z .

In Sect. 7, we establish the functoriality properties of categories $\mathrm{IndCoh}_Y(Z)$ mentioned in Sect. 1.3.10.

In Sect. 8, we develop the theory of singular support on Artin DG stacks.

In Sect. 9, we prove that if a quasi-compact algebraic DG stack \mathcal{Z} is given as a global complete intersection, then the subcategories of $\mathrm{IndCoh}(\mathcal{Z})$ defined by singular support are compactly generated.

1.4.3. In Sect. 10, we recall the definition of the DG stack of G -local systems on a given scheme X , where G is an algebraic group. The reason we decided to include this section rather than refer to some existing source is that, even for X a smooth and complete curve, the stack LocSys_G is an object of derived algebraic geometry, and the relevant definitions do not seem to be present in the literature, although seem to be well known in folklore.

In Sect. 11, we introduce the global nilpotent cone

$$\mathrm{Nilp}_{glob} \subset \mathrm{Arth}_{\check{G}} = \mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$$

and formulate the geometric Langlands conjecture.

In Sect. 12, we study how our proposed form of geometric Langlands conjecture interacts with the geometric Satake equivalence.

In Sect. 13, we study the functors of Eisenstein series on both the automorphic and the Galois side of the correspondence, and prove a consistency result (Theorem 13.3.6) with Conjecture 1.1.6.

1.4.4. In Appendix A, we list several facts pertaining to the notion of action of an algebraic group on a category, used in several places in the main body of the paper. These facts are fully documented in [31].

In Appendix B, we discuss the formation of mapping spaces in derived algebraic geometry, and how it behaves under deformation theory.

In Appendix C, we prove a version of the Thomason-Trobaugh theorem for categories defined by singular support.

In Appendix D, we prove a certain finite generation result for Exts between coherent sheaves on a quasi-smooth DG scheme.

In Appendix E, we make a brief review of the theory of \mathbb{E}_2 -algebras.

In Appendix F, we collect some facts concerning the \mathbb{E}_2 -algebra of Hochschild cochains on an affine DG scheme, and its generalization that has to do with groupoids.

In Appendix G, we study the connection between the \mathbb{E}_2 -algebra of Hochschild cochains and Lie algebras, and its generalization to the case of groupoids.

In Appendix H, we review the relation of the theory of singular support developed in this paper with several other approaches.

1.5 Conventions

1.5.1. Throughout the text, we work with a ground field k , assumed algebraically closed and of *characteristic* 0.

1.5.2 ∞ -categories and DG categories Our conventions follow completely those adopted in the paper [7], and we refer the reader to Sect. 1, where the latter are explained.

In particular:

- (i) When we say “category” by default, we mean “ $(\infty, 1)$ -category.”
- (ii) For a category \mathbf{C} and objects $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$, we will denote by $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the ∞ -groupoid of maps between them. We will denote by $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the set $\pi_0(\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2))$, i.e., Hom in the ordinary category $\mathrm{Ho}(\mathbf{C})$.
- (iii) All DG categories are assumed to be pretriangulated and, unless explicitly stated otherwise, cocomplete (that is, they contain arbitrary direct sums). All functors between DG categories are assumed to be exact and continuous (that is, commuting with arbitrary direct sums, or equivalently, with all colimits). In particular, all subcategories are by default assumed to be closed under arbitrary direct sums. Note that our terminology for functors follows [20] in that a colimit-preserving functor is called continuous, rather than co-continuous.
- (iv) For a DG category \mathbf{C} equipped with a t-structure, we will denote by $\mathbf{C}^{\leq 0}$ (resp. $\mathbf{C}^{\geq 0}$) the corresponding subcategories of connective (resp. coconnective) objects. We let \mathbf{C}^{\heartsuit} denote the heart of the t-structure. We also let \mathbf{C}^+ (resp. \mathbf{C}^-) denote the subcategory of eventually coconnective (resp. connective) objects.
- (v) We let Vect denote the DG category of complexes of vector spaces; thus, the usual category of k -vector spaces is denoted by $\mathrm{Vect}^{\heartsuit}$. The category of ∞ -groupoids is denoted by $\infty\text{-Grpd}$.
- (vi) If \mathbf{C} is a DG category, let $\mathcal{M}\mathrm{aps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ denote the corresponding object of Vect . Sometimes, we view $\mathcal{M}\mathrm{aps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ as a spectrum. In particular, $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ is recovered as the 0-th space of $\mathcal{M}\mathrm{aps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$. We can also view

$$\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \simeq \tau^{\leq 0}(\mathcal{M}\mathrm{aps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)) \in \mathrm{Vect}^{\leq 0},$$

where $\mathrm{Vect}^{\leq 0}$ maps to $\infty\text{-Grpd}$ via the Dold-Kan correspondence.

- (vi') We will denote by $\mathrm{Hom}_{\mathbf{C}}^{\bullet}(\mathbf{c}_1, \mathbf{c}_2)$ the graded vector space $H^{\bullet}(\mathcal{M}\mathrm{aps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2))$. By definition,

$$H^{\bullet}(\mathcal{M}\mathrm{aps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)) = \bigoplus_i \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(\mathbf{c}_1, \mathbf{c}_2[i]).$$

We sometimes use the notation $\mathrm{Hom}^\bullet(-, -)$ when instead of a DG category we have just a triangulated category \mathbf{T} . That is, we use $\mathrm{Hom}_{\mathbf{T}}^\bullet(\mathbf{t}_1, \mathbf{t}_2)$ in place of the more common $\mathrm{Ext}_{\mathbf{T}}^\bullet(\mathbf{t}_1, \mathbf{t}_2)$.

- (vii) Let $\mathbf{C}' \subset \mathbf{C}$ be a full subcategory of the DG category \mathbf{C} (Recall that \mathbf{C} is assumed to be cocomplete, and \mathbf{C}' is assumed to be closed under direct sums.). Suppose that the inclusion $\mathbf{C}' \rightarrow \mathbf{C}$ admits a left adjoint, which is automatically exact and continuous. We call this adjoint the *localization functor*.

In this case, we let $\mathbf{C}'' := {}^\perp(\mathbf{C}')$ be the left orthogonal complement of \mathbf{C}' . The inclusion $\mathbf{C}'' \rightarrow \mathbf{C}$ admits a right adjoint, which is easily seen to be continuous (see Lemma 3.1.5, which contains the version of this claim for triangulated categories). We call the right adjoint functor $\mathbf{C} \rightarrow \mathbf{C}''$ the *colocalization functor*. The resulting diagram

$$\mathbf{C}'' \rightleftarrows \mathbf{C} \rightleftarrows \mathbf{C}'$$

is a *short exact sequence of DG categories*.

- (viii) For a monoidal DG category \mathbf{C} , we use the terms “a DG category with an action of \mathbf{C} ” and “a DG category tensored over \mathbf{C} ” interchangeably (see [20, A.1.4] for definition).
- (ix) For a (DG) associative algebra A , we denote by $A\text{-mod}$ the corresponding DG category of A -modules.

1.5.3 DG schemes and Artin stacks Conventions and notation regarding DG schemes and DG Artin stacks (and, more generally, prestacks) follow [28]. In particular, $\mathrm{Sch}^{\mathrm{aff}}$, $\mathrm{DGSch}^{\mathrm{aff}}$, Sch , DGSch , and PreStk stand for the categories of classical affine schemes, affine DG schemes, classical schemes, DG schemes, and (DG) prestacks, respectively. A short review of the conventions can be found also in [6, Sect. 0.6.4-0.6.5].

We denote $\mathrm{pt} := \mathrm{Spec}(k)$.

By default, all schemes/Artin stacks are derived. When they are classical, we will emphasize this explicitly.

All DG schemes and DG Artin stacks in this paper will be assumed locally almost of finite type over k (see [28, Sect. 1.3.9, 2.6.5, 3.3.1 and 4.9]), unless specified otherwise. We denote by $\mathrm{DGSch}_{\mathrm{aff}} \subset \mathrm{DGSch}$ the full subcategory of DG schemes that are locally almost of finite type.

We will also use the following convention: We will not distinguish between the notions of classical scheme/Artin stack and that of 0-coconnective DG scheme/Artin stack (see [28, Sect. 4.6.3] for the latter notion).

By definition, classical schemes/Artin stacks form a full subcategory among functors $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \rightarrow \infty\text{-Grpd}$, while 0-coconnective DG schemes/Artin stacks form a full subcategory among functors $(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \rightarrow \infty\text{-Grpd}$. However, the two categories are equivalent: The equivalence is given by restriction along the fully faithful embedding $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \subset (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}}$; the inverse procedure is left Kan extension, followed by sheafification.

A more detailed discussion of the notion of n -coconnectivity can be found in [11, Sect. 0.5].

For a given DG scheme/Artin stack \mathcal{Z} , we will denote by ${}^{cl}\mathcal{Z}$ the underlying classical scheme/classical Artin stack.

1.5.4. Conventions regarding the categories $\mathrm{QCoh}(-)$ and $\mathrm{IndCoh}(-)$ on DG schemes/Artin stacks follow those of [30] and [10], respectively. Conventions regarding the category $\mathrm{D}\text{-mod}(-)$ follow [12].

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Part I: Preliminaries

2 Quasi-smooth DG schemes and the scheme of singularities

We remind that all DG schemes in this paper are assumed locally almost of finite type over the ground field k , unless explicitly stated otherwise.

In this section, we will recall the notions of quasi-smooth (DG) scheme and quasi-smooth morphism between DG schemes. To a quasi-smooth DG scheme Z , we will attach a classical scheme $\mathrm{Sing}(Z)$ that “controls” the singularities of Z .

Quasi-smoothness is the “correct” DG version of being locally a complete intersection.

2.1 The notion of quasi-smoothness

In this subsection, we define quasi-smoothness in terms of the cotangent complex. We will show that any quasi-smooth morphism is, locally on the source, a composition of a morphism that can be obtained as a base change of $\mathrm{pt} \rightarrow \mathbb{A}^n$, followed by a smooth morphism.

2.1.1. Recall the notion of smoothness for a map between DG schemes:

A map $Z_1 \rightarrow Z_2$ is called smooth if the DG scheme ${}^{cl}Z_2 \times_{Z_2} Z_1$ is classical, and the resulting map

$${}^{cl}Z_2 \times_{Z_2} Z_1 \rightarrow {}^{cl}Z_2$$

is a smooth map of classical schemes.

In particular, a DG scheme is called smooth if its map to $\mathrm{pt} := \mathrm{Spec}(k)$ is smooth. We summarize the properties of smooth maps below.

- Lemma 2.1.2** (a) A DG scheme Z is smooth if and only if its cotangent complex $T^*(Z)$ is a vector bundle, that is, $T^*(Z)$ is Zariski-locally isomorphic to \mathcal{O}_Z^n .
- (b) A smooth DG scheme is classical, and a smooth classical scheme is smooth as a DG scheme (see, e.g., [11, Sect. 8.4.2 and Proposition 9.1.4] for the proof).
- (c) A map $f : Z_1 \rightarrow Z_2$ between DG schemes is smooth if and only if its relative cotangent complex $T^*(Z_1/Z_2)$ is a vector bundle.

2.1.3 The definition of quasi-smoothness A DG scheme Z is called quasi-smooth² if its cotangent complex $T^*(Z)$ is perfect of Tor-amplitude $[-1, 0]$.

Equivalently, we require that, Zariski-locally on Z , the object $T^*(Z) \in \text{QCoh}(Z)$ could be presented by a complex

$$\mathcal{O}_Z^{\oplus m} \rightarrow \mathcal{O}_Z^{\oplus n}.$$

This is equivalent to the condition that all geometric fibers of $T^*(Z)$ are acyclic in degrees below -1 .

Remark 2.1.4 We should emphasize that if Z is a quasi-smooth DG scheme, the underlying classical scheme ${}^cl Z$ need *not* be a locally complete intersection. In fact, *any* classical affine scheme can be realized in this way for tautological reasons.

2.1.5. The following is a particular case of Corollary 2.1.11:

Corollary 2.1.6 A DG scheme Z is quasi-smooth if and only if it can be Zariski-locally presented as a fiber product (in the category of DG schemes)

$$\begin{array}{ccc} Z & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ \text{pt} & \xrightarrow{\{0\}} & \mathbb{A}^m. \end{array}$$

For future reference, we record:

Corollary 2.1.7 Let $f : Z_1 \rightarrow Z_2$ be a smooth morphism between quasi-smooth DG schemes. Then, Zariski-locally on Z_1 , there exists a Cartesian diagram

$$\begin{array}{ccc} Z_1 & \longrightarrow & \mathcal{U}_1 \\ f \downarrow & & \downarrow f_{\mathcal{U}} \\ Z_2 & \longrightarrow & \mathcal{U}_2 \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathcal{V}, \end{array}$$

where the DG schemes $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}$ are smooth, and the map $f_{\mathcal{U}}$ is smooth as well.

² A.k.a. ‘‘l.c.i.’’ = locally complete intersection.

Proof This easily follows from the fact that, given a smooth morphism $f : Z_1 \rightarrow Z_2$ between DG schemes, and a closed embedding $Z_2 \hookrightarrow \mathcal{U}_2$ with \mathcal{U}_2 smooth, we can, Zariski-locally on Z_1 , complete it to a Cartesian square

$$\begin{array}{ccc} Z_1 & \longrightarrow & \mathcal{U}_1 \\ f \downarrow & & \downarrow f_{\mathcal{U}} \\ Z_2 & \longrightarrow & \mathcal{U}_2 \end{array}$$

with $f_{\mathcal{U}}$ smooth. □

2.1.8 Quasi-smooth maps We say that a morphism of DG schemes $f : Z_1 \rightarrow Z_2$ is quasi-smooth if the relative cotangent complex $T^*(Z_1/Z_2)$ is perfect of Tor-amplitude $[-1, 0]$.

We note:

Lemma 2.1.9 *A quasi-smooth morphism can be, Zariski-locally on the source, factored as a composition of a quasi-smooth closed embedding, followed by a smooth morphism.*

Proof With no restriction of generality, we can assume that both Z_1 and Z_2 are affine. Decompose $f : Z_1 \rightarrow Z_2$ as

$$Z_1 \xrightarrow{f'} Z_2 \times \mathbb{A}^n \rightarrow Z_2,$$

where the first arrow is a closed embedding.

It follows tautologically that the fact that f is quasi-smooth implies that f' is quasi-smooth. □

The following gives an explicit description of quasi-smooth closed embeddings:

Proposition 2.1.10 *Let $f : Z_1 \rightarrow Z_2$ be a quasi-smooth closed embedding. Then, Zariski-locally on Z_2 , there exists a Cartesian diagram (in the category of DG schemes)*

$$\begin{array}{ccc} Z_1 & \xrightarrow{f} & Z_2 \\ \downarrow & & \downarrow \\ \text{pt} & \xrightarrow{\{0\}} & \mathbb{A}^m. \end{array}$$

Proof Note that $T^*(Z_1/Z_2)[-1]$ is the derived conormal sheaf $\mathcal{N}^*(Z_1/Z_2)$ to Z_1 inside Z_2 . The conditions imply that $\mathcal{N}^*(Z_1/Z_2)$ is a vector bundle.

Consider the restriction

$${}^{cl}\mathcal{N}^*(Z_1/Z_2) := \mathcal{N}^*(Z_1/Z_2)|_{{}^{cl}Z_1}.$$

This is the classical conormal sheaf to ${}^{cl}Z_1$ in ${}^{cl}Z_2$. Locally on Z_2 , we can choose sections

$$\{{}^{cl}f_1, \dots, {}^{cl}f_m\} \in \ker(\mathcal{O}_{{}^{cl}Z_2} \rightarrow f_*(\mathcal{O}_{{}^{cl}Z_1})),$$

whose differentials generate ${}^{cl}\mathcal{N}^*(Z_1/Z_2)$. Lifting the above sections to sections f_1, \dots, f_m of \mathcal{O}_{Z_2} , we obtain a map

$$Z_1 \rightarrow \mathrm{pt} \times_{\mathbb{A}^m} Z_2.$$

We claim that the latter map is an isomorphism. Indeed, it is a closed embedding and induces an isomorphism at the level of derived cotangent spaces. \square

Combining the statements of Lemma 2.1.9 and Proposition 2.1.10, we obtain:

Corollary 2.1.11 *A morphism $f : Z_1 \rightarrow Z_2$ is quasi-smooth if and only if, Zariski-locally on the source, it can be included in a diagram*

$$\begin{array}{ccc} Z_1 & \xrightarrow{f'} & Z_2 \times \mathbb{A}^n & \xrightarrow{\mathrm{pr}} & Z_2 \\ \downarrow & & \downarrow & & \\ \mathrm{pt} & \xrightarrow{\{0\}} & \mathbb{A}^m, & & \end{array}$$

in which the square is Cartesian (in the category of DG schemes), and f' is a closed embedding.

2.2 Cohomological properties of quasi-smooth maps

2.2.1. First, we note:

Corollary 2.2.2 *Let $f : Z_1 \rightarrow Z_2$ be quasi-smooth. Then, it is of bounded Tor dimension, locally in the Zariski topology on Z_1 .*

Proof Follows from Corollary 2.1.11 by base change. \square

2.2.3. Recall now the notion of *eventually coconnective* morphism, see [10, Definition 3.5.2]. Namely, a morphism $f : Z_1 \rightarrow Z_2$ between quasi-compact DG schemes is eventually coconnective if f^* sends $\mathrm{Coh}(Z_2)$ to $\mathrm{QCoh}(Z_1)^+$ (equivalently, to $\mathrm{Coh}(Z_1)$).

We will say a morphism f is *locally eventually coconnective* if it is eventually coconnective locally in the Zariski topology on the source.

Evidently, a morphism of bounded Tor-dimension is eventually coconnective.³ Hence, from Corollary 2.2.2, we obtain:

³ In fact, for maps between quasi-compact eventually coconnective schemes, the converse is also true, see [10, Lemma 3.6.3].

Corollary 2.2.4 *A quasi-smooth morphism is locally eventually coconnective.*

Corollary 2.2.5 *Suppose $f : Z_1 \rightarrow Z_2$ is a quasi-smooth map between DG schemes. Then, the functor*

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(Z_2) \rightarrow \mathrm{IndCoh}(Z_1),$$

left adjoint to $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z_1) \rightarrow \mathrm{IndCoh}(Z_2)$ is well defined.

Proof Follows from Corollary 2.2.4 by [10, Proposition 3.5.4 and Lemma 3.5.8].⁴ \square

2.2.6. Finally, let us recall the notion of Gorenstein morphism between DG schemes, see [10, Definition 7.3.2]. We have:

Corollary 2.2.7 *A quasi-smooth morphism $f : Z_1 \rightarrow Z_2$ between DG schemes is Gorenstein.*

Proof The claim is local in the Zariski topology on Z_1 . By Corollary 2.1.11, locally we can write f as a composition of a smooth morphism and a morphism which is obtained by base change from the embedding $\mathrm{pt} \rightarrow \mathbb{A}^n$. Now the statement follows from the combination of the following facts: (i) the property of being Gorenstein survives compositions (obvious from the definition), (ii) smooth morphisms are Gorenstein (see [10, Corollary 7.5.2]); (iii) the map $\mathrm{pt} \rightarrow \mathbb{A}^n$ is Gorenstein (direct calculation); and (iv) the base change of a Gorenstein morphism is Gorenstein (see [10, Sect. 7.5.4]). \square

As a particular case, we have:

Corollary 2.2.8 *A quasi-smooth DG scheme is Gorenstein (that is, its dualizing complex $\omega_Z \in \mathrm{IndCoh}_Z$ is a cohomologically shifted line bundle).*

Remark 2.2.9 If Z is a DG scheme and $n \in \mathbb{Z}$ is such that $\omega_Z[-n]$ is a line bundle, one can call n the “virtual dimension of Z .”

2.3 The scheme of singularities

In this subsection, we will introduce one of the main characters of this paper. Namely, if Z is a quasi-smooth DG scheme, we will attach to it a classical scheme $\mathrm{Sing}(Z)$ that measures how far Z is from being smooth.

2.3.1. Let Z be a DG scheme such that $T^*(Z) \in \mathrm{QCoh}(Z)$ is perfect (as is the case for quasi-smooth DG schemes). We define the tangent complex $T(Z) \in \mathrm{QCoh}(Z)$ to be the dual of $T^*(Z)$. Since $T^*(Z)$ is perfect, so is $T(Z)$, and the dual of $T(Z)$ identifies with $T^*(Z)$.

⁴ Technically, the corresponding assertions in [10] are made under the assumption that Z_1 and Z_2 be quasi-compact. However, this restriction is immaterial because the question of existence of $f^{\mathrm{IndCoh},*}$ is Zariski-local on Z_1 .

Remark 2.3.2 Of course, the tangent complex can be defined for a general DG scheme Z locally almost of finite type. However, to avoid losing information, one has to use Serre’s duality instead of the “naive” dual. Thus, we obtain a variant of the tangent complex that belongs to the category $\text{IndCoh}(Z)$. This will be addressed in more detail in [13].

2.3.3. Let Z be quasi-smooth. Note that in this case, $T(Z)$ is perfect of Tor-amplitude $[0, 1]$. In particular, it has cohomologies only in degrees 0 and 1; moreover, $H^1(T(Z))$ measures the degree to which Z is non-smooth.

For a quasi-smooth DG scheme Z , we define the classical scheme $\text{Sing}(Z)$ which we will refer to as “the scheme of singularities of Z ” as

$${}^{cl}(\text{Spec}_Z(\text{Sym}_{\mathcal{O}_Z}(T(Z)[1]))).$$

Note that since we are passing to the underlying classical scheme, the above is the same as

$$\text{Spec}_{{}^{cl}Z}(\text{Sym}_{\mathcal{O}_{{}^{cl}Z}}(H^1(T(Z)))) ,$$

where $H^1(T(Z))$ is considered as a coherent sheaf on ${}^{cl}Z$.

The scheme $\text{Sing}(Z)$ carries a canonical \mathbb{G}_m -action along the fibers of the projection $\text{Sing}(Z) \rightarrow {}^{cl}Z$; the action corresponds to the natural grading on the symmetric algebra $\text{Sym}_{\mathcal{O}_Z}(T(Z)[1])$.

2.3.4. By definition, k -points of $\text{Sing}(Z)$ are pairs (z, ξ) , where $z \in Z(k)$ and $\xi \in H^{-1}(T_z^*(Z))$.

2.3.5. Suppose that Z is presented as a fiber product

$$\begin{array}{ccc} Z & \xrightarrow{\iota} & \mathcal{U} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathcal{V}, \end{array}$$

where \mathcal{U} and \mathcal{V} are smooth. In this situation, we say that Z is given as a global complete intersection.

Let V denote the tangent space to \mathcal{V} at $\text{pt} \in \mathcal{V}$. We have

$$T(Z) \simeq \text{Cone}(\iota^*(T(\mathcal{U})) \rightarrow V \otimes \mathcal{O}_Z)[-1].$$

Hence, we obtain a canonical map

$$V \otimes \mathcal{O}_Z \rightarrow T(Z)[1],$$

which gives rise to a surjection of coherent sheaves

$$V \otimes \mathcal{O}_{{}^{cl}Z} \rightarrow H^1(T(Z)),$$

and, hence, we obtain a \mathbb{G}_m -equivariant closed embedding

$$\mathrm{Sing}(Z) \hookrightarrow {}^{cl}(V^* \times Z) \hookrightarrow V^* \times Z, \quad (2.1)$$

where V^* is the scheme $\mathrm{Spec}(\mathrm{Sym}(V))$.

2.4 The singular codifferential

In this subsection, to a map between quasi-smooth DG schemes we will attach its singular codifferential, which can be thought as the “ H^{-1} -version” of the usual codifferential, the latter being the map between classical cotangent spaces.

2.4.1. Let $f : Z_1 \rightarrow Z_2$ be a map between quasi-smooth DG schemes. Define

$$\mathrm{Sing}(Z_2)_{Z_1} := {}^{cl}\left(\mathrm{Sing}(Z_2) \times_{Z_2} Z_1\right) \simeq {}^{cl}\left(\mathrm{Spec}_{Z_1}\left(\mathrm{Sym}_{\mathcal{O}_{Z_1}}\left(f^*(T(Z_2)[1])\right)\right)\right).$$

Note that f induces a morphism $T(Z_1) \rightarrow f^*(T(Z_2))$. Taking the Zariski spectra of the corresponding symmetric algebras, we obtain a map

$$\mathrm{Sing}(f) : \mathrm{Sing}(Z_2)_{Z_1} \rightarrow \mathrm{Sing}(Z_1). \quad (2.2)$$

We will refer to this map as the “singular codifferential.”

2.4.2. We have the following characterization of quasi-smooth maps between quasi-smooth DG schemes.

Lemma 2.4.3 *Let $f : Z_1 \rightarrow Z_2$ be a morphism between quasi-smooth DG schemes. Then, f is quasi-smooth if and only if the singular codifferential $\mathrm{Sing}(f)$ is a closed embedding.*

Proof The relative cotangent complex $T^*(Z_1/Z_2)$ is the cone of the codifferential

$$f^*(T^*(Z_2)) \rightarrow T^*(Z_1).$$

Thus, f is quasi-smooth if and only if the induced map

$$(df_x)^* : H^{-1}(T^*(Z_2)_{f(x)}) \rightarrow H^{-1}(T^*(Z_1)_x)$$

is injective for every $x \in Z_1$. The latter condition is equivalent to surjectivity of the morphism $H^1(T(Z_1)) \rightarrow H^1(f^*(T(Z_2)))$, which is equivalent to $\mathrm{Sing}(f)$ being a closed embedding. \square

Lemma 2.4.4 *Let $f : Z_1 \rightarrow Z_2$ be a morphism between quasi-smooth DG schemes. Then, f is smooth if and only if the following two conditions are satisfied:*

- The (classical) differential $df_x : H^0(T(Z_1)_x) \rightarrow H^0(T(Z_2)_{f(x)})$ is surjective for all k -points $x \in X$;
- The singular codifferential $\text{Sing}(f)$ is an isomorphism.

Proof The argument is similar to the proof of Lemma 2.4.3; we leave it to the reader. \square

3 Support in triangulated and DG categories

In this section, we will review the following construction. Given a triangulated category \mathbf{T} and a commutative graded algebra A mapping to its center, we will define full subcategories in \mathbf{T} corresponding to closed (resp. open) subsets of $\text{Spec}(A)$ in the Zariski topology.

This construction is a variant of that from [4]. Unlike [4], we do not assume that the categories are compactly generated. Also, we use a coarser notion of support; see Remark 3.3.5.

3.1 Localization with respect to homogeneous elements

3.1.1. Let \mathbf{T} be a cocomplete triangulated category. Let A be a commutative algebra, graded by even integers, and equipped with a homomorphism to the graded center of \mathbf{T} . That is, for every $\mathbf{t} \in \mathbf{T}$, we have a homomorphism of graded algebras

$$A \rightarrow \bigoplus_n \text{Hom}_{\mathbf{T}}(\mathbf{t}, \mathbf{t}[2n]), \quad (3.1)$$

and for every $\phi : \mathbf{t}' \rightarrow \mathbf{t}''[m]$, the diagram

$$\begin{array}{ccc} A & \longrightarrow & \bigoplus_{n \geq 0} \text{Hom}_{\mathbf{T}}(\mathbf{t}', \mathbf{t}'[2n]) \\ \downarrow & & \downarrow \phi \circ - \\ \bigoplus_{n \geq 0} \text{Hom}_{\mathbf{T}}(\mathbf{t}'', \mathbf{t}''[2n]) & \xrightarrow{- \circ \phi} & \bigoplus_{n \geq 0} \text{Hom}_{\mathbf{T}}(\mathbf{t}', \mathbf{t}''[2n + m]) \end{array}$$

commutes. In this situation, we say that A acts on the triangulated category \mathbf{T} .

3.1.2. Let $a \in A$ be a homogeneous element. We let $Y_a \subset \text{Spec}(A)$ be the conical (i.e., \mathbb{G}_m -invariant) Zariski-closed subset of $\text{Spec}(A)$ cut out by a . Here, $\text{Spec}(A)$ is the Zariski spectrum of A (where A is viewed as a plain commutative algebra).

We define the full subcategory $\mathbf{T}_{\text{Spec}(A) - Y_a} \subset \mathbf{T}$ to consist of those objects $\mathbf{t} \in \mathbf{T}$ for which the map

$$a : \mathbf{t} \rightarrow \mathbf{t}[2k]$$

is an isomorphism, where $2k = \deg(a)$.

Clearly, the subcategory $\mathbf{T}_{\mathrm{Spec}(A)-Y_a}$ is thick (i.e., triangulated and closed under direct summands) and closed under taking arbitrary direct sums.

The inclusion $\mathbf{T} \hookrightarrow \mathbf{T}_{\mathrm{Spec}(A)-Y_a}$ admits a left adjoint, explicitly given by

$$\mathbf{t} \mapsto \mathrm{hocolim} \left(\mathbf{t} \xrightarrow{a} \mathbf{t}[2k] \xrightarrow{a} \dots \right). \quad (3.2)$$

We denote the resulting endofunctor

$$\mathbf{T} \rightarrow \mathbf{T}_{\mathrm{Spec}(A)-Y_a} \rightarrow \mathbf{T}$$

by Loc_a .

Remark 3.1.3 Although taking a homotopy colimit in a triangulated category is an operation that is defined only up to a non-canonical isomorphism, the expression in (3.2) is canonical by virtue of being a left adjoint.

3.1.4. Recall that a full thick subcategory $\mathbf{T}' \subset \mathbf{T}$ is said to be *left admissible* if the inclusion $\mathbf{T}' \hookrightarrow \mathbf{T}$ admits a left adjoint. If this is the case, we let $\mathbf{T}'' := {}^{\perp}(\mathbf{T}')$ be its left orthogonal; the inclusion $\mathbf{T}'' \hookrightarrow \mathbf{T}$ admits a right adjoint. We say that the resulting diagram

$$\mathbf{T}'' \rightleftarrows \mathbf{T} \rightleftarrows \mathbf{T}'$$

is a *short exact sequence of triangulated categories* if \mathbf{T}' is closed under direct sums. Note that \mathbf{T}'' , being a left orthogonal, is automatically closed under direct sums.

Lemma 3.1.5 *Let*

$$\mathbf{T}'' \begin{array}{c} \xrightarrow{F''} \\ \rightleftarrows \mathbf{T} \rightleftarrows \\ \xrightarrow{G''} \end{array} \begin{array}{c} \xrightarrow{F'} \\ \rightleftarrows \mathbf{T}' \rightleftarrows \\ \xrightarrow{G'} \end{array}$$

be a short exact sequence of categories. Then, all four functors F' , F'' , G' , and G'' are triangulated (preserve exact triangles and shifts) and continuous (preserve arbitrary direct sums).

Proof The inclusion functors F'' , G' are triangulated and continuous for tautological reasons. The functor F' is continuous because it is a left adjoint; it is triangulated because its right adjoint G' is triangulated.

We now see that the composition $G' \circ F'$ is continuous. Therefore, the composition $F'' \circ G''$ is continuous as well, being the cone of the adjunction map between the identity functor and $G' \circ F'$. This implies that G'' is continuous. Finally, G'' is triangulated because it is the right adjoint of a triangulated functor. \square

3.1.6. Let

$$\mathbf{T}_{Y_a} := {}^{\perp}(\mathbf{T}_{\mathrm{Spec}(A)-Y_a}) \subset \mathbf{T}$$

be the left orthogonal of $\mathbf{T}_{\mathrm{Spec}(A)-Y_a}$. We obtain an exact sequence of triangulated categories

$$\mathbf{T}_{Y_a} \rightleftarrows \mathbf{T} \rightleftarrows \mathbf{T}_{\mathrm{Spec}(A)-Y_a}.$$

Denote the composition

$$\mathbf{T} \rightarrow \mathbf{T}_{Y_a} \rightarrow \mathbf{T}$$

by co-Loc_a .

3.1.7. Suppose \mathbf{T}_1 and \mathbf{T}_2 are two cocomplete triangulated categories equipped with actions of A . Let $F : \mathbf{T}_1 \rightarrow \mathbf{T}_2$ be a continuous triangulated functor between triangulated categories, compatible with the A -actions (in the obvious sense). It is clear that it sends $(\mathbf{T}_1)_{\text{Spec}(A)-Y_a}$ to $(\mathbf{T}_2)_{\text{Spec}(A)-Y_a}$.

In addition, since F is continuous and triangulated, it preserves homotopy colimits. Thus, if $\mathbf{t}_1 \in \mathbf{T}_1$ satisfies $\text{Loc}_a(\mathbf{t}_1) = 0$, then $\text{Loc}_a(F(\mathbf{t}_1)) = 0$. Hence, the functor F sends $(\mathbf{T}_1)_{Y_a}$ to $(\mathbf{T}_2)_{Y_a}$.

This formally implies that the diagram

$$\begin{array}{ccccc} (\mathbf{T}_1)_{Y_a} & \rightleftarrows & \mathbf{T}_1 & \rightleftarrows & (\mathbf{T}_1)_{\text{Spec}(A)-Y_a} \\ F \downarrow & & \downarrow F & & \downarrow F \\ (\mathbf{T}_2)_{Y_a} & \rightleftarrows & \mathbf{T}_2 & \rightleftarrows & (\mathbf{T}_2)_{\text{Spec}(A)-Y_a} \end{array}$$

is commutative.

3.2 Zariski localization

In this subsection, we will show how localization with respect to individual homogeneous elements can be organized into localization with respect to the Zariski topology on $\text{Spec}(A)$. This will not involve much beyond the usual constructions in commutative algebra.

3.2.1. Let $a_1, a_2 \in A$ be two homogeneous elements.

Lemma 3.2.2 *The functor Loc_{a_2} preserves both $\mathbf{T}_{Y_{a_1}}$ and $\mathbf{T}_{\text{Spec}(A)-Y_{a_1}}$*

Proof Follows from Sect. 3.1.7 applied to the tautological embeddings $\mathbf{T}_{Y_{a_1}} \hookrightarrow \mathbf{T}$ and $\mathbf{T}_{\text{Spec}(A)-Y_{a_1}} \hookrightarrow \mathbf{T}$ for the action of a_2 . \square

Remark 3.2.3 Note that Lemma 3.2.2 did not use the fact that the actions of a_1 and a_2 commute.

3.2.4. From Lemma 3.2.2, we obtain that the short exact sequences

$$\mathbf{T}_{Y_{a_1}} \rightleftarrows \mathbf{T} \rightleftarrows \mathbf{T}_{\text{Spec}(A)-Y_{a_1}} \quad \text{and} \quad \mathbf{T}_{Y_{a_2}} \rightleftarrows \mathbf{T} \rightleftarrows \mathbf{T}_{\text{Spec}(A)-Y_{a_2}}$$

are compatible in the sense of [5], Sect. 1.3. Thus, we obtain a commutative diagram in which every row and every column is a short exact sequence:

$$\begin{array}{ccccc}
 \mathbf{T}_{Y_{a_1}} \cap \mathbf{T}_{Y_{a_2}} & \rightleftarrows & \mathbf{T}_{Y_{a_1}} & \rightleftarrows & \mathbf{T}_{Y_{a_1}} \cap \mathbf{T}_{\mathrm{Spec}(A)-Y_{a_2}} \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 \mathbf{T}_{Y_{a_2}} & \rightleftarrows & \mathbf{T} & \rightleftarrows & \mathbf{T}_{\mathrm{Spec}(A)-Y_{a_2}} \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 \mathbf{T}_{\mathrm{Spec}(A)-Y_{a_1}} \cap \mathbf{T}_{Y_{a_2}} & \rightleftarrows & \mathbf{T}_{\mathrm{Spec}(A)-Y_{a_1}} & \rightleftarrows & \mathbf{T}_{\mathrm{Spec}(A)-Y_{a_1}} \cap \mathbf{T}_{\mathrm{Spec}(A)-Y_{a_2}}.
 \end{array} \tag{3.3}$$

In particular, the functors Loc_{a_1} , Loc_{a_2} , $\mathrm{co}\text{-}\mathrm{Loc}_{a_1}$ and $\mathrm{co}\text{-}\mathrm{Loc}_{a_2}$ pairwise commute.

3.2.5. Note that the fact that a_1 and a_2 commute implies that

$$\mathbf{T}_{\mathrm{Spec}(A)-Y_{a_1}} \cap \mathbf{T}_{\mathrm{Spec}(A)-Y_{a_2}} = \mathbf{T}_{\mathrm{Spec}(A)-Y_{a_1 a_2}}.$$

Our next goal is to prove the following:

Proposition 3.2.6 *If $a \in A$ is a homogeneous element contained in the radical of the ideal generated by a_1, \dots, a_n , then*

$$\mathbf{T}_{Y_{a_1}} \cap \dots \cap \mathbf{T}_{Y_{a_n}} \subset \mathbf{T}_{Y_a}.$$

Prior to giving the proof, we will need the following more explicit description of the category \mathbf{T}_{Y_a} .

3.2.7. We start with a remark about A -modules, valid for an arbitrary commutative graded ring A .

Consider the (DG) category of graded A -modules, i.e., $(A\text{-mod})^{\mathbb{G}_m}$. Fix a homogeneous element $a \in A$. We identify the DG category of graded modules over the localization, i.e., $(A_a\text{-mod})^{\mathbb{G}_m}$, with a full subcategory of $(A\text{-mod})^{\mathbb{G}_m}$.

Lemma 3.2.8 *For $M \in (A\text{-mod})^{\mathbb{G}_m}$, the following conditions are equivalent:*

- (a) $\mathrm{Hom}_A(A_a(j), M[i]) = 0, \forall i, j \in \mathbb{Z}$. Here, $A_a(j)$ refers to A_a with grading shifted by j .
- (b) For any $N \in (A_a\text{-mod})^{\mathbb{G}_m}$, $\mathrm{Hom}_A(N, M) = 0$.
- (c) The map

$$\prod_{i=0}^{\infty} M \rightarrow \prod_{i=0}^{\infty} M : (m_0, m_1, \dots) \mapsto (m_0 - a(m_1), m_1 - a(m_2), \dots)$$

is an isomorphism. Here, \prod stands for the product in the category of graded A -modules.

(d) *The homotopy limit*

$$\mathit{holim} \left(M \xleftarrow{a} M \xleftarrow{a} \dots \right)$$

vanishes.

Proof Since the objects $A_a(j)$ generate $(A_a\text{-mod})^{\mathbb{G}_m}$, (a) and (b) are equivalent. Moreover, the space $\mathrm{Hom}_A(A_a(j), M[i])$ identifies with the j -th graded component of the i -th cohomology of the cone of the map from (c); therefore, (a) and (c) are equivalent. Finally, (c) and (d) are equivalent by definition. \square

3.2.9. Let us denote the full subcategory of $(A\text{-mod})^{\mathbb{G}_m}$, satisfying the equivalent conditions of Lemma 3.2.8 by

$$(A\text{-mod})_{\langle a \rangle}^{\mathbb{G}_m} \subset (A\text{-mod})^{\mathbb{G}_m}.$$

By condition (b), this subcategory is the right orthogonal of $(A_a\text{-mod})^{\mathbb{G}_m}$. Note that $(A\text{-mod})_{\langle a \rangle}^{\mathbb{G}_m}$ is a thick subcategory that is closed under products, but not coproducts.

Remark 3.2.10 One should think of $(A\text{-mod})_{\langle a \rangle}^{\mathbb{G}_m}$ as the category of a -adically complete A -modules.

We now return to the setting of Sect. 3.1.1.

Lemma 3.2.11 *For an object $\mathfrak{t} \in \mathbf{T}$, the following conditions are equivalent:*

- (a) $\mathfrak{t} \in \mathbf{T}_{Y_a}$.
- (b) For any $\mathfrak{t}' \in \mathbf{T}$, the graded A -module $\mathrm{Hom}_{\mathbf{T}}^{\bullet}(\mathfrak{t}, \mathfrak{t}')$ belongs to $(A\text{-mod})_{\langle a \rangle}^{\mathbb{G}_m}$.

Proof Indeed, using Lemma 3.2.8(c), we see that (b) is equivalent to

$$\mathrm{Hom}_{\mathbf{T}}^{\bullet}(\mathrm{Loc}_a(\mathfrak{t}), \mathfrak{t}') = 0.$$

\square

3.2.12. We are now ready to prove Proposition 3.2.6:

Proof By Lemma 3.2.11, it suffices to show that

$$(A\text{-mod})_{\langle a_1 \rangle}^{\mathbb{G}_m} \cap \dots \cap (A\text{-mod})_{\langle a_n \rangle}^{\mathbb{G}_m} \subset (A\text{-mod})_{\langle a \rangle}^{\mathbb{G}_m}.$$

Using Lemma 3.2.8(b), we see that it is enough to prove that $(A_a\text{-mod})^{\mathbb{G}_m}$ is contained in the full subcategory of $(A\text{-mod})^{\mathbb{G}_m}$ generated by the subcategories $(A_{a_i}\text{-mod})^{\mathbb{G}_m}$ for $i = 1, \dots, n$. But this is obvious from the Čech resolution (In fact, the latter subcategory identifies with the category of \mathbb{G}_m -equivariant modules on the scheme $\mathrm{Spec}(A) - \bigcap_i Y_{a_i}$.) \square

3.3 The definition of support

In this subsection, we will finally define the support of an object, and study how this notion behaves under functors between triangulated categories and morphisms between algebras.

3.3.1. Let Y be a conical (i.e., \mathbb{G}_m -invariant) Zariski-closed subset of $\text{Spec}(A)$.

We define the full subcategory

$$\mathbf{T}_Y := \bigcap_a \mathbf{T}_{Y_a},$$

where the intersection is taken over the set of homogeneous elements $a \in A$ that vanish on Y .

Suppose $Y_1, Y_2 \subset \text{Spec}(A)$ are two closed conical subsets. By Proposition 3.2.6,

$$\mathbf{T}_{Y_1 \cap Y_2} = \mathbf{T}_{Y_1} \cap \mathbf{T}_{Y_2}. \quad (3.4)$$

3.3.2. We give the following definitions:

Definition 3.3.3 Given a conical Zariski-closed subset $Y \subset \text{Spec}(A)$ and $\mathbf{t} \in \mathbf{T}$, we say that

$$\text{supp}_A(\mathbf{t}) \subset Y$$

if $\mathbf{t} \in \mathbf{T}_Y$.

Definition 3.3.4 Given $\mathbf{t} \in \mathbf{T}$, we define $\text{supp}_A(\mathbf{t})$ to be the minimal conical Zariski-closed subset $Y \subset \text{Spec}(A)$ such that $\mathbf{t} \in \mathbf{T}_Y$.

Remark 3.3.5 The definition of support given in Definition 3.3.3 differs from the one in [4]. When \mathbf{T} is compactly generated, so that the definition of [4] applies, what we call “support” is the Zariski closure of the support from [4].

3.3.6. It is clear that

$$\text{supp}_A(\mathbf{t}) = \bigcap_a Y_a,$$

where the intersection is taken over the set of homogeneous elements a such that $\mathbf{t} \in \mathbf{T}_{Y_a}$.

Lemma 3.3.7 *Let $Y \subset \text{Spec}(A)$ be a conical Zariski-closed subset whose complement $\text{Spec}(A) - Y$ is quasi-compact (If A is Noetherian, this condition is automatic.). Then, the embedding $\mathbf{T}_Y \hookrightarrow \mathbf{T}$ admits a continuous right adjoint.*

Proof By the assumption, there exists a finite collection of homogeneous elements $a_1, \dots, a_n \in A$ such that

$$Y = Y_{a_1} \cap \dots \cap Y_{a_n}. \quad (3.5)$$

By Proposition 3.2.6, we then have:

$$\mathbf{T}_Y = \mathbf{T}_{Y_{a_1}} \cap \cdots \cap \mathbf{T}_{Y_{a_n}}.$$

Iterating the diagram (3.3), we see that the embedding

$$\mathbf{T}_{Y_{a_1}} \cap \cdots \cap \mathbf{T}_{Y_{a_n}} \hookrightarrow \mathbf{T}$$

admits a continuous right adjoint such that the composed functor

$$\mathbf{T} \rightarrow \mathbf{T}_{Y_{a_1}} \cap \cdots \cap \mathbf{T}_{Y_{a_n}} \hookrightarrow \mathbf{T}$$

is isomorphic to the composition

$$\text{co-Loc}_{a_1} \circ \cdots \circ \text{co-Loc}_{a_n}.$$

□

3.3.8. Let $Y \subset \text{Spec}(A)$ be a conical Zariski-closed subset whose complement is quasi-compact. From Lemma 3.3.7, we obtain a short exact sequence of categories

$$\mathbf{T}_Y \rightleftarrows \mathbf{T} \rightleftarrows \mathbf{T}_{\text{Spec}(A)-Y},$$

where $\mathbf{T}_{\text{Spec}(A)-Y}$ is the right orthogonal to \mathbf{T}_Y . We also see that $\mathbf{T}_{\text{Spec}(A)-Y}$ is generated by categories $\mathbf{T}_{\text{Spec}(A)-Y_a}$, where $a \in A$ runs over homogeneous elements such that $Y_a \supset Y$ (In fact, it suffices to consider $a = a_i$ for a finite collection of homogeneous elements $a_1, \dots, a_n \in A$ satisfying (3.5).).

Corollary 3.3.9 *Suppose $Y_1, Y_2 \subset \text{Spec}(A)$ are conical Zariski-closed subsets whose complements are quasi-compact. Then, the category $\mathbf{T}_{Y_1 \cup Y_2}$ is generated by \mathbf{T}_{Y_1} and \mathbf{T}_{Y_2} .*

Proof Similar to (3.3), we have a diagram

$$\begin{array}{ccccc}
 \mathbf{T}_{Y_1} \cap \mathbf{T}_{Y_2} & \rightleftarrows & \mathbf{T}_{Y_1} & \rightleftarrows & \mathbf{T}_{Y_1} \cap \mathbf{T}_{\text{Spec}(A)-Y_2} \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 \mathbf{T}_{Y_2} & \rightleftarrows & \mathbf{T} & \rightleftarrows & \mathbf{T}_{\text{Spec}(A)-Y_2} \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 \mathbf{T}_{\text{Spec}(A)-Y_1} \cap \mathbf{T}_{Y_{a_2}} & \rightleftarrows & \mathbf{T}_{\text{Spec}(A)-Y_1} & \rightleftarrows & \mathbf{T}_{\text{Spec}(A)-Y_1} \cap \mathbf{T}_{\text{Spec}(A)-Y_2}
 \end{array} \tag{3.6}$$

with exact rows and columns. In order to prove the corollary, it suffices to check that

$$\mathbf{T}_{\text{Spec}(A)-Y_1} \cap \mathbf{T}_{\text{Spec}(A)-Y_2} = \mathbf{T}_{\text{Spec}(A)-Y_1 \cup Y_2}.$$

Clearly, the right-hand side is contained in the left-hand side. On the other hand,

$$\mathbf{T}_{\mathrm{Spec}(A)-Y_1} \cap \mathbf{T}_{\mathrm{Spec}(A)-Y_2} = (\mathbf{T}_{\mathrm{Spec}(A)-Y_1})_{\mathrm{Spec}(A)-Y_2}$$

is generated by the essential images

$$\mathrm{Loc}_{a_1, a_2}(\mathbf{T}) = \mathrm{Loc}_{a_2} \circ \mathrm{Loc}_{a_1}(\mathbf{T}),$$

where $a_1, a_2 \in A$ run over homogeneous elements such that $Y_1 \subset Y_{a_1}$ and $Y_2 \subset Y_{a_2}$. This proves the converse inclusion. \square

3.3.10. Let $F : \mathbf{T}_1 \rightarrow \mathbf{T}_2$ be a continuous triangulated functor compatible with the actions of A . Let $Y \subset \mathrm{Spec}(A)$ be a conical Zariski-closed subset whose complement is quasi-compact. It is clear from Sect. 3.1.7 that F induces a commutative diagram of functors:

$$\begin{array}{ccccc} (\mathbf{T}_1)_Y & \rightleftarrows & \mathbf{T}_1 & \rightleftarrows & (\mathbf{T}_1)_{\mathrm{Spec}(A)-Y} \\ F \downarrow & & \downarrow F & & \downarrow F \\ (\mathbf{T}_2)_Y & \rightleftarrows & \mathbf{T}_2 & \rightleftarrows & (\mathbf{T}_2)_{\mathrm{Spec}(A)-Y} \end{array}$$

Thus, for any $\mathbf{t} \in \mathbf{T}_1$, we have $\mathrm{supp}_A(\mathbf{t}) \supset \mathrm{supp}_A(F(\mathbf{t}))$. If we assume that F is conservative, then $\mathrm{supp}_A(\mathbf{t}) = \mathrm{supp}_A(F(\mathbf{t}))$.

In particular if $\mathbf{T}' \subset \mathbf{T}$ is a full triangulated subcategory closed under direct sums, then

$$\mathbf{T}'_Y = \mathbf{T}_Y \cap \mathbf{T}' \quad \text{and} \quad \mathbf{T}'_{\mathrm{Spec}(A)-Y} = \mathbf{T}_{\mathrm{Spec}(A)-Y} \cap \mathbf{T}'$$

as subcategories of \mathbf{T} .

3.3.11. The notion of support behaves functorially under homomorphisms of algebras. Namely, let $\phi : A' \rightarrow A$ be a homomorphism of evenly graded algebras. Let Φ denote the resulting map $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A')$. For \mathbf{T} as above, the algebra A' maps to the graded center of \mathbf{T} by composing with ϕ .

We have:

Lemma 3.3.12 For $\mathbf{t} \in \mathbf{T}$ and $Y' \subset \mathrm{Spec}(A')$ and $Y := \Phi^{-1}(Y')$,

$$\mathrm{supp}_{A'}(\mathbf{t}) \subset Y' \Leftrightarrow \mathrm{supp}_A(\mathbf{t}) \subset Y.$$

Equivalently for $\mathbf{t} \in \mathbf{T}$,

$$\mathrm{supp}_{A'}(\mathbf{t}) = \overline{\Phi(\mathrm{supp}_A(\mathbf{t}))}.$$

3.4 The compactly generated case

In this subsection, we will show that if \mathbf{T} is compactly generated, we can measure supports of objects more explicitly.

3.4.1. Assume that \mathbf{T} is compactly generated.

Lemma 3.4.2 *Let $Y \subset \text{Spec}(A)$ be a conical Zariski-closed subset whose complement $\text{Spec}(A) - Y$ is quasi-compact. Then, the category \mathbf{T}_Y is compactly generated.*

Proof By induction and (3.3), we can assume that Y is cut out by one homogeneous element a . It is easy to see that the objects

$$\text{Cone}(\mathbf{t} \xrightarrow{a} \mathbf{t}), \quad \mathbf{t} \in \mathbf{T}^c$$

generate \mathbf{T}_{Y_a} . Indeed, the right orthogonal to the class of these objects coincides with $\mathbf{T}_{\text{Spec}(A) - Y_a}$. \square

3.4.3. One can use compact objects to rewrite the definition of support:

Lemma 3.4.4 *Let Y be an arbitrary conical Zariski-closed subset of $\text{Spec}(A)$.*

(a) *For $\mathbf{t} \in \mathbf{T}$, its support is contained in Y if and only if for a set of compact generators $\mathbf{t}_\alpha \in \mathbf{T}$, the support of the A -module*

$$\text{Hom}_{\mathbf{T}}^\bullet(\mathbf{t}_\alpha, \mathbf{t})$$

is contained in Y for all α (cf. [4, Corollary 5.3].)

(b) *If \mathbf{t} is compact, its support is contained in Y if and only if the support of the A -module $\text{Hom}_{\mathbf{T}}^\bullet(\mathbf{t}, \mathbf{t})$ is contained in Y .*

(c) *If \mathbf{t} is compact, and $a \in A$ is a homogeneous element that vanishes on $\text{supp}_A(\mathbf{t})$, then there exists an integer i such that $\mathbf{t} \xrightarrow{a^i} \mathbf{t}[2k \cdot i]$ vanishes. Here, $2k = \deg(a)$.*

Proof Let a be a homogeneous element of A of degree $2k$. Suppose that a vanishes on Y . The fact that $\text{supp}(\mathbf{t}) \subset Y_a$ is equivalent to the colimit

$$\mathbf{t} \xrightarrow{a} \mathbf{t}[2k] \xrightarrow{a} \dots$$

being zero, which can be tested by mapping the generators $\mathbf{t}_\alpha[m]$, $m \in \mathbb{Z}$ into this colimit. Since the \mathbf{t}_α 's are compact, the above Hom is isomorphic to the colimit

$$\text{Hom}_{\mathbf{T}}(\mathbf{t}_\alpha, \mathbf{t}[-m]) \xrightarrow{a} \text{Hom}_{\mathbf{T}}(\mathbf{t}_\alpha, \mathbf{t}[2k - m]) \xrightarrow{a} \dots,$$

taken in the category Vect^\heartsuit . The vanishing of the latter is equivalent to

$$\text{Hom}_{\mathbf{T}}^\bullet(\mathbf{t}_\alpha, \mathbf{t})$$

being supported over Y as an A -module, which is the assertion of point (a) of the lemma.

For point (b), the “only if” direction follows from point (a). The “if” direction holds tautologically for any \mathfrak{t} (with no compactness hypothesis).

Point (c) follows from point (b): The unit element in $\mathrm{Hom}_{\mathbf{T}}(\mathfrak{t}, \mathfrak{t})$ is annihilated by some power of a . \square

3.5 Support in DG categories

From now on, we will assume that \mathbf{T} is the homotopy category of a DG category \mathbf{C} , equipped with an action of an \mathbb{E}_2 -algebra \mathcal{A} (see Sect. E.3.2, where the notion of action of an \mathbb{E}_2 -algebra on a DG category is discussed).

We will show that the notion of support in \mathbf{C} can be expressed in terms of the universal situation, namely for $\mathbf{C} = \mathcal{A}\text{-mod}$. In addition, we will study the behavior of support under tensor products of the \mathbf{C} 's.

3.5.1. Set

$$A := \bigoplus_n H^{2n}(\mathcal{A}).$$

Since \mathcal{A} has an \mathbb{E}_2 -structure, the algebra A is commutative. The action of \mathcal{A} on \mathbf{C} gives rise to a homomorphism from A to the graded center of \mathbf{T} .

3.5.2. For a conical Zariski-closed subset $Y \subset \mathrm{Spec}(A)$, we let

$$\mathbf{C}_Y \subset \mathbf{C}$$

be the full DG subcategory of \mathbf{C} defined as the preimage of $\mathbf{T}_Y \subset \mathbf{T}$.

3.5.3. In particular, we can consider $\mathbf{C} = \mathcal{A}\text{-mod}$. It is clear that the resulting subcategory

$$\mathcal{A}\text{-mod}_Y \subset \mathcal{A}\text{-mod}$$

is a (two-sided) monoidal ideal (In fact, any full cocomplete subcategory of $\mathcal{A}\text{-mod}$ is a monoidal ideal, since $\mathcal{A}\text{-mod}$ is generated by \mathcal{A} , which is the unit object.).

3.5.4. The following assertion will play a crucial role:

Proposition 3.5.5 *Let Y be such that its complement is quasi-compact. Then for any DG category \mathbf{C} equipped with an action of \mathcal{A} , we have*

$$\mathbf{C}_Y = \mathcal{A}\text{-mod}_Y \otimes_{\mathcal{A}\text{-mod}} \mathbf{C}$$

as full subcategories of

$$\mathbf{C} \simeq \mathcal{A}\text{-mod} \otimes_{\mathcal{A}\text{-mod}} \mathbf{C}.$$

Proof Note that if

$$\mathbf{C}_1 \rightleftarrows \mathbf{C}_2 \rightleftarrows \mathbf{C}_3$$

is a short exact sequence of right modules over a DG monoidal category \mathbf{O} , and \mathbf{C}' is a left module, then

$$\mathbf{C}_1 \otimes_{\mathbf{O}} \mathbf{C}' \rightleftarrows \mathbf{C}_2 \otimes_{\mathbf{O}} \mathbf{C}' \rightleftarrows \mathbf{C}_3 \otimes_{\mathbf{O}} \mathbf{C}'$$

is a short exact sequence of DG categories.

This observation together with (3.3) for \mathbf{C} and $\mathcal{A}\text{-mod}$ reduces the proposition to the case when $Y = Y_a$ for some homogeneous element $a \in A$. In this case, it is sufficient to show that

$$\mathbf{C}_{\text{Spec}(A)-Y_a} \quad \text{and} \quad \mathcal{A}\text{-mod}_{\text{Spec}(A)-Y_a} \otimes_{\mathcal{A}\text{-mod}} \mathbf{C}$$

coincide as subcategories of \mathbf{C} .

First, let us show the inclusion \supset , i.e., we have to show that the element a acts as an isomorphism on objects from $\mathcal{A}\text{-mod}_{\text{Spec}(A)-Y_a} \otimes_{\mathcal{A}\text{-mod}} \mathbf{C}$. This property is enough to establish on the generators, which we can take to be of the form $\mathcal{M} \otimes \mathbf{c}$, where $\mathbf{c} \in \mathbf{C}$ and $\mathcal{M} \in \mathcal{A}\text{-mod}_{\text{Spec}(A)-Y_a}$. The action of a on such an object equals

$$a_{\mathcal{M}} \otimes \text{id}_{\mathbf{c}},$$

and the assertion follows from the fact that a is an isomorphism on \mathcal{M} .

In particular, we obtain a natural transformation of endofunctors

$$\text{Loc}_{a, \mathbf{C}} \rightarrow \text{Loc}_{a, \mathcal{A}\text{-mod}} \otimes \text{Id}_{\mathbf{C}},$$

viewed as acting on

$$\mathbf{C} \simeq \mathcal{A}\text{-mod} \otimes_{\mathcal{A}\text{-mod}} \mathbf{C}.$$

It suffices to show that this natural transformation is an isomorphism. The latter follows immediately from (3.2). \square

3.5.6. Suppose now we have two \mathbb{E}_2 -algebras \mathcal{A}_i acting on DG categories \mathbf{C}_i , respectively ($i = 1, 2$). Let $Y_i \in \text{Spec}(\mathcal{A}_i)$ be conical Zariski-closed subsets whose complements are quasi-compact.

Set $\mathbf{C} := \mathbf{C}_1 \otimes \mathbf{C}_2$, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. We then have a natural graded homomorphism

$$\phi : A_1 \otimes A_2 \rightarrow A,$$

where

$$A_i := \bigoplus_n H^{2n}(\mathcal{A}_i) \quad (i = 1, 2).$$

It induces a map

$$\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A_1) \times \mathrm{Spec}(A_2);$$

let $Y \subset \mathrm{Spec}(A)$ be the preimage of $Y_1 \times Y_2 \subset \mathrm{Spec}(A_1) \times \mathrm{Spec}(A_2)$.

As in Proposition 3.5.5, one shows:

Proposition 3.5.7 *The subcategories*

$$(\mathbf{C}_1)_{Y_1} \otimes (\mathbf{C}_2)_{Y_2} \text{ and } \mathbf{C}_Y$$

of $\mathbf{C}_1 \otimes \mathbf{C}_2 = \mathbf{C}$ coincide.

3.5.8. Let $\mathbf{C}_i, \mathcal{A}_i, A_i$ ($i = 1, 2$) be as in Sect. 3.5.6. Suppose that \mathbf{C}_1 is dualizable. Let $F : \mathbf{C}_1 \rightarrow \mathbf{C}_2$ be a continuous functor. Such functors are in a bijection with objects

$$\mathbf{c}' \in \mathbf{C}' := \mathbf{C}_1^\vee \otimes \mathbf{C}_2.$$

Note that \mathbf{C}_1^\vee is acted on by $\mathcal{A}_1^{\mathrm{op}}$ (see Sect. E.3.3).

We can regard $\mathbf{C}_1^\vee \otimes \mathbf{C}_2$ as acted on by the \mathbb{E}_2 -algebra $\mathcal{A}' := \mathcal{A}_1^{\mathrm{op}} \otimes \mathcal{A}_2$. Let A be the corresponding graded algebra; we have a natural morphism $\phi : A_1 \otimes A_2 \rightarrow A$ (Note that the graded algebra corresponding to $\mathcal{A}_1^{\mathrm{op}}$ coincides with A_1). Let p_1, p_2 be the two components of the corresponding map

$$(p_1, p_2) : \mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A_1) \times \mathrm{Spec}(A_2).$$

We have:

Proposition 3.5.9 *Let $Y_i \subset \mathrm{Spec}(A_i)$ (for $i = 1, 2$) be conical Zariski-closed subsets such that the complement of Y_1 is quasi-compact. Let \mathbf{c}' be the object of $\mathbf{C}_1^\vee \otimes \mathbf{C}_2$ corresponding to F . Suppose that*

$$p_2(\mathrm{supp}_A(\mathbf{c}') \cap p_1^{-1}(Y_1)) \subset Y_2.$$

Then, the functor F maps $(\mathbf{C}_1)_{Y_1}$ to $(\mathbf{C}_2)_{Y_2}$.

Proof Set $Y'_1 = p_1^{-1}(Y_1) \subset \mathrm{Spec}(A)$. It is a conical Zariski-closed subset whose complement is quasi-compact. Consider the corresponding exact sequence of categories

$$\mathbf{C}'_{Y'_1} \rightleftarrows \mathbf{C}' \rightleftarrows \mathbf{C}'_{\mathrm{Spec}(A)-Y'_1}.$$

It is clear that the objects of $\mathbf{C}'_{\mathrm{Spec}(A)-Y'_1}$ correspond to functors $\mathbf{C}_1 \rightarrow \mathbf{C}_2$ that vanish on $(\mathbf{C}_1)_{Y_1}$. Therefore, we may replace \mathbf{c}' by its colocalization and assume that $\mathbf{c}' \in \mathbf{C}'_{Y'_1}$. We then have $p_2(\mathrm{supp}_A(\mathbf{c}')) \subset Y_2$. For such \mathbf{c}' , it is clear that the essential image of the corresponding functor $\mathbf{C}_1 \rightarrow \mathbf{C}_2$ is contained in $(\mathbf{C}_2)_{Y_2}$. \square

3.6 Grading shift for \mathbb{E}_2 -algebras

In this subsection, we will show how to relate the notion of support developed in the previous subsections to the more algebro-geometric notion of support *over* an algebraic stack.

This subsection may be skipped on the first reading and returned to when necessary.

3.6.1. Suppose that the \mathbb{E}_2 -algebra \mathcal{A} carries an action of \mathbb{G}_m such that the corresponding \mathbb{E}_2 -algebra $\mathcal{A}^{\text{shift}}$ (see Sect. A.2.2) is classical.

In particular, $\mathcal{A}^{\text{shift}}$ has a canonical E_∞ (i.e., commutative algebra) structure, which restricts to the initial \mathbb{E}_2 -structure. Hence, the same is true for \mathcal{A} .

We thus obtain a canonical isomorphism

$$\mathcal{A}^{\text{shift}} \simeq \mathcal{A}$$

as classical commutative algebras, which is compatible with the grading after scaling the grading on the left-hand side by 2.

3.6.2. Consider the stack $\mathcal{S}_{\mathcal{A}} = \text{Spec}(\mathcal{A}^{\text{shift}}/\mathbb{G}_m)$. Given a conical Zariski-closed subset

$$Y \subset \text{Spec}(A),$$

we regard Y/\mathbb{G}_m as a closed substack in $\mathcal{S}_{\mathcal{A}}$.

Recall that by Sect. A.2.2, we have a canonical equivalence of DG categories

$$(\mathcal{A}\text{-mod})^{\mathbb{G}_m} \simeq \text{QCoh}(\mathcal{S}_{\mathcal{A}}).$$

This equivalence naturally extends to an equivalence of (symmetric) monoidal categories.

Proposition 3.6.3 *We have*

$$\mathcal{A}\text{-mod}_Y = \mathcal{A}\text{-mod} \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{QCoh}(\mathcal{S}_{\mathcal{A}})_{Y/\mathbb{G}_m}$$

as full subcategories of

$$\mathcal{A}\text{-mod} = \mathcal{A}\text{-mod} \otimes_{(\mathcal{A}\text{-mod})^{\mathbb{G}_m}} (\mathcal{A}\text{-mod})^{\mathbb{G}_m} \simeq \mathcal{A}\text{-mod} \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{QCoh}(\mathcal{S}_{\mathcal{A}}).$$

Proof Note first that an action of \mathbb{G}_m on an associative DG algebra \mathcal{A} defines a bi-grading on \mathcal{A} . Suppose that \mathcal{A} is an \mathbb{E}_2 -algebra, and let $Y \subset \text{Spec}(A)$ be a Zariski-closed subset conical with respect to the cohomological grading. Then, the corresponding subcategory $\mathcal{A}\text{-mod}_Y$ is \mathbb{G}_m -invariant (see Sect. A.1.2) if and only if Y is conical with respect to both gradings.

Note, however, that the assumptions of the proposition imply that the two gradings on A coincide. Hence, for any conical Y , the subcategory $\mathcal{A}\text{-mod}_Y$ is \mathbb{G}_m -invariant.

By Sect. A.1.4, it suffices to show that

$$(\mathcal{A}\text{-mod}_Y)^{\mathbb{G}_m} \quad \text{and} \quad \text{QCoh}(\mathcal{S}_{\mathcal{A}})_{Y/\mathbb{G}_m}$$

coincide as subcategories of $(\mathcal{A}\text{-mod})^{\mathbb{G}_m} \simeq \text{QCoh}(\mathcal{S}_{\mathcal{A}})$.

Note that by (A.1), the category $(\mathcal{A}\text{-mod}_Y)^{\mathbb{G}_m}$ identifies with the full subcategory of $(\mathcal{A}\text{-mod})^{\mathbb{G}_m}$ consisting of modules supported on Y as plain \mathcal{A} -modules. This makes the required assertion manifest. \square

3.6.4. Let \mathbf{C} be a DG category acted on by \mathcal{A} , where \mathcal{A} is as in Sect. 3.6.1. In particular, we obtain that \mathbf{C} is a module category over $\text{QCoh}(\mathcal{S}_{\mathcal{A}})$. Let $Y \subset \text{Spec}(A)$ be a conical Zariski-closed subset such that its complement is quasi-compact.

Combining Propositions 3.5.5 and 3.6.3, we obtain:

Corollary 3.6.5 $\mathbf{C}_Y \simeq \mathbf{C} \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{QCoh}(\mathcal{S}_{\mathcal{A}})_{Y/\mathbb{G}_m}$ as subcategories of \mathbf{C} .

In particular, for $\mathbf{c} \in \mathbf{C}$ we can express $\text{supp}_A(\mathbf{c})$ in terms of the more familiar notion of support of an object in a category tensored over QCoh of an algebraic stack.

3.6.6. Suppose that the algebra A is Noetherian. Let us show that in this case, we can use fibers to study supports of objects.

Let $i_s : \text{Spec}(k') \rightarrow \text{Spec}(A)$ be a geometric point of $\text{Spec}(A)$. We have natural monoidal functors

$$\text{QCoh}(\mathcal{S}_{\mathcal{A}}) \rightarrow \mathcal{A}\text{-mod} \rightarrow \text{Vect}_{k'},$$

where $\text{Vect}_{k'}$ is the category of vector spaces over k' . This defines an action of the monoidal category $\text{QCoh}(\mathcal{S}_{\mathcal{A}})$ on $\text{Vect}_{k'}$.

Given $\mathbf{c} \in \mathbf{C}$, we define $i_s^*(\mathbf{c})$ to be the object

$$\mathbf{c} \otimes k' \in \mathbf{C} \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{Vect}_{k'}.$$

By Noetherian induction, one proves the following.

Lemma 3.6.7 *If $i_s^*(\mathbf{c}) = 0$ for all geometric points s of A , then $\mathbf{c} = 0$.*

As a consequence, we obtain:

Corollary 3.6.8 *Let $Y \subset \text{Spec}(A)$ be a conical Zariski-closed subset. Fix $\mathbf{c} \in \mathbf{C}$. Then,*

- (a) $\mathbf{c} \in \mathbf{C}_Y$ if and only if $i_s^*(\mathbf{c}) = 0$ for all $s \notin Y$;
- (b) $\mathbf{c} \in \mathbf{C}_{\text{Spec}(A)-Y}$ if and only if $i_s^*(\mathbf{c}) = 0$ for all $s \in Y$;

(c) $\text{supp}_A(\mathbf{c})$ is the Zariski closure of the set

$$\{s \in \text{Spec}(A) : i_s^*(\mathbf{c}) \neq 0\}.$$

Proof Recall that we have an exact sequence of categories

$$\mathbf{C}_Y \rightleftarrows \mathbf{C} \rightleftarrows \mathbf{C}_{\text{Spec}(A)-Y}, \quad (3.7)$$

which identifies with

$$\begin{aligned} \mathbf{C} \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{QCoh}(\mathcal{S}_{\mathcal{A}})_{Y/\mathbb{G}_m} &\rightleftarrows \mathbf{C} \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{QCoh}(\mathcal{S}_{\mathcal{A}}) \\ &\rightleftarrows \mathbf{C} \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{QCoh}(\mathcal{S}_{\mathcal{A}} - Y/\mathbb{G}_m). \end{aligned}$$

The “only if” direction in part (a) follows because

$$\text{QCoh}(\mathcal{S}_{\mathcal{A}})_{Y/\mathbb{G}_m} \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{Vect}_{k'} = 0$$

for any point $i_s : \text{Spec}(k') \rightarrow \text{Spec}(A)$ not contained in Y . Similarly, the “only if” direction in part (b) follows because

$$\text{QCoh}(\mathcal{S}_{\mathcal{A}} - Y/\mathbb{G}_m) \otimes_{\text{QCoh}(\mathcal{S}_{\mathcal{A}})} \text{Vect}_{k'} = 0$$

for any point $i_s : \text{Spec}(k') \rightarrow \text{Spec}(A)$ contained in Y . Now the “if” directions in both parts follow from the sequence (3.7) and Lemma 3.6.7. Part (c) follows from part (a). \square

Part II: The theory of singular support

4 Singular support of ind-coherent sheaves

For the rest of the paper, we will be working with DG schemes locally almost of finite type over a ground field k , which is assumed to have characteristic 0.

In this section, we introduce the notion of singular support for objects of $\text{IndCoh}(Z)$, where Z is a quasi-smooth DG scheme, and study the basic properties of the corresponding categories $\text{IndCoh}_Y(Z)$, where $Y \subset \text{Sing}(Z)$ is a conical Zariski-closed subset.

4.1 The definition of singular support

Throughout this section, Z will be a quasi-smooth DG scheme. It will be assumed *affine* until Sect. 4.5.11.

4.1.1. Consider the \mathbb{E}_2 -algebra of Hochschild cochains $\mathrm{HC}(Z)$; see Sect. F.1 where the definition of $\mathrm{HC}(Z)$ is recalled (and see also Sect. E for some background material on \mathbb{E}_2 -algebras).

As is explained in Sect. F.1, the \mathbb{E}_2 -algebra $\mathrm{HC}(Z)$ identifies canonically with the \mathbb{E}_2 -algebra of Hochschild cochains $\mathrm{HC}(Z)^{\mathrm{IndCoh}}$ of the category $\mathrm{IndCoh}(Z)$.

Let $\mathrm{HH}^\bullet(Z)$ denote the classical graded associative algebra

$$\bigoplus_n H^n(\mathrm{HC}(Z)). \quad (4.1)$$

Let $\mathrm{HH}^{\mathrm{even}}(Z)$ denote the even part of $\mathrm{HH}^\bullet(Z)$, viewed as a classical graded associative algebra. As was mentioned in Sect. 3.5.1, the algebra $\mathrm{HH}^{\mathrm{even}}(Z)$ is commutative, and $\mathrm{HH}^{\mathrm{even}}(Z)$ maps to the graded center of $\mathrm{Ho}(\mathrm{IndCoh}(Z))$.

4.1.2. Since Z was assumed quasi-smooth, $T^*(Z)$ is perfect, and we can regard $T(Z)[-1]$ as a Lie algebra⁵ in $\mathrm{QCoh}(Z)$, see Corollary G.2.7. Note that from Corollary G.2.7, we obtain a canonical map of commutative algebras

$$\Gamma(Z, \mathcal{O}_{clZ}) \rightarrow \mathrm{HH}^0(Z),$$

and of $\Gamma(Z, \mathcal{O}_{clZ})$ -modules

$$\Gamma\left(Z, H^1(T(Z))\right) \rightarrow \mathrm{HH}^2(Z).$$

It induces a homomorphism of graded algebras

$$\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)}) = \Gamma\left(Z, \mathrm{Sym}_{\mathcal{O}_{clZ}}\left(H^1(T(Z))\right)\right) \rightarrow \mathrm{HH}^{\mathrm{even}}(Z), \quad (4.2)$$

where we assign to $\Gamma(Z, H^1(T(Z)))$ degree 2.

4.1.3. We are now ready to give the main definitions of this paper:

Definition 4.1.4 The *singular support* of $\mathcal{F} \in \mathrm{IndCoh}(Z)$, denoted $\mathrm{SingSupp}(\mathcal{F})$, is

$$\mathrm{supp}_{\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})}(\mathcal{F}) \subset \mathrm{Sing}(Z).$$

Definition 4.1.5 Let Y be a conical Zariski-closed subset of $\mathrm{Sing}(Z)$. We let

$$\mathrm{IndCoh}_Y(Z) \subset \mathrm{IndCoh}(Z)$$

denote the full subcategory spanned by objects whose singular supports are contained in Y .

⁵ For reasons of tradition, while we call Lie algebras in an arbitrary symmetric monoidal ∞ -category \mathbf{O} “Lie algebras,” we refer to Lie algebras in Vect as “DG Lie algebras.”

4.1.6. The following assertion (borrowed from [4, Theorem 11.3]) gives an explicit expression for singular support:

Lemma 4.1.7 *For $\mathcal{F} \in \text{IndCoh}(Z)^c := \text{Coh}(Z)$, its singular support is equal to the support of the graded $\Gamma(\text{Sing}(Z), \mathcal{O}_{\text{Sing}(Z)})$ -module $\text{End}_{\text{Coh}(Z)}^\bullet(\mathcal{F})$.*

Proof Follows immediately from Lemma 3.4.4(b). \square

In addition, we have the following result:

Theorem 4.1.8 *For two objects $\mathcal{F}_1, \mathcal{F}_2 \in \text{Coh}(Z)$, the graded vector space $\text{Hom}_{\text{Coh}(Z)}^\bullet(\mathcal{F}_1, \mathcal{F}_2)$, regarded as a module over $\Gamma(\text{Sing}(Z), \mathcal{O}_{\text{Sing}(Z)})$, is finitely generated.*

In particular, for $\mathcal{F} \in \text{Coh}(Z)$, the $\Gamma(\text{Sing}(Z), \mathcal{O}_{\text{Sing}(Z)})$ -module $\text{End}_{\text{Coh}(Z)}^\bullet(\mathcal{F})$ appearing in Lemma 4.1.7 is finitely generated.

Remark 4.1.9 If Z is a classical scheme, the assertion of Theorem 4.1.8 is due to Gulliksen [15]; also see references in the proof of [4, Theorem 11.3]. For completeness, we will present a proof of Theorem 4.1.8 in Appendix D.

4.2 Basic properties

4.2.1. First, we note:

Lemma 4.2.2 *The subcategory $\text{IndCoh}_Y(Z) \subset \text{IndCoh}(Z)$ is stable under the monoidal action of $\text{QCoh}(Z)$ on $\text{IndCoh}(Z)$.*

Proof For any module category \mathbf{C} over $\text{QCoh}(Z)$, any full cocomplete subcategory $\mathbf{C}' \subset \mathbf{C}$ is stable under the action, since $\text{QCoh}(Z)$ is generated by its unit object, \mathcal{O}_Z . \square

4.2.3. It is easy to see that the dualizing sheaf $\omega_Z \in \text{IndCoh}(Z)$ belongs to $\text{IndCoh}_{\{0\}}(Z)$, where $\{0\} \subset \text{Sing}(Z)$ denotes the zero-section.

Indeed, the construction of the isomorphism of Corollary G.2.7 shows that the action of $\Gamma(Z, T(Z)[-1]) \rightarrow \text{HC}(Z)$ on $\omega_Z \in \text{IndCoh}(Z)$ is trivial.

4.2.4. Recall the fully faithful functor

$$\Xi_Z : \text{QCoh}(Z) \rightarrow \text{IndCoh}(Z)$$

(see [10, Proposition 1.5.3]), which is defined since Z is eventually coconnective.

Note that since Z is quasi-smooth, and hence Gorenstein (see Corollary 2.2.8), ω_Z is the image of a line bundle under Ξ_Z . Hence, by Lemma 4.2.2, the essential image of all of $\text{QCoh}(Z)$ under Ξ_Z is contained in $\text{IndCoh}_{\{0\}}(Z)$.

4.2.5. In Sect. 5.7, we will prove the converse inclusion:

Theorem 4.2.6 *The subcategory $\text{IndCoh}_{\{0\}}(Z) \subset \text{IndCoh}(Z)$ coincides with the essential image of $\text{QCoh}(Z)$ under Ξ_Z .*

4.3 Compact generation

4.3.1. Lemma 3.4.2 (or [4, Theorem 6.4]) immediately implies the following claim.

Corollary 4.3.2 *Let Z be a quasi-smooth affine DG scheme. For any conical Zariski-closed subset $Y \subset \text{Sing}(Z)$, the category $\text{IndCoh}_Y(Z)$ is compactly generated.*

Define

$$\text{Coh}_Y(Z) := \text{IndCoh}_Y(Z) \cap \text{Coh}(Z).$$

Corollary 4.3.2 can be rephrased as follows:

Corollary 4.3.3 $\text{IndCoh}_Y(Z) \simeq \text{Ind}(\text{Coh}_Y(Z))$.

4.3.4. Let $Y_1 \subset Y_2$ be two conical Zariski-closed subsets.

We have a tautologically defined fully faithful functor

$$\Xi_Z^{Y_1, Y_2} : \text{IndCoh}_{Y_1}(Z) \rightarrow \text{IndCoh}_{Y_2}(Z)$$

that sends $\text{Coh}_{Y_1}(Z)$ identically into $\text{Coh}_{Y_2}(Z)$.

Since this functor sends compact objects to compacts, it admits a continuous right adjoint. We denote this right adjoint by $\Psi_Z^{Y_1, Y_2}$. Thus, the functor $\Psi_Z^{Y_1, Y_2}$ realizes $\text{IndCoh}_{Y_1}(Z)$ as a colocalization of $\text{IndCoh}_{Y_2}(Z)$.

Note that the functor $\Xi_Z^{Y_1, Y_2}$ is (tautologically) compatible with the action of $\text{QCoh}(Z)$. Hence, the functor $\Psi_Z^{Y_1, Y_2}$ acquires a structure of being lax compatible. However, we claim:

Lemma 4.3.5 *The functor $\Psi_Z^{Y_1, Y_2}$ is compatible with the action of $\text{QCoh}(Z)$, i.e., the lax compatibility is strict.*

Proof Same as that of Lemma 4.2.2. □

4.3.6. In particular, we can take Y_2 to be all of $\text{Sing}(Z)$, in which case $\text{IndCoh}_{Y_2}(Z)$ is all of $\text{IndCoh}(Z)$.

We will denote the resulting pair of adjoint functors

$$\text{IndCoh}_Y(Z) \rightleftarrows \text{IndCoh}(Z)$$

by $(\Xi_Z^{Y, \text{all}}, \Psi_Z^{Y, \text{all}})$.

4.3.7. Similarly, for any Y that contains the zero-section, we obtain the corresponding pair of adjoint functors

$$\Xi_Z^Y : \text{QCoh}(Z) \rightleftarrows \text{IndCoh}_Y(Z) : \Psi_Z^Y$$

with Ξ_Z^Y being fully faithful.

By definition, the functor Ξ_Z^Y is the ind-extension of the natural embedding

$$\mathrm{QCoh}(Z)^c = \mathrm{QCoh}(Z)^{\mathrm{perf}} \hookrightarrow \mathrm{Coh}_Y(Z) \hookrightarrow \mathrm{IndCoh}_Y(Z),$$

and Ψ_Z^Y is the ind-extension of

$$\mathrm{Coh}_Y(Z) \hookrightarrow \mathrm{Coh}(Z) \hookrightarrow \mathrm{QCoh}(Z).$$

Note that according to Theorem 4.2.6, the latter functors are a particular case of $(\Xi_Z^{Y_1, Y_2}, \Psi_Z^{Y_1, Y_2})$ for $Y_1 = \{0\}$ and $Y_2 = Y$.

4.4 The t-structure

Let Y be a conical Zariski-closed subset of $\mathrm{Sing}(Z)$ containing the zero-section.

4.4.1. We define a t-structure on $\mathrm{IndCoh}_Y(Z)$ by declaring that

$$\mathcal{F} \in \mathrm{IndCoh}_Y(Z)^{\leq 0} \Leftrightarrow \Psi_Z^Y(\mathcal{F}) \in \mathrm{QCoh}(Z)^{\leq 0}.$$

Note that for $Y = \mathrm{Sing}(Z)$, this t-structure coincides with the canonical t-structure on $\mathrm{IndCoh}(Z)$ of [10, Sect. 1.2].

Lemma 4.4.2

$$\mathcal{F} \in \mathrm{IndCoh}_Y(Z)^{\leq 0} \Leftrightarrow \Xi_Z^{Y, \mathrm{all}}(\mathcal{F}) \in \mathrm{IndCoh}(Z)^{\leq 0}.$$

Proof Follows from the fact that

$$\Psi_Z^Y \simeq \Psi_Z^Y(\mathcal{F}) \circ \Psi_Z^{Y, \mathrm{all}} \circ \Xi_Z^{Y, \mathrm{all}} \simeq \Psi_Z \circ \Xi_Z^{Y, \mathrm{all}}.$$

□

Corollary 4.4.3 *The functors $\Psi_Z^{Y, \mathrm{all}}$ and Ψ_Z^Y are t-exact.*

Proof The previous lemma implies that $\Xi_Z^{Y, \mathrm{all}}$ is right t-exact. Hence, $\Psi_Z^{Y, \mathrm{all}}$ is left t-exact by adjunction. The fact that $\Psi_Z^{Y, \mathrm{all}}$ is right t-exact follows from the fact that

$$\Psi_Z^Y \circ \Psi_Z^{Y, \mathrm{all}} \simeq \Psi_Z.$$

The functor Ψ_Z^Y is right t-exact by definition. To show that it is left t-exact, it is enough to show that Ξ_Z^Y is right t-exact. The latter is equivalent, by definition, to the fact that $\Psi_Z^Y \circ \Xi_Z^Y$ is right t-exact. However, the latter functor is isomorphic to the identity. □

4.4.4. We will now prove:

Proposition 4.4.5 *The functors $\Psi_Z^{Y,\text{all}}$ and Ψ_Z^Y induce equivalences*

$$\text{IndCoh}(Z)^{\geq 0} \rightarrow \text{IndCoh}_Y(Z)^{\geq 0} \rightarrow \text{QCoh}(Z)^{\geq 0}.$$

Proof Note that the fact that the composed functor $\Psi_Z^Y \circ \Psi_Z^{Y,\text{all}} \simeq \Psi_Z$ induces an equivalence

$$\text{IndCoh}(Z)^{\geq 0} \rightarrow \text{QCoh}(Z)^{\geq 0}$$

is [10, Proposition 1.2.4]. In particular, the functor

$$\Psi_Z^{Y,\text{all}} : \text{IndCoh}(Z)^{\geq 0} \rightarrow \text{IndCoh}_Y(Z)^{\geq 0}$$

is conservative.

The left adjoint to $\Psi_Z^{Y,\text{all}}|_{\text{IndCoh}(Z)^{\geq 0}}$ is given by

$$\mathcal{F} \mapsto \tau^{\geq 0} \left(\Xi_Z^{Y,\text{all}}(\mathcal{F}) \right).$$

It is enough to show that this left adjoint is fully faithful, i.e., that the adjunction map

$$\mathcal{F} \rightarrow \Psi_Z^{Y,\text{all}} \left(\tau^{\geq 0} \left(\Xi_Z^{Y,\text{all}}(\mathcal{F}) \right) \right)$$

is an isomorphism.

Since $\Psi_Z^{Y,\text{all}}$ is t-exact, we have:

$$\Psi_Z^{Y,\text{all}} \left(\tau^{\geq 0} \left(\Xi_Z^{Y,\text{all}}(\mathcal{F}) \right) \right) \simeq \tau^{\geq 0} \left(\Psi_Z^{Y,\text{all}} \circ \Xi_Z^{Y,\text{all}}(\mathcal{F}) \right).$$

However, since $\Xi_Z^{Y,\text{all}}$ is fully faithful, the latter expression is isomorphic to $\tau^{\geq 0}(\mathcal{F}) \simeq \mathcal{F}$, as required. \square

4.4.6. Recall that a t-structure on a triangulated category \mathbf{T} is said to be *compactly generated* if

$$\mathcal{F} \in \mathbf{T}^{>0} \Leftrightarrow \text{Hom}_{\mathbf{T}}(\mathcal{F}_1, \mathcal{F}) = 0, \quad \forall \mathcal{F}_1 \in \mathbf{T}^{\leq 0} \cap \mathbf{T}^c.$$

Proposition 4.4.7 *The t-structure on $\text{IndCoh}_Y(Z)$ is compactly generated.*

Proof Let $\mathcal{F} \in \text{IndCoh}_Y(Z)$ be an object that is right orthogonal to

$$\text{Coh}_Y(Z) \cap \text{Coh}(Z)^{\leq 0}.$$

Let us prove that $\mathcal{F} \in \text{IndCoh}_Y(Z)^{>0}$. Truncating, we may assume that $\mathcal{F} \in \text{IndCoh}_Y(Z)^{\leq 0}$; we need to show that $\mathcal{F} = 0$.

By assumption, \mathcal{F} is right orthogonal to the essential image of $\text{QCoh}(Z)^{\text{perf}} \cap \text{Coh}(Z)^{\leq 0}$ under Ξ_Z^Y . Hence, $\Psi_Z^Y(\mathcal{F}) \in \text{QCoh}(Z)^{>0}$. However, since $\mathcal{F} \in \text{IndCoh}_Y(Z)^{\leq 0}$, we have also that $\Psi_Z^Y(\mathcal{F}) \in \text{QCoh}(Z)^{\leq 0}$, so $\Psi_Z^Y(\mathcal{F}) = 0$.

Thus, \mathcal{F} is right orthogonal to the essential image of all of $\text{QCoh}(Z)^{\text{perf}}$ under Ξ_Z^Y . To prove that $\mathcal{F} = 0$, we need to show that

$$\text{Hom}_{\text{IndCoh}_Y(Z)}(\mathcal{F}_1, \mathcal{F}) = 0$$

for any $\mathcal{F}_1 \in \text{Coh}_Y(Z)$. However, for any such \mathcal{F}_1 , there exists $\mathcal{F}_2 \in \text{QCoh}(Z)^{\text{perf}}$ and a map $\mathcal{F}_2 \rightarrow \mathcal{F}_1$, such that

$$\text{Cone}(\mathcal{F}_2 \rightarrow \mathcal{F}_1) \in \text{Coh}_Y(Z) \cap \text{Coh}(Z)^{\leq 0}.$$

This implies the required assertion by the long exact sequence. \square

4.5 Localization with respect to Z

4.5.1. Let $V \xrightarrow{i} Z$ be a closed DG subscheme. Let $\text{IndCoh}(Z)_V$ be the corresponding full subcategory of $\text{IndCoh}(Z)$ (see [10, Sect. 4.1.2]), i.e., $\text{IndCoh}(Z)_V$ consists of those objects that vanish when restricted to $Z - V$.

Equivalently, we can then produce $\text{IndCoh}(Z)_V$ in the framework of Sect. 3.5, using the action of $\Gamma(Z, \mathcal{O}_Z)$ on the category $\text{IndCoh}(Z)$.

Consider the scheme

$$\text{Sing}(Z)_V = {}^{cl}(\text{Sing}(Z) \times_Z V) \subset \text{Sing}(Z).$$

Consider the corresponding subcategory

$$\text{IndCoh}_{\text{Sing}(Z)_V}(Z) \subset \text{IndCoh}(Z).$$

The next assertion results immediately from Lemma 3.3.12:

Corollary 4.5.2 *The subcategories $\text{IndCoh}_{\text{Sing}(Z)_V}(Z)$ and $\text{IndCoh}(Z)_V$ of $\text{IndCoh}(Z)$ coincide.*

4.5.3. Let $Y \subset \text{Sing}(Z)$ be a conical Zariski-closed subset. Set

$$Y_V := {}^{cl}(Y \times_Z V).$$

From Corollary 4.5.2 and (3.4), we obtain:

Corollary 4.5.4 *The subcategories*

$$\mathrm{IndCoh}_{Y_V}(Z) \quad \text{and} \quad \mathrm{IndCoh}(Z)_V \cap \mathrm{IndCoh}_Y(Z)$$

of $\mathrm{IndCoh}(Z)$ coincide.

4.5.5. Let now $U \xrightarrow{j} Z$ be an open affine. By [10, Lemma 4.1.1], we have a pair of adjoint functors

$$j^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(Z) \rightleftarrows \mathrm{IndCoh}(U) : j_*^{\mathrm{IndCoh}},$$

which realize $\mathrm{IndCoh}(U)$ as a localization of $\mathrm{IndCoh}(Z)$. By [10, Corollary 4.4.5], we have a commutative diagram with vertical arrows being equivalences:

$$\begin{array}{ccc} \mathrm{IndCoh}(Z) \otimes_{\mathrm{QCoh}(Z)} \mathrm{QCoh}(U) & \xrightarrow{\mathrm{Id} \otimes j_*} & \mathrm{IndCoh}(Z) \otimes_{\mathrm{QCoh}(Z)} \mathrm{QCoh}(Z) \\ \downarrow & & \downarrow \\ \mathrm{IndCoh}(U) & \xrightarrow{j_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(Z). \end{array}$$

In particular, we obtain that $\mathrm{IndCoh}(U)$ can be interpreted as the full subcategory of $\mathrm{IndCoh}(Z)$ corresponding to ${}^{cl}U \subset {}^{cl}Z$ with respect to the action of $\Gamma(Z, \mathcal{O}_{{}^{cl}Z})$ on $\mathrm{IndCoh}(Z)$ in the sense of Sect. 3.1.2.

Let $Y \subset \mathrm{Sing}(Z)$ be a conical Zariski-closed subset. Set

$$Y_U := {}^{cl}(Y \times_U U) \subset \mathrm{Sing}(Z)_U \simeq \mathrm{Sing}(U).$$

From (3.3), we obtain:

Corollary 4.5.6 (a) *The functors j_*^{IndCoh} and $j^{\mathrm{IndCoh},*}$ define an equivalence between $\mathrm{IndCoh}_{Y_U}(U)$ and the intersection of $\mathrm{IndCoh}_Y(Z)$ with the essential image of $\mathrm{IndCoh}(U)$ under j_*^{IndCoh} .*

(b) *The functors $(j^{\mathrm{IndCoh},*}, j_*^{\mathrm{IndCoh}})$ map the categories*

$$\mathrm{IndCoh}_Y(Z) \rightleftarrows \mathrm{IndCoh}_{Y_U}(U)$$

to one another, and are mutually adjoint.

(c) *We have a commutative diagram with vertical arrows being isomorphisms:*

$$\begin{array}{ccc} \mathrm{IndCoh}_Y(Z) \otimes_{\mathrm{QCoh}(Z)} \mathrm{QCoh}(U) & \xrightarrow{\mathrm{Id} \otimes j_*} & \mathrm{IndCoh}_Y(Z) \otimes_{\mathrm{QCoh}(Z)} \mathrm{QCoh}(Z) \\ \downarrow & & \downarrow \\ \mathrm{IndCoh}_{Y_U}(U) & \xrightarrow{j_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}_Y(Z). \end{array}$$

Corollary 4.5.7 *Let U_i be a cover of Z by open affine subsets. An object $\mathcal{F} \in \text{IndCoh}(Z)$ belongs to $\text{IndCoh}_Y(Z)$ if and only if $\mathcal{F}|_{U_i}$ belongs to $\text{IndCoh}_{Y_{U_i}}(U_i)$ for every i .*

Proof The “only if” direction follows immediately from Corollary 4.5.6(b). The “if” direction follows from the Čech complex. \square

4.5.8. For U as above, let V be a complementary closed DG subscheme. By [10, Corollary 4.1.5], we have a short exact sequence of categories

$$\text{IndCoh}(Z)_V \rightleftarrows \text{IndCoh}(Z) \xrightarrow{j^{\text{IndCoh},*}} \text{IndCoh}(U). \quad (4.3)$$

It can be obtained from the short exact sequence of categories

$$\text{QCoh}(Z)_V \rightleftarrows \text{QCoh}(Z) \xrightarrow{j^*} \text{QCoh}(U), \quad (4.4)$$

by the operation

$$\text{IndCoh}(Z) \otimes_{\text{QCoh}(Z)} -.$$

(Here, $\text{QCoh}(Z)_V \subset \text{QCoh}(Z)$ is the full subcategory consisting of objects set-theoretically supported on V .)

Hence, we obtain:

Corollary 4.5.9 *There exists a short exact sequence of DG categories*

$$\text{IndCoh}_{Y_V}(Z) \rightleftarrows \text{IndCoh}_Y(Z) \rightleftarrows \text{IndCoh}_{Y_U}(U),$$

which can be obtained from the short exact sequence (4.4) by the operation

$$\text{IndCoh}_Y(Z) \otimes_{\text{QCoh}(Z)} -.$$

Corollary 4.5.10 *Let $Y_1 \subset Y_2$ be two conical Zariski-closed subsets. Then, the functors*

$$\Xi_Z^{Y_1, Y_2} : \text{IndCoh}_{Y_1}(Z) \rightleftarrows \text{IndCoh}_{Y_2}(Z) : \Psi_Z^{Y_1, Y_2}$$

induce (mutually adjoint) functors

$$\text{IndCoh}_{(Y_1)_V}(Z) \rightleftarrows \text{IndCoh}_{(Y_2)_V}(Z) \quad \text{and} \quad \text{IndCoh}_{(Y_1)_U}(U) \rightleftarrows \text{IndCoh}_{(Y_2)_U}(U).$$

4.5.11. From Corollary 4.5.7, we obtain that the notion of singular support of an object of $\text{IndCoh}(Z)$ makes sense for any quasi-smooth DG scheme Z (not necessarily affine).

Namely, we choose an affine cover U_α , and we set

$$\text{SingSupp}(\mathcal{F}) \cap \text{Sing}(Z)_{U_\alpha} := \text{SingSupp}(\mathcal{F}|_{U_\alpha}),$$

where we identify

$$\text{Sing}(Z)_{U_\alpha} \simeq \text{Sing}(U_\alpha),$$

and $\mathcal{F}|_{U_\alpha} := j_\alpha^{\text{IndCoh},*}(\mathcal{F})$, where $j_\alpha : U_\alpha \hookrightarrow Z$.

Corollary 4.5.7 implies that $\text{SingSupp}(\mathcal{F})$ is well defined, in particular, independent of the choice of the cover.

Furthermore, to $Y \subset \text{Sing}(Z)$ we can attach a full subcategory

$$\text{IndCoh}_Y(Z) \subset \text{IndCoh}(Z),$$

by the requirement

$$\mathcal{F} \in \text{IndCoh}_Y(Z) \Leftrightarrow \mathcal{F}|_{U_\alpha} \in \text{IndCoh}_{Y|_{U_\alpha}}(U_\alpha), \forall \alpha.$$

4.6 Behavior with respect to products

4.6.1. Recall (see [10, Proposition 4.6.2]) that if $Z_i, i = 1, 2$ are two quasi-compact DG schemes, the natural functor

$$\text{IndCoh}(Z_1) \otimes \text{IndCoh}(Z_2) \rightarrow \text{IndCoh}(Z_1 \times Z_2) \quad (4.5)$$

is an equivalence.

Remark 4.6.2 This simple statement uses the assumption that k is perfect (recall that we assume $\text{char}(k) = 0$).

4.6.3. Assume now that Z_i are both quasi-smooth. It is easy to see that we have a natural isomorphism

$$\text{Sing}(Z_1) \times \text{Sing}(Z_2) \simeq \text{Sing}(Z_1 \times Z_2).$$

Let $Y_i \subset \text{Sing}(Z_i)$ be conical Zariski-closed subsets, and consider the corresponding subset

$$Y_1 \times Y_2 \subset \text{Sing}(Z_1 \times Z_2).$$

Lemma 4.6.4 *We have:*

$$\mathrm{IndCoh}_{Y_1}(Z_1) \otimes \mathrm{IndCoh}_{Y_2}(Z_2) = \mathrm{IndCoh}_{Y_1 \times Y_2}(Z_1 \times Z_2)$$

as subcategories of $\mathrm{IndCoh}(Z_1 \times Z_2)$.

Proof This follows immediately from Proposition 3.5.7. \square

4.7 Compatibility with Serre duality

4.7.1. Recall (see, e.g., [10, Sect. 9.5]) that the category $\mathrm{Coh}(Z)$ carries a canonical anti-involution given by Serre duality, denoted $\mathbb{D}_Z^{\mathrm{Serre}}$.

Proposition 4.7.2 *For any conical Zariski-closed subset $Y \subset \mathrm{Sing}(Z)$, the anti-involution $\mathbb{D}_Z^{\mathrm{Serre}}$ preserves the subcategory $\mathrm{Coh}_Y(Z) \subset \mathrm{Coh}(Z)$.*

One proof is given in Sect. G.3.4. Another (in a sense more hands-on, but logically equivalent) proof is given in Sect. 5.5.

Corollary 4.7.3 *For $\mathcal{F} \in \mathrm{Coh}(Z)$, we have*

$$\mathrm{SingSupp}(\mathcal{F}) = \mathrm{SingSupp}(\mathbb{D}_Z(\mathcal{F})).$$

4.7.4. We obtain that there exists a canonical identification

$$(\mathrm{IndCoh}_Y(Z))^\vee \simeq \mathrm{IndCoh}_Y(Z),$$

obtained by extending $\mathbb{D}_Z^{\mathrm{Serre}}|_{\mathrm{Coh}_Y(Z)}$.

Let $Y_1 \subset Y_2$ be two conical Zariski-closed subsets, and consider the pair of adjoint functors

$$(\Psi_Z^{Y_1, Y_2})^\vee : \mathrm{IndCoh}_{Y_1}(Z) \rightleftarrows \mathrm{IndCoh}_{Y_2}(Z) : (\Xi_Z^{Y_1, Y_2})^\vee,$$

obtained from

$$\Xi_Z^{Y_1, Y_2} : \mathrm{IndCoh}_{Y_1}(Z) \rightleftarrows \mathrm{IndCoh}_{Y_2}(Z) : \Psi_Z^{Y_1, Y_2}$$

by passing to the dual functors.

Lemma 4.7.5 *We have canonical isomorphisms*

$$(\Psi_Z^{Y_1, Y_2})^\vee \simeq \Xi_Z^{Y_1, Y_2} \quad \text{and} \quad (\Xi_Z^{Y_1, Y_2})^\vee \simeq \Psi_Z^{Y_1, Y_2}.$$

Proof Follows from [29], Lemma 2.3.3 using the fact that

$$\Xi_Z^{Y_1, Y_2} \circ \mathbb{D}_Z^{\mathrm{Serre}} \simeq \mathbb{D}_Z^{\mathrm{Serre}} \circ \Xi_Z^{Y_1, Y_2}.$$

\square

Remark 4.7.6 Note that if $Y = \{0\}$ is the zero-section, the resulting self-duality on $\text{IndCoh}_{\{0\}}(Z) \simeq \text{QCoh}(Z)$ is different from the “naive” self-duality: The two differ by the automorphism of $\text{QCoh}(Z)$ given by tensoring with ω_Z .

5 Singular support on a global complete intersection and Koszul duality

In this section, we analyze the behavior of singular support on a DG scheme Z which is a “global complete intersection.” Recall that this means that Z is presented as a Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\iota} & \mathcal{U} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathcal{V}, \end{array}$$

where \mathcal{U} and \mathcal{V} are smooth affine schemes. In this case, we will reinterpret the notion of singular support in terms of Koszul duality.

Our basic tool will be the group DG scheme $\mathcal{G}_{\text{pt}/\mathcal{V}} := \text{pt} \times_{\mathcal{V}} \text{pt}$.

5.1 Koszul duality

5.1.1. Consider the groupoid

$$\mathcal{G}_{\text{pt}/\mathcal{V}} := \text{pt} \times_{\mathcal{V}} \text{pt}$$

over pt , that is, a group DG scheme:

$$\begin{array}{ccc} & \mathcal{G}_{\text{pt}/\mathcal{V}} & \\ p_1 \swarrow & & \searrow p_2 \\ \text{pt} & & \text{pt}. \end{array} \tag{5.1}$$

Let Δ_{pt} denote the diagonal map

$$\text{pt} \rightarrow \text{pt} \times_{\mathcal{V}} \text{pt} = \mathcal{G}_{\text{pt}/\mathcal{V}}.$$

The object

$$(\Delta_{\text{pt}})_*^{\text{IndCoh}}(k) \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})$$

is the unit in the monoidal category $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})$.

5.1.2. By Sect. F.4, the above groupoid gives rise to an \mathbb{E}_2 -algebra

$$\mathcal{A}_{\mathcal{G}_{\text{pt}/\mathcal{V}}} =: \text{HC}(\text{pt}/\mathcal{V}),$$

whose underlying associative DG algebra identifies canonically with

$$\mathcal{M}aps_{\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})} \left((\Delta_{\text{pt}})_*^{\text{IndCoh}}(k), (\Delta_{\text{pt}})_*^{\text{IndCoh}}(k) \right).$$

Remark 5.1.3 The \mathbb{E}_2 -algebra that we denote here by $\mathcal{A}_{\mathcal{G}_{\text{pt}/\mathcal{V}}}$ is what should be properly denoted $\mathcal{A}_{\mathcal{G}_{\text{pt}/\mathcal{V}}}^{\text{IndCoh}}$. This is done for the purpose of unburdening the notation. As was explained in Remark F.4.5, it is the IndCoh version (and not the QCoh one) that we use in this paper.

Let V denote the tangent space to \mathcal{V} at pt . We have:

Lemma 5.1.4 *The associative DG algebra underlying $\text{HC}^\bullet(\text{pt}/\mathcal{V})$ identifies canonically with $\text{Sym}(V[-2])$.*

Proof This is a special case of Corollary G.4.6, since a groupoid over pt is canonically a group DG scheme. \square

In particular, we see that

$$\text{HH}^\bullet(\text{pt}/\mathcal{V}) := \bigoplus_n H^n(\text{HC}(\text{pt}/\mathcal{V}))$$

identifies canonically with $\text{Sym}(V)$ as a classical graded algebra, where the elements of V have degree 2. Geometrically,

$$\text{Spec}(\text{HH}^\bullet(\text{pt}/\mathcal{V})) \simeq V^*.$$

5.1.5. Note that according to [10, Proposition 4.1.7(b)], $(\Delta_{\text{pt}})_*^{\text{IndCoh}}(k)$ is a compact generator of $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})$.

Hence, from Sect. E.2.4 we obtain a natural monoidal equivalence

$$\text{HC}(\text{pt}/\mathcal{V})^{\text{op-mod}} \rightarrow \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}). \quad (5.2)$$

We will denote the inverse functor $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \rightarrow \text{HC}(\text{pt}/\mathcal{V})^{\text{op-mod}}$ by $\text{KD}_{\text{pt}/\mathcal{V}}$, and refer to it as the Koszul duality functor. Explicitly,

$$\text{KD}_{\text{pt}/\mathcal{V}} = \mathcal{M}aps_{\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})}((\Delta_{\text{pt}})_*^{\text{IndCoh}}(k), -). \quad (5.3)$$

The functor $\text{KD}_{\text{pt}/\mathcal{V}}$ intertwines the forgetful functor $\text{HC}(\text{pt}/\mathcal{V})\text{-mod} \rightarrow \text{Vect}$ and

$$\Delta_{\text{pt}}^! : \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \rightarrow \text{IndCoh}(\text{pt}) = \text{Vect}.$$

5.1.6. Recall the set-up of Sect. 3.5.3 with the \mathbb{E}_2 -algebra being $\mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}}$.

In particular, to an object

$$\mathcal{F} \in \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \simeq \mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}\text{-mod}}$$

we can associate its support, which is a conical Zariski-closed subset of V^* . Conversely, to a conical Zariski-closed subset $Y \subset V^*$ we associate a full subcategory

$$(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}))_Y \subset \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}).$$

Lemma 5.1.7 *The support of $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})$ is equal to the support of the graded $\mathrm{Sym}(V[-2])$ -module*

$$\mathrm{Hom}^\bullet((\Delta_{\mathrm{pt}})_*^{\mathrm{IndCoh}}(k), \mathcal{F}) = H^\bullet(\mathrm{KD}_{\mathrm{pt}/\mathcal{V}}(\mathcal{F})).$$

Proof Since $(\Delta_{\mathrm{pt}})_*^{\mathrm{IndCoh}}(k)$ is a compact generator of $\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})$, this follows from Lemma 3.4.4. \square

Remark 5.1.8 The DG scheme $\mathcal{G}_{\mathrm{pt}/\mathcal{V}}$ is quasi-smooth and $V^* = \mathrm{Sing}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})$. It is easy to see that the support of $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})$ is nothing but $\mathrm{SingSupp}(\mathcal{F}) \subset V^*$ (see Lemma 5.3.4 for a more general statement).

5.1.9. Note that by combining Lemma 5.1.4 and (5.2), we obtain:

Corollary 5.1.10 *The categories $\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})$ and $\mathrm{Sym}(V[-2])\text{-mod}$ are canonically equivalent as plain DG categories.*

Remark 5.1.11 Both sides in Corollary 5.1.10 are naturally monoidal categories. However, the equivalence of Corollary 5.1.10 does not come with a monoidal structure (cf. Remark G.4.7). We will see in Corollary 5.4.3 that a choice of a *parallelization* of \mathcal{V} at pt upgrades the above equivalence to a monoidal one.

5.2 Functoriality of Koszul duality

The material in this section will be needed for the proof of some key properties of singular support in Sect. 7.

However, as it will not be used for the discussion of singular support in the rest of this section, the reader might choose to skip it on the first pass.

5.2.1. Let $f = f_{\mathcal{V}} : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ be a map between smooth classical schemes. Fix a point $\mathrm{pt} \rightarrow \mathcal{V}_1$. Set $V_i = T_{\mathrm{pt}}(\mathcal{V}_i)$ and $g = (df_{\mathrm{pt}})^* : V_2^* \rightarrow V_1^*$. Now consider the morphism of DG group schemes

$$f_{\mathcal{G}} : \mathcal{G}_{\mathrm{pt}/\mathcal{V}_1} \rightarrow \mathcal{G}_{\mathrm{pt}/\mathcal{V}_2}.$$

(Note that $g = \mathrm{Sing}(f_{\mathcal{G}})$.) The following lemma is obvious.

Lemma 5.2.2 *The following three conditions are equivalent:*

- (a) g is injective;
- (b) f is smooth at pt;
- (c) $f_{\mathcal{G}}$ is quasi-smooth.

The following two conditions are also equivalent:

- (a') g is surjective;
- (b') f is unramified (and then a regular immersion) at pt.

5.2.3. The map $f_{\mathcal{G}}$ induces a monoidal functor

$$(f_{\mathcal{G}})_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_1}) \rightarrow \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_2}),$$

and a homomorphism of \mathbb{E}_2 -algebras

$$f_{\text{HC}} : \text{HC}(\text{pt}/\mathcal{V}_1) \rightarrow \text{HC}(\text{pt}/\mathcal{V}_2).$$

It is easy to see that the corresponding homomorphism of graded algebras

$$\text{HH}^\bullet(\text{pt}/\mathcal{V}_1) \rightarrow \text{HH}^\bullet(\text{pt}/\mathcal{V}_2)$$

corresponds to the homomorphism

$$g^* : \Gamma(V_1^*, \mathcal{O}_{V_1^*}) \rightarrow \Gamma(V_2^*, \mathcal{O}_{V_2^*})$$

under the isomorphism

$$\text{HH}^\bullet(\text{pt}/\mathcal{V}_i) \simeq \text{Sym}(V_i) \simeq \Gamma(V_i^*, \mathcal{O}_{V_i^*}) \quad (i = 1, 2).$$

In terms of the equivalence of (5.2), the functor $(f_{\mathcal{G}})_*^{\text{IndCoh}}$ corresponds to the extension of scalars functor

$$\text{HC}(\text{pt}/\mathcal{V}_1)^{\text{op-mod}} \rightarrow \text{HC}(\text{pt}/\mathcal{V}_2)^{\text{op-mod}}.$$

5.2.4. Consider now the right adjoint functor

$$(f_{\mathcal{G}})^! : \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_2}) \rightarrow \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_1}),$$

which can also be thought of as the forgetful functor

$$\text{HC}(\text{pt}/\mathcal{V}_2)^{\text{op-mod}} \rightarrow \text{HC}(\text{pt}/\mathcal{V}_1)^{\text{op-mod}} \quad (5.4)$$

along the homomorphism f_{HC} .

We have:

Proposition 5.2.5 (a) *Suppose the support of $\mathcal{F}_2 \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_2})$ equals $Y_2 \subset V_2^*$. Then, the support of*

$$(f_{\mathcal{G}})^!(\mathcal{F}_2) \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_1})$$

equals $\overline{g(Y_2)} \subset V_1^$.*

(b) *Suppose the support of $\mathcal{F}_1 \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_1})$ equals $Y_1 \subset V_1^*$. Then, the support of*

$$(f_{\mathcal{G}})_*^{\text{IndCoh}}(\mathcal{F}_1) \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_2})$$

is contained in $g^{-1}(Y_1) \subset V_2^$.*

(b') *The support in (b) equals all of $g^{-1}(Y_1)$ if $Y_1 \subset g(V_2^*)$.*

(b'') *The support in (b) equals all of $g^{-1}(Y_1)$ if $\mathcal{F}_1 \in \text{Coh}(\mathcal{G}_{\text{pt}/\mathcal{V}_1})$.*

Remark 5.2.6 As we will see in the proof, point (b'') of the proposition is the only non-tautological one, but it is not essential for the main results of this paper.

Proof By Lemma 5.1.4, the statement reduces the corresponding assertion about modules over symmetric algebras. Let $f : W_1 \rightarrow W_2$ be a map of finite-dimensional vector spaces, and consider the corresponding homomorphism

$$\text{Sym}(W_1[-2]) \rightarrow \text{Sym}(W_2[-2]).$$

Let M_2 be an object of $\text{Sym}(W_2[-2])\text{-mod}$, and let M_1 be its image under the forgetful functor

$$\text{Sym}(W_2[-2])\text{-mod} \rightarrow \text{Sym}(W_1[-2])\text{-mod}.$$

It is clear that

$$\text{supp}_{\text{Sym}(W_1)}(H^\bullet(M_1)) \subset W_1^*$$

equals the closure of the image of

$$\text{supp}_{\text{Sym}(W_2)}(H^\bullet(M_2)) \subset W_2^*$$

under the map $g : W_2^* \rightarrow W_1^*$. This proves point (a) of the proposition.

Let now M_1 be an object of $\text{Sym}(W_1[-2])\text{-mod}$ and set

$$M_2 := \text{Sym}(W_2[-2]) \otimes_{\text{Sym}(W_1[-2])} M_1.$$

First, it is clear that

$$\text{supp}_{\text{Sym}(W_2)}(H^\bullet(M_2)) \subset g^{-1} \left(\text{supp}_{\text{Sym}(W_1)}(H^\bullet(M_1)) \right). \quad (5.5)$$

It remains to show that the above containment is an equality if either

$$\mathrm{supp}_{\mathrm{Sym}(W_1)}(H^\bullet(M_1)) \subset g(W_2^*)$$

or if M_1 is perfect.

Both assertions are easy when g is surjective. Moreover, if they hold for two composable maps f' and f'' , then they hold for their composition. This allows us to reduce the statement to the case when g is an embedding of codimension one. Thus, let us assume that W_2 is the cokernel of $t : k \rightarrow W_1$.

Denote $N_1 := H^\bullet(M_1)$, viewed as an object in $\mathrm{Sym}(W_1)\text{-mod}^\heartsuit$. Denote

$$N_2 := \mathrm{Sym}(W_2) \otimes_{\mathrm{Sym}(W_1)} N_1 \in \mathrm{Sym}(W_2)\text{-mod}.$$

The object N_2 has cohomologies in two degrees:

$$N_2' := H^0(N_2) = \mathrm{coker}(t : N_1 \rightarrow N_1) \quad \text{and} \quad N_2'' := H^{-1}(N_2) = \mathrm{ker}(t : N_1 \rightarrow N_1).$$

We claim that

$$\mathrm{supp}_{\mathrm{Sym}(W_2)}(H^\bullet(M_2)) = \mathrm{supp}_{\mathrm{Sym}(W_2)}(N_2') \cup \mathrm{supp}_{\mathrm{Sym}(W_2)}(N_2'').$$

This follows from the short exact sequence in $\mathrm{Sym}(W_2)\text{-mod}^\heartsuit$

$$0 \rightarrow N_2' \rightarrow H^\bullet(M_2) \rightarrow N_2'' \rightarrow 0.$$

Now, if

$$\mathrm{supp}_{\mathrm{Sym}(W_1)}(H^\bullet(M_1)) = \mathrm{supp}_{\mathrm{Sym}(W_1)}(N_1) \subset W_2^*,$$

we have

$$\mathrm{supp}_{\mathrm{Sym}(W_1)}(N_1) = \mathrm{supp}_{\mathrm{Sym}(W_2)}(N_2') \cup \mathrm{supp}_{\mathrm{Sym}(W_2)}(N_2''),$$

which implies the desired equality in (5.5) in this case.

If M_1 is perfect, it is easy to see that the module N_1 is finitely generated (this is in fact a particular case of Theorem 4.1.8). Then, by the Nakayama Lemma,

$$\mathrm{supp}_{\mathrm{Sym}(W_2)}(N'_2) = \left(\mathrm{supp}_{\mathrm{Sym}(W_1)}(H^\bullet(M_1)) \right) \cap W_2^*,$$

which again implies the equality in (5.5). \square

Corollary 5.2.7 (a) $(f_{\mathcal{G}})^{\dagger} : \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_2}) \rightarrow \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_1})$ is conservative.

(b) Set $Y_{1,\mathrm{can}} = g(V_2^*) \subset V_1^*$. Then, the restriction of $(f_{\mathcal{G}})^{\mathrm{IndCoh}}_{*}$ to $(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_1}))_{Y_{1,\mathrm{can}}}$ is conservative.

Proof Follows immediately from Proposition 5.2.5 (It is easy also to give a direct proof.) \square

Corollary 5.2.8 (a) For a conical Zariski-closed subset $Y_1 \subset V_1^*$, set $Y_2 := g^{-1}(Y_1) \subset V_2^*$. Then, the essential image of $(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_1}))_{Y_1}$ under the functor $(f_{\mathcal{G}})^{\mathrm{IndCoh}}_{*}$ is contained in the subcategory

$$(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_2}))_{Y_2} \subset \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_2})$$

and generates it.

(b) Suppose f is a smooth morphism at pt , so that g is injective. For a conical Zariski-closed subset $Y_2 \subset V_2^*$, set $Y_1 := g(Y_2) \subset V_1^*$. Then, the essential image of $(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_2}))_{Y_2}$ under the functor $(f_{\mathcal{G}})^{\dagger}$ is contained in the subcategory

$$(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_1}))_{Y_1} \subset \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_1})$$

and generates it.

Proof For part (a), note that we have a pair of adjoint functors

$$(f_{\mathcal{G}})^{\mathrm{IndCoh}}_{*} : \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_1}) \rightleftarrows \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_2}) : (f_{\mathcal{G}})^{\dagger}.$$

By Proposition 5.2.5, they restrict to a pair of adjoint functors

$$(f_{\mathcal{G}})^{\mathrm{IndCoh}}_{*} : \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_1})_{Y_1} \rightleftarrows \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_2})_{Y_2} : (f_{\mathcal{G}})^{\dagger}.$$

Since $(f_{\mathcal{G}})^{\dagger}$ is conservative by Corollary 5.2.7(a), the claim follows (Note that the categories involved are compactly generated.)

For part (b), note that $f_{\mathcal{G}}$ is quasi-smooth; therefore, we have a pair of adjoint functors

$$(f_{\mathcal{G}})^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_2}) \rightleftarrows \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}_1}) : (f_{\mathcal{G}})^{\mathrm{IndCoh}}_{*}.$$

Moreover, $f_{\mathcal{G}}^{\text{IndCoh},*}$ can be obtained from $f_{\mathcal{G}}^!$ by twisting by a cohomologically shifted line bundle (by Corollary 2.2.7). By Proposition 5.2.5, restriction produces a pair of adjoint functors

$$(f_{\mathcal{G}})^{\text{IndCoh},*} : \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_2})_{Y_2} \rightleftarrows \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_1})_{Y_1} : (f_{\mathcal{G}})_{*}^{\text{IndCoh}}.$$

Note that $Y_1 \subset g(V_2^*) = Y_{1,\text{can}}$, so by Corollary 5.2.7(b'), $(f_{\mathcal{G}})_{*}^{\text{IndCoh}}$ is conservative on $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}_1})_{Y_1}$. The claim follows. \square

5.2.9. The particular usefulness of Proposition 5.2.5 and Corollaries 5.2.7 and 5.2.8 for us is explained by the following observation:

Lemma 5.2.10 (a) *Let Z be a quasi-smooth DG scheme such that ${}^{cl}Z \simeq \text{pt}$. Then, Z is (non-canonically) isomorphic to $\text{pt} \times_{\mathcal{V}} \text{pt}$ for some smooth classical scheme \mathcal{V} .*
 (b) *If $Z_i = \text{pt} \times_{V_i} \text{pt}$, $i = 1, 2$ where V_i are vector spaces, then any map $Z_1 \rightarrow Z_2$ can be realized as coming from a linear map $V_1 \rightarrow V_2$.*

Proof We have

$$\mathcal{O}_Z \simeq C(L),$$

where L is the DG Lie algebra $T_z(Z)[-1]$, where z is the unique k -point of Z . By assumption, L has only cohomology in degree $+2$. Hence,

$$H^{\bullet}(\mathcal{O}_Z) \simeq \text{Sym}(V[1]),$$

where V is the vector space dual to $H^2(L)$. This implies that \mathcal{O}_Z is itself non-canonically isomorphic to $\text{Sym}(V[1])$. This establishes point (a).

Point (b) follows from the fact that the space of maps of commutative DG algebras

$$\text{Sym}(V_1[1]) \rightarrow \text{Sym}(V_2[1])$$

is isomorphic to

$$\text{Maps}(V_1[1], \text{Sym}(V_2[1])).$$

In particular, the set of homotopy classes of such maps is in bijection with $\text{Hom}(V_1, V_2)$. \square

5.3 Singular support via Koszul duality

In this subsection, we let Z be a quasi-smooth DG scheme, presented as a fiber product in the category of DG schemes

$$\begin{array}{ccc} Z & \xrightarrow{\iota} & \mathcal{U} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathcal{V}, \end{array} \quad (5.6)$$

with smooth \mathcal{U} and \mathcal{V} , as in Sect. 2.3.5. We will also assume that \mathcal{U} and \mathcal{V} are affine.

5.3.1. Note that we have a Cartesian diagram

$$\begin{array}{ccccc} & & Z \times Z & & \\ & & \mathcal{U} & & \\ & \swarrow & \downarrow & \searrow & \\ Z & & \text{pt} \times \text{pt} & & Z \\ \downarrow & & \mathcal{V} & & \downarrow \\ \text{pt} & & \mathcal{V} & & \text{pt}. \end{array} \quad (5.7)$$

In particular, we obtain that the group DG scheme $\mathcal{G}_{\text{pt}/\mathcal{V}}$ canonically acts on Z , preserving its map to \mathcal{U} .

In other words, we have a canonical isomorphism of groupoids

$$\mathcal{G}_{Z/\mathcal{U}} \simeq \mathcal{G}_{\text{pt}/\mathcal{V}} \times Z$$

acting on Z , commuting with the map to \mathcal{U} :

$$\begin{array}{ccccc} & & \mathcal{G}_{Z/\mathcal{U}} & & \\ & & \downarrow \sim & & \\ & & \mathcal{G}_{\text{pt}/\mathcal{V}} \times Z & & \\ & \swarrow & \downarrow & \searrow & \\ Z & & \mathcal{G}_{\text{pt}/\mathcal{V}} \times Z & & Z \\ \downarrow \text{id} & \swarrow \text{pr} & \downarrow \text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z} & \searrow \text{id} & \downarrow \text{id} \\ Z & & \mathcal{G}_{\text{pt}/\mathcal{V}} \times Z & & Z. \end{array}$$

Here, $\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z} : \mathcal{G}_{\text{pt}/\mathcal{V}} \times Z \rightarrow Z$ is the action morphism.

5.3.2. In particular, we obtain a canonical homomorphism of monoidal categories

$$\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{QCoh}(\mathcal{U}) \rightarrow \mathrm{IndCoh}(\mathcal{G}_{Z/\mathcal{U}}), \quad (5.8)$$

and hence a homomorphism of \mathbb{E}_2 -algebras

$$A := \mathrm{HC}(\mathrm{pt}/\mathcal{V}) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \rightarrow \mathrm{HC}(Z/U) \rightarrow \mathrm{HC}(Z), \quad (5.9)$$

where $\mathrm{HC}(Z/U) := \mathcal{A}_{\mathcal{G}_{Z/U}}$, see Sect. F.4.10.

Note that

$$A = \bigoplus_k H^{2k}(A) = \mathrm{HH}^{\mathrm{even}}(\mathrm{pt}/\mathcal{V}) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) = \mathrm{Sym}(V) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}).$$

5.3.3. Thus, we find ourselves in the paradigm of Sect. 3.5.1 with the DG category in question being $\mathrm{IndCoh}(Z)$.

In particular, to an object $\mathcal{F} \in \mathrm{IndCoh}(Z)$, we can assign a conical Zariski-closed subset

$$\mathrm{supp}_A(\mathcal{F}) \subset \mathrm{Spec}(A) = V^* \times \mathcal{U}.$$

The following lemma shows that this recovers the singular support of \mathcal{F} .

Lemma 5.3.4 *For any \mathcal{F} , the support $\mathrm{supp}_A(\mathcal{F}) \subset V^* \times \mathcal{U}$ is the image of*

$$\mathrm{SingSupp}(\mathcal{F}) \subset \mathrm{Sing}(Z)$$

under the embedding $\mathrm{Sing}(Z) \hookrightarrow V^ \times Z \hookrightarrow V^* \times \mathcal{U}$, where the first map is given by (2.1).*

Proof It suffices to verify that the diagram

$$\begin{array}{ccc} \mathrm{Sym}(V) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) & \longrightarrow & \Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)}) \\ & \searrow & \swarrow \\ & \mathrm{HH}^{\mathrm{even}}(Z) & \end{array}$$

commutes. This is straightforward. □

From Proposition 3.5.5, we obtain:

Corollary 5.3.5 *For a conical Zariski-closed subset $Y \subset \mathrm{Sing}(Z)$, we have:*

$$\mathrm{IndCoh}_Y(Z) \simeq \mathrm{IndCoh}(Z) \otimes_{\mathrm{HC}(\mathrm{pt}/\mathcal{V})\text{-mod} \otimes \mathrm{QCoh}(\mathcal{U})} (\mathrm{HC}(\mathrm{pt}/\mathcal{V})\text{-mod} \otimes \mathrm{QCoh}(\mathcal{U}))_Y.$$

5.4 Koszul duality in the parallelized situation

In this subsection, we will assume that the formal completion of \mathcal{V} at pt has been parallelized, i.e., that it is identified with the formal completion of V at 0 .

5.4.1. In this case, we have:

Lemma 5.4.2 *The \mathbb{E}_2 -algebra structure $\text{HC}^\bullet(\text{pt}/\mathcal{V})$ is canonically commutative (i.e., comes by restriction from a canonically defined \mathbb{E}_∞ -structure), and, as such, identifies with $\text{Sym}(V[-2])$.*

Proof The \mathbb{G}_m -action on V by dilations gives rise to a \mathbb{G}_m -action on the \mathbb{E}_2 -algebra $\text{HC}(\text{pt}/\mathcal{V})$. The computation of the cohomology of $\text{HC}(\text{pt}/\mathcal{V})$ puts us in the framework of Sect. 3.6.1. \square

Corollary 5.4.3 *A parallelization of \mathcal{V} upgrades the monoidal structure on the category $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \simeq \text{HC}(\text{pt}/\mathcal{V})^{\text{op-mod}}$ to a symmetric monoidal structure, and as such it is canonically equivalent to $\text{Sym}(V[-2])\text{-mod}$.*

5.4.4. Thus, we see that in the parallelized situation, we can use Corollary 3.6.5 to study support in the category $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})$ as follows.

The category

$$\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \simeq \text{HC}(\text{pt}/\mathcal{V})^{\text{op-mod}} \simeq \text{Sym}(V[-2])\text{-mod}$$

is naturally a module category over $\text{QCoh}(V^*/\mathbb{G}_m)$. For a conical Zariski-closed subset $Y \subset V^*$, we have:

Corollary 5.4.5

$$\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})_Y = \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes_{\text{QCoh}(V^*/\mathbb{G}_m)} \text{QCoh}(V^*/\mathbb{G}_m)_Y.$$

5.4.6. Let Z be as in Sect. 5.3. By Sect. 3.6.1, the category $\text{IndCoh}(Z)$ carries an action of the monoidal category $\text{QCoh}(\mathcal{S}_A)$ for the stack

$$\mathcal{S}_A = \text{Spec}(\text{Sym}(V) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}))/\mathbb{G}_m = V^*/\mathbb{G}_m \times \mathcal{U}.$$

This allows us to study the singular support of objects in $\text{IndCoh}(Z)$ using Corollaries 3.6.5 and 3.6.8. In particular, we have:

Corollary 5.4.7 *For a conical Zariski-closed subset $Y \subset \text{Sing}(Z)$, we have:*

$$\text{IndCoh}_Y(Z) \simeq \text{IndCoh}(Z) \otimes_{\text{QCoh}(V^*/\mathbb{G}_m \times \mathcal{U})} \text{QCoh}(V^*/\mathbb{G}_m \times \mathcal{U})_Y.$$

In the rest of this section, we will use the viewpoint on singular support via Koszul duality and prove some results stated earlier in the paper.

5.5 Proof of Proposition 4.7.2

Recall that Proposition 4.7.2 says that for $\mathcal{F} \in \text{Coh}(Z)$, its singular support equals that of $\mathbb{D}_Z^{\text{Serre}}(\mathcal{F})$.

5.5.1. With no restriction of generality, we can assume that Z is affine and fits into the Cartesian square (5.6). By Corollary 5.3.4, it suffices to show that the supports of \mathcal{F} and $\mathbb{D}_Z^{\text{Serre}}(\mathcal{F})$ for the action of the graded algebra $A := \text{Sym}(V) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ are equal, where the grading on $\text{Sym}(V)$ is such that $\deg(V) = 2$.

5.5.2. By Lemma 3.4.4, it suffices to show that the supports of the A -modules $\text{Hom}_{\text{Coh}(Z)}^{\bullet}(\mathcal{F}, \mathcal{F})$ and $\text{Hom}_{\text{Coh}(Z)}^{\bullet}(\mathbb{D}_Z^{\text{Serre}}(\mathcal{F}), \mathbb{D}_Z^{\text{Serre}}(\mathcal{F}))$ are equal. This follows from the next assertion:

Lemma 5.5.3 *The diagram*

$$\begin{array}{ccc} \text{Sym}(V) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) & \longrightarrow & \text{Hom}_{\text{Coh}(Z)}^{\bullet}(\mathcal{F}, \mathcal{F}) \\ \downarrow & & \downarrow \\ \text{Sym}(V) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) & \longrightarrow & \text{Hom}_{\text{Coh}(Z)}^{\bullet}(\mathbb{D}_Z^{\text{Serre}}(\mathcal{F}), \mathbb{D}_Z^{\text{Serre}}(\mathcal{F})) \end{array}$$

is commutative, where the right vertical arrow is the isomorphism given by the anti-equivalence

$$\mathbb{D}_Z^{\text{Serre}} : \text{Coh}(Z)^{\text{op}} \rightarrow \text{Coh}(Z),$$

and the left vertical arrow is the automorphism that acts as identity on $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ and as -1 on $V \subset \text{Sym}(V)$.

5.5.4. Since the action map $\mathcal{G}_{\text{pt}/\mathcal{V}} \times Z \rightarrow Z$ is proper, and hence commutes with Serre duality, the assertion of Lemma 5.5.3 follows from the next one:

Lemma 5.5.5 *The equivalence $\mathbb{D}_{\mathcal{G}_{\text{pt}/\mathcal{V}}}^{\text{Serre}} : \text{Coh}(\mathcal{G}_{\text{pt}/\mathcal{V}})^{\text{op}} \rightarrow \text{Coh}(\mathcal{G}_{\text{pt}/\mathcal{V}})$ induces the automorphism of*

$$\text{Hom}_{\mathcal{G}_{\text{pt}/\mathcal{V}}}^{\bullet} \left((\Delta_{\text{pt}})_{*}^{\text{IndCoh}}(k), (\Delta_{\text{pt}})_{*}^{\text{IndCoh}}(k) \right) \simeq \text{Sym}(V[-2]),$$

given by $v \mapsto -v : V \rightarrow V$.

5.6 Constructing objects with a given singular support

The material in this subsection is not strictly speaking necessary for the rest of the paper.

Let Z fit into a Cartesian square as in (5.6). In this subsection, we will give an explicit procedure for producing compact objects in $\text{IndCoh}_Y(Z)$.

5.6.1. Consider the action of the group DG scheme $\mathcal{G}_{\text{pt}/\mathcal{V}}$ on Z as in Sect. 5.3.1, and recall that $\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z}$ denotes the corresponding action map

$$\mathcal{G}_{\text{pt}/\mathcal{V}} \times Z \rightarrow Z.$$

Clearly, the map $\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z}$ is proper. This gives rise to a pair of adjoint functors:

$$(\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z})_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}} \times Z) \rightleftarrows \text{IndCoh}(Z) : (\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z})^! \quad (5.10)$$

We will also use the notation

$$\mathbf{F} := (\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z})_*^{\text{IndCoh}} \text{ and } \mathbf{G} := (\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z})^!,$$

and identify

$$\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}} \times Z) \simeq \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z).$$

The functor \mathbf{G} is conservative because it admits a retract (given by the pullback along the unit of $\mathcal{G}_{\text{pt}/\mathcal{V}}$).

Therefore, the essential image of \mathbf{F} generates $\text{IndCoh}(Z)$.

5.6.2. Let us regard $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z)$ as a module category over $\text{QCoh}(\mathcal{U})$ via the second factor.

In addition, we can regard $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z)$ as acted on by $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})$ via the first factor.

Combining, we obtain that the monoidal category

$$\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{QCoh}(\mathcal{U})$$

acts on $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z)$.

In particular, we obtain a map of \mathbb{E}_2 -algebras

$$\mathcal{A} = \text{HC}(\text{pt}/\mathcal{V}) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \rightarrow \text{HC}(\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z)).$$

Thus, by Sect. 3.5.1, to an object

$$\mathcal{F} \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z)$$

we can assign its support

$$\text{supp}_{\mathcal{A}}(\mathcal{F}) \subset \text{Spec}(\mathcal{A}) \simeq V^* \times \mathcal{U}.$$

Conversely, a conical Zariski-closed subset $Y \subset V^* \times \mathcal{U}$ yields a full subcategory

$$(\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z))_Y = \{\mathcal{F} \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}} \times Z), \text{supp}_{\mathcal{A}}(\mathcal{F}) \subset Y\}.$$

5.6.3 Warning The above notation may seem abusive, because $\mathcal{G}_{\text{pt}/\mathcal{V}} \times Z$ is itself a quasi-smooth affine DG scheme, which has its own notion of singular support. Note that

$$\text{Sing}(\mathcal{G}_{\text{pt}/\mathcal{V}} \times Z) \simeq \text{Sing}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \times \text{Sing}(Z) \simeq V^* \times \text{Sing}(Z) \subset V^* \times V^* \times \mathcal{U}.$$

Thus, for

$$\mathcal{F} \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}} \times Z) \simeq \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z),$$

its singular support is a conical subset

$$\text{SingSupp}(\mathcal{F}) \subset V^* \times V^* \times \mathcal{U}.$$

It follows from Lemma 3.3.12 that $\text{supp}_A(\mathcal{F})$ is the closure of the projection $p_{13}(\text{SingSupp}(\mathcal{F}))$. To avoid confusion, we never consider this singular support for objects of

$$\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z)$$

and only deal with the “coarse” support contained in $V^* \times \mathcal{U}$.

5.6.4. It is clear that the functor \mathbf{F} is compatible with the action of the monoidal category $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{QCoh}(\mathcal{U})$ on $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z)$ given above, and the action of $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{QCoh}(\mathcal{U})$ on $\text{IndCoh}(Z)$ given by (5.8).

Hence, the functor \mathbf{G} , being the right adjoint of \mathbf{F} , is lax compatible with the above actions.

Lemma 5.6.5 *The lax compatibility of \mathbf{G} with the actions of $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{QCoh}(\mathcal{U})$ on $\text{IndCoh}(Z)$ and $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z)$ is strict.*

Proof It is easy to see that the monoidal category $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{QCoh}(\mathcal{U})$ is rigid (see [29], Sect. 6, where the notion of rigidity is discussed). Now, the assertion follows from the fact that if $\mathbf{F} : \mathbf{C}_1 \rightarrow \mathbf{C}_2$ is a functor between module categories over a rigid monoidal category, which admits a continuous right adjoint as a functor between plain DG categories, then the lax compatibility structure on this right adjoint is automatically strict. \square

From Sect. 3.3.10, we obtain:

Corollary 5.6.6 *Let $Y \subset V^* \times \mathcal{U}$ be a conical Zariski-closed subset. Then, the functors \mathbf{F} and \mathbf{G} restrict to an adjoint pair of functors*

$$(\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \otimes \text{IndCoh}(Z))_Y \rightleftarrows \text{IndCoh}_{Y \cap \text{Sing}(Z)}(Z).$$

Moreover, the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z) & \xrightleftharpoons{\quad} & \mathrm{IndCoh}(Z) \\ \downarrow & & \downarrow \Psi_Z^{Y,\mathrm{all}} \\ (\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_Y & \xrightleftharpoons{\quad} & \mathrm{IndCoh}_{Y \cap \mathrm{Sing}(Z)}(Z). \end{array}$$

commutes as well, where the left vertical arrow is the right adjoint to the inclusion

$$(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_Y \hookrightarrow \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z).$$

Corollary 5.6.7 *Suppose Y is a conical Zariski-closed subset of $\mathrm{Sing}(Z) \subset V^* \times \mathcal{U}$.*

(a) *For any $\mathcal{F} \in \mathrm{IndCoh}(Z)$, we have:*

$$\mathcal{F} \in \mathrm{IndCoh}_Y(Z) \Leftrightarrow \mathbf{G}(\mathcal{F}) \in (\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_Y.$$

(b) *The essential image under \mathbf{F} of the category $(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_Y$ generates $\mathrm{IndCoh}_Y(Z)$.*

Proof Both claims formally follow from the conservativeness of \mathbf{G} . Indeed, $\mathcal{F} \in \mathrm{IndCoh}_Y(Z)$ if and only if the natural morphism $\Psi_Z^{Y,\mathrm{all}}(\mathcal{F}) \rightarrow \mathcal{F}$ is an isomorphism. Since \mathbf{G} is conservative, this happens if and only if the morphism $\mathbf{G}(\Psi_Z^{Y,\mathrm{all}}(\mathcal{F})) \rightarrow \mathbf{G}(\mathcal{F})$ is an isomorphism; by Corollary 5.6.6, this is equivalent to $\mathbf{G}(\mathcal{F}) \in (\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_Y$. We have therefore proved part (a). Now note that the restriction

$$\mathbf{F} : (\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_Y \rightarrow \mathrm{IndCoh}_{Y \cap \mathrm{Sing}(Z)}(Z)$$

is left adjoint to a conservative functor; this proves part (b). \square

5.6.8. Note that by (5.2), we can identify the category $\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})$ with the category $\mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}\text{-mod}}$. Hence, we obtain an equivalence

$$\begin{aligned} \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z) &\simeq \mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}\text{-mod}} \otimes \mathrm{IndCoh}(Z) \\ &\simeq \mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}\text{-mod}}(\mathrm{IndCoh}(Z)). \end{aligned} \quad (5.11)$$

Using the equivalence (5.11), we can translate the functors (5.10) into the language of \mathbb{E}_2 -algebras. Recall homomorphisms of \mathbb{E}_2 -algebras

$$\mathrm{HC}(\mathrm{pt}/\mathcal{V}) \rightarrow \mathrm{HC}(Z/\mathcal{U}) \rightarrow \mathrm{HC}(Z).$$

Thus, any $\mathcal{F} \in \mathrm{IndCoh}(Z)$ carries a natural structure of $\mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}\text{-mod}}$ -module, i.e., we have a functor

$$\mathrm{IndCoh}(Z) \rightarrow \mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}\text{-mod}}(\mathrm{IndCoh}(Z)).$$

It follows from the definitions that this functor identifies with the functor \mathbf{G} .

Since $\mathrm{IndCoh}(Z)$ is tensored over $\mathrm{HC}(\mathrm{pt}/\mathcal{V})$, we obtain a functor

$$\mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}\text{-mod}} \otimes \mathrm{IndCoh}(Z) \rightarrow \mathrm{IndCoh}(Z).$$

This is our functor \mathbf{F} .

5.7 Proof of Theorem 4.2.6

The proof of Theorem 4.2.6, given in this subsection, relies on the material of Sect. 5.6. The reader who skipped Sect. 5.6 will find a proof of the most essential point of the argument in Remark 7.4.4. Yet another proof, which uses a different idea, is given in Sect. 6.1.4.

5.7.1. Let us recall that Theorem 4.2.6 asserts that for an affine quasi-smooth DG scheme Z , the essential image of

$$\Xi_Z : \mathrm{QCoh}(Z) \rightarrow \mathrm{IndCoh}(Z)$$

coincides with $\mathrm{IndCoh}_{\{0\}}(Z)$. Here by a slight abuse of notation, $\{0\}$ denotes the zero-section of $\mathrm{Sing}(Z)$. We are now ready to prove this theorem, using Corollary 5.6.7.

5.7.2. Note that the statement is local on Z . Indeed, recall that Ξ_Z is fully faithful and has a right adjoint

$$\Psi_Z : \mathrm{IndCoh}(Z) \rightarrow \mathrm{QCoh}(Z).$$

Theorem 4.2.6 is equivalent to conservativeness of the restriction

$$\Psi_Z|_{\mathrm{IndCoh}_{\{0\}}(Z)} : \mathrm{IndCoh}_{\{0\}}(Z) \rightarrow \mathrm{QCoh}(Z),$$

which can be verified locally. Thus, we may assume that Z fits into a Cartesian diagram (5.6).

By Corollary 5.6.7(b), it is enough to show that the essential image of Ξ_Z contains the essential image of the functor

$$\mathbf{F} : (\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_{\{0\} \times \mathcal{U}} \rightarrow \mathrm{IndCoh}_{\{0\}}(Z).$$

5.7.3. Consider the projection $p_{\mathcal{G}_{\mathrm{pt}/\mathcal{V}}} : \mathcal{G}_{\mathrm{pt}/\mathcal{V}} \rightarrow \mathrm{pt}$. We claim that the essential image of the functor

$$\begin{aligned} (p_{\mathcal{G}_{\mathrm{pt}/\mathcal{V}}} \times \mathrm{id}_Z)^! : \mathrm{IndCoh}(Z) &\rightarrow \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}} \times Z) \\ &\simeq \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z) \end{aligned}$$

is contained in the category $(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_{\{0\}}$ and generates it.

By Proposition 3.5.7,

$$(\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z))_{\{0\}} = \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})_{\{0\}} \otimes \mathrm{IndCoh}(Z).$$

So, it is sufficient to see that the essential image of

$$p_{\mathcal{G}_{\mathrm{pt}/\mathcal{V}}}^! : \mathrm{Vect} = \mathrm{IndCoh}(\mathrm{pt}) \rightarrow \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})$$

is contained in the category $\mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}})_{\{0\}}$ and generates it. However, this is a particular case of Corollary 5.2.8(b).

5.7.4. Hence, we obtain that it is sufficient to show that the essential image of the composed functor

$$\mathrm{IndCoh}(Z) \xrightarrow{(p_{\mathcal{G}_{\mathrm{pt}/\mathcal{V}}} \times \mathrm{id}_Z)^!} \mathrm{IndCoh}(\mathcal{G}_{\mathrm{pt}/\mathcal{V}}) \otimes \mathrm{IndCoh}(Z) \xrightarrow{\mathbf{F}} \mathrm{IndCoh}(Z) \quad (5.12)$$

is contained in the essential image of Ξ_Z .

We have the following assertion:

Lemma 5.7.5 *The composition (5.12) is canonically isomorphic to $\iota^! \circ \iota_*^{\mathrm{IndCoh}}$, where $\iota : Z \hookrightarrow \mathcal{U}$.*

Proof By the definition of the functor \mathbf{F} , the lemma follows by base change along the Cartesian square

$$\begin{array}{ccc} \mathcal{G}_{Z/\mathcal{U}} & \longrightarrow & Z \\ \downarrow & & \downarrow \iota \\ Z & \xrightarrow{\iota} & \mathcal{U}. \end{array}$$

□

5.7.6. By Lemma 5.7.5, it is sufficient to show that the essential image of the functor $\iota^!$ is contained in the essential image of Ξ_Z .

However, since \mathcal{U} is smooth, the monoidal action of $\mathrm{QCoh}(\mathcal{U})$ on $\omega_{\mathcal{U}} \in \mathrm{IndCoh}(\mathcal{U})$ generates the latter category. So, it is enough to show that $\iota^!(\omega_{\mathcal{U}})$ belongs to the essential image of Ξ_Z . However, $\iota^!(\omega_{\mathcal{U}}) = \omega_Z$, and the assertion follows from the fact that Z is Gorenstein.

6 A point-wise approach to singular support

6.1 The functor of enhanced fiber

6.1.1. Let Z be an affine DG scheme with a perfect cotangent complex, and let $i_z : \mathrm{pt} \hookrightarrow Z$ be a k -point.

Consider the functor

$$i_z^! : \text{IndCoh}(Z) \rightarrow \text{Vect}.$$

We claim that this functor can be naturally enhanced to a functor

$$i_z^{\text{enh},!} : \text{IndCoh}(Z) \rightarrow T_z(Z)[-1]\text{-mod}, \quad (6.1)$$

where the DG Lie algebra $T_z(Z)[-1]$ is the fiber of $T(Z)[-1]$ at z , and where we remind that $T(Z)[-1]$ is a Lie algebra by Corollary G.2.7.

Indeed, let us interpret $i_z^!$ as

$$\mathcal{M}aps_{\text{IndCoh}}(i_*^{\text{IndCoh}}(k), -). \quad (6.2)$$

Now, it is easy to see that the canonical action of the DG Lie algebra $\Gamma(Z, T(Z)[-1])$ on $i_*^{\text{IndCoh}}(k)$, given by Corollary G.2.7, factors through

$$k \otimes_{\Gamma(Z, \mathcal{O}_Z)} \Gamma(Z, T(Z)[-1]) \simeq T_z(Z)[-1].$$

This endows the functor in (6.2) with an action of the DG Lie algebra $T_z(Z)[-1]$, as desired.

We will refer to the functor (6.1) as that of *enhanced fiber*.

6.1.2. Let us reinstate the assumption that Z is quasi-smooth.

For an object $M \in T_z(Z)[-1]\text{-mod}$, we can consider the graded vector space of its cohomologies $H^\bullet(M)$ as a module over the graded Lie algebra $H^\bullet(T_z(Z)[-1]\text{-mod})$.

In particular, $H^\bullet(M)$ is a module over $\text{Sym}(H^1(T_z(Z)))$, viewed as a graded commutative algebra whose generators are placed in degree 2. Therefore, to M we can associate the support

$$\text{supp}_{\text{Sym}(H^1(T_z(Z)))}(H^\bullet(M)) \subset \text{Spec}\left(\text{Sym}\left(H^1(T_z(Z))\right)\right).$$

Note that

$$\text{Spec}\left(\text{Sym}\left(H^1(T_z(Z))\right)\right) \simeq \text{Sing}(Z)_{\{z\}} := {}^{cl}\left(\{z\} \times_Z \text{Sing}(Z)\right).$$

Lemma 6.1.3 For $\mathcal{F} \in \text{IndCoh}(Z)$, set $M = i_z^{\text{enh},!}(\mathcal{F}) \in T_z(Z)[-1]\text{-mod}$. Then,

- (a) $\text{SingSupp}(\mathcal{F}) \cap \text{Sing}(Z)_{\{z\}} \supset \text{supp}_{\text{Sym}(H^1(T_z(Z)))}(H^\bullet(M))$.
- (b) If $\mathcal{F} \in \text{IndCoh}(Z)_{\{z\}}$, then

$$\text{SingSupp}(\mathcal{F}) = \text{supp}_{\text{Sym}(H^1(T_z(Z)))}(H^\bullet(M)).$$

Proof With no restriction of generality, we can replace Z by an open affine that contains the point z .

Consider the graded vector space $H^\bullet(i_z^!(\mathcal{F}))$ as acted on by $\mathrm{HH}^{\mathrm{even}}(Z)$. Since the action of the subalgebra $\Gamma(Z, \mathcal{O}_{cl_Z}) \subset \mathrm{HH}^{\mathrm{even}}(Z)$ on this space factors through the morphism

$$\Gamma(Z, \mathcal{O}_{cl_Z}) \rightarrow k,$$

the action of $\mathrm{HH}^{\mathrm{even}}(Z)$ factors through the quotient

$${}^{cl}(\mathrm{HH}^{\mathrm{even}}(Z) \otimes_{\Gamma(Z, \mathcal{O}_{cl_Z})} k)$$

of $\mathrm{HH}^{\mathrm{even}}(Z)$. Here, k is considered as a $\Gamma(Z, \mathcal{O}_{cl_Z})$ -module via i_z .

Similarly, the resulting action of $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$ on $H^\bullet(i_z^!(\mathcal{F}))$ factors through

$$\Gamma(\mathrm{Sing}(Z)_{\{z\}}, \mathcal{O}_{\mathrm{Sing}(Z)_{\{z\}}}) \simeq \mathrm{Sym}\left(H^1(T_z(Z))\right),$$

which is equal to the action of the latter on

$$H^\bullet(i_z^!(\mathcal{F})) \simeq H^\bullet(M).$$

Now, point (a) of the lemma follows from the interpretation of $H^\bullet(i_z^!(\mathcal{F}))$ as

$$\mathrm{Hom}_{\mathrm{IndCoh}(Z)}^\bullet((i_z)_*(k), \mathcal{F}),$$

since $(i_z)_*(k)$ is compact in $\mathrm{IndCoh}(\mathcal{F})$ (see Lemma 3.4.4).

Suppose now that $\mathcal{F} \in \mathrm{IndCoh}(Z)_{\{z\}}$. Then, $\mathrm{SingSupp}(\mathcal{F})$ coincides with the support of \mathcal{F} computed using the action of $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$ on $\mathrm{IndCoh}(Z)_{\{z\}}$ (by Sect. 3.3.10). But $(i_z)_*(k)$ generates $\mathrm{IndCoh}(Z)_{\{z\}}$ (see [10, Proposition 4.1.7(b)]), and the required equality follows from Lemma 3.4.4. \square

6.1.4 An alternative proof of Theorem 4.2.6 Let us sketch an alternative proof of Theorem 4.2.6.

By Corollary 4.3.2, $\mathrm{IndCoh}_{\{0\}}(Z)$ is generated by $\mathrm{Coh}_{\{0\}}(Z)$. Therefore, it suffices to check that $\mathrm{Coh}_{\{0\}}(Z)$ coincides with the essential image $\Xi_Z(\mathrm{QCoh}(Z)^{\mathrm{perf}})$. It suffices to show that for $\mathcal{F} \in \mathrm{Coh}_{\{0\}}(Z)$ and every k -point z of Z , the object $i_z^!(\mathcal{F}) \in \mathrm{Vect}$ is perfect.

Consider the action of $\mathrm{Sym}(H^1(T_z(Z)))$ on $H^\bullet(i_z^!(\mathcal{F}))$. On the one hand, this module is finitely generated by Theorem 4.1.8. On the other hand, by assumption and Lemma 6.1.3(a), it is supported at

$$0 \in H^1(T_z(Z))^* = \mathrm{Spec}\left(\mathrm{Sym}(H^1(T_z(Z)))\right).$$

Hence, it is finite dimensional, as desired. \square

6.2 Estimates from below

Lemma 6.1.3 says that the support of $i_z^{\text{enh},!}(\mathcal{F})$ is bounded from above by the singular support of \mathcal{F} . In this subsection, we will prove some converse estimates.

6.2.1. The next assertion describes the singular support of an arbitrary object $\mathcal{F} \in \text{IndCoh}(Z)$ in terms of its $!$ -fibers.

Note that for every geometric point $i_z : \text{Spec}(k') \rightarrow Z$, we can consider the DG Lie algebra $T_z(Z)[-1] \in \text{Vect}_{k'}$ and the functor

$$i_z^{\text{enh},!} : \text{IndCoh}(Z) \rightarrow T_z(Z)[-1]\text{-mod.}$$

We can do this by viewing $Z' := Z \times_{\text{Spec}(k)} \text{Spec}(k')$ as a quasi-smooth DG scheme over k' and viewing z' as a k' -rational point $i_{z'} : \text{Spec}(k') \rightarrow Z'$. The functor $i_z^!$ is the composition of $i_{z'}^!$ preceded by the tensoring-up functor $\text{IndCoh}(Z) \rightarrow \text{IndCoh}(Z')$.

Proposition 6.2.2 *Let Y be a conical Zariski-closed subset $Y \subset \text{Sing}(Z)$. An object $\mathcal{F} \in \text{IndCoh}(Z)$ belongs to $\text{IndCoh}_Y(Z)$ if and only if for every geometric point $i_z : \text{Spec}(k') \rightarrow Z$, the object*

$$i_z^{\text{enh},!}(\mathcal{F}) \in U(T_z(Z)[-1])\text{-mod}$$

is such that the subset

$$\text{supp}_{\text{Sym}(H^1(T_z(Z)))} \left(H^\bullet(i_z^{\text{enh},!}(\mathcal{F})) \right) \subset \text{Sing}(Z)_{\{z\}}$$

is contained in

$$Y_{\{z\}} := \{z\} \times_Z Y \subset \text{Sing}(Z)_{\{z\}}.$$

Proof The “only if” direction was established in Lemma 6.1.3. Let us prove the “if” direction. With no restriction of generality, we can assume that Y is cut out by one homogeneous element $a \in \Gamma(\text{Sing}(Z), \mathcal{O}_{\text{Sing}(Z)})$. Set $2n = \deg(a)$.

Recall that we have a pair of adjoint functors

$$\Xi_Z^{Y,\text{all}} : \text{IndCoh}_Y(Z) \rightleftarrows \text{IndCoh}(Z) : \Psi_Z^{Y,\text{all}}.$$

Without loss of generality, we may replace \mathcal{F} with the cone of the morphism

$$\Xi_Z^{Y,\text{all}} \circ \Psi_Z^{Y,\text{all}}(\mathcal{F}) \rightarrow \mathcal{F}.$$

Note that this cone is the localization $\text{Loc}_a(\mathcal{F})$ introduced in Sect. 3.1.2. Thus, we can assume that a acts as an isomorphism

$$\mathcal{F} \xrightarrow{a} \mathcal{F}[2n].$$

Let us show that in this case $\mathcal{F} = 0$.

Fix a point z as above, and consider $H^\bullet(i_z^{\text{enh},!}(\mathcal{F}))$ as a quasi-coherent sheaf on $\text{Sing}(Z)_{\{z\}}$. On the one hand, the action of a on it is invertible; on the other, its support is contained in $Y_{\{z\}}$, which is the zero locus of $a \in \Gamma(\text{Sing}(Z)_{\{z\}}, \mathcal{O}_{\text{Sing}(Z)_{\{z\}}})$. Hence, the quasi-coherent sheaf vanishes and $i_z^!(\mathcal{F}) = 0$.

The assertion now follows from the next lemma (which in turn follows from [10, Proposition 4.1.7(a)]). \square

Lemma 6.2.3 *If $\mathcal{F} \in \text{IndCoh}(Z)$ is such that $i_z^!(\mathcal{F}) = 0$ for all geometric points z , then $\mathcal{F} = 0$.*

6.2.4 *The coherent case* We have the following variant of Proposition 6.2.2:⁶

Proposition 6.2.5 *For $\mathcal{F} \in \text{Coh}(Z)$, and $z \in Z(k)$, the inclusion*

$$\text{supp}_{\text{Sym}(H^1(T_z(Z)))} \left(H^\bullet(i_z^{\text{enh},!}(\mathcal{F})) \right) \subset \text{SingSupp}(\mathcal{F}) \cap \text{Sing}(Z)_{\{z\}}$$

is an equality.

We note that Proposition 6.2.5 is *not* essential for the main results of this paper.

Remark 6.2.6 We can reformulate Propositions 6.2.2 and 6.2.5 as follows. Fix $\mathcal{F} \in \text{IndCoh}(Z)$, and consider the union

$$Y' := \bigcup_{z \in Z} \text{supp}_{\text{Sym}(H^1(T_z(Z)))} \left(H^\bullet(i_z^{\text{enh},!}(\mathcal{F})) \right) \subset \text{Sing}(Z).$$

Here, the union is over all (not necessarily closed) points of Z . Proposition 6.2.2 says that

$$\text{SingSupp}(\mathcal{F}) = \overline{Y'}.$$

Proposition 6.2.5 says that $\text{SingSupp}(\mathcal{F}) = Y'$ for $\mathcal{F} \in \text{Coh}(Z)$.

6.2.7. The rest of this subsection is devoted to the proof of Proposition 6.2.5

Let f be a function on Z such that z belongs to the set of its zeros; let Z' denote the corresponding DG subscheme of Z ,

$$Z' := \text{pt} \times_{\mathbb{A}^1} Z,$$

where $Z \rightarrow \mathbb{A}^1$ is given by f .

Consider the closed subset

$$\text{Sing}(Z)_{Z'} := \text{cl}(\text{Sing}(Z) \times_Z Z') \subset \text{Sing}(Z).$$

For $\mathcal{F} \in \text{Coh}(Z)$, let $\mathcal{F}' \in \text{Coh}(Z)$ denote the object $\text{Cone}(f : \mathcal{F} \rightarrow \mathcal{F}) \in \text{Coh}(Z)$.

⁶ Which also follows from [4, Theorem 11.3 and Remark 11.4].

Taking into account Lemma 6.1.3(b), the statement of the proposition follows by induction from the next assertion:

Lemma 6.2.8 For $\mathcal{F} \in \text{Coh}(Z)$,

$$\text{SingSupp}(\mathcal{F}) \cap \text{Sing}(Z)_{Z'} = \text{SingSupp}(\mathcal{F}')$$

as subsets of $\text{Sing}(Z)_{Z'}$.

6.2.9 Proof of Lemma 6.2.8 The assertion is local, so with no restriction of generality, we can assume that Z fits into a Cartesian diagram as in (5.6).

Note that the map

$$\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z} : \mathcal{G}_{\text{pt}/\mathcal{V}} \times Z \rightarrow Z$$

is quasi-smooth; therefore, its Tor-dimension is bounded. Hence, for $\mathcal{F} \in \text{Coh}(Z)$, the object

$$\text{act}_{\mathcal{G}_{\text{pt}/\mathcal{V}}, Z}^!(\mathcal{F}) =: \mathbf{G}(\mathcal{F}) \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}} \times \text{IndCoh}(Z))$$

is compact by [10, Lemma 7.1.2].

Now, the required assertion follows from the next one:

Lemma 6.2.10 For a compact object $\mathcal{M} \in \text{Sym}(V[-2])\text{-mod} \otimes \text{IndCoh}(Z)$, the support of $\text{Cone}(f : \mathcal{M} \rightarrow \mathcal{M})$ in $V^* \times Z'$ equals

$$(V^* \times Z') \cap \text{supp}_{V^* \times Z}(\mathcal{M}).$$

The lemma is proved by the argument given in the proof of Proposition 5.2.5(b’’).

Remark 6.2.11 Denote by i the closed embedding $Z' \hookrightarrow Z$. Clearly, i is quasi-smooth. In particular, the map

$$\text{Sing}(i) : \text{Sing}(Z)_{Z'} \rightarrow \text{Sing}(Z')$$

is a closed embedding. Clearly, the object \mathcal{F}' above is canonically isomorphic to $i_*^{\text{IndCoh}}(i^!(\mathcal{F}))[1]$.

Thus, Lemma 6.2.8 computes the singular support of $i_*^{\text{IndCoh}}(i^!(\mathcal{F}))$.

More generally, for any morphism of quasi-smooth DG schemes $f : Z' \rightarrow Z$ and any $\mathcal{F} \in \text{Coh}(Z)$ (resp., $\mathcal{F}' \in \text{Coh}(Z')$), there is a formula for $\text{SingSupp}(i^!(\mathcal{F}))$ (resp., $\text{SingSupp}(i_*(\mathcal{F}'))$), assuming that f is finite, see Theorems 7.7.2 and 7.3.3, respectively.

6.3 Enhanced fibers and Koszul duality

Let $i_z : \text{pt} \rightarrow Z$ be a quasi-smooth DG scheme and a k -point. In this subsection, we will assume that Z is written as a fiber product

$$\begin{array}{ccc} Z & \xrightarrow{\iota} & \mathcal{U} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathcal{V}, \end{array} \quad (6.3)$$

as in (5.6).

6.3.1. The action of $\mathcal{G}_{\text{pt}/\mathcal{V}}$ on Z gives rise to a map of Lie algebras

$$V[-2] \otimes \mathcal{O}_Z \rightarrow T(Z)[-1]$$

in $\text{QCoh}(Z)$, where V is the tangent space to \mathcal{V} at pt . This follows from the functoriality of the construction in Corollary G.2.4 with respect to the groupoid.

In particular, we obtain a map of DG Lie algebras $V[-2] \rightarrow T_z(Z)[-1]$.

Composing, from $i_z^{\text{enh},!}$ we obtain a functor

$$\text{IndCoh}(Z) \rightarrow V[-2]\text{-mod}. \quad (6.4)$$

In this subsection, we will give a different interpretation of the functor (6.4).

6.3.2. Consider the morphism

$$(\text{id} \times (\iota \circ i_z)) : \mathcal{G}_{\text{pt}/\mathcal{V}} = \text{pt} \times_{\mathcal{V}} \text{pt} \rightarrow \text{pt} \times_{\mathcal{V}} \mathcal{U} = Z;$$

we denote it by $i_{z,\mathcal{V}}$. It can be viewed as the action of the group DG scheme $\mathcal{G}_{\text{pt}/\mathcal{V}}$ on the point z . It is easy to see that $i_{z,\mathcal{V}}$ is quasi-smooth.

Thus, restriction defines a functor

$$i_{z,\mathcal{V}}^! : \text{IndCoh}(Z) \rightarrow \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}).$$

Note that i_z can be written as a composition

$$i_z = i_{z,\mathcal{V}} \circ \Delta_{\text{pt}}, \quad (6.5)$$

where

$$\Delta_{\text{pt}} : \text{pt} \rightarrow \text{pt} \times_{\mathcal{V}} \text{pt} = \mathcal{G}_{\text{pt}/\mathcal{V}}$$

is the diagonal. Let us observe the following:

Lemma 6.3.3 For $\mathcal{F} \in \text{IndCoh}(Z)$, $i_z^!(\mathcal{F}) = 0$ if and only if $i_{z,\mathcal{V}}^!(\mathcal{F}) = 0$.

Proof Indeed, by (6.5), we have $i_z^! = \Delta_{\text{pt}}^! \circ i_{z,\mathcal{V}}^!$. Therefore, it suffices to check that the functor $\Delta_{\text{pt}}^!$ is conservative. Equivalently, we need to prove that the essential image of $(\Delta_{\text{pt}})_*^{\text{IndCoh}}$ generates the category $\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}})$. This follows from [10, Proposition 4.1.7(b)] (or, in the case at hand, from Corollary 5.2.8(a)). \square

6.3.4. Combining the functor $i_{z,\mathcal{V}}^!$ with the equivalence

$$\text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \simeq \text{Sym}(V[-2])\text{-mod} \simeq V[-2]\text{-mod},$$

we thus obtain a functor

$$\text{IndCoh}(Z) \rightarrow \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}) \rightarrow V[-2]\text{-mod}. \quad (6.6)$$

Proposition 6.3.5 The functors (6.6) and (6.4) are canonically isomorphic.

Proof The lemma easily reduces to the case when $Z = \mathcal{G}_{\text{pt}/\mathcal{V}}$, and z is given by Δ_{pt} . In this case, the assertion is tautological from the definitions. \square

6.3.6. As was mentioned in Remark 5.1.8, $\mathcal{G}_{\text{pt}/\mathcal{V}}$ is a quasi-smooth DG scheme and

$$\text{Sing}(\mathcal{G}_{\text{pt}/\mathcal{V}}) = V^*,$$

where V is the tangent space to \mathcal{V} at pt . In addition, for

$$\mathcal{F} \in \text{IndCoh}(\text{pt} \times_{\mathcal{V}} \text{pt}) \simeq \text{HC}(\text{pt}/\mathcal{V})^{\text{op}},$$

we have an equality of Zariski-closed conical subsets:

$$\text{SingSupp}(\mathcal{F}) = \text{supp}_{\text{Sym}(V[-2])}(\mathcal{F}) \subset V^*.$$

6.3.7. Note that the diagram (6.3) yields an embedding

$$\text{Sing}(Z)_{\{z\}} \hookrightarrow V^*,$$

which can be viewed as the singular codifferential of $i_{z,\mathcal{V}}$ (The fact that $\text{Sing}(i_{z,\mathcal{V}})$ is an embedding also follows from quasi-smoothness of $i_{z,\mathcal{V}}$ by Lemma 2.4.3.).

From Proposition 6.3.5, we obtain that for $\mathcal{F} \in \text{IndCoh}(Z)$, the support of $H^\bullet(i_z^!(\mathcal{F}))$ as a module over $\text{Sym}(H^1(T_z(Z)))$, considered as a subset of

$$\text{Spec}\left(\text{Sym}\left(H^1(T_z(Z))\right)\right) = \text{Sing}(Z)_{\{z\}} \subset V^*,$$

equals the singular support of

$$i_{z,\mathcal{V}}^!(\mathcal{F}) \in \text{IndCoh}(\mathcal{G}_{\text{pt}/\mathcal{V}}).$$

7 Functorial properties of the category $\text{IndCoh}_Y(Z)$

So far, we have studied the category $\text{IndCoh}_Y(Z)$ for a given quasi-smooth DG scheme Z . In this section, we will establish a number of results on how these categories interact under pullback and pushforward functors for maps between quasi-smooth DG schemes. The main results of this section are stated in the introduction as Theorems 1.3.11, 1.3.12, and 1.3.13, corresponding to Proposition 7.1.3, Corollary 7.6.2, and Theorem 7.8.2 below.

7.1 Behavior under direct and inverse images

Recall that a map f between DG schemes induces the following functors between the categories of ind-coherent sheaves: the “ordinary” pushforward, which is denoted by f_*^{IndCoh} (the notation f^* is reserved for the pushforward on the category of quasi-coherent sheaves), and the “extraordinary” pullback $f^!$ (Actually, we need f to be quasi-compact for f_*^{IndCoh} to exist.). If f is eventually coconnective, the “ordinary” pullback, denoted by $f^{\text{IndCoh},*}$, makes sense as well.

7.1.1. Let $f : Z_1 \rightarrow Z_2$ be a map between quasi-smooth DG schemes. Consider the functor

$$f^! : \text{IndCoh}(Z_2) \rightarrow \text{IndCoh}(Z_1),$$

(see [10, Sect. 5.2.3]) and, assuming that f is quasi-compact, the functor

$$f_*^{\text{IndCoh}} : \text{IndCoh}(Z_1) \rightarrow \text{IndCoh}(Z_2)$$

(see [10, Sect. 3.1]).

Recall also that if f is proper, the above functors $(f_*^{\text{IndCoh}}, f^!)$ are naturally adjoint.

7.1.2. Recall that f gives rise to the singular codifferential

$$\text{Sing}(f) : \text{Sing}(Z_2)_{Z_1} \rightarrow \text{Sing}(Z_1),$$

where

$$\text{Sing}(Z_2)_{Z_1} = {}^{cl} \left(\text{Sing}(Z_2) \times_{Z_2} Z_1 \right).$$

Proposition 7.1.3 *Let $Y_i \subset \text{Sing}(Z_i)$ be conical Zariski-closed subsets.*

(a) *Suppose*

$$\text{Sing}(f)(Y_2 \times_{Z_2} Z_1) \subset Y_1.$$

Then, $f^!$ sends $\text{IndCoh}_{Y_2}(Z_2)$ to $\text{IndCoh}_{Y_1}(Z_1)$.

(b) Suppose that f is quasi-compact, and that

$$\mathrm{Sing}(f)^{-1}(Y_1) \subset Y_2 \times_{Z_2} Z_1.$$

Then, f_*^{IndCoh} sends $\mathrm{IndCoh}_{Y_1}(Z_1)$ to $\mathrm{IndCoh}_{Y_2}(Z_2)$.

Proof First of all, in both claims we may assume that Z_1 and Z_2 are affine. Indeed, claim (a) is clearly local on both Z_1 and Z_2 . On the other hand, claim (b) is clearly local on Z_2 . By Corollary 4.5.6(b), it is also local on Z_1 : An open cover of Z_1 can be used to compute the direct image f_*^{IndCoh} using the Čech resolution.

Since the functor

$$f^! : \mathrm{IndCoh}(Z_2) \rightarrow \mathrm{IndCoh}(Z_1)$$

is continuous, it corresponds to an object of

$$\mathrm{IndCoh}(Z_2)^\vee \otimes \mathrm{IndCoh}(Z_1).$$

By Serre's duality,

$$\mathrm{IndCoh}(Z_2)^\vee \otimes \mathrm{IndCoh}(Z_1) \simeq \mathrm{IndCoh}(Z_2) \otimes \mathrm{IndCoh}(Z_1) \simeq \mathrm{IndCoh}(Z_1 \times Z_2),$$

and it is clear that $f^!$ corresponds to the object

$$\Gamma(f)_*^{\mathrm{IndCoh}}(\omega_{Z_1}) \in \mathrm{IndCoh}(Z_1 \times Z_2),$$

where $\Gamma(f) : Z_1 \rightarrow Z_1 \times Z_2$ is the graph of f . Similarly, the continuous functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z_1) \rightarrow \mathrm{IndCoh}(Z_2)$$

corresponds to the same object under the identification

$$\mathrm{IndCoh}(Z_1)^\vee \otimes \mathrm{IndCoh}(Z_2) \simeq \mathrm{IndCoh}(Z_1) \otimes \mathrm{IndCoh}(Z_2) \simeq \mathrm{IndCoh}(Z_1 \times Z_2).$$

Now the assertion follows from Proposition 3.5.9 and Lemma 7.1.4 below. \square

Lemma 7.1.4 *The singular support*

$$\mathrm{SingSupp}\left(\Gamma(f)_*^{\mathrm{IndCoh}}(\omega_{Z_1})\right) \subset \mathrm{Sing}(Z_1 \times Z_2) = \mathrm{Sing}(Z_1) \times \mathrm{Sing}(Z_2)$$

is contained in the image of

$$\mathrm{Sing}(Z_2)_{Z_1} = Z_1 \times_{Z_2} \mathrm{Sing}(Z_2)$$

under the natural map of the latter to $\mathrm{Sing}(Z_1) \times \mathrm{Sing}(Z_2)$.

Proof The statement is clearly local on both Z_1 and Z_2 , so we may assume that Z_1 and Z_2 are affine without losing generality. Since $\Gamma(f)_*^{\text{IndCoh}}(\omega_{Z_1})$ is compact, by Lemma 3.4.4(b), it is enough to show that the homomorphisms

$$\Gamma(\text{Sing}(Z_i), \mathcal{O}_{\text{Sing}(Z_i)}) \rightarrow \text{End}_{\text{IndCoh}(Z_1 \times Z_2)}^{\bullet} \left(\Gamma(f)_*^{\text{IndCoh}}(\omega_{Z_1}) \right)$$

for $i = 1, 2$ factor through a map

$$\Gamma(\text{Sing}(Z_2)_{Z_1}, \mathcal{O}_{\text{Sing}(Z_2)_{Z_1}}) \rightarrow \text{End}_{\text{IndCoh}(Z_1 \times Z_2)}^{\bullet} \left(\Gamma(f)_*^{\text{IndCoh}}(\omega_{Z_1}) \right)$$

and the natural homomorphisms

$$\Gamma(\text{Sing}(Z_i), \mathcal{O}_{\text{Sing}(Z_i)}) \rightarrow \Gamma(\text{Sing}(Z_2)_{Z_1}, \mathcal{O}_{\text{Sing}(Z_2)_{Z_1}}).$$

We have

$$\begin{aligned} \mathcal{M}aps_{\text{IndCoh}(Z_1 \times Z_2)} \left(\Gamma(f)_*^{\text{IndCoh}}(\omega_{Z_1}), \Gamma(f)_*^{\text{IndCoh}}(\omega_{Z_1}) \right) \\ \simeq \Gamma \left(Z_1, U_{\mathcal{O}_{Z_1}}(T(Z_2)[-1]|_{Z_1}) \right) \end{aligned}$$

(established in the course of the proof of Proposition G.2.2 due to the retraction $Z_1 \times Z_2 \rightarrow Z_1$).

Moreover, the homomorphisms of \mathbb{E}_1 -algebras

$$\text{HC}(Z_i) \rightarrow \mathcal{M}aps_{\text{IndCoh}(Z_1 \times Z_2)} \left(\Gamma(f)_*^{\text{IndCoh}}(\omega_{Z_1}), \Gamma(f)_*^{\text{IndCoh}}(\omega_{Z_1}) \right)$$

identify with the naturally defined maps

$$\Gamma \left(Z_i, U_{\mathcal{O}_{Z_i}}(T(Z_i)[-1]) \right) \rightarrow \Gamma \left(Z_1, U_{\mathcal{O}_{Z_1}}(T(Z_2)[-1]|_{Z_1}) \right).$$

This establishes the desired assertion. \square

7.1.5. Assume now that both Z_1 and Z_2 are quasi-compact. Recall (see [10, Sect. 9.2.3]) that under the self-duality

$$\mathbf{D}_{Z_i}^{\text{Serre}} : \text{IndCoh}(Z_i)^{\vee} \simeq \text{IndCoh}(Z_i),$$

the dual of the functor $f^!$ is f_*^{IndCoh} , and vice versa.

Hence, from Proposition 7.1.3 and Lemma 4.7.5, we obtain:

Proposition 7.1.6 *Let $Y_i \subset \text{Sing}(Z_i)$ be conical Zariski-closed subsets.*

(a) *Suppose*

$$\text{Sing}(f)(Y_2 \times_{Z_2} Z_1) \subset Y_1.$$

Then, we have a commutative diagram of functors:

$$\begin{array}{ccc} \mathrm{IndCoh}(Z_1) & \xrightarrow{\Psi_{Z_1}^{Y_1, \mathrm{all}}} & \mathrm{IndCoh}_{Y_1}(Z_1) \\ f_*^{\mathrm{IndCoh}} \downarrow & & \downarrow \\ \mathrm{IndCoh}(Z_2) & \xrightarrow{\Psi_{Z_2}^{Y_2, \mathrm{all}}} & \mathrm{IndCoh}_{Y_2}(Z_2). \end{array}$$

That is, the counter-clockwise composition functor factors through the colocalization $\Psi_{Z_1}^{Y_1, \mathrm{all}}$.

(b) Suppose that

$$\mathrm{Sing}(f)^{-1}(Y_1) \subset Y_2 \times_{Z_2} Z_1.$$

Then, we have a commutative diagram of functors:

$$\begin{array}{ccc} \mathrm{IndCoh}(Z_1) & \xrightarrow{\Psi_{Z_1}^{Y_1, \mathrm{all}}} & \mathrm{IndCoh}_{Y_1}(Z_1) \\ f^! \uparrow & & \uparrow \\ \mathrm{IndCoh}(Z_2) & \xrightarrow{\Psi_{Z_2}^{Y_2, \mathrm{all}}} & \mathrm{IndCoh}_{Y_2}(Z_2). \end{array}$$

That is, the clockwise composition functor factors through the colocalization $\Psi_{Z_2}^{Y_2, \mathrm{all}}$.

7.2 Singular support and preservation of coherence

In this subsection, all DG schemes will be quasi-compact.

7.2.1. Let Z be a quasi-smooth DG scheme. It turns out that the knowledge of the singular support of an object $\mathcal{F} \in \mathrm{Coh}(Z)$ allows one to predict when certain functors applied to it produce a coherent object. Namely, we will prove the following assertion:

Proposition 7.2.2 (a) For $\mathcal{F}', \mathcal{F}'' \in \mathrm{Coh}(Z)$ such that, set-theoretically,

$$\mathrm{SingSupp}(\mathcal{F}') \cap \mathrm{SingSupp}(\mathcal{F}'') \subset \{0\},$$

their internal Hom object

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(Z)}(\mathcal{F}', \mathcal{F}'') \in \mathrm{QCoh}(Z)$$

belongs to $\mathrm{Coh}(Z)$ (equivalently, is cohomologically bounded above).

(b) Under the assumptions of point (a), the tensor product

$$\mathcal{F}' \otimes \mathcal{F}'' \in \mathrm{QCoh}(Z)$$

belongs to $\mathrm{Coh}(Z)$ (equivalently, is cohomologically bounded below).

(c) Let $f : Z_1 \rightarrow Z_2$ be a morphism of quasi-smooth DG schemes. Let us denote by $\ker(\mathrm{Sing}(f)) \subset \mathrm{Sing}(Z_2)_{Z_1}$ the preimage of the zero section under

$$\mathrm{Sing}(f) : \mathrm{Sing}(Z_2)_{Z_1} \rightarrow \mathrm{Sing}(Z_1).$$

For any $\mathcal{F}_2 \in \mathrm{Coh}(Z_2)$ such that, set-theoretically,

$$\left(\mathrm{SingSupp}(\mathcal{F}_2) \times_{Z_2} Z_1 \right) \cap \ker(\mathrm{Sing}(f)) \subset \{0\},$$

we have $f^!(\mathcal{F}_2) \in \mathrm{Coh}(Z_1)$.

(d) Under the assumptions of point (c), we have $f^*(\mathcal{F}_2) \in \mathrm{Coh}(Z_1)$ (equivalently, $f^*(\mathcal{F}_2) \in \mathrm{QCoh}(Z_1)$ is bounded below).

(e) Under the assumptions of point (c), the partially defined left adjoint $f^{\mathrm{IndCoh},*}$ to

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z_1) \rightarrow \mathrm{IndCoh}(Z_2),$$

is defined on \mathcal{F}_2 .

Remark 7.2.3 One can show, mimicking the proof of Theorem 7.7.2 below, that the assertions in the above proposition are actually “if and only if.”

The rest of this subsection is devoted to the proof of the above proposition. First, we notice that all assertions are local in the Zariski topology, so we can assume that the DG schemes involved are affine.

7.2.4 Proof of point (a) Since Z is affine, it suffices to show that the graded vector space

$$\mathrm{Hom}^\bullet(\mathcal{F}', \mathcal{F}'')$$

is cohomologically bounded above.

By Theorem 4.1.8, $\mathrm{Hom}^\bullet(\mathcal{F}', \mathcal{F}'')$ is finitely generated as a module over $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$. Note that the $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$ -action on $\mathrm{Hom}^\bullet(\mathcal{F}', \mathcal{F}'')$ factors through both its action on $\mathrm{End}^\bullet(\mathcal{F}')$ and $\mathrm{End}^\bullet(\mathcal{F}'')$. Hence, we obtain that

$$\mathrm{supp}_{\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})}(\mathrm{Hom}^\bullet(\mathcal{F}', \mathcal{F}'')) \subset \mathrm{SingSupp}(\mathcal{F}') \cap \mathrm{SingSupp}(\mathcal{F}'') \subset \{0\}.$$

The latter implies that $\mathrm{Hom}^\bullet(\mathcal{F}', \mathcal{F}'')$ is finitely generated as a module over $\Gamma(Z, \mathcal{O}_Z)$. This implies the desired assertion. \square

7.2.5 Proof of point (c) Replacing Z_2 by $Z_1 \times Z_2$ and \mathcal{F}_2 by $\omega_{Z_1} \boxtimes \mathcal{F}_2$, we can assume that f is a closed embedding.

It suffices to show that $f^!(\mathcal{F}_2)$ is cohomologically bounded above. The latter is equivalent to

$$\mathrm{Hom}_{\mathrm{Coh}(Z_2)}^\bullet(f_*(\mathcal{O}_{Z_1}), \mathcal{F}_2)$$

living in finitely many cohomological degrees.

Now, Proposition 7.1.3(b) implies that $\mathrm{SingSupp}(f_*(\mathcal{O}_{Z_1}))$ is contained in the image of $\ker(\mathrm{Sing}(f))$ under the projection

$$\mathrm{Sing}(Z_2)_{Z_1} \rightarrow Z_2.$$

Therefore, the condition on $\mathrm{SingSupp}(\mathcal{F}_2)$ implies that

$$\mathrm{SingSupp}(\mathcal{F}_2) \cap \mathrm{SingSupp}(f_*(\mathcal{O}_{Z_1})) = \{0\}_{Z_2}.$$

Hence, the required assertion follows from point (a) of the proposition. \square

7.2.6 Proof of point (e) By Corollary 4.7.3, the object $\mathbb{D}_{Z_2}^{\mathrm{Serre}}(\mathcal{F}_2) \in \mathrm{Coh}(Z_2)$ satisfies the condition of point (c). We claim that the object

$$\mathbb{D}_{Z_1}^{\mathrm{Serre}}\left(f^!(\mathbb{D}_{Z_2}^{\mathrm{Serre}}(\mathcal{F}_2))\right) \in \mathrm{Coh}(Z_1)$$

satisfies the required adjunction property. Indeed, for $\mathcal{F}_1 \in \mathrm{IndCoh}(Z_1)$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{IndCoh}(Z_1)}(\mathbb{D}_{Z_1}^{\mathrm{Serre}}\left(f^!(\mathbb{D}_{Z_2}^{\mathrm{Serre}}(\mathcal{F}_2))\right), \mathcal{F}_1) &\simeq \langle f^!(\mathbb{D}_{Z_2}^{\mathrm{Serre}}(\mathcal{F}_2)), \mathcal{F}_1 \rangle_{\mathrm{IndCoh}(Z_1)} \\ &\simeq \langle \mathbb{D}_{Z_2}^{\mathrm{Serre}}(\mathcal{F}_2), f_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \rangle \simeq \mathrm{Hom}_{\mathrm{IndCoh}(Z_2)}(\mathcal{F}_2, f_*^{\mathrm{IndCoh}}(\mathcal{F}_1)), \end{aligned}$$

where

$$\langle -, - \rangle_{\mathrm{IndCoh}(Z_1)} \text{ and } \langle -, - \rangle_{\mathrm{IndCoh}(Z_2)}$$

denote the canonical pairings corresponding to the Serre duality equivalences

$$\mathbf{D}_{Z_i}^{\mathrm{Serre}} : \mathrm{IndCoh}(Z_i)^\vee \simeq \mathrm{IndCoh}(Z_i),$$

$i = 1, 2$. \square

7.2.7 Proof of point (d) Consider the object

$$f^{\mathrm{IndCoh},*}(\mathcal{F}_2) \in \mathrm{IndCoh}(Z_1),$$

whose existence is guaranteed by point (e). In particular, for $\mathcal{F}_1 \in \text{IndCoh}(\mathcal{F}_1)^+$ we have a functorial isomorphism

$$\text{Hom}_{\text{IndCoh}(Z_1)}(f^{\text{IndCoh},*}(\mathcal{F}_2), \mathcal{F}_1) \simeq \text{Hom}_{\text{IndCoh}(Z_2)}(\mathcal{F}_2, f_*^{\text{IndCoh}}(\mathcal{F}_1)).$$

By construction, $f^{\text{IndCoh},*}(\mathcal{F}_2) \in \text{Coh}(Z_1)$ (this also follows because it is the value on a compact object of a partially defined left adjoint to a continuous functor).

From the commutative diagram

$$\begin{array}{ccc} \text{IndCoh}(Z_1)^+ & \xrightarrow{\sim} & \text{QCoh}(Z_1)^+ \\ f^{\text{IndCoh},*} \downarrow & & \downarrow f^* \\ \text{IndCoh}(Z_2)^+ & \xrightarrow{\sim} & \text{QCoh}(Z_2)^+ \end{array}$$

we obtain an adjunction

$$\text{Hom}_{\text{QCoh}(Z_1)}(f^{\text{IndCoh},*}(\mathcal{F}_2), \mathcal{F}'_1) \simeq \text{Hom}_{\text{QCoh}(Z_2)}(\mathcal{F}_2, f_*(\mathcal{F}'_1))$$

for $\mathcal{F}'_1 \in \text{QCoh}(Z_1)^+$. Now, the fact that $\text{QCoh}(Z_i)$, $i = 1, 2$ is left-complete in its t-structure implies that the above adjunction remains valid for any $\mathcal{F}'_1 \in \text{QCoh}(Z_1)$. Hence, $f^{\text{IndCoh},*}(\mathcal{F}_2)$, viewed as an object of

$$\text{Coh}(Z_1) \subset \text{QCoh}(Z_1),$$

is isomorphic to $f^*(\mathcal{F}_2)$. In particular, the latter belongs to $\text{Coh}(Z_1)$, as desired. \square

7.2.8 Proof of point (b) This follows formally from point (d) applied to the diagonal morphism $Z \rightarrow Z \times Z$ and

$$\mathcal{F}' \boxtimes \mathcal{F}'' \in \text{Coh}(Z \times Z).$$

\square

7.3 Direct image for finite morphisms

In this subsection, we let $f : Z_1 \rightarrow Z_2$ be a finite morphism between quasi-smooth DG schemes (Z_1 and Z_2 need not be quasi-compact). For instance, f may be a closed embedding.

7.3.1. Define $Y_{1,\text{can}} \subset \text{Sing}(Z_1)$ to be the image of the singular codifferential

$$\text{Sing}(f) : \text{Sing}(Z_2)_{Z_1} = \text{Sing}(Z_2) \times_{Z_2} Z_1 \rightarrow \text{Sing}(Z_1).$$

Note that $Y_{1,\text{can}}$ is constructible, but not necessarily Zariski closed. If f is quasi-smooth, $\text{Sing}(f)$ is a closed embedding and $Y_{1,\text{can}}$ is closed.

7.3.2. Let \mathcal{F} be an object of $\text{IndCoh}(Z_1)$, and let $Y_1 \subset \text{Sing}(Z_1)$ be its singular support. Let $Y_2 \subset \text{Sing}(Z_2)$ be the conical Zariski-closed subset equal to the projection of

$$\text{Sing}(f)^{-1}(Y_1) \subset \text{Sing}(Z_2) \times_{Z_2} Z_1$$

under $p : \text{Sing}(Z_2) \times_{Z_2} Z_1 \rightarrow \text{Sing}(Z_2)$. It is automatically closed since p is finite and therefore proper.

Consider the object $f_*^{\text{IndCoh}}(\mathcal{F}) \in \text{IndCoh}(Z_2)$. Note that by Proposition 7.1.3(b), we have:

$$\text{SingSupp}(f_*^{\text{IndCoh}}(\mathcal{F})) \subset Y_2.$$

Theorem 7.3.3 (a) Suppose that $Y_1 \subset Y_{1,\text{can}}$. Then, $\text{SingSupp}(f_*^{\text{IndCoh}}(\mathcal{F})) = Y_2$.
 (b) Suppose that $\mathcal{F} \in \text{Coh}(Z_1)$. Then, $\text{SingSupp}(f_*^{\text{IndCoh}}(\mathcal{F})) = Y_2$.

Remark 7.3.4 Point (b) of the theorem will be used in Corollary 7.3.6 to give an explicit characterization of singular support of coherent sheaves, due to Drinfeld. However, it is not essential for the main results of this paper.

Proof As in Remark 6.2.6, consider the union

$$Y'_1 := \bigcup_{z_1 \in Z_1} \text{supp}_{\text{Sym}(H^1(T_{z_1}(Z_1)))} \left(H^\bullet(i_{z_1}^{\text{enh},!}(\mathcal{F})) \right) \subset \text{Sing}(Z_1).$$

By Proposition 6.2.2, $Y_1 = \overline{Y'_1}$. Similarly, consider the union

$$Y'_2 := \bigcup_{z_2 \in Z_2} \text{supp}_{\text{Sym}(H^1(T_{z_2}(Z_2)))} \left(H^\bullet(i_{z_2}^{\text{enh},!}(f_*^{\text{IndCoh}}(\mathcal{F}))) \right) \subset \text{Sing}(Z_2);$$

then $\text{SingSupp}(f_*^{\text{IndCoh}}(\mathcal{F})) = \overline{Y'_2}$. It suffices to verify that under the hypotheses of the theorem, Y'_2 is equal to the projection of $\text{Sing}(f)^{-1}(Y'_1)$.

Let us reduce the assertion of the theorem to the case when ${}^c Z_2$ is a single point. Let $z_2 \in Z_2$ be a point of Z_2 , which we may assume to be a k -point after extending scalars. Choose a quasi-smooth map

$$i_2 : Z'_2 \rightarrow Z_2,$$

as in Sect. 6.3.2, so that Z'_2 is a DG scheme of the form $\text{pt} \times_{\mathcal{V}_2} \text{pt}$, with \mathcal{V}_2 smooth, and such that the unique k -point of Z'_2 goes to z_2 .

Denote

$$Z'_1 := Z_1 \times_{Z_2} Z'_2,$$

and let i_1 denote the corresponding map $Z'_1 \rightarrow Z_1$. The map i_1 is also quasi-smooth by base change. Since Z_1 itself is quasi-smooth, we obtain that Z'_1 is quasi-smooth. Note also that Z'_1 is finite; therefore, by Lemma 5.2.10(a), Z'_1 is isomorphic to a finite disjoint union of DG schemes of the form $\text{pt} \times_{\mathcal{V}_1} \text{pt}$.

By Sect. 6.3.6, we know that

$$\begin{aligned} \text{SingSupp}(i_2^!(f_*^{\text{IndCoh}}(\mathcal{F}))) &= Y'_2 \cap \text{Sing}(Z_2)_{Z'_2} \subset \text{Sing}(Z_2)_{Z'_2} \\ &= \text{Sing}(Z_2)_{\{z_2\}} \subset \text{Sing}(Z'_2), \end{aligned}$$

and

$$\text{SingSupp}(i_1^!(\mathcal{F})) = Y'_1 \cap \text{Sing}(Z_1)_{Z'_1} \subset \text{Sing}(Z_1)_{Z'_1} \subset \text{Sing}(Z'_1).$$

Base change allows us to replace Z_1 , Z_2 , and \mathcal{F} by Z'_1 , Z'_2 , and $i_1^!(\mathcal{F})$, respectively. Note that $i_1^!(\mathcal{F})$ satisfies the hypotheses of the theorem (this relies on i_1 being eventually coconnective, so that $i_1^!$ preserves coherence).

Thus, we assume that ${}^{cl}Z_2$ is a single point. It suffices to check the claim with Z_1 replaced by each of its connected components, so we may assume that ${}^{cl}Z_1$ is a single point as well. Now the claim follows from Lemma 5.2.10(b) and Proposition 5.2.5(b' and b''). \square

7.3.5. From Theorem 7.3.3(b), we can derive an explicit characterization of singular support for objects of $\text{Coh}(Z) \subset \text{IndCoh}(Z)$.

Let (z, ξ) be a point of $\text{Sing}(Z)$, where $z \in Z(k)$ and $0 \neq \xi \in H^{-1}(T_z^*(Z))$. We would like to determine when this point belongs to $\text{SingSupp}(\mathcal{F})$ for a given $\mathcal{F} \in \text{Coh}(Z)$.

Let Z be written as

$$\begin{array}{ccc} Z & \xrightarrow{\iota} & \mathcal{U} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathcal{V}, \end{array}$$

with smooth \mathcal{U} and \mathcal{V} , as in Sect. 2.3.5.

Using the embedding $\text{Sing}(Z) \hookrightarrow V^* \times Z$, we can view ξ as a cotangent vector to \mathcal{V} at pt . Choose a function $\mathcal{V} \rightarrow \mathbb{A}^1$ that sends $\text{pt} \mapsto 0$, and whose differential equals ξ . Let Z' be the fiber product

$$\begin{array}{ccc} Z' & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathbb{A}^1. \end{array}$$

Let f denote the closed embedding $Z \hookrightarrow Z'$.

We have the following characterization of singular support, suggested to us by V. Drinfeld.

Corollary 7.3.6 *The element (z, ξ) belongs to $\text{SingSupp}(\mathcal{F})$ if and only if $f_*(\mathcal{F}) \in \text{Coh}(Z')$ is not perfect on a Zariski neighborhood of z .*

Proof Note that $\text{Sing}(Z')_{\{z\}} = \text{Span}(\xi)$.

Let us first prove the “only if” direction. As was mentioned above, Proposition 7.1.3(b) implies that if $(z, \xi) \notin \text{SingSupp}(\mathcal{F})$, then $(z, \xi) \notin \text{SingSupp}(f_*^{\text{IndCoh}}(\mathcal{F}))$. Hence, on a Zariski neighborhood of z , we have

$$\text{SingSupp}(f_*^{\text{IndCoh}}(\mathcal{F})) \subset \{0\}.$$

Therefore, by Theorem 4.2.6, $f_*^{\text{IndCoh}}(\mathcal{F})$ belongs to the essential image of the functor

$$\Xi_{Z'} : \text{QCoh}(Z') \rightarrow \text{IndCoh}(Z').$$

Now, the assertion follows from the following general lemma ([10, Lemma 1.5.8]):

Lemma 7.3.7 *For an eventually coconnective DG scheme Z , the intersection*

$$\text{Coh}(Z) \cap \Xi_Z(\text{QCoh}(Z)) \subset \text{IndCoh}(Z)$$

equals $\Xi_Z(\text{QCoh}(Z)^{\text{perf}})$.

Proof The assertion is local, so we can assume that Z is quasi-compact. Since the functor Ξ_Z is fully faithful and continuous, if $\Xi_Z(\mathcal{F})$ is compact in $\text{IndCoh}(Z)$, then \mathcal{F} is compact in $\text{QCoh}(Z)$, i.e., $\mathcal{F} \in \text{QCoh}(Z)^{\text{perf}}$. \square

For the “if” direction, assume that $(z, \xi) \in \text{SingSupp}(\mathcal{F})$. By Theorem 7.3.3, we obtain that (z, ξ) belongs to $\text{SingSupp}(f_*^{\text{IndCoh}}(\mathcal{F}))$, considered as an object of $\text{Coh}(Z')$. Hence, $f_*^{\text{IndCoh}}(\mathcal{F})$ is not perfect on any Zariski neighborhood of z by the easy direction in Theorem 4.2.6. \square

Remark 7.3.8 We note that the assertion of Corollary 7.3.6 makes sense also when $\xi = 0$. It is easy to adapt the proof to show that it is valid in this case as well.

7.3.9. Let $Y_1 \subset \text{Sing}(Z_1)$ be conical Zariski-closed subset, and assume that Y_1 is contained in the image of $\text{Sing}(Z_1)_{Z_2}$ under $\text{Sing}(f)$ (Recall that the image is constructible, but not necessarily closed.)

From Theorem 7.3.3(a), we obtain the following corollary.

Corollary 7.3.10 *The restricted functor*

$$f_*^{\text{IndCoh}}|_{\text{IndCoh}_{Y_1}(Z_1)} : \text{IndCoh}_{Y_1}(Z_1) \rightarrow \text{IndCoh}(Z_2)$$

is conservative.

7.4 Conservativeness for finite quasi-smooth maps

7.4.1. Let us remain in the setting of Sect. 7.3, and let us assume in addition that f is quasi-smooth. For instance, f could be a quasi-smooth closed embedding, so that Z_1 is a “locally complete intersection in Z_2 .”

In this case, $\text{Sing}(f)$ is a closed embedding. As before, let $Y_{1,\text{can}} \subset \text{Sing}(Z_1)$ be the image of $\text{Sing}(f)$, which is a conical Zariski-closed subset.

7.4.2. Recall now that f is eventually coconnective, so by Corollary 2.2.5, the functor f_*^{IndCoh} admits a left adjoint, $f^{\text{IndCoh},*}$. Moreover, f is Gorenstein by Corollary 2.2.7, so the functors $f^{\text{IndCoh},*}$ and $f^!$ can be obtained from one another by tensoring by a cohomologically shifted line bundle (see [10, Proposition 7.3.8]).

By Proposition 7.1.3(a), we have two pairs of adjoint functors

$$f_*^{\text{IndCoh}} : \text{IndCoh}_{Y_{1,\text{can}}}(Z_1) \rightleftarrows \text{IndCoh}(Z_2) : f^!$$

and

$$f^{\text{IndCoh},*} : \text{IndCoh}(Z_2) \rightleftarrows \text{IndCoh}_{Y_{1,\text{can}}}(Z_1) : f_*^{\text{IndCoh}}.$$

Proposition 7.4.3 *Suppose $f : Z_1 \rightarrow Z_2$ is a finite quasi-smooth morphism between quasi-smooth DG schemes. Then, the essential image of $\text{IndCoh}(Z_2)$ under the functor $f^!$ generates $\text{IndCoh}_{Y_{1,\text{can}}}(Z_1)$.*

Proof Since the functors $f^!$ and $f^{\text{IndCoh},*}$ differ by tensoring by a cohomologically shifted line bundle, the statement is equivalent to the claim that the restriction $f_*^{\text{IndCoh}}|_{\text{IndCoh}_{Y_{1,\text{can}}}(Z_1)}$ is conservative. This is a particular case of Corollary 7.3.10. \square

Remark 7.4.4 Proposition 7.4.3 is a generalization of Theorem 4.2.6. Indeed, if we assume that Z is a global complete intersection, it admits a quasi-smooth closed embedding $\iota : Z \rightarrow \mathcal{U}$, where \mathcal{U} is smooth. The key step in the proof of Theorem 4.2.6 (see Sect. 5.7) is to show that the essential image $\iota^!(\text{IndCoh}(\mathcal{U}))$ generates $\text{IndCoh}_{\{0\}}(Z)$. But this is exactly the assertion of Proposition 7.4.3 applied to ι .

7.4.5. Let now Y_2 be an arbitrary conical Zariski-closed subset of $\text{Sing}(Z_2)$. Let $Y_1 \subset \text{Sing}(Z_1)$ be the image of $Y_2 \times_{Z_2} Z_1$ under the singular codifferential

$$\text{Sing}(f) : \text{Sing}(Z_2)_{Z_1} \rightarrow \text{Sing}(Z_1).$$

By Proposition 7.1.3, we have two pairs of adjoint functors:

$$f_*^{\text{IndCoh}} : \text{IndCoh}_{Y_1}(Z_1) \rightleftarrows \text{IndCoh}_{Y_2}(Z_2) : f^!$$

and

$$f^{\text{IndCoh},*} : \text{IndCoh}_{Y_2}(Z_2) \rightleftarrows \text{IndCoh}_{Y_1}(Z_1) : f_*^{\text{IndCoh}}.$$

Then from Corollary 7.3.10, we obtain:

Corollary 7.4.6 *Under the above circumstances,*

- (a) *The functor $f_*^{\text{IndCoh}} : \text{IndCoh}_{Y_1}(Z_1) \rightarrow \text{IndCoh}_{Y_2}(Z_2)$ is conservative.*
- (b) *The essential image of $\text{IndCoh}_{Y_2}(Z_2)$ under $f^!$ (or under $f^{\text{IndCoh},*}$) generates $\text{IndCoh}_{Y_1}(Z_1)$.*

7.4.7 A digression Let $f : W_1 \rightarrow W_2$ be a locally eventually coconnective morphism and that W_2 is quasi-compact. Let $\text{QCoh}(W_2)$ act on $\text{IndCoh}(W_1)$ via the homomorphism of monoidal categories $f^* : \text{QCoh}(W_2) \rightarrow \text{QCoh}(W_1)$. The functors $f^!$ and $f^{\text{IndCoh},*}$ are $\text{QCoh}(W_2)$ -linear, and therefore induce two functors

$$\text{QCoh}(W_1) \otimes_{\text{QCoh}(W_2)} \text{IndCoh}(W_2) \rightrightarrows \text{IndCoh}(W_1).$$

Lemma 7.4.8 *Let $f : W_1 \rightarrow W_2$ be a locally eventually coconnective morphism with W_2 quasi-compact. Then, the functors*

$$\text{QCoh}(W_1) \otimes_{\text{QCoh}(W_2)} \text{IndCoh}(W_2) \rightrightarrows \text{IndCoh}(W_1)$$

induced by the functors $f^!$ and $f^{\text{IndCoh},}$ are fully faithful.*

Proof This is [10, Propositions 4.4.2 and 7.5.9]. □

7.4.9. Let $f : Z_1 \rightarrow Z_2$ be as above, with Z_1 (and hence Z_2) quasi-compact. As in Sect. 7.4.5, we let Y_2 be a conical Zariski-closed subset of $\text{Sing}(Z_2)$, and let $Y_1 \subset \text{Sing}(Z_1)$ be the image of $Y_2 \times_{Z_2} Z_1$ under $\text{Sing}(f)$.

Since the restriction

$$f^! : \text{IndCoh}_{Y_2}(Z_2) \rightarrow \text{IndCoh}_{Y_1}(Z_1)$$

is $\text{QCoh}(Z_2)$ -linear, it induces a functor

$$\text{QCoh}(Z_1) \otimes_{\text{QCoh}(Z_2)} \text{IndCoh}_{Y_2}(Z_2) \rightarrow \text{IndCoh}_{Y_1}(Z_1). \quad (7.1)$$

Corollary 7.4.10 *The functor (7.1) is an equivalence.*

Proof The functor is fully faithful by Lemma 7.4.8, and its essential image generates $\text{IndCoh}_{Y_1}(Z_1)$ by Corollary 7.4.6(b). □

7.5 Behavior under smooth morphisms

7.5.1. Let $f : Z_1 \rightarrow Z_2$ be a smooth map between DG schemes, and assume that Z_2 is quasi-compact. Recall (see [10, Proposition 4.5.3]) that the functor

$$f^{\text{IndCoh},*} : \text{IndCoh}(Z_2) \rightarrow \text{IndCoh}(Z_1)$$

gives rise to an equivalence of categories

$$\mathrm{IndCoh}(Z_2) \otimes_{\mathrm{QCoh}(Z_2)} \mathrm{QCoh}(Z_1) \rightarrow \mathrm{IndCoh}(Z_1).$$

Similarly, the functor $f^! : \mathrm{IndCoh}(Z_2) \rightarrow \mathrm{IndCoh}(Z_1)$ gives rise to an equivalence

$$\mathrm{IndCoh}(Z_2) \otimes_{\mathrm{QCoh}(Z_2)} \mathrm{QCoh}(Z_1) \rightarrow \mathrm{IndCoh}(Z_1) \quad (7.2)$$

see [10, Corollary 7.5.7].

7.5.2. Assume now that Z_2 (and, hence, Z_1) is quasi-smooth. Recall from Lemma 2.4.4 that in this case, the singular codifferential

$$\mathrm{Sing}(f) : \mathrm{Sing}(Z_2)_{Z_1} := {}^{cl}(\mathrm{Sing}(Z_2) \times_{Z_2} Z_1) \simeq \mathrm{Sing}(Z_2) \times_{Z_2} Z_1 \rightarrow \mathrm{Sing}(Z_1)$$

is an isomorphism.

Fix a conical Zariski-closed subset $Y_2 \subset \mathrm{Sing}(Z_2)$, and let $Y_1 \subset \mathrm{Sing}(Z_1)$ be the image

$$\mathrm{Sing}(f) \left(Y_2 \times_{Z_2} Z_1 \right).$$

We have:

Proposition 7.5.3 *Under the equivalence of (7.2), we have*

$$\mathrm{IndCoh}_{Y_2}(Z_2) \otimes_{\mathrm{QCoh}(Z_2)} \mathrm{QCoh}(Z_1) = \mathrm{IndCoh}_{Y_1}(Z_1)$$

as subcategories of $\mathrm{IndCoh}(Z_1)$.

Proof Since $\mathrm{QCoh}(Z_2)$ is rigid and $\mathrm{IndCoh}_{Y_2}(Z_2)$ is dualizable, the formation of

$$\mathrm{IndCoh}_{Y_2}(Z_2) \otimes_{\mathrm{QCoh}(Z_2)} -$$

commutes with limits (see [29, Corollary 4.3.2 and 6.4.2]). Hence, the assertion is local on Z_1 . Similarly, it is easy to see that the assertion is local on Z_2 .

Hence, by Corollary 2.1.7, we can assume that f fits into a commutative diagram

$$\begin{array}{ccc} Z_1 & \longrightarrow & \mathcal{U}_1 \\ f \downarrow & & \downarrow f_{\mathcal{U}} \\ Z_2 & \longrightarrow & \mathcal{U}_2 \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathcal{V}, \end{array}$$

where $\mathcal{U}_1, \mathcal{U}_2$, and \mathcal{V} are smooth affine schemes, $f_{\mathcal{U}}$ is a smooth morphism, and all squares are Cartesian.

We can view

$$\mathrm{IndCoh}(Z_2) \otimes_{\mathrm{QCoh}(Z_2)} \mathrm{QCoh}(Z_1) \simeq \mathrm{IndCoh}(Z_1)$$

as a category tensored over

$$\mathrm{QCoh}(\mathcal{U}_1) \otimes \mathrm{HC}(\mathrm{pt}/\mathcal{V})^{\mathrm{op}\text{-mod}},$$

and both subcategories in the proposition correspond to the condition that the support be contained in $Y_1 \subset \mathcal{U}_1 \times V^*$ (see Proposition 3.5.7). \square

7.5.4. Let Y_2 and Y_1 be as above. From Proposition 7.5.3, we obtain:

Corollary 7.5.5 *We have the following commutative diagrams:*

$$\begin{array}{ccc} \mathrm{IndCoh}_{Y_1}(Z_1) & \begin{array}{c} \xrightarrow{\Xi_{Z_1}^{Y_1, \mathrm{all}}} \\ \xleftarrow{\Psi_{Z_1}^{Y_1, \mathrm{all}}} \end{array} & \mathrm{IndCoh}(Z_1) \\ \downarrow & \begin{array}{c} \Psi_{Z_1}^{Y_1, \mathrm{all}} \\ \Xi_{Z_1}^{Y_2, \mathrm{all}} \end{array} & \downarrow f^{\mathrm{IndCoh}} \\ \mathrm{IndCoh}_{Y_2}(Z_2) & \begin{array}{c} \xrightarrow{\Xi_{Z_1}^{Y_2, \mathrm{all}}} \\ \xleftarrow{\Psi_{Z_1}^{Y_2, \mathrm{all}}} \end{array} & \mathrm{IndCoh}(Z_2) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{IndCoh}_{Y_1}(Z_1) & \begin{array}{c} \xrightarrow{\Xi_{Z_1}^{Y_1, \mathrm{all}}} \\ \xleftarrow{\Psi_{Z_1}^{Y_1, \mathrm{all}}} \end{array} & \mathrm{IndCoh}(Z_1) \\ \uparrow & \begin{array}{c} \Psi_{Z_1}^{Y_1, \mathrm{all}} \\ \Xi_{Z_1}^{Y_2, \mathrm{all}} \end{array} & \uparrow f^! \\ \mathrm{IndCoh}_{Y_2}(Z_2) & \begin{array}{c} \xrightarrow{\Xi_{Z_1}^{Y_2, \mathrm{all}}} \\ \xleftarrow{\Psi_{Z_1}^{Y_2, \mathrm{all}}} \end{array} & \mathrm{IndCoh}(Z_2). \end{array}$$

7.5.6. As another corollary of Proposition 7.5.3, we obtain the following. Let $f : Z_1 \rightarrow Z_2$ be as above, and assume moreover that it is surjective, i.e., f is a smooth cover.

Let Z^\bullet denote the Čech nerve of f . Fix $Y_2 \subset \mathrm{Sing}(Z_2)$, and for each i , let $Y^i \subset \mathrm{Sing}(Z^i)$ be the corresponding subset of $\mathrm{Sing}(Z^i)$.

We can form the cosimplicial category $\mathrm{IndCoh}_{Y^\bullet}(Z^\bullet)$ using either the $!$ -pullback or $(\mathrm{IndCoh}, *)$ -pullback functors. In each case, the resulting cosimplicial category is augmented by $\mathrm{IndCoh}_{Y_2}(Z_2)$.

Proposition 7.5.7 *Under the above circumstances, the augmentation functor*

$$\mathrm{IndCoh}_{Y_2}(Z_2) \rightarrow \mathrm{Tot}(\mathrm{IndCoh}_{Y^\bullet}(Z^\bullet))$$

is an equivalence.

Proof This follows from the fact that

$$\mathrm{QCoh}(Z_2) \rightarrow \mathrm{Tot}(\mathrm{QCoh}(Z^\bullet))$$

is an equivalence, combined with the fact that the operation

$$\mathrm{IndCoh}_{Y_1}(Z_2) \otimes_{\mathrm{QCoh}(Z_2)} -$$

commutes with limits. □

Corollary 7.5.8 *For $\mathcal{F} \in \mathrm{IndCoh}(Z_2)$, we have*

$$\mathrm{SingSupp}(\mathcal{F}) \subset Y_2 \Leftrightarrow \mathrm{SingSupp}(f^!(\mathcal{F})) \subset Y_2 \times_{Z_2} Z_1,$$

and also

$$\mathrm{SingSupp}(\mathcal{F}) \subset Y_2 \Leftrightarrow \mathrm{SingSupp}(f^{\mathrm{IndCoh},*}(\mathcal{F})) \subset Y_2 \times_{Z_2} Z_1.$$

Remark 7.5.9 From Theorem 7.7.2, one can derive a more precise statement: If $f : Z_1 \rightarrow Z_2$ is a smooth map and $\mathcal{F} \in \mathrm{IndCoh}(Z_2)$, then

$$\mathrm{SingSupp}(f^!(\mathcal{F})) = \mathrm{SingSupp}(f^{\mathrm{IndCoh},*}(\mathcal{F})) = \mathrm{SingSupp}(\mathcal{F}) \times_{Z_2} Z_1.$$

7.6 Quasi-smooth morphisms, revisited

In this subsection, we will establish a generalization of Corollary 7.4.10 for arbitrary quasi-smooth maps. We will treat the case of the $!$ -pullback, while the $(\mathrm{IndCoh}, *)$ -pullback is similar.

7.6.1. Let $f : Z_1 \rightarrow Z_2$ be a quasi-smooth morphisms between quasi-smooth DG schemes, and assume that Z_2 is quasi-compact. For a conical Zariski-closed $Y_2 \subset \mathrm{Sing}(Z_2)$, let

$$Y_1 = \mathrm{Sing}(f)(Y_2 \times_{Z_2} Z_1) \subset \mathrm{Sing}(Z_1),$$

where we regard $Y_2 \times_{Z_2} Z_1$ as a subset of $\mathrm{Sing}(Z_2)_{Z_1}$.

By Proposition 7.1.3(a), we have a well-defined functor

$$f^! : \mathrm{IndCoh}_{Y_2}(Z_2) \rightarrow \mathrm{IndCoh}_{Y_1}(Z_1).$$

It extends by $\mathrm{QCoh}(Z_2)$ -linearity to a functor

$$\mathrm{QCoh}(Z_1) \otimes_{\mathrm{QCoh}(Z_2)} \mathrm{IndCoh}_{Y_2}(Z_2) \rightarrow \mathrm{IndCoh}_{Y_1}(Z_1). \quad (7.3)$$

Corollary 7.6.2 *The functor (7.3) is an equivalence.*

Proof As in the proof of Proposition 7.5.3, the statement is local on both Z_1 and Z_2 . By Lemma 2.1.9, locally, the map f can be decomposed as a composition of a quasi-smooth closed embedding followed by a smooth map. Now the assertion follows by combining Proposition 7.5.3 and Corollary 7.4.10. \square

7.6.3. We can now generalize the results of Sect. 7.4 as follows.

Proposition 7.6.4 *Suppose that Z_1 and Z_2 are quasi-smooth DG schemes, and let $f : Z_1 \rightarrow Z_2$ be an affine quasi-smooth morphism. For a conical Zariski-closed $Y_2 \subset \text{Sing}(Z_2)$, let*

$$Y_1 = \text{Sing}(f)(Y_2 \times_{Z_2} Z_1) \subset \text{Sing}(Z_1),$$

where we regard $Y_2 \times_{Z_2} Z_1$ as a subset of $\text{Sing}(Z_2)_{Z_1}$. Then, Corollary 7.4.6 holds:

- (a) *The functor $f_*^{\text{IndCoh}} : \text{IndCoh}_{Y_1}(Z_1) \rightarrow \text{IndCoh}_{Y_2}(Z_2)$ is conservative.*
- (b) *The essential image of $\text{IndCoh}_{Y_2}(Z_2)$ under $f^!$ (or under $f^{\text{IndCoh},*}$) generates $\text{IndCoh}_{Y_1}(Z_1)$.*

Proof As in the proof of Corollary 7.4.6, the claim is local on Z_2 , so we may assume that Z_2 (and, therefore, Z_1) is quasi-compact. Assertions (a) and (b) are equivalent, so it suffices to verify (b). But it follows from Corollary 7.6.2, because the essential image of $\text{QCoh}(Z_2)$ under f^* generates $\text{QCoh}(Z_1)$. \square

7.7 Inverse image

Let us now consider the behavior of singular support under the operation of inverse image. Let $f : Z_1 \rightarrow Z_2$ be a morphism between quasi-smooth DG schemes (Z_1 and Z_2 need not be quasi-compact).

7.7.1. Let \mathcal{F} be an object of $\text{IndCoh}(Z_2)$, and let $Y_2 \subset \text{Sing}(Z_2)$ be its singular support. Let $Y_1 \subset \text{Sing}(Z_1)$ be the Zariski closure of $\text{Sing}(f)(p^{-1}(Y_2))$, where

$$p : \text{Sing}(Z_2)_{Z_1} = \text{Sing}(Z_2) \times_{Z_2} Z_1 \rightarrow \text{Sing}(Z_2)$$

is the projection.

Consider the object $f^!(\mathcal{F}) \in \text{IndCoh}(Z_1)$. Note that by Proposition 7.1.3(a), we have:

$$\text{SingSupp}(f^!(\mathcal{F})) \subset Y_1.$$

Theorem 7.7.2 *Suppose that either $\mathcal{F} \in \text{Coh}(Z_2)$ or f is a topologically open morphism (e.g., flat). Then,*

$$\text{SingSupp}(f^!(\mathcal{F})) = Y_1.$$

Remark 7.7.3 The assertion of Theorem 7.7.2 is not necessary for the main results of this paper.

Proof As in Remark 6.2.6, consider the union

$$Y'_2 := \bigcup_{z_2 \in Z_2} \text{supp}_{\text{Sym}(H^1(T_{z_2}(Z_2)))} \left(H^\bullet(i_{z_2}^{\text{enh},!}(\mathcal{F})) \right) \subset \text{Sing}(Z_2).$$

By Proposition 6.2.2, $Y_2 = \overline{Y'_2}$; if $\mathcal{F} \in \text{Coh}(Z_2)$, then $Y_2 = Y'_2$ by Proposition 6.2.5. Similarly, consider the union

$$Y'_1 := \bigcup_{z_1 \in Z_1} \text{supp}_{\text{Sym}(H^1(T_{z_1}(Z_1)))} \left(H^\bullet(i_{z_1}^{\text{enh},!}(f^!(\mathcal{F}))) \right) \subset \text{Sing}(Z_1);$$

then $\text{SingSupp}(f^!(\mathcal{F})) = \overline{Y'_1}$.

Let us show that

$$Y'_1 = \text{Sing}(f)(p^{-1}(Y'_2)),$$

which would imply the assertion of the theorem.

Let $z_1 \in Z_1$ be a point of Z_1 , which we may assume to be a k -point after extending scalars. Set $z_2 = f(z_2) \in Z_2$. Choose a quasi-smooth map

$$i_2 : Z'_2 \rightarrow Z_2$$

as in Sect. 6.3.2, so that Z'_2 is a DG scheme of the form $\text{pt} \times_{\mathcal{V}_2} \text{pt}$, with \mathcal{V}_2 smooth, and such that the unique k -point of Z'_2 goes to z_2 .

Set

$$Z'_1 := Z_1 \times_{Z_2} Z'_2.$$

Since DG scheme Z_1 and the morphism $Z'_1 \rightarrow Z_1$ are quasi-smooth, Z'_1 is quasi-smooth. Also, $z_1 \in Z'_1$. We can therefore choose a quasi-smooth map

$$Z''_1 \rightarrow Z'_1$$

as in Sect. 6.3.2, so that Z''_1 is a DG scheme of the form $\text{pt} \times_{\mathcal{V}_1} \text{pt}$, with \mathcal{V}_1 smooth, and such that the unique k -point of Z''_1 goes to z_1 . Let i_1 be the composition $Z''_1 \rightarrow Z'_1 \rightarrow Z_1$. Being a composition of quasi-smooth maps, i_1 is quasi-smooth.

By Sect. 6.3.6, we know that

$$\text{SingSupp}(i_1^!(f^!(\mathcal{F}))) = Y'_1 \cap \text{Sing}(Z_1)_{Z''_1} \subset \text{Sing}(Z_1)_{Z''_1} = \text{Sing}(Z_1)_{\{z_1\}} \subset \text{Sing}(Z''_1),$$

and

$$\mathrm{SingSupp}(i_2^!(\mathcal{F})) = Y_2' \cap \mathrm{Sing}(Z_2)_{Z_2'} \subset \mathrm{Sing}(Z_2)_{Z_2'} = \mathrm{Sing}(Z_2)_{\{z_2\}} \subset \mathrm{Sing}(Z_2').$$

Now we can replace Z_1 , Z_2 , and \mathcal{F} with Z_1'' , Z_2' , and $i_2^!(\mathcal{F})$, respectively. Note that if \mathcal{F} is coherent, then so is $i_2^!(\mathcal{F})$.

Thus, we assume that ${}^{cl}Z_1$ and ${}^{cl}Z_2$ are both isomorphic to pt. The claim now follows from Lemma 5.2.10(b) and Proposition 5.2.5(a). \square

7.8 Conservativeness for proper maps

7.8.1. Suppose now that $f : Z_1 \rightarrow Z_2$ is a proper morphism between quasi-smooth DG schemes. Let $Y_1 \subset \mathrm{Sing}(Z_1)$ be a conical Zariski-closed subset, and let $Y_2 \subset \mathrm{Sing}(Z_2)$ be the image of $(\mathrm{Sing}(f))^{-1}(Y_1)$ under the projection

$$\mathrm{Sing}(Z_2)_{Z_1} \rightarrow \mathrm{Sing}(Z_2).$$

The subset Y_2 is automatically closed since the above map is proper.

By Proposition 7.1.3(b), the functor f_*^{IndCoh} sends $\mathrm{IndCoh}_{Y_1}(Z_1)$ to $\mathrm{IndCoh}_{Y_2}(Z_2)$. Our goal is to prove the following result.

Theorem 7.8.2 *Under the above circumstances, the essential image of $\mathrm{IndCoh}_{Y_1}(Z_1)$ under f_*^{IndCoh} generates $\mathrm{IndCoh}_{Y_2}(Z_2)$.*

We will derive Theorem 7.8.2 from the following more general statement:

Proposition 7.8.3 *Let $f : Z_1 \rightarrow Z_2$ be a (not necessarily proper) morphism of quasi-smooth DG schemes. Let $Y_1 \subset \mathrm{Sing}(Z_1)$ and $Y_2 \subset \mathrm{Sing}(Z_2)$ be conical Zariski-closed subsets. Suppose that Y_2 is contained in the image of $(\mathrm{Sing}(f))^{-1}(Y_1)$ under the projection*

$$\mathrm{Sing}(Z_2)_{Z_1} \rightarrow \mathrm{Sing}(Z_2).$$

Suppose $\mathcal{F} \in \mathrm{IndCoh}(Z_2)$ is such that $\Psi_{Z_2}^{Y_2, \mathrm{all}}(\mathcal{F}) \neq 0$. Then,

$$\Psi_{Z_1}^{Y_1, \mathrm{all}} \circ f^!(\mathcal{F}) \neq 0.$$

Proof of Theorem 7.8.2 The statement of the theorem is equivalent to the claim that the functor right adjoint to

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}_{Y_1}(Z_1) \rightarrow \mathrm{IndCoh}_{Y_2}(Z_2)$$

is conservative.

The right adjoint in question is equal to the composition

$$\mathrm{IndCoh}_{Y_2}(Z_2) \xrightarrow{\Xi_{Z_2}^{Y_2, \mathrm{all}}} \mathrm{IndCoh}(Z_2) \xrightarrow{f^!} \mathrm{IndCoh}(Z_1) \xrightarrow{\Psi_{Z_1}^{Y_1, \mathrm{all}}} \mathrm{IndCoh}_{Y_1}(Z_1). \quad (7.4)$$

Suppose $\mathcal{F} \in \text{IndCoh}_{Y_2}(Z_2)$ is annihilated by the composition (7.4). But by Proposition 7.8.3, the vanishing

$$\Psi_{Z_1}^{Y_1, \text{all}} \circ f^! \circ \Xi_{Z_2}^{Y_2, \text{all}}(\mathcal{F}) = 0$$

implies

$$0 = \Psi_{Z_2}^{Y_2, \text{all}} \circ \Xi_{Z_2}^{Y_2, \text{all}}(\mathcal{F}) = \mathcal{F},$$

as required. \square

The rest of this subsection is devoted to the proof of Proposition 7.8.3.

7.8.4 Step 1 We are going to reduce the statement of the proposition to the case when Z_2 is of the form $\text{pt} \times_{\mathcal{V}_2} \text{pt}$ for a smooth scheme \mathcal{V}_2 and a point $\text{pt} \hookrightarrow \mathcal{V}_2$.

Indeed, by Lemma 6.2.3, there exists a geometric point z_2 of Z_2 such that $i_{z_2}^!(\Psi_{Z_2}^{Y_2, \text{all}}(\mathcal{F})) \neq 0$. Extending the ground field, we may assume that $i_{z_2} : \text{pt} \hookrightarrow Z_2$ is a rational point.

As in Sect. 6.3.2, we now extend the morphism i_{z_2} to a quasi-smooth morphism of quasi-smooth DG schemes

$$i'_2 = i_{z_2, \mathcal{V}_2} : Z'_2 \rightarrow Z_2,$$

where

$$Z'_2 = \text{pt} \times_{\mathcal{V}_2} \text{pt}$$

for a smooth scheme \mathcal{V}_2 with a marked point $\text{pt} \hookrightarrow \mathcal{V}_2$. Then,

$$(i'_2)^!(\Psi_{Z_2}^{Y_2, \text{all}}(\mathcal{F})) \neq 0$$

by Lemma 6.3.3.

Recall that the singular codifferential $\text{Sing}(i'_2)$ is an embedding

$$\text{Sing}(Z_2)_{\{z_2\}} \hookrightarrow V_2^* = \text{Sing}(Z'_2),$$

where $V_2 = T_{\text{pt}}(\mathcal{V}_2)$. Set

$$Y'_2 = \text{Sing}(i'_2)(Y_2 \cap \text{Sing}(Z_2)_{\{z_2\}}) \subset V_2^*.$$

Set now $Z'_1 = Z_1 \times_{Z_2} Z'_2$. The morphism $i'_1 : Z'_1 \rightarrow Z_1$ is quasi-smooth, so by Lemma 2.4.3,

$$\text{Sing}(i'_1) : \text{Sing}(Z_1)_{Z'_1} = \text{Sing}(Z_1) \times_{Z_1} Z'_1 \rightarrow \text{Sing}(Z'_1)$$

is a closed embedding. Set

$$Y'_1 = \text{Sing}(i'_1) \left(Y_1 \times_{Z_1} Z'_1 \right) \subset \text{Sing}(Z'_1).$$

From Proposition 7.1.6(b), we obtain a commutative diagram of functors

$$\begin{array}{ccccccc} \text{IndCoh}_{Y_2}(Z_2) & \xleftarrow{\Psi_{Z_2}^{Y_2, \text{all}}} & \text{IndCoh}(Z_2) & \xrightarrow{f^!} & \text{IndCoh}(Z_1) & \xrightarrow{\Psi_{Z_1}^{Y_1, \text{all}}} & \text{IndCoh}_{Y_1}(Z_1) \\ (i_2)^\downarrow & & (i_2)^\downarrow & & (i_1)^\downarrow & & (i_1)^\downarrow \\ \text{IndCoh}_{Y'_2}(Z'_2) & \xleftarrow{\Psi_{Z'_2}^{Y'_2, \text{all}}} & \text{IndCoh}(Z'_2) & \xrightarrow{(f')^!} & \text{IndCoh}(Z'_1) & \xrightarrow{\Psi_{Z'_1}^{Y'_1, \text{all}}} & \text{IndCoh}_{Y'_1}(Z'_1), \end{array}$$

where $f' : Z'_1 \rightarrow Z'_2$ is the natural morphism. Hence, it suffices to show that

$$\Psi_{Z'_1}^{Y'_1, \text{all}} \circ (f')^! \circ (i'_2)^\downarrow(\mathcal{F}) \neq 0.$$

Note that f' satisfies the conditions of Proposition 7.8.3 with respect $Y'_2 \subset \text{Sing}(Z'_2)$ and $Y'_1 \subset \text{Sing}(Z'_1)$.

Thus, we obtain that the statement of proposition is reduced to the case when Z_2 is replaced by Z'_2 , Y_2 with Y'_2 , \mathcal{F} by $(i'_2)^\downarrow(\mathcal{F})$, Z_1 by Z'_1 , and Y_1 by Y'_1 .

In other words, we can assume that $Z_2 = \text{pt} \times_{\mathcal{V}_2} \text{pt}$, as desired.

7.8.5 Step 2 We are now going to reduce the assertion of the proposition to the case when Z_1 is also of the form $\text{pt} \times_{\mathcal{V}_1} \text{pt}$.

To do so, let us fix a parallelization of the formal neighborhood of pt in \mathcal{V}_2 . As explained in Sect. 5.4.6, this equips $\text{IndCoh}(Z_2)$ with an action of the monoidal category $\text{QCoh}(V_2^*/\mathbb{G}_m)$. By Lemma 3.6.7, there exists a geometric point $y_2 \in V_2^*$ such that the fiber $i_{y_2}^*(\Psi_{Z_2}^{Y_2, \text{all}}(\mathcal{F})) \neq 0$. By Corollary 3.6.8(a), we see that

$$y_2 \in Y_2 \subset V_2^*.$$

Extending the ground field, we may assume that

$$i_{y_2} : \text{pt} \hookrightarrow V_2^* = \text{Sing}(Z_2)$$

is a k -rational point.

Since Y_2 is contained in the image of $\text{Sing}(f)^{-1}(Y_1)$ under the projection

$$\text{Sing}(Z_2)_{Z_1} \rightarrow \text{Sing}(Z_2),$$

there is a k -point $z_1 \in Z_1$ such that $\text{Sing}(f)$ sends

$$(y_2, z_1) \in V_2^* \times^{cl} Z_1 = {}^{cl} \left(\text{Sing}(Z_2) \times_{Z_2} Z_1 \right) = \text{Sing}(Z_2)_{Z_1}$$

into $Y_1 \subset \text{Sing}(Z_1)$.

Now extend the morphism $i_{z_1} : \text{pt} \rightarrow Z_1$ to a quasi-smooth morphism of quasi-smooth DG schemes

$$\tilde{i}_1 = i_{z_1, \mathcal{V}_1} : \tilde{Z}_1 \rightarrow Z_1,$$

where

$$\tilde{Z}_1 = \text{pt} \times_{\mathcal{V}_1} \text{pt}$$

for a smooth scheme \mathcal{V}_1 with a marked point $\text{pt} \hookrightarrow \mathcal{V}_1$. Set $\tilde{f} = f \circ \tilde{i}_1$.

The singular codifferential $\text{Sing}(\tilde{i}_1)$ is an embedding

$$\text{Sing}(Z_1)_{\{z_1\}} \hookrightarrow V_1^* = \text{Sing}(\tilde{Z}_1).$$

Let \tilde{Y}_1 be the image

$$\text{Sing}(\tilde{i}_1)(Y_1 \cap \text{Sing}(Z_1)_{\{z_1\}}) \subset V_1^*.$$

The singular codifferential $\text{Sing}(\tilde{f})$ is a linear map $V_2^* \rightarrow V_1^*$. Set

$$\tilde{Y}_2 := Y_2 \cap \text{Sing}(\tilde{f})^{-1}(\tilde{Y}_1) \subset V_2^*.$$

By construction, $y_2 \in \tilde{Y}_2$. By Corollary 3.6.8(b),

$$i_{y_2}^*(\Psi_{Z_2}^{\tilde{Y}_2, \text{all}}(\mathcal{F})) \simeq i_{y_2}^*(\mathcal{F}) \simeq i_{y_2}^*(\Psi_{Z_2}^{Y_2, \text{all}}(\mathcal{F})) \neq 0,$$

and hence

$$\Psi_{Z_2}^{\tilde{Y}_2, \text{all}}(\mathcal{F}) \neq 0.$$

Thus, it suffices to prove the assertion of the proposition after replacing Y_2 by \tilde{Y}_2 , Z_1 by \tilde{Z}_1 , and Y_1 by \tilde{Y}_1 , while keeping \mathcal{F} and Z_2 the same.

7.8.6 Step 3 Thus, we can assume that $Z_i \simeq \text{pt} \times_{\mathcal{V}_i} \text{pt}$ for $i = 1, 2$. In this case, the assertion of the proposition follows from Lemma 5.2.10(b) and Corollary 5.2.8(a).

□[Proposition 7.8.3]

8 Singular support on stacks

In this section, we develop the notion of singular support for objects of $\text{IndCoh}(\mathcal{Z})$, where \mathcal{Z} is a quasi-smooth Artin stack. This will not be difficult, given the good functorial properties of $\text{IndCoh}(-)$ on DG schemes under smooth maps.

Essentially, all this section amounts to is showing that for Artin stacks things work just as well as for DG schemes. For this reason, this section, as well as Sect. 9, may be skipped on the first pass.

Recall that all schemes and stacks are assumed derived by default. To simplify the terminology, from now on we discard the words “differential graded” for stacks. Thus, “Artin stack” stands for “DG Artin stack.”

8.1 Quasi-smoothness for stacks

8.1.1. Let \mathcal{Z} be an Artin stack (see [28], Sect. 4). We say that it is quasi-smooth if for every affine DG scheme Z equipped with a smooth map $Z \rightarrow \mathcal{Z}$, the DG scheme Z is quasi-smooth.

Equivalently, \mathcal{Z} is quasi-smooth if for some (equivalently, every) smooth atlas $f : Z \rightarrow \mathcal{Z}$, the DG scheme Z is quasi-smooth.

Recall that for a k -Artin stack \mathcal{Z} , its cotangent complex $T^*(\mathcal{Z})$ is an object of $\text{QCoh}(\mathcal{Z})^{\leq k}$. We have:

Lemma 8.1.2 *A k -Artin stack \mathcal{Z} is quasi-smooth if and only if $T^*(\mathcal{Z})$ is perfect of Tor-amplitude $[-1, k]$.*

Proof Let $f : Z \rightarrow \mathcal{Z}$ be a smooth atlas. Then, $T^*(\mathcal{Z})$ is perfect of Tor-amplitude $[-1, k]$ if and only if $f^*(T^*(\mathcal{Z}))$ has this property. Besides, we have an exact triangle

$$f^*(T^*(\mathcal{Z})) \rightarrow T^*(Z) \rightarrow T^*(Z/Z),$$

where $T^*(Z/Z)$ is perfect of Tor-amplitude $[0, k - 1]$. Thus, $f^*(T^*(\mathcal{Z}))$ is perfect of Tor-amplitude $[-1, k]$ if and only if $T^*(Z)$ is perfect of Tor-amplitude $[-1, k]$. The latter condition is equivalent to Z being quasi-smooth. \square

We say that a map $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ between Artin stacks is quasi-smooth if $T^*(\mathcal{Z}_1/\mathcal{Z}_2)$ is perfect of Tor-amplitude bounded from below by -1 .

8.1.3. Recall the property of local eventual coconnectivity for a morphism between DG schemes (see Sect. 2.2.3). Clearly, this property is local in the smooth topology on the source and on the target. Hence, it makes sense for morphisms between Artin stacks.

Lemma 8.1.4 *A quasi-smooth morphism $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ of Artin stacks is locally eventually coconnective. In particular, a quasi-smooth Artin stack is locally eventually coconnective.*

Proof Follows from Corollary 2.2.4. \square

8.1.5. If \mathcal{Z} is a quasi-smooth Artin stack, we introduce a classical Artin stack $\text{Sing}(\mathcal{Z})$, equipped with an affine (and, in particular, schematic) map to ${}^{cl}\mathcal{Z}$. We call $\text{Sing}(\mathcal{Z})$ the stack of singularities of \mathcal{Z} . It is constructed as follows:

For every affine DG scheme Z with a smooth map to \mathcal{Z} , we set

$$\text{Sing}(\mathcal{Z}) \times_{\mathcal{Z}} Z := \text{Sing}(Z),$$

and this assignment satisfies the descent conditions because of Lemma 2.4.4. Equivalently, $\text{Sing}(\mathcal{Z})$ can be defined as

$${}^{cl}(\text{Spec}_{\mathcal{Z}}(\text{Sym}_{\mathcal{O}_{\mathcal{Z}}}(T(\mathcal{Z})[1]))) . \quad (8.1)$$

8.1.6 *The singular codifferential for stacks* Let $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ be a map between quasi-smooth Artin stacks. We claim that we have a naturally defined singular codifferential map

$$\text{Sing}(f) : \text{Sing}(\mathcal{Z}_2)_{\mathcal{Z}_1} := \text{Sing}(\mathcal{Z}_2) \times_{\mathcal{Z}_2} \mathcal{Z}_1 \rightarrow \text{Sing}(\mathcal{Z}_1). \quad (8.2)$$

It can be obtained from the differential of f using (8.1).

As in the case of DG schemes, it is easy to see that a map f is quasi-smooth (resp., smooth) if and only if the map $\text{Sing}(f)$ is a closed embedding (resp., isomorphism).

8.2 Definition of the category with supports for stacks

8.2.1. Recall ([10, Sect. 11.2]) that we have a well-defined category $\text{IndCoh}(\mathcal{Z})$, and that it can be recovered as

$$\text{IndCoh}(\mathcal{Z}) \simeq \lim_{Z \in \text{DGSch}_{/\mathcal{Z}, \text{smooth}}^{\text{aff}}} \text{IndCoh}(Z), \quad (8.3)$$

where $\text{DGSch}_{/\mathcal{Z}, \text{smooth}}^{\text{aff}}$ denotes the non-full subcategory of $(\text{DGSch}_{\text{aft}}^{\text{aff}})_{/\mathcal{Z}}$, where the objects are restricted to pairs $(Z \in \text{DGSch}_{\text{aft}}^{\text{aff}}, f : Z \rightarrow \mathcal{Z})$ where f is smooth, and where 1-morphisms are restricted to maps $g : Z_1 \rightarrow Z_2$ that are smooth as well.

In the formation of the above limit, we can use either the $!$ -pullback functors or the $(\text{IndCoh}, *)$ -pullback functors, as the two differ by the twist by a cohomologically shifted line bundle (this is due to the smoothness assumption on the morphisms).

8.2.2. We let $\text{Coh}(\mathcal{Z}) \subset \text{IndCoh}(\mathcal{Z})$ be the full (but not cocomplete) subcategory defined as

$$\text{Coh}(\mathcal{Z}) \simeq \lim_{Z \in \text{DGSch}_{/\mathcal{Z}, \text{smooth}}^{\text{aff}}} \text{Coh}(Z).$$

Note that we can think of $\mathrm{Coh}(\mathbb{Z})$ also as a full subcategory of $\mathrm{QCoh}(\mathbb{Z})$, where the latter, according to [30, Proposition 5.1.2], is isomorphic to

$$\lim_{Z \in \mathrm{DGSch}_{/\mathbb{Z}, \mathrm{smooth}}^{\mathrm{aff}}} \mathrm{QCoh}(Z).$$

8.2.3. Let Y be a conical Zariski-closed subset of $\mathrm{Sing}(\mathbb{Z})$. We define the full subcategory

$$\mathrm{IndCoh}_Y(\mathbb{Z}) \subset \mathrm{IndCoh}(\mathbb{Z})$$

as

$$\mathrm{IndCoh}_Y(\mathbb{Z}) \simeq \lim_{Z \in \mathrm{DGSch}_{/\mathbb{Z}, \mathrm{smooth}}^{\mathrm{aff}}} \mathrm{IndCoh}_{Y \times_{\mathbb{Z}} Z}(Z), \quad (8.4)$$

where we view $Y \times_{\mathbb{Z}} Z$ as a closed subset of

$$\mathrm{Sing}(\mathbb{Z}) \times_{\mathbb{Z}} Z \simeq \mathrm{Sing}(Z).$$

From Lemmas 4.2.2 and 4.3.5, we obtain:

Corollary 8.2.4 *The action of $\mathrm{QCoh}(\mathbb{Z})$ on $\mathrm{IndCoh}(\mathbb{Z})$ preserves $\mathrm{IndCoh}_Y(\mathbb{Z})$.*

8.2.5. From Corollary 7.5.5, we obtain:

Corollary 8.2.6 *There exists a pair of adjoint functors*

$$\Xi_{\mathbb{Z}}^{Y, \mathrm{all}} : \mathrm{IndCoh}_Y(\mathbb{Z}) \rightleftarrows \mathrm{IndCoh}(Z) : \Psi_{\mathbb{Z}}^{Y, \mathrm{all}},$$

with $\Xi_{\mathbb{Z}}^{Y, \mathrm{all}}$ being fully faithful. Moreover, for a smooth map $f : Z \rightarrow \mathbb{Z}$, we have commutative diagrams

$$\begin{array}{ccc} \mathrm{IndCoh}_{Y \times_{\mathbb{Z}} Z}(Z) & \xrightarrow{\Xi_{\mathbb{Z}}^{Y \times Z, \mathrm{all}}} & \mathrm{IndCoh}(Z) \\ \uparrow & & \uparrow f^! \\ \mathrm{IndCoh}_Y(\mathbb{Z}) & \xrightarrow{\Xi_{\mathbb{Z}}^{Y, \mathrm{all}}} & \mathrm{IndCoh}(\mathbb{Z}) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{IndCoh}_{Y \times_{\mathbb{Z}} Z}(Z) & \xleftarrow{\Psi_{\mathbb{Z}}^{Y \times Z, \mathrm{all}}} & \mathrm{IndCoh}(Z) \\ \uparrow & & \uparrow f^! \\ \mathrm{IndCoh}_Y(\mathbb{Z}) & \xleftarrow{\Psi_{\mathbb{Z}}^{Y, \mathrm{all}}} & \mathrm{IndCoh}(\mathbb{Z}) \end{array}$$

8.2.7. Recall from [10, Sect. 11.7.3] that for an eventually coconnective Artin stack, we have a fully faithful functor

$$\Xi_{\mathcal{Z}} : \mathrm{QCoh}(\mathcal{Z}) \rightarrow \mathrm{IndCoh}(\mathcal{Z}).$$

From Theorem 4.2.6, we obtain:

Corollary 8.2.8 *If Y is the zero-section, the subcategory*

$$\mathrm{IndCoh}_{\{0\}}(\mathcal{Z}) \subset \mathrm{IndCoh}(\mathcal{Z})$$

coincides with the essential image of $\mathrm{QCoh}(\mathcal{Z})$ under the functor

$$\Xi_{\mathcal{Z}} : \mathrm{QCoh}(\mathcal{Z}) \rightarrow \mathrm{IndCoh}(\mathcal{Z}).$$

8.2.9. Let $\mathcal{V} \hookrightarrow \mathcal{Z}$ be a closed substack (not necessarily quasi-smooth), and let $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ be the complementary open.

Corollary 8.2.10 *Let $Y \subset \mathrm{Sing}(\mathcal{Z})$ be a closed conical subset. Set*

$$Y_{\mathcal{V}} = {}^{cl}\left(Y \times_{\mathcal{Z}} \mathcal{V}\right) \subset \mathrm{Sing}(\mathcal{Z}).$$

(a) *The subcategory*

$$\mathrm{IndCoh}_Y(\mathcal{Z}) \cap \mathrm{IndCoh}(\mathcal{Z}_{\mathcal{V}}) \subset \mathrm{IndCoh}(\mathcal{Z})$$

is equal to $\mathrm{IndCoh}_{Y_{\mathcal{V}}}(\mathcal{Z})$.

(b) *We have a short exact sequence of categories*

$$\mathrm{IndCoh}_{Y_{\mathcal{V}}}(\mathcal{Z}) \rightleftarrows \mathrm{IndCoh}_Y(\mathcal{Z}) \rightleftarrows \mathrm{IndCoh}_{Y \times_{\mathcal{Z}} \mathcal{U}}(\mathcal{U}).$$

Proof The two claims follow from Corollaries 4.5.2 and 4.5.9 □

8.2.11. We have no reason to expect that the category $\mathrm{IndCoh}_Y(\mathcal{Z})$ is compactly generated for an arbitrary \mathcal{Z} .

Assume now that \mathcal{Z} is a QCA algebraic stack in the sense of [6] Definition 1.1.8⁷ (in particular, \mathcal{Z} is a 1-Artin stack).

It is shown in *loc.cit.*, Theorem 3.3.4, that in this case the category $\mathrm{IndCoh}(\mathcal{Z})$ is compactly generated by $\mathrm{Coh}(\mathcal{Z})$. In particular, $\mathrm{IndCoh}(\mathcal{Z})$ is dualizable.

By Corollary 8.2.6, the category $\mathrm{IndCoh}_Y(\mathcal{Z})$ is a retract of $\mathrm{IndCoh}(\mathcal{Z})$. Hence, by [6, Lemma 4.3.3], we obtain:

Corollary 8.2.12 *Under the above circumstances, the category $\mathrm{IndCoh}_Y(\mathcal{Z})$ is dualizable.*

⁷ QCA means quasi-compact, and the automorphism group of any geometric point is affine.

Remark 8.2.13 We do not know whether under the assumptions of Corollary 8.2.12, the category $\text{IndCoh}_Y(\mathcal{Z})$ is compactly generated. In fact, we do not know this even for $Y = \{0\}$, i.e., we do not know whether $\text{QCoh}(\mathcal{Z})$ is compactly generated. We will describe two cases when this holds: One is proved in Appendix C (when $\mathcal{Z} = Z$ is a quasi-compact DG scheme) and the other in Sect. 9.2.

8.3 Smooth descent

8.3.1 Smooth maps of stacks It follows from Lemma 2.4.3 that if $\mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ is smooth, the singular codifferential

$$\text{Sing}(f) : \text{Sing}(\mathcal{Z}_2)_{\mathcal{Z}_1} = \text{Sing}(\mathcal{Z}_2) \times_{\mathcal{Z}_2} \mathcal{Z}_1 \rightarrow \text{Sing}(\mathcal{Z}_1)$$

is an isomorphism.

For a conical closed subset $Y_2 \subset \text{Sing}(\mathcal{Z}_2)$, set

$$Y_1 := \text{Sing}(f) \left(Y_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1 \right) \subset \text{Sing}(\mathcal{Z}_1).$$

Lemma 8.3.2 *Let $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ be a smooth map between quasi-smooth Artin stacks. Then, we have the following commutative diagram*

$$\begin{array}{ccc} \text{IndCoh}_{Y_1}(\mathcal{Z}_1) & \begin{array}{c} \xleftarrow{\Xi_{\mathcal{Z}_1}^{Y_1, \text{all}}} \\ \xrightarrow{\Psi_{\mathcal{Z}_1}^{Y_1, \text{all}}} \end{array} & \text{IndCoh}(\mathcal{Z}_1) \\ \uparrow & \begin{array}{c} \Psi_{\mathcal{Z}_1}^{Y_2, \text{all}} \\ \Xi_{\mathcal{Z}_1}^{Y_2, \text{all}} \end{array} & \uparrow f^! \\ \text{IndCoh}_{Y_2}(\mathcal{Z}_2) & \begin{array}{c} \xleftarrow{\Psi_{\mathcal{Z}_1}^{Y_2, \text{all}}} \\ \xrightarrow{\Xi_{\mathcal{Z}_1}^{Y_2, \text{all}}} \end{array} & \text{IndCoh}(\mathcal{Z}_2) \end{array}$$

If in addition f is quasi-compact and schematic, we also have a commutative diagram

$$\begin{array}{ccc} \text{IndCoh}_{Y_1}(\mathcal{Z}_1) & \begin{array}{c} \xleftarrow{\Xi_{\mathcal{Z}_1}^{Y_1, \text{all}}} \\ \xrightarrow{\Psi_{\mathcal{Z}_1}^{Y_1, \text{all}}} \end{array} & \text{IndCoh}(\mathcal{Z}_1) \\ \downarrow & \begin{array}{c} \Xi_{\mathcal{Z}_1}^{Y_2, \text{all}} \\ \Psi_{\mathcal{Z}_1}^{Y_2, \text{all}} \end{array} & \downarrow f_*^{\text{IndCoh}} \\ \text{IndCoh}_{Y_2}(\mathcal{Z}_2) & \begin{array}{c} \xleftarrow{\Psi_{\mathcal{Z}_1}^{Y_2, \text{all}}} \\ \xrightarrow{\Xi_{\mathcal{Z}_1}^{Y_2, \text{all}}} \end{array} & \text{IndCoh}(\mathcal{Z}_2) \end{array}$$

Proof Both assertions follow formally from Corollary 7.5.5. □

8.3.3. Let $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ be a smooth map between quasi-smooth Artin stacks. Let \mathcal{Z}_1^\bullet denote its Čech nerve. Consider the co-simplicial DG category $\text{IndCoh}(\mathcal{Z}_1^\bullet)$ formed by using either $!-$ or $(\text{IndCoh}, *)$ -pullback functors, augmented by $\text{IndCoh}(\mathcal{Z}_2)$.

Let $Y_2 \subset \text{Sing}(\mathcal{Z}_2)$ be a conical Zariski-closed subset. Set $Y_1^\bullet \subset \mathcal{Z}_1^\bullet$ to be equal to

$$\mathcal{Z}_1^\bullet \times_{\mathcal{Z}_2} \text{Sing}(\mathcal{Z}_2).$$

According to Lemma 8.3.2, we have a well-defined full cosimplicial subcategory

$$\text{IndCoh}_{Y_1^\bullet}(\mathcal{Z}_1^\bullet) \subset \text{IndCoh}(\mathcal{Z}_1^\bullet),$$

augmented by $\text{IndCoh}_{Y_2}(\mathcal{Z}_2)$.

Proposition 8.3.4 *Suppose that f is surjective on k -points. Then, the augmentation functor*

$$\text{IndCoh}_{Y_2}(\mathcal{Z}_2) \rightarrow \text{Tot} \left(\text{IndCoh}_{Y_1^\bullet}(\mathcal{Z}_1^\bullet) \right)$$

is an equivalence.

Proof The statement formally follows from Proposition 7.5.7: Smooth descent for schemes implies smooth descent for stacks. To make the argument precise, we consider an auxiliary category of “inputs for IndCoh .” Namely, the objects of the category are pairs (Z, Y) , where Z is a quasi-smooth affine DG scheme, and $Y \subset \text{Sing}(Z)$ is a conical Zariski-closed subset. Morphisms (Z_1, Y_1) to (Z_2, Y_2) are smooth maps $Z_1 \rightarrow Z_2$ whose singular codifferential induces an isomorphism $Y_2 \times_{Z_2} Z_1 \rightarrow Y_1$.

Denote this category by $\text{DGSch}_{\text{aft}, \text{q-smooth} + \text{supp}}^{\text{aff}}$.

Let \mathcal{Z}' be a quasi-smooth Artin stack, and let $Y' \subset \text{Sing}(\mathcal{Z}')$ be a conical Zariski-closed subset. This pair defines a presheaf (\mathcal{Z}', Y') on $\text{DGSch}_{\text{aft}, \text{q-smooth} + \text{supp}}^{\text{aff}}$. Namely, for any $(Z, Y) \in \text{DGSch}_{\text{aft}, \text{q-smooth} + \text{supp}}^{\text{aff}}$, the groupoid

$$\text{Maps}((Z, Y), (\mathcal{Z}', Y'))$$

is the full subgroupoid in $\text{Maps}(Z, \mathcal{Z}')$ consisting of maps $Z \rightarrow \mathcal{Z}'$ that are smooth and whose singular codifferential induces an isomorphism

$$Z \times_{\mathcal{Z}'} Y' \rightarrow Y.$$

The assignment $(Z, Y) \mapsto \text{IndCoh}_Y(Z)$ is a functor

$$\text{IndCoh}_{\text{supp}} : (\text{DGSch}_{\text{aft}, \text{q-smooth} + \text{supp}}^{\text{aff}})^{\text{op}} \rightarrow \text{DGCat}_{\text{cont}}.$$

It follows from [10, Proposition 11.2.2] that the category $\text{IndCoh}_{Y'}(\mathcal{Z}')$ identifies with the value on

$$(\mathcal{Z}', Y') \in \text{PreShv}(\text{DGSch}_{\text{aft}, \text{q-smooth} + \text{supp}}^{\text{aff}})$$

of the right Kan extension of $\text{IndCoh}_{\text{supp}}$ along the Yoneda embedding

$$(\text{DGSch}_{\text{aft},q\text{-smooth}+\text{supp}}^{\text{aff}})^{\text{op}} \hookrightarrow (\text{PreShv}(\text{DGSch}_{\text{aft},q\text{-smooth}+\text{supp}}^{\text{aff}}))^{\text{op}}.$$

Now, Proposition 7.5.7 says that the functor $\text{IndCoh}_{\text{supp}}$ satisfies descent with respect to surjective maps. This implies the assertion of the lemma by [20, 6.2.3.5]. \square

8.3.5. Proposition 8.3.4 allows us to reduce statements concerning morphisms of Artin stacks $f : \mathcal{Z}' \rightarrow \mathcal{Z}$ to the case when \mathcal{Z} is a DG scheme. Such proofs proceed by induction along the hierarchy

$$\text{DGSch}_{\text{aft}} \rightarrow \text{Alg. Spaces} \rightarrow \text{Stk}_{1\text{-Artin}} \rightarrow \text{Stk}_{2\text{-Artin}} \rightarrow \dots$$

Namely, we choose an atlas $Z \rightarrow \mathcal{Z}$ with Z being a DG scheme that is locally almost of finite type. Now if \mathcal{Z} is a k -Artin stack, then the terms of the Čech nerve Z^\bullet are $(k-1)$ -Artin stacks.

8.4 Functorial properties

Let \mathcal{Z}_1 and \mathcal{Z}_2 be two quasi-smooth Artin stacks, and let

$$f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$$

be a map.

8.4.1 *Functoriality under pullbacks* Let $Y_i \subset \text{Sing}(\mathcal{Z}_i)$ be conical Zariski-closed subsets.

Lemma 8.4.2 *Assume that the image of $Y_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1$ under the singular codifferential (8.2) is contained in Y_1 . Then, the functor $f^!$ sends $\text{IndCoh}_{Y_2}(\mathcal{Z}_2)$ to $\text{IndCoh}_{Y_1}(\mathcal{Z}_1)$.*

Proof By Sect. 8.3.5, we reduce the statement to the case when \mathcal{Z}_2 is a DG scheme. In the latter case, the statement from Proposition 7.1.3(a). \square

Similarly, we have:

Lemma 8.4.3 *Assume that the preimage of Y_1 under the singular codifferential (8.2) is contained in $Y_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1$. Then, the functor*

$$\text{IndCoh}(\mathcal{Z}_2) \xrightarrow{f^!} \text{IndCoh}(\mathcal{Z}_1) \xrightarrow{\Psi_{\mathcal{Z}_1}^{Y_1, \text{all}}} \text{IndCoh}_{Y_1}(\mathcal{Z}_1)$$

factors through the colocalization

$$\text{IndCoh}(\mathcal{Z}_2) \xrightarrow{\Psi_{\mathcal{Z}_2}^{Y_2, \text{all}}} \text{IndCoh}_{Y_2}(\mathcal{Z}_2).$$

Proof Again, by Sect. 8.3.5, we reduce the statement to the case when \mathcal{Z}_2 is a DG scheme. In the latter case, the statement from Proposition 7.1.6(b). \square

8.4.4 *Functoriality under pushforwards* Let now

$$f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$$

be schematic and quasi-compact. Recall (see [10, Sect. 10.6]) that in this case, we have a well-defined functor

$$f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{Z}_1) \rightarrow \text{IndCoh}(\mathcal{Z}_2),$$

which satisfies a base-change property with respect to $!$ -pullbacks for maps $\mathcal{Z}'_2 \rightarrow \mathcal{Z}_2$.

Lemma 8.4.5 *Let $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ be schematic and quasi-compact. Assume that the preimage of Y_1 under the singular codifferential (8.2) is contained in $Y_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1$. Then, the functor f_*^{IndCoh} sends $\text{IndCoh}_{Y_1}(\mathcal{Z}_1)$ to $\text{IndCoh}_{Y_2}(\mathcal{Z}_2)$.*

Proof Follows from Proposition 7.1.3(b) by base change. \square

Similarly, we have:

Lemma 8.4.6 *Let $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ be schematic and quasi-compact. Assume that the image of $Y_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1$ under the singular codifferential (8.2) is contained in Y_1 . Then, the functor*

$$\text{IndCoh}(\mathcal{Z}_1) \xrightarrow{f_*^{\text{IndCoh}}} \text{IndCoh}(\mathcal{Z}_2) \xrightarrow{\Psi_{\mathcal{Z}_2}^{Y_2, \text{all}}} \text{IndCoh}_{Y_2}(\mathcal{Z}_2)$$

factors through the colocalization

$$\text{IndCoh}(\mathcal{Z}_1) \xrightarrow{\Psi_{\mathcal{Z}_1}^{Y_1, \text{all}}} \text{IndCoh}_{Y_1}(\mathcal{Z}_1).$$

Proof Follows from Proposition 7.1.6(a) by base change. \square

8.4.7 *Preservation of coherence* We have the following generalization of Proposition 7.2.2:

Corollary 8.4.8 (a) *Let $\mathcal{F}', \mathcal{F}'' \in \text{Coh}(\mathcal{Z})$ be such that, set-theoretically,*

$$\text{SingSupp}(\mathcal{F}') \cap \text{SingSupp}(\mathcal{F}'') = \{0\}.$$

Then, both

$$\mathcal{F}' \otimes \mathcal{F}'' \text{ and } \underline{\text{Hom}}(\mathcal{F}', \mathcal{F}'')$$

belong to $\text{Coh}(\mathcal{Z})$.

(b) Let $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ be a morphism and $\mathcal{F}_2 \in \text{Coh}(\mathcal{Z}_2)$ such that, set-theoretically,

$$\left(\text{SingSupp}(\mathcal{F}_2) \times_{\mathcal{Z}_2} \mathcal{Z}_1 \right) \cap \ker \left(\text{Sing}(f) : \text{Sing}(\mathcal{Z}_2)_{\mathcal{Z}_1} \rightarrow \text{Sing}(\mathcal{Z}_1) \right) \subset \{0\} \times_{\mathcal{Z}_2} \mathcal{Z}_1.$$

Then, $f^!(\mathcal{F}_2) \in \text{IndCoh}(\mathcal{Z}_1)$ belongs to $\text{Coh}(\mathcal{Z}_1) \subset \text{IndCoh}(\mathcal{Z}_1)$, and $f^*(\mathcal{F}_2) \in \text{QCoh}(\mathcal{Z}_1)$ belongs to $\text{Coh}(\mathcal{Z}_1) \subset \text{QCoh}(\mathcal{Z}_1)$.

8.4.9 Conservativeness for finite maps Let now $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ be a finite (and, in particular, affine) map of quasi-smooth Artin stacks. Let $Y_1 \subset \text{Sing}(\mathcal{Z}_1)$ be a conical Zariski-closed subset contained in the image of $\text{Sing}(f)$.

From Corollary 7.3.10, we obtain:

Corollary 8.4.10 *The functor $f_*^{\text{IndCoh}}|_{\text{IndCoh}_{Y_1}(\mathcal{Z}_1)} : \text{IndCoh}_{Y_1}(\mathcal{Z}_1) \rightarrow \text{IndCoh}(\mathcal{Z}_2)$ is conservative.*

8.4.11 Conservativeness for quasi-smooth affine maps Let us prove an extension of Proposition 7.6.4. Suppose \mathcal{Z}_1 and \mathcal{Z}_2 are quasi-smooth Artin stacks, and $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ is a quasi-smooth affine map; in particular, f is schematic and quasi-compact. As in the case of schemes, the singular codifferential

$$\text{Sing}(f) : \text{Sing}(\mathcal{Z}_2)_{\mathcal{Z}_1} \rightarrow \text{Sing}(\mathcal{Z}_1)$$

is a closed embedding; this follows from Lemma 2.4.3 by base change.

Proposition 8.4.12 *Let $Y_2 \subset \text{Sing}(\mathcal{Z}_2)$ be a conical closed subset. Set*

$$Y_1 = \text{Sing}(f) \left(Y_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1 \right) \subset \text{Sing}(\mathcal{Z}_1).$$

(a) *The essential image of $\text{IndCoh}_{Y_2}(\mathcal{Z}_2)$ under the functor $f^!$ generates $\text{IndCoh}_{Y_1}(\mathcal{Z}_1)$.*

(b) *The restriction of the functor f_*^{IndCoh} to $\text{IndCoh}_{Y_1}(\mathcal{Z}_1)$ is conservative.*

Proof By [10, Proposition 10.7.7], we have a pair of adjoint functors

$$f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{Z}_1) \rightleftarrows \text{IndCoh}(\mathcal{Z}_2) : f^!$$

From Lemmas 8.4.2 and 8.4.5, we see that they restrict to a pair of functors

$$f_*^{\text{IndCoh}} : \text{IndCoh}_{Y_1}(\mathcal{Z}_1) \rightleftarrows \text{IndCoh}_{Y_2}(\mathcal{Z}_2) : f^!$$

Moreover, since f is locally eventually coconnective and Gorenstein, we have another pair of adjoint functors

$$f_*^{\text{IndCoh},*} : \text{IndCoh}_{Y_2}(\mathcal{Z}_2) \rightleftarrows \text{IndCoh}_{Y_1}(\mathcal{Z}_1) : f_*^{\text{IndCoh}},$$

where $f^!$ differs from $f^{\text{IndCoh},*}$ by tensoring with the relative dualizing sheaf (see [10, Proposition 7.3.8]). Therefore, the two claims of the proposition are equivalent.

By Sect. 8.3.5, claim (b) is local in smooth topology on \mathcal{Z}_2 ; hence, we may assume that \mathcal{Z}_2 is a DG scheme. This reduces the proposition to Proposition 7.6.4. \square

8.4.13 Quasi-smooth maps of stacks Let $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ be a quasi-smooth map of Artin stacks. Assume now that \mathcal{Z}_2 is quasi-compact and has an affine diagonal. In particular \mathcal{Z}_2 is QCA, and by [6, Corollary 4.3.8], the category $\text{QCoh}(\mathcal{Z}_2)$ is rigid as a monoidal category.

Let $Y_2 \subset \text{Sing}(\mathcal{Z}_2)$ and let

$$Y_1 := \text{Sing}(f) \left(Y_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1 \right) \subset \text{Sing}(\mathcal{Z}_1).$$

Proposition 8.4.14 *Under the above circumstances, the functor*

$$\text{IndCoh}_{Y_2}(\mathcal{Z}_2) \otimes_{\text{QCoh}(\mathcal{Z}_2)} \text{QCoh}(\mathcal{Z}_1) \rightarrow \text{IndCoh}_{Y_1}(\mathcal{Z}_1),$$

induced by the $\text{QCoh}(\mathcal{Z}_2)$ -linear functor

$$f^! : \text{IndCoh}(\mathcal{Z}_2) \rightarrow \text{IndCoh}(\mathcal{Z}_1),$$

is an equivalence.

Proof By definition, we have

$$\text{IndCoh}_{Y_1}(\mathcal{Z}_1) \simeq \lim_{Z_1 \in \text{DGSch}_{/\mathcal{Z}_2, \text{smooth}}^{\text{aff}}} \text{IndCoh}_{Y_1 \times_{Z_1} \mathcal{Z}_1}(Z_1).$$

In addition, by [30, Proposition 5.1.2(b)]

$$\text{QCoh}(\mathcal{Z}_1) \simeq \lim_{Z_1 \in \text{DGSch}_{/\mathcal{Z}_2, \text{smooth}}^{\text{aff}}} \text{QCoh}(Z_1).$$

Since the category $\text{IndCoh}_{Y_2}(\mathcal{Z}_2)$ is dualizable, and $\text{QCoh}(\mathcal{Z}_2)$ is rigid, by [29, Corollaries 4.3.2 and 6.4.2], the formation of

$$\text{IndCoh}_{Y_2}(\mathcal{Z}_2) \otimes_{\text{QCoh}(\mathcal{Z}_2)} -$$

commutes with limits. This reduces the assertion of the proposition to the case when $\mathcal{Z}_1 = Z_1$ is an affine DG scheme.

Choose a smooth atlas $Z_2 \rightarrow \mathcal{Z}_2$, and let Z_2^\bullet be its Čech nerve. Note that the assumption on \mathcal{Z}_2 implies that the terms of Z_2^\bullet are DG schemes (and not Artin stacks).

By Proposition 8.3.4, we obtain that $\text{IndCoh}_{Y_2}(\mathcal{Z}_2)$ is the totalization of $\text{IndCoh}_{Y_2^\bullet}(Z_2^\bullet)$, where

$$Y_2^\bullet := Y_2 \times_{\mathcal{Z}_2} Z_2^\bullet.$$

Since $\text{QCoh}(Z_1)$ is dualizable and $\text{QCoh}(\mathcal{Z}_2)$ is rigid, we obtain that

$$\text{IndCoh}_{Y_2}(\mathcal{Z}_2) \otimes_{\text{QCoh}(\mathcal{Z}_2)} \text{QCoh}(Z_1)$$

maps isomorphically to the totalization of

$$\text{IndCoh}_{Y_2^\bullet}(Z_2^\bullet) \otimes_{\text{QCoh}(\mathcal{Z}_2)} \text{QCoh}(Z_1). \quad (8.5)$$

However,

$$\begin{aligned} & \text{IndCoh}_{Y_2^\bullet}(Z_2^\bullet) \otimes_{\text{QCoh}(\mathcal{Z}_2)} \text{QCoh}(Z_1) \\ & \simeq \text{IndCoh}_{Y_2^\bullet}(Z_2^\bullet) \otimes_{\text{QCoh}(Z_2^\bullet)} \text{QCoh}(Z_2^\bullet) \otimes_{\text{QCoh}(\mathcal{Z}_2)} \text{QCoh}(Z_1). \end{aligned}$$

Now, we claim that the natural functor

$$\text{QCoh}(Z_2^\bullet) \otimes_{\text{QCoh}(\mathcal{Z}_2)} \text{QCoh}(Z_1) \rightarrow \text{QCoh}(Z_2^\bullet \times_{\mathcal{Z}_2} Z_1)$$

is an equivalence. This follows from Lemma 8.4.15 below.

Thus, we obtain that the cosimplicial category (8.5) identifies with

$$\text{IndCoh}_{Y_2 \times_{\mathcal{Z}_2} Z_2^\bullet}(Z_2^\bullet) \otimes_{\text{QCoh}(Z_2^\bullet)} \text{QCoh}(Z_2^\bullet \times_{\mathcal{Z}_2} Z_1) \simeq \text{IndCoh}_{Y_2 \times_{\mathcal{Z}_2} (Z_2^\bullet \times_{\mathcal{Z}_2} Z_1)}(Z_2^\bullet \times_{\mathcal{Z}_2} Z_1),$$

where the last isomorphism takes place due to Proposition 7.5.3.

Now, $Z_2^\bullet \times_{\mathcal{Z}_2} Z_1$ is the Čech nerve of the smooth cover $Z_2 \times_{\mathcal{Z}_2} Z_1 \rightarrow Z_1$, and by Proposition 7.5.7, the totalization of

$$\text{IndCoh}_{Y_2 \times_{\mathcal{Z}_2} (Z_2^\bullet \times_{\mathcal{Z}_2} Z_1)}(Z_2^\bullet \times_{\mathcal{Z}_2} Z_1)$$

is isomorphic to $\text{IndCoh}_{Y_1}(Z_1)$, as required. \square

Lemma 8.4.15 *Let \mathcal{Z} be a quasi-compact stack with an affine diagonal. Then for any two prestacks \mathcal{Z}_1 and \mathcal{Z}_2 mapping to \mathcal{Z} , the naturally defined functor*

$$\text{QCoh}(\mathcal{Z}_1) \otimes_{\text{QCoh}(\mathcal{Z})} \text{QCoh}(\mathcal{Z}_2) \rightarrow \text{QCoh}(\mathcal{Z}_1 \times_{\mathcal{Z}} \mathcal{Z}_2)$$

is an equivalence, provided that one of the categories $\text{QCoh}(\mathcal{Z}_1)$ or $\text{QCoh}(\mathcal{Z}_2)$ is dualizable.

Proof This follows by combining [30, Proposition 3.3.3] and [6, Corollary 4.3.8]. \square

8.4.16. Let \mathcal{Z} be again a quasi-compact stack with an affine diagonal. Let $\mathcal{V} \subset \mathcal{Z}$ and $Y \subset \text{Sing}(\mathcal{Z})$ be as in Corollary 8.2.10.

In a way analogous to the proof of Proposition 8.4.14, one shows:

Proposition 8.4.17 *Under the above circumstances, the short exact sequence of categories*

$$\text{IndCoh}_{Y_{\mathcal{V}}}(\mathcal{Z}) \rightleftarrows \text{IndCoh}_Y(\mathcal{Z}) \rightleftarrows \text{IndCoh}_{Y \times_{\mathcal{Z}} \mathcal{U}}(\mathcal{U})$$

is obtained from

$$\text{QCoh}(\mathcal{Z})_{\mathcal{V}} \rightleftarrows \text{QCoh}(\mathcal{Z}) \xrightarrow{j^*} \text{QCoh}(\mathcal{U})$$

by tensoring with $\text{IndCoh}_Y(\mathcal{Z})$ over $\text{QCoh}(\mathcal{Z})$.

8.4.18 *Conservativeness for proper maps of stacks* Suppose now that $f : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ is a schematic proper morphism between quasi-smooth Artin stacks. Let $Y_1 \subset \text{Sing}(\mathcal{Z}_1)$ be a conical closed subset, and let Y_2 be the image of

$$(\text{Sing}(f))^{-1}(Y_1) \subset \text{Sing}(\mathcal{Z}_2)_{\mathcal{Z}_1}$$

under the projection

$$\text{Sing}(\mathcal{Z}_2)_{\mathcal{Z}_1} \rightarrow \text{Sing}(\mathcal{Z}_2).$$

Since the projection is proper, $Y_2 \subset \text{Sing}(\mathcal{Z}_2)$ is a closed subset.

By Lemma 8.4.5, the functor f_*^{IndCoh} sends $\text{IndCoh}_{Y_1}(\mathcal{Z}_1)$ to $\text{IndCoh}_{Y_2}(\mathcal{Z}_2)$. We have the following generalization of Theorem 7.8.2.

Proposition 8.4.19 *Under the above circumstances, the essential image of $\text{IndCoh}_{Y_1}(\mathcal{Z}_1)$ under f_*^{IndCoh} generates $\text{IndCoh}_{Y_2}(\mathcal{Z}_2)$.*

Proof It is enough to verify that the claim is local on \mathcal{Z}_2 in the smooth topology; one can then use Theorem 7.8.2. Indeed, the proposition is equivalent to the claim that the functor right adjoint to

$$f_*^{\text{IndCoh}} : \text{IndCoh}_{Y_1}(\mathcal{Z}_1) \rightarrow \text{IndCoh}_{Y_2}(\mathcal{Z}_2)$$

is conservative. The right adjoint in question is the composition

$$\text{IndCoh}_{Y_2}(\mathcal{Z}_2) \xrightarrow{\Xi_{\mathcal{Z}_2}^{Y_2, \text{all}}} \text{IndCoh}(\mathcal{Z}_2) \xrightarrow{f^!} \text{IndCoh}(\mathcal{Z}_1) \xrightarrow{\Psi_{\mathcal{Z}_1}^{Y_1, \text{all}}} \text{IndCoh}_{Y_1}(\mathcal{Z}_1).$$

The locality of this assertion follows from Sect. 8.3.5. \square

9 Global complete intersection stacks

In this section, we adapt the approach of Sect. 5 to stacks. Our main objective is to show that for a quasi-compact algebraic stack \mathcal{Z} , globally given as a “complete intersection,” and $Y \subset \text{Sing}(\mathcal{Z})$, the corresponding category $\text{IndCoh}_Y(\mathcal{Z})$ is compactly generated. The precise meaning of the words “global complete intersection stack” is explained in Sect. 9.2.

As was mentioned earlier, this section may be skipped on the first pass.

In this section, all Artin stacks will be quasi-compact with an affine diagonal (in particular, they all are QCA algebraic stacks in the sense of [6]).

9.1 Relative Koszul duality

9.1.1. Let us consider a relative version of the setting of Sect. 5. Let \mathcal{X} be a smooth stack, $\mathcal{V} \rightarrow \mathcal{X}$ a smooth schematic map, and let $\mathcal{X} \rightarrow \mathcal{V}$ be a section.

Consider the fiber product

$$\mathcal{G}_{\mathcal{X}/\mathcal{V}} = \mathcal{X} \times_{\mathcal{V}} \mathcal{X}.$$

As in Sect. 5.6.1, it is naturally a group DG scheme over \mathcal{X} .

9.1.2. The group structure on $\mathcal{G}_{\mathcal{X}/\mathcal{V}}$ over \mathcal{X} turns $\text{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$ into a monoidal category over the symmetric monoidal category $\text{QCoh}(\mathcal{X})$; the operation on $\text{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$ is the convolution. The unit object of $\text{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$ is

$$(\Delta_{\mathcal{X}})_*^{\text{IndCoh}}(\omega_{\mathcal{X}}) \in \text{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}),$$

where $\omega_{\mathcal{X}}$ is the dualizing complex on \mathcal{X} . Its endomorphisms naturally form an \mathbb{E}_2 -algebra in the symmetric monoidal category $\text{QCoh}(\mathcal{X})$ (see Sect. E.2). Denote this \mathbb{E}_2 -algebra by $\text{HC}(\mathcal{X}/\mathcal{V})$.

Moreover, $(\Delta_{\mathcal{X}})_*^{\text{IndCoh}}(\omega_{\mathcal{X}})$ generates $\text{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$ over $\text{QCoh}(\mathcal{X})$. Therefore, taking maps from $(\Delta_{\mathcal{X}})_*^{\text{IndCoh}}(\omega_{\mathcal{X}})$ defines an equivalence of monoidal categories:

$$\text{KD}_{\mathcal{X}/\mathcal{V}} : \text{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \rightarrow \text{HC}(\mathcal{X}/\mathcal{V})^{\text{op-mod}}.$$

This equivalence is the relative version of the Koszul duality (5.3).

Lemma 9.1.3 *The monoidal category $\text{HC}(\mathcal{X}/\mathcal{V})^{\text{op-mod}}$ is rigid and compactly generated.*

Proof First, note that $\text{QCoh}(\mathcal{X})$ is compactly generated, that is, that \mathcal{X} is a perfect stack. Indeed, \mathcal{X} is smooth, so it suffices to show that $\text{IndCoh}(\mathcal{X})$ is compactly generated. The latter statement holds because \mathcal{X} is a QCA stack, (see [6, Theorem 0.4.5]).

To show that $\text{HC}(\mathcal{X}/\mathcal{V})^{\text{op-mod}}$ is rigid and compactly generated, we must show that it admits a family of compact dualizable generators (see [29, Lemma 5.1.1 and

Proposition 5.2.3]). It is not hard to see that $\mathrm{HC}(\mathcal{X}/\mathcal{V})^{\mathrm{op}}$ -modules induced from perfect objects of $\mathrm{QCoh}(\mathcal{X})$ form such a family. Let us prove the corresponding general statement.

Let \mathbf{O} be a symmetric monoidal category that is compactly generated and rigid as a monoidal category. Then for any \mathbb{E}_2 -algebra \mathcal{A} in \mathbf{O} , the monoidal category $\mathcal{A}\text{-mod}(\mathbf{O})$ is rigid and compactly generated.

Indeed, for $\mathfrak{o} \in \mathbf{O}^c$, the object $\mathcal{A} \otimes \mathfrak{o}$ is compact in $\mathcal{A}\text{-mod}(\mathbf{O})$. Clearly, such compact objects generate $\mathcal{A}\text{-mod}(\mathbf{O})$.

Since \mathbf{O} is rigid, its compact objects are dualizable. Hence, $\mathcal{A} \otimes \mathfrak{o}$ is also dualizable: Its dual is $\mathcal{A} \otimes \mathfrak{o}^\vee$. Thus, $\mathcal{A}\text{-mod}(\mathbf{O})$ admits a family of compact dualizable generators, as required. \square

9.1.4. As in Lemma 5.1.4, we obtain that the \mathbb{E}_1 -algebra underlying $\mathrm{HC}(\mathcal{X}/\mathcal{V})$ is canonically isomorphic to $\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])$, where V is the pullback along $\mathcal{X} \rightarrow \mathcal{V}$ of the relative tangent sheaf to $\mathcal{V} \rightarrow \mathcal{X}$.

In particular, we have a canonical identification

$$\mathrm{HC}(\mathcal{X}/\mathcal{V})^{\mathrm{op}\text{-mod}} \simeq \mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])\text{-mod} \quad (9.1)$$

as module categories over $\mathrm{QCoh}(\mathcal{X})$.

Remark 9.1.5 A remark parallel to Remark 5.1.11 applies in the present situation.

9.1.6. Clearly, $\mathrm{Sing}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \simeq V^*$, where V^* denotes the total space of the corresponding vector bundle over \mathcal{X} .

Let $Y \subset V^*$ be a conical Zariski-closed subset. Let us denote by

$$\mathrm{HC}(\mathcal{X}/\mathcal{V})^{\mathrm{op}\text{-mod}}_Y \subset \mathrm{HC}(\mathcal{X}/\mathcal{V})^{\mathrm{op}\text{-mod}} \simeq \mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])\text{-mod}$$

the full subcategory of objects supported on Y . If \mathcal{X} is an affine scheme, it can be defined via the formalism of Sect. 3.5.1; in general, we define it using an affine atlas $X \rightarrow \mathcal{X}$.

Corollary 9.1.7 *The functor $\mathrm{KD}_{\mathcal{X}/\mathcal{V}}$ provides an equivalence between $\mathrm{IndCoh}_Y(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$ and $\mathrm{HC}(\mathcal{X}/\mathcal{V})^{\mathrm{op}\text{-mod}}_Y$.*

Proof The claim is local in the smooth topology on \mathcal{X} . Therefore, we may assume that \mathcal{X} is affine, and the assertion follows from the definition. \square

9.2 Explicit presentation of a quasi-smooth stack

9.2.1. Let \mathcal{Z} be an Artin stack, and assume that we have a commutative diagram

$$\begin{array}{ccc}
 \mathcal{Z} & \xrightarrow{\iota} & \mathcal{U} \\
 \downarrow & & \downarrow \\
 \mathcal{X} & \xrightarrow{\quad} & \mathcal{V} \\
 \searrow \text{id} & & \swarrow \\
 & \mathcal{X} &
 \end{array}
 \tag{9.2}$$

where the upper square is Cartesian, the lower portion of the diagram is as in Sect. 9.1, and \mathcal{U} is a smooth stack. Recall that \mathcal{X} is assumed to be smooth, and that \mathcal{V} is smooth and schematic over \mathcal{X} .

In this situation, we say that \mathcal{Z} is presented as a global complete intersection stack. It is easy to see that such \mathcal{Z} is quasi-smooth.

9.2.2. We have a commutative diagram:

$$\begin{array}{ccccc}
 & & \mathcal{G}_{\mathcal{Z}/\mathcal{U}} & & \\
 & \swarrow & \downarrow & \searrow & \\
 \mathcal{Z} & & & & \mathcal{Z} \\
 \downarrow & & \mathcal{G}_{\mathcal{X}/\mathcal{V}} & & \downarrow \\
 \mathcal{X} & \swarrow & & \searrow & \mathcal{X}
 \end{array}
 \tag{9.3}$$

in which both parallelograms are Cartesian.

In particular, as in Sect. 5.3.1, we obtain that the relative group DG scheme $\mathcal{G}_{\mathcal{X}/\mathcal{V}}$ canonically acts on \mathcal{Z} .

9.2.3. We obtain homomorphisms of monoidal categories

$$\begin{aligned}
 & \text{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\text{QCoh}(\mathcal{X})} \text{QCoh}(\mathcal{U}) \\
 & \simeq \text{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\text{QCoh}(\mathcal{X})} \text{QCoh}(\mathcal{U}) \rightarrow \text{IndCoh}(\mathcal{G}_{\mathcal{Z}/\mathcal{U}}).
 \end{aligned}$$

This allows us to view $\text{IndCoh}(\mathcal{Z})$ as a category tensored over

$$\text{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\text{QCoh}(\mathcal{X})} \text{QCoh}(\mathcal{U}).$$

9.2.4. Let $Y \subset V^* \times_{\mathcal{X}} \mathcal{U}$ be a conical Zariski-closed subset. We can attach to it a full subcategory

$$\left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{U}) \right)_Y \subset \mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{U}) \quad (9.4)$$

by interpreting

$$\begin{aligned} \mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{U}) &\simeq \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}) \\ &\simeq \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}), \end{aligned}$$

where the latter equivalence follows from Lemma 8.4.15 and Proposition 8.4.14 by

$$\begin{aligned} &\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}) \\ &= \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})} \mathrm{QCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}) \\ &\simeq \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})} \mathrm{QCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}) \simeq \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}). \end{aligned}$$

Finally, we note that

$$\mathrm{Sing}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}) \simeq V^* \times_{\mathcal{X}} \mathcal{U},$$

and we let the subcategory (9.4) correspond to

$$\mathrm{IndCoh}_Y(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}) \subset \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}).$$

9.2.5. We have a canonical closed embedding

$$\mathrm{Sing}(\mathcal{Z}) \hookrightarrow V^* \times_{\mathcal{X}} \mathcal{Z}. \quad (9.5)$$

The following assertion is parallel to Corollary 5.3.5.

Lemma 9.2.6 *For a conical Zariski-closed subset $Y \subset \mathrm{Sing}(\mathcal{Z})$,*

$$\begin{aligned} &\mathrm{IndCoh}_Y(\mathcal{Z}) \\ &\simeq \mathrm{IndCoh}(\mathcal{Z}) \otimes_{\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U})} \left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}) \right)_Y. \end{aligned}$$

Proof First, Lemma 8.4.15 reduces the assertion to the case when \mathcal{U} is an affine DG scheme.

Note that for any morphism $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ of prestacks and an associative algebra $\mathcal{A}_2 \in \mathrm{QCoh}(\mathcal{X}_2)$, the natural functor

$$\mathcal{A}_2\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X}_2)} \mathrm{QCoh}(\mathcal{X}_1) \rightarrow \mathcal{A}_1\text{-mod}$$

is an equivalence (here $\mathcal{A}_1 := f^*(\mathcal{A}_2)$). This follows from [29, Proposition 4.8.1].

This observation, combined with Lemma 8.4.15, reduces the assertion to the case when \mathcal{X} is an affine DG scheme. In the latter case, the assertion follows from Corollary 5.3.5. \square

Corollary 9.2.7 *For any conical Zariski-closed subset $Y \subset \mathrm{Sing}(\mathbb{Z})$, the category $\mathrm{IndCoh}_Y(\mathbb{Z})$ is compactly generated.*

Proof Since the monoidal category $\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U})$ is rigid, and $\mathrm{IndCoh}(\mathbb{Z})$ is compactly generated, it suffices to show that

$$\left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}) \right)_Y$$

is compactly generated. By (9.1), the latter is equivalent to

$$\left(\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}) \right)_Y$$

being compactly generated, which in turn would follow from the compact generation of

$$\left(\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}) \right)_Y^{\mathbb{G}_m},$$

where \mathbb{G}_m acts on V by dilations.

Let us now compare the categories $\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])\text{-mod}$ and $\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V)\text{-mod}$. While the two categories are not equivalent, they differ only by a shift of grading; this implies that their categories of \mathbb{G}_m -equivariant objects are naturally equivalent (we discuss the framework of the grading shift in Sect. A.2.2). The same applies to the categories

$$\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U})$$

and

$$\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V)\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}).$$

Hence,

$$\begin{aligned} & \left(\text{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])\text{-mod} \otimes_{\text{QCoh}(\mathcal{X})} \text{QCoh}(\mathcal{U}) \right)_Y^{\mathbb{G}_m} \\ & \simeq \left(\text{Sym}_{\mathcal{O}_{\mathcal{X}}}(V)\text{-mod} \otimes_{\text{QCoh}(\mathcal{X})} \text{QCoh}(\mathcal{U}) \right)_Y^{\mathbb{G}_m} \simeq \text{QCoh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U})_{Y/\mathbb{G}_m}, \end{aligned}$$

and the latter is easily seen to be compactly generated by

$$\text{Coh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U})_{Y/\mathbb{G}_m} = \text{QCoh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U})^{\text{perf}} \cap \text{QCoh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U})_{Y/\mathbb{G}_m},$$

since the stack $(V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U}$ is smooth. \square

Corollary 9.2.8 *Under the circumstances of Corollary 9.2.7, we have:*

$$\text{IndCoh}_Y(\mathcal{Z}) \simeq \text{Ind}(\text{Coh}_Y(\mathcal{Z})),$$

where $\text{Coh}_Y(\mathcal{Z}) := \text{IndCoh}_Y(\mathcal{Z}) \cap \text{Coh}(Z)$.

Proof By Corollary 9.2.7, it suffices to show that

$$(\text{IndCoh}_Y(\mathcal{Z}))^c = \text{IndCoh}_Y(\mathcal{Z}) \cap \text{Coh}(Z),$$

as subcategories of $\text{IndCoh}_Y(\mathcal{Z})$.

However, this follows from the fact that the functor $\text{IndCoh}_Y(\mathcal{Z}) \hookrightarrow \text{IndCoh}(\mathcal{Z})$ admits a continuous right adjoint and hence sends compacts to compacts, is fully faithful, and

$$\text{Coh}(\mathcal{Z}) = \text{IndCoh}(\mathcal{Z})^c$$

(the latter is [6, Proposition 3.4.2(b)]). \square

9.3 Parallelized situation

Assume now that in diagram (9.2), the map $\mathcal{V} \rightarrow \mathcal{X}$ has been parallelized. That is, assume that \mathcal{V} is a vector bundle V over \mathcal{X} , and the section $\mathcal{X} \rightarrow \mathcal{V}$ is the zero-section.

9.3.1. The diagram (9.2) can then be simplified, at least assuming that the rank of the vector bundle V is constant on \mathcal{X} (for instance, this is true if \mathcal{X} is connected). Indeed, suppose that $\text{rk}(V) = n$. By definition, the vector bundle V on \mathcal{X} defines a morphism from \mathcal{X} into the classifying stack $\text{pt}/\text{GL}(n)$. Clearly,

$$V = \mathcal{X} \times_{\text{pt}/\text{GL}(n)} (\mathbb{A}^n/\text{GL}(n)).$$

Consider the composition

$$\mathcal{U} \rightarrow V \rightarrow (\mathbb{A}^n / \mathrm{GL}(n));$$

we then have

$$\mathcal{Z} = \mathcal{U} \times_{\mathbb{A}^n / \mathrm{GL}(n)} (\mathrm{pt} / \mathrm{GL}(n)),$$

where we embed pt into \mathbb{A}^n as the origin. In other words, we may assume that $\mathcal{X} = \mathrm{pt} / \mathrm{GL}(n)$ and $V = \mathbb{A}^n / \mathrm{GL}(n)$ in (9.2).

In more explicit terms, \mathcal{U} is equipped with a rank n vector bundle and a section, and \mathcal{Z} is the zero locus of this section. That is, we may assume that $\mathcal{X} = \mathcal{U}$ in (9.2) and that the composition $\mathcal{U} \rightarrow V \rightarrow \mathcal{X}$ is the identity map.

9.3.2. As in Lemma 5.4.2, we obtain that $\mathrm{HC}(\mathcal{X}/\mathcal{V})$ is canonically isomorphic to $\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])$ as an \mathbb{E}_2 -algebra.

In particular, we obtain that for \mathcal{Z} in (9.2), the category $\mathrm{IndCoh}(\mathcal{Z})$ is tensored over the monoidal category

$$\mathrm{QCoh}(V^*/\mathbb{G}_m) \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{QCoh}(\mathcal{U}) \simeq \mathrm{QCoh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U}).$$

Moreover, we have the following version of Lemma 9.2.6:

Corollary 9.3.3 *For a conical Zariski-closed subset $Y \subset \mathrm{Sing}(\mathcal{Z})$, we have*

$$\mathrm{IndCoh}_Y(\mathcal{Z}) = \mathrm{IndCoh}(\mathcal{Z}) \otimes_{\mathrm{QCoh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U})} \mathrm{QCoh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U})_{Y/\mathbb{G}_m} \quad (9.6)$$

as full subcategories of $\mathrm{IndCoh}(\mathcal{Z})$.

Proof Follows from the fact that

$$\mathrm{Vect} \otimes_{\mathrm{QCoh}(\mathrm{pt}/\mathbb{G}_m)} \mathrm{QCoh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U}) \simeq \mathrm{QCoh}(V^* \times_{\mathcal{X}} \mathcal{U})$$

as monoidal categories, and

$$\mathrm{Vect} \otimes_{\mathrm{QCoh}(\mathrm{pt}/\mathbb{G}_m)} \mathrm{QCoh}((V^*/\mathbb{G}_m) \times_{\mathcal{X}} \mathcal{U})_{Y/\mathbb{G}_m} \simeq \mathrm{QCoh}(V^* \times_{\mathcal{X}} \mathcal{U})_Y$$

as modules over them. □

9.4 Generating the category defined by singular support on a stack

9.4.1. As in Sect. 5.6.1, we have a tautologically defined functor

$$\mathbf{G} : \mathrm{IndCoh}(\mathcal{Z}) \rightarrow \mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z}),$$

and its left adjoint

$$\mathbf{F} : \mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z}) \rightarrow \mathrm{IndCoh}(\mathcal{Z}).$$

These functors are obtained as pullback and pushforward, respectively, for the action map

$$\mathrm{act}_{\mathcal{G}_{\mathcal{X}/\mathcal{V}}, \mathcal{Z}} : \mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{Z} \rightarrow \mathcal{Z}.$$

We have the following versions of Corollaries 5.6.6 and 5.6.7.

Proposition 9.4.2 *For any conical Zariski-closed subset $Y \subset \mathcal{V}^* \times_{\mathcal{X}} \mathcal{U}$, the functors \mathbf{F} and \mathbf{G} restrict to a pair of adjoint functors*

$$\mathbf{F} : \left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z}) \right)_Y \rightleftarrows \mathrm{IndCoh}_{Y \cap \mathrm{Sing}(\mathcal{Z})}(\mathcal{Z}) : \mathbf{G}.$$

Moreover, the diagram

$$\begin{array}{ccc} \mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z}) & \rightleftarrows & \mathrm{IndCoh}(\mathcal{Z}) \\ \downarrow & & \downarrow \Psi_{\mathcal{Z}}^{Y, \mathrm{all}} \\ \left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z}) \right)_Y & \rightleftarrows & \mathrm{IndCoh}_{Y \cap \mathrm{Sing}(\mathcal{Z})}(\mathcal{Z}). \end{array}$$

commutes. Here, the left vertical arrow is the right adjoint to the inclusion

$$\left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z}) \right)_Y \hookrightarrow \mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z}).$$

Proof Reduces to the case of DG schemes as in the proof of Lemma 9.2.6. \square

Corollary 9.4.3 *Suppose Y is a conical Zariski-closed subset of $\mathrm{Sing}(\mathcal{Z}) \subset \mathcal{V}^* \times_{\mathcal{X}} \mathcal{U}$.*

(a) *For any $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Z})$, we have:*

$$\mathcal{F} \in \mathrm{IndCoh}_Y(\mathcal{Z}) \Leftrightarrow \mathbf{G}(\mathcal{F}) \in \left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z}) \right)_Y.$$

(b) *The essential image under \mathbf{F} of the category $\left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z})\right)_Y$ generates $\mathrm{IndCoh}_Y(\mathcal{Z})$.*

Proof The corollary formally follows from the conservativeness of \mathbf{G} , similarly to the proof of Corollary 5.6.7. Namely, $\mathcal{F} \in \mathrm{IndCoh}_Y(\mathcal{Z})$ if and only if the natural morphism $\Psi_{\mathcal{Z}}^{Y,\mathrm{all}}(\mathcal{F}) \rightarrow \mathcal{F}$ is an isomorphism. Since \mathbf{G} is conservative, this happens if and only if the morphism $\mathbf{G}(\Psi_{\mathcal{Z}}^{Y,\mathrm{all}}(\mathcal{F})) \rightarrow \mathbf{G}(\mathcal{F})$ is an isomorphism; by Proposition 9.4.2, this is equivalent to

$$\mathbf{G}(\mathcal{F}) \in \left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z})\right)_Y.$$

We have therefore proved part (a). Also, by Proposition 9.4.2, the restriction

$$\mathbf{F} : \left(\mathrm{HC}(\mathcal{X}/\mathcal{V})\text{-mod} \otimes_{\mathrm{QCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{Z})\right)_Y \rightarrow \mathrm{IndCoh}_{Y \cap \mathrm{Sing}(\mathcal{Z})}(\mathcal{Z})$$

is left adjoint to a conservative functor; this proves part (b). □

Part III: The geometric Langlands conjecture

10 The stack LocSys_G : recollections

In this section, G is an arbitrary affine algebraic group. Given a DG scheme X , we will recall the construction of the stack $\mathrm{LocSys}_G(X)$ of G -local systems on X . We will compute its tangent and cotangent complexes. When X is a smooth and complete curve, we will show that $\mathrm{LocSys}_G(X)$ is quasi-smooth and calculate the corresponding classical stack $\mathrm{Sing}(\mathrm{LocSys}_G(X))$. We will also show that $\mathrm{LocSys}_G(X)$ can in fact be written as a “global complete intersection” as in Sect. 9.

This section may be skipped if the reader is willing to take the existence of the stack $\mathrm{LocSys}_G(X)$ and its basic properties on faith.

10.1 Definition of LocSys_G

As the stack $\mathrm{LocSys}_G(X)$ of local systems is in general an object of derived algebraic geometry, some extra care is required. In this subsection, we give the relevant definitions. Since the discussion will be purely technical, the reader can skip this subsection and return to it when necessary.

For the duration of this subsection, we remove the a priori assumption that all prestacks are locally almost of finite type.

10.1.1. Let X be an arbitrary DG scheme almost of finite type. We define the prestacks $\mathrm{Bun}_G(X)$ and $\mathrm{LocSys}_G(X)$ using the general framework of Appendix B.

Namely, for $S \in \text{DGSch}^{\text{aff}}$, we set

$$\text{Maps}(S, \text{Bun}_G(X)) := \text{Maps}(S \times X, \text{pt}/G)$$

and

$$\text{Maps}(S, \text{LocSys}_G(X)) := \text{Maps}(S \times X_{\text{dR}}, \text{pt}/G),$$

respectively. Here, X_{dR} denotes the de Rham prestack of X , see [12, Sect. 1.1.1].

The natural projection $X \rightarrow X_{\text{dR}}$ defines the forgetful map

$$\text{LocSys}_G(X) \rightarrow \text{Bun}_G(X). \quad (10.1)$$

10.1.2. The following is tautological:

Lemma 10.1.3 *Suppose that X is classical. Then, the classical prestack ${}^{\text{cl}}\text{Bun}_G(X)$ is the usual prestack of G -bundles on X , i.e., for $S \in \text{Sch}^{\text{aff}}$,*

$$\text{Maps}(S, \text{Bun}_G(X)) = \text{Maps}_{\text{cl-PreStk}}(S \times X, \text{pt}/G).$$

Remark 10.1.4 Note that when X is *not* classical, the classical prestack ${}^{\text{cl}}\text{Bun}_G(X)$ cannot be recovered from classical algebraic geometry. For instance, take X to be a DG scheme with ${}^{\text{cl}}X = \text{pt}$. Then, the ∞ -groupoid $\text{Maps}(\text{pt}, \text{Bun}_G(X))$ is that of G -bundles on X , which can have nonzero homotopy groups up to degree $n + 1$ if X is n -coconnective.

10.1.5. Similarly, we have:

Lemma 10.1.6 *Let X be arbitrary (but still almost of finite type). Then, the classical prestack ${}^{\text{cl}}\text{LocSys}_G(X)$ is the usual prestack of G -local systems on X , i.e., for $S \in \text{Sch}^{\text{aff}}$,*

$$\text{Maps}(S, \text{LocSys}_G(X)) = \text{Maps}_{\text{cl-PreStk}}(S \times X_{\text{dR}}, \text{pt}/G).$$

Proof Follows from the fact that $X_{\text{dR}} \in \text{PreStk}$ is classical, see [12, Proposition 1.3.3(b)]. \square

Remark 10.1.7 For a more familiar description of ${}^{\text{cl}}\text{LocSys}_G(X)$ via the Tannakian formalism, see Sect. 10.2.5.

10.2 A Tannakian description

One can describe the ∞ -groupoids

$$\text{Maps}(S \times X, \text{pt}/G) \text{ and } \text{Maps}(S \times X_{\text{dR}}, \text{pt}/G)$$

in more intuitive terms using Tannakian duality.

The material in this subsection will not be used elsewhere in the paper.

10.2.1. Let \mathcal{X} be any prestack (which we will take to be either $S \times X$ or $S \times X_{\mathrm{dR}}$). Then by [23, Theorem 3.4.2], the ∞ -groupoid $\mathrm{Maps}(\mathcal{X}, \mathrm{pt}/G)$ identifies with the full subcategory of symmetric monoidal functors

$$\mathrm{Rep}(G) := \mathrm{QCoh}(\mathrm{pt}/G) \xrightarrow{\Phi} \mathrm{QCoh}(\mathcal{X})$$

that satisfy:

- Φ is continuous;
- Φ is right t-exact, i.e., sends $\mathrm{Rep}(G)^{\leq 0}$ to $\mathrm{QCoh}(\mathcal{X})^{\leq 0}$ (see [30, Sect. 1.2.3] for the definition of the t-structure on QCoh over an arbitrary prestack);
- Φ sends *flat* objects to *flat* objects (an object of QCoh on a prestack is said to be flat if its pullback to an arbitrary affine DG scheme is flat.⁸)

Remark 10.2.2 The above description of $\mathrm{Maps}(\mathcal{X}, -)$ is valid for pt/G replaced by any *geometric stack* (see [23, Definition 3.4.1]). Note that any quasi-compact algebraic stack with an affine diagonal is geometric.

10.2.3. Since $\mathrm{Rep}(G)$ identifies with the derived category of its heart, the above category of symmetric monoidal functors can be identified with that of symmetric monoidal functors

$$\mathrm{Rep}(G)^{\heartsuit, c} = \mathrm{Rep}(G)^{\heartsuit} \cap \mathrm{Rep}(G)^c \xrightarrow{\Phi} \mathrm{QCoh}(\mathcal{X})^{\leq 0},$$

such that:

- Φ takes short exact sequences in $\mathrm{Rep}(G)^{\heartsuit, c}$ to distinguished triangles in $\mathrm{QCoh}(\mathcal{X})^{\leq 0}$.

Remark 10.2.4 Note that every object of $\mathrm{Rep}(G)^{\heartsuit, c}$ is dualizable. Therefore, the same holds for its image under such Φ . From this, it is easy to see that the essential image of $\mathrm{Rep}(G)^{\heartsuit, c}$ automatically belongs to the full subcategory $\mathrm{QCoh}(\mathcal{X})^{\mathrm{loc. free}}$ of $\mathrm{QCoh}(\mathcal{X})$ spanned by locally free sheaves of finite rank (i.e., those objects whose pullback to any affine DG scheme S is a direct summand of $\mathcal{O}_S^{\oplus n}$ for some integer n).

10.2.5. Assume for a moment that X is classical. We obtain that Sect. 10.2.3 recovers the usual description of the classical prestack ${}^{\mathrm{cl}}\mathrm{Bun}_G(X)$. Namely, for $S \in \mathrm{Sch}^{\mathrm{aff}}$, the groupoid $\mathrm{Maps}(S, \mathrm{Bun}_G(X))$ is that of exact symmetric monoidal functors

$$\Phi : \mathrm{Rep}(G)^{\heartsuit, c} \rightarrow \mathrm{QCoh}(S \times X)^{\heartsuit}.$$

Note that by Remark 10.2.4, the essential image of such Φ consists of vector bundles on $S \times X$.

⁸ If A is a connective ring, an A -module M is flat if M is connective, and $H^0(A) \otimes_A M$ is acyclic off degree 0 and is flat over $H^0(A)$.

Similarly, for an arbitrary X , we obtain the usual description of the classical prestack ${}^{cl}\text{LocSys}_G(X)$. Namely, for $S \in \text{Sch}^{\text{aff}}$, the groupoid $\text{Maps}(S, \text{LocSys}_G(X))$ is that of exact symmetric monoidal functors

$$\Phi : \text{Rep}(G)^{\heartsuit, c} \rightarrow \text{QCoh}(S \times X_{\text{dR}})^{\heartsuit}.$$

By Remark 10.2.4, the essential image of such Φ belongs to the subcategory of $\text{QCoh}(S \times X_{\text{dR}})$ that consists of S -families of local systems on X .

Remark 10.2.6 To obtain the above Tannakian description of ${}^{cl}\text{Bun}_G(X)$ (for X classical) and ${}^{cl}\text{LocSys}_G(X)$ (for X arbitrary), one needs something weaker than the full strength of [23, Theorem 3.4.2]. Namely, one can make do with its classical version, given by [24, Theorem 5.11].

10.3 Basic properties of Bun_G and LocSys_G

In this subsection, we will calculate the (pro)-cotangent spaces of $\text{Bun}_G(X)$ and $\text{LocSys}_G(X)$. In addition, we will show that if X is classical and proper, the forgetful map $\text{LocSys}_G(X) \rightarrow \text{Bun}_G(X)$ is schematic and affine.

Throughout this subsection we will assume that X is eventually coconnective.

10.3.1. Let X be l -coconnected. It follows from Proposition B.3.2 that $\text{Bun}_G(X)$ admits a $(-l - 1)$ -connective deformation theory.

Similarly, since X_{dR} is classical, we obtain that $\text{LocSys}_G(X)$ admits a (-1) -connective deformation theory.

10.3.2. The pro-cotangent spaces of $\text{Bun}_G(X)$ and $\text{LocSys}_G(X)$ can be described as follows:

The stack pt/G admits co-representable (-1) -connective deformation theory, and its cotangent complex identifies with $\mathfrak{g}_{\mathcal{P}^{univ}}^*[-1]$, where \mathcal{P}^{univ} is the universal G -bundle on pt/G , and $\mathfrak{g}_{\mathcal{P}^{univ}}^*$ is the vector bundle on pt/G associated with the coadjoint representation.

Let \mathcal{P} (resp., (\mathcal{P}, ∇)) be an S -point of $\text{Bun}_G(X)$ (resp., $\text{LocSys}_G(X)$). Then by Proposition B.3.2(b), the pro-cotangent space to $\text{Bun}_G(X)$ (resp., $\text{LocSys}_G(X)$) at the above point, viewed as a functor

$$\text{QCoh}(S)^{\leq 0} \rightarrow \infty\text{-Grpd},$$

identifies with

$$\mathcal{M} \mapsto \Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}})[1] \tag{10.2}$$

and

$$\mathcal{M} \mapsto \Gamma(S \times X_{\text{dR}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}})[1], \tag{10.3}$$

respectively, where $\mathfrak{g}_{\mathcal{P}}$ denotes the bundle associated with the adjoint representation.

The relative pro-cotangent space to the map (10.1) at the above point is the functor

$$\mathcal{M} \mapsto \ker\left(\Gamma(S \times X_{\mathrm{dR}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}})[1] \rightarrow \Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}})[1]\right). \quad (10.4)$$

All of the above functors commute with colimits.

10.3.3. Note that by Corollary B.3.4, if X is proper, the pro-cotangent spaces to $\mathrm{Bun}_G(X)$ and $\mathrm{LocSys}_G(X)$ are co-representable by objects of $\mathrm{QCoh}(S)^-$.

In other words, $\mathrm{Bun}_G(X)$ (resp., $\mathrm{LocSys}_G(X)$) admits a *co-representable* $(-l-1)$ -connective (resp., (-1) -connective) deformation theory.

For $S \rightarrow \mathrm{Bun}_G(X)$ (resp., $S \rightarrow \mathrm{LocSys}_G(X)$), we will denote the resulting cotangent spaces, viewed as objects of $\mathrm{QCoh}(S)$, by

$$T^*(\mathrm{Bun}_G(X))|_S, \quad T^*(\mathrm{LocSys}_G(X))|_S, \quad \text{and} \quad T^*(\mathrm{LocSys}_G(X)/\mathrm{Bun}_G(X))|_S, \quad (10.5)$$

respectively.

10.3.4. We claim:

Lemma 10.3.5 *Assume that X is classical. Then, the relative pro-cotangent spaces of $\mathrm{LocSys}_G(X) \rightarrow \mathrm{Bun}_G(X)$ are connective.*

Proof We need to show that the functor (10.4), viewed as a functor $\mathrm{QCoh}(S) \rightarrow \mathrm{Vect}$, is left t-exact.

By [12, Proposition 3.4.3], the object $\Gamma(S \times X_{\mathrm{dR}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}}) \in \mathrm{Vect}$ can be calculated as the totalization of the co-simplicial object of Vect whose n -simplices are

$$\Gamma\left(S \times (X^n)_{\hat{X}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}}|_{S \times (X^n)_{\hat{X}}}\right),$$

where $(X^n)_{\hat{X}}$ is the DG ind-scheme equal to the formal completion of X^i along the main diagonal.

In particular, the projection onto the 0-simplices is the canonical map

$$\Gamma(S \times X_{\mathrm{dR}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}}) \rightarrow \Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}}).$$

Hence, it suffices to show that for every n and $\mathcal{M} \in \mathrm{QCoh}(S)^{\geq 0}$, we have

$$\Gamma\left(S \times (X^n)_{\hat{X}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}}|_{S \times (X^n)_{\hat{X}}}\right) \in \mathrm{Vect}^{\geq 0}. \quad (10.6)$$

The key observation is that the assumption that X be classical implies that the DG ind-scheme $(X^n)_{\hat{X}}$ is *classical*, see [11, Proposition 6.8.2]. That is, it can be written as a colimit of classical schemes Z_α .

Hence,

$$\Gamma\left(S \times (X^n)_{\hat{X}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}}|_{S \times (X^n)_{\hat{X}}}\right) \simeq \lim_{\alpha} \Gamma\left(S \times Z_\alpha, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}}|_{S \times Z_\alpha}\right).$$

Now (10.6) follows from the fact that, for each α ,

$$\Gamma(S \times Z_\alpha, \mathcal{M} \otimes \mathfrak{g}\mathcal{P}|_{S \times Z_\alpha}) \in \text{Vect}^{\geq 0}.$$

□

Corollary 10.3.6 *For a proper (classical) scheme X and any $S \rightarrow \text{LocSys}_G(X)$, the object $T^*(\text{LocSys}_G(X)/\text{Bun}_G(X))|_S$ belongs to $\text{QCoh}(S)^{\leq 0}$.*

10.3.7. Finally, we claim:

Proposition 10.3.8 (a) *If X is a classical scheme, the map (10.1) is ind-schematic and in fact ind-affine.*

(a') *If X is a proper classical scheme, the map (10.1) is schematic and affine.*

(b) *The prestacks $\text{Bun}_G(X)$ and $\text{LocSys}_G(X)$ are locally almost of finite type.*

Proof The fact that $\text{Bun}_G(X)$ is locally almost of finite type is a particular case of Corollary B.4.4.

Assume now that X is classical. Let S be an affine DG scheme almost of finite type equipped with a map to $\text{Bun}_G(X)$. We will prove that

$$S \times_{\text{Bun}_G(X)} \text{LocSys}_G(X) \tag{10.7}$$

is an ind-affine DG ind-scheme locally almost of finite type and is in fact an affine DG scheme if X proper. This implies (a) and (a'). It also implies (b): To show that $\text{LocSys}_G(X)$ is locally almost of finite type, we can replace the initial X by ${}^{cl}X$.

The fact that

$${}^{cl}(S \times_{\text{Bun}_G(X)} \text{LocSys}_G(X))$$

is an ind-affine ind-scheme (resp., affine scheme for X proper) and that it is locally of finite type follows from Lemmas 10.1.3 and 10.1.6, since the corresponding assertions in classical algebraic geometry are well known.⁹

To prove that (10.7) is a DG ind-scheme/DG scheme, we use Theorem B.2.14. Indeed, the required condition on the pro-cotangent spaces follows from Lemma 10.3.5.

The fact that (10.7) is locally almost of finite type follows from Lemma B.4.2. □

10.4 The case of curves

From now on, let us assume that X is a smooth, complete, and connected curve. In what follows, we will omit X from the notation in $\text{Bun}_G(X)$ and $\text{LocSys}_G(X)$, unless an ambiguity is likely to occur.

⁹ In the case when X is smooth, this is obvious, using the description of local systems as bundles with a connection. For a general X , it is enough to consider the case of $G = GL_n$. Then, the “locally of finite type” assertion is a general property of the category of D-modules. The (ind)-representability can be proved using the infinitesimal groupoid as in Lemma 10.3.5.

We will show that Bun_G is a smooth classical stack. We will also show that LocSys_G is quasi-smooth and compute the corresponding classical stack $\text{Sing}(\text{LocSys}_G)$.

10.4.1. First, we note that Bun_G is an algebraic stack (a.k.a. 1-Artin stack in the terminology of [28]). Indeed, the usual proof that ${}^{\text{cl}}\text{Bun}_G$ is a classical algebraic stack (see, e.g., [1]) applies in the context of derived algebraic geometry to show the corresponding property of Bun_G .¹⁰

We claim:

Lemma 10.4.2 *The stack Bun_G is smooth (and, in particular, classical).*

Proof By Lemma 2.1.2, this follows from the fact that Bun_G is an Artin stack locally almost of finite type and from the description of the cotangent spaces to Bun_G given by (10.2). \square

10.4.3. From the fact that Bun_G is an algebraic stack and the fact that the map (10.1) is schematic (see Proposition 10.3.8), we obtain that LocSys_G is also an algebraic stack. Since Bun_G has an affine diagonal, we obtain that the same is true for LocSys_G , since the map (10.1) is separated (in fact, it is affine).

Moreover, it is easy to see that the image of LocSys_G in Bun_G is contained in a quasi-compact open substack of Bun_G .¹¹ This implies that LocSys_G itself is quasi-compact. Thus, LocSys_G is a QCA stack in the terminology of [6].

Again, from Proposition 10.3.8 we obtain that LocSys_G is locally almost of finite type.

But, of course, LocSys_G is not smooth.

10.4.4. We now claim:

Proposition 10.4.5 *The stack LocSys_G is quasi-smooth.*

Proof This follows immediately from the description of the cotangent spaces given by (10.3). Namely, for any S -point of LocSys_G , the object $T^*(\text{LocSys}_G)|_S \in \text{QCoh}(S)$ is given, by Verdier duality along X , by

$$\Gamma(S \times X_{\text{dR}}, \mathfrak{g}_{\mathcal{P}}^*[1]), \quad (10.8)$$

which lives in cohomological degrees ≥ -1 , as required. \square

Note that by the same token, we obtain a description of the tangent complex of LocSys_G : For an S -point of LocSys_G , the object $T(\text{LocSys}_G)|_S \in \text{QCoh}(S)$ is given by

$$\Gamma(S \times X_{\text{dR}}, \mathfrak{g}_{\mathcal{P}})[1]. \quad (10.9)$$

¹⁰ Another way to see this is to choose sufficiently deep level structure (over every fixed quasi-compact open substack in Bun_G) and apply Theorem B.2.14.

¹¹ Here is a sketch of the proof. It is enough to consider the case of $G = GL_n$. Now, if a rank n -bundle \mathcal{E} splits as a direct sum $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, then a connection on \mathcal{E} gives rise to connections on \mathcal{E}_i . However, it follows from the Riemann-Roch Theorem that every rank n -bundle outside a certain quasi-compact open substack of Bun_n admits a direct sum decomposition as above with either $\text{deg}(\mathcal{E}_1) \neq 0$ or $\text{deg}(\mathcal{E}_2) \neq 0$.

10.4.6 The stack $\text{Sing}(\text{LocSys}_G)$ The above description of the tangent complex of LocSys_G implies the following description of the *classical* stack $\text{Sing}(\text{LocSys}_G)$:

Corollary 10.4.7 *The stack $\text{Sing}(\text{LocSys}_G)$ admits the following description: Given a classical affine scheme $S \in \text{Sch}^{\text{aff}}$,*

$$\text{Maps}(S, \text{Sing}(\text{LocSys}_G)) = (\mathcal{P}, \nabla, A),$$

where $(\mathcal{P}, \nabla) \in \text{Maps}(S, \text{LocSys}_G)$, and A is an element of

$$H^0(\Gamma(S \times X_{\text{dR}}, \mathfrak{g}_{\mathcal{P}}^*)).$$

Proof By definition, $\text{Sing}(\text{LocSys}_G)$ is the classical stack underlying

$$\text{Spec}_{\text{LocSys}_G} \left(\text{Sym}_{\mathcal{O}_{\text{LocSys}_G}} \left(T(\text{LocSys}_G[1]) \right) \right).$$

The assertion of the corollary follows from (10.9), since for (\mathcal{P}, ∇) as in the corollary, by the Serre duality,

$$\text{Hom}_{\text{Coh}(S)} \left(T(\text{LocSys}_G[1])_S, \mathcal{O}_S \right) \simeq H^0(\Gamma(S \times X_{\text{dR}}, \mathfrak{g}_{\mathcal{P}}^*)).$$

□

10.4.8. We denote the stack $\text{Sing}(\text{LocSys}_G)$ by Arth_G .

Remark 10.4.9 As was explained in the introduction, if G is reductive, the Arthur parameters for the automorphic side are supposed to correspond to points $(\mathcal{P}, \nabla, A) \in \text{Arth}_G$ where A is nilpotent; see Sect. 11 for details.

10.5 Accessing LocSys_G via an affine cover

In this subsection, we will show how to make sense in the DG world of such operations as adding a 1-form to a connection, and taking the polar part of a meromorphic connection.

10.5.1. Let $U \subset X$ be a non-empty open affine subset. Consider the prestack

$$\text{LocSys}_G(X; U) := \text{LocSys}_G(U) \times_{\text{Bun}_G(U)} \text{Bun}_G(X).$$

It is clear that if $X = U_1 \cup U_2$, we have

$$\text{LocSys}_G := \text{LocSys}_G(X) \simeq \text{LocSys}_G(X; U_1) \times_{\text{LocSys}_G(X; U_{1,2})} \text{LocSys}_G(X; U_2), \quad (10.10)$$

where $U_{1,2} = U_1 \cap U_2$.

The description of LocSys_G via (10.10) will be handy for establishing certain of its properties.

10.5.2. The main observation is:

Proposition 10.5.3 *The prestack $\mathrm{LocSys}_G(X; U)$ is classical.*

Proof Since Bun_G is a smooth classical algebraic stack, it suffices to show that for any smooth classical affine scheme S and any map $S \rightarrow \mathrm{Bun}_G$, the fiber product

$$S \times_{\mathrm{Bun}_G} \mathrm{LocSys}_G(X; U) \quad (10.11)$$

is classical.

First, we claim that the DG ind-scheme (10.11) is formally smooth. For that it suffices to show that $\mathrm{LocSys}_G(X; U)$ is formally smooth *over* Bun_G . By [11, Proposition 8.2.2], this is equivalent to showing that for any $S' \in \mathrm{DGSch}_{\mathrm{aff}}^{\mathrm{aff}}$ with a map to $\mathrm{LocSys}_G(X; U)$ and any $\mathcal{M} \in \mathrm{QCoh}(S')^{<0}$, we have

$$\pi_0(T^*(\mathrm{LocSys}_G(X; U)/\mathrm{Bun}_G)|_{S'}(\mathcal{M})) = 0,$$

where $T^*(\mathrm{LocSys}_G(X; U)/\mathrm{Bun}_G)|_S$ is viewed as a functor

$$\mathrm{QCoh}(S')^{\leq 0} \rightarrow \infty\text{-Grpd}.$$

By (10.4), we have:

$$\begin{aligned} & T^*(\mathrm{LocSys}_G(X; U)/\mathrm{Bun}_G)|_{S'}(\mathcal{M}) \\ & \simeq \mathrm{Cone}\left(\Gamma(S' \times U_{\mathrm{dR}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}}) \rightarrow \Gamma(S' \times U, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}})\right) \simeq \Gamma(U, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}} \otimes \omega_X), \end{aligned} \quad (10.12)$$

and the required vanishing follows from the fact that U is affine.¹²

Consider now the classical ind-scheme

$${}^{\mathrm{cl}}(S \times_{\mathrm{Bun}_G} \mathrm{LocSys}_G(X; U)).$$

This is the classical moduli problem corresponding to endowing a given G -bundle (given by a point of S) with a connection defined over $U \subset X$. It is easy to see that, locally on S , we have an isomorphism

$${}^{\mathrm{cl}}(S \times_{\mathrm{Bun}_G} \mathrm{LocSys}_G(X; U)) \simeq S \times \mathbb{A}^\infty,$$

¹² The fact that (10.11) is formally smooth already implies that its classical via [11, Theorem 9.1.6]. Below we give a more explicit proof which avoids (the non-trivial) [11, Theorem 9.1.6] and instead uses (the more elementary) [11, Proposition 9.1.4].

where

$$\mathbb{A}^\infty = \operatorname{colim}_{n \in \mathbb{Z}^{\geq 0}} \mathbb{A}^n.$$

In particular, ${}^{cl}(S \times_{\operatorname{Bun}_G} \operatorname{LocSys}_G(X; U))$, when viewed as a DG ind-scheme, is formally smooth.

The assertion of the proposition follows now from [11, Proposition 9.1.4].¹³ \square

10.5.4. As a first application of Proposition 10.5.3, we will prove the following. Consider the following group DG scheme over Bun_G , which we denote by Hitch_G :

For an S -point \mathcal{P} of Bun_G , the ∞ -groupoid of its lifts to an S -point of Hitch_G is by definition

$$\tau^{\leq 0} \left(\Gamma(S \times X, \mathfrak{g}_{\mathcal{P}} \otimes \omega_X) \right).$$

By Serre duality, we have

$$S \times_{\operatorname{Bun}_G} \operatorname{Hitch}_G = \operatorname{Spec} \left(\operatorname{Sym}_{\mathcal{O}_S} \left(\Gamma(S \times X, \mathfrak{g}_{\mathcal{P}}^*[1]) \right) \right).$$

Note that Hitch_G is naturally a DG vector bundle¹⁴ (and therefore a DG group scheme) over Bun_G .

Corollary 10.5.5 *There exists a canonical action of Hitch_G on $\operatorname{LocSys}(G)$ over Bun_G ; the action is simply transitive in the sense that the induced map*

$$\operatorname{Hitch}_G \times_{\operatorname{Bun}_G} \operatorname{LocSys}(G) \rightarrow \operatorname{LocSys}(G) \times_{\operatorname{Bun}_G} \operatorname{LocSys}(G)$$

is an isomorphism.

Remark 10.5.6 This corollary is a triviality for the underlying classical stacks: Any two connections on a given bundle over a curve differ by a 1-form. However, it is less obvious at the derived level, since the procedure of adding a 1-form to a connection is difficult to make sense of in the ∞ -categorical setting.

Proof As in the case of $\operatorname{LocSys}_G(X; U)$, we can define a relative ind-scheme, $\operatorname{Hitch}(X; U)$, over Bun_G , whose S -points are pairs (\mathcal{P}, α) , where \mathcal{P} is an S -point of Bun_G and α is a point of

$$\Gamma(S \times U, \mathfrak{g}_{\mathcal{P}} \otimes \omega_X),$$

¹³ The assertion if [11, Proposition 9.1.4] says that if \mathcal{Z} is a formally smooth DG ind-scheme, such that the underlying classical ind-scheme ${}^{cl}\mathcal{Z}$ is formally smooth when viewed as a derived ind-scheme, then the canonical map ${}^{cl}\mathcal{Z} \rightarrow \mathcal{Z}$ is an isomorphism, up to sheafification.

¹⁴ By a DG vector bundle over a prestack \mathcal{Z} , we mean a prestack of the form $\operatorname{Spec}(\operatorname{Sym}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{F}))$ for $\mathcal{F} \in \operatorname{QCoh}(\mathcal{Z})^{\leq 0}$.

considered as an ∞ -groupoid. As in the case of $\text{LocSys}_G(X; U)$, we show that $\text{Hitch}(X; U)$ is classical. Similarly,

$$\text{Hitch}(X; U) \times_{\text{Bun}_G} \text{LocSys}_G(X; U)$$

is classical.

Since we are dealing with classical objects, it is easy to see that $\text{Hitch}(X; U)$ acts simply transitively on $\text{LocSys}_G(X; U)$ over Bun_G . Moreover, these actions are compatible under restrictions for $U \hookrightarrow U'$.

Covering $X = U_1 \cup U_2$, we have

$$\text{Hitch}_G \simeq \text{Hitch}(X; U_1) \times_{\text{Hitch}(X; U_{1,2})} \text{Hitch}(X; U_2),$$

as prestacks. Now, the required assertion follows from (10.10). \square

10.5.7. Let now x be a k -point of X outside of U . Consider the following relative DG ind-scheme over $\text{Bun}_G(X)$, denoted $\text{Polar}(G, x)$, whose S -points are pairs $(\mathcal{P}, A_{\text{Polar}})$, where \mathcal{P} is an S -point of $\text{Bun}_G(X)$, and A_{Polar} is a point of

$$\Gamma(S \times X, \mathfrak{g}_{\mathcal{P}} \otimes \omega_X(\infty \cdot x)/\omega_X),$$

considered as an ∞ -groupoid via $\text{Vect}^{\leq 0} \rightarrow \infty\text{-Grpd}$.

One easily shows that $\text{Polar}(G, x)$ is formally smooth and classical as a prestack: Locally in the fppf topology on Bun_G , the stack $\text{Polar}(G, x)$ looks like the product of Bun_G and \mathbb{A}^∞ .

10.5.8. Note that since $\text{LocSys}_G(X; U)$ and $\text{Polar}(G, x)$ are both classical prestacks, the usual operation of taking the polar part of the connection defines a map

$$\text{LocSys}_G(X; U) \rightarrow \text{Polar}(G, x).$$

Proposition 10.5.9 *Set $U' := U \cup \{x\}$, and suppose that it is still affine. Then, there exists a canonical isomorphism*

$$\text{LocSys}_G(X; U') \simeq \text{LocSys}_G(X; U) \times_{\text{Polar}(G, x)} \text{Bun}_G(X),$$

where $\text{Bun}_G(X) \rightarrow \text{Polar}(G, x)$ is the zero-section.

Proof Note that since U' is affine and hence $\text{LocSys}_G(X; U')$ is classical, there exists a canonically defined map

$$\text{LocSys}_G(X; U') \rightarrow \text{LocSys}_G(X; U) \times_{\text{Polar}(G, x)} \text{Bun}_G(X),$$

which is an isomorphism at the classical level. To show that this map is an isomorphism, it is enough to show that it induces an isomorphism at the level of cotangent spaces at

S -points for a classical affine scheme $S \in \text{Sch}^{\text{aff}}$. The latter, in turn, follows from the computation of the cotangent spaces in Sect. 10.3.1. \square

10.5.10. Let now $U = X - x$. We claim:

Corollary 10.5.11 *There exists a canonical isomorphism*

$$\text{LocSys}_G(X) \simeq \text{LocSys}_G(X; U) \times_{\text{Polar}(G, x)} \text{Bun}_G(X).$$

Proof Let U' be another open affine of X that contains the point x . Applying Proposition 10.5.9, we obtain:

$$\text{LocSys}_G(X; U') \simeq \text{LocSys}_G(X; U \cap U') \times_{\text{Polar}(G, x)} \text{Bun}_G(X),$$

so

$$\begin{aligned} \text{LocSys}_G &\simeq \text{LocSys}_G(X; U) \times_{\text{LocSys}_G(X; U \cap U')} \text{LocSys}_G(X; U') \\ &\simeq \text{LocSys}_G(X; U) \times_{\text{LocSys}_G(X; U \cap U')} \left(\text{LocSys}_G(X; U \cap U') \times_{\text{Polar}(G, x)} \text{Bun}_G(X) \right) \\ &\simeq \text{LocSys}_G(X; U) \times_{\text{Polar}(G, x)} \text{Bun}_G(X), \end{aligned}$$

as required.

It is equally easy to see that the constructed map does not depend on the choice of U' : For $U'' \subset U'$, the corresponding diagram commutes. \square

10.6 Presentation of LocSys_G as a fiber product

In this subsection, we will show how to define the notion of connection that has a pole of order ≤ 1 at a given point, and how to represent LocSys_G as a fiber product of smooth stacks.

10.6.1. Fix a point $x \in X$, and let

$$\text{Polar}^{\leq 1}(G, x) \subset \text{Polar}(G, x)$$

be the closed substack corresponding to

$$\mathfrak{g}_{\mathcal{P}} \otimes \omega_X(x) / \omega_X \subset \mathfrak{g}_{\mathcal{P}} \otimes \omega_X(\infty \cdot x) / \omega_X.$$

That is, this substack corresponds to pairs $(\mathcal{P}, A_{\text{Polar}})$ where A_{Polar} has at most a simple pole.

It is easy to see that we have a canonical identification (the residue map)

$$\text{Polar}^{\leq 1}(G, x) \simeq \mathfrak{g}/G \times_{\text{pt}/G} \text{Bun}_G,$$

where $\text{Bun}_G \rightarrow \text{pt}/G$ is the canonical map corresponding to the restriction of a G -bundle to $x \in X$.

10.6.2. We define the stack $\text{LocSys}_G^{\text{R.S.}}$ of local systems with (at most) a simple pole at x by

$$\text{LocSys}_G^{\text{R.S.}} := \text{LocSys}_G(X; X - x) \times_{\text{Polar}(G, x)} \text{Polar}^{\leq 1}(G, x).$$

By Corollary 10.5.11, we have a canonical map

$$\iota : \text{LocSys}_G \hookrightarrow \text{LocSys}_G^{\text{R.S.}}$$

and a canonical map

$$\text{res} : \text{LocSys}_G^{\text{R.S.}} \rightarrow \mathfrak{g}/G \times_{\text{pt}/G} \text{Bun}_G$$

that fit into a Cartesian square

$$\begin{array}{ccc} \text{LocSys}_G & \longrightarrow & \text{LocSys}_G^{\text{R.S.}} \\ \downarrow & & \downarrow \text{res} \\ \text{Bun}_G & \longrightarrow & \mathfrak{g}/G \times_{\text{pt}/G} \text{Bun}_G, \end{array} \quad (10.13)$$

where the bottom horizontal arrows comes from the zero-section map $\text{pt}/G \rightarrow \mathfrak{g}/G$.

10.6.3. From (10.12), we obtain the following description of the relative cotangent spaces of $\text{LocSys}_G^{\text{R.S.}}$ over Bun_G :

For an S -point (\mathcal{P}, ∇, A) , the cotangent space $T^*(\text{LocSys}_G^{\text{R.S.}}/\text{Bun}_G)|_S$, viewed as a functor

$$\text{QCoh}(S)^{\leq 0} \rightarrow \infty\text{-Grpd}$$

is given by

$$\mathcal{M} \mapsto \Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}} \otimes \omega_X(x)).$$

In particular,

$$T^*(\text{LocSys}_G^{\text{R.S.}}/\text{Bun}_G)|_S \in \text{QCoh}(S)^{\leq 0}.$$

In fact, by the Serre duality,

$$T^*(\mathrm{LocSys}_G^{\mathrm{R.S.}} / \mathrm{Bun}_G)|_S \simeq \Gamma(S \times X, \mathfrak{g}_{\mathcal{P}}^*(-x))[1]. \quad (10.14)$$

10.6.4. By Theorem B.2.14, we obtain that the map

$$\mathrm{LocSys}_G^{\mathrm{R.S.}} \rightarrow \mathrm{Bun}_G$$

is schematic (The same argument applies to connections with poles of any fixed order instead of simple poles.). This map is also easily seen to be separated (and, in fact, affine). This implies that $\mathrm{LocSys}_G^{\mathrm{R.S.}}$ has an affine diagonal.

Note that, unlike LocSys_G , the stack $\mathrm{LocSys}_G^{\mathrm{R.S.}}$ is not quasi-compact (unless G is unipotent). However, for our applications the stack $\mathrm{LocSys}_G^{\mathrm{R.S.}}$ may be replaced by a Zariski neighborhood of the image $\iota(\mathrm{LocSys}_G) \subset \mathrm{LocSys}_G^{\mathrm{R.S.}}$; we can choose such a neighborhood to be quasi-compact, and therefore QCA.

10.6.5. From (10.14), we obtain that the map

$$\mathrm{LocSys}_G^{\mathrm{R.S.}} \rightarrow \mathrm{Bun}_G$$

is quasi-smooth.

Since Bun_G is smooth, we obtain that the stack $\mathrm{LocSys}_G^{\mathrm{R.S.}}$ is quasi-smooth. We now claim:

Proposition 10.6.6 (a) *The stack $\mathrm{LocSys}_G^{\mathrm{R.S.}}$ is smooth in a Zariski neighborhood of the image of the closed embedding $\iota : \mathrm{LocSys}_G \hookrightarrow \mathrm{LocSys}_G^{\mathrm{R.S.}}$.*

(b) *If G is unipotent,¹⁵ then $\mathrm{LocSys}_G^{\mathrm{R.S.}}$ is smooth.*

Proof Let \mathcal{Z} be an Artin stack with a perfect cotangent complex (For instance, this is the case if \mathcal{Z} is quasi-smooth.). It is easy to see that smoothness of \mathcal{Z} can be verified at k -points. Namely, a point $z : \mathrm{Spec}(k) \rightarrow \mathcal{Z}$ belongs to the smooth locus of \mathcal{Z} if and only if $T_z^*(\mathcal{Z}) \in \mathrm{Vect}^{\geq 0}$.

A k -point of $\mathrm{LocSys}_G^{\mathrm{R.S.}}$ is a pair $z = (\mathcal{P}, \nabla)$, where \mathcal{P} is a G -bundle on X , and ∇ is a connection on \mathcal{P} with a simple pole at x .

We have the following description of $T_z^*(\mathrm{LocSys}_G^{\mathrm{R.S.}})$, parallel to that of (10.8):

$$T_z^*(\mathrm{LocSys}_G^{\mathrm{R.S.}}) \simeq \mathrm{Cone}(\nabla : \Gamma(X, \mathfrak{g}_{\mathcal{P}}^*(-x)) \rightarrow \Gamma(X, \mathfrak{g}_{\mathcal{P}}^* \otimes \omega_X)). \quad (10.15)$$

Therefore, a point z belongs to the smooth locus if and only if the map of (classical) vector spaces

$$\nabla : H^0(\Gamma(X, \mathfrak{g}_{\mathcal{P}}^*(-x))) \rightarrow H^0(\Gamma(X, \mathfrak{g}_{\mathcal{P}}^* \otimes \omega_X))$$

is injective.

¹⁵ S. Raskin has observed that the assertion and its proof remain valid under the weaker assumption that the identity connected component of G is solvable.

In other words, smooth points correspond to pairs (\mathcal{P}, ∇) such that $\mathfrak{g}_{\mathcal{P}}^*$ has no nonzero horizontal sections that vanish at x . Recall that the connection on $\mathfrak{g}_{\mathcal{P}}^*$ has a simple pole at x ; the condition automatically holds if none of the eigenvalues of the coadjoint action of the residue $\text{res}(\nabla) \in \mathfrak{g}/G$ is a negative integer.

In particular, if (\mathcal{P}, ∇) is a point of $\iota(\text{LocSys}_G)$, then $\text{res}(\nabla) = 0$ and the condition trivially holds; this proves part (a). On the other hand, if G is unipotent, the coadjoint action of \mathfrak{g} is nilpotent, and the condition is satisfied as well; this proves part (b). \square

10.6.7. By (9.5), from Proposition 10.6.6(a), we obtain a canonical closed embedding

$$\text{Arth}_G \hookrightarrow \mathfrak{g}^*/G \times_{\text{pt}/G} \text{LocSys}_G. \quad (10.16)$$

Recall that by Corollary 10.4.7, Arth_G is isomorphic to the moduli stack (in the classical sense) of triples (\mathcal{P}, ∇, A) , where $(\mathcal{P}, \nabla) \in \text{LocSys}_G$ and $A \in H^0(\Gamma(X_{\text{dR}}, \mathfrak{g}_{\mathcal{P}}^*))$. It is easy to see that (10.16) is given by

$$(\mathcal{P}, \nabla, A) \mapsto (A(x), (\mathcal{P}, \nabla)).$$

11 The global nilpotent cone and formulation of the conjecture

As before, let X be a connected smooth projective curve. From now on, we assume that the algebraic group G is reductive. Let \check{G} be its Langlands dual.

In this section, we will formulate the geometric Langlands conjecture, whose automorphic (a.k.a. geometric) side involves the category $\text{D-mod}(\text{Bun}_G)$, and the Galois (a.k.a. spectral) side, an appropriate modification of the category $\text{QCoh}(\text{LocSys}_{\check{G}})$.

11.1 The global nilpotent cone

11.1.1. Recall that Proposition 10.4.7 provides an isomorphism between $\text{Sing}(\text{LocSys}_{\check{G}})$ and the (classical) moduli stack $\text{Arth}_{\check{G}}$, which parametrizes triples (\mathcal{P}, ∇, A) . Here, \mathcal{P} is a \check{G} -bundle on X , ∇ is a connection on \mathcal{P} , and A is a horizontal section of $\check{\mathfrak{g}}_{\mathcal{P}}^*$.

We define a Zariski-closed subset

$$\text{Nilp}_{\text{glob}} \subset \text{Arth}_{\check{G}} \quad (11.1)$$

to correspond to triples (\mathcal{P}, ∇, A) with nilpotent A .

That is, we require that for every local trivialization of \mathcal{P} , the element A viewed (locally) as a map $S \times X \rightarrow \check{\mathfrak{g}}^*$ hit the locus of nilpotent elements $\text{Nilp}(\check{\mathfrak{g}}^*) \subset \check{\mathfrak{g}}^*$. The latter is defined as the image of the locus of nilpotent elements $\text{Nilp}(\check{\mathfrak{g}}) \subset \check{\mathfrak{g}}$ under some (or any) \check{G} -invariant identification $\check{\mathfrak{g}} \simeq \check{\mathfrak{g}}^*$.

11.1.2. Let $c(\check{\mathfrak{g}})$ denote the characteristic variety of $\check{\mathfrak{g}}$, i.e.,

$$c(\check{\mathfrak{g}}) := \text{Spec}(\text{Sym}(\check{\mathfrak{g}})^{\check{G}}),$$

and let ϖ denote the Chevalley map

$$\varpi : \check{\mathfrak{g}}^* = \text{Spec}(\text{Sym}(\check{\mathfrak{g}})) \rightarrow \text{Spec}(\text{Sym}(\check{\mathfrak{g}})^{\check{G}}) = \mathfrak{c}(\check{\mathfrak{g}}).$$

For $(\mathcal{P}, \nabla, A) \in \text{Maps}(S, \text{Arth}_{\check{G}})$, we thus obtain a map

$$\varpi(A) : S \times X \rightarrow \mathfrak{c}(\check{\mathfrak{g}}).$$

The nilpotence condition can be phrased as the requirement that $\varpi(A)$ should factor through

$$\{0\} \subset \mathfrak{c}(\check{\mathfrak{g}}).$$

11.1.3. We can also express the nilpotence condition locally:

Lemma 11.1.4 *For an S -point (\mathcal{P}, ∇, A) of $\text{Arth}_{\check{G}}$, the element A is nilpotent if and only if for some (and then any) point $x \in X$, the value $A|_{S \times \{x\}}$ of A at x is nilpotent as a section of $\check{\mathfrak{g}}_{\mathcal{P}, x}^* := \check{\mathfrak{g}}_{\mathcal{P}}^*|_{S \times \{x\}}$.*

Proof The fact that A is horizontal implies that the map $\varpi(A)$ is infinitesimally constant along X (i.e., factors through a map $S \times X_{\text{dR}} \rightarrow \mathfrak{c}(\check{\mathfrak{g}})$), and therefore is constant (since X is connected). This implies the assertion of the lemma. \square

Recall that by (10.16), we have a canonical closed embedding

$$\text{Arth}_{\check{G}} \hookrightarrow \check{\mathfrak{g}}^*/\check{G} \times_{\text{pt}/\check{G}} \text{LocSys}_{\check{G}}.$$

Thus, Lemma 11.1.4 can be reformulated as the equality between

$$\text{Nilp}_{\text{glob}} \subset \text{Arth}_{\check{G}}$$

and the preimage of the closed subset

$$\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G} \times_{\text{pt}/\check{G}} \text{LocSys}_{\check{G}} \subset \check{\mathfrak{g}}^*/\check{G} \times_{\text{pt}/\check{G}} \text{LocSys}_{\check{G}}$$

under the above map.

11.1.5 *The spectral side of the geometric Langlands conjecture* Our main object of study is the category

$$\text{IndCoh}_{\text{Nilp}_{\text{glob}}}(\text{LocSys}_{\check{G}}).$$

By definition, this is a full subcategory of $\text{IndCoh}(\text{LocSys}_{\check{G}})$, which contains the essential image of $\text{QCoh}(\text{LocSys}_{\check{G}})$ under the functor

$$\Xi_{\text{LocSys}_{\check{G}}} : \text{QCoh}(\text{LocSys}_{\check{G}}) \rightarrow \text{IndCoh}(\text{LocSys}_{\check{G}}).$$

We propose the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}}(\mathrm{LocSys}_{\check{G}})$ as the category appearing on the spectral side of the geometric Langlands conjecture.

11.1.6. By Corollary 9.2.7 and Proposition 10.6.6, the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}}(\mathrm{LocSys}_{\check{G}})$ is compactly generated.

By Sect. 9.3.2, Proposition 10.6.6 allows us to view $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ as tensored over the monoidal category $\mathrm{QCoh}(\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m))$. We emphasize that the latter structure depends on the choice of a point $x \in X$.

By Lemma 11.1.4 and Corollary 9.3.3, we have:

$$\begin{aligned} & \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}}(\mathrm{LocSys}_{\check{G}}) \\ & \simeq \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) \otimes_{\mathrm{QCoh}(\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m))} \mathrm{QCoh}(\mathrm{Nilp}(\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m))). \end{aligned}$$

11.2 Formulation of the Geometric Langlands conjecture

11.2.1. We propose the following form of the geometric Langlands conjecture:

Conjecture 11.2.2 *There exists an equivalence of DG categories*

$$\mathrm{D}\text{-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}}(\mathrm{LocSys}_{\check{G}}).$$

Since the DG categories appearing on both sides of Conjecture 11.2.2 are compactly generated, it can be tautologically rephrased as follows:

Conjecture 11.2.3 *There exists an equivalence of non-cocomplete DG categories*

$$\mathrm{D}\text{-mod}(\mathrm{Bun}_G)^c \simeq \mathrm{Coh}_{\mathrm{Nilp}_{glob}}(\mathrm{LocSys}_{\check{G}}).$$

11.2.4. In what follows, we will refer to the essential image in $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ of

$$\Xi_{\mathrm{LocSys}_{\check{G}}}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})) \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}}(\mathrm{LocSys}_{\check{G}})$$

under the above conjectural equivalence as the “tempered part” of $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, and denote it by $\mathrm{D}\text{-mod}_{\mathrm{temp}}(\mathrm{Bun}_G)$.

11.2.5. Of course, one needs to specify a lot more data to fix the equivalence of Conjecture 11.2.2 uniquely. This will be done over the course of several papers following this one. In the present paper, we will discuss the following aspects:

- (i) The case when G is a torus;
- (ii) Compatibility with the geometric Satake equivalence (see Sect. 12);
- (iii) Compatibility with the Eisenstein series (see Sect. 13).

11.2.6 The case of a torus Let G be a torus T . This case offers nothing new. The subset $\mathrm{Nilp}_{\mathrm{glob}}$ is the zero-section of $\mathrm{Sing}(\mathrm{LocSys}_{\check{T}})$, so by Corollary 8.2.8,

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{T}}) = \Xi_{\mathrm{LocSys}_{\check{G}}}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})),$$

as subcategories of $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{T}})$.

In this case, the equivalence

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{T}}) \simeq \mathrm{D}\text{-mod}(\mathrm{Bun}_T)$$

is a particular case of the Fourier transform for D-modules on an abelian variety (see [18, 19] and [35, 36]), appropriately adjusted to the DG setting.

Remark 11.2.7 In more detail, for $G = \mathbb{G}_m$, a choice of $x \in X$ identifies

$$\mathrm{Bun}_{\mathbb{G}_m} \simeq \mathrm{Pic} \times \mathrm{pt} / \mathbb{G}_m \times \mathbb{Z} \quad \text{and} \quad \mathrm{LocSys}_{\mathbb{G}_m} \simeq \widetilde{\mathrm{Pic}} \times (\mathrm{pt} \times \mathrm{pt}) \times \mathrm{pt} / \mathbb{G}_m,$$

where Pic is the Picard *scheme* and $\widetilde{\mathrm{Pic}}$ is its universal additive extension. Then, the classical Fourier-Mukai-Laumon transform identifies

$$\mathrm{D}\text{-mod}(\mathrm{Pic}) \simeq \mathrm{QCoh}(\widetilde{\mathrm{Pic}}),$$

and we have explicit equivalences of categories

$$\mathrm{D}\text{-mod}(\mathrm{pt} / \mathbb{G}_m) \simeq \mathrm{QCoh}(\mathrm{pt} \times \mathrm{pt}) \quad \text{and} \quad \mathrm{D}\text{-mod}(\mathbb{Z}) \simeq \mathrm{QCoh}(\mathrm{pt} / \mathbb{G}_m).$$

Note that the compact generator of $\mathrm{D}\text{-mod}(\mathrm{pt} / \mathbb{G}_m)$ is the direct image with compact supports of $k \in \mathrm{Vect}$ under the map $\mathrm{pt} \rightarrow \mathrm{pt} / \mathbb{G}_m$. It corresponds to the structure sheaf of $\mathrm{QCoh}(\mathrm{pt} \times \mathrm{pt})$.

Note also that under this equivalence, the constant sheaf $k_{\mathrm{pt} / \mathbb{G}_m} \in \mathrm{D}\text{-mod}(\mathrm{pt} / \mathbb{G}_m)$ is not compact, and it corresponds to the sky-scraper on $\mathrm{pt} \times \mathrm{pt}$, which is an object of $\mathrm{Coh}(\mathrm{pt} \times \mathrm{pt})$, but not of $\mathrm{QCoh}(\mathrm{pt} \times \mathrm{pt})^{\mathrm{perf}}$, i.e., it is *not* compact in $\mathrm{QCoh}(\mathrm{pt} \times \mathrm{pt})$.

12 Compatibility with geometric satake equivalence

One of the key properties of the geometric Langlands equivalence is its behavior with respect to the Hecke functors on both sides of the correspondence. In this section, we will study how this is compatible with the proposed candidate for the spectral side: the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$.

12.1 Main results of this section

12.1.1 The geometric Satake equivalence As before, X is a smooth connected projective curve, G is a reductive group, and \check{G} is its Langlands dual. Fix a point $x \in X$. The category of Hecke functors at x (“the spherical Hecke category at x ”) is the category of $G(\check{\mathcal{O}}_x)$ -equivariant D-modules on the affine Grassmannian $\mathrm{Gr}_{G,x}$, which we denote by

$$\mathrm{Sph}(G, x) := \mathrm{D}\text{-mod}(\mathrm{Gr}_{G,x})^{G(\check{\mathcal{O}}_x)}.$$

We regard it as a monoidal category with respect to the convolution product.

In Corollary 12.5.5, we will construct a monoidal equivalence

$$\mathrm{Sat} : \mathrm{IndCoh}_{\mathrm{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}) \simeq \mathrm{Sph}(G, x)$$

between $\mathrm{Sph}(G, x)$ and a certain category constructed from the group \check{G} . Explicitly, $\mathrm{IndCoh}_{\mathrm{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$ is the category of ind-coherent sheaves on the stack

$$\mathrm{Hecke}(\check{G})_{\mathrm{spec}} := \mathrm{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \mathrm{pt}/\check{G}$$

whose singular support is contained in

$$\mathrm{Nilp}(\check{\mathfrak{g}}^*)/\check{G} \subset \check{\mathfrak{g}}^*/\check{G} \simeq \mathrm{Sing}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}),$$

where $\mathrm{Nilp}(\check{\mathfrak{g}}^*) \subset \check{\mathfrak{g}}^*$ is the nilpotent cone.

We refer to Sat as the geometric Satake equivalence. It is naturally related to the other versions of the Satake equivalence, constructed in [26] and [2] (which are given below as (12.2) and Theorem 12.3.3, respectively).

12.1.2 The geometric Langlands conjecture and the geometric satake equivalence

We show that the “modified” geometric Langlands conjecture (Conjecture 11.2.2) agrees with the geometric Satake equivalence Sat : There is a natural action of the monoidal category

$$\mathrm{IndCoh}_{\mathrm{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\mathrm{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \mathrm{pt}/\check{G})$$

on the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$, see Corollary 12.7.3. Under the equivalence of Conjecture 11.2.2, this action should correspond to the action of the monoidal category $\mathrm{Sph}(G, x)$ on $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ by the Hecke functors; this is Conjecture 12.7.6.

12.1.3 Tempered D-modules Recall that Conjecture 11.2.2 implies the existence of a certain full subcategory $\mathrm{D}\text{-mod}_{\mathrm{temp}}(\mathrm{Bun}_G) \subset \mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, the “tempered part” of $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, defined as the essential image in $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ of

$$\mathbb{E}_{\text{LocSys}_{\check{G}}}(\text{QCoh}(\text{LocSys}_{\check{G}})) \subset \text{IndCohNilp}_{\text{glob}}(\text{LocSys}_{\check{G}}).$$

Assuming the compatibility of Conjecture 12.7.6, we will be able to describe the subcategory $\text{D-mod}_{\text{temp}}(\text{Bun}_G)$ in purely “geometric” terms, using the Hecke functors at a fixed point $x \in X$. The description is independent of the Langlands conjecture; we denote the resulting full subcategory by

$$\text{D-mod}_{\text{temp}}^x(\text{Bun}_G) \subset \text{D-mod}(\text{Bun}_G).$$

However, it is not clear that the subcategory $\text{D-mod}_{\text{temp}}^x(\text{Bun}_G)$ is independent of the choice of the point x . This is the content of Conjecture 12.8.5, which follows from Conjectures 11.2.2 and 12.7.6.

12.2 Preliminaries on the spherical Hecke category

12.2.1. Recall that the spherical Hecke category at a point $x \in X$ is defined as

$$\text{Sph}(G, x) := \text{D-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)},$$

regarded as a monoidal category with respect to the convolution product. We claim that as a DG category, $\text{Sph}(G, x)$ is compactly generated.

Indeed, we can represent $\text{Gr}_{G,x}$ as a union of $G(\widehat{\mathcal{O}}_x)$ -invariant finite-dimensional closed subschemes Z_α . We have

$$\text{D-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)} \simeq \underset{\alpha}{\text{colim}} \text{D-mod}(Z_\alpha)^{G(\widehat{\mathcal{O}}_x)},$$

where for $\alpha_1 \geq \alpha_2$, the functor

$$\text{D-mod}(Z_{\alpha_1})^{G(\widehat{\mathcal{O}}_x)} \rightarrow \text{D-mod}(Z_{\alpha_2})^{G(\widehat{\mathcal{O}}_x)}$$

is given by direct image along the corresponding closed embedding. In particular, for every α , the functor

$$\text{D-mod}(Z_\alpha)^{G(\widehat{\mathcal{O}}_x)} \rightarrow \text{D-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)}$$

sends compacts to compacts. By [29, Lemma 1.3.3], this reduces the assertion to showing that each $\text{D-mod}(Z_\alpha)^{G(\widehat{\mathcal{O}}_x)}$ is compactly generated.

Let G_α be a finite-dimensional quotient of $G(\widehat{\mathcal{O}}_x)$ through which it acts on Z_α . With no restriction of generality, we can assume that $\ker(G(\widehat{\mathcal{O}}_x) \rightarrow G_\alpha)$ is pro-unipotent. Hence, the forgetful functor

$$\text{D-mod}(Z_\alpha)^{G_\alpha} \rightarrow \text{D-mod}(Z_\alpha)^{G(\widehat{\mathcal{O}}_x)}$$

is an equivalence.

Now, Z_α/G_α is a QCA algebraic stack, and the compact generation of $\mathbf{D}\text{-mod}(Z_\alpha)^{G_\alpha}$ follows from [6, Theorem 0.2.2] (Since Z_α/G_α is a global quotient, the compact generation follows more easily from the results of [3].).

Note that the monoidal operation on $\mathbf{Sph}(G, x)$ preserves the subcategory of compact objects (this follows from the properness of $\text{Gr}_{G,x}$). Hence, $\mathbf{Sph}(G, x)^c$ acquires a structure of non-cocomplete monoidal DG category.

12.2.2. Consider the heart $(\mathbf{D}\text{-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)})^\heartsuit$ of the natural t-structure on the category $\mathbf{D}\text{-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)}$. $(\mathbf{D}\text{-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)})^\heartsuit$ is the abelian category of $G(\widehat{\mathcal{O}}_x)$ -equivariant D-modules on $\text{Gr}_{G,x}$. The geometric Satake isomorphism of [26] gives an equivalence between $(\mathbf{D}\text{-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)})^\heartsuit$ and the abelian category of representations of \check{G} , which we denote by $\text{Rep}(\check{G})^\heartsuit$.

We let $\mathbf{Sph}(G, x)^{\text{naive}}$ be the derived category (considered as a DG category) of the abelian category

$$(\mathbf{D}\text{-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)})^\heartsuit.$$

We have a canonical (but not fully faithful) monoidal functor

$$\mathbf{Sph}(G, x)^{\text{naive}} \rightarrow \mathbf{Sph}(G, x) \quad (12.1)$$

(see [21, Theorem 1.3.2.2]).

The Satake equivalence of [26] induces a canonical t-exact equivalence of monoidal categories

$$\text{Sat}^{\text{naive}} : \text{Rep}(\check{G}) \simeq \mathbf{Sph}(G, x)^{\text{naive}}. \quad (12.2)$$

We will refer to it as the “naive” version of the geometric Satake equivalence.

In order to describe $\mathbf{Sph}(G, x)$, it is convenient to first introduce and describe its “renormalized” version. Here, “renormalization” refers to the process of changing (in this case, enlarging) the class of compact objects of the category.

12.2.3. Let $\mathbf{Sph}(G, x)^{\text{loc.c}}$ denote the full subcategory of $\mathbf{Sph}(G, x)$ consisting of those objects of $\mathbf{Sph}(G, x) \simeq \mathbf{D}\text{-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)}$ that become compact after applying the forgetful functor

$$\mathbf{D}\text{-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)} \rightarrow \mathbf{D}\text{-mod}(\text{Gr}_{G,x}).$$

(The superscript “loc.c” stands for “locally compact.”)

The category $\mathbf{Sph}(G, x)^{\text{loc.c}} \subset \mathbf{Sph}(G, x)$ is stable under the monoidal operation, and hence acquires a structure of (non-cocomplete) monoidal DG category. We define

$$\mathbf{Sph}(G, x)^{\text{ren}} := \text{Ind}(\mathbf{Sph}(G, x)^{\text{loc.c}}),$$

which thus acquires a structure of monoidal DG category.

12.2.4. We have a canonically defined monoidal functor

$$\Psi^{\text{Sph}} : \text{Sph}(G, x)^{\text{ren}} \rightarrow \text{Sph}(G, x),$$

obtained by ind-extending the tautological embedding $\text{Sph}(G, x)^{\text{loc.c}} \hookrightarrow \text{Sph}(G, x)$.

Since $\text{Sph}(G, x)^{\text{loc.c}}$ is closed under the truncations with respect to the (usual) t-structure on $\text{Sph}(G, x)$, we obtain that $\text{Sph}(G, x)^{\text{ren}}$ acquires a unique t-structure, compatible with colimits,¹⁶ for which the functor Ψ^{Sph} is t-exact.

The functor Ψ^{Sph} admits a left adjoint, denoted Ξ^{Sph} , obtained by ind-extending the tautological embedding $\text{Sph}(G, x)^c \hookrightarrow \text{Sph}(G, x)^{\text{loc.c}}$. By construction, the functor Ξ^{Sph} is fully faithful. So, Ψ^{Sph} makes $\text{Sph}(G, x)$ into a colocalization of $\text{Sph}(G, x)^{\text{ren}}$.

Finally, note that Ξ^{Sph} has a natural structure of a monoidal functor. Indeed, it is clearly co-lax monoidal, being the left adjoint of a monoidal functor. Since the category $\text{Sph}(G, x)$ is compactly generated, it suffices to check that Ξ^{Sph} is (strictly) monoidal after restriction to the category of compact objects $\text{Sph}(G, x)^c$. However, this restriction identifies with the embedding of monoidal categories $\text{Sph}(G, x)^c \hookrightarrow \text{Sph}(G, x)^{\text{loc.c}}$.

12.2.5. We claim that the tautological functor $\text{Sph}(G, x)^{\text{naive}} \rightarrow \text{Sph}(G, x)$ of (12.1) canonically factors as

$$\text{Sph}(G, x)^{\text{naive}} \rightarrow \text{Sph}(G, x)^{\text{ren}} \xrightarrow{\Psi^{\text{Sph}}} \text{Sph}(G, x).$$

Indeed, we construct the functor

$$\text{Sph}(G, x)^{\text{naive}} \rightarrow \text{Sph}(G, x)^{\text{ren}} \tag{12.3}$$

as the ind-extension of a functor $(\text{Sph}(G, x)^{\text{naive}})^c \rightarrow \text{Sph}(G, x)^{\text{loc.c}}$. The latter is obtained by noticing that the essential image of $(\text{Sph}(G, x)^{\text{naive}})^c$ under the functor (12.1) is contained in $\text{Sph}(G, x)^{\text{loc.c}}$.

By construction, the functor (12.3) sends compact objects to compact ones. By contrast, the functor (12.1) does not have this property.

We will denote by $\text{Sat}^{\text{naive,ren}}$ the resulting functor

$$\text{Rep}(\check{G}) \rightarrow \text{Sph}(G, x)^{\text{ren}}.$$

12.3 The Hecke category on the spectral side

12.3.1. Consider now the stack

$$\text{Hecke}(\check{G})_{\text{spec}} := \text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G},$$

¹⁶ A t-structure on a cocomplete DG category is called compatible with colimits if the subcategory of coconnective objects is closed under filtered colimits.

where both maps $\text{pt} \rightarrow \check{\mathfrak{g}}$ correspond to $0 \in \check{\mathfrak{g}}$. In the notation of Sect. 9.1.1,

$$\text{Hecke}(\check{G})_{\text{spec}} = \mathcal{G}_{(\text{pt}/\check{G})/(\check{\mathfrak{g}}/\check{G})}.$$

The stack $\text{Hecke}(\check{G})_{\text{spec}}$ is naturally a groupoid acting on pt/\check{G} . This groupoid structure equips

$$\text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}})$$

with a structure of monoidal category via convolution.

We can also consider the subcategory

$$\text{Coh}(\text{Hecke}(\check{G})_{\text{spec}}) \subset \text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}}),$$

which is stable under the monoidal operation, and thus acquires a structure of (non-cocomplete) monoidal category, whose ind-completion identifies with $\text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}})$.

12.3.2. The following description of $\text{Sph}(G, x)^{\text{loc.c}}$ is given by [2, Theorem 5]¹⁷:

Theorem 12.3.3 *There is a canonical equivalence of (non-cocomplete) monoidal categories*

$$\text{Coh}(\text{Hecke}(\check{G})_{\text{spec}}) \simeq \text{Sph}(G, x)^{\text{loc.c}}.$$

This equivalence tautologically extends to an equivalence between the ind-completions of these categories, giving the following “renormalized” geometric Satake equivalence.

Corollary 12.3.4 *There exists a canonical equivalence of monoidal categories*

$$\text{Sat}^{\text{ren}} : \text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}}) \simeq \text{Sph}(G, x)^{\text{ren}}.$$

Under this equivalence, the functor $\text{Sat}^{\text{naive,ren}} : \text{Rep}(\check{G}) \rightarrow \text{Sph}(G, x)^{\text{ren}}$ corresponds to the canonical functor

$$\text{Rep}(\check{G}) \rightarrow \text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}}),$$

given by the direct image along the diagonal map

$$\Delta_{\text{pt}/\check{G}} : \text{pt}/\check{G} \rightarrow \text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G} = \text{Hecke}(\check{G})_{\text{spec}}.$$

¹⁷ The statement in *loc.cit.* is the combination of Theorem 12.3.3 as stated below and Proposition 12.4.2.

Remark 12.3.5 The category $\mathrm{Sph}(G, x)$, as well as $\mathrm{Sph}(G, x)^{\mathrm{ren}}$, has a richer structure, namely that of *factorizable* monoidal category, when we allow the point x to move along X . One can see this structure on the category $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$ as well, and one can show that the equivalence of (12.3.4) can be naturally upgraded to an equivalence of factorizable monoidal categories.¹⁸

12.4 A Koszul dual description

12.4.1. Consider the commutative DG algebra $\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}$, which is acted on canonically by \check{G} . Consider the category

$$(\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}$$

as a monoidal category via the usual tensor product operation of modules over a commutative algebra.

We claim:

Proposition 12.4.2 *There exists a canonical equivalence of monoidal categories*

$$\mathrm{KD}_{\mathrm{Hecke}(\check{G})_{\mathrm{spec}}} : \mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}) \simeq (\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}.$$

Proof Consider $V = \check{\mathfrak{g}}/\check{G}$ as a vector bundle over $\mathcal{X} = \mathrm{pt}/\check{G}$. Then, $\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])$ is a commutative (i.e., \mathbb{E}_{∞}) algebra in $\mathrm{QCoh}(\mathcal{X})$, and we have a natural equivalence

$$(\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}} \simeq \mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2])\text{-mod}.$$

Note that we are in the setting of Sect. 9.3; therefore, we have an isomorphism of \mathbb{E}_2 -algebras in $\mathrm{QCoh}(\mathcal{X})$:

$$\mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}}(V[-2]) \simeq \mathrm{HC}(\mathcal{X}/V).$$

Now the claim follows from Koszul duality of Corollary 9.1.7. \square

Remark 12.4.3 Being a symmetric monoidal category, $(\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}$ also has a structure of factorizable monoidal category over X . However, the equivalence of Proposition 12.4.2 is only between mere monoidal categories: It is not compatible with the factorizable structure. In fact, one can show that $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$ does not admit an \mathbb{E}_2 -structure which is compatible with the factorizable structure, even if G is a torus.¹⁹

¹⁸ The latter statement is known as “derived Satake”; it was conjectured by V. Drinfeld and proved by J. Lurie and the second author (unpublished) by interpreting $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$ as the \mathbb{E}_3 -center of $\mathrm{Rep}(\check{G})$, viewed as an \mathbb{E}_2 -category.

¹⁹ This is more convenient to see on the geometric side. The key fact is that the transgression map $H^{\bullet}(\mathrm{pt}/\mathbb{G}_m) \otimes H_{\bullet}(X) \rightarrow H^{\bullet}(\mathrm{Bun}_{\mathbb{G}_m})$ does not commute with the maps $H^{\bullet}(\mathrm{Bun}_{\mathbb{G}_m}) \rightarrow H^{\bullet}(\mathrm{Bun}_{\mathbb{G}_m})$ given by translation by points of X . Here, $H^{\bullet}(\mathrm{pt}/\mathbb{G}_m)$ appears as the endomorphism algebra of the unit object of $\mathrm{Sph}(\mathbb{G}_m, x)$.

12.4.4. Combining Corollary 12.3.4 and Proposition 12.4.2, we obtain:

Corollary 12.4.5 *There exists a canonical equivalence of monoidal categories*

$$\mathrm{Sat}^{\mathrm{ren}} \circ (\mathrm{KD}_{\mathrm{Hecke}(\check{G})_{\mathrm{spec}}})^{-1} : (\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}} \simeq \mathrm{Sph}(G, x)^{\mathrm{ren}}.$$

Moreover, from Sect. 12.2.4, we obtain:

Corollary 12.4.6 *There exists a canonically defined monoidal functor*

$$(\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}} \rightarrow \mathrm{Sph}(G, x),$$

which is, moreover, a colocalization.

12.4.7. In fact, the equivalence of Corollary 12.4.5 can be made more explicit:

Let $\delta_{\mathbf{1}}$ be the unit object of

$$\mathrm{Sph}(G, x)^{\mathrm{loc.c}} \subset \mathrm{Sph}(G, x),$$

given by the delta-function at $\mathbf{1} \in \mathrm{Gr}_{G,x}$. Theorem 12.3.3 implies that there exists a canonical isomorphism of \mathbb{E}_2 -algebras

$$\mathcal{M}\mathrm{aps}_{\mathrm{Sph}(G,x)^{\mathrm{loc.c}}}(\delta_{\mathbf{1}}, \delta_{\mathbf{1}}) \simeq \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}, \quad (12.4)$$

and that for any $\mathcal{M} \in \mathrm{Sph}(G, x)^{\mathrm{loc.c}}$, we have an isomorphism of $\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}$ -modules

$$\left(\mathrm{KD}_{\mathrm{Hecke}(\check{G})_{\mathrm{spec}}} \left((\mathrm{Sat}^{\mathrm{ren}})^{-1}(\mathcal{M}) \right) \right)^{\check{G}} \simeq \mathcal{M}\mathrm{aps}_{\mathrm{Sph}(G,x)^{\mathrm{loc.c}}}(\delta_{\mathbf{1}}, \mathcal{M}). \quad (12.5)$$

12.4.8. Thus, $\mathrm{KD}_{\mathrm{Hecke}(\check{G})_{\mathrm{spec}}} \left((\mathrm{Sat}^{\mathrm{ren}})^{-1}(\mathcal{M}) \right)$ is a \check{G} -equivariant module over $\mathrm{Sym}(\check{\mathfrak{g}}[-2])$, and (12.5) recovers \check{G} -invariants in this module.

One can reconstruct the entire module $\mathrm{KD}_{\mathrm{Hecke}(\check{G})_{\mathrm{spec}}} \left((\mathrm{Sat}^{\mathrm{ren}})^{-1}(\mathcal{M}) \right)$ by considering convolutions of \mathcal{M} with objects of the form $\mathrm{Sat}^{\mathrm{naive,ren}}(\rho)$ for $\rho \in \mathrm{Rep}(\check{G})^c$:

$$\left(\mathrm{KD}_{\mathrm{Hecke}(\check{G})_{\mathrm{spec}}} \left((\mathrm{Sat}^{\mathrm{ren}})^{-1}(\mathcal{M}) \right) \otimes \rho \right)^{\check{G}} \simeq \mathcal{M}\mathrm{aps}_{\mathrm{Sph}(G,x)^{\mathrm{loc.c}}}(\delta_{\mathbf{1}}, \mathcal{M} \star \mathrm{Sat}^{\mathrm{naive,ren}}(\rho)).$$

12.4.9. Note that since $\mathbf{1}$ is a closed $G(\widehat{\mathcal{O}}_x)$ -invariant point of $\mathrm{Gr}_{G,x}$,

$$\begin{aligned} \mathcal{M}\mathrm{aps}_{\mathrm{Sph}(G,x)^{\mathrm{loc.c}}}(\delta_{\mathbf{1}}, \delta_{\mathbf{1}}) &\simeq \mathcal{M}\mathrm{aps}_{\mathrm{Sph}(G,x)}(\delta_{\mathbf{1}}, \delta_{\mathbf{1}}) \\ &\simeq \mathcal{M}\mathrm{aps}_{\mathrm{D}\text{-mod}(\mathrm{pt})^{G(\widehat{\mathcal{O}}_x)}}(\delta, \delta) \simeq \mathcal{M}\mathrm{aps}_{\mathrm{D}\text{-mod}(\mathrm{pt})^G}(\delta, \delta), \end{aligned} \quad (12.6)$$

where δ denotes the generator $k \in \mathrm{Vect} \simeq \mathrm{pt}$, which is naturally equivariant with respect to any group.

We note that the last isomorphism in (12.6) is due to the fact that $\ker(G(\widehat{\mathcal{O}}_x) \rightarrow G)$ is pro-unipotent.

By definition, the algebra $\mathcal{M}aps_{\mathcal{D}\text{-mod}(\text{pt})^G}(\delta, \delta)$ is the equivariant cohomology of G , which we denote by $H_{\text{dR}}(\text{pt}/G)$, and we have a canonical isomorphism

$$H_{\text{dR}}(\text{pt}/G) \simeq \text{Sym}(\mathfrak{h}^*[-2])^W \simeq \text{Sym}(\check{\mathfrak{h}}[-2])^W \simeq \text{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}.$$

Now, it follows from the construction of the isomorphism of Theorem 12.3.3 that the resulting isomorphism

$$\text{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \simeq H_{\text{dR}}(\text{pt}/G) \simeq \mathcal{M}aps_{\mathcal{D}\text{-mod}(\text{pt})^G}(\delta, \delta) \simeq \mathcal{M}aps_{\text{Sph}(G,x)^{\text{loc.c}}}(\delta_{\mathbf{1}}, \delta_{\mathbf{1}})$$

equals one given by (12.4).

12.5 A description of $\text{Sph}(G, x)$

12.5.1. Recall that the stack

$$\text{Hecke}(\check{G})_{\text{spec}} = \text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G} = \mathcal{G}_{(\text{pt}/\check{G})/(\check{\mathfrak{g}}/\mathcal{G})}$$

is quasi-smooth, and

$$\text{Sing}(\text{Hecke}(\check{G})_{\text{spec}}) \simeq \check{\mathfrak{g}}^*/\check{G},$$

see Sect. 9.1.6.

Moreover, by Corollary 9.1.7, the equivalence of Proposition 12.4.2 calculates the singular support of objects of $\text{Hecke}(\check{G})_{\text{spec}}$:

For $\mathcal{F} \in \text{Hecke}(\check{G})_{\text{spec}}$, we have

$$\text{SingSupp}(\mathcal{F}) = \text{supp}(\text{KD}_{\text{Hecke}(\check{G})_{\text{spec}}}(\mathcal{F})), \quad (12.7)$$

as subsets of $\check{\mathfrak{g}}^*/\check{G}$.

12.5.2. We will prove:

Theorem 12.5.3 *Under the equivalence*

$$\text{Sat}^{\text{ren}} : \text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}}) \simeq \text{Sph}(G, x)^{\text{ren}}$$

of Corollary 12.3.4, the colocalization

$$\Xi^{\text{Sph}} : \text{Sph}(G, x) \rightleftarrows \text{Sph}(G, x)^{\text{ren}} : \Psi^{\text{Sph}}$$

identifies with

$$\text{IndCoh}_{\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\text{Hecke}(\check{G})_{\text{spec}}) \rightleftarrows \text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}}).$$

We remind that the functors $(\Xi^{\text{Sph}}, \Psi^{\text{Sph}})$ appearing in Theorem 12.5.3 are those from Sect. 12.2.4.

In terms of Corollary 12.4.6, the assertion of Theorem 12.5.3 can be reformulated as follows:

Corollary 12.5.4 *The colocalization*

$$\text{Sph}(G, x) \rightleftarrows (\text{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}$$

of Corollary 12.4.6 identifies with

$$\left((\text{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}} \right)_{\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G}} \rightleftarrows (\text{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}.$$

Theorem 12.5.3, in particular, implies:

Corollary 12.5.5 *There exists a canonical equivalence of monoidal categories*

$$\text{Sat} : \text{IndCoh}_{\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\text{Hecke}(\check{G})_{\text{spec}}) \simeq \text{Sph}(G, x),$$

and of non-cocomplete monoidal categories²⁰

$$\text{Coh}_{\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\text{Hecke}(\check{G})_{\text{spec}}) \simeq \text{Sph}(G, x)^c.$$

12.6 Proof of Theorem 12.5.3

By (12.7), we need to show that the essential image of $\text{Sph}(G, x)$ under the equivalence

$$\text{KD}_{\text{Hecke}(\check{G})_{\text{spec}}} \circ (\text{Sat}^{\text{ren}})^{-1} : \text{Sph}(G, x)^{\text{ren}} \simeq (\text{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}$$

coincides with the subcategory

$$(\text{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}_{\text{Nilp}(\check{\mathfrak{g}}^*)})^{\check{G}} \subset (\text{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}.$$

12.6.1. Let

$$\text{D-mod}(\text{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)\text{-mon}} \subset \text{D-mod}(\text{Gr}_{G,x})$$

be the full subcategory generated by the essential image of the forgetful functor

$$\text{Sph}(G, x)^{\text{loc.c}} \rightarrow \text{D-mod}(\text{Gr}_{G,x}).$$

²⁰ The fact that $\text{Coh}_{\text{Nilp}(\check{\mathfrak{g}}^*)/\check{G}}(\text{Hecke}(\check{G})_{\text{spec}})$ is preserved under the monoidal operation follows, e.g., from Proposition 12.4.2.

Since $\mathrm{Sph}(G, x)^{loc.c}$ is closed under the truncation functors, we obtain that the category $\mathrm{D-mod}(\mathrm{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)\text{-mon}}$ is compactly generated by the essential image of

$$(\mathrm{Sph}(G, x)^{loc.c})^\heartsuit \subset \mathrm{Sph}(G, x)^{loc.c}.$$

Since the generators of $\mathrm{D-mod}(\mathrm{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)\text{-mon}}$ are holonomic, the forgetful functor

$$\mathbf{oblv}_{G(\widehat{\mathcal{O}}_x)} : \mathrm{Sph}(G, x) \rightarrow \mathrm{D-mod}(\mathrm{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)\text{-mon}}$$

admits a *left* adjoint, given by $!$ -averaging with respect to $G(\widehat{\mathcal{O}}_x)$. We denote this functor by $\mathrm{Av}_{G(\widehat{\mathcal{O}}_x),!}$.

12.6.2. Since the functor $\mathbf{oblv}_{G(\widehat{\mathcal{O}}_x)}$ is conservative, the essential image of $\mathrm{Av}_{G(\widehat{\mathcal{O}}_x),!}$ generates $\mathrm{Sph}(G, x)$. Moreover, being a left adjoint of a continuous functor, $\mathrm{Av}_{G(\widehat{\mathcal{O}}_x),!}$ sends compact objects to compact ones.

Thus, we obtain that $\mathrm{Sph}(G, x)$ is compactly generated by the objects

$$\mathrm{Av}_{G(\widehat{\mathcal{O}}_x),!} \left(\mathbf{oblv}_{G(\widehat{\mathcal{O}}_x)}(\mathcal{M}) \right)$$

for $\mathcal{M} \in (\mathrm{Sph}(G, x)^{loc.c})^\heartsuit$.

12.6.3. Note also that for

$$\mathcal{M}_1 \in \mathrm{D-mod}(\mathrm{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)\text{-mon}} \quad \text{and} \quad \mathcal{M}_2 \in \mathrm{Sph}(G, x),$$

we have:

$$\mathrm{Av}_{G(\widehat{\mathcal{O}}_x),!}(\mathcal{M}_1) \star \mathcal{M}_2 \simeq \mathrm{Av}_{G(\widehat{\mathcal{O}}_x),!}(\mathcal{M}_1 \star \mathcal{M}_2). \quad (12.8)$$

In particular, if $\mathcal{M}_1 \in \mathrm{D-mod}(\mathrm{Gr}_{G,x})^c$, and $\mathcal{M}_2 \in \mathrm{Sph}(G, x)^{loc.c}$, then

$$\mathcal{M}_1 \star \mathcal{M}_2 \in \mathrm{D-mod}(\mathrm{Gr}_{G,x})^c,$$

and therefore in this case

$$\mathrm{Av}_{G(\widehat{\mathcal{O}}_x),!}(\mathcal{M}_1) \star \mathcal{M}_2 \in \mathrm{Sph}(G, x)^c.$$

12.6.4. Let $\widetilde{\delta}$ be the object of $\mathrm{D-mod}(\mathrm{pt})^G$ equal to $\mathrm{Av}_{G,!}(k)$, where $\mathrm{Av}_{G,!}$ is the *left* adjoint to the forgetful functor

$$\mathrm{D-mod}(\mathrm{pt})^G \rightarrow \mathrm{D-mod}(\mathrm{pt}) = \mathrm{Vect}. \quad (12.9)$$

The following is well known:

Lemma 12.6.5

$$\tilde{\delta} \simeq \delta \otimes_{\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}} \mathfrak{l}, \quad (12.10)$$

where δ is as in Sect. 12.4.9, and \mathfrak{l} is a graded line (placed in the cohomological degree $-\dim(G)$), acted on trivially by $\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}$.

Proof Let \mathfrak{a} be the object of Vect such that $A := \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \simeq \mathrm{Sym}(\mathfrak{a})$. It is well known (see, e.g., [6, Example 6.5.5]) that $\mathrm{D}\text{-mod}(\mathrm{pt})^{\check{G}}$, equipped with the forgetful functor (12.9), identifies with the category $B\text{-mod}$, where $B = \mathrm{Sym}(\mathfrak{a}^*[-1])\text{-mod}$. In particular, the object $\tilde{\delta}$ corresponds to B itself, and δ corresponds to the augmentation $B \rightarrow k$.

This makes the assertion of the lemma manifest, where \mathfrak{l} is the graded line such that

$$B \simeq B^* \otimes \mathfrak{l},$$

where B^* is the linear dual of B regarded as an object of $B\text{-mod}$. □

12.6.6. Let $\tilde{\delta}_1$ denote the corresponding object of $\mathrm{Sph}(G, x)$ obtained via

$$\begin{aligned} \mathrm{D}\text{-mod}(\mathrm{pt}/G) &\simeq \mathrm{D}\text{-mod}(\mathrm{pt})^G \simeq \mathrm{D}\text{-mod}(\mathrm{pt})^{G(\widehat{\mathcal{O}}_x)} \xrightarrow{\mathbf{1}} \mathrm{D}\text{-mod}(\mathrm{Gr}_{G,x})^{G(\widehat{\mathcal{O}}_x)} \\ &= \mathrm{Sph}(G, x). \end{aligned}$$

via the inclusion of the point $\mathbf{1} \in \mathrm{Gr}_{G,x}$.

By construction,

$$\tilde{\delta}_1 \simeq \mathrm{Av}_{G(\widehat{\mathcal{O}}_x),!}(\delta_1),$$

so from (12.8), we obtain that the category $\mathrm{Sph}(G, x)$ is compactly generated by objects of the form

$$\tilde{\delta}_1 \star \mathcal{M}$$

for $\mathcal{M} \in (\mathrm{Sph}(G, x)^{\mathrm{loc.c}})^{\heartsuit}$. Such \mathcal{M} are of the form $\mathrm{Sat}^{\mathrm{naive,ren}}(\rho)$ for $\rho \in (\mathrm{Rep}(\check{G})^c)^{\heartsuit}$, by the construction of $\mathrm{Sat}^{\mathrm{naive,ren}}(\rho)$.

12.6.7. By (12.10) and Sects. 12.4.7 and 12.4.9, for $\rho \in \mathrm{Rep}(\check{G})^c$ we have

$$\mathrm{KD}_{\mathrm{Hecke}(\check{G})_{\mathrm{spec}}} \circ (\mathrm{Sat}^{\mathrm{ren}})^{-1} \left(\tilde{\delta}_1 \star \mathrm{Sat}^{\mathrm{naive,ren}}(\rho) \right) \simeq \left(\mathrm{Sym}(\check{\mathfrak{g}}[-2]) \otimes_{\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}} \mathfrak{l} \right) \otimes \rho,$$

regarded as an object of $(\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}$.

So, the essential image of $\text{Sph}(G, x)$ under $\text{KD}_{\text{Hecke}(\check{G})_{\text{spec}}} \circ (\text{Sat}^{\text{ren}})^{-1}$ is compactly generated by objects of form

$$\left(\text{Sym}(\check{\mathfrak{g}}[-2]) \otimes_{\text{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}} k \right) \otimes \rho, \quad \rho \in \text{Rep}(\check{G})^c.$$

However, as

$$\text{Sym}(\check{\mathfrak{g}}) \otimes_{(\text{Sym}(\check{\mathfrak{g}}))^{\check{G}}} k \simeq \mathcal{O}_{\text{Nilp}(\check{\mathfrak{g}}^*)},$$

it is clear that the subcategory generated by such objects is exactly

$$(\text{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}_{\text{Nilp}(\check{\mathfrak{g}}^*)})^{\check{G}}.$$

□[Theorem 12.5.3]

12.7 The action on $\text{D-mod}(\text{Bun}_G)$ and $\text{IndCoh}_{\text{Nilp}_{\text{glob}}}(\text{LocSys}_{\check{G}})$

Recall that the monoidal category $\text{Sph}(G, x)$ canonically acts on $\text{D-mod}(\text{Bun}_G)$. In this subsection, we will study the corresponding action on the spectral side.

12.7.1. First, we claim that the monoidal category $\text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}})$ canonically acts on $\text{IndCoh}(\text{LocSys}_{\check{G}})$. This follows from the fact that we have a commutative diagram in which both parallelograms are Cartesian:

$$\begin{array}{ccccc}
 & & \text{LocSys}_{\check{G}} & \times_{\text{LocSys}_{\check{G}}^{\text{R.S.}}} & \text{LocSys}_{\check{G}} \\
 & \swarrow & & \downarrow & \searrow \\
 \text{LocSys}_{\check{G}} & & & & \text{LocSys}_{\check{G}} \\
 \downarrow & & & & \downarrow \\
 & & \text{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \text{pt}/\check{G} & & \\
 & \swarrow & & \searrow & \\
 \text{pt}/\check{G} & & & & \text{pt}/\check{G}
 \end{array} \tag{12.11}$$

indeed, this a special case of diagram (9.3).

12.7.2. The next proposition shows that the several different ways to define an action of the monoidal category $\mathrm{IndCoh}_{\mathrm{Nilp}(\check{g}^*)/\check{G}}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$ on $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ give the same result.

Proposition 12.7.3 (a) *For any conical Zariski-closed subset $Y \subset \mathrm{Arth}_{\check{G}}$, the action of the monoidal category $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$ sends $\mathrm{IndCoh}_Y(\mathrm{LocSys}_{\check{G}})$ to $\mathrm{IndCoh}_Y(\mathrm{LocSys}_{\check{G}})$. Moreover, the diagram*

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}) \otimes \mathrm{IndCoh}_Y(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\text{action}} & \mathrm{IndCoh}_Y(\mathrm{LocSys}_{\check{G}}) \\ \mathrm{Id} \otimes \Psi_{\mathrm{LocSys}_{\check{G}}}^{Y, \mathrm{all}} \uparrow & & \uparrow \Psi_{\mathrm{LocSys}_{\check{G}}}^{Y, \mathrm{all}} \\ \mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}) \otimes \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\text{action}} & \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) \end{array}$$

commutes as well (i.e., the functor $\Psi^{Y, \mathrm{all}}$, which is a priori lax compatible with the action of $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$, is strictly compatible).

- (b) *The action of $\mathrm{IndCoh}_{\mathrm{Nilp}(\check{g}^*)/\check{G}}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$ sends $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ to the subcategory $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$.*
(c) *The composed functor*

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}) \otimes \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\text{action}} & \\ & \xrightarrow{\text{colocalization}} & \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \end{array}$$

factors through the colocalization

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}) \otimes \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) & & \\ \rightarrow \mathrm{IndCoh}_{\mathrm{Nilp}(\check{g}^*)/\check{G}}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}) \otimes \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}). & & \end{array}$$

Proof To prove point (a), it is enough to do so on the generators of $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$, i.e., on the essential image of

$$(\Delta_{\mathrm{pt}/\check{G}}^{\mathrm{IndCoh}})_* : \mathrm{IndCoh}(\mathrm{pt}/\check{G}) \rightarrow \mathrm{IndCoh}(\mathrm{pt}/\check{G} \times_{\check{g}/\check{G}} \mathrm{pt}/\check{G}).$$

However, for $\mathcal{F} \in \mathrm{IndCoh}(\mathrm{pt}/\check{G})$, the action of $(\Delta_{\mathrm{pt}/\check{G}}^{\mathrm{IndCoh}})_*(\mathcal{F})$ on $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ is given by tensor product with the pullback of

$$\mathcal{F} \in \mathrm{IndCoh}(\mathrm{pt}/\check{G}) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

under the map

$$\mathrm{LocSys}_{\check{G}} \rightarrow \mathrm{pt}/\check{G},$$

corresponding to the point x . Hence, the assertion follows from Corollary 8.2.4.

Points (b) and (c) are a particular case of Corollary 9.4.2. \square

As a corollary, we obtain:

Corollary 12.7.4 *There exists a canonically defined action of $\text{IndCoh}_{\text{Nilp}(\check{g}^*)/\check{G}}$ ($\text{Hecke}(\check{G})_{\text{spec}}$) on $\text{IndCoh}_{\text{Nilp}_{\text{glob}}}(\text{LocSys}_{\check{G}})$, which is compatible with the $\text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}})$ -action on $\text{IndCoh}(\text{LocSys}_{\check{G}})$ via any of the functors*

$$\Xi_{\text{LocSys}_{\check{G}}}^{\text{Nilp}_{\text{glob}}} : \text{IndCoh}_{\text{Nilp}_{\text{glob}}}(\text{LocSys}_{\check{G}}) \rightleftarrows \text{IndCoh}(\text{LocSys}_{\check{G}}) : \Psi_{\text{LocSys}_{\check{G}}}^{\text{Nilp}_{\text{glob}}}$$

and

$$\begin{aligned} \Xi_{\text{Hecke}(\check{G})_{\text{spec}}}^{\text{Nilp}(\check{g}^*)/\check{G}, \text{all}} &: \text{IndCoh}_{\text{Nilp}(\check{g}^*)/\check{G}}(\text{Hecke}(\check{G})_{\text{spec}}) \\ &\rightleftarrows \text{IndCoh}(\text{Hecke}(\check{G})_{\text{spec}}) : \Psi_{\text{Hecke}(\check{G})_{\text{spec}}}^{\text{Nilp}(\check{g}^*)/\check{G}, \text{all}}. \end{aligned}$$

12.7.5. The compatibility of Conjecture 11.2.2 with the geometric Satake equivalence reads

Conjecture 12.7.6 *The action of $\text{Sph}(G, x)$ on $D\text{-mod}(\text{Bun}_G)$ corresponds via*

$$\text{Sat} : \text{IndCoh}_{\text{Nilp}(\check{g}^*)/\check{G}}(\text{Hecke}(\check{G})_{\text{spec}}) \simeq \text{Sph}(G, x)$$

to the action of $\text{IndCoh}_{\text{Nilp}(\check{g}^)/\check{G}}(\text{Hecke}(\check{G})_{\text{spec}})$ on $\text{IndCoh}_{\text{Nilp}_{\text{glob}}}(\text{LocSys}_{\check{G}})$.*

Remark 12.7.7 The above conjecture is not yet the full compatibility of the geometric Satake equivalence with the geometric Langlands equivalence. The full version amounts to formulating Conjecture 12.7.6 in a way that takes into account the factorizable structure of

$$\text{IndCoh}_{\text{Nilp}(\check{g}^*)/\check{G}}(\text{Hecke}(\check{G})_{\text{spec}}) \simeq \text{Sph}(G, x)$$

as x moves along X .

12.8 Singular support via the Hecke action

12.8.1. If Conjecture 11.2.2 holds, an object $\mathcal{M} \in D\text{-mod}(\text{Bun}_G)$ can be assigned its singular support, which by definition is equal to the singular support of the corresponding object of $\text{IndCoh}_{\text{Nilp}_{\text{glob}}}(\text{LocSys}_{\check{G}})$. The singular support is a conical Zariski-closed subset $\text{SingSupp}(\mathcal{M}) \subset \text{Nilp}_{\text{glob}}$.

It turns out that Conjecture 12.7.6 implies certain relation between $\text{SingSupp}(\mathcal{M})$ and the action of the Hecke category $\text{Sph}(G, x)$ on \mathcal{M} . Let us explain this in more detail.

12.8.2. The equivalence

$$\mathrm{KD}_{\mathrm{Hecke}(\check{G})_{\mathrm{spec}}} : \mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}}) \simeq (\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod})^{\check{G}}$$

of Proposition 12.4.2 and Proposition 12.7.3(a) makes the categories $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ and $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ into categories tensored over $\mathrm{QCoh}(\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m))$.

By construction, this is the same structure as that given by the embedding

$$\iota : \mathrm{LocSys}_{\check{G}} \hookrightarrow \mathrm{LocSys}_{\check{G}}^{\mathrm{R.S.}}$$

in terms of Sect. 9.3.2.

Thus, we can determine the singular support of objects of $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ via the above action of $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G})_{\mathrm{spec}})$.

12.8.3. On the other hand, Corollary 12.4.6 defines on $\mathrm{D}\text{-mod}(\mathrm{Bun}_{\check{G}})$ a structure of category tensored over

$$\mathrm{QCoh}(\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m)).$$

Hence, we can attach to an object $\mathcal{M} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_{\check{G}})$ its support

$$\mathrm{supp}^x(\mathcal{M}) \subset \check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m),$$

which is a Zariski-closed subset (The superscript x indicates that this support depends on the choice of the point $x \in X$).

Conjectures 11.2.2 and 12.7.6 would imply that $\mathrm{supp}^x(\mathcal{M})$ is the Zariski closure of the image of

$$\mathrm{SingSupp}(\mathcal{M}) \subset \mathrm{Arth}_{\check{G}} = \mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$$

under the map

$$\mathrm{Arth}_{\check{G}} \rightarrow \check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m) : (\mathcal{P}, \nabla, A) \mapsto A(x).$$

(This easily follows from Lemma 3.3.12.). Here, we use the explicit description of $\mathrm{Arth}_{\check{G}}$ given in Corollary 10.4.7.

12.8.4. In particular, consider the full subcategory

$$\mathrm{D}\text{-mod}_{\mathrm{temp}}^x(\mathrm{Bun}_{\check{G}}) := \{\mathcal{M} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_{\check{G}}) : \mathrm{supp}^x(\mathcal{M}) = \{0\}\}.$$

Equivalently, we can define it as the tensor product

$$\mathrm{D}\text{-mod}_{\mathrm{temp}}^x(\mathrm{Bun}_{\check{G}}) = \mathrm{D}\text{-mod}(\mathrm{Bun}_{\check{G}}) \otimes_{\mathrm{QCoh}(\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m))} \mathrm{QCoh}(\check{\mathfrak{g}}^*/(\mathcal{G} \times \mathbb{G}_m))_{\{0\}}. \quad (12.12)$$

Conjectures 11.2.2 and 12.7.6 imply that under the equivalence

$$D\text{-mod}(\text{Bun}_G) \simeq \text{IndCoh}_{\text{Nilp}_{glob}}(\text{LocSys}_{\check{G}}),$$

the category $D\text{-mod}_{\text{temp}}^x(\text{Bun}_G)$ corresponds to the subcategory

$$\text{IndCoh}_{\{0\}}(\text{LocSys}_{\check{G}}) \subset \text{IndCoh}_{\text{Nilp}_{glob}}(\text{LocSys}_{\check{G}}),$$

which is the same as the essential image of $\text{QCoh}(\text{LocSys}_{\check{G}})$ under the functor

$$\Psi_{\text{LocSys}_{\check{G}}} : \text{QCoh}(\text{LocSys}_{\check{G}}) \rightarrow \text{IndCoh}(\text{LocSys}_{\check{G}}).$$

Thus, $D\text{-mod}_{\text{temp}}^x(\text{Bun}_G)$ should be equal to the subcategory

$$D\text{-mod}_{\text{temp}}(\text{Bun}_G) \subset D\text{-mod}(\text{Bun}_G)$$

of Sect. 11.2.4.

In particular, we obtain:

Conjecture 12.8.5 *The subcategory $D\text{-mod}_{\text{temp}}^x(\text{Bun}_G) \subset D\text{-mod}(\text{Bun}_G)$ is independent of the choice of the point $x \in X$.*

12.8.6. Let us provide a more explicit description of the subcategory $D\text{-mod}_{\text{temp}}^x(\text{Bun}_G)$.

Recall that $D\text{-mod}(\text{Bun}_G)$ is compactly generated (see [7]). Now (12.12) implies that $D\text{-mod}_{\text{temp}}^x(\text{Bun}_G)$ is compactly generated by

$$D\text{-mod}_{\text{temp}}^x(\text{Bun}_G)^c = D\text{-mod}_{\text{temp}}^x(\text{Bun}_G) \cap D\text{-mod}(\text{Bun}_G)^c.$$

For this reason, it suffices to describe the compact objects of $D\text{-mod}_{\text{temp}}(\text{Bun}_G)$.

By Theorem 12.3.3, there exists a canonical map in $\text{Sph}(G, x)$:

$$\alpha : \text{Sat}^{naive}(\check{\mathfrak{g}})[-2] \rightarrow \text{Sat}^{naive}(k) = \delta_{\mathbf{1}},$$

where $k \in \text{Rep}(\check{G})$ is the trivial representation.

Moreover, for any $n \in \mathbb{N}$ we can consider its ‘‘symmetric power’’

$$\alpha_n : \text{Sat}(\text{Sym}^n(\check{\mathfrak{g}}))[-2n] \rightarrow \delta_{\mathbf{1}}.$$

From Lemma 3.4.4(c), we obtain:

Corollary 12.8.7 *For $\mathcal{M} \in D\text{-mod}(\text{Bun}_G)^c$, the following conditions are equivalent:*

(a) $\mathcal{M} \in D\text{-mod}_{\text{temp}}^x(\text{Bun}_G)$;

(b) *The induced map*

$$(\alpha_n \star \text{id}_{\mathcal{M}}) : \text{Sat}^{\text{naive}}(\text{Sym}^n(\check{\mathfrak{g}})) \star \mathcal{M}[-2n] \rightarrow \mathcal{M}$$

vanishes for some integer $n \geq 0$;

(c) *The map $\alpha_n \star \text{id}_{\mathcal{M}}$ vanishes for all sufficiently large integers n .*

Remark 12.8.8 One can show *unconditionally* that when we view $\text{D-mod}(\text{Bun}_G)$ as tensored over $\text{QCoh}(\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m))$, i.e., as of category over the stack $\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m)$, it is supported over

$$\text{Nilp}(\check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m)) \subset \check{\mathfrak{g}}^*/(\check{G} \times \mathbb{G}_m).$$

This fact is equivalent to the following. In the situation of Corollary 12.8.7, consider the map of \check{G} -representations

$$\text{Sym}^n(\check{\mathfrak{g}})^{\check{G}} \otimes k \rightarrow \text{Sym}^n(\check{\mathfrak{g}}),$$

where $\text{Sym}^n(\check{\mathfrak{g}})^{\check{G}}$ is regarded as a mere vector space and k is the trivial representation. Composing, for $\mathcal{M} \in \text{D-mod}(\text{Bun}_G)$, we obtain a map

$$\text{Sym}^n(\check{\mathfrak{g}})^{\check{G}} \otimes \mathcal{M}[-2n] \rightarrow \mathcal{M}. \quad (12.13)$$

We claim that this map vanishes for any $\mathcal{M} \in \text{D-mod}(\text{Bun}_G)^c$ whenever n is sufficiently large.

To prove this, we note that by Sect. 12.4.9, the map (12.13) comes from the composition

$$\text{Sym}^n(\check{\mathfrak{g}})^{\check{G}} \hookrightarrow H_{\text{dR}}^{2n}(\text{pt}/G) \rightarrow H_{\text{dR}}^{2n}(\text{Bun}_G),$$

where $H_{\text{dR}}(\text{pt}/G) \rightarrow H_{\text{dR}}(\text{Bun}_G)$ is the homomorphism given by the map $\text{Bun}_G \rightarrow \text{pt}/G$, given, corresponding to the restriction along $x \rightarrow X$.

Now, one can show that for any (connected) algebraic stack \mathcal{Y} and any $\mathcal{M} \in \text{D-mod}(\mathcal{Y})^c$, the map

$$H_{\text{dR}}(\mathcal{Y}) \otimes \mathcal{M} \rightarrow \mathcal{M}$$

vanishes on a sufficiently high power of the augmentation ideal of $H_{\text{dR}}(\mathcal{Y})$.

12.8.9. Using Lemma 3.4.4(a), one can give the following characterization of the entire subcategory $\text{D-mod}_{\text{temp}}^x(\text{Bun}_G) \subset \text{D-mod}(\text{Bun}_G)$ (and not just its compact objects):

Lemma 12.8.10 *An object $\mathcal{M} \in D\text{-mod}(\text{Bun}_G)$ belongs to $D\text{-mod}_{\text{temp}}^x(\text{Bun}_G)$ if and only for a set of compact generators $\mathcal{M}_\alpha \in D\text{-mod}(\text{Bun}_G)$ and any $\mathcal{M}_\alpha \rightarrow \mathcal{M}$, for all sufficiently large n the composition*

$$\mathcal{M}_\alpha \rightarrow \mathcal{M} \rightarrow \text{Sat}^{\text{naive}}(\text{Sym}^n(\check{\mathfrak{g}}^*)) \star \mathcal{M}[2n]$$

vanishes.

From here, we obtain:

Corollary 12.8.11 *The constant sheaf $k_{\text{Bun}_G} \in D\text{-mod}(\text{Bun}_G)$ is not tempered.*

Proof Follows from the fact that for any n , the map

$$\begin{aligned} k_{\text{Bun}_G}[-2n] &\rightarrow \text{Sat}^{\text{naive}}(\text{Sym}^n(\check{\mathfrak{g}}^*)) \star k_{\text{Bun}_G} \\ &\simeq H\left(\text{Gr}_{G,x}, \text{Sat}^{\text{naive}}(\text{Sym}^n(\check{\mathfrak{g}}^*))\right) \otimes k_{\text{Bun}_G} \end{aligned}$$

is the inclusion of a direct summand. □

13 Compatibility with Eisenstein series

A crucial ingredient in the formulation of the geometric Langlands equivalence is the interaction of G with its Levi subgroups. Such interaction is given by the functors of Eisenstein series on both sides of the correspondence. In this section, we will study how these functors act on our category $\text{IndCoh}_{\text{NilP}_{\text{glob}}^G}(\text{LocSys}_{\check{G}})$.

13.1 The Eisenstein series functor on the geometric side

13.1.1. Let P be a parabolic subgroup of G with the Levi quotient M . Let us recall the definition of the Eisenstein series functor

$$\text{Eis}_!^P : D\text{-mod}(\text{Bun}_M) \rightarrow D\text{-mod}(\text{Bun}_G)$$

(see [8]).

By definition,

$$\text{Eis}_!^P = (\mathfrak{p}^P)_! \circ (\mathfrak{q}^P)^*,$$

where \mathfrak{p}^P and \mathfrak{q}^P are the maps in the diagram

$$\begin{array}{ccc} & \text{Bun}_P & \\ \mathfrak{p}^P \swarrow & & \searrow \mathfrak{q}^P \\ \text{Bun}_G & & \text{Bun}_M. \end{array} \tag{13.1}$$

We note that the functor $(\mathfrak{q}^P)^*$ is defined because the morphism $\mathfrak{q}^P : \text{Bun}_P \rightarrow \text{Bun}_M$ is smooth and that the functor $(\mathfrak{p}^P)_!$, left adjoint to $(\mathfrak{p}^P)^!$, is defined on the essential image of $(\mathfrak{q}^P)^*$, as is shown in [8, Proposition 1.2].

Note that the functor $\text{Eis}_!^P$ sends compact objects in $\text{D-mod}(\text{Bun}_M)^c$ to compact objects in $\text{D-mod}(\text{Bun}_G)^c$, since it admits a continuous right adjoint

$$\text{CT}_*^P = (\mathfrak{q}^P)_* \circ (\mathfrak{p}^P)^!$$

13.1.2. Let $\text{D-mod}_{\text{Eis}}(\text{Bun}_G)$ be the full subcategory of $\text{D-mod}(\text{Bun}_G)$ generated by the essential images of the functors $\text{Eis}_!^P$ for all proper parabolic subgroups P . Let $\text{D-mod}_{\text{cusp}}(\text{Bun}_G)$ denote the full subcategory of $\text{D-mod}(\text{Bun}_G)$ equal to the right orthogonal of $\text{D-mod}_{\text{Eis}}(\text{Bun}_G)$.

Since the functors $\text{Eis}_!^P$ preserve compactness, the category $\text{D-mod}_{\text{Eis}}(\text{Bun}_G)$ is compactly generated. Therefore, $\text{D-mod}_{\text{cusp}}(\text{Bun}_G)$ is a localization of $\text{D-mod}(\text{Bun}_G)$ with respect to $\text{D-mod}_{\text{Eis}}(\text{Bun}_G)$, so we obtain a short exact sequence of DG categories

$$\text{D-mod}_{\text{Eis}}(\text{Bun}_G) \rightleftarrows \text{D-mod}(\text{Bun}_G) \rightleftarrows \text{D-mod}_{\text{cusp}}(\text{Bun}_G).$$

13.2 Eisenstein series on the spectral side

13.2.1. Fix a parabolic subgroup $P \subset G$, and consider the corresponding parabolic subgroup $\check{P} \subset \check{G}$, whose Levi quotient \check{M} identifies with the Langlands dual of M . Consider the diagram:

$$\begin{array}{ccc}
 & \text{LocSys}_{\check{P}} & \\
 \mathfrak{p}_{spec}^P \swarrow & & \searrow \mathfrak{q}_{spec}^P \\
 \text{LocSys}_{\check{G}} & & \text{LocSys}_{\check{M}}.
 \end{array} \tag{13.2}$$

We define the functor

$$\text{Eis}_{spec}^P : \text{IndCoh}(\text{LocSys}_{\check{M}}) \rightarrow \text{IndCoh}(\text{LocSys}_{\check{G}}),$$

to be

$$(\mathfrak{p}_{spec}^P)_*^{\text{IndCoh}} \circ (\mathfrak{q}_{spec}^P)^!$$

First, we note:

Lemma 13.2.2 (a) *The map \mathfrak{q}_{spec}^P is quasi-smooth.*

(b) *The map \mathfrak{p}_{spec}^P is schematic and proper.*

Proof For part (a), we claim that for any surjective homomorphism of algebraic groups

$$\check{G}_1 \rightarrow \check{G}_2,$$

the corresponding map $\mathrm{LocSys}_{\check{G}_1} \rightarrow \mathrm{LocSys}_{\check{G}_2}$ is quasi-smooth. This relative version of Proposition 10.4.5 can be proved in the same way as Proposition 10.4.5. Part (b) is straightforward (and well known). \square

Corollary 13.2.3 *The functor Eis_{spec}^P sends $\mathrm{Coh}(\mathrm{LocSys}_{\check{M}})$ to $\mathrm{Coh}(\mathrm{LocSys}_{\check{G}})$.*

Proof First, we claim that the functor $(\mathfrak{q}_{spec}^P)^1$ sends $\mathrm{Coh}(\mathrm{LocSys}_{\check{M}})$ to $\mathrm{Coh}(\mathrm{LocSys}_{\check{p}})$. This follows from [10, Lemma 7.1.2] and Lemma 13.2.2(a).

Now, $(\mathfrak{p}_{spec}^P)_{*}^{\mathrm{IndCoh}}$ sends $\mathrm{Coh}(\mathrm{LocSys}_{\check{p}})$ to $\mathrm{Coh}(\mathrm{LocSys}_{\check{G}})$ by [10, Lemma 3.3.5] and Lemma 13.2.2(b). \square

13.2.4. Let us now analyze the singular codifferential of the morphisms \mathfrak{p}_{spec}^P and \mathfrak{q}_{spec}^P . To avoid confusion, let us introduce superscripts and write

$$\mathrm{Nilp}_{glob}^G \subset \mathrm{Arth}_{\check{G}} \quad \text{and} \quad \mathrm{Nilp}_{glob}^M \subset \mathrm{Arth}_{\check{M}}$$

to distinguish between the global nilpotent cones for \check{M} and \check{G} .

By Lemma 13.2.2(a), the singular codifferential

$$\mathrm{Sing}(\mathfrak{q}_{spec}^P) : \mathrm{Arth}_{\check{M}} \times_{\mathrm{LocSys}_{\check{M}}} \mathrm{LocSys}_{\check{p}} \rightarrow \mathrm{Arth}_{\check{p}}$$

is a closed embedding. Consider the subset

$$\mathrm{Nilp}_{glob}^M \times_{\mathrm{LocSys}_{\check{M}}} \mathrm{LocSys}_{\check{p}} \subset \mathrm{Arth}_{\check{M}} \times_{\mathrm{LocSys}_{\check{M}}} \mathrm{LocSys}_{\check{p}}$$

and let

$$\mathrm{Nilp}_{glob}^P := \mathrm{Sing}(\mathfrak{q}_{spec}^P) \left(\mathrm{Nilp}_{glob}^M \times_{\mathrm{LocSys}_{\check{M}}} \mathrm{LocSys}_{\check{p}} \right) \subset \mathrm{Arth}_{\check{p}}$$

be its image. Here is an explicit description.

Lemma 13.2.5 *Let us identify $\mathrm{Arth}_{\check{p}}$ with the moduli stack (in the classical sense) of triples $(\mathcal{P}^{\check{p}}, \nabla, A^{\check{p}} \in H^0(\Gamma(X_{\mathrm{dR}}, \check{\mathfrak{p}}_{\mathcal{P}^{\check{p}}}^*))$ using Corollary 10.4.7. Then,*

$$\mathrm{Nilp}_{glob}^P = \{(\mathcal{P}^{\check{p}}, \nabla, A^{\check{p}}), A^{\check{p}} \text{ is a nilpotent section of } \check{\mathfrak{m}}_{\mathcal{P}^{\check{p}}}^* \subset \check{\mathfrak{p}}_{\mathcal{P}^{\check{p}}}^*\}.$$

Proof Indeed, $\mathrm{Arth}_{\check{M}} \times_{\mathrm{LocSys}_{\check{M}}} \mathrm{LocSys}_{\check{p}}$ is identified with the classical moduli stack of triples $(\check{\mathcal{P}}^{\check{p}}, \nabla, A^{\check{M}})$, where $(\check{\mathcal{P}}^{\check{p}}, \nabla) \in \mathrm{LocSys}_{\check{p}}$ and $A^{\check{M}} \in H^0(\Gamma(X_{\mathrm{dR}}, \check{\mathfrak{m}}_{\check{\mathcal{P}}^{\check{p}}}^*))$. Here, we use the natural action of \check{P} on $\check{\mathfrak{m}}$ (and $\check{\mathfrak{m}}^*$).

Under this identification,

$$\mathrm{Sing}(\mathfrak{q}_{spec}^P)(\check{\mathcal{P}}^{\check{p}}, \nabla, A^{\check{M}}) = (\mathcal{P}^{\check{p}}, \nabla, A^{\check{p}}),$$

where $A^{\check{P}}$ is the image of $A^{\check{M}}$ under the natural embedding $\check{m}^* \hookrightarrow \check{p}^*$. The claim follows. \square

Proposition 13.2.6 *The functor Eis_{spec}^P sends the subcategory*

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}}) \subset \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}})$$

to the subcategory

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$$

The proposition provides a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}}) & \longrightarrow & \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}}) \\ \downarrow & & \downarrow \mathrm{Eis}_{spec}^P \\ \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}) & \longrightarrow & \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}), \end{array}$$

where the horizontal arrows are the tautological embeddings. In particular, the resulting functor

$$\mathrm{Eis}_{spec}^P : \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}}) \rightarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$$

also sends compact objects to compact objects, that is, it restricts to a functor

$$\mathrm{Coh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}}) \rightarrow \mathrm{Coh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}).$$

13.2.7 Proof of Proposition 13.2.6 By Lemma 8.4.2, we see that

$$(\mathfrak{q}_{spec}^P)^! \left(\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}}) \right) \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^P}(\mathrm{LocSys}_{\check{P}}).$$

Therefore, it is enough to check that

$$(\mathfrak{p}_{spec}^P)_*^{\mathrm{IndCoh}} \left(\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^P}(\mathrm{LocSys}_{\check{P}}) \right) \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}).$$

By Lemma 8.4.5, it suffices to show that the preimage of Nilp_{glob}^P under the singular codifferential

$$\mathrm{Sing}(\mathfrak{p}_{spec}^P) : \mathrm{Arth}_{\check{G}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}} \rightarrow \mathrm{Arth}_{\check{P}}$$

is contained in

$$\mathrm{Nilp}_{\mathrm{glob}}^G \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}} \subset \mathrm{Arth}_{\check{G}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}}.$$

Using Corollary 10.4.7, we can identify $\mathrm{Arth}_{\check{G}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}}$ with the classical moduli stack of triples $(\mathcal{P}^{\check{P}}, \nabla, A^{\check{G}})$, where $(\mathcal{P}^{\check{P}}, \nabla) \in \mathrm{LocSys}_{\check{P}}$ and $A^{\check{G}} \in H^0(\Gamma(X_{\mathrm{dR}}, \check{\mathfrak{g}}_{\mathcal{P}^{\check{P}}}^*))$. Under this identification, $\mathrm{Sing}(\mathfrak{p}_{\mathrm{spec}}^P)$ sends such a triple to the triple

$$(\mathcal{P}^{\check{P}}, \nabla, A^{\check{P}}) \in H^0(\Gamma(X_{\mathrm{dR}}, \check{\mathfrak{p}}_{\mathcal{P}^{\check{P}}}^*)),$$

where $A^{\check{P}}$ is obtained from $A^{\check{G}}$ via the natural projection $\check{\mathfrak{g}}^* \rightarrow \check{\mathfrak{p}}^*$.

It remains to notice that if $a \in \check{\mathfrak{g}}^*$ is such that its projection to $\check{\mathfrak{p}}^*$ is a nilpotent element of $\check{\mathfrak{m}}^* \subset \check{\mathfrak{p}}^*$, then a itself is nilpotent. \square

13.2.8 Compatibility between geometric Langlands correspondence and Eisenstein series The following is one of the key requirements on the equivalence of Conjecture 11.2.2:

Conjecture 13.2.9 *For every parabolic P , the following diagram of functors*

$$\begin{array}{ccc} D\text{-mod}(\mathrm{Bun}_G) & \longrightarrow & \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}^G}(\mathrm{LocSys}_{\check{G}}) \\ \mathrm{Eis}_t^P \uparrow & & \uparrow \mathrm{Eis}_{\mathrm{spec}}^P \\ D\text{-mod}(\mathrm{Bun}_M) & \longrightarrow & \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}^M}(\mathrm{LocSys}_{\check{M}}) \end{array}$$

commutes, up to an auto-equivalence of $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}^M}(\mathrm{LocSys}_{\check{M}})$ given by tensoring by a line bundle.

13.3 The main result

13.3.1. Let $\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}$ denote the closed substack of $\mathrm{LocSys}_{\check{G}}$ equal to the union of the images of the maps $\mathfrak{p}_{\mathrm{spec}}^P$ for all proper parabolics P , considered, say, with the reduced structure. Let $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$ be the complementary open; we denote by j the open embedding

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}} \hookrightarrow \mathrm{LocSys}_{\check{G}}.$$

By Corollary 8.2.10, we obtain a diagram of short exact sequences of DG categories

$$\begin{array}{ccc}
 \mathrm{IndCoh}_{(\mathrm{Nilp}_{glob}^G)_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}}}(\mathrm{LocSys}_{\check{G}}) & \longrightarrow & \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}} \\
 \downarrow & & \downarrow \\
 \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}) & \longrightarrow & \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) \\
 \downarrow J^{\mathrm{IndCoh},*} & & \downarrow J^{\mathrm{IndCoh},*} \\
 \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}) & \longrightarrow & \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}})
 \end{array} \quad (13.3)$$

obtained from $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}) \hookrightarrow \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ by tensoring over $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ with the short exact sequence

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}} \rightleftarrows \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \rightleftarrows \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}).$$

In particular, the vertical arrows in the diagram (13.3) admit right adjoints, and the horizontal arrows are fully faithful embeddings. Moreover, all the categories involved are compactly generated; in particular,

$$\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}} \quad \text{and} \quad \mathrm{IndCoh}_{(\mathrm{Nilp}_{glob}^G)_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}}}(\mathrm{LocSys}_{\check{G}})$$

are compactly generated by

$$\mathrm{Coh}(\mathrm{LocSys}_{\check{G}})_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}} \quad \text{and} \quad \mathrm{Coh}(\mathrm{LocSys}_{\check{G}})_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}} \cap \mathrm{Coh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}),$$

respectively.

13.3.2. We have:

Proposition 13.3.3 *The inclusion*

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}) \hookrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}})$$

is an equality.

Proof This follows from Corollary 8.2.8 using the following observation: □

Lemma 13.3.4 *The preimage of $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$ in Nilp_{glob}^G consists of the zero-section.*

Proof Indeed, an irreducible \check{G} -local system admits no non-trivial horizontal nilpotent sections of the associated bundle of Lie algebras. □

13.3.5. We are now ready to state the main result of this paper:

Theorem 13.3.6 *The subcategory*

$$\mathrm{IndCoh}_{(\mathrm{Nilp}_{glob}^G)_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red}}}}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$$

is generated by the essential images of the functors

$$\mathrm{Eis}_{spec}^P : \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}}) \rightarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$$

for all proper parabolics P .

13.3.7. Note that from Theorem 13.3.6, combined with Proposition 13.3.3 and (13.3), we obtain:

Corollary 13.3.8 *The subcategory $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ and the essential images of*

$$\mathrm{Eis}_{spec}^P \mid \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}})$$

for all proper parabolics P , generate the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$.

Now, the transitivity property of Eisenstein series and induction on the semi-simple rank imply:

Corollary 13.3.9 *The subcategory $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, together with the essential images of the subcategories $\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}})$ under the functors Eis_{spec}^P for all proper parabolics P , generate $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$.*

Still equivalently, we have:

Corollary 13.3.10 *The essential images of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}})$ under the functors Eis_{spec}^P for all parabolic subgroups P (including the case $P = G$) generate $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$.*

13.3.11. Let us explain the significance of this theorem from the point of view of Conjecture 11.2.2. We are going to show that $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$ is the *smallest* subcategory of $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ that contains $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, which can be equivalent to $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, if we assume compatibility with the Eisenstein series as in Conjecture 13.2.9.

More precisely, let us assume that there exists an equivalence between $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ and *some* subcategory

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{IndCoh}_?(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}),$$

for which the diagrams

$$\begin{array}{ccc}
 \mathrm{D}\text{-mod}(\mathrm{Bun}_G) & \xleftarrow{\sim} & \mathrm{IndCoh}_?(\mathrm{LocSys}_{\check{G}}) \\
 \mathrm{Eis}_!^P \uparrow & & \uparrow \mathrm{Eis}_{spec}^P \\
 \mathrm{D}\text{-mod}(\mathrm{Bun}_M) & \xleftarrow{\quad} & \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})
 \end{array} \tag{13.4}$$

commute for all proper parabolics P , up to tensoring by a line bundle as in Conjecture 13.2.9.

We claim that in this case, $\mathrm{IndCoh}_?(\mathrm{LocSys}_{\check{G}})$ necessarily contains $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$. Indeed, this follows from Corollary 13.3.9.

13.3.12. Recall (Sect. 13.1.2) that $\mathrm{D}\text{-mod}_{\mathrm{cusp}}(\mathrm{Bun}_G) \subset \mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ is the full subcategory of D -modules that are right orthogonal to the essential images of the Eisenstein series functors $\mathrm{Eis}_!^P$ for all proper parabolic subgroups $P \subset G$. Let us note the following corollary of Conjecture 13.2.9 and Theorem 13.3.6:

Corollary 13.3.13 *The equivalence of Conjecture 11.2.2 gives rise to an equivalence*

$$\mathrm{D}\text{-mod}_{\mathrm{cusp}}(\mathrm{Bun}_G) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}).$$

13.4 Proof of Theorem 13.3.6

13.4.1. We will prove a more precise result. For a given parabolic P , let $\mathrm{LocSys}_{\check{G}}^{\mathrm{red},P}$ denote the closed substack of $\mathrm{LocSys}_{\check{G}}$ equal to the image of the map \mathfrak{p}_{spec}^P , considered, say, with the reduced structure. Let

$$\mathrm{IndCoh}_{(\mathrm{Nilp}_{glob}^G)_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red},P}}}(\mathrm{LocSys}_{\check{G}})$$

denote the corresponding full subcategory of $\mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$.

Clearly, the functor

$$\mathrm{Eis}_{spec}^P : \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}}) \rightarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^G}(\mathrm{LocSys}_{\check{G}})$$

factors through $\mathrm{IndCoh}_{(\mathrm{Nilp}_{glob}^G)_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red},P}}}(\mathrm{LocSys}_{\check{G}})$.

We will prove:

Theorem 13.4.2 *The essential image of the functor*

$$\mathrm{Eis}_{spec}^P : \mathrm{IndCoh}_{\mathrm{Nilp}_{glob}^M}(\mathrm{LocSys}_{\check{M}}) \rightarrow \mathrm{IndCoh}_{(\mathrm{Nilp}_{glob}^G)_{\mathrm{LocSys}_{\check{G}}^{\mathrm{red},P}}}(\mathrm{LocSys}_{\check{G}})$$

generates the target category.

Theorem 13.4.2 implies Theorem 13.3.6 by Corollary 3.3.9.

13.4.3. Theorem 13.4.2 follows from the combination of the following two statements:

Proposition 13.4.4 *The essential image of the functor*

$$(\mathfrak{q}_{spec}^P)^{\dagger} : \text{IndCoh}_{\text{Nilp}_{glob}^M}(\text{LocSys}_{\check{M}}) \rightarrow \text{IndCoh}_{\text{Nilp}_{glob}^P}(\text{LocSys}_{\check{P}})$$

generates the target category.

Proposition 13.4.5 *The essential image of the functor*

$$(\mathfrak{p}_{spec}^P)_{*}^{\text{IndCoh}} : \text{IndCoh}_{\text{Nilp}_{glob}^P}(\text{LocSys}_{\check{P}}) \rightarrow \text{IndCoh}_{(\text{Nilp}_{glob}^G)_{\text{LocSys}_{\check{G}}^{\text{red}_P}}}(\text{LocSys}_{\check{G}})$$

generates the target category.

13.4.6 *Proof of Proposition 13.4.4* Recall that the map \mathfrak{q}_{spec}^P is quasi-smooth, so that its singular codifferential

$$\text{Sing}(\mathfrak{q}_{spec}^P) : \text{Arth}_{\check{M}} \times_{\text{LocSys}_{\check{M}}} \text{LocSys}_{\check{P}} \rightarrow \text{Arth}_{\check{P}}$$

is a closed embedding. Moreover, $\text{Nilp}_{glob}^P \subset \text{Arth}_{\check{P}}$ is equal to the image of the closed subset

$$\text{Nilp}_{glob}^M \times_{\text{LocSys}_{\check{M}}} \text{LocSys}_{\check{P}} \subset \text{Arth}_{\check{M}} \times_{\text{LocSys}_{\check{M}}} \text{LocSys}_{\check{P}}$$

under $\text{Sing}(\mathfrak{q}_{spec}^P)$. Therefore, Proposition 8.4.14 implies that $(\mathfrak{q}_{spec}^P)^{\dagger}$ induces an equivalence

$$\text{QCoh}(\text{LocSys}_{\check{P}}) \otimes_{\text{QCoh}(\text{LocSys}_{\check{M}})} \text{IndCoh}_{\text{Nilp}_{glob}^M}(\text{LocSys}_{\check{M}}) \rightarrow \text{IndCoh}_{\text{Nilp}_{glob}^P}(\text{LocSys}_{\check{P}}).$$

It remains to show that the essential image of the usual pullback functor

$$(\mathfrak{q}_{spec}^P)^{*} : \text{QCoh}(\text{LocSys}_{\check{M}}) \rightarrow \text{QCoh}(\text{LocSys}_{\check{P}})$$

generates the target category.

Since $(\mathfrak{q}_{spec}^P)^{*}$ is the left adjoint to $(\mathfrak{q}_{spec}^P)_{*}$, we need to show that the pushforward functor

$$(\mathfrak{q}_{spec}^P)_{*} : \text{QCoh}(\text{LocSys}_{\check{P}}) \rightarrow \text{QCoh}(\text{LocSys}_{\check{M}})$$

is conservative. But this is true because the map \mathfrak{q}_{spec}^P can be presented as a quotient of an schematic affine map by an action of a unipotent group scheme (i.e., \mathfrak{q}_{spec}^P is cohomologically affine). □[Proposition 13.4.4]

13.4.7 *Proof of Proposition 13.4.5* We will deduce the proposition from Proposition 8.4.19.

We need to show that the map

$$(\text{Sing}(\mathfrak{p}_{\text{spec}}^P))^{-1}(\text{Nilp}_{\text{glob}}^P) \rightarrow \text{Nilp}_{\text{glob}}^G$$

is surjective at the level of k -points.

Concretely, this means the following: let $(\mathcal{P}^{\check{G}}, \nabla)$ be a \check{G} -bundle on X , equipped with a connection ∇ , which admits a horizontal reduction to the parabolic \check{P} . Let A be a horizontal section of $\check{\mathfrak{g}}_{\mathcal{P}^{\check{G}}}$, which is nilpotent (here, we have chosen some \check{G} -invariant identification of $\check{\mathfrak{g}}$ with $\check{\mathfrak{g}}^*$). We need to show that there exists a horizontal reduction of $\mathcal{P}^{\check{G}}$ to \check{P} such that A belongs to $\check{\mathfrak{p}}_{\mathcal{P}^{\check{G}}}$.

Let $\text{Sect}^{\nabla}(X, \mathcal{P}^{\check{G}}/\check{P})$ be the (classical) scheme of all horizontal reductions of $\mathcal{P}^{\check{G}}$ to \check{P} . By assumption, this scheme is non-empty, and it is also proper, since it embeds as a closed subscheme into $\mathcal{P}_x^{\check{G}}/\check{P}$ for any/some $x \in X$.

The algebraic group $\text{Aut}(\mathcal{P}^{\check{G}}, \nabla)$ acts naturally on $\text{Sect}^{\nabla}(X, \mathcal{P}^{\check{G}}/\check{P})$. Note that the Lie algebra of $\text{Aut}(\mathcal{P}^{\check{G}}, \nabla)$ identifies with $H^0(\Gamma(X_{\text{dR}}, \check{\mathfrak{g}}_{\mathcal{P}^{\check{G}}}))$. By assumption, the element

$$A \in H^0(\Gamma(X_{\text{dR}}, \check{\mathfrak{g}}_{\mathcal{P}^{\check{G}}}))$$

is nilpotent (as a linear operator on the algebra of functions on $\text{Aut}(\mathcal{P}^{\check{G}}, \nabla)$). Hence, it comes from a homomorphism $\mathbb{G}_a \rightarrow \text{Aut}(\mathcal{P}^{\check{G}}, \nabla)$.

By properness, the resulting action of \mathbb{G}_a on $\text{Sect}^{\nabla}(X, \mathcal{P}^{\check{G}}/\check{P})$ has a fixed point, which is the desired reduction.²¹ □[Proposition 13.4.5]

Appendix A: Action of groups on categories

Operations explained in this appendix have been used several times in the main body of the paper. They are applicable to any affine algebraic group G over a ground field of characteristic 0.

A.1 Equivariantization and de-equivariantization

A.1.1. Let BG^{\bullet} be the standard simplicial model of the classifying space of G . Quasi-coherent sheaves on BG^{\bullet} form a cosimplicial monoidal category, which we denote by $\text{QCoh}(BG^{\bullet})$. By definition, a DG category acted on by G is a cosimplicial category \mathbf{C}^{\bullet} tensored over $\text{QCoh}(BG^{\bullet})$ which is co-Cartesian in the sense that for every face

²¹ This argument was inspired by the proof that every nilpotent element in a Lie algebra (over a not necessarily algebraically closed field) is contained in a minimal parabolic that we learned from J. Lurie. It can also be used to reprove [14, Lemma 6] which is a key ingredient of the proof in *loc.cit.* that the global nilpotent cone in $T^*(\text{Bun}_G)$ is Lagrangian.

map $[k] \rightarrow [l]$, the functor

$$\mathrm{QCoh}(BG^l) \otimes_{\mathrm{QCoh}(BG^k)} \mathbf{C}^k \rightarrow \mathbf{C}^l$$

is an equivalence.

We will regard this as an additional structure over a plain DG category $\mathbf{C} := \mathbf{C}^0$. We denote the 2-category of DG categories acted on by G (regarded as an $(\infty, 1)$ -category) by $G\text{-mod}$.

For \mathbf{C} as above, we let \mathbf{C}^G denote the category $\mathrm{Tot}(\mathbf{C}^\bullet)$.

A.1.2. It is easy to see that for \mathbf{C} acted on by G and a full subcategory $\mathbf{C}' \subset \mathbf{C}$, there is at most one way to define a G -action on \mathbf{C}' in a way compatible with the embedding to \mathbf{C} ; this condition is enough to check at the level of the underlying triangulated categories and for 1-simplices. If this is the case, we will say that \mathbf{C}' is invariant under the action of G .

It is easy to see that in this case $(\mathbf{C}')^G$ is a full subcategory of \mathbf{C}^G that fits into the pullback square

$$\begin{array}{ccc} \mathbf{C}'^G & \longrightarrow & \mathbf{C}^G \\ \downarrow & & \downarrow \\ \mathbf{C}' & \longrightarrow & \mathbf{C}. \end{array} \tag{A.1}$$

A.1.3. Let \mathbf{C} be acted on by G . By construction, \mathbf{C}^G is a module category over

$$\mathrm{Tot}(\mathrm{QCoh}(BG^\bullet)) \simeq \mathrm{Rep}(G).$$

Thus, we obtain a functor

$$\mathbf{C} \mapsto \mathbf{C}^G : G\text{-mod} \rightarrow \mathrm{Rep}(G)\text{-mod}, \tag{A.2}$$

where $\mathrm{Rep}(G)\text{-mod}$ is the 2-category of module categories over $\mathrm{Rep}(G)$.

The above functor admits a left adjoint given by

$$\tilde{\mathbf{C}} \mapsto \mathrm{de}\text{-Eq}^G(\tilde{\mathbf{C}}) := \mathrm{Vect} \otimes_{\mathrm{Rep}(G)} \tilde{\mathbf{C}}, \tag{A.3}$$

where Vect is naturally regarded as a DG category endowed with the trivial G -action and the trivial structure of a $\mathrm{Rep}(G)$ -module with the natural compatibility structure between the two.

Note that we also have the naturally defined functors between plain DG categories $\mathbf{C}^G \rightarrow \mathbf{C}$ or, equivalently, $\tilde{\mathbf{C}} \rightarrow \mathrm{de}\text{-Eq}^G(\tilde{\mathbf{C}})$.

A.1.4. We have the following assertion ([31, Theorem 2.2.2]):

Theorem A.1.5 *The two functors (A.2) and (A.3) are mutually inverse.*

A.1.6 Several comments are in order:

The fact that for $\mathbf{C} \in G\text{-mod}$, the adjunction map

$$\text{de-Eq}^G(\mathbf{C}^G) \rightarrow \mathbf{C}$$

is an equivalence is easy. It follows from the fact that the functor de-Eq^G commutes with both *colimits* and *limits*, which in turn follows from the fact that the monoidal category $\text{Rep}(G)$ is rigid (see [29, Corollaries 4.3.2 and 6.4.2]).

For $\tilde{\mathbf{C}} \in \text{Rep}(G)\text{-mod}$, the fact that the adjunction map

$$\tilde{\mathbf{C}} \rightarrow (\text{de-Eq}^G(\tilde{\mathbf{C}}))^G$$

is an isomorphism is also easy to see when $\tilde{\mathbf{C}}$ is dualizable.

The above two observations are the only two cases of Theorem A.1.5 that have been used in the main body of the text.

The difficult direction in Theorem A.1.5 implies that if \mathbf{C} is dualizable (as an abstract DG category), then so is \mathbf{C}^G .²²

A.2 Shift of grading

A.2.1. Consider the symmetric monoidal category $\text{Rep}(\mathbb{G}_m) \simeq \text{QCoh}(\text{pt}/\mathbb{G}_m)$. We may view its objects as bigraded vector spaces equipped with differential of bidegree $(1, 0)$. Here in the grading (i, k) , the first index refers to the cohomological grading, and the second index to the grading coming from the \mathbb{G}_m -action.

The symmetric monoidal category $\text{Rep}(\mathbb{G}_m)$ carries a canonical automorphism which we will refer to as the “grading shift”

$$M \mapsto M^{\text{shift}}, \quad \text{where } M_{(i,k)}^{\text{shift}} = M_{i+2k,k}.$$

In particular, the 2-category $\text{Rep}(\mathbb{G}_m)\text{-mod}$ of DG categories tensored over $\text{Rep}(\mathbb{G}_m)$ carries a canonical auto-equivalence, which commutes with the forgetful functor to the 2-category $\text{DGCat}_{\text{cont}}$ of plain DG categories. We denote it by

$$\tilde{\mathbf{C}} \rightsquigarrow \tilde{\mathbf{C}}^{\text{shift}}. \tag{A.4}$$

A.2.2. For example, suppose A is a \mathbb{Z} -graded associative DG algebra, and set $\mathbf{C} = (A\text{-mod})^{\mathbb{G}_m}$. That is, \mathbf{C} is the DG category of *graded* A -modules.

In this case,

$$((A\text{-mod})^{\mathbb{G}_m})^{\text{shift}} \simeq (A^{\text{shift}}\text{-mod})^{\mathbb{G}_m}.$$

²² We do not know whether the fact that \mathbf{C} is compactly generated implies the corresponding fact for \mathbf{C}^G .

The categories $(A^{\text{shift}}\text{-mod})^{\mathbb{G}_m}$ and $(A\text{-mod})^{\mathbb{G}_m}$ are equivalent as DG categories (but not as categories tensored over $\text{Rep}(\mathbb{G}_m)$) with the equivalence given by

$$M \mapsto M^{\text{shift}} \quad \text{for } M \in (A\text{-mod})^{\mathbb{G}_m}.$$

A.2.3. By Theorem A.1.5, the shift of grading auto-equivalence of $\text{Rep}(\mathbb{G}_m)\text{-mod}$ induces an auto-equivalence of the 2-category $\mathbb{G}_m\text{-mod}$, which we denote by

$$\mathbf{C} \rightsquigarrow \mathbf{C}^{\text{shift}}. \quad (\text{A.5})$$

Note, however, that the auto-equivalence (A.4) of $\mathbb{G}_m\text{-mod}$ *does not* commute with the forgetful functor to $\text{DGCat}_{\text{cont}}$.

For example, for a graded associative DG algebra A as above, we have:

$$(A\text{-mod})^{\text{shift}} \simeq A^{\text{shift}}\text{-mod}.$$

Appendix B: Spaces of maps and deformation theory

In this appendix, we drop the assumption that our DG schemes/Artin stacks/prestacks be locally almost of finite type.

B.1 Spaces of maps

Let $\mathcal{Z} \in \text{PreStk}$ be an arbitrary prestack (see [28, Sect. 1.1.1]), thought of as the target, and let \mathcal{X} be another object of PreStk , thought of as the source.

B.1.1. We define a new prestack $\mathbf{Maps}(\mathcal{X}, \mathcal{Z}) \in \text{PreStk}$, by

$$\mathbf{Maps}(S, \mathbf{Maps}(\mathcal{X}, \mathcal{Z})) := \mathbf{Maps}(S \times \mathcal{X}, \mathcal{Z})$$

for $S \in \text{DGSch}^{\text{aff}}$.

Remark B.1.2 Note that the above procedure is a particular case of restriction of scalars à la Weil: We can start with a map $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ in PreStk and $\mathcal{Z}_1 \in \text{PreStk}/_{\mathcal{X}_1}$, and define

$$\mathcal{Z}_2 = \text{Res}_{\mathcal{X}_2}^{\mathcal{X}_1}(\mathcal{Z}_1) \in \text{PreStk}/_{\mathcal{X}_2}$$

by

$$\mathbf{Maps}(S, \mathcal{Z}_2) := \mathbf{Maps}_{\text{PreStk}/_{\mathcal{X}_1}}(S \times_{\mathcal{X}_2} \mathcal{X}_1, \mathcal{Z}_1).$$

In our case $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_2 = \text{pt}$.

B.1.3. For example, we define:

$$\mathrm{Bun}_G(\mathcal{X}) := \mathbf{Maps}(\mathcal{X}, \mathrm{pt}/G) \quad \text{and} \quad \mathrm{LocSys}_G(\mathcal{X}) := \mathbf{Maps}(\mathcal{X}_{\mathrm{dR}}, \mathrm{pt}/G),$$

where $\mathcal{X}_{\mathrm{dR}}$ is the de Rham prestack of \mathcal{X} (see [12, Sect. 1.1.1]).

B.2 Deformation theory

Let us recall some basic definitions from deformation theory. We refer the reader to [22, Sect. 2.1] or [11, Sect. 4] for a more detailed treatment.

B.2.1. Recall that for $S \in \mathrm{DGSch}^{\mathrm{aff}}$, we have a canonically defined functor

$$\mathrm{QCoh}(S)^{\leq 0} \rightarrow \mathrm{DGSch}_{S'}^{\mathrm{aff}}$$

that assigns to $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$ the corresponding split square-zero extension $S_{\mathcal{F}}$ of S , i.e.,

$$\mathcal{F} \mapsto S_{\mathcal{F}} := \mathrm{Spec}(\Gamma(S, \mathcal{O}_S) \oplus \Gamma(S, \mathcal{F})).$$

B.2.2. Let \mathcal{Z} be an object of PreStk , and let z be a point of $\mathrm{Maps}(S, \mathcal{Z})$. Consider the following functor $\mathrm{QCoh}(S)^{\leq 0} \rightarrow \infty\text{-Grpd}$

$$\mathcal{F} \mapsto \mathrm{Maps}(S_{\mathcal{F}}, \mathcal{Z}) \times_{\mathrm{Maps}(S, \mathcal{Z})} \{z\}. \quad (\mathrm{B.1})$$

Definition B.2.3 Let k be a non-negative integer.

- (a) We will say that \mathcal{Z} admits $(-k)$ -connective pro-cotangent spaces if for any (S, x) the functor (B.1) is pro-representable by an object of $\mathrm{Pro}(\mathrm{QCoh}(S)^{\leq k})$.
- (b) We will say that \mathcal{Z} admits $(-k)$ -connective cotangent spaces if for any (S, x) the functor (B.1) is co-representable by an object of $\mathrm{QCoh}(S)^{\leq k}$.

For \mathcal{Z} as in Definition B.2.3(a) (resp., (b)), we will denote the resulting object of $\mathrm{Pro}(\mathrm{QCoh}(S)^{\leq k})$ (resp., $\mathrm{QCoh}(S)^{\leq k}$) by $T_z^*(\mathcal{Z})$ and refer to it as the ‘‘cotangent space to \mathcal{Z} at the point z .’’

That is, if we regard

$$\mathcal{M}aps_{\mathrm{QCoh}(S)}(T_z^*(\mathcal{Z}), \mathcal{F})$$

as an object of Vect , and consider its truncation

$$\mathrm{Maps}_{\mathrm{QCoh}(S)}(T_z^*(\mathcal{Z}), \mathcal{F}) := \tau^{\leq 0}(\mathcal{M}aps_{\mathrm{QCoh}(S)}(T_z^*(\mathcal{Z}), \mathcal{F}))$$

as an ∞ -groupoid via $\mathrm{Vect}^{\leq 0} \rightarrow \infty\text{-Grpd}$, the result is canonically isomorphic to (B.1).

B.2.4. Let $\alpha : S_1 \rightarrow S$ be a map in $\text{DGSch}^{\text{aff}}$. Consider the corresponding functor

$$\text{Pro}(\alpha^*) : \text{Pro}(\text{QCoh}(S)) \rightarrow \text{Pro}(\text{QCoh}(S_1)).$$

By definition, for an object $\Phi \in \text{Pro}(\text{QCoh}(S))$, viewed as a functor $\text{QCoh}(S) \rightarrow \text{Vect}$, the object $\text{Pro}(\alpha^*)(\Phi) \in \text{Pro}(\text{QCoh}(S_1))$, viewed as a functor $\text{QCoh}(S_1) \rightarrow \text{Vect}$, is given by the left Kan extension of Φ along $\alpha^* : \text{QCoh}(S) \rightarrow \text{QCoh}(S_1)$.

Let \mathcal{Z} be an object of PreStk that admits $(-k)$ -connective pro-cotangent spaces. Then for $z : S \rightarrow \mathcal{Z}$ and $z_1 := z \circ \alpha$, we obtain a map

$$T_{z_1}^*(\mathcal{Z}) \rightarrow \text{Pro}(\alpha^*)(T_z^*(\mathcal{Z})) \tag{B.2}$$

in $\text{Pro}(\text{QCoh}(S_1))$.

If \mathcal{Z} admits $(-k)$ -connective cotangent spaces, then

$$T_z^*(\mathcal{Z}) \in \text{QCoh}(S)^{\leq k} \quad \text{and} \quad T_{z_1}^*(\mathcal{Z}) \in \text{QCoh}(S_1)^{\leq k},$$

and the map (B.2) is a map

$$T_{z_1}^*(\mathcal{Z}) \rightarrow \alpha^*(T_z^*(\mathcal{Z})) \in \text{QCoh}(S_1).$$

Definition B.2.5 (a) We will say that \mathcal{Z} admits a $(-k)$ -connective pro-cotangent complex if it admits $(-k)$ -connective pro-cotangent spaces and the map (B.2) is an isomorphism for any z and α .

(b) We will say that \mathcal{Z} admits a $(-k)$ -connective cotangent complex if it admits $(-k)$ -connective cotangent spaces and the map (B.2) is an isomorphism for any z and α .

If \mathcal{Z} admits a $(-k)$ -connective cotangent complex, it gives rise to a well-defined object in $\text{QCoh}(\mathcal{Z})^{\leq -k}$ that we will denote by $T^*(\mathcal{Z})$. For a given $z : S \rightarrow \mathcal{Z}$, we will also use the notation

$$T^*(Z)|_S := T_z^*(Z).$$

B.2.6. Let now \mathcal{S} be an arbitrary object of PreStk and let $z : S \rightarrow \mathcal{Z}$. Consider the functor

$$\mathcal{F} \mapsto \mathcal{S}_{\mathcal{F}} : \text{QCoh}(S)^{\leq 0} \rightarrow \text{PreStk}_{\mathcal{S}/},$$

defined by

$$\text{Maps}(U, \mathcal{S}_{\mathcal{F}}) = \{s : U \rightarrow \mathcal{S}, f : U \rightarrow U_{s^*(\mathcal{F})}\}$$

for $U \in \text{DGSch}^{\text{aff}}$ (here, the notation $U_{s^*(\mathcal{F})}$ is as in Sect. B.2.1).

The following is tautological from the definitions:

Lemma B.2.7 *Assume that \mathcal{Z} admits a $(-k)$ -connective cotangent complex. For any $\mathcal{S} \in \text{PreStk}$ and a map $z : \mathcal{S} \rightarrow \mathcal{Z}$, consider the functor $\text{QCoh}(\mathcal{S})^{\leq 0} \rightarrow \infty\text{-Grpd}$ given by*

$$\text{Maps}(\mathcal{S}_{\mathcal{F}}, \mathcal{Z}) \times_{\text{Maps}(\mathcal{S}, \mathcal{Z})} \{z\}.$$

This functor is canonically isomorphic to

$$\tau^{\leq 0} (\text{Maps}_{\text{QCoh}(\mathcal{S})}(T_z^*(\mathcal{Z}), \mathcal{F})).$$

B.2.8. Let S' be a square-zero extension of $S \in \text{DGSch}^{\text{aff}}$, not necessarily split. Such S' corresponds to an object $\mathcal{J} \in \text{QCoh}(S)^{\leq 0}$ (the ideal of $S \subset S'$) and a map

$$T^*(S) \rightarrow \mathcal{J}[1],$$

where $T^*(S)$ is the cotangent complex of S (see [11, Sect. 4.5]).

Let \mathcal{Z} admit $(-k)$ -connective pro-cotangent spaces. Let $z : S \rightarrow \mathcal{Z}$ be a map. As in [11, Lemma 4.6.6], we obtain a map

$$\text{Maps}(S', \mathcal{Z}) \times_{\text{Maps}(S, \mathcal{Z})} \{z\} \rightarrow \tau^{\leq 0} \left(\text{Maps}_{\text{QCoh}(S)} \left(\text{Cone}(T_z^*(\mathcal{Z}) \xrightarrow{(dz)^*} T^*(S)), \mathcal{J}[1] \right) \right), \tag{B.3}$$

where $(dz)^* : T_z^*(\mathcal{Z}) \rightarrow T^*(S)$ is the dual of the differential, see [11, Sect. 4.4.5].

Definition B.2.9 We will say that \mathcal{Z} is infinitesimally cohesive if the map (B.3) is an isomorphism for any S, z and S' .

Remark B.2.10 The meaning of the above definition is that (pro)-cotangent spaces control not only maps out of split square-zero extensions, but from all square-zero extensions. Iterating, we obtain that infinitesimal cohesiveness of \mathcal{Z} implies that its (pro)-cotangent spaces effectively control extensions of a given map $S \rightarrow \mathcal{Z}$ to maps $S' \rightarrow \mathcal{Z}$ for any nil-immersion $S \hookrightarrow S'$.

B.2.11. Finally, we define:

Definition B.2.12 We will say that $\mathcal{Z} \in \text{PreStk}$ admits a $(-k)$ -connective deformation theory (resp., co-representable $(-k)$ -connective deformation theory) if:

- \mathcal{Z} is convergent (see [28], Sect. 1.2.1);
- \mathcal{Z} admits a $(-k)$ -connective pro-cotangent complex (resp., $(-k)$ -connective cotangent complex);
- \mathcal{Z} is infinitesimally cohesive.

B.2.13. We record the following result for use in the main body of the paper.

Theorem B.2.14 *An object $\mathcal{Z} \in \text{PreStk}$ is a DG ind-scheme (resp., DG scheme) if and only if the following conditions hold:*

- The classical prestack ${}^{cl}\mathcal{Z}$ is a classical ind-scheme (resp., scheme).
- \mathcal{Z} admits a 0-connective (resp., co-representable 0-connective) deformation theory.

The above theorem for DG ind-schemes is [11, Theorem 5.1.1]. The case of DG schemes can be proved similarly (see [22, Theorem 3.1.2]).

The same proof also shows that if ${}^{cl}\mathcal{Z}$ is an ind-affine ind-scheme (i.e., is a filtered colimit of *affine* classical schemes under closed embeddings), or an affine scheme, then \mathcal{Z} is an ind-affine DG ind-scheme, or an affine DG scheme, respectively.

B.3 Deformation theory of spaces of maps

B.3.1. Let \mathcal{X} and \mathcal{Z} be as in Sect. B.1. We are going to show:

Proposition B.3.2 *Assume that \mathcal{Z} admits a co-representable $(-k)$ -connective deformation theory. Assume that \mathcal{X} is l -coconnective for some $l \in \mathbb{Z}^{\geq 0}$. Then:*

- The prestack $\mathbf{Maps}(\mathcal{X}, \mathcal{Z})$ admits a $(-k - l)$ -connective deformation theory.
- For $\tilde{z} \in \mathbf{Maps}(S, \mathbf{Maps}(\mathcal{X}, \mathcal{Z}))$ corresponding to $z \in \mathbf{Maps}(S \times \mathcal{X}, \mathcal{Z})$, the pro-cotangent space $T_{\tilde{z}}^*(\mathbf{Maps}(\mathcal{X}, \mathcal{Z}))$, viewed as a functor $\mathrm{QCoh}(S)^{\leq 0} \rightarrow \mathrm{Vect}^{\leq 0}$, identifies with

$$\mathcal{F} \mapsto \tau^{\leq 0}(\mathcal{M}aps_{\mathrm{QCoh}(S \times \mathcal{X})}(T_z^*(\mathcal{Z}), \mathcal{F} \boxtimes \mathcal{O}_{\mathcal{X}})). \quad (\text{B.4})$$

Proof The fact that the pro-cotangent spaces of $\mathbf{Maps}(\mathcal{X}, \mathcal{Z})$ are given by the functor (B.4) follows from Lemma B.2.7.

Let us show that these pro-cotangent spaces are $(-k - l)$ -connective. That is, we need to show that the functor

$$\mathcal{F} \mapsto \mathcal{M}aps_{\mathrm{QCoh}(S \times \mathcal{X})}(T_z^*(\mathcal{Z}), \mathcal{F} \boxtimes \mathcal{O}_{\mathcal{X}}), \quad \mathrm{QCoh}(S) \rightarrow \mathrm{Vect}$$

sends $\mathrm{QCoh}(S)^{\geq 0}$ to $\mathrm{Vect}^{\geq -k-l}$.

Since \mathcal{X} is l -coconnective, we can write \mathcal{X} as $\mathop{\mathrm{colim}}_{U \in {}^{<l}\mathrm{DGSch}_{/\mathcal{X}}^{\mathrm{aff}}} U$, and hence

$$\begin{aligned} & \mathcal{M}aps_{\mathrm{QCoh}(S \times \mathcal{X})}(T_z^*(\mathcal{Z}), \mathcal{F} \boxtimes \mathcal{O}_{\mathcal{X}}) \\ & \simeq \lim_{U \in ({}^{<l}\mathrm{DGSch}_{/\mathcal{X}}^{\mathrm{aff}})^{\mathrm{op}}} \mathcal{M}aps_{\mathrm{QCoh}(S \times U)}(T_z^*(\mathcal{Z})|_{S \times U}, \mathcal{F} \boxtimes \mathcal{O}_U). \end{aligned}$$

However, $\mathcal{F} \in \mathrm{QCoh}(S)^{\geq 0}$ implies $\mathcal{F} \boxtimes \mathcal{O}_U \in \mathrm{QCoh}(S \times U)^{\geq -l}$, and the assertion follows.

The fact that (B.2) and (B.3) are isomorphisms follows from the definitions. Finally, the fact that $\mathbf{Maps}(\mathcal{X}, \mathcal{Z})$ is convergent follows tautologically from the fact that \mathcal{Z} is convergent. \square

B.3.3. Let now X be an l -coconnective DG scheme, proper over $\mathrm{Spec}(k)$. Let \mathcal{Z} be again a prestack that admits a co-representable $(-k)$ -connective deformation theory.

Corollary B.3.4 *The prestack $\mathbf{Maps}(X, \mathcal{Z})$ admits a co-representable $(-k - l)$ -connective deformation theory and the prestack admits $\mathbf{Maps}(X_{\mathrm{dR}}, \mathcal{Z})$ admits a co-representable $(-k)$ -connective deformation theory.*

Proof Let \mathcal{X} be either X or X_{dR} .

We need to show the existence of a left adjoint functor to

$$\mathcal{F} \mapsto \mathcal{F} \boxtimes \mathcal{O}_{\mathcal{X}} : \mathrm{QCoh}(S) \rightarrow \mathrm{QCoh}(S \times \mathcal{X}).$$

Since $\mathrm{QCoh}(S \times \mathcal{X}) \simeq \mathrm{QCoh}(S) \otimes \mathrm{QCoh}(\mathcal{X})$ (see [30, Proposition 1.4.4]), it is sufficient to consider the case $S = \mathrm{pt}$.

Since the category $\mathrm{QCoh}(\mathcal{X})$ is compactly generated, it suffices to construct the left adjoint on compact objects of $\mathrm{QCoh}(\mathcal{X})$. This amounts to showing that for $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})^c$, the object

$$\mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{E}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{Vect} \tag{B.5}$$

is compact (i.e., has finitely many nonzero cohomologies, all of which are finite-dimensional); then the left adjoint in question sends $k \in \mathrm{Vect}$ to the dual of (B.5).

The compactness of (B.5) follows easily from the assumption on \mathcal{X} . □

B.4 The “locally almost of finite type” condition

B.4.1. Recall the notion of prestack locally almost of finite type, see [28, Sect. 1.3.9]). We have the following assertion:

Lemma B.4.2 *Let $\mathcal{Z} \in \mathrm{PreStk}$ be a prestack that admits a $(-k)$ -connective deformation theory for some k . Then, \mathcal{Z} is locally almost of finite type if and only if:*

- *The underlying classical prestack ${}^{\mathrm{cl}}\mathcal{Z}$ is locally of finite type (see [28, Sect. 1.3.2] for what this means).*
- *Given any classical scheme of finite type $S \in \mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$, a morphism $z : S \rightarrow \mathcal{Z}$, and an integer $k \geq 0$, the cotangent space $T_z^*(\mathcal{Z})$, viewed as a functor*

$$\mathrm{QCoh}(S)^{\geq -k, \leq 0} \rightarrow \infty\text{-Grpd},$$

commutes with filtered colimits.

The proof is essentially the same as that of [11, Proposition 5.3.2].

B.4.3. Let \mathcal{Z} be a prestack that is locally almost of finite type. Let X be an eventually coconnective quasi-compact DG scheme almost of finite type.

Corollary B.4.4 *The prestack $\mathbf{Maps}(X, \mathcal{Z})$ is locally almost of finite type, provided that the classical prestack ${}^{\mathrm{cl}}\mathcal{Z}$ satisfies Zariski descent.*

Proof The description of the cotangent spaces to $\mathbf{Maps}(X, \mathcal{Z})$ given by Proposition B.3.2(b) implies that the second condition of Lemma B.4.2 is satisfied. So, it remains to show that the classical prestack ${}^{cl}\mathbf{Maps}(X, \mathcal{Z})$ is locally of finite type. By definition, this means that the functor

$$S \mapsto \mathbf{Maps}(S, \mathbf{Maps}(X, \mathcal{Z})) \quad (\text{B.6})$$

on the category $(\text{Sch}^{\text{aff}})^{\text{op}}$ should commute with filtered colimits.

Any quasi-compact and quasi-separated scheme can be expressed as a *finite* colimit of affine schemes in the category of Zariski sheaves (i.e., the full subcategory of PreStk consisting of objects that satisfy Zariski descent). Since finite limits commute with filtered colimits, the assumption on \mathcal{Z} reduces the assertion to the case when X is affine.

Let n be such that $X \in \leq^n \text{DGSch}^{\text{aff}}$. The functor

$$S \mapsto S \times X : \text{Sch}^{\text{aff}} \rightarrow \leq^n \text{DGSch}^{\text{aff}}$$

commutes with filtered limits, and the required property of (B.6) follows from the fact that \mathcal{Z} is locally almost of finite type (namely, that for every n , it takes filtered colimits in $(\leq^n \text{DGSch}^{\text{aff}})^{\text{op}}$ to colimits in $\infty\text{-Grpd}$). \square

Appendix C: The Thomason-Trobaugh theorem “with supports”

In this appendix, we will prove the following result:

Theorem C *Let Z be a quasi-compact DG scheme, and $Y \subset \text{Sing}(Z)$ a conical Zariski-closed subset. Then, the category $\text{IndCoh}_Y(Z)$ is compactly generated.*

The proof is an easy adaptation of the argument of [38] for the compact generation of $\text{QCoh}(Z)$. Let us sketch the proof following [27].

Proof Recall that the objects of $\text{Coh}_Y(Z)$ are compact in $\text{IndCoh}_Y(Z)$. Hence, it suffices to check that $\text{Coh}_Y(Z)$ generates $\text{IndCoh}_Y(Z)$.

The proof proceeds by induction on the number of affine open subsets covering Z . The base case is when Z itself is affine. In this case, the assertion follows from Corollary 4.3.2.

Suppose now Z is arbitrary, and let W_i be an affine cover of Z . Fix $\mathcal{F} \in \text{IndCoh}_Y(Z)$, $\mathcal{F} \neq 0$. Set

$$U_i = \bigcup_{j \neq i} W_j,$$

and choose i so that $\mathcal{F}|_{U_i} \neq 0$. We now drop the index i and write simply U and W for U_i and W_i .

By the induction hypothesis, we can assume that $\text{IndCoh}_{Y \times_Z U}(U)$ is compactly generated. Therefore, there exists a compact object

$$\tilde{\mathcal{G}}_U \in \text{Coh}_{Y \times_Z U}(U)$$

together with a nonzero map $\tilde{\iota}_U : \tilde{\mathcal{G}}_U \rightarrow \mathcal{F}|_U$. Set $\mathcal{G}_U := \tilde{\mathcal{G}}_U \oplus \tilde{\mathcal{G}}_U[1]$ and let $\iota_U : \mathcal{G}_U \rightarrow \mathcal{F}|_U$ be equal to

$$\tilde{\mathcal{G}}_U \oplus \tilde{\mathcal{G}}_U[1] \rightarrow \tilde{\mathcal{G}}_U \xrightarrow{\tilde{\iota}_U} \mathcal{F}|_U.$$

We will extend the pair (\mathcal{G}_U, ι_U) to $\mathcal{G} \in \text{Coh}_Y(Z)$ and a map $\iota : \mathcal{G} \rightarrow \mathcal{F}$. Indeed, by [27, Theorem 3.1], there exists an object

$$\mathcal{G}_W \in \text{Coh}_{Y \times_Z W}(W)$$

together with an isomorphism

$$(\mathcal{G}_W)|_{U \cap W} \simeq (\mathcal{G}_U)|_{U \cap W}$$

and a map $\iota_W : \mathcal{G}_W \rightarrow \mathcal{F}|_W$ whose restriction to $U \cap W$ equals $\iota_U|_{U \cap W}$.

Define

$$\mathcal{G} := \text{Cone}\left(j_{U,*}(\mathcal{G}_U) \oplus j_{W,*}(\mathcal{G}_W) \rightarrow j_{U \cap W,*}((\mathcal{G}_W)|_{U \cap W})\right)[-1].$$

Here j_U , j_W , and $j_{U \cap W}$ are the natural embeddings $U \hookrightarrow Z$, $W \hookrightarrow Z$, and $U \cap W \hookrightarrow Z$, respectively. We have

$$\mathcal{G}|_U \simeq \mathcal{G}_U \text{ and } \mathcal{G}|_W \simeq \mathcal{G}_W,$$

so $\mathcal{G} \in \text{Coh}_Y(Z)$. The morphisms ι_U , ι_W induce a map $\iota : \mathcal{G} \rightarrow \mathcal{F}$, whose restrictions to U and W identify with ι_U and ι_W , respectively. \square

Appendix D: Finite generation of Exts

In this appendix, we will prove Theorem 4.1.8. Let us recall its formulation:

Theorem D *Let Z be a quasi-smooth affine DG scheme Z . Given $\mathcal{F}_1, \mathcal{F}_2 \in \text{Coh}(Z)$, consider the graded vector space $\text{Hom}_{\text{Coh}(Z)}^\bullet(\mathcal{F}_1, \mathcal{F}_2)$ as a module over the graded algebra $\Gamma(\text{Sing}(Z), \mathcal{O}_{\text{Sing}(Z)})$. We claim that the module is finitely generated.*

If Z is classical, this is due to Gulliksen [15], and the extension to DG schemes is straightforward.

D.1. It is easy to see that the statement is Zariski-local on Z . So, we can assume that Z is as in (6.3). Let

$$\text{pt} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_n = \mathcal{V}$$

be a flag of smooth closed subschemes whose dimensions increase by one. With no restriction of generality, we can assume that \mathcal{V}_{i-1} is cut out by one function inside \mathcal{V}_i .

Set

$$Z_i := \mathcal{V}_i \times_{\mathcal{V}} \mathcal{U}.$$

All these DG schemes are quasi-smooth, and Z_{i-1} is cut out inside of Z_i by one function. We have $Z_0 = Z$ and $Z_n = \mathcal{U}$. Let g_i denote the closed embedding $Z \rightarrow Z_i$.

D.2. We will argue by descending induction on i , assuming that

$$\mathrm{Hom}_{\mathrm{Coh}(Z_i)}^{\bullet}((g_i)_*(\mathcal{F}_1), (g_i)_*(\mathcal{F}_2))$$

is finitely generated as a module over $\Gamma(\mathrm{Sing}(Z_i), \mathcal{O}_{\mathrm{Sing}(Z_i)})$.

The base of induction is $i = n$. In this case, $Z_n = \mathcal{U}$ is smooth, and the assertion is obvious.

To carry out the induction step, we can thus assume that we have a quasi-smooth closed embedding

$$g : Z \hookrightarrow Z',$$

that fits into a Cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & Z' \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^1. \end{array}$$

By induction, we can assume that

$$\mathrm{Hom}_{\mathrm{Coh}(Z')}^{\bullet}(g_*(\mathcal{F}), g_*(\mathcal{F}))$$

is finitely generated as a module over $\Gamma(\mathrm{Sing}(Z'), \mathcal{O}_{\mathrm{Sing}(Z')})$.

D.3. Note that the generator of $T_{\{0\}}(\mathbb{A}^1)$ gives rise to an element $\eta \in \mathrm{HH}^2(Z)$.

Since the grading on the $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$ -module $\mathrm{Hom}_{\mathrm{Coh}(Z)}^{\bullet}(\mathcal{F}_1, \mathcal{F}_2)$ is bounded below, it is sufficient to show that

$$\mathrm{coker} \left(\eta : \mathrm{Hom}_{\mathrm{Coh}(Z)}^{\bullet}(\mathcal{F}_1[2], \mathcal{F}_2) \rightarrow \mathrm{Hom}_{\mathrm{Coh}(Z)}^{\bullet}(\mathcal{F}_1, \mathcal{F}_2) \right)$$

is finitely generated.

However, from the long exact sequence we obtain that the above cokernel is a submodule in

$$\mathrm{Hom}_{\mathrm{Coh}(Z)}^{\bullet} \left(\mathrm{Cone}(\mathcal{F}_1 \xrightarrow{\eta} \mathcal{F}_1[2]), \mathcal{F}_2[1] \right). \quad (\text{D.1})$$

Since the algebra $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$ is Noetherian, it is enough to show that the graded $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$ -module given by (D.1), is finitely generated.

D.4 Note that for any $\mathcal{F} \in \mathrm{QCoh}(Z)$, the corresponding object

$$\mathrm{Cone}(\mathcal{F} \xrightarrow{\eta} \mathcal{F}[2])$$

identifies with $g^* \circ g_*(\mathcal{F})$.

Hence, the module (D.1) identifies with

$$\mathrm{Hom}_{\mathrm{Coh}(Z)}^\bullet(g^* \circ g_*(\mathcal{F}_1), \mathcal{F}_2),$$

up to a shift of grading.

We have an isomorphism of vector spaces:

$$\mathrm{Hom}_{\mathrm{Coh}(Z)}^\bullet(g^* \circ g_*(\mathcal{F}_1), \mathcal{F}_2) \simeq \mathrm{Hom}_{\mathrm{Coh}(Z')}^\bullet(g_*(\mathcal{F}_1), g_*(\mathcal{F}_2)). \quad (\mathrm{D}.2)$$

Note that the right-hand side in (D.2) is acted on by $\Gamma(\mathrm{Sing}(Z'), \mathcal{O}_{\mathrm{Sing}(Z')})$, and this action factors through the surjection

$$\Gamma(\mathrm{Sing}(Z'), \mathcal{O}_{\mathrm{Sing}(Z')}) \twoheadrightarrow \Gamma(\mathrm{Sing}(Z')_Z, \mathcal{O}_{\mathrm{Sing}(Z')_Z}).$$

In particular, this gives an action of $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$ on right-hand side in (D.2) via the closed embedding

$$\mathrm{Sing}(g) : \mathrm{Sing}(Z')_Z \hookrightarrow \mathrm{Sing}(Z).$$

Now, it follows from the construction that the above action of $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$ on the right-hand side in (D.2) is compatible with the canonical action on the left-hand side.

Now, as was mentioned above, by the induction hypothesis, $\mathrm{Hom}_{\mathrm{Coh}(Z')}^\bullet(g_*(\mathcal{F}_1), g_*(\mathcal{F}_2))$ is finitely generated over $\Gamma(\mathrm{Sing}(Z'), \mathcal{O}_{\mathrm{Sing}(Z')})$, which implies that $\mathrm{Hom}_{\mathrm{Coh}(Z)}^\bullet(g^* \circ g_*(\mathcal{F}_1), \mathcal{F}_2)$ is finitely generated over $\Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)})$, as required. \square

Appendix E: Recollections on \mathbb{E}_2 -algebras

In this appendix, we recall some basic facts regarding \mathbb{E}_2 -algebras and their actions on categories.

E.1 \mathbb{E}_2 -algebras

The main reference to the theory of \mathbb{E}_2 -algebras is [21, Sect. 5.1]. Here, we will summarize some basic facts. All monoidal categories, \mathbb{E}_1 -algebras and \mathbb{E}_2 -algebras will be assumed unital.

We will use the terms “ \mathbb{E}_1 -algebra” and “associative DG algebra” interchangeably; and similarly for the terms “monoidal functor” and “homomorphism of monoidal DG categories.”

E.1.1. For the purposes of the paper, we set

$$\mathbb{E}_2\text{-Alg} := \mathbb{E}_1\text{-Alg}(\mathbb{E}_1\text{-Alg}).$$

An \mathbb{E}_2 -algebra is an associative algebra object in the category $\mathbb{E}_1\text{-Alg}$ of associative algebras.

The fact that this definition is equivalent to the definition of \mathbb{E}_2 -algebras as modules over the little disks operad is the Dunn Additivity Theorem, see [21, Theorem 5.1.2.2].

E.1.2. For an object $\mathcal{A} \in \mathbb{E}_2\text{-Alg}$, we will refer to the associative algebra structure on the underlying object of $\mathbb{E}_1\text{-Alg}$ as *the interior multiplication*.

We will refer to the associative algebra structure on the image of \mathcal{A} under the forgetful functor

$$\mathbb{E}_2\text{-Alg} = \mathbb{E}_1\text{-Alg}(\mathbb{E}_1\text{-Alg}) \rightarrow \mathbb{E}_1\text{-Alg}(\text{Vect}) = \mathbb{E}_1\text{-Alg}$$

as *the exterior multiplication*.

Unless specified otherwise, for $\mathcal{A} \in \mathbb{E}_2\text{-Alg}$, by the *underlying \mathbb{E}_1 -algebra* we will mean the result of the application of the forgetful functor that remembers *the interior multiplication*.

In particular, we will denote by $\mathcal{A}\text{-mod}$ the category of left \mathcal{A} -modules, where \mathcal{A} is regarded as an \mathbb{E}_1 -algebra with the interior multiplication.

E.1.3. We will denote by $\mathcal{A}^{\text{int-op}}$ (resp., $\mathcal{A}^{\text{ext-op}}$) the \mathbb{E}_2 -algebra obtained by reversing the first (resp., second) multiplication. Note, however, because of [21, Theorem 5.1.2.2], the choice of orientation on S^1 (the real circle) gives rise to a canonical isomorphism

$$\mathcal{A}^{\text{int-op}} \simeq \mathcal{A}^{\text{ext-op}}$$

of \mathbb{E}_2 -algebras.

Having made the choice of orientation once and for all, we will sometimes use the notation \mathcal{A}^{op} for the resulting \mathbb{E}_2 -algebra, with one of the multiplications reversed.

E.2 \mathbb{E}_2 -algebras and monoidal categories

E.2.1. We recall the following construction, explained to us by J. Lurie.

On the one hand, we consider the symmetric monoidal ∞ -category $\mathbb{E}_1\text{-Alg}$ of \mathbb{E}_1 -algebras. On the other hand, we consider the symmetric monoidal ∞ -category

$$(\text{DGCat}_{\text{cont}})_{\text{Vect}/},$$

i.e., the category of pairs (\mathbf{c}, \mathbf{C}) , where $\mathbf{C} \in \text{DGCat}_{\text{cont}}$ and $\mathbf{c} \in \mathbf{C}$.

We have a fully faithful symmetric monoidal functor

$$\mathbb{E}_1\text{-Alg} \rightarrow (\text{DGCat}_{\text{cont}})_{\text{Vect}/}, \quad A \mapsto (A, A^{\text{op}}\text{-mod}). \quad (\text{E.1})$$

The essential image of this functor consists of those (\mathbf{c}, \mathbf{C}) for which \mathbf{c} is a compact generator of \mathbf{C} .

The functor (E.1) has a right adjoint, given by

$$(\mathbf{c}, \mathbf{C}) \mapsto \mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, \mathbf{c}). \quad (\text{E.2})$$

Tautologically, the functor (E.2) is *right-lax* symmetric monoidal. In particular, it naturally upgrades to a functor between the categories of algebras (for any operad).

E.2.2. Since the functor (E.1) is symmetric monoidal, we obtain that for $\mathcal{A} \in \mathbb{E}_2\text{-Alg}$, the category $\mathcal{A}^{\text{int-op-mod}}$ acquires a monoidal structure, for which \mathcal{A} is the unit.

We denote this operation by

$$\mathcal{M}_1, \mathcal{M}_2 \mapsto \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2,$$

Informally, if we view \mathcal{A} as an associative DG algebra with respect to the interior multiplication, the tensor product is induction with respect to the homomorphism

$$\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$$

given by the exterior multiplication.

In fact, since the functor (E.1) is fully faithful, the structure of \mathbb{E}_2 -algebra on a given $\mathcal{A} \in \mathbb{E}_1\text{-Alg}$ is equivalent to a structure on $\mathcal{A}^{\text{int-op-mod}}$ of monoidal category with the unit being \mathcal{A} .

E.2.3. For a given symmetric monoidal category \mathbf{O} , the functor (E.2) sends it to the \mathbb{E}_2 -algebra whose underlying \mathbb{E}_1 -algebra is $\mathcal{M}aps_{\mathbf{O}}(\mathbf{1}_{\mathbf{O}}, \mathbf{1}_{\mathbf{O}})$. We denote this \mathbb{E}_2 -algebra by $\mathcal{A}_{\mathbf{O}}$.

It is easy to see that we have a canonical isomorphism of \mathbb{E}_2 -algebras:

$$\mathcal{A}_{\mathbf{O}^{\text{op-mon}}} \simeq \mathcal{A}_{\mathbf{O}}^{\text{ext-op}}, \quad (\text{E.3})$$

where $\mathbf{O}^{\text{op-mon}}$ denotes the monoidal category obtained from \mathbf{O} by reversing the monoidal structure.

E.2.4. By adjunction, a datum of a homomorphism of \mathbb{E}_2 -algebras

$$\mathcal{A} \rightarrow \mathcal{A}_{\mathbf{O}}$$

is equivalent to that of a continuous monoidal functor

$$\mathcal{A}^{\text{int-op-mod}} \rightarrow \mathbf{O}.$$

In particular, the unit of the adjunction is the tautological isomorphism

$$\mathcal{A} \rightarrow \mathcal{M}aps_{\mathcal{A}^{\text{int-op}}}(\mathcal{A}, \mathcal{A}).$$

The co-unit of the adjunction is a canonically defined monoidal functor

$$\mathcal{M}aps_{\mathbf{O}}(\mathbf{1}_{\mathbf{O}}, \mathbf{1}_{\mathbf{O}})^{\text{int-op-mod}} \rightarrow \mathbf{O}.$$

The latter functor is an equivalence if and only if $\mathbf{1}_{\mathbf{O}}$ is a compact generator of \mathbf{O} .

E.3 Hochschild cochains of a DG category

E.3.1. For $\mathbf{C} \in \text{DGCat}_{\text{cont}}$, we consider the monoidal category

$$\text{Funct}_{\text{cont}}(\mathbf{C}, \mathbf{C}).$$

We let $\text{HC}(\mathbf{C})$ denote the resulting \mathbb{E}_2 -algebra.

By definition, the \mathbb{E}_1 -algebra underlying $\text{HC}(\mathbf{C})$ is

$$\mathcal{M}aps_{\text{Funct}_{\text{cont}}(\mathbf{C}, \mathbf{C})}(\text{Id}_{\mathbf{C}}, \text{Id}_{\mathbf{C}}).$$

E.3.2. Let \mathcal{A} be an \mathbb{E}_2 -algebra. By adjunction, the following pieces of data are equivalent:

- an action of the monoidal category $\mathcal{A}\text{-mod}$ on \mathbf{C} ;
- a homomorphism of monoidal categories $\mathcal{A}\text{-mod} \rightarrow \text{Funct}_{\text{cont}}(\mathbf{C}, \mathbf{C})$;
- a map of \mathbb{E}_2 -algebras $\mathcal{A}^{\text{int-op}} \rightarrow \text{HC}(\mathbf{C})$.

We will refer to such data as a (left) *action of \mathcal{A} on \mathbf{C}* .

By a right action of \mathcal{A} on \mathbf{C} , we will mean a left action of $\mathcal{A}^{\text{int-op}}$, or, equivalently, a homomorphism of \mathbb{E}_2 -algebras $\mathcal{A} \rightarrow \text{HC}(\mathbf{C})$.

E.3.3. Assume now that \mathbf{C} is dualizable. In this case, we have a canonical equivalence:

$$\text{Funct}_{\text{cont}}(\mathbf{C}^{\vee}, \mathbf{C}^{\vee}) \simeq (\text{Funct}_{\text{cont}}(\mathbf{C}, \mathbf{C}))^{\text{op-mon}}.$$

Hence, from (E.3), we obtain a canonical isomorphism of \mathbb{E}_2 -algebras

$$\text{HC}(\mathbf{C}^{\vee}) \simeq \text{HC}(\mathbf{C})^{\text{ext-op}}.$$

In particular, if an \mathbb{E}_2 -algebra \mathcal{A} acts on \mathbf{C} , then $\mathcal{A}^{\text{ext-op}}$ naturally acts on \mathbf{C}^{\vee} .

Appendix F: Hochschild cochains of a DG scheme and groupoids

In this appendix, we collect some facts and constructions regarding the \mathbb{E}_2 -algebra of Hochschild cochains of a DG scheme Z . We will also review a variant of this construction, when we produce an \mathbb{E}_2 -algebra out of a groupoid acting on Z .

F.1 Hochschild cochains of a DG scheme

Let Z be a quasi-compact DG scheme. We put

$$\mathrm{HC}(Z) := \mathrm{HC}(\mathrm{QCoh}(Z)).$$

The main result of this subsection is that we can equivalently realize $\mathrm{HC}(Z)$ as the \mathbb{E}_2 -algebra of Hochschild cochains of the category $\mathrm{IndCoh}(Z)$.

Remark F.1.1 The assertion regarding the comparison of the two versions of $\mathrm{HC}(Z)$ is *not* essential for the contents of this paper. The object that we are really interested in is the \mathbb{E}_2 -algebra of Hochschild cochains of the category $\mathrm{IndCoh}(Z)$, denoted below by $\mathrm{HC}^{\mathrm{IndCoh}}(Z)$.

So, the reader can skip this subsection, substituting $\mathrm{HC}^{\mathrm{IndCoh}}(Z)$ for $\mathrm{HC}(Z)$ in any occurrence of the latter in the main body of the paper, in particular, in Sect. 4.

F.1.2. Recall that the category $\mathrm{QCoh}(Z)$ is canonically self-dual

$$\mathbf{D}_Z^{\mathrm{naive}} : \mathrm{QCoh}(Z)^\vee \simeq \mathrm{QCoh}(Z), \tag{F.1}$$

where the corresponding functor on the compact objects

$$(\mathrm{QCoh}(Z)^c)^{\mathrm{op}} \rightarrow \mathrm{QCoh}(Z)^c$$

is the “naive” duality functor $\mathbb{D}_Z^{\mathrm{naive}}(-) = \underline{\mathrm{Hom}}_{\mathrm{QCoh}(Z)}(-, \mathcal{O}_Z)$ on $\mathrm{QCoh}(Z)^c = \mathrm{QCoh}(Z)^{\mathrm{perf}}$.

In particular, from Sect. E.3.3, we obtain a canonical identification

$$\mathrm{HC}(Z) \simeq \mathrm{HC}(Z)^{\mathrm{ext-op}}. \tag{F.2}$$

F.1.3. Consider now the category $\mathrm{IndCoh}(Z)$. Denote

$$\mathrm{HC}^{\mathrm{IndCoh}}(Z) := \mathrm{HC}(\mathrm{IndCoh}(Z)).$$

Recall (see [10, Sect. 9.2.1]) that the category $\mathrm{IndCoh}(Z)$ is also canonically self-dual

$$\mathbf{D}_Z^{\mathrm{Serre}} : \mathrm{IndCoh}(Z)^\vee \simeq \mathrm{IndCoh}(Z), \tag{F.3}$$

where the corresponding functor on the compact objects

$$(\mathrm{IndCoh}(Z)^c)^{\mathrm{op}} \rightarrow \mathrm{IndCoh}(Z)^c$$

is the Serre duality functor $\mathbb{D}_Z^{\mathrm{Serre}}$ on $\mathrm{IndCoh}(Z)^c = \mathrm{Coh}(Z)$.

Hence, we obtain a canonical isomorphism:

$$\mathrm{HC}^{\mathrm{IndCoh}}(Z) \simeq \mathrm{HC}^{\mathrm{IndCoh}}(Z)^{\mathrm{ext-op}}. \tag{F.4}$$

F.1.4. Recall now that there exists a canonically defined functor

$$\Psi_Z : \text{IndCoh}(Z) \rightarrow \text{QCoh}(Z),$$

obtained by ind-extending the tautological embedding $\text{Coh}(Z) \hookrightarrow \text{QCoh}(Z)$, see [10, Sect. 1.1.5].

We let

$$\Upsilon_Z : \text{QCoh}(Z) \rightarrow \text{IndCoh}(Z)$$

be its dual with respect to the identifications (F.1) and (F.3).

Recall also (see [10, Proposition 9.3.3]) that

$$\Upsilon_Z(\mathcal{F}) = \mathcal{F} \otimes \omega_Z, \quad \mathcal{F} \in \text{QCoh}(Z),$$

where $\omega_Z \in \text{IndCoh}(Z)$ is the dualizing complex and “ \otimes ” stands for the action of the monoidal category $\text{QCoh}(Z)$ on $\text{IndCoh}(Z)$, see [10, Sect. 1.4].

Proposition F.1.5 *There exist uniquely defined isomorphisms*

$$\Psi_{\text{HC}} : \text{HC}(Z) \rightarrow \text{HC}^{\text{IndCoh}}(Z) \text{ and } \Upsilon_{\text{HC}} : \text{HC}^{\text{IndCoh}}(Z) \rightarrow \text{HC}(Z),$$

the former compatible with the functor Ψ_Z , and the latter compatible with the functor Υ_Z . Moreover, the following diagram commutes:

$$\begin{array}{ccc} \text{HC}(Z)^{\text{ext-op}} & \xrightarrow{\sim} & \text{HC}(Z) \\ (\Upsilon_{\text{HC}})^{\text{ext-op}} \uparrow & & \downarrow \Psi_{\text{HC}} \\ \text{HC}^{\text{IndCoh}}(Z)^{\text{ext-op}} & \xleftarrow{\sim} & \text{HC}^{\text{IndCoh}}(Z) \end{array} \quad (\text{F.5})$$

(in the sense that the composition map starting from any corner is canonically isomorphic to the identity map).

F.2 Proof of Proposition F.1.5

F.2.1. We will need the following general construction.

Let \mathbf{C}_1 and \mathbf{C}_2 be two DG categories, and let $\Phi : \mathbf{C}_1 \rightarrow \mathbf{C}_2$ be a continuous functor. Assume that \mathbf{C}_1 is compactly generated, and assume that $\Phi|_{\mathbf{C}_1^c}$ is fully faithful.

We claim that in this case, there exists a unique map of \mathbb{E}_2 -algebras

$$\Phi_{\text{HC}} : \text{HC}(\mathbf{C}_2) \rightarrow \text{HC}(\mathbf{C}_1),$$

compatible with Φ , i.e., the functor Φ intertwines the action of $\text{HC}(\mathbf{C}_2)^{\text{int-op}}$ on \mathbf{C}_1 , induced by Φ_{HC} , and the tautological action of $\text{HC}(\mathbf{C}_2)^{\text{int-op}}$ on \mathbf{C}_2 .

Indeed, consider the object

$$(\Phi, \text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)) \in (\text{DGCat}_{\text{cont}})_{\text{Vect}/}$$

It is acted on the right and on the left by

$$(\text{Id}_{\mathbf{C}_i}, \text{Funct}_{\text{cont}}(\mathbf{C}_i, \mathbf{C}_i)), \quad i = 1, 2,$$

respectively.

Applying the functor (E.2), we obtain the object

$$\mathcal{M}aps_{\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)}(\Phi, \Phi) \in \mathbb{E}_1\text{-Alg},$$

equipped with an action of $\text{HC}(\mathbf{C}_2) \in \mathbb{E}_1\text{-Alg}(\mathbb{E}_1\text{-Alg})$, as well as a *right* commuting action of $\text{HC}(\mathbf{C}_1) \in \mathbb{E}_1\text{-Alg}(\mathbb{E}_1\text{-Alg})$.

Furthermore, the assumption on Φ implies that the action on the unit defines an isomorphism of right modules over $\text{HC}(\mathbf{C}_1)$:

$$\text{HC}(\mathbf{C}_1) \rightarrow \mathcal{M}aps_{\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)}(\Phi, \Phi).$$

Since

$$\text{HC}(\mathbf{C}_1) \rightarrow \text{End}_{\text{HC}(\mathbf{C}_1)^{\text{ext-op-mod}}}(\text{HC}(\mathbf{C}_1))$$

is an isomorphism of \mathbb{E}_2 -algebras, the left action of $\text{HC}(\mathbf{C}_2)$ defines the desired homomorphism Φ_{HC} .

The following assertion results from the construction:

Lemma F.2.2 *Assume that in the above situation the category \mathbf{C}_2 is also compactly generated and that the dual functor $\Phi^\vee : \mathbf{C}_2^\vee \rightarrow \mathbf{C}_1^\vee$ is such that $\Phi^\vee|_{(\mathbf{C}_2^\vee)^c}$ is also fully faithful. Then, the resulting diagram*

$$\begin{array}{ccc} \text{HC}(\mathbf{C}_1)^{\text{ext-op}} & \xrightarrow{\sim} & \text{HC}(\mathbf{C}_1^\vee) \\ \Phi_{\text{HC}}^{\text{ext-op}} \uparrow & & \downarrow \Phi_{\text{HC}}^\vee \\ \text{HC}(\mathbf{C}_2)^{\text{ext-op}} & \xleftarrow{\sim} & \text{HC}(\mathbf{C}_2^\vee) \end{array}$$

commutes.

Corollary F.2.3 *In the situation of Lemma F.2.2, the maps Φ_{HC} and Φ_{HC}^\vee are isomorphisms.*

Proof of Proposition F.1.5 We apply Lemma F.2.2 to $\mathbf{C}_1 := \text{QCoh}(Z)$, $\mathbf{C}_2 := \text{IndCoh}(Z)$, $\Phi := \Upsilon_Z$, and so $\Phi^\vee = \Psi_Z$. By definition, $\Psi|_{\text{Coh}(Z)}$ is fully faithful. It also easy to see that $\Upsilon_Z|_{\text{QCoh}(Z)^{\text{perf}}}$ is fully faithful, as required. \square

F.3 Further remarks on the relation between $\mathrm{HC}(Z)$ and $\mathrm{HC}^{\mathrm{IndCoh}}(Z)$

In what follows we will identify $\mathrm{HC}(Z)$ and $\mathrm{HC}^{\mathrm{IndCoh}}(Z)$, and unless specified otherwise, we will do so using the isomorphism Υ_{HC} .

F.3.1. Assume that Z is eventually coconnective. Recall that in this case the functor Ψ_Z admits a fully faithful left adjoint, denoted Ξ_Z .

In particular, by the first paragraph of the proof of Proposition F.1.5, we obtain that there exists a unique homomorphism

$$\Xi_{\mathrm{HC}} : \mathrm{HC}^{\mathrm{IndCoh}}(Z) \rightarrow \mathrm{HC}(Z),$$

compatible with Ξ_Z .

Note, however, that the isomorphism Ψ_{HC} is also compatible with Ξ_Z , by adjunction. So, we obtain that Ξ_{HC} provides an explicit inverse to Ψ_{HC} .

F.3.2. Assume now that Z is eventually coconnective and Gorenstein (that is, $\omega_Z \in \mathrm{Coh}(Z)$ is a cohomologically shifted line bundle). For example, this is the case for quasi-smooth DG schemes.

In this case, we can regard the tensor product by ω_Z as a self-equivalence of the category $\mathrm{QCoh}(Z)$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Z) & \xrightarrow{\Xi_Z} & \mathrm{IndCoh}(Z) \\ \omega_Z \otimes - \uparrow & & \uparrow \mathrm{Id} \\ \mathrm{QCoh}(Z) & \xrightarrow{\Upsilon_Z} & \mathrm{IndCoh}(Z). \end{array}$$

Let ω_{HC} denote the automorphism of $\mathrm{HC}(Z)$ compatible with the functor $\omega_Z \otimes -$. Thus, we obtain:

Lemma F.3.3 *We have*

$$\Upsilon_{\mathrm{HC}} \simeq \omega_{\mathrm{HC}} \circ \Xi_{\mathrm{HC}}$$

as isomorphisms $\mathrm{HC}^{\mathrm{IndCoh}}(Z) \rightarrow \mathrm{HC}(Z)$.

F.4 \mathbb{E}_2 -algebras arising from groupoids

F.4.1. Let Z be an affine DG scheme, and let

$$\begin{array}{ccc} & \mathcal{G} & \\ p_1 \swarrow & \uparrow \text{unit} & \searrow p_2 \\ Z & & Z \end{array} \tag{F.6}$$

be a groupoid acting on Z , where \mathcal{G} is itself an affine DG scheme.

The category $\mathrm{QCoh}(\mathcal{G})$ acquires a natural monoidal structure via the convolution product, and as such it acts on $\mathrm{QCoh}(Z)$. The unit object in $\mathrm{QCoh}(\mathcal{G})$ is

$$\mathrm{unit}_*(\mathcal{O}_Z) \in \mathrm{QCoh}(\mathcal{G}).$$

Hence, its endomorphism algebra

$$\mathcal{A}_{\mathcal{G}} := \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{G})}(\mathrm{unit}_*(\mathcal{O}_Z), \mathrm{unit}_*(\mathcal{O}_Z))$$

is naturally an \mathbb{E}_2 -algebra.

F.4.2. The category $\mathrm{IndCoh}(\mathcal{G})$ also acquires a natural monoidal structure, and as such it acts on $\mathrm{IndCoh}(Z)$, where we use the functor $f^!$ for pullback, f_*^{IndCoh} for pushforward and $\overset{!}{\otimes}$ for tensor product, where

$$\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2 := \Delta^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

The unit object in $\mathrm{IndCoh}(\mathcal{G})$ is

$$\mathrm{unit}_*^{\mathrm{IndCoh}}(\omega_Z) \in \mathrm{IndCoh}(\mathcal{G}),$$

and we let

$$\mathcal{A}_{\mathcal{G}}^{\mathrm{IndCoh}} = \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{G})}(\mathrm{unit}_*^{\mathrm{IndCoh}}(\omega_Z), \mathrm{unit}_*^{\mathrm{IndCoh}}(\omega_Z))$$

denote the \mathbb{E}_2 -algebra of its endomorphisms.

F.4.3. We have the following assertion:

Proposition F.4.4 *There exists a canonical isomorphism of \mathbb{E}_2 -algebras.*

$$\mathcal{A}_{\mathcal{G}} \rightarrow \mathcal{A}_{\mathcal{G}}^{\mathrm{IndCoh}}.$$

The proof will be given in Sect. F.5.

Remark F.4.5 As is the case with $\mathrm{HC}(Z)$ vs. $\mathrm{HC}^{\mathrm{IndCoh}}(Z)$, the assertion of Proposition F.4.4 is *not* essential for the contents of the paper. Namely, the object that we need to work with is $\mathcal{A}_{\mathcal{G}}^{\mathrm{IndCoh}}$. So, the reader who is not interested in the proof of Proposition F.4.4 can simply substitute $\mathcal{A}_{\mathcal{G}}^{\mathrm{IndCoh}}$ for $\mathcal{A}_{\mathcal{G}}$ for any occurrence of the latter in the main body of the paper.

F.4.6. Hochschild cochains via groupoids Let Z be a quasi-compact DG scheme. Consider the groupoid $\mathcal{G} = Z \times Z$; the unit section is the diagonal morphism

$$\Delta_Z : Z \rightarrow Z \times Z.$$

The resulting \mathbb{E}_2 -algebra $\mathcal{A}_{Z \times Z}$ (resp., $\mathcal{A}_{Z \times Z}^{\text{IndCoh}}$) identifies with $\text{HC}(Z)$ (resp., $\text{HC}^{\text{IndCoh}}(Z)$), using the identifications of the monoidal categories

$$\text{QCoh}(Z \times Z) \rightarrow \text{Funct}_{\text{cont}}(\text{QCoh}(Z), \text{QCoh}(Z))$$

and

$$\text{IndCoh}(Z \times Z) \rightarrow \text{Funct}_{\text{cont}}(\text{IndCoh}(Z), \text{IndCoh}(Z)),$$

respectively.

Remark F.4.7 It will follow from the proof of Proposition F.4.4 that the resulting isomorphism

$$\text{HC}(Z) \rightarrow \text{HC}^{\text{IndCoh}}(Z)$$

identifies with the one given by Υ_{HC} .

F.4.8. For an arbitrary groupoid \mathcal{G} , the algebra $\mathcal{A}_{\mathcal{G}}$ (resp., $\mathcal{A}_{\mathcal{G}}^{\text{IndCoh}}$) naturally maps to $\text{HC}(Z)$ (resp., $\text{HC}^{\text{IndCoh}}(Z)$).

This can be viewed as a corollary of the functoriality of the assignment $\mathcal{G} \rightsquigarrow \mathcal{A}_{\mathcal{G}}$. Namely, a homomorphism of groupoids $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ induces homomorphisms of monoidal categories

$$f_* : \text{QCoh}(\mathcal{G}_1) \rightarrow \text{QCoh}(\mathcal{G}_2) \quad \text{and} \quad f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{G}_1) \rightarrow \text{IndCoh}(\mathcal{G}_2),$$

and hence of \mathbb{E}_2 -algebras

$$\mathcal{A}_{\mathcal{G}_1} \rightarrow \mathcal{A}_{\mathcal{G}_2} \quad \text{and} \quad \mathcal{A}_{\mathcal{G}_1}^{\text{IndCoh}} \rightarrow \mathcal{A}_{\mathcal{G}_2}^{\text{IndCoh}}.$$

We apply this to $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_2 = Z \times Z$ and use Sect. F.4.6.

Equivalently, the monoidal category $(\mathcal{A}_{\mathcal{G}})^{\text{int-op-mod}}$ (resp., $(\mathcal{A}_{\mathcal{G}}^{\text{IndCoh}})^{\text{int-op-mod}}$) acts on the category $\text{QCoh}(Z)$ (resp., $\text{IndCoh}(Z)$) via the canonical action of $\text{QCoh}(\mathcal{G})$ on $\text{QCoh}(Z)$ (resp., of $\text{IndCoh}(\mathcal{G})$ on $\text{IndCoh}(Z)$) by convolution.

Remark F.4.9 Again, from the construction of the isomorphism of Proposition F.4.4, it will follow that for a homomorphism of groupoids $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$, the following diagram of \mathbb{E}_2 -algebras naturally commutes:

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{G}_1} & \xrightarrow{\sim} & \mathcal{A}_{\mathcal{G}_1}^{\text{IndCoh}} \\ f_* \downarrow & & \downarrow f_*^{\text{IndCoh}} \\ \mathcal{A}_{\mathcal{G}_2} & \xrightarrow{\sim} & \mathcal{A}_{\mathcal{G}_2}^{\text{IndCoh}}. \end{array}$$

F.4.10. Relative Hochschild cochains Let $Z \rightarrow \mathcal{U}$ be a morphism of affine DG schemes. Consider the groupoid

$$\mathcal{G}_{Z/\mathcal{U}} := Z \times_{\mathcal{U}} Z.$$

By a slight abuse of notation, we will continue to denote by Δ_Z the diagonal map $Z \rightarrow Z \times_{\mathcal{U}} Z$, which is the unit of the above groupoid.

The groupoid gives rise to \mathbb{E}_2 -algebras $\mathcal{A}_{\mathcal{G}_{Z/\mathcal{U}}}$ and $\mathcal{A}_{\mathcal{G}_{Z/\mathcal{U}}}^{\text{IndCoh}}$ of endomorphisms of the unit objects $\Delta_{Z,*} \mathcal{O}_Z \in \text{QCoh}(\mathcal{G}_{Z/\mathcal{U}})$ and $\Delta_{Z,*}^{\text{IndCoh}} \omega_Z \in \text{IndCoh}(\mathcal{G}_{Z/\mathcal{U}})$, respectively. We put

$$\begin{aligned} \text{HC}(Z/C\mathcal{U}) &:= \mathcal{A}_{\mathcal{G}_{Z/\mathcal{U}}} \\ \text{HC}^{\text{IndCoh}}(Z/\mathcal{U}) &:= \mathcal{A}_{\mathcal{G}_{Z/\mathcal{U}}}^{\text{IndCoh}} \end{aligned}$$

and refer to these \mathbb{E}_2 -algebras as “the algebras of Hochschild cochains on Z relative to \mathcal{U} .”

It is easy to see that $\text{HC}(Z/\mathcal{U})$ identifies with the \mathbb{E}_2 -algebra

$$\text{HC}(\text{QCoh}(Z))_{\text{QCoh}(\mathcal{U})}$$

of endomorphisms of the identity functor on $\text{QCoh}(Z)$ as a DG category tensored over $\text{QCoh}(\mathcal{U})$, i.e., the \mathbb{E}_2 -algebra of endomorphisms of the unit in the monoidal category

$$\text{Funct}_{\text{QCoh}(\mathcal{U})}(\text{QCoh}(Z), \text{QCoh}(Z)).$$

(This is because the natural homomorphism from $\text{QCoh}(\mathcal{G}_{Z/\mathcal{U}})$ to the above category is an equivalence.)

By essentially repeating the proof of Proposition F.1.5, one can construct an isomorphism of \mathbb{E}_2 -algebras

$$\text{HC}(Z/\mathcal{U}) = \text{HC}(\text{QCoh}(Z))_{\text{QCoh}(\mathcal{U})} \xrightarrow{\sim} \text{HC}(\text{IndCoh}(Z))_{\text{QCoh}(\mathcal{U})}.$$

Here, $\text{HC}(\text{IndCoh}(Z))_{\text{QCoh}(\mathcal{U})}$ is the \mathbb{E}_2 -algebra of endomorphisms of the identity functor on $\text{IndCoh}(Z)$ as a DG category tensored over $\text{QCoh}(\mathcal{U})$, that is, the \mathbb{E}_2 -algebra of endomorphisms of the unit in the monoidal category

$$\text{Funct}_{\text{QCoh}(\mathcal{U})}(\text{IndCoh}(Z), \text{IndCoh}(Z)).$$

The content of Proposition F.4.4 in this case is that the natural map

$$\mathcal{A}_{\mathcal{G}_{Z/\mathcal{U}}}^{\text{IndCoh}} \rightarrow \text{Funct}_{\text{QCoh}(\mathcal{U})}(\text{IndCoh}(Z), \text{IndCoh}(Z))$$

is an isomorphism.

F.5 Proof of Proposition F.4.4

F.5.1. We will need the following variant of the construction of the assignment $\mathbf{O} \mapsto \mathcal{A}_{\mathbf{O}}$ of Sect. E.2.

Let $'\mathbf{O}$ be a *non-cocomplete* DG category equipped with a monoidal structure. The datum of such a monoidal structure is, by definition, equivalent to a datum of monoidal structure on the cocomplete DG category $\mathbf{O} := \text{Ind}(' \mathbf{O})$ such that the monoidal operation $\mathbf{O} \times \mathbf{O} \rightarrow \mathbf{O}$ sends $'\mathbf{O} \times '\mathbf{O} \subset \mathbf{O} \times \mathbf{O}$ to $'\mathbf{O} \subset \mathbf{O}$.

We set, by definition, $\mathcal{A}_{'\mathbf{O}} := \mathcal{A}_{\mathbf{O}}$. The assignment $'\mathbf{O} \rightarrow \mathcal{A}_{'\mathbf{O}}$ is clearly functorial. In addition, it has the following property:

Let $'\Phi : '\mathbf{O} \rightarrow '\mathbf{O}_1$ be a monoidal functor, where \mathbf{O}_1 is another monoidal DG category, not necessarily cocomplete. Then, $'\Phi$ induces a continuous monoidal functor

$$\Phi : \mathbf{O} \rightarrow \mathbf{O}_1,$$

where $\mathbf{O}_1 = \text{Ind}(' \mathbf{O}_1)$, and provides a homomorphism

$$\mathcal{A}_{\mathbf{O}} = \mathcal{A}_{\mathbf{O}} \rightarrow \mathcal{A}_{\mathbf{O}_1} = \mathcal{A}_{'\mathbf{O}_1}.$$

Assume now that the original functor $'\Phi$ is fully faithful (but Φ does not have to be). Then, the above homomorphism $\mathcal{A}_{\mathbf{O}} \rightarrow \mathcal{A}_{'\mathbf{O}_1}$ is an isomorphism. In fact, it suffices to assume that Φ' induces an isomorphism

$$\text{Maps}_{'\mathbf{O}}(\mathbf{1}_{'\mathbf{O}}, \mathbf{1}_{'\mathbf{O}}) \rightarrow \text{Maps}_{'\mathbf{O}_1}(\mathbf{1}_{'\mathbf{O}_1}, \mathbf{1}_{'\mathbf{O}_1}).$$

Equivalently, Φ' needs to be fully faithful on the full subcategory of $'\mathbf{O}$ that is *strongly generated* by the unit object $\mathbf{1}_{'\mathbf{O}} \in '\mathbf{O}$ (that is, on the smallest full DG subcategory of $'\mathbf{O}$ containing $\mathbf{1}_{'\mathbf{O}}$).

F.5.2. Consider the full, but not cocomplete, subcategory

$$\text{QCoh}(\mathcal{G})^{\text{bdd.Tor}} \subset \text{QCoh}(\mathcal{G}),$$

consisting of the objects whose Tor-amplitude with respect to p_2 is bounded on the left. That is, $\mathcal{F} \in \text{QCoh}(\mathcal{G})$ if and only if the functor

$$\mathcal{F}' \mapsto \mathcal{F} \otimes p_2^*(\mathcal{F}'), \quad \text{QCoh}(Z) \rightarrow \text{QCoh}(\mathcal{G})$$

is of *bounded cohomological amplitude on the left*. For example, the object $\text{unit}_*(\mathcal{O}_Z)$ belongs to $\text{QCoh}(\mathcal{G})^{\text{bdd.Tor}}$ (In what follows, we could actually replace the category $\text{QCoh}(\mathcal{G})^{\text{bdd.Tor}}$ by its full subcategory strongly generated by the unit object $\text{unit}_*(\mathcal{O}_Z)$).

The monoidal structure on $\text{QCoh}(\mathcal{G})$ preserves $\text{QCoh}(\mathcal{G})^{\text{bdd.Tor}}$, which therefore acquires a structure of a monoidal (non-cocomplete) DG category. Similarly, the monoidal structure on $\text{IndCoh}(\mathcal{G})$ preserves $\text{IndCoh}(\mathcal{G})^+$, and therefore $\text{IndCoh}(\mathcal{G})^+$ acquires a structure of a monoidal (non-cocomplete) DG category.

By Sect. F.5.1, it is sufficient to construct a monoidal functor

$$\mathrm{QCoh}(\mathcal{G})^{\mathrm{bdd.Tor}} \rightarrow \mathrm{IndCoh}(\mathcal{G})^+$$

and verify that it is fully faithful on the subcategory strongly generated by the unit object.

F.5.3. The operation of convolution in $\mathrm{QCoh}(\mathcal{G})$ defines an action of the monoidal category $\mathrm{QCoh}(\mathcal{G})^{\mathrm{bdd.Tor}}$ on $\mathrm{QCoh}(\mathcal{G})^+$.

Recall that the functor $\Psi_{\mathcal{G}}$ defines an equivalence $\mathrm{IndCoh}(\mathcal{G})^+ \rightarrow \mathrm{QCoh}(\mathcal{G})^+$. Hence, we obtain a monoidal action of $\mathrm{QCoh}(\mathcal{G})^{\mathrm{bdd.Tor}}$ on $\mathrm{IndCoh}(\mathcal{G})^+$.

Lemma F.5.4 *The above action of $\mathrm{QCoh}(\mathcal{G})^{\mathrm{bdd.Tor}}$ on $\mathrm{IndCoh}(\mathcal{G})^+$ commutes with the right action of $\mathrm{IndCoh}(\mathcal{G})^+$ on itself that is induced by the monoidal structure on $\mathrm{IndCoh}(\mathcal{G})$.*

Proof Follows from [10, Lemma 3.6.13]. □

On the other hand, the monoidal category of endomorphisms of the right $\mathrm{IndCoh}(\mathcal{G})^+$ -module $\mathrm{IndCoh}(\mathcal{G})^+$ is identified with $\mathrm{IndCoh}(\mathcal{G})^+$ via its left action on itself. Thus, we obtain a monoidal functor

$$\mathrm{QCoh}(\mathcal{G})^{\mathrm{bdd.Tor}} \rightarrow \mathrm{IndCoh}(\mathcal{G})^+.$$

If one ignores the monoidal structure, the functor can be given explicitly by

$$\mathcal{F} \mapsto \mathcal{F} \otimes p_2^*(\Psi_Z(\omega_Z)) \in \mathrm{QCoh}(\mathcal{G})^+ \simeq \mathrm{IndCoh}(\mathcal{G})^+ \quad (\mathcal{F} \in \mathrm{QCoh}(\mathcal{G})^{\mathrm{bdd.Tor}}). \quad (\mathrm{F.7})$$

The advantage of using Lemma F.5.4 to construct the functor (rather than treating (F.7) as its definition) is that the monoidal structure appears naturally.

F.5.5. It remains to show that the map

$$\begin{aligned} & \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{G})}(\mathrm{unit}_*(\mathcal{O}_Z), \mathrm{unit}_*(\mathcal{O}_Z)) \\ & \rightarrow \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{G})}(\mathrm{unit}_*^{\mathrm{IndCoh}}(\omega_Z), \mathrm{unit}_*^{\mathrm{IndCoh}}(\omega_Z)) \end{aligned} \quad (\mathrm{F.8})$$

induced by (F.7) is an isomorphism.

F.5.6. More generally, let $f : Z' \rightarrow Z$ be a map between affine DG schemes, and consider the corresponding subcategory

$$\mathrm{QCoh}(Z')^{\mathrm{bdd.Tor}/Z} \subset \mathrm{QCoh}(Z),$$

consisting of objects $\mathcal{F}' \in \mathrm{QCoh}(Z')$, for which the functor

$$\mathcal{F} \mapsto \mathcal{F}' \otimes f^*(\mathcal{F}), \quad \mathrm{QCoh}(Z) \rightarrow \mathrm{QCoh}(Z')$$

is of bounded cohomological amplitude on the left (i.e., these are objects whose Tor-dimension over Z is bounded on the left).

Consider the corresponding functor

$$\Phi : \mathrm{QCoh}(Z')^{\mathrm{bdd.Tor}/Z} \rightarrow \mathrm{IndCoh}(Z'),$$

defined by

$$\Phi(\mathcal{F}') := \mathcal{F}' \otimes f^*(\Psi_Z(\omega_Z)) \in \mathrm{QCoh}(Z')^+ \simeq \mathrm{IndCoh}(Z')^+ \subset \mathrm{IndCoh}(Z').$$

We will claim:

Proposition F.5.7 *The functor Φ induces an isomorphism*

$$\mathcal{M}aps_{\mathrm{QCoh}(Z')}(\mathcal{F}'_1, \mathcal{F}'_2) \rightarrow \mathcal{M}aps_{\mathrm{IndCoh}(Z')}(\Phi(\mathcal{F}'_1), \Phi(\mathcal{F}'_2)),$$

whenever $\mathcal{F}'_2 \in \mathrm{QCoh}(Z')^{\mathrm{bdd.Tor}/Z} \cap \mathrm{QCoh}(Z')^-$.

Proof Since Z' was assumed affine, we can take $\mathcal{F}'_1 = \mathcal{O}_{Z'}$. Denote \mathcal{F}'_2 by \mathcal{F}' . By the projection formula, it suffices to show that the map

$$\Gamma(Z, f_*(\mathcal{F}')) \rightarrow \mathcal{M}aps_{\mathrm{QCoh}(Z)}(\Psi_Z(\omega_Z), f_*(\mathcal{F}') \otimes \Psi_Z(\omega_Z))$$

is an isomorphism.

The assumption that $\mathcal{F}' \in \mathrm{QCoh}(Z')^{\mathrm{bdd.Tor}/Z}$ implies that the right-hand side identifies with $\mathcal{M}aps_{\mathrm{IndCoh}(Z)}(\omega_Z, \mathcal{F} \otimes \omega_Z)$, where $\mathcal{F} := f_*(\mathcal{F}')$, and \otimes denotes the monoidal action of $\mathrm{QCoh}(Z)$ on $\mathrm{IndCoh}(Z)$. \square

Hence, the required assertion follows from the next general lemma:

Lemma F.5.8 *Let Z be a DG scheme. Then the map*

$$\Gamma(Z, \mathcal{F}) \rightarrow \mathcal{M}aps_{\mathrm{IndCoh}(Z)}(\omega_Z, \mathcal{F} \otimes \omega_Z) \tag{F.9}$$

is an isomorphism, provided that $\mathcal{F} \in \mathrm{QCoh}(Z)^-$.

Proof of Lemma F.5.8 Recall that if Z is eventually coconnective, (F.9) is an isomorphism for any $\mathcal{F} \in \mathrm{QCoh}(Z)$, see [10, Corollary 9.6.3]. Let $i_k : Z_k \rightarrow Z$ be the k -coconnective truncation of Z . Then

$$\omega_Z = \underset{\rightarrow}{\mathrm{colim}} (i_k)_*^{\mathrm{IndCoh}}(\omega_{Z_k}).$$

Therefore, the right-hand side of (F.9) can be evaluated as

$$\begin{aligned} \mathrm{Hom}_{\mathrm{IndCoh}(Z)}(\omega_Z, \mathcal{F} \otimes \omega_Z) &\simeq \underset{\leftarrow}{\mathrm{lim}} \mathrm{Hom}_{\mathrm{IndCoh}(Z_k)}(\omega_{Z_k}, (i_k)^!(\mathcal{F} \otimes \omega_Z)) \\ &\simeq \underset{\leftarrow}{\mathrm{lim}} \mathrm{Hom}_{\mathrm{IndCoh}(Z_k)}(\omega_{Z_k}, (i_k)^*(\mathcal{F}) \otimes \omega_{Z_k}) \simeq \underset{\leftarrow}{\mathrm{lim}} \mathrm{Hom}_{\mathrm{QCoh}(Z_k)}(\mathcal{O}_{Z_k}, (i_k)^*(\mathcal{F})) \\ &\simeq \underset{\leftarrow}{\mathrm{lim}} \mathrm{Hom}_{\mathrm{QCoh}(Z)}(\mathcal{O}_Z, (i_k)_* \circ (i_k)^*(\mathcal{F})). \end{aligned}$$

It remains to notice that \mathcal{F} maps isomorphically to the inverse limit

$$\varprojlim (i_k)_* \circ (i_k)^*(\mathcal{F}),$$

which is true because $\mathcal{F} \in \mathrm{QCoh}(Z)^-$: indeed, if $\mathcal{F} \in \mathrm{QCoh}(Z)^{\leq 0}$, then the map

$$\mathcal{F} \rightarrow (i_k)_* \circ (i_k)^*(\mathcal{F})$$

induces an isomorphism in the cohomological degrees $\geq -k$. \square

Appendix G: Hochschild cohomology and Lie algebras

In this appendix, we connect \mathbb{E}_2 -algebras arising from groupoids as in Sect. F to Lie algebras in the symmetric monoidal category IndCoh .²³

G.1 Lie algebras arising from group DG schemes

In this subsection, we quote (without proof) two facts (Propositions G.1.2 and G.1.5) about the relationship between group DG schemes and Lie algebras. The full exposition appears in [13, Part IV.3, Sect. 3].

G.1.1. Let Z be an affine DG scheme, and let

$$p : \mathcal{G} \rightrightarrows Z : \mathrm{unit}$$

be a group DG scheme over Z . We claim:

Proposition G.1.2 *The object $\mathrm{unit}^*(T^*(\mathcal{G}/Z)) \in \mathrm{QCoh}(Z)$ has a natural structure of Lie co-algebra in the symmetric monoidal category $\mathrm{QCoh}(Z)$.*

G.1.3. Recall that all DG schemes are assumed to be almost of finite type. We have

$$T^*(\mathcal{G}/Z) \in \mathrm{QCoh}(Z)^-,$$

and note that it has *coherent cohomologies*. Serre duality identifies this category with a full subcategory of $\mathrm{QCoh}(Z)^+ \simeq \mathrm{IndCoh}(Z)^+$ consisting of objects with coherent cohomologies.

Hence,

$$T^{\mathrm{IndCoh}}(\mathcal{G}/Z) := \mathbf{D}_Z^{\mathrm{Serre}}(T^*(\mathcal{G}/Z)) \in \mathrm{IndCoh}(Z)$$

²³ As we will be dealing with Lie algebras, the assumption that our ground field k has characteristic 0 is crucial.

acquires a structure of Lie algebra, where $\text{IndCoh}(Z)$ is regarded as a symmetric monoidal category under the $\overset{!}{\otimes}$ tensor product.

G.1.4. Let $\text{IndCoh}(\mathcal{G})_Z \subset \text{IndCoh}(\mathcal{G})$ be the full subcategory of objects set-theoretically supported on the unit section. The embedding $\text{IndCoh}(\mathcal{G})_Z \hookrightarrow \text{IndCoh}(\mathcal{G})$ admits a right adjoint, which we denote by

$$\mathcal{F} \mapsto \mathcal{F}^\wedge : \text{IndCoh}(\mathcal{G}) \rightarrow \text{IndCoh}(\mathcal{G})_Z.$$

Consider the object

$$\omega_{\mathcal{G}}^\wedge \in \text{IndCoh}(\mathcal{G}),$$

and its direct image $p_*^{\text{IndCoh}}(\omega_{\mathcal{G}}^\wedge) \in \text{IndCoh}(Z)$.

The group structure on \mathcal{G} makes $p_*^{\text{IndCoh}}(\omega_{\mathcal{G}}^\wedge)$ into an associative algebra object in $\text{IndCoh}(Z)$ (where $\text{IndCoh}(Z)$ is, as always, considered as a symmetric monoidal category under the $\overset{!}{\otimes}$ tensor product). We claim:

Proposition G.1.5 *There is a canonical isomorphism of associative algebras in $\text{IndCoh}(Z)$*

$$p_*^{\text{IndCoh}}(\omega_{\mathcal{G}}^\wedge) \simeq U(T^{\text{IndCoh}}(\mathcal{G}/Z)).$$

G.1.6. Assume now that \mathcal{G} is such that $T^*(\mathcal{G}/Z)$ belongs to $\text{QCoh}(Z)^{\text{perf}}$. Denote by $T(\mathcal{G}/Z) \in \text{QCoh}(Z)^{\text{perf}}$ its monoidal dual.

By Proposition G.1.2, the object $T(\mathcal{G}/Z)$ acquires a structure of Lie algebra in the symmetric monoidal category $\text{QCoh}(Z)$.

Recall also that the functor

$$\Upsilon_Z = - \otimes \omega_Z : \text{QCoh}(Z) \rightarrow \text{IndCoh}(Z)$$

is symmetric monoidal. By construction, we have

$$T^{\text{IndCoh}}(\mathcal{G}/Z) \simeq \Upsilon_Z(T(\mathcal{G}/Z)),$$

as Lie algebras in $\text{IndCoh}(Z)$, and hence

$$U(T^{\text{IndCoh}}(\mathcal{G}/Z)) \simeq \Upsilon_Z(U_{\mathcal{O}_Z}(T(\mathcal{G}/Z))),$$

as associative algebras in $\text{IndCoh}(Z)$.

G.2 Lie algebras arising from groupoids

In this subsection, we will relate the \mathbb{E}_2 -algebra $\mathrm{HC}^{\mathrm{IndCoh}}(Z)$ to the tangent sheaf of Z . This relationship is convenient for the direct invariant definition of singular support, as is given in Sect. 4.1.

G.2.1. Consider now the following situation. Let $i : Z \rightarrow W$ be a proper map of affine DG schemes, equipped with a retraction, i.e., a map $s : W \rightarrow Z$ and an isomorphism $s \circ i \sim \mathrm{id}_Z$.

Note that in this case, the groupoid $\mathcal{G}_{Z/W} := Z \times_{\underset{W}{Z}} Z$ over Z is actually a group DG scheme over Z .

Assume now that $T^*(Z/W)$ is perfect, and consider its monoidal dual $T(Z/W) \in \mathrm{QCoh}(Z)^{\mathrm{perf}}$. We will prove:

Proposition G.2.2 (a) *The object $T(Z/W) \in \mathrm{QCoh}(Z)$ has a structure of Lie algebra.*

(b) *There is a canonically defined homomorphism of DG associative algebras*

$$\Gamma(Z, U_{\mathcal{O}_Z}(T(Z/W))) \rightarrow \mathrm{Maps}_{\mathrm{IndCoh}(W)}(i_*^{\mathrm{IndCoh}}(\omega_Z), i_*^{\mathrm{IndCoh}}(\omega_Z)),$$

which is an isomorphism if Z is eventually coconnective.

G.2.3. Before proving Proposition G.2.2 let us show how we are going to apply it. Suppose we are in situation of Sect. F.4.1; thus, \mathcal{G} is a groupoid acting on Z , where both Z and \mathcal{G} are affine DG schemes. Assume that the relative cotangent complex $T^*(\mathcal{G}/Z)$ (with respect to, say, projection p_1) is perfect; let $T(\mathcal{G}/Z) \in \mathrm{QCoh}(\mathcal{G})^{\mathrm{perf}}$ denote its monoidal dual.

From Proposition G.2.2, we obtain:

Corollary G.2.4 (a) *The object $\mathrm{unit}^*(T(\mathcal{G}/Z))[-1] \in \mathrm{QCoh}(Z)$ has a natural structure of Lie algebra.*

(b) *There exists a canonically defined homomorphism*

$$\Gamma(Z, U_{\mathcal{O}_Z}(\mathrm{unit}^*(T(\mathcal{G}/Z))[-1])) \rightarrow \mathcal{A}_{\mathcal{G}}^{\mathrm{IndCoh}},$$

which is an isomorphism if Z is eventually coconnective.

Proof Apply Proposition G.2.2 to $W := \mathcal{G}$ with the map i being the unit map $Z \rightarrow \mathcal{G}$ and the retraction being, say, the first projection $p_1 : \mathcal{G} \rightarrow Z$. It remains to use the canonical identification

$$\mathrm{unit}^*(T(\mathcal{G}/Z))[-1] \simeq T(Z/\mathcal{G}).$$

□

Remark G.2.5 Note that the structure of Lie algebra on $\text{unit}^*(T(\mathcal{G}/Z)[-1])$ depends on the choice of the retraction of the map $\text{unit} : Z \rightarrow \mathcal{G}$. If we chose a different retraction, namely p_2 instead of p_1 , the resulting Lie algebra structure would be different. In general, there does not exist an isomorphism between the two resulting Lie algebras that induces the identity map on the underlying object of $\text{QCoh}(Z)$.

G.2.6. Consider the particular case when $\mathcal{G} = Z \times Z$. Assume that the cotangent complex of Z is perfect. We obtain the following basic identification:

- Corollary G.2.7** (a) *The object $T(Z)[-1]$ has a natural structure of Lie algebra in the symmetric monoidal category $\text{QCoh}(Z)$.*
 (b) *The associative DG algebra underlying the \mathbb{E}_2 -algebra $\text{HC}^{\text{IndCoh}}(Z)$ receives a canonical homomorphism from $\Gamma(Z, U_{\mathcal{O}_Z}(T(Z)[-1]))$. This homomorphism is an isomorphism if Z is eventually coconnective.*

More generally, for a map $Z \rightarrow \mathcal{U}$ whose relative cotangent complex $T^*(Z/\mathcal{U})$ is perfect, we obtain:

- Corollary G.2.8** (a) *The object $T(Z/\mathcal{U})[-1]$ has a natural structure of Lie algebra in the symmetric monoidal category $\text{QCoh}(Z)$.*
 (b) *The associative DG algebra underlying the \mathbb{E}_2 -algebra $\text{HC}^{\text{IndCoh}}(Z/\mathcal{U})$ receives a canonical homomorphism from $\Gamma(Z, U_{\mathcal{O}_Z}(T(Z/U)[-1]))$. This homomorphism is an isomorphism if Z is eventually coconnective.*

G.2.9. *Proof of Proposition G.2.2* Let us start with a map $i : Z \rightarrow W$. Consider the groupoid $\mathcal{G}_{Z/W} = Z \times_W Z$:

$$\begin{array}{ccc}
 & \mathcal{G}_{Z/W} & \\
 p_1 \swarrow & \uparrow \text{unit} & \searrow p_2 \\
 Z & & Z.
 \end{array}
 \tag{G.1}$$

By proper base change (see also [13, Part II.2, Sect. 5.3]), the monad $i^! \circ i_*^{\text{IndCoh}}$ acting on $\text{IndCoh}(Z)$ identifies with the monad $(p_2)_*^{\text{IndCoh}} \circ p_1^!$.

In particular, for $\mathcal{F} \in \text{IndCoh}(Z)$, the structure of associative DG algebra on

$$\mathcal{M}aps_{\text{IndCoh}(Z)}(\mathcal{F}, (p_2)_*^{\text{IndCoh}} \circ p_1^!(\mathcal{F}))
 \tag{G.2}$$

identifies canonically with

$$\mathcal{M}aps_{\text{IndCoh}(W)}(i_*^{\text{IndCoh}}(\mathcal{F}), i_*^{\text{IndCoh}}(\mathcal{F})).$$

In particular, we obtain an identification of associative DG algebras

$$\mathcal{M}aps_{\text{IndCoh}(Z)}(\omega_Z, (p_2)_*^{\text{IndCoh}} \circ p_1^!(\omega_Z)) \simeq \mathcal{M}aps_{\text{IndCoh}(W)}(i_*^{\text{IndCoh}}(\omega_Z), i_*^{\text{IndCoh}}(\omega_Z)).
 \tag{G.3}$$

Assume now that the map $i : Z \rightarrow W$ is equipped with a retraction. In this case, $\mathcal{G}_{Z/W}$ is a group DG scheme over Z , and by Sect. G.1.6, the object

$$\text{unit}^*(T(\mathcal{G}_{Z/W}/Z)) \in \text{QCoh}(Z)$$

acquires a Lie algebra structure. Point (a) of Proposition G.2.2 follows now by noticing that

$$\text{unit}^*(T(\mathcal{G}_{Z/W}/Z)) \simeq T(Z/W).$$

Denote $\mathcal{G} := \mathcal{G}_{Z/W}$ and let p denote its projection to Z (which is canonically identified with both p_1 and p_2). To prove point (b), we note that the associative algebra structure on (G.2) comes from the structure of associative algebra on $\text{IndCoh}(Z)$ on $p_*^{\text{IndCoh}}(\omega_{\mathcal{G}})$, induced by the group structure on \mathcal{G} .

Note now that the existence of the retraction s implies that the map $i : Z \rightarrow W$ is a closed embedding. Hence, the maps $p : \mathcal{G} \rightarrow Z$ and $\text{unit} : Z \rightarrow \mathcal{G}$ induce isomorphisms of the underlying reduced classical schemes. In particular, $\omega_{\mathcal{G}} \simeq \omega_{\mathcal{G}}^{\wedge}$. Therefore, combining Proposition G.1.5 and (G.3), we obtain an identification of associative DG algebras

$$\mathcal{M}aps_{\text{IndCoh}(Z)}(\omega_Z, U(T^{\text{IndCoh}}(\mathcal{G}/Z))) \simeq \mathcal{M}aps_{\text{IndCoh}(W)}(i_*^{\text{IndCoh}}(\omega_Z), i_*^{\text{IndCoh}}(\omega_Z)). \quad (\text{G.4})$$

Finally, the symmetric monoidal functor Υ_Z defines a homomorphism

$$\begin{aligned} \Gamma(Z, U(\text{unit}^*(T(\mathcal{G}/Z))[-1])) &\simeq \mathcal{M}aps_{\text{QCoh}(Z)}(\mathcal{O}_Z, U_{\mathcal{O}_Z}(T(\mathcal{G}/Z))) \\ &\rightarrow \mathcal{M}aps_{\text{IndCoh}(Z)}(\omega_Z, U(T^{\text{IndCoh}}(\mathcal{G}/Z))). \end{aligned} \quad (\text{G.5})$$

Composing (G.4) and (G.5), we obtain the desired map

$$\Gamma(Z, U(\text{unit}^*(T(\mathcal{G}/Z))[-1])) \rightarrow \mathcal{M}aps_{\text{IndCoh}(W)}(i_*^{\text{IndCoh}}(\omega_Z), i_*^{\text{IndCoh}}(\omega_Z)). \quad (\text{G.6})$$

If Z is eventually coconnective, the functor Υ_Z is fully faithful, which implies that (G.5) is an isomorphism. Hence, (G.6) is an isomorphism as well. \square

G.3 Compatibility with duality

G.3.1. Consider the homomorphism of \mathbb{E}_1 -algebras

$$\Gamma(Z, U_{\mathcal{O}_Z}(T(Z)[-1])) \rightarrow \text{HC}^{\text{IndCoh}}(Z),$$

and the resulting map in Vect

$$\Gamma(Z, T(Z))[-1] \rightarrow \text{HC}^{\text{IndCoh}}(Z). \quad (\text{G.7})$$

G.3.2. We claim:

Lemma G.3.3 *The following diagram commutes*

$$\begin{array}{ccc} \Gamma(Z, T(Z))[-1] & \xrightarrow{\text{(G.7)}} & \mathbf{HC}^{\text{IndCoh}}(Z) \\ \xi \mapsto -\xi \downarrow & & \downarrow \text{(F.4)} \\ \Gamma(Z, T(Z))[-1] & \xrightarrow{\text{(G.7)}} & \mathbf{HC}^{\text{IndCoh}}(Z)^{\text{ext-op}} \end{array}$$

in \mathbf{Vect} .

Proof The map (G.7) is the composition of the natural morphism

$$\begin{aligned} \Gamma(Z, T(Z))[-1] &\simeq \Gamma(Z, N_{Z/Z \times Z})[-1] \\ &\rightarrow \mathcal{M}aps_{\text{IndCoh}(Z \times Z)}((\Delta_Z)_*^{\text{IndCoh}}(\omega_Z), (\Delta_Z)_*^{\text{IndCoh}}(\omega_Z)), \end{aligned}$$

and the homomorphism of objects of \mathbf{Vect} (in fact, \mathbb{E}_1 -algebras)

$$\mathcal{M}aps_{\text{IndCoh}(Z \times Z)}((\Delta_Z)_*^{\text{IndCoh}}(\omega_Z), (\Delta_Z)_*^{\text{IndCoh}}(\omega_Z)) \simeq \mathbf{HC}^{\text{IndCoh}}(Z).$$

Unwinding the definitions, we obtain that the isomorphism (F.4) corresponds to the automorphism of $\mathcal{M}aps_{\text{IndCoh}(Z \times Z)}((\Delta_Z)_*^{\text{IndCoh}}(\omega_Z), (\Delta_Z)_*^{\text{IndCoh}}(\omega_Z))$, induced by the transposition involution σ on $Z \times Z$.

The assertion of the lemma follows now from the commutativity of the diagram

$$\begin{array}{ccc} T(Z) & \longrightarrow & N_{Z/Z \times Z} \\ \xi \mapsto -\xi \downarrow & & \downarrow \sigma \\ T(Z) & \longrightarrow & N_{Z/Z \times Z}. \end{array}$$

□

G.3.4. We are now ready to give a more direct proof of Proposition 4.7.2:

Proof The assertion is local, so we can assume that Z is affine. Fix $\mathcal{F} \in \text{Coh}(Z)$. It suffices to show that the action maps

$$\Gamma(Z, T(Z))[-1] \otimes \mathcal{F} \rightarrow \mathcal{F} \text{ and } \Gamma(Z, T(Z))[-1] \otimes \mathbb{D}_Z^{\text{Serre}}(\mathcal{F}) \rightarrow \mathbb{D}_Z^{\text{Serre}}(\mathcal{F})$$

correspond to each other under the automorphism of $\Gamma(Z, T(Z))[-1]$ given by $\xi \mapsto -\xi$ (In this statement, $\Gamma(Z, T(Z))[-1]$ appears merely as an object of \mathbf{Vect} : we make no statement about compatibility with the Lie algebra structure.).

The required statement follows from Lemma G.3.3 and the following observation:

Let \mathbf{C} be a dualizable DG category, and $\mathbf{c} \in \mathbf{C}$ a compact object. Let \mathbf{c}^\vee denote the corresponding compact object of \mathbf{C}^\vee . Then under the isomorphism of \mathbb{E}_2 -algebras

$$\mathbf{HC}(\mathbf{C})^{\text{ext-op}} \simeq \mathbf{HC}(\mathbf{C}^\vee)$$

(see Sect. E.3.3), the diagram

$$\begin{array}{ccc}
 \mathrm{HC}(\mathbf{C})^{\mathrm{op}} & \longrightarrow & \mathcal{M}\mathrm{aps}_{\mathbf{C}}(\mathbf{c}, \mathbf{c})^{\mathrm{op}} \\
 \downarrow & & \downarrow \\
 \mathrm{HC}(\mathbf{C}^{\vee}) & \longrightarrow & \mathcal{M}\mathrm{aps}_{\mathbf{C}^{\vee}}(\mathbf{c}^{\vee}, \mathbf{c}^{\vee}),
 \end{array}$$

commutes, where the horizontal arrows use the *external* forgetful functor $\mathbb{E}_2\text{-Alg} \rightarrow \mathbb{E}_1\text{-Alg}$. □

G.4 The case of group schemes

G.4.1. Suppose that in the setting of Sect. F.4.1, \mathcal{G} is a group DG scheme over Z . In addition, we continue to assume that the relative cotangent complex $T^*(\mathcal{G}/Z)$ is perfect. In this case, we claim:

Proposition G.4.2 *The Lie algebra structure on $\mathrm{unit}^*(T(\mathcal{G}/Z))[-1] \in \mathrm{QCoh}(Z)$, given by Corollary G.2.4(a), is canonically trivial.*

Proof We will deduce the assertion of the proposition from the following lemma:²⁴

Lemma G.4.3 *Let L be a Lie algebra in a symmetric monoidal category \mathbf{O} over a field of characteristic 0. Then the loop object $\Omega(L)$, considered as a plain Lie algebra in \mathbf{O} is canonically abelian, i.e., identifies with object $L[-1] \in \mathbf{O}$ with the trivial Lie algebra structure.*

Namely, we claim that the Lie algebra $\mathrm{unit}^*(T(\mathcal{G}/Z))[-1] \in \mathrm{QCoh}(Z)$, given by Corollary G.2.4(a), is canonically isomorphic to

$$\Omega(\mathrm{unit}^*(T(\mathcal{G}/Z))),$$

where $\mathrm{unit}^*(T(\mathcal{G}/Z))$ is the Lie algebra of Sect. G.1.6.

For the latter, it suffices to notice that we have an identification of group DG schemes over Z :

$$Z \times_{\mathcal{G}} Z \simeq \Omega(\mathcal{G}).$$

Remark G.4.4 We emphasize that the isomorphism between $\Omega(L)$ and the abelian Lie algebra $L[-1]$ given by Lemma G.4.3 does not respect the structure of group objects in the category of Lie algebras.

²⁴ This lemma is probably well known; the proof is given in [13, Part IV.2, Theorem 2.2.2].

G.4.5. Combining Proposition G.4.2 with Corollary G.2.4(b), we obtain:

Corollary G.4.6 *The associative DG algebra underlying the \mathbb{E}_2 -algebra $\mathcal{A}_{\mathcal{G}}^{\text{IndCoh}}$ receives a canonically defined homomorphism from $\Gamma(Z, \text{Sym}_{\mathcal{O}_Z}(L_{\mathcal{G}}[-1]))$. This homomorphism is an isomorphism if Z is eventually coconnective.*

Remark G.4.7 Let us note that each side in the homomorphism

$$\Gamma(Z, \text{Sym}_{\mathcal{O}_Z}(L_{\mathcal{G}}[-1])) \rightarrow \mathcal{A}_{\mathcal{G}}^{\text{IndCoh}}$$

of Corollary G.4.6 has a structure of \mathbb{E}_2 -algebra. However, this homomorphism does not respect this structure: It is a homomorphism of mere \mathbb{E}_1 -algebras.

Appendix H: Other approaches to singular support

H.1 IndCoh(Z) via the category of singularities

H.1.1. Let us assume that Z is an affine DG scheme, which is a global complete intersection. That is, Z can be included in a Cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{t} & \mathcal{U} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathcal{V}, \end{array}$$

where \mathcal{U} and \mathcal{V} are smooth. Moreover, we assume that \mathcal{V} is parallelized; this allows us to replace \mathcal{V} with its tangent space at the fixed point. Thus, we will assume that $\mathcal{V} = V$ is a finite-dimensional vector space.

Remark H.1.2 In fact, the construction of this section remains valid in the setting of Sect. 9.3: That is, we may replace Z with the zero locus of a section of a vector bundle on a smooth stack. However, one can use affine charts of a stack to deduce this more general case from the special case that we consider here.

H.1.3. Consider the product $V^* \times \mathcal{U}$. The map $s : \mathcal{U} \rightarrow V$ defines a function

$$V^* \times \mathcal{U} \rightarrow \mathbb{A}^1 : (u, \phi) \mapsto \langle \phi, s(u) \rangle,$$

which we still denote by s . Let

$$H := (V^* \times \mathcal{U}) \times_{\mathbb{A}^1} \text{pt}$$

be the zero locus of s . Note that H is a classical scheme (and then a closed hypersurface in $V^* \times \mathcal{U}$) unless s vanishes identically on a connected component of \mathcal{U} . Clearly, H is conical, that is, it carries a natural action of \mathbb{G}_m lifting its action on $V^* \times \mathcal{U}$ by

dilations. Moreover, it is easy to see that the singular locus of H is identified with $\text{Sing}(Z) \subset V^* \times \mathcal{U}$.

Let

$$\mathcal{H} := H/\mathbb{G}_m \simeq ((V^*/\mathbb{G}_m) \times \mathcal{U})_{\mathbb{A}^1/\mathbb{G}_m} \times (\text{pt}/\mathbb{G}_m)$$

be the quotient stack. The following theorem is due to M. Umut Isik:

Theorem H.1.4 *There is a natural equivalence*

$$\text{IndCoh}(Z) \simeq \text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H}), \quad (\text{H.1})$$

where $\text{QCoh}(\mathcal{H})$ is viewed as a full subcategory of $\text{IndCoh}(\mathcal{H})$ using the functor $\Xi_{\mathcal{H}}$.

Theorem H.1.4 is a version of [39, Theorem 3.6]. In [39], it is assumed that Z is classical (that is, that the coordinates of the map s form a regular sequence of functions), but this assumption is not used in the proof.

Remark H.1.5 Let S be a quasi-compact DG scheme. The category $\text{IndCoh}(S)/\text{QCoh}(S)$ is compactly generated by the quotient

$$\text{Coh}(S)/\text{QCoh}(S)^{\text{perf}}.$$

The category $\text{Coh}(S)/\text{QCoh}(S)^{\text{perf}}$ is known as the *category of singularities* of S , introduced in [32]. Then $\text{IndCoh}(S)/\text{QCoh}(S)$ identifies with the ind-completion of the category of singularities. The category $\text{IndCoh}(S)/\text{QCoh}(S)$ was introduced in [16] under the name “stable derived category.” (As a minor detail, both [32] and [16] work with Noetherian classical schemes.)

H.1.6 Theorem H.1.4 is about the “stable derived category” $\text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})$ of the stack \mathcal{H} .

Note that both $\text{IndCoh}(\mathcal{H})$ and $\text{QCoh}(\mathcal{H})$ are compactly generated: The former because \mathcal{H} is QCA (see [6]), the latter because \mathcal{H} is a quotient of an affine DG scheme by a linear group, which is a perfect stack (see [3]). (Alternatively, the two categories are compactly generated by Corollary 9.2.7, since \mathcal{H} is a global complete intersection.) Therefore, $\text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})$ is equivalent to the ind-completion of the “category of singularities”

$$\text{Coh}(\mathcal{H})/\text{QCoh}(\mathcal{H})^{\text{perf}}.$$

In fact, [39, Theorem 3.6] gives an equivalence between the categories of compact objects

$$\text{Coh}(Z) \simeq \text{Coh}(\mathcal{H})/\text{QCoh}(\mathcal{H})^{\text{perf}},$$

rather than between their ind-completions, as in Theorem H.1.4.

H.1.7. Singular support via category of singularities The category $\text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})$ is naturally tensored over the monoidal category $\text{QCoh}(\mathcal{H})$. Using the natural morphism

$$\mathcal{H} \rightarrow (V^*/\mathbb{G}_m) \times \mathcal{U},$$

we can consider $\text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})$ as a category tensored over $\text{QCoh}((V^*/\mathbb{G}_m) \times \mathcal{U})$. Recall from Sect. 5.4.6 that the category $\text{IndCoh}(Z)$ is tensored over the category $\text{QCoh}((V^*/\mathbb{G}_m) \times \mathcal{U})$ as well. It is not hard to check that (H.1) is an equivalence of $\text{QCoh}((V^*/\mathbb{G}_m) \times \mathcal{U})$ -modules.

In particular, fix $\mathcal{F} \in \text{IndCoh}(Z)$ and let $\mathcal{F}' \in \text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})$ be its image under (H.1). We claim that

$$\text{SingSupp}(\mathcal{F}) = \text{supp}(\mathcal{F}'). \quad (\text{H.2})$$

Note that

$$\text{SingSupp}(\mathcal{F}) \subset \text{Sing}(Z) \subset V^* \times \mathcal{U},$$

while the support of \mathcal{F}' can be defined naively, as the minimal closed subset of \mathcal{H} (that is, a conical Zariski-closed subset of $H \subset V^* \times \mathcal{U}$) such that \mathcal{F}' restricts to zero on its complement. It is clear that for any $\mathcal{F}' \in \text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})$, its support is a conical Zariski-closed subset of the singular locus of H (recall that the singular locus of H is identified with $\text{Sing}(Z)$).

H.2 The category of singularities of Z

H.2.1. Denote by

$$(\text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H}))_{\{0\}} \subset \text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})$$

the full subcategory of objects of $\text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})$ supported on the zero-section

$$\{0\} \times \mathcal{U} \subset V^* \times \mathcal{U}.$$

Under the equivalence (H.1), it corresponds to the full subcategory $\text{QCoh}(Z) \subset \text{IndCoh}(Z)$ (where we identify $\text{QCoh}(Z)$ with its image under Ξ_Z). This claim is not hard to check directly, but it also follows from (H.2): indeed, $\mathcal{F} \in \text{IndCoh}(Z)$ belongs to the essential image of $\text{QCoh}(Z)$ if and only if its singular support is contained in the zero-section (Theorem 4.2.6).

H.2.2. Therefore, Theorem H.1.4 induces an equivalence between the quotients

$$\text{IndCoh}(Z)/\text{QCoh}(Z) \simeq (\text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H})) / (\text{IndCoh}(\mathcal{H})/\text{QCoh}(\mathcal{H}))_{\{0\}}.$$

Set

$$\mathcal{H}' := \mathcal{H} - (\{0\}/\mathbb{G}_m) \times \mathcal{U} \subset \mathcal{H}.$$

Note that \mathcal{H}' is a DG scheme rather than a stack (in fact, \mathcal{H}' is a classical scheme unless the map $s : \mathcal{U} \rightarrow V$ vanishes on a connected component of \mathcal{U}). We can identify

$$\mathrm{IndCoh}(\mathcal{H}')/\mathrm{QCoh}(\mathcal{H}') \simeq (\mathrm{IndCoh}(\mathcal{H})/\mathrm{QCoh}(\mathcal{H})) / (\mathrm{IndCoh}(\mathcal{H})/\mathrm{QCoh}(\mathcal{H}))_{\{0\}}$$

(cf. [16, Proposition 6.9]). Therefore, Theorem H.1.4 implies the following equivalence, due to D. Orlov.

Theorem H.2.3 *There is a natural equivalence*

$$\mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z) \simeq \mathrm{IndCoh}(\mathcal{H}')/\mathrm{QCoh}(\mathcal{H}').$$

Theorem H.2.3 is a variant of [33, Theorem 2.1]. Some minor differences include that [33] works with the category of compact objects, and assumes that Z is classical. Besides, the equivalence of Theorem H.2.3 is constructed in a different and more explicit way than the equivalence of Theorem H.1.4; in fact, while the introduction to [39] mentions the similarity between the two results, it also states that the agreement between the two constructions is not immediately clear.

Remark H.2.4 Fix $\mathcal{F} \in \mathrm{IndCoh}(Z)$. Just like Theorem H.1.4 can be used to determine $\mathrm{SingSupp}(\mathcal{F})$ (using (H.2)), Theorem H.2.3 determines

$$\mathrm{SingSupp}(\mathcal{F}) \cap (V^* - \{0\}) \times \mathcal{U},$$

that is, the complement to the zero-section in $\mathrm{SingSupp}(\mathcal{F})$. However, one can easily reconstruct the entire singular support, because

$$\mathrm{SingSupp}(\mathcal{F}) \cap \{0\} \times \mathcal{U} = \{0\} \times \mathrm{supp}(\mathcal{F}).$$

H.2.5. Let Y be a conical Zariski-closed subset of $\mathrm{Sing}(Z)$ that contains the zero-section. Such subsets are in one-to-one correspondence with Zariski-closed subsets of the singular locus of \mathcal{H}' : The correspondence sends Y to

$$Y' := (Y - \{0\} \times \mathcal{U})/\mathbb{G}_m \subset \mathcal{H}'.$$

Since Y contains the zero-section, the corresponding full subcategory $\mathrm{IndCoh}_Y(Z)$ contains $\mathrm{QCoh}(Z)$. Therefore, we can consider the quotient $\mathrm{IndCoh}_Y(Z)/\mathrm{QCoh}(Z)$, which embeds as a full subcategory into $\mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z)$.

G. Stevenson provides a complete classification of localizing subcategories of the triangulated category $\mathrm{Ho}(\mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z))$ in [37, Corollary 10.5] (a triangulated subcategory is localizing if it is closed under arbitrary direct sums). Such subcategories are in one-to-one correspondence with subsets of \mathcal{H}' . Subcategories that are generated by objects that are compact in $\mathrm{Ho}(\mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z))$

correspond to specialization-closed subsets. Under this correspondence, the category $\text{Ho}(\text{IndCoh}_Y(Z)/\text{QCoh}(Z))$ corresponds to the subset $Y' \subset \mathcal{H}'$ (This is almost obvious because [37] uses Orlov's equivalence of Theorem H.2.3 to study $\text{IndCoh}(Z)/\text{QCoh}(Z)$).

To summarize, we can also reconstruct the singular support of an object $\mathcal{F} \in \text{IndCoh}(Z)$ using the results of [37]. Technically, this only allows us to reconstruct the support of the image of \mathcal{F} in $\text{IndCoh}(Z)/\text{QCoh}(Z)$, that is, the complement to the zero-section in $\text{SingSupp}(\mathcal{F})$; the entire SingSupp can be recovered using Remark H.2.4.

H.3 $\text{IndCoh}(Z)$ as the coderived category

Let us comment on the relation between the results in the main body of the present paper and the nonlinear Koszul transform introduced by L. Positselski in [34].

H.3.1. The key notion that we need from [34] is that of coderived category of modules over a curved DG algebra. To simplify the exposition, we do not give the definitions in maximal generality, and only work with curved DG algebras whose curvature is central. This class of curved DG algebra suffices for our purposes.

Let A be a DG algebra. Fix a central element $c \in A^2$ such that $d(c) = 0$. We refer to the pair (A, c) as a “curved DG algebra”; c is called the curvature of A .

A (left) module over the curved DG algebra (A, c) is by definition a graded vector space M equipped with an action of A and a degree one map $d : M \rightarrow M$ that satisfies the Leibniz rule and the identity $d^2 = c$. Modules over (A, c) form a DG category, which we denote by $A\text{-mod}_c$. Consider the corresponding triangulated category $\text{Ho}(A\text{-mod}_c)$.

Definition H.3.2 The full subcategory of coacyclic modules

$$\text{Acycl}^{co}(A\text{-mod}_c) \subset \text{Ho}(A\text{-mod}_c)$$

is the subcategory generated by the total complexes of exact sequences

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

of (A, c) -modules. The coderived category $D^{co}(A\text{-mod}_c)$ is defined to be the quotient

$$D^{co}(A\text{-mod}_c) := \text{Ho}(A\text{-mod}_c) / \text{Acycl}^{co}(A\text{-mod}_c)$$

H.3.3. Suppose Z is as in Sect. H.1.1. We consider two curved DG algebras: one is

$$A := \text{Sym}(V^*) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$$

with differential given by $s \in \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}} \otimes V)$ and curvature zero. The other is

$$B := \mathrm{Sym}(V[-2]) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$$

with zero differential and curvature s .

The following is a variant of a special case of [34, Theorem 6.5a] (see also [34, Theorem 6.3a])

Theorem H.3.4 *There is a natural equivalence between the coderived categories*

$$D^{co}(A\text{-mod}_0) \simeq D^{co}(B\text{-mod}_s).$$

Remark H.3.5 We state Theorem H.3.4 with algebras on both sides of the equivalence. This is one point of difference from [34], where the equivalence relates algebras and coalgebras. However, note that A is free of finite rank over $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$, so it is easy to pass from modules over A to comodules over the dual coalgebra. Another point of difference with [34] is that Theorem H.3.4 is “relative” the correspondence is linear over the algebra $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$.

H.3.6. Let us explain the relation between Theorem H.3.4 and Theorem H.1.4.

Indeed, $Z = \mathrm{Spec}(A)$, and it follows from [34, Theorem 3.11.2] that $D^{co}(A\text{-mod}_0)$ can be identified with $\mathrm{Ho}(\mathrm{IndCoh}(Z))$. On the other hand, (B, s) -modules are similar to “equivariant matrix factorization” (cf. [34, Example 3.11]), and it is natural that they can be used to study the “equivariant category of singularities” $\mathrm{IndCoh}(\mathcal{H})/\mathrm{QCoh}(\mathcal{H})$. Finally, note that, just as the equivalence of Theorem H.3.4 is given by a (nonlinear) Koszul transform, the equivalence of Theorem H.1.4 (constructed in [39]) is derived using a (linear) Koszul transform, namely the linear Koszul transform of I. Mirković and S. Riche [25].

References

1. Avni, N.: Why $\mathrm{Bun}(G)$ is an algebraic stack. http://www.math.harvard.edu/~gaitsgde/grad_2009/
2. Bezrukavnikov, R., Finkelberg, M.: Equivariant Satake category and Kostant-Whittaker reduction. *Mosc. Math. J.*, **8**(1), 39–72, 183 (2008)
3. Ben-Zvi, D., Francis, J., Nadler, D.: Integral transforms and Drinfeld centers in derived algebraic geometry. *J. Am. Math. Soc.* **23**(4), 909–966 (2010)
4. Benson, D., Iyengar, S.B., Krause, H.: Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér.* **41**(4), 573–619 (2008)
5. Beilinson, A., Vologodsky, V.: A DG guide to Voevodsky’s motives. *Geom. Funct. Anal.* **17**(6), 1709–1787 (2008)
6. Drinfeld, V., Gaitsgory, D.: On some finiteness questions for algebraic stacks. *GAGA* **23**, 149–294 (2013)
7. Drinfeld, V., Gaitsgory, D.: Compact generation of the category of D-modules on the stack of G-bundles on a curve. Preprint [arXiv:1112.2402](https://arxiv.org/abs/1112.2402)
8. Drinfeld, V., Gaitsgory, D.: Geometric constant term functor(s). Preprint [arXiv:1311.2071](https://arxiv.org/abs/1311.2071)
9. Francis, J., Gaitsgory, D.: Chiral Koszul duality. *Sel. Math. (New Series)* **18**(1), 2–87 (2012)
10. Gaitsgory, D.: Sheaves of categories and the notion of 1-affineness. Preprint [arXiv:1306.4304](https://arxiv.org/abs/1306.4304)
11. Gaitsgory, D.: Ind-coherent sheaves. *Mosc. Math. J.* **13**, 399–528 (2013)
12. Gaitsgory, D., Rozenblyum, N.: DG indschemes. *Contemp. Math.* **610**, 139–251 (2014)
13. Gaitsgory, D., Rozenblyum, N.: Crystals and D-modules. Preprint [arXiv:1111.2087](https://arxiv.org/abs/1111.2087)
14. Gaitsgory, D., Rozenblyum, N.: Notes on Geometric Langlands: studies in derived algebraic geometry. Book in preparation. <http://www.math.harvard.edu/~gaitsgde/Nantes14/>
15. Ginzburg, V.: The global nilpotent variety is Lagrangian. *Duke Math. J.* **109**(3), 511–519 (2001)

16. Gulliksen, T.H.: A change of ring theorem with applications to Poincaré series and intersection multiplicity. *Math. Scand.* **34**, 167–183 (1974)
17. Krause, H.: The stable derived category of a Noetherian scheme. *Compos. Math.* **141**(5), 1128–1162 (2005)
18. Lafforgue, V.: Quelques calculs reliés à la correspondance de Langlands géométrique pour \mathbb{P}^1 (version provisoire). <http://www.math.jussieu.fr/~vlafforg/>
19. Laumon, G.: Transformation de Fourier généralisée. Preprint alg-geom/9603004
20. Laumon, G.: Transformation de Fourier géométrique. Preprint IHES/85/M/52 (1985)
21. Lurie, J.: Higher topos theory, Volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ (2009)
22. Lurie, J.: Higher algebra. <http://www.math.harvard.edu/~lurie>
23. Lurie, J.: DAG-XIV: Representability theorems. <http://www.math.harvard.edu/~lurie>
24. Lurie, J.: DAG-XVIII: Quasi-coherent sheaves and Tannaka duality theorems. <http://www.math.harvard.edu/~lurie>
25. Lurie, J.: Tannaka duality for geometric stacks. <http://www.math.harvard.edu/~lurie>
26. Mirković, I., Riche, S.: Linear Koszul duality. *Compos. Math.* **146**(1), 233–258 (2010)
27. Mirković, I., Vilonen, K.: Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. Math.* **166**(1), 95–143 (2007)
28. Neeman, A.: The Grothendieck duality theorem via Bousfield’s techniques and Brown representability. *J. Am. Math. Soc.* **9**(1), 205–236 (1996)
29. Notes on Geometric Langlands: Generalities on DG categories. <http://www.math.harvard.edu/~gaitsgde/GL/>
30. Notes on Geometric Langlands: Stacks. <http://www.math.harvard.edu/~gaitsgde/GL/>
31. Notes on Geometric Langlands: Quasi-coherent sheaves on stacks. <http://www.math.harvard.edu/~gaitsgde/GL/>
32. Orlov, D.O.: Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.), pp. 240–262 (2004)
33. Orlov, D.O.: Triangulated categories of singularities, and equivalences between Landau-Ginzburg models. *Math. Sb.* **197**(12), 117–132 (2006)
34. Positselski, L.: Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Am. Math. Soc.* **212**(996), vi+133 (2011)
35. Rothstein, M.: Sheaves with connection on abelian varieties. *Duke Math. J.* **84**(3), 565–598 (1996)
36. Rothstein, M.: Correction to: “Sheaves with connection on abelian varieties”. *Duke Math. J.* **87**(1), 205–211 (1997)
37. Stevenson, G.: Subcategories of singularity categories via tensor actions. Preprint [arXiv:1105.4698](https://arxiv.org/abs/1105.4698)
38. Thomason, R.W., Trobaugh, T.: Higher algebraic K -theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pp. 247–435. Birkhäuser Boston, Boston, MA (1990)
39. Umut Isik, M.: Equivalence of the derived category of a variety with a singularity category. Preprint [arXiv:1011.1484](https://arxiv.org/abs/1011.1484)