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# Elliptic associators

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**Abstract** We construct a genus one analogue of the theory of associators and the Grothendieck–Teichmüller (GT) group. The analogue of the Galois action on the profinite braid groups is an action of the arithmetic fundamental group of a moduli space of elliptic curves on the profinite braid groups in genus one. This action factors through an explicit profinite group $\widehat{\mathrm{GT}}_{ell}$, which admits an interpretation in terms of decorations of braided monoidal categories. This group acts on the tower of profinite braid groups in genus one and has the structure of a semidirect product of the profinite GT group $\widehat{\mathrm{GT}}$ by an explicit radical. We relate $\widehat{\mathrm{GT}}_{ell}$ to its prounipotent group scheme version $\mathrm{GT}_{ell}(-)$, which also has a semidirect product structure. We construct a torsor over this group, the scheme of elliptic associators. An explicit family of elliptic associators is constructed, based on earlier joint work with Calaque and Etingof on the universal KZB connexion. The existence of elliptic associators enables one to show that the Lie algebra of $\mathrm{GT}_{ell}(-)$ is isomorphic to a graded Lie algebra, on which we obtain several results: it is a semidirect product of the graded GT Lie algebra $\mathfrak{grt}$ by an explicit radical; we exhibit an explicit Lie subalgebra. Elliptic associators also allow one to compute the Zariski closure of the mapping class group in genus one (isomorphic to the braid group $B_3$) in the automorphism groups of the prounipotent completions of braid groups in genus one. The analytic study of the family of elliptic associators produces relations between MZVs and iterated integrals of Eisenstein series.

Dedicated to the memory of my father Albert Abraham Enriquez (1921–2010).

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## 1 Introduction

The theory of associators and of the Grothendieck–Teichmüller group has been developed by Drinfeld [9] in relation to certain problems of quantum groups. This theory was based on several previous pieces of work: on the one hand, on the approach proposed by Grothendieck to the study of $G_{\mathbb{Q}}$, the absolute Galois group of $\mathbb{Q}$, via its action on the Teichmüller tower in genus zero, and in particular on the profinite completions of the braid groups [10]; on the other hand, on rational homotopy theory, in particular the computation by Kohno of the prounipotent completions of the pure braid groups, based on the study of a particular connexion on the configuration spaces of the plane, which may be identified with a universal version of the Knizhnik–Zamolodchikov (KZ) connection.

The main actors of associator theory are as follows: a profinite group $\widehat{\mathrm{GT}}$, of categorical origin, containing $G_{\mathbb{Q}}$; pro-$l$, proalgebraic variants of this group, and the associated Lie algebra $\mathfrak{gt}$; a principal homogeneous space, the space of associators, which enables one to prove that $\mathfrak{gt}$ is isomorphic to a graded Lie algebra $\mathfrak{grt}$; a particular associator, the KZ associator, whose study allows one both to derive a system of relations between multizeta values (MZVs) and a collection of generators for $\mathfrak{grt}$. The theory of associators is therefore related to the theory of MZVs and motives [1]; it allows one to exhibit conditions satisfied by elements of motivic Lie algebras.

The purpose of the present work is to construct the analogous theory in genus one. On the Galois side, the object of interest is the arithmetic fundamental group of the moduli space of elliptic curves with $n$ marked points $M_{1,n}^{\mathbb{Q}}$, which gives rise to an action of the arithmetic fundamental group of the moduli space of elliptic curves $M_{1,1}^{\mathbb{Q}}$ on the profinite completions of braid groups in genus one; when $n = 2$, this action is studied in Nakamura [24], Sect. 5.1, and a higher genus analogue is studied in Oda [27], on the basis of [11]. The analogue of the rational homotopy part is the computation of the prounipotent completion of braid groups in genus one, first obtained by Bezrukavnikov using minimal model theory, and later rederived in Calaque et al. [7] using an analogue of the KZ connection, the universal KZB connection (this connection was independently obtained in Levin and Racinet [19]). A new feature of the KZB connection is its horizontal part (related to variation of the elliptic modulus), which corresponds to an extension of the holonomy Lie algebra $\mathfrak{t}_{1,n}$ by a Lie algebra of derivations $\langle \delta_{2n}, n \geq -1 \rangle$.

Our construction of the genus one analogue of Grothendieck–Teichmüller theory consists of several steps. We first construct a genus one analogue of the theory of braided monoidal categories (BMCs). This enables us to define the genus one analogue $\widehat{\mathrm{GT}}_{ell}$ of $\widehat{\mathrm{GT}}$, which is a profinite group containing $\pi_1(M_{1,\vec{1}}^{\mathbb{Q}})$. We construct the pro-$l$ and proalgebraic variants of this group; the associated Lie algebra is denoted as $\mathfrak{gt}_{ell}$.

We construct a torsor under this proalgebraic group: the scheme of elliptic associators. We present two constructions of elliptic associators: (a) we define an explicit map from the set of associators to its elliptic analogue; (b) the KZB connection gives rise to a map $\tau \to e(\tau)$ from the Poincaré half-plane to the set of elliptic associators. We study the properties of this map: differential system, modular behaviour, and behaviour at infinity; this shows in particular that the constructions (a) and (b) are related to each other by suitable specializations and limiting procedures. The existence of elliptic associators then enable us to construct an isomorphism between $\mathfrak{gt}_{ell}$ and an explicit Lie algebra $\mathfrak{grt}_{ell}$. We prove several results on $\mathfrak{grt}_{ell}$: (a) $\mathfrak{grt}_{ell}$ is a semidirect product of $\mathfrak{grt}$ by a Lie algebra $\mathfrak{r}_{ell}$, which is therefore acted upon by $\mathfrak{grt}$; (b) we construct an explicit Lie subalgebra of $\mathfrak{r}_{ell}$.

Beside these results, which may be viewed as internal to the theory, our work leads to the following results:

(a) The outer action of the arithmetic fundamental group of $M_{1,\vec{1}}^{\mathbb{Q}}$ on the $\mathbb{Q}_l$-points of the prounipotent completions of the braid group in genus one with various numbers of strands factors through the action of the group of $\mathbb{Q}_l$-points of one and the same proalgebraic group, which is $\mathrm{GT}_{ell}(-)$;
(b) The mapping class group of surfaces of genus one with one boundary component, which is isomorphic to the group $B_3$ of braids with three strands, naturally acts on the pure braid groups in genus one. We compute the Zariski closure of $B_3$ in the automorphism group of their prounipotent completions, in terms of the Lie algebra $\langle \delta_{2n}, n \geq -1 \rangle$;
(c) The study of the above-mentioned map from the Poincaré half-plane to the space of elliptic associators leads to relations between MZVs and iterated integrals of Eisenstein series.

This paper is organized as follows. In Sect. 2, we define the genus one counterpart of the notion of braided monoidal category. This enables us to define the group $\widehat{\mathrm{GT}}_{ell}$ in Sect. 3, as well as its pro-$l$ and prounipotent variants. In Sect. 4, we introduce the space of elliptic associators, prove its nonemptiness, and study its torsor structure. This leads us to the definition of the group scheme $\mathrm{GRT}_{ell}(-)$ in Sect. 5; we prove the announced results on its Lie algebra $\mathfrak{grt}_{ell}$: isomorphism with $\mathfrak{gt}$, generators, semidirect product structure. In Sect. 6, we introduce the map $\tau \mapsto e(\tau)$ and study its properties. In Sect. 7, we carry out the computation of Zariski closure of $B_3$ explained above. We define the iterated integrals of Eisenstein series in Sect. 8 and prove there their relations with MZVs. In Sect. 9, we recall the relations between $G_{\mathbb{Q}}$, $\widehat{\mathrm{GT}}$ and the Teichmüller groupoid in genus zero, and generalize these results to genus one. Section 10 raises a question on the structure of the kernel $\mathfrak{r}_{ell}$ of a natural morphism $\mathfrak{grt}_{ell} \to \mathfrak{grt}$, and its relation with a transcendence conjecture on the KZ associator (which is related to the Grothendieck period conjecture); namely, it is shown that an affirmative answer to both questions imply the same (also conjectural) statement on the behaviour of certain isomorphisms arising from associators (see Propositions 10.4 and 10.5).

Let us now mention some works and projects related to the present work. Hain and Matsumoto [14] construct a theory of "mixed elliptic motives". This gives rise to a proalgebraic $\mathbb{Q}$-group scheme $G_{MEM}(-)$, equipped with a morphism $G_{MEM}(-) \to G_{MTM}(-)$. One may expect a commutative diagram from this morphism to $\mathrm{GT}_{ell}(-) \to \mathrm{GT}(-)$. The Lie algebra $\langle \delta_{2n}, n \geq -1 \rangle$ is a Lie subalgebra of the graded version of the kernels of both morphisms and was studied in Pollack's Ph.D. thesis [28]. On the other hand, Brown and Levin [6] develop a parallel theory of elliptic motives; the elliptic multiple zeta values arising from this theory could be related to the family $\tau \mapsto e(\tau)$ of elliptic associators studied here.

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## 2 Elliptic structures over braided monoidal categories

In Calaque et al. [7], we introduced a notion of elliptic structure over a braided monoidal category (BMC) $\mathcal{C}$. It consists in a category $\mathcal{E}$, a functor $\mathcal{E} \to \mathcal{C}$, and additional data. In this section, we introduce a variant of this notion, which consists in a category $\tilde{\mathcal{C}}$, a functor $\mathcal{C} \to \tilde{\mathcal{C}}$, and additional data. The two definitions can be related by adjunction, as will be explained in forthcoming joint work with P. Etingof. As is the notion from [7], the variant presented here is related with elliptic braid groups in the same way as BMCs are related to usual braid groups.

### 2.1 Definition

Let $(\mathcal{C}, \otimes, \beta_{\dots}, a_{\dots}, \mathbf{1})$ be a braided monoidal category (see e.g. [17]). Here $\otimes : \mathcal{C} \times \mathcal{C}$ is the tensor product, $\beta_{X,Y} : X \otimes Y \to Y \otimes X$ and $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ are the braiding and associativity isomorphisms and $\mathbf{1}$ is the unit object. They satisfy in particular the pentagon and hexagon identities

$$a_{X,Y,Z\otimes T} a_{X\otimes Y,Z,T} = (\mathrm{id}_X \otimes a_{Y,Z,T}) a_{X,Y\otimes Z,T} (a_{X,Y,Z} \otimes \mathrm{id}_T),$$
$$(\mathrm{id}_Y \otimes \beta^{\pm}_{X,Z}) a_{Y,X,Z} (\beta^{\pm}_{X,Y} \otimes \mathrm{id}_Z) = a_{Y,Z,X} \beta^{\pm}_{X,Y\otimes Z} a_{X,Y,Z},$$

where $\beta^{+}_{X,Y} = \beta_{X,Y}$, $\beta^{-}_{X,Y} = \beta^{-1}_{Y,X}$.

**Definition 2.1** An elliptic structure over the braided monoidal category $\mathcal{C}$ is a set $(\tilde{\mathcal{C}}, F, A^{+}_{\dots}, A^{-}_{\dots})$, where $\mathcal{C}$ is a category, $F : \mathcal{C} \to \tilde{\mathcal{C}}$ is a functor,[1] and $A^{\pm}_{\dots}$ are natural[2] assignments $(\mathrm{Ob}\,\mathcal{C})^2 \ni (X, Y) \mapsto A^{\pm}_{X,Y} \in \mathrm{Aut}_{\tilde{\mathcal{C}}}(F(X \otimes Y))$, such that:

$$\alpha^{\pm}_{Z,X,Y} \alpha^{\pm}_{Y,Z,X} \alpha^{\pm}_{X,Y,Z} = \mathrm{id}_{(X\otimes Y)\otimes Z}, \tag{1}$$

[1] For $\mathcal{C}$ a category, $\mathrm{Ob}\,\mathcal{C}$ is its class of objects; for $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Iso}_{\mathcal{C}}(X, Y) \subset \mathcal{C}(X, Y)$ are the sets of isomorphisms and morphisms $X \to Y$; $\mathrm{Aut}_{\mathcal{C}}(X) = \mathrm{Iso}_{\mathcal{C}}(X, X)$.

[2] Natural means that if $\varphi \in \mathcal{C}_0(X, X')$, $\psi \in \mathcal{C}_0(Y, Y')$, then $A^{\pm}_{X',Y'} F(\varphi \otimes \psi) = F(\varphi \otimes \psi) A^{\pm}_{X,Y}$.

where $\alpha^{\pm}_{X,Y,Z} = F(\beta^{\pm}_{X,Y\otimes Z})A^{\pm}_{X,Y\otimes Z}F(a_{X,Y,Z})$,

$$F(\beta_{Y,X}\beta_{X,Y}\otimes \mathrm{id}_Z) = \Big(F(a^{-1}_{X,Y,Z})A^{-}_{X,Y\otimes Z}F(a_{X,Y,Z}),\\ F\big((\beta^{-1}_{X,Y}\otimes \mathrm{id}_Z)a^{-1}_{Y,X,Z}\big)(A^{+}_{Y,X\otimes Z})^{-1}F\big(a_{Y,X,Z}(\beta^{-1}_{Y,X}\otimes \mathrm{id}_Z)\big)\Big), \tag{2}$$

(identities[3] in $\mathrm{Aut}_{\tilde{\mathcal{C}}}(F((X\otimes Y)\otimes Z))$), for any $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$, and

$$A^{\pm}_{\mathbf{1},X} = \mathrm{id}_{F(\mathbf{1}\otimes X)} \qquad \text{for any } X \in \mathrm{Ob}\,\mathcal{C}. \tag{3}$$

Dropping associativity constraints and the functor $F$ (which can be put in automatically), the two first conditions mean that the cycles

$$\begin{array}{ccccc} X\otimes Y\otimes Z & \xrightarrow{A^{\pm}_{X,Y\otimes Z}} & X\otimes Y\otimes Z & \xrightarrow{\beta^{\pm}_{X,Y\otimes Z}} & Y\otimes Z\otimes X \\ \uparrow{\scriptstyle \beta^{\pm}_{Z,X\otimes Y}} & & & & \downarrow{\scriptstyle A^{\pm}_{Y,Z\otimes X}} \\ Z\otimes X\otimes Y & \xleftarrow[A^{\pm}_{Z,X\otimes Y}]{} & Z\otimes X\otimes Y & \xleftarrow[\beta^{\pm}_{Y,Z\otimes X}]{} & Y\otimes Z\otimes X \end{array} \qquad \text{and}$$

$$\begin{array}{ccccc} Y\otimes X\otimes Z & \xrightarrow{A_{Y,X\otimes Z}} & Y\otimes X\otimes Z & \xrightarrow{\beta_{YX}\otimes \mathrm{id}_Z} & X\otimes Y\otimes Z \\ \uparrow{\scriptstyle \beta^{-1}_{YX}\otimes \mathrm{id}_Z} & & & & \downarrow{\scriptstyle B^{-1}_{X,Y\otimes Z}} \\ X\otimes Y\otimes Z & & & & X\otimes Y\otimes Z \\ \uparrow{\scriptstyle B_{X,Y\otimes Z}} & & & & \downarrow{\scriptstyle \beta^{-1}_{YX}\otimes \mathrm{id}_Z} \\ X\otimes Y\otimes Z & \xleftarrow[\beta^{-1}_{XY}\otimes \mathrm{id}_Z]{} & Y\otimes X\otimes Z & \xleftarrow[A^{-1}_{Y,X\otimes Z}]{} & Y\otimes X\otimes Z \end{array}$$

are identity morphisms, where $A_{\dots} = A^{+}_{\dots}$, $B_{\dots} = A^{-}_{\dots}$.

A morphism $(\mathcal{C}, \tilde{\mathcal{C}}, F, A^{\pm}_{\dots}) \to (\mathcal{C}', \tilde{\mathcal{C}}', F', A'^{\pm}_{\dots})$ is then the data of a tensor functor $\mathcal{C} \overset{\varphi}{\to} \mathcal{C}'$ and a functor $\tilde{\mathcal{C}} \overset{\tilde{\varphi}}{\to} \tilde{\mathcal{C}}'$, such that $\begin{array}{ccc} \mathcal{C} & \to & \mathcal{C}' \\ \downarrow & & \downarrow \\ \tilde{\mathcal{C}} & \to & \tilde{\mathcal{C}}' \end{array}$ commutes, and $\tilde{\varphi}(A^{\pm}_{X,Y}) = A'^{\pm}_{\varphi(X),\varphi(Y)}$.

*Remark 2.2* By setting $Z = \mathbf{1}$, the axioms (1)–(3) imply

$$F(\beta^{\pm}_{Y,X})A^{\pm}_{Y,X}F(\beta^{\pm}_{X,Y})A^{\pm}_{X,Y} = \mathrm{id}_{F(X\otimes Y)}, \quad F(\beta_{Y,X}\beta_{X,Y}) = (A^{-}_{X,Y}, A^{+}_{X,Y}), \tag{4}$$

which in their turn imply

$$A^{\pm}_{X,\mathbf{1}} = \mathrm{id}_{F(X\otimes \mathbf{1})}\,. \tag{5}$$

[3] In (2) and later, we set $(g, h) := ghg^{-1}h^{-1}$.

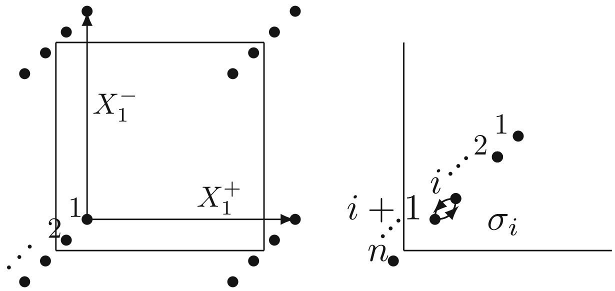



**Fig. 1** Generators of the elliptic braid group $B_{1,n}$

Taking these identities, (3) and the hexagon identities into account, axiom (1) can be replaced by

$$A^{\pm}_{X\otimes Y,Z} = F((\beta^{\pm}_{Y,X}\otimes \mathrm{id}_Z)a^{-1}_{Y,X,Z})A^{\pm}_{Y,X\otimes Z}$$
$$F(a_{Y,X,Z}(\beta^{\pm}_{X,Y}\otimes \mathrm{id}_Z)a^{-1}_{X,Y,Z})A^{\pm}_{X,Y\otimes Z}F(a_{X,Y,Z}). \qquad (6)$$

2.2 Relation with elliptic braid groups

For $n \geq 1$, the reduced pure elliptic braid group on $n$ strands $P_{1,n}$ is the fundamental group of the reduced configuration space $\overline{\mathrm{Cf}}_n(T) := \mathrm{Cf}_n(T)/T$, where $\mathrm{Cf}_n(T) = T^n - (\text{diagonals})$ is the configuration space of $n$ points on the topological torus $T := \mathbb{R}^2/\mathbb{Z}^2$, on which $T$ acts diagonally. The reduced elliptic braid group $B_{1,n}$ is the fundamental group of the quotient $\overline{\mathrm{Cf}}_{[n]}(T) := \overline{\mathrm{Cf}}_n(T)/S_n$. We then have an exact sequence

$$1 \to P_{1,n} \to B_{1,n} \to S_n \to 1.$$

These definitions are extended by $P_{1,0} = B_{1,0} = \{1\}$.

The group $B_{1,n} (n \geq 1)$ can be presented by generators $\sigma_i (i = 1, \dots, n-1)$, $X_1^{\pm}$, and relations

$$(\sigma_1^{\pm 1}X_1^{\pm})^2 = (X_1^{\pm}\sigma_1^{\pm 1})^2, \quad (X_1^{\pm}, \sigma_i) = 1 \quad \text{for } i = 2, \dots, n-1,$$
$$(X_1^-, (X_2^+)^{-1}) = \sigma_1^2, \quad X_1^{\pm} \cdots X_n^{\pm} = 1, \quad (\sigma_i, \sigma_j) = 1 \quad \text{for } |i-j| > 1,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } i = 1, \dots, n-2, \qquad (7)$$

where $X^{\pm}_{i+1} = \sigma_i^{\pm 1}X_i^{\pm}\sigma_i^{\pm 1}$ for $i = 1, \dots, n-1$ (see [4] and Fig. 1). In particular, $P_{1,1} = B_{1,1} = \{1\}$, and $P_{1,2}$ is the free group with two generators $X_1^{\pm}$.

The braid group $B_n$ on $n$ strands ($n \geq 1$) is presented by generators $\sigma_i$, $i = 1, \dots, n-1$ and the Artin relations (7). Its definition is extended to $n = 0$ by $B_0 = \{1\}$. There is a unique morphism $B_n \to B_{1,n}$ such that $\sigma_i \mapsto \sigma_i$. If $\mathcal{C}$ is a braided monoidal category and $X \in \mathrm{Ob}\,\mathcal{C}$, then there is a unique group morphism $\varphi : B_n \to \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})$ ($X^{\otimes n}$ is defined by right parenthesization, so $X^{\otimes n} = X \otimes X^{\otimes n-1}$), such that

$$\sigma_i \mapsto a_i((\mathrm{id}_{X^{\otimes i-1}} \otimes \beta_{X,X}) \otimes \mathrm{id}_{X^{\otimes n-i-1}})a_i^{-1},$$

where $a_i : (X^{\otimes i-1} \otimes X^{\otimes 2}) \otimes X^{\otimes n-i-1} \to X^{\otimes n}$ is the associativity constraint.

**Proposition 2.3** *If $(\tilde{\mathcal{C}}, F, A^{\pm}_{\dots})$ is an elliptic structure over $\mathcal{C}$ and $X \in \mathrm{Ob}\,\mathcal{C}$, then there is a unique group morphism $B_{1,n} \to \mathrm{Aut}_{\tilde{\mathcal{C}}}(F(X^{\otimes n}))$, such that*

$$X_1^{\pm} \mapsto A^{\pm}_{X,X^{\otimes n-1}}, \quad \sigma_i \mapsto F(\varphi(\sigma_i)).$$

*Proof* Let us check that $(\sigma_1 X_1^+)^2 = (X_1^+\sigma_1)^2$, i.e., $(X_1^+, X_2^+) = 1$ is preserved (for simplicity, we omit the associativity constraints). By naturality, $(\beta_{X,Y} \otimes \mathrm{id}_Z)A^+_{X\otimes Y,Z} = A^+_{Y\otimes X,Z}(\beta_{X,Y} \otimes \mathrm{id}_Z)$. Plugging in this equality the relation (6) and its analogue with $X, Y$ exchanged, we obtain

$$(A^+_{X,Y\otimes Z}, F(\beta_{Y,X} \otimes \mathrm{id}_Z)A^+_{Y,X\otimes Z}F(\beta_{X,Y} \otimes \mathrm{id}_Z)) = 1;$$

if we set $Y := X,\ Z := X^{\otimes n-2}$, this says that $(X_1^+, X_2^+) = 1$ is preserved. Similarly, one proves that (1) with sign implies that $(\sigma_1^{-1}X_1^-)^2 = (X_1^-\sigma_1^{-1})^2$ is preserved. (2) immediately implies that $(X_1^-, (X_2^+)^{-1}) = \sigma_1^2$ is preserved. The naturality assumption implies that $(X_1^{\pm}, \sigma_i) = 1$ $(i > 1)$ is preserved. One shows by induction that the image of $X_k^{\pm}$ is $F(\beta^{\pm}_{X^{\otimes k-1},X} \otimes \mathrm{id}_{X^{\otimes n-k}})A^{\pm}_{X,X^{\otimes n-1}}F(\beta^{\pm}_{X,X^{\otimes k-1}} \otimes \mathrm{id}_{X^{\otimes n-k}})$; therefore, the image of $X_1^{\pm} \cdots X_k^{\pm}$ is $A^{\pm}_{X^{\otimes k},X^{\otimes n-k}}$. It follows that the image of $X_1^{\pm} \cdots X_n^{\pm}$ is $A^{\pm}_{X^{\otimes n},\mathbf{1}}$, which is $\mathrm{id}_{F(X^{\otimes n})}$ by (5). Finally, as $B_n \to \mathrm{Aut}_{\mathcal{C}}(F(X^{\otimes n}))$ is a group morphism, the Artin relations are preserved. □

## 2.3 Universal elliptic structures

Let **PaB** be the braided monoidal category of parenthesized braids (see [3,16]). Its set of objects is $\mathbf{Par} := \sqcup_{n\geq 0}\mathbf{Par}_n$, where $\mathbf{Par}_n = \{$parenthesizations of the word $\bullet \dots \bullet$ of length $n\}$, so $\mathbf{Par}_0 = \{\mathbf{1}\}$, $\mathbf{Par}_1 = \{\bullet\}$, $\mathbf{Par}_2 = \{\bullet\bullet\}$, $\mathbf{Par}_3 = \{(\bullet\bullet)\bullet, \bullet(\bullet\bullet)\}$, $\mathbf{Par}_4 = \{((\bullet\bullet)\bullet)\bullet, (\bullet(\bullet\bullet))\bullet, (\bullet\bullet)(\bullet\bullet), \bullet((\bullet\bullet)\bullet), \bullet(\bullet(\bullet\bullet))\}$, etc. For $O, O' \in \mathbf{Par}$, we set $|O| :=$ the integer such that $O \in \mathbf{Par}_{|O|}$, and $\mathcal{C}_0(O, O') := \begin{cases} B_{|O|} & \text{if } |O| = |O'| \\ \emptyset & \text{otherwise} \end{cases}$. The composition is the product in $B_{|O|}$. The tensor product is defined at the level of objects, as the juxtaposition, and at the level of morphisms, by the group morphism $B_n \times B_{n'} \to B_{n+n'}$, $(\sigma_i, 1) \mapsto \sigma_i$, $(1, \sigma_j) \mapsto \sigma_{n+j}$. We set $a_{O,O',O''} := 1 \in B_{|O|+|O'|+|O''|} = \mathbf{PaB}((O \otimes O') \otimes O'', O \otimes (O' \otimes O''))$ and $\beta_{O,O'} := \sigma_{|O|,|O'|} \in B_{|O|+|O'|} = \mathbf{PaB}(O \otimes O', O' \otimes O)$, where $\sigma_{n,n'} := (\sigma_n \cdots \sigma_1)(\sigma_{n+1} \cdots \sigma_2) \cdots (\sigma_{n+n'-1} \cdots \sigma_{n'}) \in B_{n+n'}$.

Let now $\mathbf{PaB}_{ell}$ be the category with the same objects, $\mathbf{PaB}_{ell}(O, O') := \begin{cases} B_{1,|O|} & \text{if } |O| = |O'| \\ \emptyset & \text{otherwise} \end{cases}$ and whose product is the composition in $B_{1,|O|}$. Let $F : \mathbf{PaB} \to \mathbf{PaB}_{ell}$ be the functor induced by the identity at the level of objects, and by $B_n \to B_{1,n}$, $\sigma_i \mapsto \sigma_i$ at the level of morphisms. For $O, O' \in \mathbf{Par}$, set

$A^{\pm}_{O,O'} := X_1^{\pm}\cdots X_{|O|}^{\pm} \in B_{1,|O|+|O'|}$. Then $(\mathbf{PaB}_{ell}, F, A^{\pm}_{\dots})$ is an elliptic structure over **PaB**. Indeed, relations (1) and (2) for objects $O$, $O'$, $O''$ are consequences of the identities $(\sigma_2^{\pm 1}\sigma_1^{\pm 1}X_1^{\pm})^3 = 1$ and $(X_1^-, (\sigma_1 X_1^+ \sigma_1)^{-1}) = \sigma_1^2$ in $P_{1,3}$ under the morphism $P_{1,3} \to P_{1,|O|+|O'|+|O''|}$ induced by the replacement of the first (respectively, second, third) strand by $|O|$ (respectively, $|O'|$, $|O''|$) consecutive strands.

The pair $(\mathbf{PaB}, \bullet)$ has the following universal property: for any pair $(\mathcal{C}, M)$, where $\mathcal{C}$ is a braided monoidal category and $M \in \mathrm{Ob}\,\mathcal{C}$, there exists a unique tensor functor $\varphi_0 : \mathbf{PaB} \to \mathcal{C}$, such that $F(\bullet) = M$. Proposition 2.3 immediately implies that this property extends as follows.

**Proposition 2.4** *If $\tilde{\mathcal{C}}$ is an elliptic structure over $\mathcal{C}$, then there exists a unique morphism $(\mathbf{PaB}, \mathbf{PaB}_{ell}) \to (\mathcal{C}, \tilde{\mathcal{C}})$, extending $\varphi$.*

## 3 The elliptic Grothendieck–Teichmüller group

In this section, we introduce the group $\mathrm{GT}_{ell}$ of universal automorphisms of elliptic structures over BMCs, which we call the elliptic Grothendieck-Teichmüller group. We compute the "naive" version of this group and then introduce its variants (profinite, pro-$l$, proalgebraic) by playing on the classes of considered BMCs. We study the relations between these groups and the corresponding variants of GT; we construct in particular, in the various frameworks, a section of the natural morphism $\mathrm{GT}_{ell} \to \mathrm{GT}$. This shows that $\mathrm{GT}_{ell}$ and its variants have semidirect product structures.

### 3.1 Reminders about $\underline{\mathrm{GT}}$ and its variants

According to [9], $\underline{\mathrm{GT}}$ is the set of pairs $(\lambda, f) \in (1 + 2\mathbb{Z}) \times F_2$, $F_2$ being the free group with generators $X$ and $Y$, such that

$$f(X_3, X_1)X_3^m f(X_2, X_3)X_2^m f(X_1, X_2)X_1^m = 1, \quad m = \frac{\lambda - 1}{2}, \quad X_1X_2X_3 = 1,$$
$$f(Y, X) = f(X, Y)^{-1}, \quad \partial_3(f)\partial_1(f) = \partial_0(f)\partial_2(f)\partial_4(f), \tag{8}$$

where[4] $\partial_i : F_2 \subset P_3 \to P_4$ are simplicial morphisms. It is equipped with a semigroup structure with $(\lambda, f)(\lambda', f') = (\lambda'', f'')$, with

$$\lambda'' := \lambda\lambda', \quad f''(X, Y) := f(f'(X, Y)X^{\lambda'} f'(X, Y)^{-1}, Y^{\lambda'})f'(X, Y).$$

One defines similarly semigroups $\underline{\widehat{\mathrm{GT}}}$, $\underline{\mathrm{GT}}_l$, $\underline{\mathrm{GT}}(\mathbf{k})$ by replacing in the above definition $(\mathbb{Z}, F_2)$ by their profinite, pro-$l$, $\mathbf{k}$-prounipotent versions (where $\mathbf{k}$ is a $\mathbb{Q}$-ring). We then have morphisms $\underline{\mathrm{GT}} \hookrightarrow \underline{\widehat{\mathrm{GT}}} \to \underline{\mathrm{GT}}_l \hookrightarrow \underline{\mathrm{GT}}(\mathbb{Q}_l)$ and $\underline{\mathrm{GT}} \to \underline{\mathrm{GT}}(\mathbf{k})$ for any $\mathbf{k}$.

[4] $P_n = \mathrm{Ker}(B_n \to S_n, \sigma_i \mapsto (i, i+1))$ is the pure braid group on $n$ strands.

$\underline{\mathrm{GT}}$ acts on {braided monoidal categories (BMCs)} by $(\lambda, f)(\mathcal{C}_0, \beta_{\dots}, a_{\dots}) := (\mathcal{C}_0, \beta'_{\dots}, a'_{\dots})$, where

$$\beta'_{X,Y} := \beta_{X,Y}(\beta_{Y,X}\beta_{X,Y})^m,$$
$$a'_{X,Y,Z} := a_{X,Y,Z} f(\beta_{YX}\beta_{XY} \otimes \mathrm{id}_Z, \quad a^{-1}_{X,Y,Z}(\mathrm{id}_X \otimes \beta_{ZY}\beta_{YZ})a_{X,Y,Z}).$$

Similarly, $\underline{\widehat{\mathrm{GT}}}$ (respectively, $\underline{\mathrm{GT}}_l$, $\underline{\mathrm{GT}}(\mathbf{k})$) act on {BMCs $\mathcal{C}_0$ such that $\mathrm{Aut}_{\mathcal{C}_0}(X)$ is finite for any $X \in \mathrm{Ob}\,\mathcal{C}_0$} (respectively, such that the image of $P_n \to \mathrm{Aut}_{\mathcal{C}_0}(X_1 \otimes \cdots \otimes X_n)$ is an $l$-group and is contained in a unipotent group).

## 3.2 The semigroup $\underline{\mathrm{GT}}_{ell}$ and its variants

Let us define $\underline{\mathrm{GT}}_{ell}$ as the set of all $(\lambda, f, g_\pm)$, where $(\lambda, f) \in \underline{\mathrm{GT}}$, $g_\pm \in F_2$ are such that

$$(\sigma_2^{\pm 1}\sigma_1^{\pm 1}(\sigma_1\sigma_2^2\sigma_1)^{\pm m} g_\pm(X_1^+, X_1^-) f(\sigma_1^2, \sigma_2^2))^3 = 1, \tag{9}$$
$$u^2 = (g_-, u^{-1}g_+^{-1}u^{-1}) \tag{10}$$

(identities in $B_{1,3}$) where $u = f(\sigma_1^2, \sigma_2^2)\sigma_1^\lambda f(\sigma_1^2, \sigma_2^2)^{-1}$, $g_\pm = g_\pm(X_1^+, X_1^-)$.

If $\mathcal{C}$ is a BMC and $(\tilde{\mathcal{C}}, F, A^\pm_{\dots})$ is an elliptic structure over $\mathcal{C}$, then $(\tilde{\mathcal{C}}, F, A'^\pm_{\dots})$ is an elliptic structure over $\mathcal{C}'$, where

$$\mathcal{C}' := (\lambda, f) * \mathcal{C}, \quad A'^\pm_{X,Y} = g_\pm(A^+_{X,Y}, A^-_{X,Y})(\in \mathrm{Aut}_{\mathcal{C}}(X \otimes Y)). \tag{11}$$

The following statement is then the analogue of Eqs. (4).

**Lemma 3.1** *The conditions (9), (10) imply the identities*

$$(\sigma_1^{\pm\lambda} g_\pm(X_1^+, X_1^-))^2 = 1, \quad \sigma_1^{2\lambda} = (g_-(X_1^+, X_1^-), g_+(X_1^+, X_1^-)) \tag{12}$$

*in* $B_{1,2}$.

*Proof* Let $\sigma_\pm := \sigma_2^{\pm 1}\sigma_1^{\pm 1}(\sigma_1\sigma_2^2\sigma_1)^{\pm m}$, $g_\pm := g_\pm(X_1^+, X_1^-)$, $f := f(\sigma_1^2, \sigma_2^2)$, then the first equation of (9) is rewritten as $\mathrm{Ad}(\sigma_\pm)^{-1}(g_\pm f) \cdot g_\pm f \cdot \mathrm{Ad}(\sigma_\pm)(g_\pm f) = \sigma_\pm^{-3}$, an identity in $P_{1,3}$. There is a unique morphism $P_{1,3} \to P_{1,2}$, corresponding to the erasing of the third point, i.e. to the map $\mathrm{Cf}_3(T) \to \mathrm{Cf}_2(T)$, $(x_1, x_2, x_3) \mapsto (x_1, x_2)$. It is given by $X_1^\pm \mapsto X_1^\pm$, $X_2^\pm \mapsto 1$, $X_3^\pm \mapsto (X_1^\pm)^{-1}$, $\sigma_1^2 \mapsto \sigma_1^2$, $\sigma_2^2 \mapsto 1$, $(\sigma_1\sigma_2)^3 \mapsto \sigma_1^2$. The image of the above identity in $P_{1,3}$ by this morphism is the identity $g_\pm(X_1^+, X_1^-) \cdot \mathrm{Ad}(\sigma_1^{\pm\lambda})(g_+(X_1^+, X_1^-)) = \sigma_1^{\mp 2\lambda}$ in $P_{1,2}$, which is equivalent to the first equations of (12). The same morphism similarly takes (10) to the last equation of (12). □

For $(\lambda, f, g_\pm), (\lambda', f', g'_\pm) \in \underline{\mathrm{GT}}_{ell}$, we set

$$(\lambda, f, g_\pm)(\lambda', f', g'_\pm) := (\lambda'', f'', g''_\pm), \text{ where } g''_\pm(X, Y) = g_\pm(g'_+(X, Y), g'_-(X, Y)).$$

**Proposition 3.2** *This defines a semigroup structure on* $\underline{\mathrm{GT}}_{ell}$*. We have a semigroup inclusion* $\underline{\mathrm{GT}}_{ell} \subset \underline{\mathrm{GT}} \times \mathrm{End}(F_2)^{op}$, $(\lambda, f, g_\pm) \mapsto ((\lambda, f), \theta_{g_\pm})$*, where* $\theta_{g_\pm} = (X \mapsto g_+(X, Y), Y \mapsto g_-(X, Y))$.

*Proof* We first prove: □

**Lemma 3.3** *If* $(\lambda, f, g_\pm) \in \underline{\mathrm{GT}}_{ell}$*, then there is a unique endomorphism of* $B_{1,3}$*, such that*

$$\sigma_1 \mapsto \tilde{\sigma}_1 := f(\sigma_1^2, \sigma_2^2)\sigma_1^\lambda f(\sigma_1^2, \sigma_2^2)^{-1}, \quad \sigma_2 \mapsto \tilde{\sigma}_2 := \sigma_2^\lambda, \quad X_1^\pm \mapsto g_\pm(X_1^+, X_1^-).$$

*For any* $\lambda' \in 2\mathbb{Z} + 1$*, we then have*

$$f(\sigma_1^2, \sigma_2^2)\sigma_2^{\pm 1}\sigma_1^{\pm 1}(\sigma_1\sigma_2^2\sigma_1)^{\pm\frac{\lambda\lambda'-1}{2}} = \tilde{\sigma}_2^{\pm 1}\tilde{\sigma}_1^{\pm 1}(\tilde{\sigma}_1\tilde{\sigma}_2^2\tilde{\sigma}_1)^{\pm\frac{\lambda'-1}{2}}. \tag{13}$$

*Proof* Recall that we have an elliptic structure $(\mathbf{PaB}, \mathbf{PaB}_{ell}, F, A_{\dots}^{\pm})$. Applying $(\lambda, f, g_\pm)$ to it, we get an elliptic structure $(\underline{\mathbf{PaB}}, \underline{\mathbf{PaB}}^{ell}, F, \underline{A}_{\dots}^{\pm})$. An endomorphism of $B_{1,3}$ is given by the composition

$$B_{1,3} \to \mathrm{Aut}_{\underline{\mathbf{PaB}}^{ell}}(\bullet(\bullet\bullet)) \simeq B_{1,3},$$

where the first morphism arises from the elliptic structure of $\underline{\mathbf{PaB}}^{ell}$, and the second morphism arises from the isofunctor $\underline{\mathbf{PaB}}^{ell} \simeq \mathbf{PaB}^{ell}$. One checks that this endomorphism of $B_{1,3}$ is given by the above formulas.

We now prove (13). The hexagon identity implies

$$(\sigma_2^2)^m f(\sigma_1^2, \sigma_2^2)(\sigma_1^2)^m f((\sigma_1^2\sigma_2^2)^{-1}, \sigma_1^2)(\sigma_1^2\sigma_2^2)^{-m} f(\sigma_2^2, (\sigma_1^2\sigma_2^2)^{-1}) = 1.$$

Now, since $(\sigma_1^2\sigma_2^2)^{-1} \equiv \sigma_1\sigma_2^2\sigma_1^{-1} \equiv \sigma_2^{-1}\sigma_1^2\sigma_2 \bmod Z(B_3)$, since $f(a, b) = f(a', b')$ for any group $G$ and any $a, a', b, b' \in G$ with $a \equiv a'$, $b \equiv b' \bmod Z(G)$ (as $f \in F_2' = (F_2, F_2)$), and by the duality identity, this is rewritten

$$(\sigma_2^2)^m f(\sigma_1^2, \sigma_2^2)\sigma_1^{2m+1} f(\sigma_1^2, \sigma_2^2)^{-1}\sigma_1^{-1}(\sigma_1^2\sigma_2^2)^{-m}\sigma_2^{-1} f(\sigma_1^2, \sigma_2^2)^{-1}\sigma_2 = 1,$$

which yields (13) with $(\pm, \lambda') = (+, 1)$.

(13) with $\pm = +$ then follows from

$$\tilde{\sigma}_1\tilde{\sigma}_2^2\tilde{\sigma}_1 = (\sigma_1\sigma_2^2\sigma_1)^\lambda, \tag{14}$$

which is proved as follows. The hexagon identity (8) implies that if $X_1X_2X_3$ commutes with all the $X_i$, then

$$f(X_3, X_1)X_3^m f(X_2, X_3)X_2^m f(X_1, X_2) = (X_2X_3)^m.$$

Applying this to $X_1 = \sigma_2^2$, $X_2 = \sigma_1\sigma_2^2\sigma_1^{-1}$, $X_3 = \sigma_1^2$, and using $\sigma_1^2 = \mathrm{Ad}(\sigma_2^{-1}\sigma_1^{-1})(\sigma_2^2)$, this implies

$$\begin{aligned}\tilde\sigma_1\tilde\sigma_2 f(\sigma_1^2,\sigma_2^2) &= f(\sigma_1^2,\sigma_2^2)(\sigma_1^2)^m f(\sigma_1\sigma_2^2\sigma_1^{-1},\sigma_2^2)(\sigma_1\sigma_2^2\sigma_1^{-1})^m f(\sigma_2^2,\sigma_1\sigma_2^2\sigma_1^{-1})\sigma_1\sigma_2\\ &= (\sigma_1\sigma_2^2\sigma_1)^m\sigma_1\sigma_2.\end{aligned}$$

Using the same identity with $X_1 = \sigma_2^2$, $X_2 = \sigma_1^2$, $X_3 = \sigma_1^{-1}\sigma_2^2\sigma_1$, one proves similarly that

$$f(\sigma_2^2,\sigma_1^2)\tilde\sigma_2\tilde\sigma_1 = \sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^m.$$

The product of these identities yields (14).

Each side of (13) with $\pm = -$ identifies with the same side of (13) with $\pm = +$ and $\lambda'$ replaced by $-\lambda'$. This implies (13) with $\pm = -$. □

*End of proof of Proposition 3.3* It suffices to prove that $(\lambda'', f'', g''_\pm) \in \underline{\mathrm{GT}}_{ell}$, i.e. that it satisfies conditions (9) and (10). □

Condition (9) is expressed as follows

$$\begin{aligned}&\big(\sigma_2^{\pm1}\sigma_1^{\pm1}(\sigma_1\sigma_2^2\sigma_1)^{\pm\frac{\lambda\lambda'-1}{2}} g_\pm(g'_+(X_1^+,X_1^-),g'_-(X_1^+,X_1^-)) f(\mathrm{Ad}(f'(\sigma_1^2,\sigma_2^2))(\sigma_1^{2\lambda'}),\\ &\quad \sigma_2^{2\lambda'}) f'(\sigma_1^2,\sigma_2^2)\big)^3 = 1,\end{aligned}$$

i.e. according to (13), as follows

$$\big(g_\pm(g'_+(X_1^+,X_1^-),g'_-(X_1^+,X_1^-)) f(\tilde\sigma_1'^2,\tilde\sigma_1'^2)\tilde\sigma_2'^{\pm1}\tilde\sigma_1'^{\pm1}(\tilde\sigma_1'\tilde\sigma_2'^2\tilde\sigma_1')^{\pm m}\big)^3 = 1,$$

where $\tilde\sigma_1', \tilde\sigma_2'$ are the analogues of $\tilde\sigma_1, \tilde\sigma_2$ from Lemma 3.3 with $(\lambda', f')$ instead of $(\lambda, f)$. The latter identity is the image of identity (9) satisfied by $(\lambda, f, g_\pm)$ by the endomorphism of $B_{1,3}$ attached to $(\lambda', f', g'_\pm)$ by Lemma 3.3.

Condition (10) is the image of identity (10) satisfied by $(\lambda, f, g_\pm)$ under the endomorphism of $B_{1,3}$ attached to $(\lambda', f', g'_\pm)$ by Lemma 3.3. □

The operation $(\lambda, f, g_\pm)(\mathcal{C}, \tilde{\mathcal{C}}, F, A^\pm_{\dots}) := (\mathcal{C}', \tilde{\mathcal{C}}, F, A'^\pm_{\dots})$, where $\mathcal{C}', A'^\pm_{\dots}$ are as in (11) defines an action of $\underline{\mathrm{GT}}_{ell}$ on $\{(\mathcal{C}, \tilde{\mathcal{C}}, F, A^\pm_{\dots}) | \mathcal{C}$ is a BMC, $(\tilde{\mathcal{C}}, F, A^\pm_{\dots})$ is an elliptic structure over it$\}$.

As before, we define semigroups $\widehat{\underline{\mathrm{GT}}}_{ell}$, $\underline{\mathrm{GT}}_l^{ell}$, $\underline{\mathrm{GT}}(\mathbf{k})$ by replacing in the definition of $\underline{\mathrm{GT}}_{ell}$, $(\underline{\mathrm{GT}}, F_2)$ by[5] $(\widehat{\underline{\mathrm{GT}}}, \widehat{F}_2)$, $(\underline{\mathrm{GT}}_l, (F_2)_l)$, $(\underline{\mathrm{GT}}(\mathbf{k}), F_2(\mathbf{k}))$. They act on the sets of pairs $(\mathcal{C}, \tilde{\mathcal{C}})$, such that $\mathcal{C}$ satisfies the same conditions as above, together with: $\mathrm{Aut}_{\tilde{\mathcal{C}}}(F(X))$ is finite for any $X \in \mathrm{Ob}\,\mathcal{C}$ (respectively, the image of

[5] For $G$ a group (other than GT, $\mathrm{GT}_{ell}$ or $R_{ell}$), $\widehat{G}$ is its profinite completion. If $G$ is a free or pure (elliptic) braid group, $G_l$, $G(\mathbf{k})$ are its pro-$l$, $\mathbf{k}$-prounipotent completions. Here $G(-)$ is the prounipotent $\mathbb{Q}$-group scheme associated to $G$; it is characterized by $\mathrm{Hom}_{groups}(G, U(\mathbb{Q})) \simeq \mathrm{Hom}_{gp\ schemes}(G(-), U)$ for any unipotent group scheme $U$. If $G = B_n$ or $B_{1,n}$, then $G_l := P_l *_P G$, $G(\mathbf{k}) := P(\mathbf{k}) *_P G$, where $P = \mathrm{Ker}(G \to S_n)$ and $*_P$ denotes the amalgamated product over the group $P$.

$P_{1,n} \to \mathrm{Aut}_{\tilde{\mathcal{C}}}(F(X_1 \otimes \cdots \otimes X_n))$ is an $l$-group and is contained in a unipotent group). We have morphisms $\underline{\mathrm{GT}}_{ell} \hookrightarrow \widehat{\underline{\mathrm{GT}}}_{ell} \to \underline{\mathrm{GT}}_l^{ell} \hookrightarrow \underline{\mathrm{GT}}_{ell}(\mathbb{Q}_l)$ and $\underline{\mathrm{GT}}_{ell} \to \underline{\mathrm{GT}}_{ell}(\mathbf{k})$ compatible with the similar 'non-elliptic' morphisms.

*Remark 3.4* Specializing the morphism in Proposition 5.22, 2) to the object $\bullet(\dots(\bullet\bullet))$, one shows that the formulas from Lemma 2.3 generalize to an action of $\underline{\mathrm{GT}}_{ell}$ on the tower of elliptic braid groups $B_{1,n}$, given by

$$\begin{aligned}(\lambda, f, g_\pm)\cdot X_1^\pm &:= g_\pm(X_1^+, X_1^-),\\ (\lambda, f, g_\pm)\cdot \sigma_i &:= f(\sigma_i^2, \sigma_{i+1}\sigma_{i+2}\cdots\sigma_n^2\cdots\sigma_{i+2}\sigma_{i+1})\cdot\sigma_i^\lambda\\ &\quad\cdot f(\sigma_i^2, \sigma_i\sigma_{i+1}\cdots\sigma_n^2\cdots\sigma_{i+1}\sigma_i)^{-1}.\end{aligned}$$

Composing this action with the morphism $\underline{\mathrm{GT}} \to \underline{\mathrm{GT}}_{ell}$ (Proposition 2.20), one obtains an action of $\underline{\mathrm{GT}}$ on the tower of elliptic braid groups, given by

$$\begin{aligned}(\lambda, f)\cdot X_1^+ &:= f(X_1^+, (X_1^-, X_1^+))\cdot (X_1^+)^\lambda \cdot f(X_1^+, (X_1^-, X_1^+))^{-1},\\ (\lambda, f)\cdot X_1^- &:= (X_1^-, X_1^+)^{\frac{\lambda-1}{2}}\\ &\quad\cdot f(X_1^-(X_1^+)^{-1}(X_1^-)^{-1}, (X_1^-, X_1^+))\cdot X_1^- \cdot f(X_1^+, (X_1^-, X_1^+))^{-1},\\ (\lambda, f)\cdot\sigma_i &:= f(\sigma_i^2, \sigma_{i+1}\sigma_{i+2}\cdots\sigma_n^2\cdots\sigma_{i+2}\sigma_{i+1})\cdot\sigma_i^\lambda\\ &\quad\cdot f(\sigma_i^2, \sigma_{i+1}\sigma_{i+2}\cdots\sigma_n^2\cdots\sigma_{i+2}\sigma_{i+1})^{-1}.\end{aligned}$$

The profinite, pro-$l$, and prounipotent versions of $\underline{\mathrm{GT}}_{ell}$ and $\underline{\mathrm{GT}}$ acts on the profinite, pro-$l$, and prounipotent versions of the tower of elliptic braid groups by the same formulas.

### 3.3 Computation of $\underline{\mathrm{GT}}_{ell}$

Recall that the braid group $B_3$ is presented by generators $\Psi_\pm$ and relations $\Psi_+\Psi_-\Psi_+ = \Psi_-\Psi_+\Psi_-$ ($\Psi_\pm$ are the $\sigma_1, \sigma_2$ of the standard presentation and are used in order to avoid confusion with previous notation). Its centre $Z(B_3)$ is isomorphic to $\mathbb{Z}$ and generated by $(\Psi_+\Psi_-)^3$. There is a central exact sequence

$$1 \to 2Z(B_3) \to B_3 \to \mathrm{SL}_2(\mathbb{Z}) \to 1,$$

given by $\Psi_+ \mapsto \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\Psi_- \mapsto \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$.

**Proposition 3.5** *Let $\tilde{B}_3$ be the group generated by $\Psi_\pm$, $\varepsilon$ and relations*

$$\Psi_+\Psi_-\Psi_+ = \Psi_-\Psi_+\Psi_-, \quad \varepsilon\Psi_+\varepsilon\Psi_- = 1, \quad \varepsilon^2 = 1.$$

*There is an exact sequence $1 \to B_3 \to \tilde{B}_3 \to \mathbb{Z}/2 \to 1$, where $\tilde{B}_3 \to \mathbb{Z}/2$ is given by $\Psi_\pm \mapsto 1$, $\varepsilon \to -1$. There is also a (noncentral) exact sequence $1 \to 2Z(B_3) \to$*

$\tilde{B}_3 \to \mathrm{GL}_2(\mathbb{Z}) \to 1$, *where* $\tilde{B}_3 \to \mathrm{GL}_2(\mathbb{Z})$ *extends* $B_3 \to \mathrm{SL}_2(\mathbb{Z})$ *by* $\varepsilon \mapsto\!\!\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *All these morphisms fit in the diagram*

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 2Z(B_3) & \longrightarrow & B_3 & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 2Z(B_3) & \longrightarrow & \tilde{B}_3 & \longrightarrow & \mathrm{GL}_2(\mathbb{Z}) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}/2 & = & \mathbb{Z}/2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

The proof is straightforward.

**Proposition 3.6** *1) There is a unique semigroup morphism* $\tilde{B}_3 \to \underline{\mathrm{GT}}_{ell}$, *such that:*

$$\begin{aligned}
\Psi_+ &\mapsto (\lambda, f, g_+, g_-) = (1, 1, g_+(X, Y) = X, g_-(X, Y) = YX), \\
\Psi_- &\mapsto (\lambda, f, g_+, g_-) = (1, 1, g_+(X, Y) = XY^{-1}, g_-(X, Y) = Y), \\
\varepsilon &\mapsto (\lambda, f, g_+, g_-) = (-1, 1, g_+(X, Y) = Y, g_-(X, Y) = X),
\end{aligned}$$

*It fits in a commutative diagram*

$$\begin{array}{ccc}
\tilde{B}_3 & \to & \underline{\mathrm{GT}}_{ell} \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 & \to & \underline{\mathrm{GT}}
\end{array}$$

*2) The horizontal maps in this diagram are isomorphisms.*

*Proof* Set $X_i := X_i^+$, $Y_i := X_i^-$. Using the commutation of $\sigma_2$ with $X_1$ and the braid relation between $\sigma_1$ and $\sigma_2$, one obtains $(\sigma_2\sigma_1 X_1)^3 = X_3X_2X_1 = 1$ (relation in $B_{1,3}$). In the same way, $(\sigma_2^{-1}\sigma_1^{-1}Y_1X_1)^3$ expresses as an element of $P_{1,3}$ as

$$Y_3X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}Y_2X_2\sigma_1^{-2}Y_1X_1.$$

Since $(Y_1, X_2^{-1}) = \sigma_1^2$, $X_2\sigma_1^{-2}Y_1$ can be replaced by $Y_1X_2$; in the resulting expression, $Y_2Y_1X_2X_1$ can then be replaced by $Y_3^{-1}X_3^{-1}$. The above expression is therefore equal to

$$Y_3X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}Y_3^{-1}X_3^{-1}.$$

One has $(Y_3^{-1}, X_3^{-1}) = (Y_2Y_1, X_3^{-1}) = Y_1(Y_2, X_3^{-1})Y_1^{-1}(Y_1, X_3^{-1})$; one computes $(Y_2, X_3^{-1}) = \sigma_2^2$, $(Y_1, X_3^{-1}) = \sigma_2^{-1}\sigma_1^2\sigma_2$, which implies that $(Y_3^{-1}, X_3^{-1}) = \sigma_2\sigma_1^2\sigma_2$ and therefore that

$$(\sigma_2^{-1}\sigma_1^{-1}Y_1X_1)^3 = 1 \quad \text{(equality in } B_{1,3}).$$

Finally, $(Y_1X_1, \sigma_1^{-1}X_1^{-1}\sigma_1^{-1}) = (Y_1X_1, X_2^{-1}) = (Y_1, X_2^{-1}) = \sigma_1^2$ (equality in $B_{1,3}$), where the second equality uses the commutation of $X_1$ and $X_2$. All this implies that $(1, 1, X, YX) \in \underline{\mathrm{GT}}_{ell}$. □

If $(\lambda, f, g_+, g_-) = (-1, 1, Y, X)$, then $m = -1$, therefore

$$\begin{aligned}(\sigma_2^{\pm 1}\sigma_1^{\pm 1}(\sigma_1\sigma_2^2\sigma_1)^{\pm m}g_\pm(X_1^+, X_1^-)f(\sigma_1^2, \sigma_2^2))^3 \\ = (\sigma_2^{\mp 1}\sigma_1^{\mp 1}X_1^{\mp})^3 = 1 \quad \text{(relation in } B_{1,3}).\end{aligned}$$

The relation $u^2 = (g_-, u^{-1}g_+^{-1}u^{-1})$ follows from $\sigma_1^{-2} = (X_1, Y_2^{-1})$ (relation in $B_{1,3}$). All this implies that $(-1, 1, Y, X) \in \underline{\mathrm{GT}}_{ell}$.

One checks that $(1, 1, XY^{-1}, Y) = (-1, 1, Y, X)(1, 1, X, YX)^{-1}(-1, 1, Y, X)$, therefore

$$(1, 1, XY^{-1}, Y) \in \underline{\mathrm{GT}}_{ell}.$$

Finally, one checks that the relations between $\Psi_+$, $\Psi_-$ and $\varepsilon$ are also satisfied by their images in $\underline{\mathrm{GT}}_{ell}$. All this proves 1).

Let us prove 2). The bijectivity of $\mathbb{Z}/2 \to \underline{\mathrm{GT}}$ is proved in Drinfeld [9], Proposition 4.1. Set $\underline{R}_{ell} := \mathrm{Ker}(\underline{\mathrm{GT}}_{ell} \to \underline{\mathrm{GT}})$, then the commutativity of the above diagram implies that its upper map restricts to a morphism $B_3 \to \underline{R}_{ell}$, and we need to prove that it is bijective. According to the second identity in (12), $\underline{R}_{ell} \subset \{(g_+, g_-) \in (F_2)^2 | (g_-(X, Y), g_+(X, Y)) = (Y, X)\}$. We now recall some results due to Nielsen.

**Theorem 3.7** ([25])

*1) The morphism* $\mathrm{Out}(F_2) \to \mathrm{GL}_2(\mathbb{Z})$ *induced by abelianization is an isomorphism.*
*2)* $\mathrm{Im}(\mathrm{Aut}(F_2) \to (F_2)^2) = \{(g_+, g_-) \in (F_2)^2 | \exists k \in F_2, \exists \epsilon \in \{\pm 1\}, (g_-(X, Y), g_+(X, Y)) = k(Y, X)^\epsilon k^{-1}\}$, *where the map* $\mathrm{Aut}(F_2) \to (F_2)^2$ *is* $\theta \mapsto (\theta(X), \theta(Y))$.

The bijectivity of $B_3 \to \underline{R}_{ell}$, together with the equality $\underline{R}_{ell} = \{(g_+, g_-) \in (F_2)^2 | (g_-(X, Y), g_+(X, Y)) = (Y, X)\}$ are then proven in the following corollary to Theorem 3.7:

**Corollary 3.8** *We have bijections*

$$B_3 \to \mathrm{Aut}_{(X,Y)}(F_2) \to \{(g_+, g_-) \in (F_2)^2 | (g_-(X, Y), g_+(X, Y)) = (Y, X)\},$$

*where* $\mathrm{Aut}_{(X,Y)}(F_2) = \{\theta \in \mathrm{Aut}(F_2) | \theta((X, Y)) = (X, Y)\}$, *the first map is as in Proposition 3.6 and the second map is* $\theta \mapsto (\theta(X), \theta(Y))$.

*Proof of Corollary 3.8* The bijectivity of the second map follows from the injectivity of $\mathrm{Aut}(F_2) \to (F_2)^2$, $\theta \mapsto (\theta(X), \theta(Y))$ and from Theorem 3.7, 2). Let us now prove the bijectivity of the map $B_3 \to \mathrm{Aut}_{(X,Y)}(F_2)$. The kernel of $B_3 \to \mathrm{Aut}_{(X,Y)}(F_2)$ is contained in $\mathrm{Ker}(B_3 \to \mathrm{Aut}_{(X,Y)}(F_2) \to \mathrm{Out}(F_2) \to \mathrm{GL}_2(\mathbb{Z})) = \langle(\Psi_+\Psi_-)^6\rangle$. On the other hand, $B_3 \to \mathrm{Aut}_{(X,Y)}(F_2)$ takes $(\Psi_+\Psi_-)^6$ to[6] $\mathrm{Ad}((X,Y)^{-1})$, so the restriction of $B_3 \to \mathrm{Aut}_{(X,Y)}(F_2)$ to $\langle(\Psi_+\Psi_-)^6\rangle$ is injective. It follows that $B_3 \to \mathrm{Aut}_{(X,Y)}(F_2)$ is injective.

Let us now show that $B_3 \to \mathrm{Aut}_{(X,Y)}(F_2)$ is surjective. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Aut}(F_2) & \longrightarrow & \mathrm{Out}(F_2) & \xrightarrow{\sim} & \mathrm{GL}_2(\mathbb{Z}) \\ \uparrow & & & & \uparrow \\ \mathrm{Aut}_{(X,Y)}(F_2) & & \longrightarrow & & \mathrm{SL}_2(\mathbb{Z}) \end{array}$$

where the isomorphism follows from Theorem 3.7, 1), and the bottom map is given by abelianization. It follows that $\mathrm{Ker}(\mathrm{Aut}_{(X,Y)}(F_2) \to \mathrm{SL}_2(\mathbb{Z})) = \mathrm{Ker}(\mathrm{Aut}_{(X,Y)}(F_2) \to \mathrm{Out}(F_2)) = \mathrm{Aut}_{(X,Y)}(F_2) \cap \mathrm{Inn}(F_2) = \{\theta \in \mathrm{Aut}(F_2) | \exists k \in F_2, \theta = \mathrm{Ad}(k)$ and $k$ commutes with $(X,Y)\}$. The subgroup of $F_2$ generated by $k$ and $(X,Y)$ is abelian and, according to [26], free, and therefore isomorphic to $\mathbb{Z}$. If $(X,Y)$ is a power of an element $h$ of $F_2$, then the sum of the degrees of $h$ in $X$ and in $Y$ is zero, and comparing coefficients in $[\log X, \log Y]$ in $\log(X,Y)$ and $\log h$ in the Lie algebra of the prounipotent completion of $F_2$, one sees that $h$ is $(X,Y)$ or its inverse; therefore, $(X,Y)$ is not the power of an element of $F_2$ other that itself or its inverse. All this implies that $k$ should be a power of $(X,Y)$; therefore, $\mathrm{Ker}(\mathrm{Aut}_{(X,Y)}(F_2) \to \mathrm{SL}_2(\mathbb{Z})) = \langle \mathrm{Ad}(X,Y)\rangle = \langle(\Psi_+\Psi_-)^6\rangle$. On the other hand, as the composition $B_3 \to \mathrm{Aut}_{(X,Y)}(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ is surjective, so is the morphism $\mathrm{Aut}_{(X,Y)}(F_2) \to \mathrm{SL}_2(\mathbb{Z})$. All this implies that there is an exact sequence

$$1 \to \langle(\Psi_+\Psi_-)^6\rangle \to \mathrm{Aut}_{(X,Y)}(F_2) \to \mathrm{SL}_2(\mathbb{Z}) \to 1.$$

Let us denote this exact sequence as $1 \to K \to G \to H \to 1$, and let $G' := \mathrm{Im}(B_3 \to G) \subset G$. To prove that $G' = G$, it suffices to prove that $\mathrm{Im}(G' \subset G \to H) = H$ and that $G' \supset K$. The first statement follows from the surjectivity to $B_3 \to \mathrm{Aut}_{(X,Y)}(F_2) \to \mathrm{SL}_2(\mathbb{Z})$, while the second statement follows from the fact that $(\Psi_+\Psi_-)^6 \in \mathrm{Im}(B_3 \to \mathrm{Aut}_{(X,Y)}(F_2))$. □

*Remark 3.9* As $\mathrm{GL}_2(\mathbb{Z})$ is the nonoriented mapping class group of the topological torus, we have a morphism $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{Out}(B_{1,n})$, obtained by applying mapping class group elements to elliptic braids; its target is an outer automorphism group because the mapping class group does not preserve a base point of the elliptic configuration space. This morphism lifts to a morphism

$$\tilde{B}_3 \to \mathrm{Aut}(B_{1,n}), \tag{15}$$

[6] Here and later, $\mathrm{Ad}(g)$ is the inner automorphism $x \mapsto gxg^{-1}$.

given by $\Psi_+ \mapsto (X_1 \mapsto X_1, Y_1 \mapsto Y_1X_1, \sigma_i \mapsto \sigma_i)$, $\Psi_- \mapsto (X_1 \mapsto X_1Y_1^{-1}, \sigma_i \mapsto \sigma_i)$, $\varepsilon \mapsto (X_1 \leftrightarrow Y_1, \sigma_i \mapsto \sigma_i^{-1})$. It is such that $(\Psi_+\Psi_-)^6 \mapsto$ (conjugation by the image of $z \in P_n \to B_{1,n}$), where $z$ is a generator of $Z(P_n) \simeq \mathbb{Z}$. The assignment {elliptic structures over BMCs} $\to$ {representations of $B_{1,n}$} is then $\tilde{B}_3$-equivariant.

*Remark 3.10* The morphisms $\tilde{B}_3 \to \mathrm{Aut}(B_{1,n})$ and $\underline{\mathrm{GT}}_{ell} \to \mathrm{End}(B_{1,3})$ from Lemma 3.3 admit a common generalization to a morphism $\underline{\mathrm{GT}}_{ell} \to \mathrm{End}(B_{1,n})$, taking $(\lambda, f, g_+, g_-)$ to the endomorphism $X_1 \mapsto g_+(X_1, Y_1)$, $Y_1 \mapsto g_-(X_1, Y_1)$, $\sigma_i \mapsto \mathrm{Ad}(f(\sigma_i^2, \sigma_{i+1} \ldots \sigma_{n-1}^2 \ldots \sigma_{i+1}))(\sigma_i^{\lambda})$; this corresponds to the identification of $B_{1,n}$ with $\mathrm{Aut}_{\mathbf{PaB}_{ell}}(\bullet(\bullet \cdots (\bullet\bullet)))$. This morphism extends to the various setups (profinite, etc.).

## 3.4 The semigroup scheme $\underline{\mathrm{GT}}_{ell}(-)$

For $\mathbf{k}$ a $\mathbb{Q}$-ring, we set[7] $R_{ell}(\mathbf{k}) := \mathrm{Ker}(\underline{\mathrm{GT}}_{ell}(\mathbf{k}) \to \underline{\mathrm{GT}}(\mathbf{k}))$. The assignments $\mathbf{k} \mapsto \underline{\mathrm{GT}}_{(ell)}(\mathbf{k})$, $R_{ell}(\mathbf{k})$ are functors {$\mathbb{Q}$-rings} $\to$ {semigroups}, i.e. semigroup schemes over $\mathbb{Q}$.

**Proposition 3.11** *We have a commutative diagram of morphisms of semigroup schemes*

$$\begin{array}{ccccc} R_{ell}(-) & \to & \underline{\mathrm{GT}}_{ell}(-) & \to & \underline{\mathrm{GT}}(-) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{SL}_2(-) & \to & \mathrm{M}_2(-) & \stackrel{\det}{\to} & \mathbb{A}^1(-) \end{array}$$

*where* $\underline{\mathrm{GT}}(\mathbf{k}) \to \mathbf{k}$ *is* $(\lambda, f) \mapsto \lambda$ *and* $\underline{\mathrm{GT}}_{ell}(\mathbf{k}) \to \mathrm{M}_2(\mathbf{k})$ *is* $(\lambda, f, g_\pm) \mapsto \begin{pmatrix} \alpha_+ & \beta_+ \\ \alpha_- & \beta_- \end{pmatrix}$, *where* $\log g_\pm(X, Y) = \alpha_\pm \log X + \beta_\pm \log Y$ *mod*[8] $[\hat{\mathfrak{f}}_2^{\mathbf{k}}, \hat{\mathfrak{f}}_2^{\mathbf{k}}]$.

*Proof* It suffices to show that the right square is commutative, which follows by abelianization from the second part of (12). □

Recall that[9] $\mathrm{GT}(\mathbf{k}) = \underline{\mathrm{GT}}(\mathbf{k})^\times$. We set

**Definition 3.12** $\mathrm{GT}_{ell}(\mathbf{k}) := \underline{\mathrm{GT}}_{ell}(\mathbf{k})^\times$.

**Proposition 3.13** *1)* $\mathrm{GT}_{ell}(\mathbf{k}) = \underline{\mathrm{GT}}_{ell}(\mathbf{k}) \times_{\mathrm{M}_2(\mathbf{k})} \mathrm{GL}_2(\mathbf{k})$ *(Cartesian product in the category of proalgebraic varieties).*
*2)* $R_{ell}(\mathbf{k})$ *is a group.*

*Proof* Let $(\lambda, f, g_\pm) \in \underline{\mathrm{GT}}_{ell}(\mathbf{k})$ be invertible as an element of $\underline{\mathrm{GT}}(\mathbf{k}) \times \mathrm{End}(F_2(\mathbf{k}))^{op}$, with inverse $(\lambda', f', g'_\pm)$. Then, the endomorphism of Lemma 3.3 attached to

[7] The kernel of a morphism of semigroups with unit is the preimage of the unit of the target semigroup; it is again a semigroup with unit.

[8] Recall that $F_2(\mathbf{k}) = \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$, where $\hat{\mathfrak{f}}_2^{\mathbf{k}}$ is the topologically free $\mathbf{k}$-Lie algebra in two generators $\log X$ and $\log Y$.

[9] If $S$ is a semigroup with unit, $S^\times$ is the group of its invertible elements.

$(\lambda, f, g_\pm)$ is an automorphism of $B_{1,3}(\mathbf{k})$. The identities $(\sigma_2^{\pm 1}\sigma_1^{\pm 1}X_1^{\pm})^3 = 1$, $\sigma_1^2 = (X_1^-, (X_2^+)^{-1})$ in $B_{1,3}(\mathbf{k})$ are the images by this automorphism of the identities expressing that $(\lambda', f', g'_\pm)$ belongs to $\underline{\mathrm{GT}}_{ell}(\mathbf{k})$. It follows that $(\lambda', f', g'_\pm) \in \underline{\mathrm{GT}}_{ell}(\mathbf{k})$. The element $(\lambda, f, g_\pm)$ is invertible iff the image of $(\lambda, f, g_\pm) \in \underline{\mathrm{GT}}_{ell}(\mathbf{k}) \to \mathrm{M}_2(\mathbf{k})$ lies in $\mathrm{GL}_2(\mathbf{k})$. All this proves 1). 2) is then immediate. □

Recall that for any $\mathbb{Q}$-ring $\mathbf{k}$, $\mathrm{GT}_1(\mathbf{k}) = \mathrm{Ker}(\underline{\mathrm{GT}}(\mathbf{k}) \to \mathbf{k})$. We also set

$$\mathrm{GT}^{ell}_{I_2}(\mathbf{k}) := \mathrm{Ker}(\underline{\mathrm{GT}}_{ell}(\mathbf{k}) \to \mathrm{M}_2(\mathbf{k})), \quad R^{ell}_{I_2}(\mathbf{k}) := \mathrm{Ker}(R_{ell}(\mathbf{k}) \to \mathrm{SL}_2(\mathbf{k})).$$

Then, $\mathbf{k} \mapsto \mathrm{GT}^{(ell)}_{I_2}(\mathbf{k})$, $R^{ell}_{I_2}(\mathbf{k})$ are $\mathbb{Q}$-group schemes. It is known that $\mathrm{GT}_1(-)$ is prounipotent.

**Proposition 3.14** *The group schemes* $\mathrm{GT}^{ell}_{I_2}(-)$ *and* $R^{ell}_{I_2}(-)$ *are prounipotent.*

*Proof* $\mathrm{GT}^{ell}_{I_2}(\mathbf{k}) \subset \mathrm{GT}_1(\mathbf{k}) \times \mathrm{Aut}_{I_2}(F_2(\mathbf{k}))^{op}$, where $\mathrm{Aut}_{I_2}(F_2(\mathbf{k})) = \mathrm{Ker}(\mathrm{Aut}(F_2(\mathbf{k})) \to \mathrm{GL}_2(\mathbf{k}))$; $\mathbf{k} \mapsto \mathrm{Aut}_{I_2}(F_2(\mathbf{k}))$ is prounipotent, so $\mathbf{k} \mapsto \mathrm{GT}^{ell}_{I_2}(\mathbf{k})$ is prounipotent as the subgroup of a prounipotent group scheme. The same argument implies that $R^{ell}_{I_2}(-)$ is prounipotent. □

**Proposition 3.15** *We have exact sequences* $1 \to R^{ell}_{I_2}(\mathbf{k}) \to R_{ell}(\mathbf{k}) \to \mathrm{SL}_2(\mathbf{k}) \to 1$ *and* $1 \to \mathrm{GT}^{ell}_{I_2}(\mathbf{k}) \to \mathrm{GT}_{ell}(\mathbf{k}) \to \mathrm{GL}_2(\mathbf{k}) \to 1$.

*Proof* We need to prove that $R_{ell}(\mathbf{k}) \to \mathrm{SL}_2(\mathbf{k})$ is surjective. Set $G(\mathbf{k}) := \mathrm{Im}(R_{ell}(\mathbf{k}) \to \mathrm{SL}_2(\mathbf{k}))$, then $\mathbf{k} \mapsto G(\mathbf{k})$ is a group subscheme of $\mathrm{SL}_2$. We have two morphisms $\mathbb{G}_a \to R_{ell}(-)$, extending $\mathbb{Z} \to B_3$, $1 \mapsto \Psi_\pm$

in the sense that

$$\begin{array}{ccc} B_3 & \to & R_{ell}(\mathbf{k}) \\ \uparrow & & \uparrow \\ \mathbb{Z} & \to & \mathbb{G}_a(\mathbf{k}) \end{array}$$

commutes; then, $\mathbb{G}_a \to R_{ell} \to \mathrm{SL}_2$ are the morphisms $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$. So the Lie algebra of $G(-)$ contains both $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and hence is equal to $\mathfrak{sl}_2$, so $G = \mathrm{SL}_2$.

Let us now prove that $\mathrm{GT}_{ell}(\mathbf{k}) \to \mathrm{GL}_2(\mathbf{k})$ is surjective. Set $\tilde{G}(\mathbf{k}) := \mathrm{Im}(\mathrm{GT}_{ell}(\mathbf{k}) \to \mathrm{GL}_2(\mathbf{k}))$, then $\mathrm{SL}_2 \subset \tilde{G}(-) \subset \mathrm{GL}_2$. We will construct in Sect. 3.6 a semigroup scheme morphism $\underline{\mathrm{GT}}(-) \overset{\sigma}{\to} \underline{\mathrm{GT}}_{ell}(-)$, such that

$$\begin{array}{ccc} \underline{\mathrm{GT}}(-) & \to & \mathbb{A}^1(-) \\ \sigma\downarrow & & \downarrow \\ \underline{\mathrm{GT}}_{ell}(-) & \to & \mathrm{M}_2(-) \end{array}$$

commutes, where $\mathbb{A}^1 \to \mathrm{M}_2$ is $t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. Then $\tilde{G}(-)$ contains the image of $\mathbb{G}_m \to \mathrm{GT}(-) \overset{\sigma}{\to} \mathrm{GT}_{ell}(-) \to \mathrm{GL}_2$, where $\mathbb{G}_m \to \mathrm{GT}(-)$ is a section of $\mathrm{GT}(-) \to \mathbb{G}_m$

(see [9]), which is the image of $\mathbb{G}_m \to \mathrm{GL}_2$, $t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. Then $\mathrm{Lie}(\tilde{G}) = \mathfrak{gl}_2$, so $\tilde{G} = \mathrm{GL}_2$, as wanted. □

### 3.5 The Zariski closure $\langle B_3 \rangle \subset R_{ell}(-)$

Recall that we have a group morphism $B_3 = R_{ell} \to R_{ell}(\mathbb{Q})$. The Zariski closure $\langle B_3 \rangle \subset R_{ell}(-)$ is then the subgroup scheme[10] defined as

$$\langle B_3 \rangle := \bigcap_{\substack{G \subset R_{ell}(-) \text{ subgroup scheme} \,| \\ G(\mathbb{Q}) \supset \mathrm{Im}(B_3 \to R_{ell}(\mathbb{Q}))}} G$$

Let us compute the Lie algebra[11] inclusion $\mathrm{Lie}\langle B_3 \rangle \subset \mathrm{Lie}\, R_{ell}(-)$. First, $\mathrm{Lie}\, R_{ell}(-)$ is a Lie subalgebra of

$$\mathrm{Lie\,Aut}(F_2(-))^{op} \simeq \mathrm{Lie\,Aut}(\hat{\mathfrak{f}}_2)^{op} \simeq (\mathrm{Der}\, \hat{\mathfrak{f}}_2)^{op} \simeq \hat{\mathfrak{f}}_2^2,$$

where:

- $\hat{\mathfrak{f}}_2 := \hat{\mathfrak{f}}_2^{\mathbb{Q}}$ is the Lie algebra freely generated by $\xi := \log X$ and $\eta := \log Y$;
- the first map is based on the isomorphism $F_2(\mathbf{k}) \simeq \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$;
- the Lie algebra structure on $\hat{\mathfrak{f}}_2^2$ is given by

$$[(\alpha, \beta), (\alpha', \beta')] := (D_{\alpha',\beta'}(\alpha), D_{\alpha',\beta'}(\beta)) - (D_{\alpha,\beta}(\alpha'), D_{\alpha,\beta}(\beta')),$$

  where $D_{\alpha,\beta} \in \mathrm{Der}(\hat{\mathfrak{f}}_2)$ is given by $\xi \mapsto \alpha$, $\eta \mapsto \beta$;
- the last isomorphism $(\mathrm{Der}\, \hat{\mathfrak{f}}_2)^{op} \simeq \hat{\mathfrak{f}}_2^2$ has inverse $(\alpha, \beta) \mapsto D_{\alpha,\beta}$.

**Lemma 3.16** $\mathrm{Lie}\, R_{ell}(-) \subset \mathrm{Lie\,Aut}(F_2(-))^{op}$ *identifies with the set of* $(\alpha, \beta) \in \hat{\mathfrak{f}}_2^2$ *such that*

$$\tilde{\alpha}(X_1, Y_1) + \tilde{\alpha}(X_2\sigma_1^{-2}, Y_2) + \tilde{\alpha}(X_3\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}, Y_3) = 0,$$
$$\tilde{\beta}(X_1, Y_1) + \tilde{\beta}(X_2, Y_2\sigma_1^2) + \tilde{\beta}(X_3, Y_3\sigma_1\sigma_2^2\sigma_1) = 0,$$
$$(\mathrm{Ad}\, X_2^{-1} - 1)\tilde{\beta}(X_1, Y_1) + (1 - \mathrm{Ad}\, Y_1^{-1})\tilde{\alpha}(X_2\sigma_1^{-2}, \sigma_1^2 Y_2) = 0$$

*(relations in* $\mathrm{Lie}\, P_{1,3}(-)$*). Here* $\tilde{\alpha}(X_1, Y_1), \ldots$ *are the images of the elements*

$$\tilde{\alpha}(e^\xi, e^\eta) := \frac{1 - e^{-\mathrm{ad}\,\xi}}{\mathrm{ad}\,\xi}(\alpha(\xi, \eta)), \quad \tilde{\beta}(e^\xi, e^\eta) := \frac{1 - e^{-\mathrm{ad}\,\eta}}{\mathrm{ad}\,\eta}(\beta(\xi, \eta))$$

---

10 According to Conjecture 10.1, the inclusion $\langle B_3 \rangle \subset R_{ell}(-)$ is an equality (see Proposition 10.2).

11 Recall that the Lie algebra of a $\mathbb{Q}$-group scheme $G$ is $\mathrm{Ker}(G(\mathbb{Q}[\varepsilon]/(\varepsilon^2)) \to G(\mathbb{Q}))$.

*of* $\hat{\mathfrak{f}}_2$ *by the morphism* $\hat{\mathfrak{f}}_2 \to \mathrm{Lie}\, P_{1,3}(-)$, $\xi \mapsto \log X_1$, $\eta \mapsto \log Y_1$, *etc., and* $X_i := X_i^+$, $Y_i := X_i^-$ *(elements of* $P_{1,\cdot}$*).*

*The above relations imply the relations*

$$\tilde{\alpha}(X_1, Y_1) + \tilde{\alpha}(X_1^{-1}\sigma_1^{-2}, Y_1^{-1}) = 0, \quad \tilde{\beta}(X_1, Y_1) + \tilde{\beta}(X_1^{-1}, Y_1^{-1}\sigma_1^2) = 0,$$
$$(\mathrm{Ad}\, X_1 - 1)\tilde{\beta}(X_1, Y_1) + (1 - \mathrm{Ad}\, Y_1^{-1})\tilde{\alpha}(X_1^{-1}\sigma_1^{-2}, \sigma_1^2 Y_1^{-1}) = 0$$

*in* $\mathrm{Lie}\, P_{1,2}(-)$.

*Proof* $(\alpha, \beta) \in \hat{\mathfrak{f}}_2^2 \simeq (\mathrm{Der}\, \hat{\mathfrak{f}}_2)^{op}$ induces the infinitesimal automorphism of $F_2(\mathbb{Q})$ given by $X \mapsto g_+(X, Y) = X(1 + \epsilon\tilde{\alpha}(X, Y))$, $Y \mapsto g_-(X, Y) = Y(1 + \epsilon\tilde{\beta}(X, Y))$, where $\epsilon^2 = 0$. The condition that $(1, 1, g_+, g_-)$ belongs to $R_{ell}(\mathbb{Q}[\epsilon]/(\epsilon^2))$ linearizes as follows

$$\big(\mathrm{id} + \mathrm{Ad}(\sigma_2\sigma_1 X_1) + \mathrm{Ad}(\sigma_2\sigma_1 X_1)^2\big)(\tilde{\alpha}(X_1, Y_1)) = 0,$$
$$\big(\mathrm{id} + \mathrm{Ad}(\sigma_2^{-1}\sigma_1^{-1} Y_1) + \mathrm{Ad}(\sigma_2^{-1}\sigma_1^{-1} Y_1)^2\big)(\tilde{\beta}(X_1, Y_1)) = 0,$$
$$\big(Y_1(1 + \epsilon\tilde{\beta}(X_1, Y_1)), (1 - \epsilon\tilde{\alpha}(\sigma_1^{-2}X_2, Y_2\sigma_1^2))X_2^{-1}\big) = (Y_1, X_2^{-1}),$$

which are equivalent to the announced identities using the relations in $P_{1,3} : (X_i, X_j) = (Y_i, Y_j) = 1$,

$$(Y_1, X_1) = \sigma_1\sigma_2^2\sigma_1, \quad (Y_1, X_2^{-1}) = \sigma_1^2 = (Y_2^{-1}, X_1), \quad (Y_1, X_3^{-1}) = \sigma_2^{-1}\sigma_1^2\sigma_2,$$
$$(X_1, Y_3^{-1}) = \sigma_2\sigma_1^{-2}\sigma_2^{-1}, \quad (Y_2, X_3^{-1}) = \sigma_2^2 = (Y_3^{-1}, X_2), \quad (Y_3^{-1}, X_3^{-1}) = \sigma_2\sigma_1^2\sigma_2.$$

□

We now compute $\mathrm{Lie}\langle B_3\rangle \subset \mathrm{Lie}\, R_{ell}(-)$.

**Lemma 3.17** *Let* $u := \Big(0, \frac{\mathrm{ad}\,\eta}{1 - e^{-\mathrm{ad}\,\eta}}(\xi)\Big)$, $v := \Big(\frac{\mathrm{ad}\,\xi}{1 - e^{-\mathrm{ad}\,\xi}}(\eta), 0\Big)$ *in* $\mathrm{Lie}\, \mathrm{Aut}(F_2(-))^{op} \simeq \hat{\mathfrak{f}}_2^2$*, then* $u, v \in \mathrm{Lie}\langle B_3\rangle$.

*Proof* We have morphisms $\mathbb{G}_a \to \langle B_3\rangle \subset \mathrm{Aut}(F_2(-))^{op}$, extending $\mathbb{Z} \to B_3$, $1 \mapsto \Psi_\pm^{\pm 1}$. The corresponding morphisms $(\mathbf{k}, +) \to \mathrm{Aut}(F_2(\mathbf{k}))^{op}$ are $t \mapsto (X \mapsto X, Y \mapsto YX^t)$ and $t \mapsto (X \mapsto XY^t, Y \mapsto Y)$. The equality $XY^t = e^\xi e^{t\eta} = \exp\Big(\xi + t\frac{\mathrm{ad}\,\xi}{1 - e^{-\mathrm{ad}\,\xi}}(\eta)\Big)$, valid for $t^2 = 0$, and the similar equality for $YX^t$, imply that the associated Lie algebra morphisms are $\mathbb{Q} \to \mathrm{Lie}\, \mathrm{Aut}(F_2(-))^{op}$, $1 \mapsto u, v$, which proves that $u, v \in \mathrm{Lie}\langle B_3\rangle$. □

**Proposition 3.18** $\mathrm{Lie}\langle B_3\rangle \subset \mathrm{Lie}\, \mathrm{Aut}(F_2(-))^{op} \simeq \hat{\mathfrak{f}}_2^2$ *is the smallest closed Lie subalgebra containing* $u$ *and* $v$*. In particular, the image of* $\mathrm{Lie}\langle B_3\rangle$ *by the morphism* $\mathrm{Der}(\hat{\mathfrak{f}}_2)^{op} \to \mathfrak{gl}_2$ *induced by the abelianization map* $\mathfrak{f}_2 \to \mathbb{Q}^2$ *is* $\mathfrak{sl}_2$.

We first prove:

**Lemma 3.19** *Let G be a proalgebraic group over* $\mathbb{Q}$ *fitting in* $1 \to U \to G \to G_0 \to 1$*, where* $G_0$ *is semisimple and* $U$ *is prounipotent. Let* $0 \to \mathfrak{u} \to \mathfrak{g} \to \mathfrak{g}_0 \to 0$ *be the corresponding exact sequence of Lie algebras. Then* $H \mapsto \mathrm{Lie}\, H$ *sets up a bijection* $\{$*proalgebraic subgroups* $H \subset G$*, such that* $\mathrm{Im}(H \subset G \to G_0) = G_0\} \xrightarrow{\sim} \{$*closed Lie subalgebras* $\mathfrak{h} \subset \mathfrak{g}$*, such that* $\mathrm{Im}(\mathfrak{h} \subset \mathfrak{g} \to \mathfrak{g}_0) = \mathfrak{g}_0\}$.

*Proof* If $H$ is in the first set, then we have an exact sequence $1 \to H \cap U \to H \to G_0 \to 1$, where $H \cap U$ is necessarily prounipotent, hence connected, which implies that $H$ is connected. According to [31], Prop. 24.3.5, ii), if $\tilde{G}$ is an algebraic group, then the map $\{$connected algebraic subgroups of $\tilde{G}\} \to \{$Lie subalgebras of Lie $\tilde{G}\}$ defined by taking Lie algebras is injective. Applying this to the algebraic quotients of $G$, one derives the injectivity of the map $H \mapsto \mathrm{Lie}\, H$. □

Let us prove its surjectivity. Let $\mathfrak{h}$ belong to the second set.

Firstly, note that according to the Levi–Mostow decomposition ([5,23] prop. 5.1), there exists a section $\tilde{\sigma} : G_0 \to G$ of $G \to G_0$. We denote by $\sigma : \mathfrak{g}_0 \to \mathfrak{g}$ its infinitesimal. Any section of $\mathfrak{g} \to \mathfrak{g}_0$ is then conjugate to $\sigma$ by an element of $U(\mathbb{Q})$.

Then, we have an exact sequence $0 \to \mathfrak{h} \cap \mathfrak{u} \to \mathfrak{h} \to \mathfrak{g}_0 \to 0$; applying the Levi decomposition theorem for Lie algebras, and we obtain a section $\tau : \mathfrak{g}_0 \to \mathfrak{h}$ of $\mathfrak{h} \to \mathfrak{g}_0$. Now the composite map $\mathfrak{g}_0 \xrightarrow{\tau} \mathfrak{h} \hookrightarrow \mathfrak{g}$ is a section of $\mathfrak{g} \to \mathfrak{g}_0$, hence of the form $\mathrm{Ad}(x) \circ \sigma$, where $x \in U(\mathbb{Q})$. If $\mathfrak{v} := \mathfrak{h} \cap \mathfrak{u}$, we then have $[\mathrm{Ad}(x)(\sigma(\mathfrak{g}_0)), \mathfrak{v}] \subset \mathfrak{v}$.

Let then $V \subset U$ be the subgroup with Lie algebra $\mathfrak{v}$; if we set $H := V \cdot \mathrm{Ad}(x)(\tilde{\sigma}(G_0)) = \mathrm{Ad}(x)(\tilde{\sigma}(G_0)) \cdot V$, then $H$ is in the first set, and has Lie algebra $\mathfrak{h}$.

*Proof of Proposition 3.18* Let $\mathrm{Lie}(u, v)$ be the smallest closed Lie subalgebra of

$$\mathrm{Lie}\, \mathrm{Aut}(F_2(-))^{op}$$

containing $u$ and $v$. Then $\mathrm{Lie}\langle B_3 \rangle \supset \mathrm{Lie}(u, v)$. Apply now Lemma 3.19 with $G = R_{ell}(-)$, $G_0 = \mathrm{SL}_2$. The map $\mathfrak{g} \to \mathfrak{g}_0 = \mathfrak{sl}_2$ is such that $u \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $v \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so if $\mathfrak{h} := \mathrm{Lie}(u, v)$, then $\mathrm{Im}(\mathfrak{h} \subset \mathfrak{g} \to \mathfrak{g}_0) = \mathfrak{g}_0$. Let then $H \subset R_{ell}(-)$ be the proalgebraic subgroup corresponding to $\mathfrak{h}$ by Lemma 3.19; then $\langle B_3 \rangle \supset H$. On the other hand, we have group morphisms $\mathbb{G}_a \to H$ corresponding to $\mathbb{Q} \to \mathfrak{h}$, $1 \mapsto u, v$, whose versions over $\mathbb{Q}$ are $(\mathbb{Q}, +) \to H(\mathbb{Q}) \subset \mathrm{Aut}(F_2(\mathbb{Q}))^{op}$, $t \mapsto \Psi_{\pm}^t$. Setting $t = 1$, we obtain $H(\mathbb{Q}) \ni \Psi_{\pm}$, and as $\Psi_+, \Psi_-$ generate $B_3$, $H(\mathbb{Q}) \supset B_3$. So $\langle B_3 \rangle = H$. Taking Lie algebras, we obtain Proposition 3.18. □

*Remark 3.20* Let $d := [[u, v], u] + 2u$, $e := [[u, v], v] - 2v$. Then for any $(\alpha, \beta, \gamma) \in \mathbb{N}^3$,

$$x_{\alpha,\beta,\gamma} := \mathrm{ad}(u)^\alpha\, \mathrm{ad}(v)^\beta\, \mathrm{ad}([u, v])^\gamma (d),$$
$$y_{\alpha,\beta,\gamma} := \mathrm{ad}(u)^\alpha\, \mathrm{ad}(v)^\beta\, \mathrm{ad}([u, v])^\gamma (e) \in \mathrm{Ker}(\mathrm{Lie}\langle B_3 \rangle \to \mathfrak{sl}_2).$$

Then, $\mathrm{Ker}(\mathrm{Lie}\langle B_3 \rangle \to \mathfrak{sl}_2)$ is topologically generated by these elements, and more precisely, it is equal to $\left\{\sum_{n \geq 1} P_n((x_{\alpha,\beta,\gamma})_{\alpha,\beta,\gamma}, (y_{\alpha,\beta,\gamma})_{\alpha,\beta,\gamma}) | (P_n)_n \in \prod_{n \geq 1} \mathfrak{f}_n\right\}$, where $\mathfrak{f}_n$ is the part of degree $n$ of the free Lie algebra with generators indexed by $\mathbb{N}^3 \sqcup \mathbb{N}^3$ (each generator having degree 1). Then, $\mathrm{Lie}\langle B_3\rangle = \mathrm{Ker}(\mathrm{Lie}\langle B_3\rangle \to \mathfrak{sl}_2) \oplus \mathrm{Span}_{\mathbb{Q}}(u, v, [u, v])$.

### 3.6 A morphism $\underline{\mathrm{GT}} \to \underline{\mathrm{GT}}_{ell}$ and its variants

We now construct a section of the semigroup morphism $\underline{\mathrm{GT}}_{ell} \to \underline{\mathrm{GT}}$ and of its variants.

**Proposition 3.21** *There exists a unique semigroup morphism $\underline{\mathrm{GT}} \to \underline{\mathrm{GT}}_{ell}$, defined by $(\lambda, f) \mapsto (\lambda, f, g_\pm)$, where*

$$g_+(X, Y) = f(X, (Y, X))X^\lambda f(X, (Y, X))^{-1},$$
$$g_-(X, Y) = (Y, X)^{\frac{\lambda-1}{2}} f(YX^{-1}Y^{-1}, (Y, X))Yf(X, (Y, X))^{-1}.$$

*The same formulas define semigroup morphisms $\widehat{\underline{\mathrm{GT}}} \to \widehat{\underline{\mathrm{GT}}}_{ell}$, $\underline{\mathrm{GT}}_l \to \underline{\mathrm{GT}}_l^{ell}$, and a semigroup scheme morphism $\underline{\mathrm{GT}}(-) \to \underline{\mathrm{GT}}_{ell}(-)$, compatible with the natural maps between the various versions of $\underline{\mathrm{GT}}_{(ell)}$.*

*There are commutative diagrams*

$$\begin{array}{ccc} \underline{\mathrm{GT}} & \longrightarrow & \underline{\mathrm{GT}}_{ell} \\ \sim\downarrow & & \sim\downarrow \\ \mathbb{Z}/2 & \longrightarrow & \tilde{B}_3 \end{array} \quad \text{and} \quad \begin{array}{ccc} \underline{\mathrm{GT}}(-) & \longrightarrow & \underline{\mathrm{GT}}_{ell}(-) \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathrm{M}_2 \end{array}$$

*where the bottom morphisms are $\bar{1} \mapsto \varepsilon\Psi_+\Psi_-\Psi_+$ and $\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$.*

*Proof* 1) As the centre $Z(B_{n+1})$ of $B_{n+1}$ is contained in the pure braid group $P_{n+1} := \mathrm{Ker}(B_{n+1} \to S_{n+1})$, the morphism $B_{n+1} \to S_{n+1}$ descends to a morphism $B_{n+1}/Z(B_{n+1}) \to S_{n+1}$. Identify $S_{n+1} \simeq \mathrm{Perm}(\{0, \dots, n\})$ and let $S_n \subset S_{n+1}$ be $\{\sigma | \sigma(0) = 0\}$. The Cartesian product

$$(B_{n+1}/Z(B_{n+1})) \times_{S_{n+1}} S_n$$

then identifies with the quotient $(B_{n+1} \times_{S_{n+1}} S_n)/Z(B_{n+1})$ relative to the sequence of inclusions $Z(B_{n+1}) \subset B_{n+1} \times_{S_{n+1}} S_n \subset B_{n+1}$. The middle subgroup identifies with a type B braid group and is generated by $\sigma_0^2, \sigma_1, \dots, \sigma_{n-1}$, where the generators of $B_{n+1}$ are labelled $\sigma_0, \dots, \sigma_{n-1}$. Using the presentation of the type B group, one proves that there is a unique morphism $B_{n+1} \times_{S_{n+1}} S_n \to B_{n+1}$, such that $\sigma_0^2 \mapsto X_1^+$, $\sigma_i \mapsto \sigma_i$ $(i > 1)$. Moreover, this morphism takes a generator of $Z(B_{n+1}) \simeq \mathbb{Z}$ to $X_1^+ \cdots X_n^+ = 1 \in B_{1,n}$. It follows that it factors though a morphism $(B_{n+1} \times_{S_{n+1}} S_n)/Z(B_{n+1}) \to B_{1,n}$, i.e.

$$(B_{n+1}/Z(B_{n+1})) \times_{S_{n+1}} S_n \to B_{1,n}. \tag{16}$$

This morphism admits the following interpretation. If $X$ is a topological additive group, let $C_{[n]}(X) := \mathrm{Inj}([n], X)/S_n$, where Inj means the space of injections, $[n] := \{1, \dots, n\}$, and $\overline{C}_{[n]}(X) := C_{[n]}(X)/X$, where $X$ acts by addition of a constant function. We then have the identifications

$$\pi_1(C_{[n]}(\mathbb{C}^\times)) \simeq B_{n+1} \times_{S_{n+1}} S_n, \quad \pi_1(\overline{C}_n(\mathbb{C}^\times)) \simeq (B_{n+1} \times_{S_{n+1}} S_n)/Z(B_{n+1}),$$
$$\pi_1(\overline{C}_n(\mathbb{C}^\times/q^{\mathbb{Z}})) \simeq B_{1,n},$$

where $q$ is a real number with $0 < q < 1$. The canonical projection $\mathbb{C}^\times \to \mathbb{C}^\times/q^{\mathbb{Z}}$ and then induces a group morphism $\pi_1(\overline{C}_n(\mathbb{C}^\times)) \to \pi_1(\overline{C}_n(\mathbb{C}^\times/q^{\mathbb{Z}}))$, which turns out to coincide with (16).

Any $(\lambda, f) \in \underline{\mathrm{GT}}$ induces an endomorphism $F_{\lambda,f}$ of **PaB**, such that for any object $O$ and $z \in \mathbf{PaB}(O)$,

$$F_{\lambda,f}(z) = z^\lambda$$

if $z$ corresponds to an element of $Z(B_{|O|})$.

The element $\sigma_2\sigma_1\sigma_0^2 \in B_4$ corresponds to

$$(\mathrm{id}_\bullet \otimes \beta_{\bullet,\bullet\bullet}) a_{\bullet,\bullet,\bullet\bullet} (\beta^2_{\bullet,\bullet} \otimes \mathrm{id}_{\bullet\bullet}) a^{-1}_{\bullet,\bullet,\bullet\bullet} (\mathrm{id}_\bullet \otimes a_{\bullet,\bullet,\bullet}) \in \mathbf{PaB}(\bullet((\bullet\bullet)\bullet)). \quad (17)$$

The image of (17) by this endomorphism is the product of the images of its factors, namely

$$\begin{aligned}
F_{\lambda,f}(\mathrm{id}_\bullet \otimes \beta_{\bullet,\bullet\bullet}) &= \mathrm{id}_\bullet \otimes \beta_{\bullet,\bullet\bullet}(\beta_{\bullet\bullet,\bullet}\beta_{\bullet,\bullet\bullet})^m \in \mathbf{PaB}(\bullet(\bullet(\bullet\bullet)), \bullet((\bullet\bullet)\bullet)) \\
&\leftrightarrow \sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^m \in B_4, \\
F_{\lambda,f}(a_{\bullet,\bullet,\bullet\bullet}) &= a_{\bullet,\bullet,\bullet\bullet} f(\beta^2_{\bullet,\bullet} \otimes \mathrm{id}_{\bullet\bullet}, a^{-1}(\mathrm{id}_\bullet \otimes \beta_{\bullet\bullet,\bullet}\beta_{\bullet,\bullet\bullet})a) \\
&\quad \in \mathbf{PaB}((\bullet\bullet)(\bullet\bullet), \bullet(\bullet(\bullet\bullet))) \\
&\leftrightarrow f(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1) \in B_4, \\
F_{\lambda,f}(\beta^2_{\bullet,\bullet} \otimes \mathrm{id}_{\bullet\bullet}) &= \beta^{2\lambda}_{\bullet,\bullet} \otimes \mathrm{id}_{\bullet\bullet} \in \mathbf{PaB}((\bullet\bullet)(\bullet\bullet)) \leftrightarrow \sigma_0^{2\lambda} \in B_4, \\
F_{\lambda,f}(\mathrm{id}_\bullet \otimes a_{\bullet,\bullet,\bullet}) &= \mathrm{id}_\bullet \otimes a_{\bullet,\bullet,\bullet} f(\beta^2_{\bullet\bullet} \otimes \mathrm{id}_\bullet, a^{-1}(\mathrm{id}_\bullet \otimes \beta^2_{\bullet\bullet})a) \\
&\quad \in \mathbf{PaB}(\bullet((\bullet\bullet)\bullet), \bullet(\bullet(\bullet\bullet))) \\
&\leftrightarrow f(\sigma_1^2, \sigma_2^2) \in B_4.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&F_{\lambda,f}((17)) \in \mathbf{PaB}(\bullet((\bullet\bullet)\bullet)) \\
&\quad \leftrightarrow \sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^m f(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1)\sigma_0^{2\lambda} f^{-1}(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1) f(\sigma_1^2, \sigma_2^2) \in B_4.
\end{aligned}$$

Now, $(\sigma_2\sigma_1\sigma_0^2)^3$ generates $Z(B_4)$, therefore

$$(17)^3 \in \mathbf{PaB}(\bullet((\bullet\bullet)\bullet)) \leftrightarrow (\sigma_2\sigma_1\sigma_0^2)^3 \in Z(B_4).$$

It follows that $F_{\lambda,f}((17)^3) = (17)^{3\lambda}$. The image of this equality in $B_4$ is

$$\Big(\sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^m f(\sigma_0^2,\sigma_1\sigma_2^2\sigma_1)\sigma_0^{2\lambda} f^{-1}(\sigma_0^2,\sigma_1\sigma_2^2\sigma_1) f(\sigma_1^2,\sigma_2^2)\Big)^3 = (\sigma_2\sigma_1\sigma_0^2)^{3\lambda}.$$

As $Z(B_4)$ is in the kernel of $B_4 \times_{S_4} S_4 \to B_{1,3}$, the image of the left-hand side of this equality under this morphism is $1 \in B_{1,3}$. It follows that

$$\Big(\sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^m f(X_1^+,(X_1^-,X_1^+))(X_1^+)^\lambda f^{-1}(X_1^+,(X_1^-,X_1^+)) f(\sigma_1^2,\sigma_2^2)\Big)^3 = 1 \tag{18}$$

in $B_{1,3}$. This means that identity (9) is satisfied with $\pm = +$.

2) We show that $g_-$ satisfies (9) with $\pm = -$, i.e.

$$\Big(\sigma_2^{-1}\sigma_1^{-1}(\sigma_1\sigma_2^2\sigma_1)^{-m} g_-(X_1,Y_1) f(\sigma_1^2,\sigma_2^2)\Big)^3 = 1$$

in $B_{1,3}$ (we set $X_i := X_i^+$, $Y_i := X_i^-$). Substituting the given expression for $g_-(X_1,Y_1)$, using $(Y_1,X_1) = \sigma_1\sigma_2^2\sigma_1$, the identities $(\sigma_2^{-1}\sigma_1^{-1})Y_1X_1^{-1}Y_1^{-1} = X_3^{-1}(\sigma_2^{-1}\sigma_1^{-1})$, $(\sigma_2^{-1}\sigma_1^{-1})\sigma_1\sigma_2^2\sigma_1 = \sigma_2\sigma_1^2\sigma_2(\sigma_2^{-1}\sigma_1^{-1})$, and after a suitable conjugation, this equality is equivalent to

$$\Big(Y_1 f^{-1}(X_1,\sigma_1\sigma_2^2\sigma_1) f(\sigma_1^2,\sigma_2^2) f(X_3^{-1},\sigma_2\sigma_1^2\sigma_2)\sigma_2^{-1}\sigma_1^{-1}\Big)^3 = 1. \tag{19}$$

As $f \in F_2'$, $f(a\alpha,b) = f(a,b)$ if $\alpha$ commutes with both $a$ and $b$. In particular, $\sigma_1^2$ commutes (in $B_4$) with both $\sigma_0\sigma_1^2\sigma_0$ and $\sigma_2\sigma_1^2\sigma_2$. It follows that $f(\sigma_0\sigma_1^2\sigma_0,\sigma_2\sigma_1^2\sigma_2) = f((\sigma_1^2\sigma_0)^2,\sigma_2\sigma_1^2\sigma_2)$. Since $(\sigma_1^2\sigma_0)^2 = (\sigma_1\sigma_0^2)^2$, $f(\sigma_0\sigma_1^2\sigma_0, \sigma_2\sigma_1^2\sigma_2) = f((\sigma_1\sigma_0^2)^2,\sigma_2\sigma_1^2\sigma_2)$. Substituting this identity in the pentagon identity

$$f(\sigma_1^2,\sigma_2^2) f(\sigma_0\sigma_1^2\sigma_0,\sigma_2\sigma_1^2\sigma_2) f(\sigma_0^2,\sigma_1^2) = f(\sigma_0^2,\sigma_1\sigma_2^2\sigma_1) f(\sigma_1\sigma_0^2\sigma_1,\sigma_2^2)$$

in $P_4 := \mathrm{Ker}(B_4 \to S_4)$, taking the image of the resulting identity by the morphism $P_4 \subset B_4 \times_{S_4} S_3 \to B_{1,3}$, and using the identity $X_2X_1 = X_3^{-1}$ in $B_{1,3}$, one obtains

$$f(\sigma_1^2,\sigma_2^2) f(X_3^{-1},\sigma_2\sigma_1^2\sigma_2) f(X_1,\sigma_1^2) = f(X_1,\sigma_1\sigma_2^2\sigma_1) f(X_2,\sigma_2^2)$$

(identity in $B_{1,3}$). Using this identity, (19) is equivalent to

$$(Y_1A\sigma_2^{-1}\sigma_1^{-1})^3 = 1,$$

where $A := f(X_2,\sigma_2^2) f^{-1}(X_1,\sigma_1^2)$. Using $Y_3 = \sigma_2^{-1}\sigma_1^{-1}Y_1\sigma_1^{-1}\sigma_2^{-1}$, $Y_2 = \sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}Y_1\sigma_2^{-1}\sigma_1^{-1}$, the latter identity is equivalent to

$$Y_1AY_3(\sigma_2\sigma_1A\sigma_1^{-1}\sigma_2^{-1})Y_2(\sigma_1\sigma_2A\sigma_2^{-1}\sigma_1^{-1}) = 1. \tag{20}$$

As $Y_1X_2 = (X_2\sigma_1^{-2})Y_1$, $Y_1\sigma_2^2 = \sigma_2^2Y_1$, $X_1Y_3 = Y_3(\sigma_2\sigma_1^2\sigma_2^{-1}X_1)$, $\sigma_1^2Y_3 = Y_3\sigma_1^2$, and $Y_1Y_3 = Y_2^{-1}$,

$$Y_1AY_3 = f(X_2\sigma_1^{-2}, \sigma_2^2)Y_2^{-1}f^{-1}(\sigma_2\sigma_1^2\sigma_2^{-1}X_1, \sigma_1^2).$$

As $\mathrm{Ad}(\sigma_2\sigma_1)(X_2) = \sigma_2\sigma_1^2\sigma_2^{-1}X_1$, $\mathrm{Ad}(\sigma_2\sigma_1)(\sigma_2^2) = \sigma_1^2$, $\mathrm{Ad}(\sigma_2\sigma_1)(X_1) = X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}$, $\mathrm{Ad}(\sigma_2\sigma_1)(\sigma_1^2) = \sigma_2\sigma_1^2\sigma_2^{-1}$,

$$\sigma_2\sigma_1A\sigma_1^{-1}\sigma_2^{-1} = f(\sigma_2\sigma_1^2\sigma_2^{-1}X_1, \sigma_1^2)f^{-1}(X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}, \sigma_2\sigma_1^2\sigma_2^{-1}).$$

As $\mathrm{Ad}(\sigma_1\sigma_2)(X_2) = X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2$, $\mathrm{Ad}(\sigma_1\sigma_2)(\sigma_2^2) = \sigma_2\sigma_1^2\sigma_2^{-1}$, $\mathrm{Ad}(\sigma_1\sigma_2)(X_1) = X_2\sigma_1^{-2}$, $\mathrm{Ad}(\sigma_1\sigma_2)(\sigma_1^2) = \sigma_2^2$,

$$\sigma_1\sigma_2A\sigma_2^{-1}\sigma_1^{-1} = f(X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2, \sigma_2\sigma_1^2\sigma_2^{-1})f^{-1}(X_2\sigma_1^{-2}, \sigma_2^2).$$

Taking these equalities into account and after simplification and conjugation, (20) is equivalent to

$$Y_2^{-1}f^{-1}(X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}, \sigma_2\sigma_1^2\sigma_2^{-1})Y_2f(X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2, \sigma_2\sigma_1^2\sigma_2^{-1}) = 1,$$

which follows from $Y_2^{-1} \cdot X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1} \cdot Y_2 = X_3\sigma_2^{-1}\sigma_1^{-2}\sigma_2$ and from the fact that $Y_2$ commutes with $\sigma_2\sigma_1^2\sigma_2^{-1}$.

3) Since $\sigma_1^2$ commutes with both $\sigma_2\sigma_1^2\sigma_2$ and $\sigma_0\sigma_1^2\sigma_0$ and since $f \in F_2'$, one has

$$f(\sigma_2\sigma_1^2\sigma_2, \sigma_0\sigma_1^2\sigma_0) = f(\sigma_2\sigma_1^2\sigma_2, (\sigma_0\sigma_1^2)^2)$$

(equality in $B_4$). Since $(\sigma_0\sigma_1^2)^2 \equiv (\sigma_2\sigma_1\sigma_0^2\sigma_1\sigma_2)^{-1} \bmod Z(P_4)$ and $f \in F_2'$, one has

$$f(\sigma_2\sigma_1^2\sigma_2, (\sigma_0\sigma_1^2)^2) = f(\sigma_2\sigma_1^2\sigma_2, (\sigma_2\sigma_1\sigma_0^2\sigma_1\sigma_2)^{-1})$$

(equality in $B_4$). Plugging these equalities in the pentagon equation

$$f(\sigma_1\sigma_0^2\sigma_1, \sigma_2^2)f^{-1}(\sigma_0^2, \sigma_1^2)f(\sigma_2\sigma_1^2\sigma_2, \sigma_0\sigma_1^2\sigma_0)f^{-1}(\sigma_1^2, \sigma_2^2)f(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1) = 1$$

(in $B_4$) and multiplying by $f^{-1}(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1)$ from the right, one obtains

$$f(\sigma_1\sigma_0^2\sigma_1, \sigma_2^2)f^{-1}(\sigma_0^2, \sigma_1^2)f(\sigma_2\sigma_1^2\sigma_2, (\sigma_2\sigma_1\sigma_0^2\sigma_1\sigma_2)^{-1})f^{-1}(\sigma_1^2, \sigma_2^2)$$
$$= f^{-1}(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1)$$

(in $B_4$). As $\sigma_2$ commutes with both $\sigma_0^2$ and $\sigma_1\sigma_2^2\sigma_1$, the right side of this equality, and therefore also its left side, commutes with $\sigma_2^\lambda$. It follows that the equality also holds with the left side replaced by its conjugation of $\sigma_2^\lambda$; multiplying the resulting equality by $f(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1)$ from the right, and using the identities $\sigma_2\sigma_1^2\sigma_2 =$

$\sigma_2^{-1}\sigma_1^{-1}(\sigma_1\sigma_2^2\sigma_1)\sigma_1\sigma_2,\ (\sigma_2\sigma_1\sigma_0^2\sigma_1\sigma_2)^{-1} = \sigma_2^{-1}\sigma_1^{-1}(\sigma_1\sigma_2^2\sigma_1\sigma_0^2)^{-1}\sigma_1\sigma_2$, one obtains

$$\sigma_2^{\lambda} f(\sigma_1\sigma_0^2\sigma_1, \sigma_2^2) f^{-1}(\sigma_0^2, \sigma_1^2)\sigma_2^{-1}\sigma_1^{-1} f(\sigma_1\sigma_2^2\sigma_1, (\sigma_1\sigma_2^2\sigma_1\sigma_0^2)^{-1})\sigma_1\sigma_2 f^{-1}(\sigma_1^2, \sigma_2^2)\sigma_2^{-\lambda} f(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1) = 1. \tag{21}$$

On the other hand, $\sigma_1^{-1}\sigma_2^2\sigma_1 = \sigma_2\sigma_1^2\sigma_2^{-1} \equiv (\sigma_2^2\sigma_1^2)^{-1} \bmod Z(P_3)$ (equalities in $P_3$); together with $f \in F_2'$, this implies $f((\sigma_2^2\sigma_1^2)^{-1}, \sigma_2^2) = f(\sigma_2\sigma_1^2\sigma_2^{-1}, \sigma_2^2)$ and $f(\sigma_1^2, (\sigma_2^2\sigma_1^2)^{-1}) = f(\sigma_1^2, \sigma_1^{-1}\sigma_2^2\sigma_1)$ (equalities in $P_3$). Plugging these equalities in the hexagon equation

$$1 = (\sigma_2^2)^m f((\sigma_2^2\sigma_1^2)^{-1}, \sigma_2^2)(\sigma_2^2\sigma_1^2)^{-m} f(\sigma_1^2, (\sigma_2^2\sigma_1^2)^{-1})(\sigma_1^2)^m f(\sigma_2^2, \sigma_1^2)$$

(in $P_3$), using the equalities $f(\sigma_2\sigma_1^2\sigma_2^{-1}, \sigma_2^2) = \sigma_2 f(\sigma_1^2, \sigma_2^2)\sigma_2^{-1}$, $f(\sigma_1^2, \sigma_1^{-1}\sigma_2^2\sigma_1) = \sigma_1^{-1} f(\sigma_1^2, \sigma_2^2)\sigma_1$, $(\sigma_2^2)\sigma_2 = \sigma_2^{\lambda}$, $\sigma_1(\sigma_1^2)^m = \sigma_1^{\lambda}$, multiplying by $\sigma_1\sigma_2 f^{-1}(\sigma_1^2, \sigma_2^2)\sigma_2^{-\lambda}$ from the left and using $\sigma_1(\sigma_2^2\sigma_1^2)^{-m}\sigma_1^{-1} = (\sigma_1\sigma_2^2\sigma_1)^{\frac{1-\lambda}{2}}$, one obtains

$$\sigma_1\sigma_2 f^{-1}(\sigma_1^2, \sigma_2^2)\sigma_2^{-\lambda} = (\sigma_1\sigma_2^2\sigma_1)^{\frac{1-\lambda}{2}} f(\sigma_1^2, \sigma_2^2)\sigma_1^{\lambda} f^{-1}(\sigma_1^2, \sigma_2^2).$$

Plugging this equality in (21), one obtains

$$\sigma_2^{\lambda} f(\sigma_1\sigma_0^2\sigma_1, \sigma_2^2) f^{-1}(\sigma_0^2, \sigma_1^2)\sigma_2^{-1}\sigma_1^{-1} f(\sigma_1\sigma_2^2\sigma_1, \sigma_0^{-2}\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}) (\sigma_1\sigma_2^2\sigma_1)^{\frac{1-\lambda}{2}} f(\sigma_1^2, \sigma_2^2)\sigma_1^{\lambda} f^{-1}(\sigma_1^2, \sigma_2^2) f(\sigma_0^2, \sigma_1\sigma_2^2\sigma_1) = 1$$

(in $B_4 \times_{S_4} S_3$). Taking the image of this equality under $B_4 \times_{S_4} S_3 \to B_{1,3}$ and multiplying the resulting equality by $\sigma_2 f^{-1}(X_2, \sigma_2^2)\sigma_2^{-\lambda}$ from the left, one obtains

$$\sigma_2 f^{-1}(X_2, \sigma_2^2)\sigma_2^{-\lambda} = \sigma_1^{-1} f^{-1}(X_2\sigma_1^{-2}, \sigma_2^2) f((Y_1, X_1), X_1^{-1} (Y_1, X_1)^{-1})(Y_1, X_1)^{\frac{1-\lambda}{2}} f(\sigma_1^2, \sigma_2^2)\sigma_1^{\lambda} f^{-1}(\sigma_1^2, \sigma_2^2) f(X_1, (Y_1, X_1))$$

(in $B_{1,3}$). As both $\sigma_2$ and $X_2$ commute with $X_1$, the left side of this equality commutes with $X_1^{\lambda}$, and therefore so does its right side. Expressing the equality of $X_1^{\lambda}$ with its conjugate by the right side, and conjugating the resulting equality, one obtains

$$\mathrm{Ad}\Big(f((Y_1, X_1), X_1^{-1}(Y_1, X_1)^{-1})(Y_1, X_1)^{\frac{1-\lambda}{2}} f(\sigma_1^2, \sigma_2^2)\sigma_1^{\lambda} f^{-1}(\sigma_1^2, \sigma_2^2) f(X_1, (Y_1, X_1))\Big)((X_1)^{\lambda}) = \mathrm{Ad}\Big(f(X_2\sigma_1^{-2}, \sigma_2^2)\sigma_1\Big)((X_1)^{\lambda}) \tag{22}$$

(in $B_{1,3}$).

The equality

$$a_{\bullet\bullet,\bullet,\bullet}(\beta_{\bullet,\bullet\bullet}\beta_{\bullet\bullet,\bullet}\otimes \mathrm{id}_\bullet)a^{-1}_{\bullet\bullet,\bullet,\bullet}$$
$$= a^{-1}_{\bullet,\bullet,\bullet\bullet}(\mathrm{id}_\bullet \otimes a_{\bullet,\bullet,\bullet})(\mathrm{id}_\bullet \otimes(\beta_{\bullet,\bullet}\otimes \mathrm{id}_\bullet))(\mathrm{id}_\bullet \otimes a^{-1}_{\bullet,\bullet,\bullet})a_{\bullet,\bullet,\bullet\bullet}\cdot(\beta^2_{\bullet,\bullet}\otimes \mathrm{id}_{\bullet\bullet})$$
$$\cdot a^{-1}_{\bullet,\bullet,\bullet\bullet}(\mathrm{id}_\bullet \otimes a_{\bullet,\bullet,\bullet})(\mathrm{id}_\bullet \otimes(\beta_{\bullet,\bullet}\otimes \mathrm{id}_\bullet))(\mathrm{id}_\bullet \otimes a^{-1}_{\bullet,\bullet,\bullet})a_{\bullet,\bullet,\bullet\bullet}$$

in $\mathbf{PaB}((\bullet\bullet)(\bullet\bullet))$ follows from the fact that both sides correspond to the element $\sigma_1\sigma_0^2\sigma_1 \in B_4$. Applying the automorphism $F_{\lambda,f}$ to this equality, one obtains an equality in $\mathbf{PaB}((\bullet\bullet)(\bullet\bullet))$, which translates into the equality

$$f(\sigma_1\sigma_0^2\sigma_1,\sigma_2^2)(\sigma_1\sigma_0^2\sigma_1)^\lambda f(\sigma_1\sigma_0^2\sigma_1,\sigma_2^2)$$
$$= f(\sigma_0^2,\sigma_1\sigma_2^2\sigma_1)^{-1}f(\sigma_1^2,\sigma_2^2)\sigma_1^\lambda f(\sigma_1^2,\sigma_2^2)^{-1}f(\sigma_0^2,\sigma_1\sigma_2^2\sigma_1)\cdot\sigma_0^{2\lambda}\cdot$$
$$\cdot f(\sigma_0^2,\sigma_1\sigma_2^2\sigma_1)^{-1}f(\sigma_1^2,\sigma_2^2)\sigma_1^\lambda f(\sigma_1^2,\sigma_2^2)^{-1}f(\sigma_0^2,\sigma_1\sigma_2^2\sigma_1)$$

in $B_4\times_{S_4} S_3\subset B_4$.
As $(Y_1,X_1)=\sigma_1\sigma_2^2\sigma_1$ (relation in $B_{1,3}$), the image of this equality in $B_{1,3}$ is

$$f(X_2,\sigma_2^2)X_2^\lambda f(X_2,\sigma_2^2)^{-1} = f(X_1,(Y_1,X_1))^{-1}ug_+uf(X_1,(Y_1,X_1)),$$

where

$$u := f(\sigma_1^2,\sigma_2^2)\sigma_1^\lambda f(\sigma_1^2,\sigma_2^2)^{-1},\quad g_+ := f(X_1,(Y_1,X_1))X_1^{2\lambda}f(X_1,(Y_1,X_1))^{-1}$$

(elements of $B_{1,3}$). Conjugating by $Y_1$ and using $Y_1X_2Y_1^{-1}=X_2\sigma_1^{-2}$, $Y_1\sigma_1^2Y_1^{-1}=\sigma_1^2$, one obtains

$$f(X_2\sigma_1^{-2},\sigma_2^2)(X_2\sigma_1^{-2})^\lambda f(X_2\sigma_1^{-2},\sigma_2^2)^{-1}$$
$$= Y_1f(X_1,(Y_1,X_1))^{-1}ug_+uf(X_1,(Y_1,X_1))Y_1^{-1}.$$

As $X_2\sigma_1^{-2}=\sigma_1X_1\sigma_1^{-1}$, the left side of this identity identifies with the right side of (22). Combining these identities, one gets

$$\mathrm{Ad}\left(f((Y_1,X_1),Y_1X_1^{-1}Y_1^{-1})(Y_1,X_1)^{\frac{1-\lambda}{2}}\right)(ug_+u^{-1})$$
$$= \mathrm{Ad}\left(Y_1f(X_1,(Y_1,X_1))^{-1}\right)(ug_+u),$$

which gives after conjugation

$$ug_+u^{-1} = g_-ug_+ug_-^{-1},$$

where $g_- := g_-(X_1,Y_1)$, which is equivalent to $u^2=(ug_+^{-1}u^{-1},g_-^{-1})$, so the pair $(g_+,g_-)$ defined in the statement of the Proposition satisfies (10).

4) The fact that $\underline{\mathrm{GT}} \to \underline{\mathrm{GT}}_{ell}$, $(\lambda, f) \mapsto (\lambda, f, g_\pm)$ is a morphism of semigroups follows from the identity $(g_-(X,Y), g_+(X,Y)) = (Y,X)^\lambda$. It is straightforward to check the commutativity of the first diagram; the second diagram follows from $((\lambda, f) \in \underline{\mathrm{GT}}(\mathbf{k})) \Rightarrow (\log f \in [\hat{\mathfrak{f}}_2^{\mathbf{k}}, \hat{\mathfrak{f}}_2^{\mathbf{k}}])$.
5) The arguments used in the case of $\underline{\mathrm{GT}}_{(ell)}$ extend *mutatis* to their profinite, pro-$l$, and prounipotent versions. □

*Remark 3.22* There are compatible group morphisms $\mathrm{GT} \to \mathrm{Aut}(R_{ell})$, $\mathrm{GT}_l \to \mathrm{Aut}(R_l^{ell})$, $\mathrm{GT}(\mathbf{k}) \to \mathrm{Aut}(R_{ell}(\mathbf{k}))$ (where $R_l^{ell} = \mathrm{Ker}(\underline{\mathrm{GT}}_l^{ell} \to \underline{\mathrm{GT}}_l)$), defined by $(\lambda, f) \mapsto \theta_{\lambda,f} :=$ conjugation by the image of $(\lambda, f) \mapsto (\lambda, f, g_\pm)$ from Proposition 3.21. One computes $\theta_{\lambda,f}(\Psi_+) = \Psi_+^{1/\lambda}$ and $\theta_{\lambda,f}((\Psi_+\Psi_-)^3) = ((\Psi_+\Psi_-)^{3/\lambda}$, where $(\Psi_+\Psi_-)^3$ is a generator of $Z(B_3) = \mathbb{Z}$ and $(\Psi_+\Psi_-)^{3(1+2m)} = (\Psi_+\Psi_-)^3(\Psi_+\Psi_-)^{6m} = (\Psi_+\Psi_-)^3 \,\mathrm{Ad}(Y,X)^m$.

The semigroup scheme morphism from Proposition 3.21 restricts to a group scheme morphism, which yields an action of $\mathrm{GT}(-)$ on $R_{ell}(-)$. The group scheme $\mathrm{GT}_{ell}(-)$ has then a semidirect product structure, fitting in the diagram

$$\begin{array}{ccc} \mathrm{GT}_{ell}(-) & \simeq & R_{ell}(-) \rtimes \mathrm{GT}(-) \\ \downarrow & & \downarrow \\ \mathrm{GL}_2 & \simeq & \mathrm{SL}_2 \rtimes \mathbb{G}_m \end{array}$$

where the bottom map is induced by $\mathbb{G}_m \to \mathrm{GL}_2$, $\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$.

## 4 Elliptic associators

In this section, we introduce the notion of elliptic associator. This notion yields particular elliptic structures over BMCs. It gives rise to a scheme of elliptic associators, which appears to be a torsor under the action of the group scheme $\mathrm{GT}_{ell}(-)$. We construct a morphism of torsors from the scheme of associators to its elliptic analogue, which enables us to establish the existence of rational elliptic associators.

### 4.1 Lie algebras $\mathfrak{t}_n$ and $\mathfrak{t}_{1,n}$

Let $\mathbf{k}$ be a $\mathbb{Q}$-ring. If $S$ is a finite set, we define $\mathfrak{t}_S^{\mathbf{k}}$ as the $\mathbf{k}$-Lie algebra with generators $t_{ij}$, $i \neq j \in S$ and relations $t_{ji} = t_{ij}$, $[t_{ij}, t_{ik} + t_{jk}] = 0$ for $i, j, k$ distinct, $[t_{ij}, t_{kl}] = 0$ for $i, j, k, l$ distinct. We define $\hat{\mathfrak{t}}_S^{\mathbf{k}}$ as its degree completion, where $\deg(t_{ij}) = 1$.

For $S' \supset D_\phi \xrightarrow{\phi} S$ a partially defined map, there is a unique Lie algebra morphism $\mathfrak{t}_S^{\mathbf{k}} \to \mathfrak{t}_{S'}^{\mathbf{k}}$, $x \mapsto x^\phi$, defined by $(t_{ij})^\phi := \sum_{i' \in \phi^{-1}(i), j' \in \phi^{-1}(j)} t_{i'j'}$. Then, $S \mapsto \mathfrak{t}_S^{\mathbf{k}}$ is a contravariant functor (finite sets, partially defined maps) $\to$ {Lie algebras}.

We also define $\mathfrak{t}_{1,S}^{\mathbf{k}}$ as the $\mathbf{k}$-Lie algebra with generators $x_i^\pm$, $i \in S$ and relations $\sum_{i \in S} x_i^\pm = 0$, $[x_i^\pm, x_j^\pm] = 0$ for $i \neq j$, $[x_i^+, x_j^-] = [x_j^+, x_i^-]$ for $i \neq j$, $[x_k^\pm, [x_i^+, x_j^-]] = 0$ for $i, j, k$ distinct. We then have a Lie algebra morphism

$\mathfrak{t}_S^{\mathbf{k}} \to \mathfrak{t}_{1,S}^{\mathbf{k}}$, $t_{ij} \mapsto [x_i^+, x_j^-]$, which we denote by $x \mapsto \{x\}$. We will also write $t_{ij} = [x_i^+, x_j^-]$. We define $\hat{\mathfrak{t}}_{1,S}^{\mathbf{k}}$ as the degree completion of $\mathfrak{t}_{1,S}^{\mathbf{k}}$, where $\deg(x_i^{\pm}) = 1$.

For $S' \stackrel{\phi}{\to} S$ a map, there is a unique Lie algebra morphism $\mathfrak{t}_{1,S}^{\mathbf{k}} \to \mathfrak{t}_{1,S'}^{\mathbf{k}}$, $x \mapsto x^{\phi}$, such that $(x_i^{\pm})^{\phi} := \sum_{i' \in \phi^{-1}(i)} x_{i'}^{\pm}$. Then, $S \mapsto \mathfrak{t}_{1,S}^{\mathbf{k}}$ is a contravariant functor (finite sets, maps) $\to$ {Lie algebras}. By restriction, $S \mapsto \mathfrak{t}_S^{\mathbf{k}}$ may be viewed as a contravariant functor of the same type, and the morphism $\mathfrak{t}_S^{\mathbf{k}} \to \mathfrak{t}_{1,S}^{\mathbf{k}}$ is then functorial; that is, we have $\{x\}^{\phi} = \{x^{\phi}\}$ for $x \in \mathfrak{t}_S$ and any map $S' \stackrel{\phi}{\to} S$.

We set $\mathfrak{t}_n^{\mathbf{k}} := \mathfrak{t}_{[n]}^{\mathbf{k}}$, $\mathfrak{t}_{1,n} := \mathfrak{t}_{1,[n]}^{\mathbf{k}}$, where $[n] = \{1, \ldots, n\}$, and we write $x^{\phi}$ as $x^{I_1,\ldots,I_n}$, where $I_i = \phi^{-1}(i)$ for $x \in \mathfrak{t}_n^{\mathbf{k}}$ or $x \in \mathfrak{t}_{1,n}^{\mathbf{k}}$.

### 4.2 Elliptic associators

Recall that the set $\underline{M}(\mathbf{k})$ of associators defined over $\mathbf{k}$ is the set of $(\mu, \Phi) \in \mathbf{k} \times \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$, such that $\Phi^{3,2,1} = \Phi^{-1}$,

$$e^{\mu t_{23}/2}\Phi^{1,2,3}e^{\mu t_{12}/2}\Phi^{3,1,2}e^{\mu t_{31}/2}\Phi^{2,3,1} = e^{\mu(t_{12}+t_{13}+t_{23})/2}, \tag{23}$$

$$\Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} = \Phi^{1,2,34}\Phi^{12,3,4}, \tag{24}$$

where $\Phi$ is viewed as an element of $\exp(\hat{\mathfrak{t}}_3^{\mathbf{k}})$ via the inclusion $\hat{\mathfrak{f}}_2^{\mathbf{k}} \subset \hat{\mathfrak{t}}_3^{\mathbf{k}}$, $A, B \mapsto t_{12}, t_{23}$.

**Definition 4.1** The set $\underline{Ell}(\mathbf{k})$ of elliptic associators defined over $\mathbf{k}$ is the set of quadruples $(\mu, \Phi, A_+, A_-)$, where $(\mu, \Phi) \in \underline{M}(\mathbf{k})$ and $A_{\pm} \in \exp(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})$, such that:

$$\alpha_{\pm}^{3,1,2}\alpha_{\pm}^{2,3,1}\alpha_{\pm}^{1,2,3} = 1, \text{ where } \alpha_{\pm} = \{e^{\pm\mu(t_{12}+t_{13})/2}\}A_{\pm}^{1,23}\{\Phi^{1,2,3}\}, \tag{25}$$

$$\{e^{\mu t_{12}}\} = \big(\{\Phi\}^{-1}A_-^{1,23}\{\Phi\}, \{e^{-\mu t_{12}/2}(\Phi^{2,1,3})^{-1}\}(A_+^{2,13})^{-1} \{\Phi^{2,1,3}e^{-\mu t_{12}/2}\}\big). \tag{26}$$

*Remark 4.2* We then have $\{e^{\pm\mu t_{12}/2}\}A_{\pm}^{2,1}\{e^{\pm\mu t_{12}/2}\}A_{\pm}^{1,2} = 1$ and $\{e^{\mu t_{12}}\} = (A_-, A_+)$; here as in (26), the notation $(g, h)$ stands for the group commutator $ghg^{-1}h^{-1}$.

Then $\mathbf{k} \mapsto \underline{M}(\mathbf{k}), \underline{Ell}(\mathbf{k})$ are functors $\{\mathbb{Q}\text{-rings}\} \to \{\text{sets}\}$, i.e. $\mathbb{Q}$-schemes. We have an obvious scheme morphism $\underline{Ell} \to \underline{M}$, $(\mu, \Phi, A_+, A_-) \mapsto (\mu, \Phi)$.

Define also a scheme morphism $\underline{Ell} \to \mathrm{M}_2$ by $(\mu, \Phi, A_+, A_-) \mapsto \begin{pmatrix} u_+ & v_+ \\ u_- & v_- \end{pmatrix}$, where $u_{\pm}, v_{\pm}$ are the coefficients arising from $\log A_{\pm} \equiv u_{\pm}x_1^+ + v_{\pm}x_1^- \mod [\hat{\mathfrak{t}}_{1,2}, \hat{\mathfrak{t}}_{1,2}]$. Then, relation (26) implies that the diagram

$$\begin{array}{ccc} \underline{Ell} & \to & \underline{M} \\ \downarrow & & \downarrow \\ \mathrm{M}_2 & \stackrel{\det}{\to} & \mathbb{A} \end{array}$$

commutes.

### 4.3 Categorical interpretations

**Definition 4.3** (see [9]) An infinitesimally braided monoidal category (IBMC) over $\mathbf{k}$ is a set $(\mathcal{C}, \otimes, c_{\dots}, a_{\dots}, U_{\dots}, t_{\dots})$, such that:

1) $(\mathcal{C}, \otimes, c_{\dots}, a_{\dots})$ is a symmetric monoidal category (i.e., $c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}$);
2) $\mathrm{Ob}\,\mathcal{C} \ni X \mapsto U_X \lhd \mathrm{Aut}_{\mathcal{C}}(X)$ is such that $U_X$ is a $\mathbf{k}$-prounipotent group, and $iU_X i^{-1} = U_Y$ for any $i \in \mathrm{Iso}_{\mathcal{C}}(X, Y)$;
3) $(\mathrm{Ob}\,\mathcal{C})^2 \ni (X, Y) \mapsto t_{X,Y} \in \mathrm{Lie}\, U_{X\otimes Y}$ is a natural assignment;
4) $t_{Y,X} = c_{X,Y} t_{X,Y} c_{X,Y}^{-1}$ and

$$t_{X\otimes Y,Z} = a_{X,Y,Z}(\mathrm{id}_X \otimes t_{Y,Z})a_{X,Y,Z}^{-1} + (c_{Y,X} \otimes \mathrm{id}_Z)a_{Y,X,Z}(\mathrm{id}_Y \otimes t_{X,Z})((c_{Y,X} \otimes \mathrm{id}_Z)a_{Y,X,Z})^{-1}.$$

A functor $f : \mathcal{C} \to \mathcal{C}'$ between IBMCs is then a tensor functor, such that $f(U_X) \subset U'_{f(X)}$ and $f(t_{X,Y}) = t'_{f(X),f(Y)}$. An example of IMBC is constructed as follows: $\mathcal{C} = \mathbf{PaCD}$ is the category with the same objects as $\mathbf{PaB}$, $\mathbf{PaCD}(O, O') := \begin{cases} \exp(\hat{\mathfrak{t}}_{|O|}) \rtimes S_{|O|} & \text{if } |O| = |O'| \\ \emptyset & \text{otherwise} \end{cases}$, $c_{O,O'} = s_{|O|,|O'|} \in S_{|O|+|O'|} \subset \mathrm{Aut}_{\mathbf{PaCD}}(O \otimes O')$ is the permutation $i \mapsto i+|O'|$ for $i \in [1, |O|]$, $i \mapsto i-|O|$ for $i \in [|O|+1, |O|+|O'|]$, $a_{O,O',O''} := 1$, $U_O = \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{|O|}) \lhd \mathrm{Aut}_{\mathcal{C}}(O)$, $t_{O,O'} := \sum_{i=1}^{|O|} \sum_{i'=|O|+1}^{|O|+|O'|} t_{ii'}$. The pair $(\mathbf{PaCD}, \bullet)$ is initial among pairs (an IBMC, a distinguished object).

We then set:

**Definition 4.4** An elliptic structure over the IBMC $\mathcal{C}$ is a set $(\tilde{\mathcal{C}}, F, \tilde{U}_{\dots}, x^{\pm}_{\dots})$, where $\tilde{\mathcal{C}}$ is a category, $F : \mathcal{C} \to \tilde{\mathcal{C}}$ is a functor, $\mathrm{Ob}\,\tilde{\mathcal{C}} \ni \tilde{X} \mapsto \tilde{U}_{\tilde{X}} \lhd \mathrm{Aut}_{\tilde{\mathcal{C}}}(\tilde{X})$ is the assignment of a $\mathbf{k}$-prounipotent group, where $\tilde{i}\tilde{U}_{\tilde{X}}\tilde{i}^{-1} = \tilde{U}_{\tilde{Y}}$ for $\tilde{i} \in \mathrm{Iso}_{\tilde{\mathcal{C}}}(\tilde{X}, \tilde{Y})$ and $F(U_X) \subset \tilde{U}_{F(X)}$, and $(\mathrm{Ob}\,\mathcal{C})^2 \ni (X, Y) \mapsto x^{\pm}_{X,Y} \in \mathrm{Lie}\,\tilde{U}_{F(X\otimes Y)}$ is a natural assignment, such that

$$x^{\pm}_{Y,X} = F(c_{X,Y})x^{\pm}_{X,Y}F(c^{-1}_{X,Y}), \quad x^{\pm}_{X,\mathbf{1}} = 0,$$
$$x^{\pm}_{X\otimes Y,Z} + F(c_{X,Y\otimes Z}a_{X,Y,Z})^{-1}x^{\pm}_{Y\otimes Z,X}F(c_{X,Y\otimes Z}a_{X,Y,Z})$$
$$+F(a^{-1}_{Z,X,Y}c_{X\otimes Y,Z})^{-1}x^{\pm}_{Z\otimes X,Y}F(a^{-1}_{Z,X,Y}c_{X\otimes Y,Z}) = 0,$$
$$F(t_{X,Y} \otimes \mathrm{id}_Z) = [F(a_{X,Y,Z})^{-1}x^{+}_{X,Y\otimes Z}F(a_{X,Y,Z}),$$
$$F((c_{X,Y} \otimes \mathrm{id}_Z)^{-1}a_{Y,X,Z})x^{-}_{Y,X\otimes Z}F(a^{-1}_{Y,X,Z}(c_{X,Y} \otimes \mathrm{id}_Z))].$$

Functors between pairs (an IBMC, an elliptic structure over it) are defined in an obvious way. An elliptic structure over $\mathbf{PaCD}$ is defined as follows: $\tilde{\mathcal{C}} := \mathbf{PaCD}_{ell}$ is the category with the same objects as $\mathbf{PaB}$, $\mathbf{PaCD}_{ell}(O, O') := \begin{cases} \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,|O|}) \rtimes S_{|O|} & \text{if } |O| = |O'| \\ \emptyset & \text{otherwise,} \end{cases}$ $\tilde{U}_O = \exp(\hat{\mathfrak{t}}_{1,|O|}) \lhd \mathrm{Aut}_{\mathbf{PaCD}_{ell}}(O)$, $F$ is induced by the morphism $\mathfrak{t}_n \to \mathfrak{t}_{1,n}$, $x \mapsto \{x\}$ and the identity between symmetric groups, $x^{\pm}_{O,O'} = \sum_{i=1}^{|O|} x^{\pm}_i \in \mathrm{Lie}\,\tilde{U}_{O\otimes O'}$. The triple $(\mathbf{PaCD}, \mathbf{PaCD}_{ell}, \bullet)$ is universal for triples (an IBMC, an elliptic structure over it, a distinguished object).

Let us say that a **k**-BMC is a braided monoidal category (BMC) $\mathcal{C}$, such that the image of each morphism $P_n \to \mathrm{Aut}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n)$ is contained in a **k**-prounipotent group. Then, each $(\mu, \Phi) \in \underline{M}(\mathbf{k})$ gives rise to a map $\{\text{IBMCs}\} \to \{\mathbf{k}\text{-BMCs}\}$, $\mathcal{C} \mapsto (\mu, \Phi) * \mathcal{C}$, where $(\mu, \Phi) * \mathcal{C} := (\mathcal{C}, \otimes, \beta_{X,Y} := c_{X,Y} e^{\mu t_{X,Y}/2}, \tilde{a}_{X,Y,Z} := \Phi(a_{X,Y,Z}(t_{X,Y} \otimes \mathrm{id}_Z) a^{-1}_{X,Y,Z}, \mathrm{id}_X \otimes t_{Y,Z}) a_{X,Y,Z})$.

In the same say, a **k**-elliptic structure over a **k**-BMC is an elliptic structure, such that the image of each morphism $P_{1,n} \to \mathrm{Aut}_{\tilde{\mathcal{C}}}(F(X_1 \otimes \cdots \otimes X_n))$ is contained in a **k**-prounipotent group. Then, each $(\mu, \Phi, A_+, A_-) \in \underline{Ell}(\mathbf{k})$ gives rise to a map $\{(\text{an IBMC, an elliptic structure over it})\} \to \{(\text{a } \mathbf{k}\text{-BMC, an elliptic structure over it})\}$, $(\mathcal{C}, \tilde{\mathcal{C}}) \mapsto (\mu, \Phi, A_+, A_-) * (\mathcal{C}, \tilde{\mathcal{C}}) = (\mathcal{C}', \tilde{\mathcal{C}}')$, where $\mathcal{C}' = (\mu, \Phi) * \mathcal{C}$ and $\tilde{\mathcal{C}}' = (\tilde{\mathcal{C}}, F, \tilde{A}^+_{X,Y}, \tilde{A}^-_{X,Y})$, where $\tilde{A}^\pm_{X,Y} := A_\pm(x^+_{X,Y}, x^-_{X,Y})$.

### 4.4 Action of $\underline{\mathrm{GT}}_{ell}(-)$ on $\underline{Ell}$

Recall first that there is an action of $\underline{\mathrm{GT}}(\mathbf{k})$ on $\underline{M}(\mathbf{k})$, defined by

$$(\lambda, f) * (\mu, \Phi) := (\lambda\mu, \Phi(A, B) f(e^{\mu A}, \Phi(A, B)^{-1} e^{\mu B} \Phi(A, B))) = (\mu', \Phi').$$

For $(\lambda, f, g_+, g_-) \in \underline{\mathrm{GT}}_{ell}(\mathbf{k})$ and $(\mu, \Phi, A_+, A_-) \in \underline{Ell}(\mathbf{k})$, we set

$$(\lambda, f, g_+, g_-) * (\mu, \Phi, A_+, A_-) := (\mu', \Phi', A'_+, A'_-)$$

where $A'_\pm := g_\pm(A_+, A_-)$.

**Proposition 4.5** *This defines an action of* $\underline{\mathrm{GT}}_{ell}(\mathbf{k})$ *on* $\underline{Ell}(\mathbf{k})$.

*Proof* For $g_{ell} \in \underline{\mathrm{GT}}_{ell}(\mathbf{k})$, and $(\mathcal{C}, \tilde{\mathcal{C}}) \in \{(\text{a } \mathbf{k}\text{-BMC, an elliptic structure over it})\}$, we have $g_{ell} * ((\mu, \Phi, A_+, A_-) * (\mathcal{C}, \tilde{\mathcal{C}})) = (g_{ell} * (\mu, \Phi, A_\pm)) * (\mathcal{C}, \tilde{\mathcal{C}})$. When $(\mathcal{C}, \tilde{\mathcal{C}}) = (\mathbf{PaCD}, \mathbf{PaCD}_{ell})$, $(\mu, \Phi, A_+, A_-)$ can be recovered uniquely from $(\mu, \Phi, A_+, A_-) * (\mathcal{C}, \tilde{\mathcal{C}})$, as $e^{\mu t_{12}} = \beta^2_{\bullet,\bullet}$, $\Phi = \tilde{a}_{\bullet,\bullet,\bullet}$, and $A_\pm = A^\pm_{\bullet,\bullet}$, which implies that the above formula defines an action. □

*Remark 4.6* The actions of $\underline{\mathrm{GT}}(\mathbf{k})$ on $\{\mathbf{k}\text{-BMCs}\}$ and on $\underline{M}(\mathbf{k})$ are compatible, in the sense that for $g \in \underline{\mathrm{GT}}(\mathbf{k})$, $g * ((\mu, \Phi) * \mathcal{C}) = (g * (\mu, \Phi)) * \mathcal{C}$. In the same way, the actions of $\underline{\mathrm{GT}}_{ell}(\mathbf{k})$ on $\{(\text{a } \mathbf{k}\text{-BMC, an elliptic structure over it})\}$ and on $\underline{Ell}(\mathbf{k})$ are compatible.

*Remark 4.7* The morphism $\underline{Ell} \to \mathrm{M}_2$ from Sect. 4.2 is compatible with the semigroup scheme morphism $\underline{\mathrm{GT}}_{ell}(-) \to \mathrm{M}_2$ from Proposition 3.11, with the action of $\underline{\mathrm{GT}}_{ell}$ on $\underline{Ell}$, and with the left multiplication action of $\mathrm{M}_2$ on itself.

### 4.5 A morphism $\underline{M} \to \underline{Ell}$

The scheme morphism $\underline{Ell} \to \underline{M}$, $(\mu, \Phi, A_+, A_-) \to (\mu, \Phi)$ is clearly compatible with the semigroup scheme morphism $\underline{\mathrm{GT}}_{ell}(-) \to \underline{\mathrm{GT}}(-)$. We now construct a section of this morphism.

**Proposition 4.8** *There is a unique scheme morphism* $\sigma : \underline{M} \to \underline{Ell}$, $(\mu, \Phi) \to (\mu, \Phi, A_+, A_-)$*, where*

$$A_+ := \Phi\left(\frac{\operatorname{ad} x_1}{e^{\operatorname{ad} x_1} - 1}(y_2), t_{12}\right) \cdot e^{\mu \frac{\operatorname{ad} x_1}{e^{\operatorname{ad} x_1}-1}(y_2)} \cdot \Phi\left(\frac{\operatorname{ad} x_1}{e^{\operatorname{ad} x_1} - 1}(y_2), t_{12}\right)^{-1},$$
$$A_- := e^{\mu t_{12}/2}\Phi\left(\frac{\operatorname{ad} x_2}{e^{\operatorname{ad} x_2} - 1}(y_1), t_{21}\right) e^{x_1}\Phi\left(\frac{\operatorname{ad} x_1}{e^{\operatorname{ad} x_1} - 1}(y_2), t_{12}\right)^{-1}$$

*(we set* $x_i := x_i^+$, $y_i := x_i^-$*). It is compatible with the semigroup scheme morphism* $\underline{\mathrm{GT}}(-) \to \underline{\mathrm{GT}}_{ell}(-)$ *from Proposition 3.21.*

One checks that $\sigma$ fits in a diagram

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\sigma} & \underline{Ell} \\ \downarrow & & \downarrow \\ \mathbb{A} & \to & \mathrm{M}_2 \end{array}$$

where the bottom map is $\mu \mapsto \begin{pmatrix} 0 & -\mu \\ 1 & 0 \end{pmatrix}$. This diagram is compatible with the last diagram of Proposition 3.21.

*Proof* By Calaque et al. [7], Prop. 5.3, $(\mu, \Phi, A_+, A_-)$ satisfies

$$A_\pm^{12,3} = \{e^{\pm\mu t_{12}/2}(\Phi^{-1})^{2,1,3}\}A_\pm^{2,13}\{\Phi^{2,1,3}e^{\pm\mu t_{12}/2}\Phi^{-1}\}A_\pm^{1,23}\{\Phi\},$$

and therefore (25).

The last identity of *loc. cit.* can be rewritten as follows (using the commutation of $\{t_{12}\}$ with $A_+^{12,3}$)

$$A_-^{2,13}\{\Phi^{2,1,3}\}A_+^{12,3}\{(\Phi^{2,1,3})^{-1}\}(A_-^{2,13})^{-1}$$
$$= \{(\Phi^{3,1,2})^{-1}e^{\mu t_{12}/2}\Phi^{3,2,1}e^{\mu t_{23}}\Phi^{1,2,3}e^{-\mu t_{12}/2}\}A_+^{12,3}\{\Phi^{3,1,2}\}.$$

Now, the hexagon and duality identities imply

$$(\Phi^{3,2,1})^{-1}e^{\mu t_{12}/2}\Phi^{3,2,1}e^{\mu t_{23}}\Phi^{1,2,3}e^{-\mu t_{12}/2} = e^{-\mu t_{13}/2}\Phi^{2,3,1}e^{\mu t_{23}/2}(\Phi^{3,2,1})^{-1}$$
$$e^{\mu t_{3,12}/2},\ \Phi^{3,1,2} = e^{\mu t_{3,21}/2}\Phi^{3,2,1}e^{-\mu t_{23}/2}(\Phi^{2,3,1})^{-1}e^{-\mu t_{13}/2}, \qquad (27)$$

and

$$\Phi^{2,1,3} = e^{\mp\mu t_{13}/2}\Phi^{2,3,1}e^{\mp\mu t_{23}/2}(\Phi^{3,2,1})^{-1}e^{\pm\mu t_{3,12}/2},$$

so (27) is rewritten (using the commutation of $\{t_{13}\}$ with $A_-^{2,13}$)

$$\begin{aligned}&\{e^{-\mu t_{13}/2}\}A_-^{2,13}\{\Phi^{2,3,1}e^{-\mu t_{23}/2}(\Phi^{3,2,1})^{-1}e^{\mu t_{3,12}/2}\}A_+^{12,3}\\&\{e^{\mu t_{3,12}/2}\Phi^{3,2,1}e^{-\mu t_{23}/2}(\Phi^{2,3,1})^{-1}\}(A_-^{2,13})^{-1}\{e^{-\mu t_{13}/2}\}\\&=\{e^{-\mu t_{13}/2}\Phi^{2,3,1}e^{\mu t_{23}/2}(\Phi^{3,2,1})^{-1}e^{\mu t_{3,21}/2}\}A_+^{12,3}\\&\{e^{\mu t_{3,21}/2}\Phi^{3,2,1}e^{-\mu t_{23}/2}(\Phi^{2,3,1})^{-1}e^{-\mu t_{13}/2}\}.\end{aligned}\tag{28}$$

As $A_+^{2,1}e^{\mu t_{12}/2}A_+e^{\mu t_{12}/2}=1$, we have $e^{\mu t_{3,12}/2}A_+^{12,3}e^{\mu t_{3,12}/2}=(A_+^{3,12})^{-1}$; using this identity and performing the transformation of indices $(1,2,3)\to(3,1,2)$, (28) yields (26). So $(\mu,\Phi,A_+,A_-)\in\underline{Ell}(\mathbf{k})$. The compatibility of $\sigma:\underline{M}(\mathbf{k})\to\underline{Ell}(\mathbf{k})$ with the semigroup morphism $\underline{\mathrm{GT}}(\mathbf{k})\to\underline{\mathrm{GT}}_{ell}(\mathbf{k})$ follows from $(A_-,A_+)=e^{\mu t_{12}}$. □

## 4.6 A subscheme $Ell\subset\underline{Ell}$ and its torsor structure under $\mathrm{GT}_{ell}(-)$

Set $M(\mathbf{k}):=\{(\mu,\Phi)|\mu\in\mathbf{k}^\times\}\subset\underline{M}(\mathbf{k})$ and $Ell(\mathbf{k}):=\{(\mu,\Phi,A_\pm)|\mu\in\mathbf{k}^\times\}\subset\underline{Ell}(\mathbf{k})$. The actions of $\underline{\mathrm{GT}}_{(ell)}$ restrict to actions of $\mathrm{GT}(\mathbf{k})$ on $M(\mathbf{k})$ and $\mathrm{GT}_{ell}(\mathbf{k})$ on $Ell(\mathbf{k})$. Recall that $M(\mathbb{Q})\neq\emptyset$ and that $M(\mathbf{k})$ is a principal homogeneous space under the action of $\mathrm{GT}(\mathbf{k})$ ([9]). Similarly:

**Proposition 4.9** *1) The map $Ell(\mathbf{k})\to M(\mathbf{k})$ is surjective ;*
*2) $Ell(\mathbf{k})\neq\emptyset$ (in particular, $Ell(\mathbb{Q})\neq\emptyset$) ;*
*3) $Ell(\mathbf{k})$ is a principal homogeneous space under the action of* $\mathrm{GT}_{ell}(\mathbf{k})$.

*Proof* The scheme morphism $\sigma:\underline{M}\to\underline{Ell}$ restricts to a morphism $M\to Ell$, which yields a map $M(\mathbf{k})\to Ell(\mathbf{k})$, which is a section of the map $Ell(\mathbf{k})\to M(\mathbf{k})$. It follows that the latter map is surjective, which proves 1). The nonemptiness of $Ell(\mathbb{Q})$ then follows from that of $M(\mathbb{Q})$ and from the surjectivity of $Ell(\mathbb{Q})\to M(\mathbb{Q})$. It follows that $Ell(\mathbf{k})$ is also nonempty. This proves 2). □

Let us show that the action of $\mathrm{GT}_{ell}(\mathbf{k})$ on $Ell(\mathbf{k})$ is free. If $(\lambda,f,g_+,g_-)\in\mathrm{Stab}(\mu,\Phi,A_+,A_-)$, then by the freeness of the action of $\mathrm{GT}(\mathbf{k})$ on $M(\mathbf{k})$, $(\lambda,f)=1$. Then, $A_\pm=g_\pm(A_+,A_-)$. Relation (26) implies that if $a_\pm,b_\pm\in\mathbf{k}$ are such that $\log A_\pm\equiv a_\pm x_1^++b_\pm x_1^-$ mod degree $\geq 2$ (where $x_1^\pm$ have degree 1), then $a_+b_--a_-b_+=\mu$, which implies that $(\log A_+,\log A_-)$ generate $\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}}$, and therefore that $g_\pm=1$.

We now prove that the action is transitive. As the action of $\mathrm{GT}(\mathbf{k})$ on $M(\mathbf{k})$ is transitive, and as $\mathrm{GT}_{ell}(\mathbf{k})\to\mathrm{GT}(\mathbf{k})$ is surjective (as the morphism defined in Proposition 3.21 restricts to a section of it), it suffices to prove that for any $(\mu,\Phi)\in Ell(\mathbf{k})$, the action of $R_{ell}(\mathbf{k})$ on $\{(A_+,A_-)|(\mu,\Phi,A_+,A_-)\in Ell(\mathbf{k})\}$ is transitive. If $(A_+,A_-)$ and $(A'_+,A'_-)$ belong to this set, then there is a unique $(g_+,g_-)\in F_2(\mathbf{k})^2\simeq P_{1,2}(\mathbf{k})^2$ such that $A'_\pm=g_\pm(A_+,A_-)$. Then,

$$\alpha_\pm^{1,2,3}\alpha_\pm^{3,1,2}\alpha_\pm^{2,3,1}=1,\ \text{ where }\alpha_\pm=g_\pm(A_+^{1,23},A_-^{1,23})\{\Phi^{1,2,3}e^{\pm\mu t_{12,3}/2}\}.$$

The canonical morphism $B_{1,3} \to \mathrm{Aut}_{(\mu,\Phi,A_+,A_-)*\mathbf{PaCD}}(\bullet(\bullet\bullet)) = \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,3}) \rtimes S_3$ extends to an isomorphism $B_{1,3}(\mathbf{k}) \simeq \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,3}) \rtimes S_3$, given by $X_1^{\pm} \mapsto A_{\pm}^{1,23}$, $\sigma_1 \mapsto \{\Phi e^{\mu t_{12}/2}\}(12)\{\Phi\}^{-1}$, $\sigma_2 \mapsto \{e^{\mu t_{23}/2}\}(23)$. It is such that $\sigma_2^{\pm 1}\sigma_1^{\pm 1} \mapsto \{\Phi e^{\pm(\mu/2)t_{3,12}}\}(23)(12)$. The preimage of the above identity by this isomorphism then yields $(g_{\pm}(X_1^+, X_1^-)\sigma_2^{\pm 1}\sigma_1^{\pm 1})^3 = 1$. Similarly, the preimage of the identity

$$\{e^{\mu t_{12}}\} = \Big(\{\Phi^{-1}\}g_-(A_+^{1,23}, A_-^{1,23})\{\Phi\}, \{e^{-(\mu/2)t_{12}} (\Phi^{2,1,3})^{-1}\}g_+^{-1}(A_+^{2,13}, A_-^{2,13})\{\Phi^{2,1,3}e^{-(\mu/2)t_{12}}\}\Big)$$

yields $\sigma_1^2 = (\sigma_1 g_+^{-1}(X_1^+, X_1^-)\sigma_1, g_-(X_1^+, X_1^-))$.

Recall the following definition:

**Definition 4.10** A $\mathbb{Q}$-torsor is the data of: $\mathbb{Q}$-group schemes $G, H$, a $\mathbb{Q}$-scheme $X$, commuting left and right actions of $G, H$ on $X$, such that: for any $\mathbf{k}$ with $X(\mathbf{k}) \neq \emptyset$, the action of $G(\mathbf{k})$ and $H(\mathbf{k})$ on $X(\mathbf{k})$ is free and transitive.

Morphisms of torsors are then defined in the obvious way.

The above $\mathbb{Q}$-scheme morphisms between $\underline{Ell}$ and $\underline{M}$ restrict to a torsor morphism $Ell \to M$ and a section of it $M \xrightarrow{\sigma} Ell$, fitting in commutative diagrams

$$\begin{array}{ccc} Ell & \to & M \\ \downarrow & & \downarrow \\ \mathrm{GL}_2 & \xrightarrow{\det} & \mathbb{G}_m \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \to & Ell \\ \downarrow & & \downarrow \\ \mathbb{G}_m & \xrightarrow{\mu \mapsto \begin{pmatrix} 0 & -1 \\ \mu & 0 \end{pmatrix}} & \mathrm{GL}_2 \end{array}$$

## 5 The group $\mathbf{GRT}_{ell}(\mathbf{k})$ and isomorphisms of Lie algebras

In this section, we study the group scheme $\mathrm{GRT}_{ell}(-)$ of $\mathrm{GT}_{ell}(-)$-automorphisms of the scheme of elliptic associators. We show that its Lie algebra $\mathfrak{grt}_{ell}$ is graded and equipped with a graded morphism $\mathfrak{grt}_{ell} \to \mathfrak{grt}$. We construct a section of this morphism, which brings to light the semidirect product structure of $\mathfrak{grt}_{ell}$. We show that the Lie subalgebra $\mathfrak{sl}_2 \subset \mathrm{Der}(\mathfrak{t}_{1,2})$ and the derivations $\delta_{2k}, k \geq 0$ of $\mathfrak{t}_{1,2}$ from [7] give rise to a family of elements of the kernel $\mathfrak{r}_{ell} := \mathrm{Ker}(\mathfrak{grt}_{ell} \to \mathfrak{grt})$ (which according to Conjecture 10.1, should generate it as a Lie algebra). The existence of rational elliptic associators enables us to construct an isomorphism between the group schemes $\mathrm{GT}_{ell}(-)$ and $\mathrm{GRT}_{ell}(-)$, compatible with their semidirect product structures and with their actions on the elliptic braid groups and their graded versions.

### 5.1 Reminders about GRT(k)

Let $\mathbf{k}$ be a $\mathbb{Q}$-ring. Recall [9] that $\mathrm{GRT}_1(\mathbf{k})$ is defined as the set of all $g \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}}) \subset \exp(\hat{\mathfrak{t}}_3^{\mathbf{k}})$, such that:

$$g^{3,2,1}=g^{-1}, g^{3,1,2}g^{2,3,1}g^{1,2,3}=1 (\text{relations in } \exp(\hat{\mathfrak{t}}_3^{\mathbf{k}})),$$
$$t_{12}+\mathrm{Ad}(g^{1,2,3})^{-1}(t_{23})+\mathrm{Ad}(g^{2,1,3})^{-1}(t_{13})=t_{12}+t_{13}+t_{23} \quad (\text{relation in } \hat{\mathfrak{t}}_3^{\mathbf{k}}),$$
$$g^{2,3,4}g^{1,23,4}g^{1,2,3}=g^{1,2,34}g^{12,3,4} \quad (\text{relation in } \exp(\hat{\mathfrak{t}}_4^{\mathbf{k}})).$$

This is a group with law $(g_1 * g_2)(A,B) := g_1(\mathrm{Ad}(g_2(A,B))(A),B)g_2(A,B)$. Note that $g \in \mathrm{GRT}_1(\mathbf{k})$ gives rise to $\theta_g \in \mathrm{Aut}(\hat{\mathfrak{t}}_3^{\mathbf{k}})$, defined by

$$\theta_g : t_{12} \mapsto t_{12}, \quad t_{23} \mapsto \mathrm{Ad}(g^{1,2,3})^{-1}(t_{23}), \quad t_{13} \mapsto \mathrm{Ad}(g^{2,1,3})^{-1}(t_{13}).$$

Then $g_1 * g_2 = g_1\theta_{g_2}(g_1)$, and $\theta_{g_1 * g_2} = \theta_{g_2}\theta_{g_1}$, so $g \mapsto \theta_g$ is a group antimorphism.

The group $\mathbf{k}^\times$ acts on $\mathrm{GRT}_1(\mathbf{k})$ by $(c\cdot g)(A,B) := g(c^{-1}A, c^{-1}B)$, and one sets $\mathrm{GRT}(\mathbf{k}) := \mathrm{GRT}_1(\mathbf{k}) \rtimes \mathbf{k}^\times$. $\mathrm{GRT}_1(-)$ is a prounipotent group scheme.

## 5.2 The group $\mathrm{GRT}_{ell}(\mathbf{k})$

Define $\mathrm{GRT}_1^{ell}(\mathbf{k})$ as the set of all $(g, u_+, u_-)$, such that $g \in \mathrm{GRT}_1(\mathbf{k})$, $u_\pm \in \hat{\mathfrak{t}}_{1,2}^{\mathbf{k}}$, and

$$\mathrm{Ad}(g^{1,2,3})^{-1}(u_\pm^{1,23}) + \mathrm{Ad}(g^{2,1,3})^{-1}(u_\pm^{2,13}) + u_\pm^{3,12} = 0, \tag{29}$$
$$[\mathrm{Ad}(g^{1,2,3})^{-1}(u_\pm^{1,23}), u_\pm^{3,12}] = 0, \tag{30}$$
$$[\mathrm{Ad}(g^{1,2,3})^{-1}(u_+^{1,23}), \mathrm{Ad}(g^{2,1,3})^{-1}(u_-^{2,13})] = t_{12} \tag{31}$$

(relations in $\hat{\mathfrak{t}}_{1,3}^{\mathbf{k}}$). Set $(g_1, u_+^1, u_-^1) * (g_2, u_+^2, u_-^2) := (g, u_+, u_-)$, where

$$u_\pm(x_1, y_1) := u_\pm^1(u_+^2(x_1,y_1), u_-^2(x_1,y_1)) \tag{32}$$

(where $\mathfrak{t}_{1,2}^{\mathbf{k}}$ is viewed as the free Lie algebra generated by $x_1, y_1$).

We first prove:

**Lemma 5.1** $(g, u_+, u_-) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$ *iff there exists an automorphism of* $\hat{\mathfrak{t}}_{1,3}^{\mathbf{k}}$ *(henceforth denoted* $\theta_{g,u_\pm}$*), such that*

$$x_1^\pm \mapsto \mathrm{Ad}(g^{1,2,3})^{-1}(u_\pm^{1,23}), \quad x_2^\pm \mapsto \mathrm{Ad}(g^{2,1,3})^{-1}(u_\pm^{2,13}), \quad x_3^\pm \mapsto u_\pm^{3,12},$$
$$t_{12} \mapsto t_{12}, \quad t_{23} \mapsto \mathrm{Ad}(g^{1,2,3})^{-1}(t_{23}), \quad t_{13} \mapsto \mathrm{Ad}(g^{2,1,3})^{-1}(t_{13}).$$

*Proof* The condition that the relations $x_1^\pm + x_2^\pm + x_3^\pm = 0$ (resp., $[x_1^\pm, x_3^\pm] = 0$, $[x_1^+, x_2^-] = t_{12}$) are preserved is equivalent to condition (29) (resp., (30), (31)), and the relation $[t_{12}, x_3^\pm] = 0$ is automatically preserved. Then, the relation $g^{3,1,2}g^{2,3,1}g^{1,2,3} = 1$ implies that $\theta_{g,u_\pm}(x^{2,3,1}) = \mathrm{Ad}(g^{1,2,3})^{-1}(\theta_{g,u_\pm}(x)^{2,3,1})$ for $x \in \{x_i^\pm, t_{ij}\}$. So the other relations $[t_{ij}, x_k^\pm] = 0$ are also preserved. □

**Proposition 5.2** $\mathrm{GRT}_1^{ell}(\mathbf{k})$*, equipped with the above product, is a group.*

*Proof* The product is that of the group $\mathrm{GRT}_1(\mathbf{k}) \times \mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,2})^{op}$, so it remains to prove that $\mathrm{GRT}_1^{ell}(\mathbf{k})$ is stable under the operations of product and inverse. If $(g_i, u^i_\pm) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$ $(i = 1, 2)$, then the action of $\theta_{g_2,u_2^\pm}\theta_{g_1,u_1^\pm}$ on the generators of $\hat{\mathfrak{t}}^{\mathbf{k}}_{1,3}$ is given by the formulas of Lemma 5.1, with $g = g_1 * g_2$ and $u_\pm$ as in (32). So $(g, u_\pm) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$, as claimed. Similarly, if $(g, u_\pm) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$, then the action of $\theta^{-1}_{g,u_\pm}$ on the generators of $\hat{\mathfrak{t}}^{\mathbf{k}}_{1,3}$ is as in Lemma 5.1, with $(g, u_\pm)$ replaced by (inverse of $g$ in $\mathrm{GRT}_1(\mathbf{k})$, inverse of $(u_+, u_-)$ in $\mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,2})$), so $(g, u_\pm)$ is invertible. □

In particular, we have

$$\theta_{(g_2,u_2^\pm)}\theta_{(g_1,u_1^\pm)} = \theta_{(g_1,u_1^\pm)*(g_2,u_2^\pm)}. \tag{33}$$

The assignments $\mathbf{k} \mapsto \mathrm{GRT}_1(\mathbf{k}), \mathrm{GRT}_1^{ell}(\mathbf{k})$ are then $\mathbb{Q}$-group schemes.

For $(g, u_\pm) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$, define $a_\pm, b_\pm \in \mathbf{k}$ by $u_\pm(x_1, y_1) = a_\pm x_1 + b_\pm y_1$ mod $[\hat{\mathfrak{t}}^{\mathbf{k}}_{1,2}, \hat{\mathfrak{t}}^{\mathbf{k}}_{1,2}]$.

**Lemma 5.3** *1) There is a unique group scheme morphism* $\mathrm{GRT}_1^{ell}(-) \to \mathrm{SL}_2$, $(g, u_\pm) \mapsto \begin{pmatrix} a_+ & b_+ \\ a_- & b_- \end{pmatrix}$.

*2) This morphism has a section* $\mathrm{SL}_2 \to \mathrm{GRT}_1^{ell}(-)$, *given by* $\begin{pmatrix} a_+ & b_+ \\ a_- & b_- \end{pmatrix} \mapsto (1, u_\pm(x_1, y_1) = a_\pm x_1 + b_\pm y_1)$.

*Proof* 1) $a_+b_- - a_-b_+ = 1$ follows from (31); the morphism property is clear. 2) is straightforward. □

We now set $\mathrm{GRT}_{I_2}^{ell}(\mathbf{k}) := \mathrm{Ker}\big(\mathrm{GRT}_1^{ell}(\mathbf{k}) \to \mathrm{SL}_2(\mathbf{k})\big)$. This defines a group scheme $\mathrm{GRT}_{I_2}^{ell}(-)$.

**Lemma 5.4** $\mathrm{GRT}_{I_2}^{ell}(-)$ *is a prounipotent group scheme; we have* $\mathrm{GRT}_1^{ell}(-) = \mathrm{GRT}_{I_2}^{ell}(-) \rtimes \mathrm{SL}_2$.

*Proof* $\mathrm{GRT}_{I_2}^{ell}(\mathbf{k})$ is a subgroup of $\mathrm{GRT}_1(\mathbf{k}) \times \mathrm{Ker}\big(\mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,2}) \to \mathrm{GL}_2(\mathbf{k})\big)$; the assignment $\mathbf{k} \mapsto$ (the latter group) is a prounipotent group scheme, hence so is $\mathrm{GRT}_{I_2}^{ell}(-)$. The second statement follows from Lemma 5.3. □

The group $\mathbf{k}^\times$ acts on $\mathrm{GRT}_1^{ell}(\mathbf{k})$ by $c \cdot (g, u_\pm) := (c \cdot g, c \cdot u_\pm)$, where $c \cdot g$ is as above, $(c \cdot u_+)(x_1^+, x_1^-) := u_+(x_1^+, c^{-1}x_1^-)$, $(c \cdot u_-)(x_1^+, x_1^-) := cu_-(x_1^+, c^{-1}x_1^-)$. We then set $\mathrm{GRT}_{ell}(\mathbf{k}) := \mathrm{GRT}_1^{ell}(\mathbf{k}) \rtimes \mathbf{k}^\times$. Then, $\mathbf{k} \mapsto \mathrm{GRT}_{ell}(\mathbf{k})$ is a $\mathbb{Q}$-group scheme, and $\mathrm{GRT}_{ell}(-) = \mathrm{GRT}_1^{ell}(-) \rtimes \mathbb{G}_m$.

There is a unique group scheme morphism $\mathrm{GRT}_1^{ell}(-) \to \mathrm{GRT}_1(-)$, given by $(g, u_\pm) \mapsto g$; it extends to a group scheme morphism

$$\mathrm{GRT}_{ell}(-) \to \mathrm{GRT}(-), \tag{34}$$

whose restriction to $\mathbb{G}_m$ is the identity.

To elucidate the structure of $\mathrm{GRT}_{ell}(-)$, we use the following statement on iterated semidirect products:

**Lemma 5.5** *Let $G_i$ be groups ($i = 1, 2, 3$). The following data are equivalent:*

*(a) actions*[12] $G_j \to \mathrm{Aut}(G_i)$ *for* $i < j$, *such that* $g_3 * (g_2 * g_1) = (g_3 * g_2) * (g_3 * g_1)$;
*(b) actions* $G_j \to \mathrm{Aut}(G_i)$ *for* $(i, j) = (1, 2)$ *and* $(2, 3)$, *and an action* $G_{23} \to \mathrm{Aut}(G_{12})$ *(where* $G_{ij} := G_i \rtimes G_j$*), compatible with the actions of* $G_j$ *on* $G_i$ *for* $(i, j) = (1, 2)$ *or* $(2, 3)$, *and with the adjoint action of* $G_2$ *on itself.*
*These equivalent data yield actions* $G_3 \to \mathrm{Aut}(G_{12})$ *and* $G_{23} \to \mathrm{Aut}(G_1)$, *and we then have a canonical isomorphism* $(G_1 \rtimes G_2) \rtimes G_3 \simeq G_1 \rtimes (G_2 \rtimes G_3)$.

*Proof* Straightforward. □

We then have an action of $\mathrm{GL}_2$ on $\mathrm{GRT}_1^{ell}(-)$, given by $\gamma \cdot (g, u_+, u_-) := (\det \gamma \cdot g, \tilde{u}_+, \tilde{u}_-)$, where $\begin{pmatrix} \tilde{u}_+(x_1, y_1) \\ \tilde{u}_-(x_1, y_1) \end{pmatrix} := \gamma^{-1} \begin{pmatrix} u_+(\tilde{x}_1, \tilde{y}_1) \\ u_-(\tilde{x}_1, \tilde{y}_1) \end{pmatrix}$ and $\begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \end{pmatrix} := \gamma \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. It satisfies the conditions of Lemma 5.5, (b), where: $G_1 = \mathrm{GRT}_{I_2}^{ell}(-)$, $G_2 = \mathrm{SL}_2$, $G_3 = \mathbb{G}_m$, the isomorphism $G_2 \rtimes G_3 \simeq \mathrm{GL}_2$ being given by $\mathbb{G}_m \to \mathrm{GL}_2$, $c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$. We have therefore an isomorphism

$$\mathrm{GRT}_{ell}(-) \simeq \mathrm{GRT}_{I_2}^{ell}(-) \rtimes \mathrm{GL}_2,$$

where we recall that $\mathrm{GRT}_{I_2}^{ell}(-)$ is prounipotent.

The morphism (34) then fits in a commutative diagram

$$\begin{array}{ccc} \mathrm{GRT}_{ell}(-) & \to & \mathrm{GRT}(-) \\ \downarrow & & \downarrow \\ \mathrm{GL}_2 & \stackrel{\det}{\to} & \mathbb{G}_m \end{array}$$

as the morphism $G_2 \rtimes G_3 \to G_3$ coincides with det.

## 5.3 A morphism $\mathrm{GRT}(\mathbf{k}) \to \mathrm{GRT}_{ell}(\mathbf{k})$

We now construct a section of the morphism (34). We first set

$$t_{0i} := -\frac{\mathrm{ad}\, x_i}{e^{\mathrm{ad}\, x_i} - 1}(y_i) \in \hat{\mathfrak{t}}_{1,n}^{\mathbf{k}} \quad \text{for } i \in \{1, \ldots, n\}. \tag{35}$$

For $g = g(A, B) \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$,
we set $g^{0,1,2} := g(t_{01}, t_{12}) \in \exp(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}}) g^{0,2,1} := g(t_{02}, t_{21}) \in \exp(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})$.

**Lemma-Definition 5.6** *For* $g \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$, *there exists* $\alpha_g \in \mathrm{Aut}(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})$, *uniquely defined by* $\alpha_g(x_1) = \log(g^{0,2,1} e^{x_1} (g^{0,1,2})^{-1})$, $\alpha_g(t_{01}) = g^{0,1,2} t_{01} (g^{0,1,2})^{-1}$. *We set*

$$(u_+^g, u_-^g) := (\alpha_g(x_1), \alpha_g(y_1)) \in (\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})^2.$$

[12] The action of $g_j \in G_j$ on $g_i \in G_i$ is denoted $g_j * g_i \in G_i$.

*Proof* This follows from the fact that $\mathfrak{t}^{\mathbf{k}}_{1,2}$ is freely generated by $x_1$ and $t_{01}$. □

**Proposition 5.7** *There exists a unique group morphism* $\mathrm{GRT}_1(\mathbf{k}) \to \mathrm{GRT}_1^{ell}(\mathbf{k})$*, given by* $g \mapsto (g, u_+^g, u_-^g)$*. It is compatible with the action of* $\mathbf{k}^\times$*, hence extends to a group morphism* $\mathrm{GRT}(\mathbf{k}) \to \mathrm{GRT}_{ell}(\mathbf{k})$*, which is a section of (34) and fits in a commutative diagram*

$$\begin{array}{ccc} \mathrm{GRT}(-) & \to & \mathrm{GRT}_{ell}(-) \\ \downarrow & & \downarrow \\ \mathbb{G}_m & \to & \mathrm{GL}_2, \end{array}$$

*where the bottom morphism is* $c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$.

*Proof* We first prove: □

**Lemma 5.8** $\hat{\mathfrak{t}}^{\mathbf{k}}_{1,n}$ *admits the following presentation: generators* $x_i, t_{\alpha\beta}$ *(*$i \in \{1, ..., n\}$*,* $\alpha \neq \beta \in \{0, ..., n\}$*); one sets* $X_i := e^{x_i}$*; relations (*$i, j, ...$ *run over* $\{1, ..., n\}$ *while* $\alpha, \beta, ...$ *run over* $\{0, \ldots, n\}$*):*

$$t_{\beta\alpha} = t_{\alpha\beta} \text{ for } \alpha \neq \beta, \quad [t_{\alpha\beta}, t_{\gamma\delta}] = [t_{\alpha\beta}, t_{\alpha\gamma} + t_{\beta\gamma}] = 0 \quad \text{for } \alpha, ..., \delta \text{ all different,} \tag{36}$$

$$\log(X_i, X_j) = \log\left(\prod_i X_i\right) = 0, \tag{37}$$

$$X_i(t_{0j} + t_{ij})X_i^{-1} = t_{0j} \quad \text{if } i \neq j, \quad X_i t_{0i} X_i^{-1} = \sum_{\alpha \neq i} t_{\alpha i}, \tag{38}$$

$$X_i t_{jk} X_i^{-1} = t_{jk} \quad \text{for } i, j, k \text{ distinct}, \quad (X_j X_k) t_{jk} (X_j X_k)^{-1} = t_{jk} \text{ for } i \neq j, \tag{39}$$

$$\sum_{0 \leq \alpha < \beta \leq n} t_{\alpha\beta} = 0. \tag{40}$$

*Proof* One first checks that if one defines $t_{0i}$ as in (35), then the above relations are satisfied; conversely, if one sets $y_i := -\frac{e^{\mathrm{ad}\, x_i} - 1}{\mathrm{ad}\, x_i}(t_{0i})$, then the above relations lead to the defining relations of $\hat{\mathfrak{t}}^{\mathbf{k}}_{1,n}$. □

**Lemma 5.9** *Let* $(g, u_\pm) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$ *and* $\alpha \in \mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,2})$ *be defined by* $\alpha(x_1^\pm) = u_\pm$*. Then* $\theta_{g,u_\pm} \in \mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,3})$ *(see Lemma 5.1) may be defined by*

$$\theta_{g,u_\pm} : X_1 \mapsto \mathrm{Ad}(g^{1,2,3})^{-1}(\alpha(X_1)^{1,23}), \quad X_2 \mapsto \mathrm{Ad}(g^{2,1,3})^{-1}(\alpha(X_1)^{2,13}), X_3 \mapsto \alpha(X_1)^{3,12}, \quad t_{01} \mapsto \mathrm{Ad}(g^{1,2,3})^{-1}(\alpha(t_{01})^{1,23}), \quad t_{02} \mapsto \mathrm{Ad}(g^{2,1,3})^{-1}(\alpha(t_{01})^{2,13}), t_{03} \mapsto \alpha(t_{01})^{3,12}, \quad t_{12} \mapsto t_{12}, \quad t_{23} \mapsto \mathrm{Ad}(g^{1,2,3})^{-1}(t_{23}), \quad t_{13} \mapsto \mathrm{Ad}(g^{2,1,3})^{-1}(t_{13}).$$

*Proof* Immediate. □

**Lemma 5.10** *Let $g \in \mathrm{GRT}_1(\mathbf{k})$. There is a unique $\tilde\theta_g \in \mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,3})$, such that*

$$\begin{aligned}
\tilde\theta_g : X_1 &\mapsto (g^{1,2,3})^{-1} g^{0,23,1} X_1 (g^{0,1,23})^{-1} g^{1,2,3},\\
X_2 &\mapsto (g^{2,1,3})^{-1} g^{0,13,2} X_2 (g^{0,2,13})^{-1} g^{2,1,3},\\
X_3 &\mapsto g^{0,12,3} X_3 (g^{0,3,12})^{-1},\\
t_{01} &\mapsto \mathrm{Ad}((g^{1,2,3})^{-1} g^{0,1,23})(t_{01}),\ t_{02} \mapsto \mathrm{Ad}((g^{2,1,3})^{-1} g^{0,2,13})(t_{02}),\\
t_{03} &\mapsto \mathrm{Ad}(g^{0,3,12})(t_{03}), \qquad (41)\\
t_{12} &\mapsto t_{12},\ t_{23} \mapsto \mathrm{Ad}(g^{1,2,3})^{-1}(t_{23}),\ t_{13} \mapsto \mathrm{Ad}(g^{2,1,3})^{-1}(t_{13}). \qquad (42)
\end{aligned}$$

*Proof* Let us first prove that relations (36) and (40) (for $n = 3$) are preserved. In Sect. 5.4, we will construct an elliptic IBMC $g * \mathbf{PaCD}$ with distinguished object $\bullet$, which gives rise to a functor $\mathbf{PaCD} \to g * \mathbf{PaCD}$. One derives from there an automorphism $\exp(\hat{\mathfrak{t}}^{\mathbf{k}}_n) \rtimes S_n = \mathrm{Aut}_{\mathbf{PaCD}}(O) \to \mathrm{Aut}_{g*\mathbf{PaCD}}(O) = \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_n) \rtimes S_n$ for any $O \in \mathbf{PaCD}(O)$, $|O| = n$. When $O = \bullet((\bullet\bullet)\bullet)$, the resulting automorphism of $\hat{\mathfrak{t}}^{\mathbf{k}}_4$ is given by (41), (42). So relations (36) are preserved. The automorphism necessarily preserves $Z(\hat{\mathfrak{t}}^{\mathbf{k}}_4) = \mathbf{k} \cdot \sum_{\alpha<\beta} t_{\alpha\beta}$, so relation (40) is also preserved.

Note for later use that

$$\tilde\theta_g(x^{2,3,1}) = \mathrm{Ad}(g^{1,2,3})^{-1}(\tilde\theta_g(x)^{2,3,1}) \quad \text{for } x \in \{x_i, t_{\alpha\beta}\}. \qquad (43)$$

We have

$$\begin{aligned}
\tilde\theta_g(X_2)\tilde\theta_g(X_3) &= (g^{2,1,3})^{-1} g^{0,13,2} X_2 (g^{0,2,13})^{-1} g^{2,1,3} g^{0,21,3} X_3 (g^{0,3,12})^{-1}\\
&= (g^{2,1,3})^{-1} g^{0,13,2} X_2 g^{02,1,3} (g^{0,2,1})^{-1} X_3 (g^{0,3,12})^{-1}\\
&= (g^{2,1,3})^{-1} g^{0,13,2} g^{0,1,3} X_2 X_3 (g^{03,2,1})^{-1} (g^{0,3,12})^{-1}\\
&= (g^{2,1,3})^{-1} (g^{1,3,2})^{-1} g^{0,1,32} g^{01,3,2} X_2 X_3 (g^{0,3,2})^{-1} (g^{0,32,1})^{-1} (g^{3,2,1})^{-1}\\
&= (g^{2,1,3})^{-1} (g^{1,3,2})^{-1} g^{0,1,32} X_2 X_3 (g^{0,32,1})^{-1} (g^{3,2,1})^{-1},
\end{aligned}$$

while

$$\begin{aligned}
\tilde\theta_g(X_3)\tilde\theta_g(X_2) &= g^{0,21,3} X_3 (g^{0,3,12})^{-1} (g^{2,1,3})^{-1} g^{0,13,2} X_2 (g^{0,2,13})^{-1} g^{2,1,3}\\
&= g^{0,21,3} X_3 g^{03,1,2} (g^{0,3,1})^{-1} X_2 (g^{0,2,13})^{-1} g^{2,1,3}\\
&= g^{0,12,3} g^{0,1,2} X_3 X_2 (g^{02,3,1})^{-1} (g^{0,2,13})^{-1} g^{2,1,3}\\
&= (g^{1,2,3})^{-1} g^{0,1,23} g^{01,2,3} X_3 X_2 (g^{0,2,3})^{-1} (g^{0,23,1})^{-1} (g^{2,3,1})^{-1} g^{2,1,3}\\
&= (g^{1,2,3})^{-1} g^{0,1,23} X_3 X_2 (g^{0,23,1})^{-1} (g^{2,3,1})^{-1} g^{2,1,3},
\end{aligned}$$

which implies $(\tilde\theta_g(X_2), \tilde\theta_g(X_3)) = 1$. Then, (43) implies that $(\tilde\theta_g(X_i), \tilde\theta_g(X_j)) = 1$ for any $i, j$.

The above computation of $\tilde\theta_g(X_2)\tilde\theta_g(X_3)$ implies that

$$\begin{aligned}&\tilde\theta_g(X_1)\tilde\theta_g(X_2)\tilde\theta_g(X_3)\\&\quad=(g^{1,2,3})^{-1}g^{0,23,1}X_1(g^{0,1,23})^{-1}g^{1,2,3}(g^{2,1,3})^{-1}(g^{1,3,2})^{-1}\\&\qquad X_2X_3(g^{0,32,1})^{-1}(g^{3,2,1})^{-1}=1\end{aligned}$$

as $X_1X_2X_3=1$. So $X_1X_2X_3=1$ is preserved.

$\tilde\theta_g(X_3)$ clearly commutes with $\tilde\theta_g(t_{12})$, which implies that $X_jt_{jk}X_i^{-1}=t_{jk}$ is preserved in view of (43), as well as $X_jX_kt_{jk}(X_jX_k)^{-1}=t_{jk}$ (as the $X_i$ commute and $X_1X_2X_3=1$).

Now,

$$\begin{aligned}\tilde\theta_g(t_{02}+t_{12})&=\mathrm{Ad}((g^{2,1,3})^{-1}g^{0,2,31})(t_{02})+t_{12}=\mathrm{Ad}(g^{0,21,3})(\mathrm{Ad}(g^{0,2,1})(t_{02})+t_{12})\\&=\mathrm{Ad}(g^{0,21,3})(t_{01}+t_{02}+t_{12}-\mathrm{Ad}(g^{0,1,2})(t_{01}))\\&=t_{12}+\mathrm{Ad}(g^{0,21,3})(t_{01}+t_{02})-\mathrm{Ad}(g^{0,12,3}g^{0,1,2})(t_{01}).\end{aligned}$$

Then,

$$\begin{aligned}&\tilde\theta_g(X_1)\tilde\theta_g(t_{02}+t_{12})\\&\quad=(g^{1,2,3})^{-1}g^{0,23,1}X_1\Big((g^{0,1,23})^{-1}g^{1,2,3}t_{12}+(g^{0,1,23})^{-1}g^{1,2,3}g^{0,12,3}\\&\qquad(t_{01}+t_{02})(g^{0,12,3})^{-1}-(g^{0,1,23})^{-1}g^{1,2,3}g^{0,12,3}g^{0,1,2}t_{01}(g^{0,1,2})^{-1}(g^{0,12,3})^{-1}\Big)\\&\quad=(g^{1,2,3})^{-1}g^{0,23,1}X_1\Big(g^{01,2,3}(g^{0,1,2})^{-1}t_{12}(g^{0,12,3})^{-1}+g^{01,2,3}\\&\qquad(g^{0,1,2})^{-1}(t_{01}+t_{02})(g^{0,12,3})^{-1}-g^{01,2,3}t_{01}(g^{0,1,2})^{-1}(g^{0,12,3})^{-1}\Big)\\&\quad=(g^{1,2,3})^{-1}g^{0,23,1}X_1g^{01,2,3}(t_{02}+t_{12})(g^{0,1,2})^{-1}(g^{0,12,3})^{-1}\\&\quad=(g^{1,2,3})^{-1}g^{0,23,1}g^{0,2,3}t_{02}X_1(g^{0,1,2})^{-1}(g^{0,12,3})^{-1},\end{aligned}$$

while

$$\begin{aligned}\tilde\theta_g(t_{02})\tilde\theta_g(X_1)&=(g^{2,1,3})^{-1}g^{0,2,31}t_{02}(g^{0,2,31})^{-1}g^{2,1,3}(g^{1,2,3})^{-1}g^{0,23,1}X_1(g^{0,1,23})^{-1}\\g^{1,2,3}&=(g^{2,1,3})^{-1}g^{0,2,31}t_{02}g^{02,3,1}(g^{0,2,3})^{-1}X_1(g^{0,1,23})^{-1}g^{1,2,3}\\&=(g^{2,1,3})^{-1}g^{0,2,31}g^{02,3,1}t_{02}X_1(g^{01,2,3})^{-1}(g^{0,1,23})^{-1}g^{1,2,3}\\&=g^{3,1,2}g^{2,3,1}g^{0,23,1}g^{0,2,3}t_{02}X_1(g^{0,1,2})^{-1}(g^{0,12,3})^{-1},\end{aligned}$$

so the relation $X_1(t_{02}+t_{12})X_1^{-1}=t_{02}$ is preserved. (43) then implies that the relations $X_i(t_{0j}+t_{ij})X_i^{-1}=t_{0j}$ are preserved. Together with the other relations, these relations imply the relations $X_it_{0i}X_i^{-1}=\sum_{\alpha\neq i}t_{\alpha i}$, which are therefore also preserved. □

*End of proof of Proposition 5.7* If $g\in\mathrm{GRT}_1(\mathbf{k})$, then one checks that the automorphisms $\tilde\theta_g$ from Lemma 5.10 and $\alpha_g\in\mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,2})$ from Lemma-Definition 5.6 are related in the same way as $\theta_{g,u_\pm}$ and $\alpha$ are in Lemma 5.9. It follows that if $u^g_\pm$

are as in Lemma-Definition 5.6, then $(g, u_+^g, u_-^g) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$. This defines a map $\mathrm{GRT}_1(\mathbf{k}) \to \mathrm{GRT}_1^{ell}(\mathbf{k})$. □

Let us show that $\mathrm{GRT}_1(\mathbf{k}) \to \mathrm{GRT}_1^{ell}(\mathbf{k})$ is a group morphism. In view of (33), it suffices to prove that $\tilde{\theta}_{g_2}\tilde{\theta}_{g_1} = \tilde{\theta}_{g_1 * g_2}$, which can be checked directly, e.g.,

$$\begin{aligned}\tilde{\theta}_{g_2}(\tilde{\theta}_{g_1}(X_1)) &= \tilde{\theta}_{g_2}(g_1^{0,2,1}X_1(g^{0,1,2})^{-1}) = \tilde{\theta}_{g_2}(g_1(t_{02}, t_{21})X_1 g_1^{-1}(t_{01}, t_{12}))\\ &= g_1(\mathrm{Ad}(g_2^{0,2,1})(t_{02}), t_{21}) g_2^{0,2,1} X_1 (g_2^{0,2,1})^{-1} g_1^{-1}(\mathrm{Ad}(g_2^{0,2,1})(t_{01}), t_{12})\\ &= (g_1 * g_2)^{0,2,1} X_1 ((g_1 * g_2)^{0,1,2})^{-1} = \tilde{\theta}_{g_1 * g_2}(X_1),\end{aligned}$$

etc.

Let us prove that $\mathrm{GRT}_1(\mathbf{k}) \to \mathrm{GRT}_1^{ell}(\mathbf{k})$ is compatible with the actions of $\mathbf{k}^\times$. If $c \cdot (g, u_\pm) = (\tilde{g}, \tilde{u}_\pm)$, then $\theta_{\tilde{g},\tilde{u}_\pm}$ and $\theta_{g,u_\pm}$ are related by $\theta_{\tilde{g},\tilde{u}_\pm} = \gamma_c \theta_{g,u_\pm} \gamma_c^{-1}$, where $\gamma_c \in \mathrm{Aut}(\hat{\mathfrak{t}}_{1,3}^{\mathbf{k}})$ is given by $\gamma_c(x_i^+) = x_i^+$, $\gamma_c(x_i^-) = c^{-1}x_i^-$. It then suffices to prove that $\tilde{\theta}_{\tilde{g}} = \gamma_c \tilde{\theta}_g \gamma_c^{-1}$, where we recall that $\tilde{g}(A, B) = g(c^{-1}A, c^{-1}B)$, which follows from $\gamma_c(x_i) = x_i$, $\gamma_c(t_{\alpha\beta}) = c^{-1}t_{\alpha\beta}$ for $0 \le \alpha \ne \beta \le 3$.

The final commutative diagram follows from

$$\begin{array}{ccc} \mathrm{GRT}_1^{ell}(\mathbf{k}) \rtimes \mathbf{k}^\times & & \\ \downarrow & \searrow & \\ \mathrm{SL}_2(\mathbf{k}) \rtimes \mathbf{k}^\times & \xrightarrow{\sim} & \mathrm{GL}_2(\mathbf{k}) \end{array}$$

We set

$$R_{ell}^{gr}(\mathbf{k}) := \mathrm{Ker}\left(\mathrm{GRT}_{ell}(\mathbf{k}) \to \mathrm{GRT}(\mathbf{k})\right). \tag{44}$$

Explicitly,

$$\begin{aligned} R_{ell}^{gr}(\mathbf{k}) = \{&(u_+, u_-) \in (\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})^2 | u_\pm^{1,23} + u_\pm^{2,31} + u_\pm^{3,12} = 0, [u_\pm^{1,23}, u_\pm^{2,13}] = 0,\\ &[u_+^{1,23}, u_-^{2,13}] = t_{12}\} \subset \mathrm{Aut}(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})^{op}. \end{aligned} \tag{45}$$

Then, $\mathbf{k} \mapsto R_{ell}^{gr}(\mathbf{k})$ is $\mathbb{Q}$-group scheme, and we have a commutative diagram

$$\begin{array}{ccccccccc} 1 \to & R_{ell}^{gr}(-) & \to & \mathrm{GRT}_{ell}(-) & \to & \mathrm{GRT}(-) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & \mathrm{SL}_2 & \to & \mathrm{GL}_2 & \overset{\det}{\to} & \mathbb{G}_m & \to 1 \end{array}$$

The lift of $\mathrm{GRT}_{ell}(-) \to \mathrm{GL}_2$ restricts to a morphism $\mathrm{SL}_2 \to R_{ell}^{gr}(-)$, and the structure of $R_{ell}^{gr}(-)$ is therefore

$$R_{ell}^{gr}(-) = \mathrm{Ker}\left(R_{ell}^{gr}(-) \to \mathrm{SL}_2\right) \rtimes \mathrm{SL}_2,$$

in which the kernel is prounipotent.

The morphism from Proposition 5.7 enables us to define an action of GRT(−) on $R^{gr}_{ell}(-)$. $\mathrm{GRT}_{ell}(-)$ has then the structure of a semidirect product, fitting in

$$\begin{array}{ccc} \mathrm{GRT}_{ell}(-) & \simeq & R^{gr}_{ell}(-) \rtimes \mathrm{GRT}(-) \\ \downarrow & & \downarrow \\ \mathrm{GL}_2 & \simeq & \mathrm{SL}_2 \rtimes \mathbb{G}_m \end{array}$$

where the bottom morphism is induced by $\mathbb{G}_m \to \mathrm{GL}_2,\ c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$.

*Remark 5.11* For any $n \geq 1$, the algebra $U(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,n}) \rtimes S_n$ is generated by $x_1^{\pm}, s_1, \ldots, s_{n-1}$, where $s_i$ is the transposition $(i, i+1)$ of $S_n$ (a presentation is $s_i^2 = 1$ for any $i$, $(s_i s_{i+1})^3 = 1$ for $i < n-1$, $s_i s_j = s_j s_i$ for $|i-j| \geq 2$, $x_1^{\pm} s_i = s_i x_1^{\pm}$ for $i > 1$, $[x_1^{\pm}, s_1 x_1^{\pm} s_1] = 0$, $[x_1^+, s_1 x_1^- s_1] = [s_1 x_1^+ s_1, x_1^-]$, $[s_2 s_1 x_1^{\pm} s_1 s_2, [x_1^+, s_1 x_1^- s_1]] = 0$, $x_1^{\pm} + s_1 x_1^{\pm} s_1 + \cdots + s_{n-1} \cdots s_1 x_1^{\pm} s_1 \cdots s_{n-1} = 0$). Specializing the morphism from Proposition 5.23, 2) to the object $\bullet(\ldots(\bullet\bullet))$, one shows that the formulas from Lemma 5.10 generalize to an action to $\mathrm{GRT}_{ell}(\mathbf{k})$ on the tower of algebras $U(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,n}) \rtimes S_n$, given by

$$(g, u_+, u_-) \cdot x_1^{\pm} := u_{\pm}^{1,2\ldots n}, \quad (g, u_+, u_-) \cdot s_i := g^{i,i+1,i+2\ldots n} \cdot s_i \cdot (g^{i,i+1,i+2\ldots n})^{-1},$$

for $i = 1, \ldots, n-1$. These actions preserve the group $\exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,n}) \rtimes S_n$ and the Lie algebra $\hat{\mathfrak{t}}^{\mathbf{k}}_{1,n}$. Composing this action with the morphism $\mathrm{GRT}(\mathbf{k}) \to \mathrm{GRT}_{ell}(\mathbf{k})$, one obtains an action of $\mathrm{GRT}(\mathbf{k})$ on the same objects given by

$$g \cdot x_1^{\pm} := \alpha_g(x_1^{\pm})^{1,2\ldots n}, \quad g \cdot s_i := g^{i,i+1,i+2\ldots n} \cdot s_i \cdot (g^{i,i+1,i+2\ldots n})^{-1},$$

where $\alpha_g(x_1^{\pm})$ are defined in Lemma-Definition 5.6.

### 5.4 Categorical interpretations

A left action of $\mathrm{GRT}(\mathbf{k})$ on {IMBCs} is defined as follows: $g \in \mathrm{GRT}_1(\mathbf{k})$ acts on $(\mathcal{C}, c_{\ldots}, a_{\ldots}, t_{\ldots})$ by only modifying $a_{XYZ}$ into $a'_{X,Y,Z} := a_{XYZ}\, g(t_{XY} \otimes \mathrm{id}_Z, a^{-1}_{XYZ}(\mathrm{id}_X \otimes t_{YZ}) a_{XYZ})$ and $c \in \mathbf{k}^{\times}$ acts by only modifying $t_{XY}$ into $c t_{XY}$.

Similarly, one can show that a left action of $\mathrm{GRT}_{ell}(\mathbf{k})$ on {(an IBMC, an elliptic structure over it)} is defined as follows: $(g, u_+, u_-) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$ acts on $(\mathcal{C}, \tilde{\mathcal{C}})$ as $(g, u_+, u_-) * (\mathcal{C}, \tilde{\mathcal{C}}) := (g * \mathcal{C}, \tilde{\mathcal{C}}')$, where for $\tilde{\mathcal{C}} = (\tilde{\mathcal{C}}, F, x^{\pm}_{\ldots})$, we set $\tilde{\mathcal{C}}' = (\tilde{\mathcal{C}}, F, \underline{x}^{\pm}_{\ldots})$, where $\underline{x}^{\pm}_{X,Y} = u^{\pm}(x^+_{X,Y}, x^-_{X,Y})$, and $c \in \mathbf{k}^{\times}$ acts on $(\mathcal{C}, \tilde{\mathcal{C}})$ as $c * (\mathcal{C}, \tilde{\mathcal{C}}) := (c * \mathcal{C}, \tilde{\mathcal{C}}')$, where $\tilde{\mathcal{C}}' = (\tilde{\mathcal{C}}, F, x^+_{X,Y}, c x^-_{X,Y})$.

### 5.5 Action of $\mathrm{GRT}_{ell}(\mathbf{k})$ on $\underline{Ell}(\mathbf{k})$

Recall that $\mathrm{GRT}(\mathbf{k})$ acts on $\underline{M}(\mathbf{k})$ from the right as follows: for $g \in \mathrm{GRT}_1(\mathbf{k})$ and $(\mu, \Phi) \in \underline{M}(\mathbf{k})$, $(\mu, \Phi) * g := (\mu, \tilde{\Phi})$, where

$$\tilde{\Phi}(t_{12}, t_{23}) = \Phi(\mathrm{Ad}(g^{1,2,3})(t_{12}), t_{23})g^{1,2,3},$$

and for $c \in \mathbf{k}^\times$, $(\mu, \Phi) * c := (c\mu, c * \Phi)$, where $(c * \Phi)(A, B) = \Phi(cA, cB)$. This action is compatible with the maps {IBMCs} $\to$ {BMCs} induced by elements of $\underline{M}(\mathbf{k})$: $\Phi * (g * \mathcal{C}_0) = (\Phi * g) * \mathcal{C}_0$ for any $\Phi \in \underline{M}(\mathbf{k})$, $g \in \mathrm{GRT}(\mathbf{k})$ and IBMC $\mathcal{C}_0$.

For $(g, u_\pm) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$ and $(\mu, \Phi, A_\pm) \in \underline{Ell}(\mathbf{k})$, we set $(\mu, \Phi, A_\pm) * (g, u_\pm) := (\mu, \tilde{\Phi}, \tilde{A}_\pm)$, where

$$\tilde{A}_\pm(x_1, y_1) := A_\pm(u_+(x_1, y_1), u_-(x_1, y_1))$$

(in other terms, $\tilde{A}_\pm = \theta(A_\pm)$, where $\theta \in \mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,2})$ is $x_1^\pm \mapsto u_\pm(x_1^+, x_1^-)$) and for $c \in \mathbf{k}^\times$, we set $(\mu, \Phi, A_\pm) * c := (\mu, c * \Phi, c\sharp A_\pm)$, where $(c\sharp A_\pm)(x_1^+, x_1^-) := A_\pm(x_1^+, cx_1^-)$.

**Proposition 5.12** *This defines a right action of* $\mathrm{GRT}_{ell}(\mathbf{k})$ *on* $\underline{Ell}(\mathbf{k})$*, commuting with the left action of* $\underline{\mathrm{GT}}_{ell}(\mathbf{k})$ *and compatible with the right action of* $\mathrm{GL}_2(\mathbf{k})$ *on* $\mathrm{M}_2(\mathbf{k})$.

*Proof* Let us show that $(\mu, \tilde{\Phi}, \tilde{A}_\pm) \in \underline{Ell}(\mathbf{k})$. If $\theta \in \mathrm{Aut}(\hat{\mathfrak{t}}^{\mathbf{k}}_{1,2})$ is defined by $\theta(x_1^\pm) = u_\pm$, and $\tilde{\theta} := \theta_{g,u_\pm}$, then one checks that

$$\begin{aligned}\tilde{\theta}(x^{1,23}) &= \mathrm{Ad}(g^{1,2,3})^{-1}(\theta(x)^{1,23}),\\ \tilde{\theta}(x^{2,31}) &= \mathrm{Ad}(g^{2,1,3})^{-1}(\theta(x)^{2,31}),\\ \tilde{\theta}(x^{3,12}) &= \theta(x)^{3,12}\end{aligned}$$

for any $x \in \hat{\mathfrak{t}}^{\mathbf{k}}_{1,2}$. Applying $\tilde{\theta}$ to (25), one gets

$$\begin{aligned}&\theta(\{e^{\pm\mu t_{12}/2}\}A_\pm)^{3,12}\tilde{\theta}(\Phi^{3,1,2})(g^{2,1,3})^{-1}\theta(\{e^{\pm\mu t_{12}/2}\}A_\pm)^{2,31}\\ &\quad g^{2,1,3}\tilde{\theta}(\Phi^{2,3,1})(g^{1,2,3})^{-1}\theta(\{e^{\pm\mu t_{12}/2}\}A_\pm)^{1,23}\\ &\quad g^{1,2,3}\tilde{\theta}(\Phi^{1,2,3}) = 1.\end{aligned}$$

Using the identities $\tilde{\theta}(\Phi^{3,1,2})(g^{2,1,3})^{-1} = \tilde{\Phi}^{3,1,2}$, $g^{2,1,3}\tilde{\theta}(\Phi^{2,3,1})(g^{1,2,3})^{-1} = \tilde{\Phi}^{2,3,1}$, $g^{1,2,3}\tilde{\theta}(\Phi^{1,2,3}) = \tilde{\Phi}^{1,2,3}$, and $\theta(\{e^{\pm\mu t_{12}/2}\}A_\pm) = \{e^{\pm\mu t_{12}/2}\}\tilde{A}_\pm$, one obtains that $(\mu, \tilde{\Phi}, \tilde{A}_\pm)$ satisfies (25).

Applying now $\tilde{\theta}$ to (26), one gets

$$\begin{aligned}e^{\mu t_{12}} = \big(&\tilde{\theta}(\Phi)^{-1}g^{-1}\theta(A_-)^{1,23}g\tilde{\theta}(\Phi), e^{-\mu t_{12}/2}\tilde{\theta}(\Phi^{2,1,3})^{-1}\\ &(g^{2,1,3})^{-1}(\theta(A)^{2,13})^{-1}g^{2,1,3}\tilde{\theta}(\Phi^{2,1,3})e^{-\mu t_{12}/2}\big).\end{aligned}$$

Using again $g\tilde{\theta}(\Phi) = \tilde{\Phi}$ and $g^{2,1,3}\tilde{\theta}(\Phi^{2,1,3}) = \tilde{\Phi}^{2,1,3}$, together with $\theta(A_\pm) = \tilde{A}_\pm$, one obtains that $(\mu, \tilde{\Phi}, \tilde{A}_\pm)$ satisfies (26).

Similarly, applying the automorphism $x_i^+ \mapsto x_i^+$, $x_i^- \mapsto cx_i^-$ to identities (25), (26), one obtains that $(\mu, \Phi, A_\pm) * c$ satisfies the same identities, hence belongs to $\underline{Ell}(\mathbf{k})$. It is then immediate to check that this defines a right action of $\mathrm{GRT}_{ell}(\mathbf{k})$, commuting with the left action of $\underline{\mathrm{GT}}_{ell}(\mathbf{k})$. □

**Proposition 5.13** *The action of* $\mathrm{GRT}_{ell}(\mathbf{k})$ *on* $\underline{Ell}(\mathbf{k})$ *restricts to an action on* $Ell(\mathbf{k}) \subset \underline{Ell}(\mathbf{k})$*, which is free and transitive.*

*Proof* Given that the action of $\mathrm{GRT}(\mathbf{k})$ on $M(\mathbf{k})$ is free and transitive, it suffices to prove that the action of $R_{ell}^{gr}(\mathbf{k})$ on $Ell_{(\mu,\Phi)}(\mathbf{k}) := Ell(\mathbf{k}) \times_{M(\mathbf{k})} \{(\mu, \Phi)\}$ is free and transitive for any $(\mu, \Phi) \in Ell(\mathbf{k})$.

Recall that $R_{ell}^{gr}(\mathbf{k})$ is explicitly described by (45); its inclusion into $\mathrm{Aut}(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})$ is given by $(u_+, u_-) \mapsto \theta_{u_+,u_-} = (x_1^{\pm} \mapsto u_{\pm})$. On the other hand, $Ell_{(\mu,\Phi)}(\mathbf{k}) = \{(A_+, A_-)$ satisfying (25), (26)$\}$. Then,

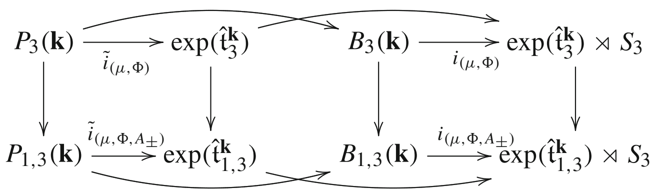
$$(A_+, A_-) * (u_+, u_-) = (\theta_{u_\pm}(A_+), \theta_{u_\pm}(A_-)). \tag{46}$$

Relation (26) implies that $(A_-, A_+) = e^{\mu t_{12}}$, which together with $\mu \in \mathbf{k}^\times$ implies that $\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}}$ is generated by $\log A_+$, $\log A_-$. Together with (46), this implies that the action of $R_{ell}^{gr}(\mathbf{k})$ on $Ell_{(\mu,\Phi)}(\mathbf{k})$ is free.

Let us now show that this action is transitive. We first observe that $R_{ell}^{gr}(\mathbf{k})$ can be described as $\{\theta \in \mathrm{Aut}(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}}) | \exists \tilde{\theta} \in \mathrm{Aut}(\hat{\mathfrak{t}}_{1,3}^{\mathbf{k}})$ with $\tilde{\theta}(t_{ij}) = t_{ij}$ for $1 \le i \ne j \le 3$ and $\tilde{\theta}(x^{i,jk}) = \theta(x)^{i,jk}$ for $\{i, j, k\} = \{1, 2, 3\}$ and $x \in \hat{\mathfrak{t}}_{1,2}^{\mathbf{k}}\}$. Let $(A_+, A_-)$ and $(\tilde{A}_+, \tilde{A}_-) \in Ell_{(\mu,\Phi)}(\mathbf{k})$ and let $\theta \in \mathrm{Aut}(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})$ be the automorphism such that $\theta(A_\pm) = \tilde{A}_\pm$. Let us show that there exists $\tilde{\theta} \in \mathrm{Aut}(\hat{\mathfrak{t}}_{1,3}^{\mathbf{k}})$, such that

$$\begin{aligned} &\tilde{\theta}(t_{ij}) = t_{ij} \text{ for } 1 \le i \ne j \le 3 \quad \text{and} \quad \tilde{\theta}(x^{i,jk}) = \theta(x)^{i,jk} \quad \text{for} \\ &\quad \{i, j, k\} = \{1, 2, 3\} \text{ and } x \in \hat{\mathfrak{t}}_{1,2}^{\mathbf{k}}. \end{aligned} \tag{47}$$

Let $i_{(\mu,\Phi)} : B_3(\mathbf{k}) \to \exp(\hat{\mathfrak{t}}_3) \rtimes S_3$, $i_{(\mu,\Phi,A_\pm)} : B_{1,3}(\mathbf{k}) \to \exp(\hat{\mathfrak{t}}_{1,3}) \rtimes S_3$ be the isomorphisms induced by $(\mu, \Phi)$, $(\mu, \Phi, A_\pm)$ and the object $\bullet(\bullet\bullet)$. We have a commutative diagram




where the maps '$i$' are isomorphisms. Note that for $\sigma \in B_3$, $i_{(\mu,\Phi)}(\sigma)[\sigma]^{-1} \in \exp(\hat{\mathfrak{t}}_3^{\mathbf{k}})$ (where $\sigma \mapsto [\sigma]$ is the canonical morphism $B_3 \to S_3$).

Then,

$$\begin{aligned} &\tilde{i}_{(\mu,\Phi,A_\pm)}(X_1^\pm) = A_\pm^{1,23}, \quad \tilde{i}_{(\mu,\Phi,A_\pm)}(X_2^\pm) = \{i_\Phi(\sigma_1^{\pm 1})s_1\} A_\pm^{2,13} \{s_1 i_\Phi(\sigma_1^{\pm 1})\}, \\ &\quad \tilde{i}_{(\mu,\Phi,A_\pm)}(X_3^\pm) = \{i_\Phi(\sigma_2^{\pm 1}\sigma_1^{\pm 1})s_1 s_2\} A_\pm^{3,12} \{s_2 s_1 i_\Phi(\sigma_1^{\pm 1}\sigma_2^{\pm 1})\}, \end{aligned}$$

where we recall that $x \mapsto \{x\}$ is induced by the canonical morphism $\mathfrak{t}_3 \to \mathfrak{t}_{1,3}$. Also, $\tilde{i}_{(\mu,\Phi,A_\pm)}(\sigma_i^2) = \{\tilde{i}_{(\mu,\Phi)}(\sigma_i^2)\}$, for $i = 1, 2$ and $\tilde{i}_{(\mu,\Phi,A_\pm)}(\sigma_1\sigma_2^2\sigma_1) = \{\tilde{i}_{(\mu,\Phi)}(\sigma_1\sigma_2^2\sigma_1)\}$.

Let $\tilde\theta := \tilde i_{(\mu,\Phi,\tilde A_\pm)} \circ \tilde i^{-1}_{(\mu,\Phi,A_\pm)}$. Then, $\tilde\theta \in \mathrm{Aut}(\hat{\mathfrak t}^{\mathbf k}_{1,3})$, and

(a) $\tilde\theta$ leaves $\{\tilde i_{(\mu,\Phi)}(\sigma_i^2)\}$ $(i = 1, 2)$ and $\{\tilde i_{(\mu,\Phi)}(\sigma_1\sigma_2^2\sigma_1)\}$ fixed, so it leaves the image of $\hat{\mathfrak t}_3 \to \hat{\mathfrak t}_{1,3}$ pointwise fixed;

(b) $\tilde\theta(A_\pm^{1,23}) = \tilde A_\pm^{1,23}$,

$$\tilde\theta\left(\left\{i_{(\mu,\Phi)}(\sigma_1^{\pm1})s_1\right\} A_\pm^{2,13} \left\{s_1 i_{(\mu,\Phi)}(\sigma_1^{\pm1})\right\}\right) = \left\{i_{(\mu,\Phi)}(\sigma_1^{\pm1})s_1\right\} \tilde A_\pm^{2,13} \left\{s_1 i_{(\mu,\Phi)}(\sigma_1^{\pm1})\right\},$$

which implies, as $\{i_{(\mu,\Phi)}(\sigma_1^{\pm1})s_1\}$ and $\{s_1 i_{(\mu,\Phi)}(\sigma_1^{\pm1})\} \in \mathrm{im}(\exp(\hat{\mathfrak t}^{\mathbf k}_3) \to \exp(\hat{\mathfrak t}^{\mathbf k}_{1,3}))$, that $\tilde\theta(A_\pm^{2,13}) = \tilde A_\pm^{2,13}$; one proves similarly that $\tilde\theta(A_\pm^{3,12}) = \tilde A_\pm^{3,12}$.

(b) implies that $\tilde\theta(x^{i,jk}) = \theta(x)^{i,jk}$ holds for $x = A_\pm$, therefore also for $x$ in the topological group generated by $A_\pm$. As $\mu \in \mathbf k^\times$, this group is equal to $\exp(\hat{\mathfrak t}^{\mathbf k}_{1,2})$, so $\tilde\theta$ satisfies (47). So $\theta \in R^{gr}_{ell}(\mathbf k)$. □

**Proposition 5.14** *The scheme morphisms* $\underline{Ell} \to \underline{M}$ *and* $\underline{M} \xrightarrow{\sigma} \underline{Ell}$ *(see Proposition 4.8) are compatible with the morphisms* $\mathrm{GRT}_{ell}(-) \to \mathrm{GRT}(-)$ *and* $\mathrm{GRT}(-) \to \mathrm{GRT}_{ell}(-)$ *(see Proposition 5.7).*

*Proof* We need to prove the second statement only. Let $\underline{M}(\mathbf k) \xrightarrow{\sigma} \underline{Ell}(\mathbf k)$ be given by $(\mu,\Phi) \mapsto (\mu,\Phi,A_\pm(\mu,\Phi))$, then we must show that for $g \in \mathrm{GRT}_1(\mathbf k)$ and $(\mu,\tilde\Phi) = (\mu,\Phi) * g$, we have $A_\pm(\mu,\tilde\Phi) = \alpha_g(A(\mu,\Phi))$, where $\alpha_g$ is as in Lemma-Definition 5.6. This follows from the fact that $\alpha_g$ satisfies $\alpha_g(t_{02}) = \mathrm{Ad}(g^{0,2,1})(t_{02})$, $\alpha_g(t_{12}) = t_{12}$. It is also clear that $\underline{M}(\mathbf k) \xrightarrow{\sigma} \underline{Ell}(\mathbf k)$ is compatible with the action of $\mathbf k^\times$. □

*Remark 5.15* In fact, the commutative diagrams $\begin{array}{ccc} \underline{Ell} & \to & \underline{M} \\ \downarrow & & \downarrow \\ \mathrm{M}_2 & \xrightarrow{\det} & \mathbb A \end{array}$ and $\begin{array}{ccc} \underline{M} & \xrightarrow{\sigma} & \underline{Ell} \\ \downarrow & & \downarrow \\ \mathbb A & \xrightarrow{c \mapsto \begin{pmatrix} 0 & -c \\ 1 & 0 \end{pmatrix}} & \mathrm{M}_2 \end{array}$ are compatible with the right actions of the diagrams

$$\begin{array}{ccc} \mathrm{GRT}_{ell}(-) & \to & \mathrm{GRT}(-) \\ \downarrow & & \downarrow \\ \mathrm{GL}_2 & \xrightarrow{\det} & \mathbb G_m \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{GRT}(-) & \to & \mathrm{GRT}_{ell}(-) \\ \downarrow & & \downarrow \\ \mathbb G_m & \xrightarrow{c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}} & \mathrm{GL}_2 \end{array}$$

## 5.6 Lie algebras

The graded Grothendieck-Teichmüller Lie algebra is[13]

$$\mathfrak{grt}_1 = \{\psi \in \mathfrak f_2 | \psi + \psi^{3,2,1} = 0, \psi + \psi^{2,3,1} + \psi^{3,1,2} = 0, [t_{23}, \psi^{1,2,3}] + [t_{13}, \psi^{2,1,3}] = 0, \psi^{2,3,4} - \psi^{12,3,4} + \psi^{1,23,4} - \psi^{1,2,34} + \psi^{1,2,3} = 0\},$$

[13] As before, $\mathfrak f_2 = \mathfrak f_2^{\mathbb Q}$, etc.

where we use the inclusion $\mathfrak{f}_2 \subset \mathfrak{t}_3$, $A \mapsto t_{12}$, $B \mapsto t_{23}$; it is equipped with the Lie bracket $\langle\psi_1, \psi_2\rangle = [\psi_1, \psi_2] + D_{\psi_2}(\psi_1) - D_{\psi_1}(\psi_2)$, where $D_\psi : A \mapsto [\psi, A]$, $B \mapsto 0$.

The Lie algebra $\mathbb{Q}$ acts on $\mathfrak{grt}_1$ by $[1, \psi] = -(\deg\psi)\psi$ (where $\deg A = \deg B = 1$), and we set $\mathfrak{grt} := \mathfrak{grt}_1 \rtimes \mathbb{Q}$.

The Lie algebras $\mathfrak{grt}$, $\mathfrak{grt}_1$ are $\mathbb{N}$-graded (where deg is extended to be 0 on $\mathbb{Q}$), we then have $\mathrm{Lie}\,\mathrm{GRT}_{(1)}(-) = \widehat{\mathfrak{grt}}_{(1)}$ (the degree completions).

Let

$$\begin{aligned}\mathfrak{grt}_1^{ell} := \big\{(\psi, \alpha_\pm) \in \mathfrak{f}_2 \times (\mathfrak{t}_{1,2})^2 | \psi \in \mathfrak{grt}_1,\\ \alpha_\pm^{1,23} + \alpha_\pm^{2,31} + \alpha_\pm^{3,12} + [x_\pm^1, \psi^{1,2,3}] + [x_\pm^2, \psi^{2,1,3}] = 0,\\ [x_\pm^1, \alpha_\pm^{3,12}] + [\alpha_\pm^{1,23}, x_\pm^3] - [x_\pm^1, [x_\pm^3, \psi^{1,2,3}]] = 0,\\ [x_+^1, \alpha_-^{2,13}] - [x_-^2, \alpha_+^{1,23}] = [x_-^2, [x_+^1, \psi^{1,2,3}]] - [x_+^1, [x_-^2, \psi^{2,1,3}]]\big\}.\end{aligned}$$

For $\alpha_\pm \in \mathfrak{t}_{1,2}$, define $D_{\alpha_\pm} \in \mathrm{Der}(\mathfrak{t}_{1,2})$ by $x_1^\pm \mapsto \alpha_\pm$. Then,

$$[(\psi_1, \alpha_1^\pm), (\psi_2, \alpha_2^\pm)] = (\langle\psi_1, \psi_2\rangle, D_{\alpha_2^\pm}(\alpha_1^\pm) - D_{\alpha_1^\pm}(\alpha_2^\pm))$$

defines a Lie bracket on $\mathfrak{grt}_1^{ell}$, and

$$\mathfrak{grt}_1^{ell} \subset \mathfrak{grt}_1 \times \mathrm{Der}(\mathfrak{t}_{1,2})^{op}.$$

The Lie algebra $\mathbb{Q}e_{22}$ acts on $\mathfrak{grt}_1^{ell}$ by

$$[e_{22}, (\psi, \alpha_+, \alpha_-)] = (-(\deg\psi)\psi, -(\deg_-\alpha_+)\alpha_+, (1 - \deg_-\alpha_-)\alpha_-),$$

where $\deg\psi$ is as above, and $\deg_-\alpha_\pm$ is defined by $\deg_- x_1^+ = 0$, $\deg_- x_1^- = 1$. We then set $\mathfrak{grt}_{ell} := \mathfrak{grt}_1^{ell} \rtimes \mathbb{Q}e_{22}$.

The Lie algebras $\mathfrak{grt}_{(1)}^{ell}$ are $\mathbb{N}$-graded, where $(\psi, \alpha_\pm)$ has degree $n$ if $2\deg\psi = \deg\alpha_\pm - 1 = n$ ($\deg\alpha_\pm$ being defined by $\deg x_1^\pm = 1$ and $\deg\psi$ by $\deg t_{12} = \deg t_{23} = 1$) and $e_{22}$ has degree 0. Then $\mathrm{Lie}\,\mathrm{GRT}_{(1)}^{ell}(-) = \widehat{\mathfrak{grt}}_{(1)}^{ell}$.

We have a morphism $\mathfrak{grt}_1^{ell} \to \mathfrak{sl}_2$, $(\psi, \alpha_+, \alpha_-) \mapsto \begin{pmatrix} a_+ & b_+ \\ a_- & b_- \end{pmatrix}$, where $\alpha_+ \equiv a_\pm x_1 + b_\pm y_1$ modulo degree $\geq 2$. It extends to a morphism $\mathfrak{grt}_{ell} \to \mathfrak{gl}_2$ via $e_{22} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We denote by $\mathfrak{grt}_{I_2}^{ell}$ the common kernel of these morphisms; it coincides with the part of $\mathfrak{grt}_{ell}$ (or $\mathfrak{grt}_1^{ell}$) of positive degree.

These morphisms admit sections $\mathfrak{sl}_2 \to \mathfrak{grt}_1^{ell}$ given by $\begin{pmatrix} a_+ & b_+ \\ a_- & b_- \end{pmatrix} \mapsto (0, a_\pm x_1 + b_\pm y_1)$ and $\mathfrak{gl}_2 \to \mathfrak{grt}_{ell}$ given by its extension by $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto e_{22}$. We then have $\mathfrak{grt}_1^{ell} \simeq \mathfrak{grt}_{I_2}^{ell} \rtimes \mathfrak{sl}_2$, $\mathfrak{grt}_{ell} \simeq \mathfrak{grt}_{I_2}^{ell} \rtimes \mathfrak{gl}_2$.

$\mathbb{Z}^2$-gradings may be defined on $\mathfrak{grt}_{(1)}^{ell}$ as follows. We have a Lie algebra inclusion $\mathfrak{grt}_1^{ell} \subset \mathfrak{grt}_1 \oplus \mathrm{Der}(\mathfrak{t}_{1,2}) =: \mathfrak{G}$. Recall that $\mathfrak{grt}_1$ is $\mathbb{N}$-graded while

$\mathrm{Der}(\mathfrak{t}_{1,2})$ is $\mathbb{Z}^2$-graded by the $\mathbb{Z}^2$-grading of $\mathfrak{t}_{1,2}$ given by $(\deg_+, \deg_-)(x_1^+) = (1,0)$, $(\deg_+, \deg_-)(x_1^-) = (0,1)$. We then define a $\mathbb{Z}^2$-grading on $\mathfrak{G}$ by $\mathfrak{G}[p,q] := \begin{cases} \mathrm{Der}(\mathfrak{t}_{1,2})[p,q] & \text{if } q \neq p \\ \mathfrak{grt}_1[p] \oplus \mathrm{Der}(\mathfrak{t}_{1,2})[p,p] & \text{if } q = p \end{cases}$. This restricts to a $\mathbb{Z}^2$-grading $(\deg_+, \deg_-)$ of $\mathfrak{grt}_1^{ell}$, which extends to $\mathfrak{grt}_{ell}$ by $(\deg_+, \deg_-)(e_{22}) = (0,0)$.

The $\mathbb{Z}^2$-grading of $\mathfrak{grt}_{ell}$ is compatible with the action of the Cartan subalgebra of $\mathfrak{gl}_2$: we have $[e_{11}, x] = -(\deg_+ x)x$, $[e_{22}, x] = -(\deg_- x)x$ for $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_2$ and $x \in \mathfrak{grt}_{ell}$ homogeneous.

We have a morphism $\mathfrak{grt}_1^{ell} \to \mathfrak{grt}_1$, $(\psi, u_\pm) \mapsto \psi$. It extends to a morphism $\mathfrak{grt}_{ell} \to \mathfrak{grt}$ by $e_{22} \mapsto 1$. Using Proposition 5.7, sections of these morphisms are constructed as follows:

**Proposition 5.16** *There is a unique Lie algebra morphism* $\mathfrak{grt}_1 \to \widehat{\mathfrak{grt}}_1^{ell}$, $\psi \mapsto (\psi, u_+^\psi, u_-^\psi)$, *where* $u_\pm^\psi := D_\psi(x_1^\pm)$ *and* $D_\psi \in \mathrm{Der}(\hat{\mathfrak{t}}_{1,2})$ *is defined by*

$$D_\psi(e^{x_1}) = \psi^{0,2,1} e^{x_1} - e^{x_1} \psi^{0,1,2}, \quad D_\psi(t_{01}) = [\psi^{0,1,2}, t_{01}];$$

*recall that*

$$\psi^{0,1,2} = \psi(t_{01}, t_{12}), \quad \psi^{0,2,1} = \psi(t_{02}, t_{21}), \quad t_{0i} = -\frac{\mathrm{ad}\, x_i}{e^{\mathrm{ad}\, x_i} - 1}(y_i), \quad i = 1, 2.$$

*It extends to a Lie algebra morphism* $\mathfrak{grt} \to \widehat{\mathfrak{grt}}_{ell}$ *by* $1 \mapsto e_{22}$. *It is homogeneous,* $\mathfrak{grt}$ *being equipped with its degree and* $\widehat{\mathfrak{grt}}_{ell}$ *with degree* $\deg_-$.

Set now $\mathfrak{r}_{ell}^{gr} := \mathrm{Ker}(\mathfrak{grt}_{ell} \to \mathfrak{grt})$. We have

$$\begin{aligned} \mathfrak{r}_{ell}^{gr} = \big\{(\alpha_+, \alpha_-) \in (\mathfrak{t}_{1,2})^2 |\ & \alpha_\pm^{1,23} + \alpha_\pm^{2,31} + \alpha_\pm^{3,12} = 0, \\ & [x_\pm^1, \alpha_\pm^{2,13}] + [\alpha_\pm^{1,23}, x_\pm^2] = 0, \\ & [x_+^1, \alpha_-^{2,13}] + [\alpha_+^{1,23}, x_-^2] = 0\big\} \subset \mathrm{Der}(\mathfrak{t}_{1,2})^{op}. \end{aligned}$$

This is a $\mathbb{Z}^2$-graded Lie subalgebra of $\mathfrak{grt}_{ell}$; it is also $\mathbb{N}$-graded by $\deg_+ + \deg_-$. We have $\mathfrak{r}_{ell}^{gr}[0] \simeq \mathfrak{sl}_2$ and $\mathfrak{r}_{ell}^{gr} \simeq (\oplus_{d>0} \mathfrak{r}_{ell}^{gr}[d]) \rtimes \mathfrak{sl}_2$. Its completion for the $\mathbb{N}$-degree is isomorphic to $\mathrm{Lie}\, R_{ell}^{gr}(-)$.

Define a partial completion $\hat{\mathfrak{r}}_{ell}^{gr} := \oplus_q (\prod_p \mathfrak{r}_{ell}^{gr}[p,q])$. Proposition 5.16 gives rise to a Lie algebra morphism $\mathfrak{grt} \to \mathrm{Der}(\hat{\mathfrak{r}}_{ell}^{gr})$. We then have $\hat{\mathfrak{grt}}_{ell} \simeq \hat{\mathfrak{r}}_{ell}^{gr} \rtimes \mathfrak{grt}$, where $\hat{\mathfrak{grt}}_{ell} := \oplus_q \prod_p \mathfrak{grt}_{ell}[p,q]$ is a partial completion.

Set $\mathfrak{gt}_{ell} := \mathrm{Lie}\,\mathrm{GT}_{ell}(-)$, $\mathfrak{gt}_1^{ell} := \mathrm{Lie}\,\mathrm{GT}_1^{ell}(-)$, then $\mathfrak{gt}_{ell} = \mathfrak{gt}_1^{ell} \rtimes \mathbb{Q}$. The Lie algebra $\mathfrak{gt}_1^{ell}$ admits a description as a subspace of $\hat{\mathfrak{f}}_2 \times (\hat{\mathfrak{t}}_{1,2})^2$ similar to that of Lemma 3.16 and is filtered as follows: $\mathfrak{gt}_{ell} = \mathfrak{gt}_1^{ell} \rtimes \mathbb{Q}$, where $\mathfrak{gt}_1^{ell} := \mathrm{Lie}\,\mathrm{GT}_1^{ell}(-) \subset \hat{\mathfrak{f}}_2 \times (\hat{\mathfrak{t}}_{1,2})^2$. We then set $(\mathfrak{gt}_1^{ell})^{\geq n} := \mathfrak{gt}_1^{ell} \cap \big(\hat{\mathfrak{f}}_2^{\geq n/2} \times ((\hat{\mathfrak{t}}_{1,2})^2)^{\geq n+1}\big)$ for $n \geq 0$, where the degree in $\hat{\mathfrak{f}}_2$ is induced by $\deg(t_{12}) = \deg(t_{23}) = 1$ and the degree in $\hat{\mathfrak{t}}_{1,2}$ by $\deg(x_1^\pm) = 1$. The Lie algebra $\mathfrak{gt}_{ell}$ is similarly filtered by $(\mathfrak{gt}_{ell})^{\geq 0} = \mathfrak{gt}_{ell}$, $(\mathfrak{gt}_{ell})^{\geq n} = (\mathfrak{gt}_1^{ell})^{\geq n}$ if $n > 0$. It follows from the form of the conditions under which $(\psi, \alpha_+, \alpha_-) \in$

$\hat{\mathfrak{f}}_2 \times (\mathfrak{t}_{1,2})^2$ belong to $\mathfrak{gt}_1^{ell}$ that there is a canonical morphism $\mathrm{gr}(\mathfrak{gt}_{ell}) \to \mathfrak{grt}_{ell}$, restricting to $\mathrm{gr}(\mathfrak{r}_{ell}) \to \mathfrak{r}_{ell}^{gr}$ and Wednesday, August 7, 2013 at 8:09 pmcompatible with $\mathrm{gr}(\mathfrak{gt}) \to \mathfrak{grt}$. In Sect. 5.8, we will see that all these morphisms are isomorphisms.

*Remark 5.17* The relations between Lie groups and algebras are summarized as follows:

$$\mathrm{GRT}_1(\mathbf{k}) = \exp(\widehat{\mathfrak{grt}_1^{\mathbf{k}}}), \quad \mathrm{GRT}(\mathbf{k}) = \exp(\widehat{\mathfrak{grt}_1^{\mathbf{k}}}) \rtimes \mathbf{k}^\times,$$
$$\mathrm{GRT}_1^{ell}(\mathbf{k}) = \exp(\widehat{\mathfrak{grt}_{I_2}^{ell,\mathbf{k}}}) \rtimes \mathrm{SL}_2(\mathbf{k}), \quad \mathrm{GRT}_{ell}(\mathbf{k}) = \exp(\widehat{\mathfrak{grt}_{I_2}^{ell,\mathbf{k}}}) \rtimes \mathrm{GL}_2(\mathbf{k}),$$
$$R_{ell}^{gr}(\mathbf{k}) = \exp\left(\prod_{d>0} \mathfrak{r}_{ell}^{gr}[d] \otimes \mathbf{k}\right) \rtimes \mathrm{SL}_2(\mathbf{k}).$$

*Remark 5.18* Any $(\alpha_+, \alpha_-) \in \mathfrak{r}_{ell}^{gr}$ satisfies $\alpha_\pm + \alpha_\pm^{2,1} = 0$, which implies that the total degree (in which $x_1^\pm$ have degree 1) of $\alpha_\pm$ is odd. So $\mathfrak{r}_{ell}^{gr}[d] = 0$ unless $d$ is even.

*Remark 5.19* (Relation with the work of H. Tsunogai.) In Tsunogai [32], a "stable derivation algebra" in genus one is described. This is a graded Lie algebra version of the intersection over $n \geq 1$ of the images of the morphisms $\mathrm{Out}^*(P_{1,n}) \to \mathrm{Out}^*(P_{1,1})$, where $\mathrm{Out}^* \subset \mathrm{Out}$ are certain subgroups. This is a Lie subalgebra $\mathcal{G}_{\mathrm{Ts}} \subset \mathrm{Der}(\mathfrak{t}_{1,2})$, which may be defined as the set of all $(\alpha_+, \alpha_-) \in (\mathfrak{t}_{1,2})^2$, such that there exists $\psi \in \mathfrak{t}_3$, such that

$$\psi^{1,2,3} + \psi^{3,2,1} = [t_{12}, \psi^{1,2,3}] + [t_{13}, \psi^{2,1,3}] = 0,$$
$$[x_+^1, \alpha_-^{1,2}] + [\alpha_+^{1,2}, x_-^1] = 0,$$
$$[x_\pm^1, \alpha_\pm^{3,12}] + [\alpha_\pm^{1,23}, x_\pm^3] = [x_3^\pm, [x_1^\pm, \psi^{1,2,3}]],$$
$$[x_+^1, \alpha_-^{3,12}] + [\alpha_+^{1,23}, x_-^2] = [t_{13}, \psi^{1,3,2}] + [x_+^1, [x_-^2, \psi^{1,3,2}]]$$

(the relation between the present formalism and that of [32] is as follows: $\mathfrak{t}_3 \leftrightarrow \mathcal{L}_1^{(2)\circ}$, $\mathfrak{t}_{1,2} \leftrightarrow \mathcal{L}_1^{(2)}$, $\alpha_+, \alpha_- \leftrightarrow S, T$, $U^{1,2,3} \leftrightarrow \psi^{2,1,3}$; the present relations are obtained from those of [32] by some changes of indices). This system of conditions is a consequence of the system expressing that $(\psi, \alpha_+, \alpha_-) \in \mathfrak{grt}_1^{ell}$; the latter is more restrictive as it contains additional conditions, namely the pentagon and hexagon conditions on $\psi$, as well as the conditions $\alpha_\pm^{1,23} + \alpha_\pm^{2,31} + \alpha_\pm^{3,12} + [x_\pm^1, \psi^{1,2,3}] + [x_\pm^2, \psi^{2,1,3}] = 0$. It follows that there is a double inclusion

$$\mathrm{im}(\mathfrak{grt}_1^{ell} \to \mathrm{Der}(\mathfrak{t}_{1,2})) \subset \mathcal{G}_{\mathrm{Ts}} \subset \mathrm{Der}(\mathfrak{t}_{1,2}).$$

## 5.7 A Lie subalgebra $\mathfrak{b}_3 \subset \mathfrak{r}_{ell}^{gr}$

**Proposition 5.20** *For $n \geq 0$, set*

$$\delta_{2n} := (\alpha_+ = \mathrm{ad}(x_1)^{2n+2}(y_1), \alpha_- = \frac{1}{2} \sum_{\substack{0 \leq p \leq 2n+1, \\ p+q=2n+1}} (-1)^p [(\mathrm{ad}\, x_1)^p(y_1), (\mathrm{ad}\, x_1)^q(y_1)]). \qquad (48)$$

*Then* $\delta_{2n} \in \mathfrak{r}^{gr}_{ell}[2n+1, 1]$. *The element* $\delta_0$ *is central in* $\mathfrak{grt}^{ell}_1$, *such that* $[e_{11}+e_{22}, \delta_0] = -2\delta_0$, *and it coincides with* $\mathrm{ad}\, t_{12}$ *as an element of* $\mathrm{Der}(\mathfrak{t}_{1,2})^{op}$.

*Proof* In Calaque et al. [7], Proposition 3.1, we constructed derivations $\dot{\delta}^{(m)}_{2n} \in \mathrm{Der}(\mathfrak{t}_{1,m})$, such that

$$\dot{\delta}^{(m)}_{2n} : x_i \mapsto 0, t_{ij} \mapsto [t_{ij}, (\mathrm{ad}\, x_i)^{2n}(t_{ij})], y_i \mapsto \sum_{j:j\neq i} \frac{1}{2} \sum_{p+q=2n-1} [(\mathrm{ad}\, x_i)^p(t_{ij}), (-\mathrm{ad}\, x_i)^q(t_{ij})].$$

Let then $\delta^{(m)}_{2n} := \dot{\delta}^{(m)}_{2n} + \left[\sum_{i<j} (\mathrm{ad}\, x_i)^{2n}(t_{ij}), -\right]$. Then

$$\delta^{(m)}_{2n}(x_i) = \left[\sum_{j\neq i} (\mathrm{ad}\, x_i)^{2n}(t_{ij}), x_i\right] = (\mathrm{ad}\, x_i)^{2n+2}(y_i) = \alpha_-^{i,1\ldots\check{i}\ldots n}, \quad \delta^{(m)}_{2n}(t_{ij}) = 0,$$

$$\begin{aligned}
\delta^{(m)}_{2n}(y_i) &= \dot{\delta}^{(m)}_{2n}(y_i) + \left[\sum_{j<k} (\mathrm{ad}\, x_j)^{2n}(t_{jk}), y_i\right] \\
&= \dot{\delta}^{(m)}_{2n}(y_i) + \sum_{j\neq i} [(\mathrm{ad}\, x_i)^{2n}(t_{ij}), y_i] + \sum_{j<k;\, j,k\neq i} [(\mathrm{ad}\, x_j)^{2n}(t_{jk}), y_i] \\
&= \dot{\delta}^{(m)}_{2n}(y_i) + \sum_{j\neq i} [(\mathrm{ad}\, x_i)^{2n}(t_{ij}), y_i] \\
&\quad + \sum_{j<k;\, j,k\neq i} \sum_{p+q=2n-1} (\mathrm{ad}\, x_j)^p [t_{ij}, (\mathrm{ad}\, x_j)^q(t_{jk})] \\
&= \dot{\delta}^{(m)}_{2n}(y_i) + \sum_{j\neq i} [(\mathrm{ad}\, x_i)^{2n}(t_{ij}), y_i] \\
&\quad - \sum_{j<k;\, j,k\neq i} \sum_{p+q=2n-1} [(-\mathrm{ad}\, x_i)^p(t_{ij}), (\mathrm{ad}\, x_i)^q(t_{ik})] \\
&= \dot{\delta}^{(m)}_{2n}(y_i) + \sum_{j\neq i} [(\mathrm{ad}\, x_i)^{2n}(t_{ij}), y_i] \\
&\quad - \frac{1}{2} \sum_{p+q=2n-1} \left[\sum_{j\neq i} (-\mathrm{ad}\, x_i)^p(t_{ij}), \sum_{k\neq i} (\mathrm{ad}\, x_i)^q(t_{ik})\right] \\
&\quad + \frac{1}{2} \sum_{j\neq i} \sum_{p+q=2n-1} [(-\mathrm{ad}\, x_i)^p(t_{ij}), (\mathrm{ad}\, x_i)^q(t_{ij})] \\
&= -[(\mathrm{ad}\, x_i)^{2n+1}(y_i), y_i] + \frac{1}{2} \sum_{p+q=2n-1} [(-\mathrm{ad}\, x_i)^{p+1}(y_i), (\mathrm{ad}\, x_i)^{q+1}(y_i)] \\
&= \frac{1}{2} \sum_{p+q=2n+1} [(-\mathrm{ad}\, x_i)^p(y_i), (\mathrm{ad}\, x_i)^q(y_i)] = \alpha_-^{i,1\ldots\check{i}\ldots n}.
\end{aligned}$$

Then $0 = \delta_{2n}^{(3)}([x_1^{\pm}, x_2^{\pm}]) = [x_1^{\pm}, \alpha_{\pm}^{2,13}] + [\alpha_{\pm}^{1,23}, x_2^{\pm}]$ and $0 = \delta_{2n}^{(3)}(t_{12}) = [x_1^{+}, \alpha_{-}^{2,13}] + [\alpha_{+}^{1,23}, x_2^{-}]$, which implies that $\delta_{2n} \in \mathfrak{r}_{ell}^{gr}$.

If $(\psi, \alpha_{\pm}) \in \mathfrak{grt}_1^{ell}$, then applying the morphism $\mathfrak{t}_{1,2} \to \mathfrak{t}_{1,3}$ corresponding to the map $\{1, 2\} \to \{1, 2, 3\}$, $1 \mapsto 1$, $2 \mapsto 2$ to the first defining condition of $\mathfrak{grt}_{ell}$, one gets $\alpha_{\pm}^{1,2} + \alpha_{\pm}^{2,1} = 0$. Applying the same morphism to the last defining condition of $\mathfrak{grt}_{ell}$, one gets $[x_+^1, \alpha_-^{2,1}] - [x_-^2, \alpha_+^{1,2}] = 0$, so $[x_+^1, \alpha_-^{1,2}] + [\alpha_+^{1,2}, x_-^1] = 0$. It follows that the derivation $D_{\alpha_{\pm}}$ of $\mathfrak{t}_{1,2}$ such that $x_1^{\pm} \mapsto \alpha_{\pm}$ is such that $D_{\alpha_{\pm}}(t_{12}) = 0$, so there is a Lie algebra inclusion $\mathfrak{grt}_1^{ell} \subset \mathfrak{grt}_1 \times \mathrm{Der}_t(\mathfrak{t}_{1,2})^{op}$ (where the index $t$ means the derivations taking $t_{12}$ to zero). Since $\delta_0 = (0, \mathrm{ad}\, t_{12}) \in \mathfrak{grt}_1 \times \mathrm{Der}_t(\mathfrak{t}_{1,2})^{op}$, $\delta_0$ is central in $\mathfrak{grt}_1 \times \mathrm{Der}_t(\mathfrak{t}_{1,2})^{op}$, therefore also in $\mathfrak{grt}_1^{ell}$. Finally, $[e_{11} + e_{22}, D] = -\deg(D) \cdot D$ for any $D \in \mathrm{Der}_t(\mathfrak{t}_{1,2})$, where the degree is $D$ corresponds to the degree on $\mathfrak{t}_{1,2}$ for which $x_1$ and $y_1$ have degree 1. Therefore $[e_{11} + e_{22}, \delta_0] = -2 \cdot \delta_0$. □

We define $\mathfrak{b}_3 := \langle \mathfrak{sl}_2, \delta_{2n}; n \geq 0 \rangle \subset \mathfrak{r}_{ell}^{gr}$ as the Lie subalgebra[14] generated by $\mathfrak{sl}_2$ and the $\delta_{2n}$. A basis of $\mathfrak{sl}_2 \subset \mathfrak{b}_3$ is

$$\begin{aligned} e_+ &:= (\alpha_+ = 0, \alpha_- = x_1), \quad e_- := (\alpha_+ = y_1, \alpha_- = 0), \\ h &:= (\alpha_+ = x_1, \alpha_- = -y_-). \end{aligned} \tag{49}$$

The Lie algebra $\mathfrak{b}_3$ is $\mathbb{N}$-graded and corresponds to the subgroup $\exp(\hat{\mathfrak{b}}_3^{+,\mathbf{k}}) \rtimes \mathrm{SL}_2(\mathbf{k}) \subset R_{ell}^{gr}(\mathbf{k})$ (where the hat denotes the degree completion and + means the positive degree part).

## 5.8 Isomorphisms of Lie algebras

Let $\mathbf{k}$ be a $\mathbb{Q}$-ring. As $Ell(\mathbf{k})$ is a torsor, each $e \in Ell(\mathbf{k})$ gives rise to an isomorphism $i_e : \mathrm{GT}_{ell}(\mathbf{k}) \to \mathrm{GRT}_{ell}(\mathbf{k})$, defined by $g * e = e * i_e(g)$ for any $g \in \mathrm{GT}_{ell}(\mathbf{k})$. Similarly, any $\tilde{\Phi} \in M(\mathbf{k})$ gives rise to an isomorphism $i_{\tilde{\Phi}} : \mathrm{GT}(\mathbf{k}) \to \mathrm{GRT}(\mathbf{k})$ defined by the same conditions. We then have a commutative diagram

$$\begin{array}{ccc} \mathrm{GT}_{ell}(\mathbf{k}) & \xrightarrow{i_e} & \mathrm{GRT}_{ell}(\mathbf{k}) \\ \downarrow & & \downarrow \\ \mathrm{GT}(\mathbf{k}) & \xrightarrow{i_{\tilde{\Phi}}} & \mathrm{GRT}(\mathbf{k}) \end{array} \tag{50}$$

where $\tilde{\Phi} = \mathrm{im}(e \in Ell(\mathbf{k}) \to M(\mathbf{k}))$. In particular, $i_e$ restricts to an isomorphism $i_e : R_{ell}(\mathbf{k}) \to R_{ell}^{gr}(\mathbf{k})$. When $e \in \mathrm{im}(M(\mathbf{k}) \xrightarrow{\sigma} Ell(\mathbf{k}))$, the isomorphism $R_{ell}(\mathbf{k}) \xrightarrow{i_e} R_{ell}^{gr}(\mathbf{k})$ is compatible with $i_{\tilde{\Phi}}$ and the actions of $\mathrm{GT}(\mathbf{k})$, $\mathrm{GRT}(\mathbf{k})$ on both sides via the lifts $\mathrm{GT}(\mathbf{k}) \xrightarrow{\sigma} \mathrm{GT}_{ell}(\mathbf{k})$, $\mathrm{GRT}(\mathbf{k}) \xrightarrow{\sigma} \mathrm{GRT}_{ell}(\mathbf{k})$.

The isomorphisms $i_e$ induce Lie algebra isomorphisms $\mathfrak{gt}_{ell}^{\mathbf{k}} \to \widehat{\mathfrak{grt}}_{ell}^{\mathbf{k}}$, restricting to $\mathfrak{r}_{ell}^{\mathbf{k}} \to \hat{\mathfrak{r}}_{ell}^{gr,\mathbf{k}}$, compatible with the filtrations and whose associated graded isomor-

[14] Conjecture 10.1 is the statement that this inclusion is an equality, and Proposition 9.2 shows that this statement is equivalent to the conjectural equality $\langle B_3 \rangle = R_{ell}(-)$ discussed in Sect. 3.5.

phisms are the canonical morphisms from the end of Sect. 5.6. Since $Ell(\mathbb{Q}) \neq \emptyset$ (e.g. because it contains $\sigma(M(\mathbb{Q}))$), we obtain:

**Proposition 5.21** *There are isomorphisms* $\mathfrak{gt}_{ell} \simeq \hat{\mathrm{gr}}(\mathfrak{gt}_{ell}) = \widehat{\mathfrak{grt}}_{ell}$ *and* $\mathfrak{r}_{ell} \simeq \hat{\mathrm{gr}}(\mathfrak{r}_{ell}) = \hat{\mathfrak{r}}^{gr}_{ell}$.

## 5.9 Actions on prounipotent completions of elliptic braid groups

Let $\mathbf{k}$ be a $\mathbb{Q}$-ring. We recall that $P_n(\mathbf{k})$ (respectively, $P_{1,n}(\mathbf{k})$) is the prounipotent completion of the pure (respectively, elliptic) braid group $P_n$ (respectively, $P_{1,n}$), where $n \geq 1$ and that $B_n(\mathbf{k})$ (respectively, $B_{1,n}(\mathbf{k})$) to be the relative completion of the full (respectively, elliptic) braid group with $n$ strands with respect to the canonical morphism to $S_n$; it identifes with the pushout $B_n *_{P_n} P_n(\mathbf{k})$ (respectively, $B_{1,n} *_{P_{1,n}} P_{1,n}(\mathbf{k})$).

**Proposition 5.22** *1) The action of* $\mathrm{GT} = \mathbb{Z}/2\mathbb{Z}$ *on* $B_n$ *via* $(-1) \cdot \sigma_i = \sigma_i^{-1}$ *extends to the following objects:*

- *a morphism* $\mu_O : \mathrm{GT}(\mathbf{k}) \to \mathrm{Aut}(B_n(\mathbf{k}))$ *for each* $O \in \mathbf{Pa}_n$*;*
- *a map*

$$\mathrm{GT}(\mathbf{k}) \times \mathbf{Pa}_n \times \mathbf{Pa}_n \to P_n(\mathbf{k}), \quad (g, O, O') \mapsto b_{OO'}(g),$$

*related by the identities*

$$\mu_{O'}(g) = \mathrm{Inn}(b_{OO'}(g)) \circ \mu_O(g), \tag{51}$$

$$b_{OO'}(gh) = b_{OO'}(g) \cdot \mu_O(g)(b_{OO'}(h)), \quad b_{OO''}(g) = b_{O'O''}(g) b_{OO'}(g). \tag{52}$$

*2) The action of* $\mathrm{GT}_{ell} = \tilde{B}_3$ *on* $B_{1,n}$ *given by (15) extends to a collection of morphisms*

$$\mu_O^{ell} : \mathrm{GT}_{ell}(\mathbf{k}) \to \mathrm{Aut}(B_{1,n}(\mathbf{k}))$$

*indexed by* $O \in \mathbf{Pa}_n$*, related to the morphisms* $\mu_O$ *by the identity*

$$\mu_O^{ell}(g_{ell})(b_{ell}) = \mu_O(g)(b)_{ell}, \tag{53}$$

*and satisfying*

$$\mu_{O'}^{ell}(g_{ell}) = \mathrm{Inn}(b_{OO'}(g)_{ell}) \circ \mu_O^{ell}(g), \tag{54}$$

*for any* $g_{ell} \in \mathrm{GT}_{ell}(\mathbf{k})$ *and* $b \in B_n(\mathbf{k})$*, where* $g := \mathrm{im}(g_{ell} \in \mathrm{GT}_{ell}(\mathbf{k}) \to \mathrm{GT}(\mathbf{k}))$ *and* $b_{ell} := \mathrm{im}(b \in B_n(\mathbf{k}) \to B_{1,n}(\mathbf{k}))$.

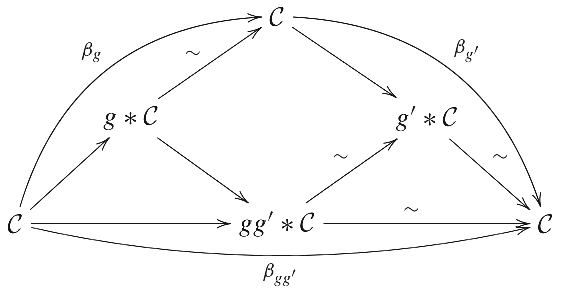
*3) The restriction* $\mu^{ell}_{O|R_{ell}(\mathbf{k})}$ *is independent of* $O \in \mathbf{Pa}_n$ *and will be denoted*

$$\mu_{ell} : R_{ell}(\mathbf{k}) \to \mathrm{Aut}(B_{1,n}(\mathbf{k})).$$

*If* $g_{ell} = (1, 1, g_+, g_-) \in R_{ell}(\mathbf{k})$*, where* $g_\pm = g_\pm(X_1, Y_1) \in P_{1,2}(\mathbf{k})$*, then the action of* $g_{ell}$ *on* $B_{1,n}(\mathbf{k})$ *induced by* $\mu_{ell}$ *is such that*

$$g_{ell} \cdot X_1^\pm = g_\pm(X_1^+, X_1^-), \quad g_{ell} \cdot \sigma_i = \sigma_i \quad \textit{for} \quad i = 1, \ldots, n-1.$$

*Proof* 1) Let $\mathcal{C} := \mathbf{PaB_k}$ be the $\mathbf{k}$-prounipotent version of the BMC $\mathbf{PaB}$, $G := \mathrm{GT}(\mathbf{k})$. For $g \in G$, $g * \mathcal{C}$ is a BMC with distinguished object $\bullet$. By the universal property of $\mathcal{C}$, one derives from there a functor $\alpha_g : \mathcal{C} \to g * \mathcal{C}$, uniquely defined by the condition that it is tensor and that it induces the identity on objects. As a category, $g*\mathcal{C}$ canonically identifies with $\mathcal{C}$; let $i_g : g*\mathcal{C} \to \mathcal{C}$ be this isomorphism. One then defines $\beta_g := i_g \circ \alpha_g : \mathcal{C} \to \mathcal{C}$. The identity $\beta_g \beta_{g'} = \beta_{g'g}$ follows from the commutativity of




in which the commutativity of the central square follows from that of

$$\begin{array}{ccc} g * \mathcal{C} & \stackrel{g*\varphi}{\to} & g * \mathcal{D} \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{C} & \stackrel{\varphi}{\to} & \mathcal{D} \end{array}$$

for any braided monoidal categories $\mathcal{C}, \mathcal{D}$ and any tensor functor $\varphi : \mathcal{C} \to \mathcal{D}$.
It follows that $g \mapsto \beta_{g^{-1}}$ defines a morphism from $G$ to the group of autofunctors of $\mathcal{C}$, i.e. an action of $G$ on $\mathcal{C}$.
Let $O, O' \in \mathbf{Pa}_n$. There is a canonical isomorphism $i_O : \mathrm{Aut}_{\mathcal{C}}(O) \to B_n(\mathbf{k})$ and a canonical element $i_{OO'} \in \mathrm{Iso}_{\mathcal{C}}(O, O')$ (corresponding to the unit in $B_n(\mathbf{k})$). Then, for $f \in \mathrm{Aut}_{\mathcal{C}}(O)$, $i_O(f) = i_{O'}(i_{OO'} f i_{OO'}^{-1})$.
Define the action $\mu_O$ of $G$ on $B_{1,n}(\mathbf{k})$ as the transport via $i_O$ of its action on $\mathrm{Aut}_{\mathcal{C}}(O)$, namely $\mu_O(g)(b) := i_O(g * i_O^{-1}(b))$. The claimed identities then hold with $b_{OO'}(g) := i_O(i_{OO'}^{-1} \circ (g * i_{OO'}))$.

2) The collection of morphisms $\mu_O^{ell}$ is then defined in the same way: $G$ is replaced by $G_{ell} := \mathrm{GT}_{ell}(\mathbf{k})$, $\mathcal{C}$ by $\mathcal{C}_{ell} := \mathbf{PaB}_{\mathbf{k}}^{ell}$, the isomorphisms $i_O$ by $i_O^{ell}$ and $i_{OO'}$ by $F(i_{OO'})$, where $F : \mathcal{C} \to \mathcal{C}_{ell}$ is the canonical functor. The claimed identity follows from $i_O^{ell}(F(x)) = i_O(F(x))_{ell}$, for $x \in \mathrm{Aut}_{\mathcal{C}}(O)$.
3) follows from identity (54), from the fact that $g = 1_{\mathrm{GT}(\mathbf{k})}$ if $g_{ell} \in R_{ell}(\mathbf{k})$, and from $b_{OO'}(1_{\mathrm{GT}(\mathbf{k})}) = 1_{P_n(\mathbf{k})}$, which follows from the first part of (52). □

**Proposition 5.23** *1) There are morphisms*

$$\mu_O^{gr} : \mathrm{GRT}(\mathbf{k}) \to \mathrm{Aut}(\exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes S_n) \quad \textit{for each } O \in \mathbf{Pa}_n$$

*and a map*

$$\mathrm{GRT}(\mathbf{k}) \times \mathbf{Pa}_n \times \mathbf{Pa}_n \to \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}), \quad (g, O, O') \mapsto b_{OO'}^{gr}(g),$$

*satisfying the analogues of the identities of Proposition 5.22, 1).*

*2) There are morphisms*

$$\mu_O^{ell,gr} : \mathrm{GRT}_{ell}(\mathbf{k}) \to \mathrm{Aut}(\exp(\hat{\mathfrak{t}}_{1,n}^{\mathbf{k}}) \rtimes S_n)$$

*for each $O \in \mathbf{Pa}_n$, satisfying the analogues of the identities of Proposition 5.22, 2).*

*3) The restriction $\mu^{ell,gr}_{O|R^{gr}_{ell}(\mathbf{k})}$ is independent of $O$ and will be denoted*

$$\mu_{ell}^{gr} : R_{ell}^{gr}(\mathbf{k}) \to \mathrm{Aut}(\exp(\hat{\mathfrak{t}}_{1,n}^{\mathbf{k}}) \rtimes S_n).$$

*This morphism factors as $R_{ell}^{gr}(\mathbf{k}) \to \mathrm{Aut}(\hat{\mathfrak{t}}_{1,n}^{\mathbf{k}})^{S_n} \to \mathrm{Aut}(\exp(\hat{\mathfrak{t}}_{1,n}^{\mathbf{k}}) \rtimes S_n)$. The Lie algebra morphism associated to the first factor is*

$$\mathfrak{r}_{ell}^{gr} \to \mathrm{Der}(\mathfrak{t}_{1,n})^{S_n}, \quad (\alpha_+, \alpha_-) \mapsto (x_i^{\pm} \mapsto \alpha_{\pm}^{i,1\cdots\check{i}\cdots n}).$$

*Proof* Similar to that of Proposition 5.22. □

*Remark 5.24* In Calaque et al. [7], we introduced the Lie algebra $\mathfrak{d} := \mathfrak{d}_+ \rtimes \mathfrak{sl}_2$, where $\mathfrak{d}_+$ is the $\mathfrak{sl}_2$-Lie algebra freely generated by a family $(\tilde{\delta}_{2m})_{m \geq 0}$, subject to the only constraint that for any $m \geq 0$, $\tilde{\delta}_{2m}$ generates a simple $(2m+1)$-dimensional $\mathfrak{sl}_2$-module, for which it is a highest weight vector. There is a surjective morphism $\mathfrak{d} \to \mathfrak{b}_3$, which is the identity on $\mathfrak{sl}_2$ and given by $\tilde{\delta}_{2m} \mapsto \delta_{2m}$. In Calaque et al. [7], we also constructed a morphism

$$\mathfrak{d} \to \mathrm{Der}(\mathfrak{t}_{1,n})^{S_n}.$$

According to Proposition 5.23, 3), this morphism factors as $\mathfrak{d} \to \mathfrak{r}_{ell}^{gr} \to \mathrm{Der}(\mathfrak{t}_{1,n})^{S_n}$. As $\mathrm{im}(\mathfrak{d} \to \mathfrak{r}_{ell}^{gr}) = \mathfrak{b}_3$, the morphism from [7] factors through $\mathfrak{b}_3$.

Let us set $B_n^{gr}(\mathbf{k}) := \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes S_n$, $B_{1,n}^{gr}(\mathbf{k}) := \exp(\hat{\mathfrak{t}}_{1,n}^{\mathbf{k}}) \rtimes S_n$. We define $P_n^{gr}(\mathbf{k})$, $P_{1,n}^{gr}(\mathbf{k})$ as the "pure" versions of these groups (i.e. the kernels of their maps to $S_n$).

**Proposition 5.25** *1) There is a family of isomorphisms $i_O^{\tilde{\Phi}} : B_n(\mathbf{k}) \to B_n^{gr}(\mathbf{k})$ for each $\tilde{\Phi} := (\mu, \Phi) \in M(\mathbf{k})$, and a family of maps*

$$M(\mathbf{k}) \times \mathbf{Pa}_n \times \mathbf{Pa}_n \to P_n^{gr}(\mathbf{k}), \quad (\tilde{\Phi}, O, O') \mapsto b_{OO'}^{gr}(\tilde{\Phi}),$$

*such that*

$$i^{\tilde\Phi}_{O'} = \mathrm{Inn}(b^{gr}_{OO'}(\tilde\Phi)) \circ i^{\tilde\Phi}_{O}, \quad b^{gr}_{OO''}(\tilde\Phi) = b^{gr}_{O'O''}(\tilde\Phi) b^{gr}_{OO'}(\tilde\Phi),$$
$$i^{\tilde\Phi}_{O} \circ \mu_O(g) = i^{g^{-1}*\tilde\Phi}_{O}, \quad \mu^{gr}_O(g_{gr}) \circ i^{\tilde\Phi}_{O} = i^{\tilde\Phi * g^{-1}_{gr}}_{O},$$

*where* $g \in \mathrm{GT}(\mathbf{k}),\ g_{gr} \in \mathrm{GRT}(\mathbf{k})$.

2) *Each* $e \in Ell(\mathbf{k})$ *gives rise to a family of isomorphisms* $i^{ell,e}_O : B_{1,n}(\mathbf{k}) \to B^{gr}_{1,n}(\mathbf{k})$, *indexed by* $O \in \mathbf{Pa}_n$. *They satisfy*

$$i^{\tilde\Phi}_O(b)_{ell} = i^{ell,e}_O(b_{ell}), \quad i^{ell,e}_{O'} = \mathrm{Inn}(b_{O,O'}(\tilde\Phi)_{ell}) \circ i^{ell,e}_O$$

*for* $b \in B_n(\mathbf{k})$, *if* $\tilde\Phi := \mathrm{im}(e \in Ell(\mathbf{k}) \to M(\mathbf{k}))$, *and*

$$i^{ell,e}_O \circ \mu^{ell}_O(g) = i^{ell,g^{-1}*e}_O, \quad \mu^{ell,gr}_O(g_{gr}) \circ i^{ell,e}_O = i^{ell,e*g^{-1}_{gr}}_O,$$

*for* $g \in \mathrm{GT}_{ell}(\mathbf{k}),\ g_{gr} \in \mathrm{GRT}_{ell}(\mathbf{k})$.
*There is a commutative diagram*

$$\begin{array}{ccc} R_{ell}(\mathbf{k}) & \xrightarrow{\mu^{gr}_{ell}} & \mathrm{Aut}(B_{1,n}(\mathbf{k})) \\ \downarrow{\scriptstyle i_e} & & \downarrow{\scriptstyle (i_{e,O})^*} \\ R^{gr}_{ell}(\mathbf{k}) & \xrightarrow{\mu^{ell}} & \mathrm{Aut}(B^{gr}_{1,n}(\mathbf{k})) \end{array}$$

*Proof* Let $\mathcal{C}^{gr} := \mathbf{PaCD}_{\mathbf{k}}$, $\mathcal{C}^{gr}_{ell} := \mathbf{PaCD}_{ell,\mathbf{k}}$, then there are compatible functors $\mathcal{C} \to \tilde\Phi * \mathcal{C}_{gr} \simeq \mathcal{C}_{gr}$, $\mathcal{C}_{ell} \to e * \mathcal{C}^{gr}_{ell} \simeq \mathcal{C}^{gr}_{ell}$, where in each case, the first functor arises from universal properties and the second tensor forgets about the IBMC (or elliptic IBMC) structures. The statements follow from the compatibility of these functors with the actions of $\mathrm{GT}_{ell}(\mathbf{k}),\ \mathrm{GRT}_{ell}(\mathbf{k})$. ◻

## 6 A family of elliptic associators, $\tau \mapsto e(\tau)$

In this section, we construct an analytic family of elliptic associators $\tau \mapsto e(\tau)$, indexed by the Poincaré half-plane. This family arises from the KZB connection [7] and may therefore be viewed as an analogue of the KZ associator. We study various functional properties of this family: modular properties, behaviour at infinity, and differential system.

### 6.1 The KZ associator

Let $G_0(z),\ G_1(z)$ be the analytic solutions of

$$G'(z) = \left(\frac{A}{z} + \frac{B}{z-1}\right) G(z)$$

in ]0, 1[, valued in $\exp(\hat{\mathfrak{f}}_2^{\mathbb{C}})$, with asymptotic behaviour $G_0(z) \sim z^A$ as $z \to 0$ and $G_1(z) \sim (1-z)^B$ as $z \to 1$. The KZ associator is defined by

$$\Phi_{KZ} := G_1(z)^{-1} G_0(z) \in \exp(\hat{\mathfrak{f}}_2).$$

Then[15] $(2\pi\,\mathrm{i}, \Phi_{KZ}) \in M(\mathbb{C})$ [9].

6.2 Definition of $e(\tau) = (A(\tau), B(\tau))$

Let $\mathfrak{H} := \{\tau \in \mathbb{C} | \Im(\tau) > 0\}$ be the Poincaré half-plane. Let $(z, \tau) \mapsto \theta(z|\tau)$ be the holomorphic function on $\mathbb{C} \times \mathfrak{H}$, such that $\theta(z+1|\tau) = -\theta(z|\tau) = \theta(-z|\tau)$, $\theta(z+\tau|\tau) = -e^{2\pi\,\mathrm{i}\,z\tau + \mathrm{i}\,\pi\tau}\theta(z|\tau)$, $\{z|\theta(z|\tau) = 0\} = \mathbb{Z} + \tau\mathbb{Z}$, $\partial_z\theta(0|\tau) = 1$.

For $\tau \in \mathfrak{H}$, let $F(z|\tau)$ be the holomorphic function on $\{z = a + b\tau | a, b \in \mathbb{R},\ a$ or $b \in ]0,1[\}$, valued in $\exp(\hat{\mathfrak{t}}_{1,2}^{\mathbb{C}}) \simeq \exp(\hat{\mathfrak{f}}_2^{\mathbb{C}})$, such that

$$\partial_z F(z|\tau) = -\frac{\theta(z + \mathrm{ad}\,x|\tau)\,\mathrm{ad}\,x}{\theta(z|\tau)\theta(\mathrm{ad}\,x|\tau)}(y) \cdot F(z|\tau) \text{ and } F(z|\tau) \sim (-2\pi\,\mathrm{i}\,z)^t \text{ as } z \to 0;$$

here $x := x_2^+,\ y := x_2^-,\ t := t_{12}$. We then set

$$A(\tau) := F(z|\tau)^{-1} F(z+1|\tau), \quad B(\tau) := F(z|\tau)^{-1} e^{2\pi\,\mathrm{i}\,x} F(z+\tau|\tau).$$

6.3 Algebraic properties of $e(\tau)$

We set $Ell_{KZ} := Ell(\mathbb{C}) \times_{M(\mathbb{C})} \{(2\pi\,\mathrm{i}, \Phi_{KZ})\}$.

**Proposition 6.1** *$\tau \mapsto e(\tau) := (A(\tau), B(\tau))$ is an analytic map $\mathfrak{H} \to Ell_{KZ}$.*

*Proof* In Calaque et al. [7], Sect. 4.3, we introduced $\tilde{A}, \tilde{B} \in \exp(\hat{\mathfrak{t}}_{1,2}^{\mathbb{C}})$. We set $\tilde{A}_+ := \tilde{A},\ \tilde{A}_- := \tilde{B},\ A_+(\tau) := A(\tau),\ A_-(\tau) := B(\tau)$, then

$$A_\pm(\tau) = \mathrm{Ad}((-2\pi\,\mathrm{i})^{-t})(\tilde{A}_\pm),$$

So $(A_+(\tau), A_-(\tau))$ satisfies (22), (23), (26) in Calaque et al. [7]. (22), (23) imply that $(A_+(\tau), A_-(\tau))$ satisfies (25). (26) implies that

$$(A_-(\tau)^{12,3}\{\Phi^{-1}\}(A_-(\tau)^{1,23})^{-1}\{\Phi\}, A_+(\tau)^{12,3}) = \{\Phi^{-1} e^{2\pi\,\mathrm{i}\,t_{23}}\Phi\}$$

and using (23) in Calaque et al. [7], we rewrite this as

$$(\{e^{-\mathrm{i}\,\pi t_{12}}\Phi^{3,2,1}\}A_-(\tau)^{2,13}\{\Phi^{2,1,3}e^{-\mathrm{i}\,\pi t_{12}}\}, A_+(\tau)^{12,3}) = \{\Phi^{-1}e^{2\pi\,\mathrm{i}\,t_{23}}\Phi\},$$

which as in the proof of Proposition 4.8 implies that $(A_+(\tau), A_-(\tau))$ satisfies (26). □

15 We set $\mathrm{i} := \sqrt{-1}$.

6.4 Analytic properties of $e(\tau)$

**Proposition 6.2** *One has*

$$2\pi \mathrm{i} \frac{\partial}{\partial \tau} e(\tau) = e(\tau) * (-e_- - \sum_{k\geq 0} (2k+1) G_{2k+2}(\tau)\delta_{2k}),$$

*where $G_k(\tau)$ are the Eisenstein series defined by*

$$G_k(\tau) = \sum_{a\in(\mathbb{Z}+\tau\mathbb{Z})-\{0\}} a^{-k} \text{ for } k \text{ even } \geq 4, \quad G_2(\tau) = \sum_{m\in\mathbb{Z}} \left( \sum_n{}' (n+m\tau)^{-2} \right),$$

*where $\sum'$ means $\sum_{n\in\mathbb{Z}}$ if $m \geq 0$ and $\sum_{n\in\mathbb{Z}-\{0\}}$ if $m = 0$ (notation as in (48), (49)).*

*Proof* $R_{ell}(\mathbb{C}) \subset \mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbb{C}})^{op}$ acts from the right on $Ell_{KZ}$ by $(A_+, A_-) * (u_+, u_-) := (A_+(u_+, u_-), A_-(u_+, u_-))$. The same formula defines a left action of $R_{ell}(\mathbb{C})^{op} \subset \mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbb{C}})$ on $Ell_{KZ}$. To prove that

$$2\pi \mathrm{i}\, \partial_\tau e(\tau) = e(\tau) * x(\tau)$$

for $x(\tau) \in \hat{\mathfrak{t}}_{ell}^{\mathbb{C}} \subset \mathrm{Der}(\hat{\mathfrak{f}}_2^{\mathbb{C}})^{op}$, it therefore suffices to prove that

$$2\pi \mathrm{i}\, \partial_\tau A(\tau) = x(\tau)(A(\tau)), \quad 2\pi \mathrm{i}\, \partial_\tau B(\tau) = x(\tau)(B(\tau)),$$

where $x(\tau)$ is now viewed as an element of $\mathrm{Der}(\hat{\mathfrak{f}}_2^{\mathbb{C}})$.

In Calaque et al. [7], Lemma 23, we constructed a function $F^{(2)}(z|\tau)$, defined on $\{(z,\tau) \in \mathbb{C} \times \mathfrak{H} | z = a + b\tau, (a,b) \in ]0,1[\times\mathbb{R} \cup \mathbb{R}\times]0,1[\}$ and valued in $\exp(\hat{\mathfrak{f}}_2^{\mathbb{C}}) \rtimes \mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbb{C}})$, such that

$$\partial_z F^{(2)}(z|\tau) = -\frac{\theta(z + \mathrm{ad}\, x|\tau)\, \mathrm{ad}\, x}{\theta(z|\tau)\theta(\mathrm{ad}\, x|\tau)}(y) \cdot F^{(2)}(z|\tau),$$

$$2\pi \mathrm{i} \frac{\partial}{\partial \tau} F^{(2)}(z|\tau) = -\Big(e_- + \sum_{k\geq 0} (2k+1) G_{2k+2}(\tau)\dot{\delta}_{2k}^{(2)} - g(z, \mathrm{ad}\, x|\tau)(t)\Big) \cdot F^{(2)}(z|\tau)$$
$$= -\Big(e_- + \sum_{k\geq 0} (2k+1) G_{2k+2}(\tau)\delta_{2k}^{(2)} - g(z|\tau)(t)\Big) \cdot F^{(2)}(z|\tau),$$

and $F^{(2)}(z|\tau) \sim z^t \exp(\frac{-\tau}{2\pi \mathrm{i}}(e_- + \sum_{k\geq 0} 2(2k+1)\zeta(2k+2)\delta_{2k}^{(2)}))$ as $z \to 0$ and $\tau \to \mathrm{i}\infty$. Here $g(z, x|\tau) = \frac{\theta(z+x|\tau)}{\theta(z|\tau)\theta(x|\tau)}(\frac{\theta'}{\theta}(z+x|\tau) - \frac{\theta'}{\theta}(z|\tau)) + \frac{1}{x^2}$, and $g(z|\tau) := g(z, \mathrm{ad}\, x|\tau)(t) - g(0, \mathrm{ad}\, x|\tau)(t)$; in the notation of *loc. cit.*, $e_- = \Delta_0$.

These conditions imply that the image of $F^{(2)}(z|\tau)$ in $\mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbb{C}})$ is independent of $z$. Then

$$A_{z_0}^{z_1}(\tau) := F^{(2)}(z_1|\tau) F^{(2)}(z_0|\tau)^{-1} \in \exp(\hat{\mathfrak{f}}_2^{\mathbb{C}})$$

and satisfies

$$2\pi\,\mathrm{i}\,\partial_\tau A_{z_0}^{z_1}(\tau) = -\left(e_- + \sum_{k\geq 0}(2k+1)G_{2k+2}(\tau)\delta_{2k}\right)(A_{z_0}^{z_1}(\tau))$$
$$+g(z_1|\tau)\cdot A_{z_0}^{z_1}(\tau) - A_{z_0}^{z_1}(\tau)\cdot g(z_0|\tau).$$

The function $F(z|\tau)$, basic to the definition of $(A(\tau), B(\tau))$, is related to the function $F^{(2)}(z|\tau)$ by $F^{(2)}(z|\tau) = F(z|\tau)\varphi(\tau)$, where $\varphi(\tau)$ takes values in $\exp(\hat{\mathfrak{f}}_2^{\mathbb{C}}) \rtimes \mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbb{C}})$, as both satisfy the same differential equation in $z$. It follows that

$$A_{z_0}^{z_1}(\tau) = F(z_1|\tau)F(z_0|\tau)^{-1}.$$

Therefore, $A(\tau) = F(z|\tau)^{-1}A_z^{z+1}(\tau)F(z|\tau)$. In the limit $z \to 0$, this gives

$$A(\tau) = \lim_{\epsilon\to 0}(-2\pi\,\mathrm{i}\,\epsilon)^{-\operatorname{ad} t}\big(A_\epsilon^{1+\epsilon}(\tau)\big).$$

$\varepsilon$ being fixed, $(-2\pi\,\mathrm{i}\,\epsilon)^{-\operatorname{ad} t}\big(A_\epsilon^{1+\epsilon}(\tau)\big)$ satisfies the same differential equation in $\tau$ as $A_{z_0}^{z_1}(\tau)$, with $g(z_0|\tau)$ replaced by $(-2\pi\,\mathrm{i}\,\epsilon)^{-\operatorname{ad}(t)}(g(\epsilon|\tau))$ and $g(z_1|\tau)$ replaced by $(-2\pi\,\mathrm{i}\,\epsilon)^{-\operatorname{ad}(t)}(g(1+\epsilon|\tau))$, which both tend to 0 as $\epsilon \to 0$. It follows that these terms disappear from the differential equation satisfied by $A(\tau)$, so

$$2\pi\,\mathrm{i}\,\partial_\tau A(\tau) = -\left(e_- + \sum_{k\geq 0}(2k+1)G_{2k+2}(\tau)\delta_{2k}\right)(A(\tau)).$$

Similarly, $B(\tau) = F(z|\tau)^{-1}e^{2\pi\,\mathrm{i}\,x}A_z^{z+\tau}(\tau)F(z|\tau)$, hence

$$B(\tau) = \lim_{\epsilon\to 0}(-2\pi\,\mathrm{i}\,\epsilon)^{-t}e^{2\pi\,\mathrm{i}\,x}A_\epsilon^{\tau+\epsilon}(\tau)(-2\pi\,\mathrm{i}\,\epsilon)^t.$$

One computes

$$\partial_\tau\left(A_\epsilon^{\tau+\epsilon}(\tau)\right) = \frac{-1}{2\pi\,\mathrm{i}}(e_- + \sum_{k\geq 0}(2k+1)G_{2k+2}(\tau)\delta_{2k})(A_\epsilon^{\tau+\epsilon}(\tau))$$
$$+\left(\frac{1}{2\pi\,\mathrm{i}}g(\tau+\epsilon|\tau) - \frac{\theta(\tau+\epsilon+\operatorname{ad}x|\tau)\operatorname{ad}x}{\theta(\tau+\epsilon|\tau)\theta(\operatorname{ad}x|\tau)}(y)\right)A_\epsilon^{\tau+\epsilon}(\tau)$$
$$- A_\epsilon^{\tau+\epsilon}(\tau)\frac{1}{2\pi\,\mathrm{i}}g(\epsilon|\tau).$$

So $X_\epsilon(\tau) := (-2\pi\,\mathrm{i}\,\epsilon)^{-t}e^{2\pi\,\mathrm{i}\,x}A_\epsilon^{\tau+\epsilon}(\tau)(-2\pi\,\mathrm{i}\,\epsilon)^t$ satisfies ($\epsilon$ being fixed)

$$2\pi\,\mathrm{i}\,\partial_\tau(X_\epsilon(\tau)) = -(e_- + \sum_{k\geq 0}(2k+1)G_{2k+2}(\tau)\delta_{2k})(X_\epsilon(\tau)) - X_\epsilon(\tau)\cdot$$

$$\big((-2\pi\,\mathrm{i}\,\epsilon)^{-t} g(\epsilon|\tau)(-2\pi\,\mathrm{i}\,\epsilon)^{t}\big)$$
$$+\left(\mathrm{Ad}((-2\pi\,\mathrm{i}\,\epsilon)^{-t} e^{2\pi\,\mathrm{i}\,x})\big(g(\tau+\epsilon|\tau) - 2\pi\,\mathrm{i}\,\frac{\theta(\tau+\epsilon+\mathrm{ad}\,x|\tau)\,\mathrm{ad}\,x}{\theta(\tau+\epsilon|\tau)\theta(\mathrm{ad}\,x|\tau)}(y)\big)\right.$$
$$\left. - (-2\pi\,\mathrm{i}\,\epsilon)^{-t} e^{2\pi\,\mathrm{i}\,x}\left(e_- + \sum_{k\geq 0}(2k+1)G_{2k+2}(\tau)\delta_{2k}\right)(e^{-2\pi\,\mathrm{i}\,x})(-2\pi\,\mathrm{i}\,\epsilon)^{t}\right)\cdot (X_\epsilon(\tau)).$$

Identity (7) in Calaque et al. [7] implies that the parenthesis in the two last lines equals $\mathrm{Ad}((-2\pi\,\mathrm{i}\,\epsilon)^{-t})(g(\epsilon|\tau))$. As before, we get in the limit $\epsilon \to 0$

$$2\pi\,\mathrm{i}\,\partial_\tau B(\tau) = -\left(e_- + \sum_{k\geq 0}(2k+1)G_{2k+2}(\tau)\delta_{2k})(B(\tau)\right).$$

□

**Proposition 6.3**

$$\sigma(\Phi_{KZ})_{\substack{|x\mapsto 2\pi\,\mathrm{i}\,x,\\ y\mapsto(2\pi\,\mathrm{i})^{-1}y}} = \lim_{\tau\to\mathrm{i}\,\infty} e(\tau) * \exp\left(\frac{\tau}{2\pi\,\mathrm{i}}(e_- + \sum_{k\geq 0}(2k+1)\zeta(2k+2)\delta_{2k})\right).$$

*Proof* In Calaque et al. [7] (proof of Prop. 24 and Lemma 29), is it proved that

$$A(\tau) = \Phi_{KZ}(\tilde{y}, t)e^{2\pi\,\mathrm{i}\,\tilde{y}}\Phi_{KZ}(\tilde{y}, t)^{-1} + O(e^{2\pi\,\mathrm{i}\,\tau}),$$
$$B(\tau) = e^{\mathrm{i}\,\pi t}\Phi_{KZ}(-\tilde{y} - t, t)e^{2\pi\,\mathrm{i}\,x}e^{2\pi\,\mathrm{i}\,\tilde{y}\tau}\Phi_{KZ}(\tilde{y}, t)^{-1} + O(e^{2\pi\,\mathrm{i}\,\tau(1-\epsilon)}),$$

for any $\epsilon > 0$, where

$$\tilde{y} := -\frac{\mathrm{ad}\,x}{e^{2\pi\,\mathrm{i}\,\mathrm{ad}\,x} - 1}(y).$$

Let $(A_{pol}(\tau), B_{pol}(\tau))$ be the principal parts of the right sides of these equalities; $A_{pol}(\tau)$ is constant in $\tau$, while each coordinate of $B_{pol}(\tau)$ in a basis of $U(\mathfrak{f}_2)$ is a polynomial in $\tau$.

It is proved in Calaque et al. [7] that $\tilde{y}, t$ are in the kernel of $e_- + \sum_{k\geq 0}(2k+1)\zeta(2k+2)\delta_{2k}$, while

$$\exp\left(\frac{\tau}{2\pi\,\mathrm{i}}(e_- + \sum_{k\geq 0}(2k+1)\zeta(2k+2)\delta_{2k}\right)(e^{2\pi\,\mathrm{i}\,x}e^{2\pi\,\mathrm{i}\,\tau\tilde{y}}) = e^{2\pi\,\mathrm{i}\,x}.$$

It follows that

$$\exp\left(\frac{\tau}{2\pi\,\mathrm{i}}(e_- + \sum_{k\geq 0}(2k+1)\zeta(2k+2)\delta_{2k}\right)(A_{pol}(\tau), B_{pol}(\tau))$$
$$= \sigma(\Phi_{KZ})\sigma(\Phi_{KZ})_{|x\mapsto 2\pi\,\mathrm{i}\,x, \atop y\mapsto(2\pi\,\mathrm{i})^{-1}y},$$

which implies that statement. □

Note that the operation $e \mapsto e_{|x\mapsto 2\pi\,\mathrm{i}\,x, \atop y\mapsto(2\pi\,\mathrm{i})^{-1}y}$ amounts the action of $\mathrm{diag}(2\pi\,\mathrm{i}, (2\pi\,\mathrm{i})^{-1}) \subset \mathrm{SL}_2(\mathbb{C}) \subset R_{ell}(\mathbb{C})$ on $Ell_{KZ}$.

## 6.5 Modularity properties of $e(\tau)$

We now describe the behaviour of the map $\tau \mapsto e(\tau)$ under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$.

Define $\log : \mathbb{C}^\times \to \mathbb{C}$ by the condition that its image is contained in $\mathbb{R} + \mathrm{i}[-\pi, \pi[$. We define group morphisms $t : \mathbb{C}^\times \to \mathrm{SL}_2(\mathbb{C})$ and $n_\pm : \mathbb{C} \to \mathrm{SL}_2(\mathbb{C})$ by $t(\lambda) := \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, $n_+(a) := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, $n_-(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

**Proposition 6.4** *1) There is a unique map*

$$f : B_3 \times \mathfrak{H} \to \mathbb{C},$$

*such that*

$$f(\sigma_1, \tau) = 0, \quad f(\sigma_2, \tau) = -\log\left(\frac{-1}{\tau - 1}\right)$$

*and with the cocycle property* $f(gg', \tau) = f(g, \overline{g'} \cdot \tau) + f(g', \tau)$, *where* $g \mapsto \overline{g}$ *is the morphism* $B_3 \to \mathrm{SL}_2(\mathbb{Z})$ *and the action on* $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathfrak{H}$ *is* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \tau = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$.

*2) For any* $g \in B_3$ *and* $\tau \in \mathfrak{H}$, *one has*

$$e(\overline{g} \cdot \tau) = \mathrm{Ad}(e^{f(g,\tau)t})\Big(g * \big(a(\overline{g}, \tau) \bullet e(\tau)\big)\Big), \tag{55}$$

*where:*

- *for* $\alpha \in \mathbb{C}$, $\mathrm{Ad}(e^{\alpha t})$ *is the self-map of* $Ell_{KZ}$ *given by* $\mathrm{Ad}(e^{\alpha t})(e) := (e^{\alpha t}Ae^{-\alpha t}, e^{\alpha t}Be^{-\alpha t})$ *for* $e = (A, B)$;
- $a : \mathrm{SL}_2(\mathbb{Z}) \times \mathfrak{H} \to \mathrm{SL}_2(\mathbb{C})$ *is given by* $a(\overline{g}, \tau) = \begin{pmatrix} \gamma\tau+\delta & 0 \\ 2\pi\,\mathrm{i}\,\gamma & (\gamma\tau+\delta)^{-1} \end{pmatrix} = n_+(\frac{2\pi\,\mathrm{i}\,\gamma}{\gamma\tau+\delta})t((\gamma\tau+\delta)^{-1})$ *if* $\overline{g} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$;
- $*$ *and* $\bullet$ *are the commuting left actions of* $B_3 = R_{ell} \subset R_{ell}(\mathbb{C})$ *and* $\mathrm{SL}_2(\mathbb{C}) \subset R^{gr}_{ell}(\mathbb{C})^{op}$ *on* $Ell_{KZ}$, *given as follows:*

- *for* $e = (A, B) \in Ell_{KZ}$ *and* $g \in B_3$, $g * e := (\theta_g(a)_{|(a,b)\mapsto(A,B)}, \theta_g(b)_{|(a,b)\mapsto(A,B)})$, *where* $\theta : B_3 \to \mathrm{Aut}(F_2)$ *is the action of* $B_3$ *on the free group* $F_2$ *generated by* $a$, $b$, *and* $x \mapsto x_{|(a,b)\mapsto(A,B)}$ *is the morphism* $F_2 \to \exp(\hat{\mathfrak{f}}_2^{\mathbb{C}})$, *given by* $a, b \mapsto A, B$;
- *for* $e = (A, B) \in Ell_{KZ}$ *and* $a \in \mathrm{SL}_2(\mathbb{C})$, $a \bullet e := (\alpha_a(A), \alpha_a(B))$, *where* $\alpha : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\exp(\hat{\mathfrak{f}}_2^{\mathbb{C}}))^{op}$ *is induced by* $\alpha_a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ *if* $a = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

*Remark 6.5* Let $g \in B_3$ and $\overline{g} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is its image in $\mathrm{SL}_2(\mathbb{Z})$, then $\exp f(g, \tau) = \gamma\tau + \delta$ for any $\tau \in \mathfrak{H}$.

*Remark 6.6* For $g = (\sigma_1\sigma_2)^6$ (a generator of the kernel of $B_3 \to \mathrm{SL}_2(\mathbb{Z})$),

$$g * e = (\mathrm{Ad}(B, A)(A), \mathrm{Ad}(B, A)(B)) = (\mathrm{Ad}(e^{2\pi\,\mathrm{i}\,t})(A), \mathrm{Ad}(e^{2\pi\,\mathrm{i}\,t})(B)),$$

while $f(g, \tau) = -2\pi\,\mathrm{i}$. One checks this way that the r.h.s. of (55) does not depend of the choice of a lift $g$ of $\overline{g}$ to $B_3$.

*Proof* Statement 1) can be checked using the presentation of $B_3$. It follows from the cocycle identity for $f(g, \tau)$ and from the cocycle identity

$$a(hh', \tau) = a(h', \tau)a(h, h' \cdot \tau), \quad h, h' \in \mathrm{SL}_2(\mathbb{Z}), \tau \in \mathfrak{H}$$

that $\Gamma := \{g \in B_3 |$ identity (55) holds for any $\tau \in \mathfrak{H}\}$ is a subgroup of $B_3$. So statement 2) follows from its particular cases $g = \sigma_1,\ g = \sigma_1\sigma_2\sigma_1$. □

Recall that

$$A(\tau) = \lim_{\epsilon\to 0^+} (-2\pi\,\mathrm{i}\,\epsilon)^{-t} A_\epsilon^{1+\epsilon}(\tau)(-2\pi\,\mathrm{i}\,\epsilon)^t,$$
$$B(\tau) = \lim_{\epsilon\to 0^+} (-2\pi\,\mathrm{i}\,\epsilon)^{-t} e^{2\pi\,\mathrm{i}\,x} A_\epsilon^{\tau+\epsilon}(\tau)(-2\pi\,\mathrm{i}\,\epsilon)^t,$$

where $A_{z_0}^{z_1}(\tau)$ be the solution of $\partial_{z_1} A_{z_0}^{z_1}(\tau) = K(z_1|\tau)A_{z_0}^{z_1}(\tau)$ such that $A_z^z(\tau) = 1$, where $K(z|\tau) = -\frac{\theta(z+\mathrm{ad}\,x|\tau)\,\mathrm{ad}\,x}{\theta(z|\tau)\theta(\mathrm{ad}\,x|\tau)}(y)$ and where the chosen branches of $A_{z_0}^{z_1}(\tau)$ are as in Fig. 2.

The identity $K(z|\tau) = K(z|\tau + 1)$ implies $A_\epsilon^{1+\epsilon}(\tau + 1) = A_\epsilon^{1+\epsilon}(\tau)$, and using the decomposition of Fig. 3, it also implies $A_\epsilon^{\tau+1+\epsilon}(\tau + 1) = A_{1+\epsilon}^{\tau+1+\epsilon}(\tau)A_\epsilon^{1+\epsilon}(\tau) = A_\epsilon^{\tau+\epsilon}(\tau)A_\epsilon^{1+\epsilon}(\tau)$. So $A(\tau+1) = A(\tau),\ B(\tau+1) = B(\tau)A(\tau)$, so $e(\tau+1) = \sigma_1 * e(\tau)$, which shows (55) in the case $g = \sigma_1$.

Let $w := -z/\tau$, then

$$\frac{\partial}{\partial w} - K(w|\frac{-1}{\tau}) = -\tau e^{-2\pi\,\mathrm{i}\,zx}\Big(\frac{\partial}{\partial z} - \begin{pmatrix} -\tau & 0 \\ -2\pi\,\mathrm{i} & \frac{-1}{\tau} \end{pmatrix} \bullet K(z|\tau)\Big)e^{2\pi\,\mathrm{i}\,zx}.$$

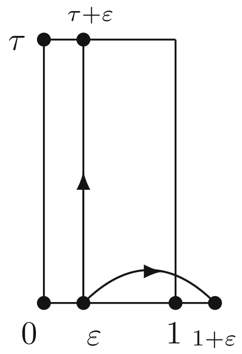
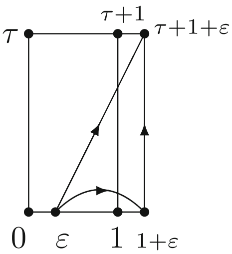



**Fig. 2** Analytic continuation of $z \mapsto A_\epsilon^z(\tau)$

**Fig. 3** Proof of $B(\tau+1) = B(\tau)A(\tau)$




So

$$A_{w_0}^{w_1}\left(\frac{-1}{\tau}\right) = e^{-2\pi \mathrm{i}\, x z_1} \cdot \begin{pmatrix} -\tau & 0 \\ -2\pi \mathrm{i} & \frac{-1}{\tau} \end{pmatrix} \bullet (A_{z_0}^{z_1}(\tau)) \cdot e^{2\pi \mathrm{i}\, x z_0}.$$

Then

$$A\left(\frac{-1}{\tau}\right) = \lim_{\varepsilon\to 0^+} (-2\pi \mathrm{i}\, \epsilon)^{-t} A_\epsilon^{1+\epsilon}(\frac{-1}{\tau})(-2\pi \mathrm{i}\, \epsilon)^t$$

$$= \lim_{\varepsilon\to 0^+} (-2\pi \mathrm{i}\, \epsilon)^{-t} \cdot \begin{pmatrix} -\tau & 0 \\ -2\pi \mathrm{i} & \frac{-1}{\tau} \end{pmatrix} \bullet \left(e^{-2\pi \mathrm{i}(1+\epsilon)x} A_{-\varepsilon\tau}^{-\tau-\varepsilon\tau}(\tau) e^{2\pi \mathrm{i}\, \epsilon x}\right) \cdot (-2\pi \mathrm{i}\, \epsilon)^t$$

$$= \exp(-\log\left(\frac{-1}{\tau}\right) t) \cdot \begin{pmatrix} -\tau & 0 \\ -2\pi \mathrm{i} & \frac{-1}{\tau} \end{pmatrix} \bullet B(\tau)^{-1} \cdot \exp\left(\log(\frac{-1}{\tau}) t\right),$$

see Fig. 4; and

$$B\left(\frac{-1}{\tau}\right) = \lim_{\epsilon\to 0^+} (-2\pi \mathrm{i}\, \epsilon)^{-t} e^{2\pi \mathrm{i}\, x} A_\epsilon^{\frac{-1}{\tau}+\epsilon}\left(\frac{-1}{\tau}\right)(-2\pi \mathrm{i}\, \epsilon)^t$$

$$= \lim_{\varepsilon\to 0^+} (-2\pi \mathrm{i}\, \epsilon)^{-t} e^{2\pi \mathrm{i}\, x\tau\epsilon} \cdot \begin{pmatrix} -\tau & 0 \\ -2\pi \mathrm{i} & \frac{-1}{\tau} \end{pmatrix} \bullet A_{-\tau\epsilon}^{1-\tau\epsilon}(\tau) \cdot e^{-2\pi \mathrm{i}\, x\tau\epsilon}(-2\pi \mathrm{i}\, \epsilon)^{-t}$$

$$= \exp(-\log\left(\frac{-1}{\tau}\right) t) \cdot \begin{pmatrix} -\tau & 0 \\ -2\pi \mathrm{i} & \frac{-1}{\tau} \end{pmatrix} \bullet \left(\lim_{\varepsilon\to 0^+} (2\pi \mathrm{i}\, \tau\varepsilon)^{-t} A_{-\tau\varepsilon}^{1-\tau\varepsilon}(\tau)(2\pi \mathrm{i}\, \tau\varepsilon)^t\right)$$

$$\cdot \exp\left(\log\left(\frac{-1}{\tau}\right) t\right)$$

**Fig. 4** Relation between $A(\frac{-1}{\tau})$ and $(A(\tau), B(\tau))$

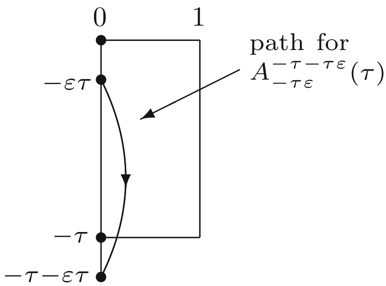
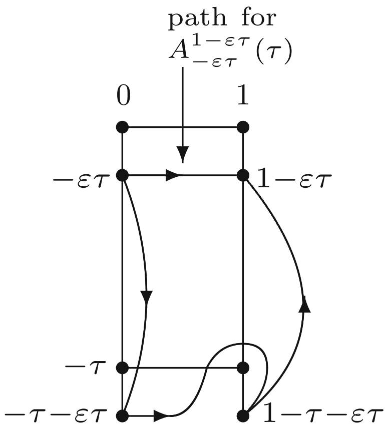



**Fig. 5** Relation between $B(\frac{-1}{\tau})$ and $(A(\tau), B(\tau))$




$$= \exp\left(-\log\left(\frac{-1}{\tau}\right)t\right)\cdot\begin{pmatrix}-\tau & 0\\ -2\pi\,\mathrm{i} & \frac{-1}{\tau}\end{pmatrix}\bullet(B(\tau)A(\tau)B(\tau)^{-1})\cdot\exp\left(\log\left(\frac{-1}{\tau}\right)t\right),$$

see Fig. 5. It follows that

$$e\left(\frac{-1}{\tau}\right) = \mathrm{Ad}\left(\exp(-\log\left(\frac{-1}{\tau}\right)t)\right)\left(\sigma_1\sigma_2\sigma_1 * \left(\begin{pmatrix}-\tau & 0\\ -2\pi\,\mathrm{i} & \frac{-1}{\tau}\end{pmatrix}\bullet e(\tau)\right)\right)$$

The result for $g = \sigma_1\sigma_2\sigma_1$ then follows.

## 7 Computations of Zariski closures

The action of the mapping class group $B_3$ in genus one on the braid groups in genus one [see (15)] restricts to an action on the pure braid subgroups. In this section, we compute the Zariski closure of the image of $B_3$ in the automorphism groups of their prounipotent completions. This computation relies on the relation between the action of $\mathrm{GT}_{ell}(-)$ on these prounipotent completions and its the graded counterpart (Sect. 5), and on the properties of the elliptic analogues of the KZ associator (Sect. 6). There properties enable us to establish the key result that the lift $e_{KZ}$ of $\Phi_{KZ}$ is compatible with the inclusion of certain subgroups in $R_{ell}(-)$ and $R_{ell}^{gr}(-)$ (see Proposition 6.3); under Conjecture 10.1, any element of $Ell(\mathbb{C})$ has the same property.

### 7.1 Automorphisms of group schemes

We will view a $\mathbb{Q}$-group scheme as a functor {$\mathbb{Q}$-algebras} → {groups}. The Lie algebra of a $\mathbb{Q}$-group scheme $G(-)$ is then $\mathrm{Lie}\, G(-) := \mathrm{Ker}(G(\mathbb{Q}[\epsilon]/(\epsilon^2)) \to G(\mathbb{Q}))$.

If $\Gamma$ is a finitely generated group, let $\Gamma(-)$ be its $\mathbb{Q}$-prounipotent completion and let $\mathrm{Lie}\,\Gamma$ be its Lie algebra (a pronilpotent $\mathbb{Q}$-Lie algebra). Let $\underline{\mathrm{Aut}\,\Gamma}(-)$ be the $\mathbb{Q}$-group scheme defined by $\underline{\mathrm{Aut}\,\Gamma}(\mathbf{k}) := \mathrm{Aut}(\mathrm{Lie}\,\Gamma\hat{\otimes}\mathbf{k})$ for $\mathbf{k}$ a $\mathbb{Q}$-ring, where $\mathrm{Lie}\,\Gamma\hat{\otimes}\mathbf{k} := \lim_{\leftarrow}(\mathrm{Lie}\,\Gamma/(\mathrm{Lie}\,\Gamma)^{\geq n}) \otimes \mathbf{k}$, and $\mathrm{Lie}\,\Gamma = (\mathrm{Lie}\,\Gamma)^{\geq 0} \supset (\mathrm{Lie}\,\Gamma)^{\geq 1} \supset \cdots$ is the lower central series filtration of $\mathrm{Lie}\,\Gamma$.

Any automorphism of $\Gamma$ gives rise to an automorphism of $\mathrm{Lie}\,\Gamma$, so there are natural morphisms

$$\mathrm{Aut}\,\Gamma \to \underline{\mathrm{Aut}\,\Gamma}(\mathbb{Q}) \to \mathrm{Aut}(\Gamma(\mathbf{k}))$$

for any $\mathbb{Q}$-ring $\mathbf{k}$. One checks that there is a morphism of $\mathbb{Q}$-group schemes

$$\mu_O : \mathrm{GT}(-) \to \underline{\mathrm{Aut}\,P_n}(-)$$

such that the resulting morphism $\mathrm{GT}(\mathbf{k}) \to \mathrm{Aut}(P_n(\mathbf{k}))$ is compatible with $\mathrm{GT}(\mathbf{k}) \to \mathrm{Aut}(B_n(\mathbf{k}))$, and morphisms

$$\begin{aligned} \mu_O^{gr} &: \mathrm{GRT}(-) \to \underline{\mathrm{Aut}\,P_n^{gr}}(-), \quad \mu_O^{ell} : \mathrm{GT}_{ell}(-) \to \underline{\mathrm{Aut}\,P_{1,n}}(-), \\ \mu_O^{ell,gr} &: \mathrm{GRT}_{ell}(-) \to \underline{\mathrm{Aut}\,P_{1,n}^{gr}}(-), \\ \mu_{ell} &: R_{ell}(-) \to \underline{\mathrm{Aut}\,P_{1,n}}(-), \quad R_{ell}^{gr}(-) \to \underline{\mathrm{Aut}\,P_{1,n}^{gr}}(-) \end{aligned}$$

with similar properties.

### 7.2 Results on Zariski closures

Define the $\mathbb{Q}$-group scheme $\langle B_3\rangle$ to be the Zariski closure of the composite group morphism $B_3 \to \mathrm{Aut}\,F_2 \to \underline{\mathrm{Aut}\,F_2}(\mathbb{Q})$; this is a group subscheme of $\underline{\mathrm{Aut}\,F_2}(-)$.

**Theorem 7.1** *Any elliptic associator of the form $e(\tau)$, $\tau \in \mathfrak{H}$, or $e_{KZ}$, gives rise to an isomorphism of $\mathbb{C}$-group schemes $\langle B_3\rangle \otimes \mathbb{C} \simeq \big(\exp(\hat{\mathfrak{b}}_3^+) \rtimes \mathrm{SL}_2\big) \otimes \mathbb{C}$. Any two isomorphisms arising in this way are related by an inner automorphism. There exists an analogous isomorphism for $\mathbb{Q}$-group schemes.*

For $n \geq 1$, define $\langle B_3\rangle_n$ to be the Zariski closure of the composite group morphism $B_3 \to \mathrm{Aut}\,P_{1,n} \to \underline{\mathrm{Aut}\,P_{1,n}}(\mathbb{Q})$; this is a group subscheme of $\underline{\mathrm{Aut}\,P_{1,n}}(-)$.

**Theorem 7.2** *For any $n \geq 1$, there is an isomorphism $\langle B_3\rangle \simeq \langle B_3\rangle_n$ of $\mathbb{Q}$-group schemes, which is compatible with the maps from $B_3$ to both sides.*

7.3 Proof of Theorem 7.1

Composing (50) with the morphism $\tilde{B}_3 \to \mathrm{GT}_{ell}(\mathbf{k})$, we obtain a commutative diagram

$$\begin{array}{ccccc} \tilde{B}_3 & \to & \mathrm{GT}_{ell}(\mathbf{k}) & \to & \mathrm{GRT}_{ell}(\mathbf{k}) \\ \downarrow & & \downarrow & & \downarrow \\ \{\pm 1\} & \to & \mathrm{GT}(\mathbf{k}) & \to & \mathrm{GRT}(\mathbf{k}) \end{array}$$

Set inducing morphisms $B_3 \to R_{ell}(\mathbf{k}) \to R_{ell}^{gr}(\mathbf{k})$.

$$e_{KZ} := \sigma(\Phi_{KZ})_{|x \mapsto 2\pi\,\mathrm{i}\,x,\ y \mapsto (2\pi\,\mathrm{i})^{-1} y}.$$

When $\mathbf{k} = \mathbb{C}$ and $e = e(\tau), e_{KZ}$, the morphism $B_3 \to R_{ell}^{gr}(\mathbb{C})$ is computed as follows.

Define $F(\tau)$ as the map $\mathfrak{H} \to \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C})$ such that

$$2\pi\,\mathrm{i}\,\partial_\tau F(\tau) = \left(e_- + \sum_{k\geq 0}(2k+1)G_{2k+2}(\tau)\delta_{2k}\right)F(\tau) \tag{56}$$

and $F(\tau) \sim \exp(\frac{\tau}{2\pi\,\mathrm{i}}(e_- + \sum_{k\geq 0}(2k+1)2\zeta(2k+2)\delta_{2k}))$ as $\tau \to \mathrm{i}\,\infty$. Then the map $\tau \mapsto e(\tau) * F(\tau)$ is a constant, and

$$e_{KZ} = e(\tau) * F(\tau) \text{ for any } \tau \in \mathfrak{H}. \tag{57}$$

Moreover, for any $\tilde{g} \in B_3$ with image $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, one has

$$\begin{aligned} e(\tau) * i_{e(\tau)}(\tilde{g}) &= \tilde{g} * e(\tau) = \mathrm{Ad}(e^{-f(\tilde{g},\tau)t})(\begin{pmatrix} \gamma\tau+\delta & 0 \\ 2\pi\,\mathrm{i}\,\gamma & (\gamma\tau+\delta)^{-1} \end{pmatrix}^{-1} \bullet e(g\tau)) \\ &= e(\tau) * F(\tau)F(g\tau)^{-1}\begin{pmatrix} \gamma\tau+\delta & 0 \\ 2\pi\,\mathrm{i}\,\gamma & (\gamma\tau+\delta)^{-1} \end{pmatrix}^{-1} e^{-f(\tilde{g},\tau)\delta_0}, \end{aligned} \tag{58}$$

where the third equality follows from (57) for $\tau$ and $g\tau$. It follows that

$$i_{e(\tau)}(\tilde{g}) = F(\tau)F(g\tau)^{-1}\begin{pmatrix} \gamma\tau+\delta & 0 \\ 2\pi\,\mathrm{i}\,\gamma & (\gamma\tau+\delta)^{-1} \end{pmatrix}^{-1} e^{-f(\tilde{g},\tau)\delta_0}. \tag{59}$$

Acting from the right by $F(\tau)$ in the equality between the second and the fourth terms of (58), one gets $\tilde{g} * e_{KZ} = e_{KZ} * F(g\tau)^{-1}\begin{pmatrix} \gamma\tau+\delta & 0 \\ 2\pi\,\mathrm{i}\,\gamma & (\gamma\tau+\delta)^{-1} \end{pmatrix}^{-1} F(\tau)e^{-f(\tilde{g},\tau)\delta_0}$, so

$$i_{e_{KZ}}(\tilde{g}) = F(g\tau)^{-1}\begin{pmatrix} \gamma\tau+\delta & 0 \\ 2\pi\,\mathrm{i}\,\gamma & (\gamma\tau+\delta)^{-1} \end{pmatrix}^{-1} F(\tau)e^{-f(\tilde{g},\tau)\delta_0} \tag{60}$$

for any $\tau \in \mathfrak{H}$. It follows that the images of $i_{e(\tau)}, i_{e_{KZ}}$ are contained in $\exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C}) \subset R_{ell}^{gr}(\mathbb{C})$. The composite morphism $B_3 \to \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ is $\tilde{g} \mapsto \begin{pmatrix} \alpha & -\beta/(2\pi\,\mathrm{i}) \\ -2\pi\,\mathrm{i}\,\gamma & \delta \end{pmatrix}$.

Recall that $B_3 \subset \mathrm{Aut}(F_2)$ is generated by $\Psi_+, \Psi_-$ given by $\Psi_+ : X, Y \mapsto X, YX$ and $\Psi_- : X, Y \mapsto XY^{-1}, Y$. Let $\Theta := (\Psi_+\Psi_-\Psi_+)^{-1}$, then $\Psi_- = \Theta\Psi_+\Theta^{-1}$ and $\Theta : X, Y \mapsto XYX^{-1}, X^{-1}$.

Then

$$\begin{aligned} i_{e_{KZ}}(\Psi_+) &= F(\tau+1)^{-1}F(\tau) \\ &= \exp(-(2\pi\,\mathrm{i})^{-1}(e_- + \sum_{k>0} 2(2k+1)\zeta(2k+2)\delta_{2k}))e^{\frac{2\pi\,\mathrm{i}}{12}\delta_0} \\ &=: \psi_+, \end{aligned}$$

and since $i_{e_{KZ}}(\Theta) \in \begin{pmatrix} 0 & -(2\pi\,\mathrm{i})^{-1} \\ 2\pi\,\mathrm{i} & 0 \end{pmatrix} \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}})$,

$$i_{e_{KZ}}(\Psi_-) = \psi_-, \quad \text{where} \quad \log\psi_- = 2\pi\,\mathrm{i}(e_+ + \text{ element of } \hat{\mathfrak{b}}_3^{\mathbb{C},+}).$$

We then prove:

**Proposition 7.3** *For $e = e_{KZ}$, the isomorphism $i_e : R_{ell}(\mathbb{C}) \to R_{ell}^{gr}(\mathbb{C})$ restricts to an isomorphism $\langle B_3\rangle(\mathbb{C}) \to \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C})$.*

*Proof* $i_e(\langle B_3\rangle(\mathbb{C}))$ is the Zariski closure of the subgroup of $R_{ell}^{gr}(\mathbb{C})$ generated by $\psi_\pm$. These are elements of the subgroup $\exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C})$, which is Zariski closed, so $i_e(\langle B_3\rangle(\mathbb{C}))$ is contained in this group. On the other hand, the Lie algebra of this Zariski closure is the topological Lie algebra generated by $\log\psi_\pm$. It then suffices to prove that this Lie algebra coincides with $\hat{\mathfrak{b}}_3^{\mathbb{C}}$.

Equip $\mathrm{Aut}(F_2(\mathbb{C}))$ with the topology for which a system of neighbourhoods of 1 is $\mathrm{Aut}^n(F_2(\mathbb{C})) = \{\theta | \forall g \in F_2(\mathbb{C}), \theta(g) \equiv g \bmod F_2^{(n)}(\mathbb{C})\} \subset \mathrm{Aut}(F_2(\mathbb{C}))$, where $F_2^{(1)}(\mathbb{C}) = F_2(\mathbb{C})$ and $F_2^{(n)}(\mathbb{C}) = (F_2^{(n-1)}(\mathbb{C}), F_2(\mathbb{C}))$. This induces a topology on $R_{ell}(\mathbb{C})$, which we call the prounipotent topology. □

**Lemma 7.4** *$\langle B_3\rangle(\mathbb{C}) \subset R_{ell}(\mathbb{C})$ is closed for this topology.*

*Proof* We have $\langle B_3\rangle(\mathbb{C}) = \cap_{G\in\mathcal{G}} G$, where $\mathcal{G} = \{G(\mathbb{C}) | G \subset R_{ell}(-)$ is a subgroup scheme such that $G(\mathbb{Q}) \supset B_3\}$. It then suffices to show that each $G(\mathbb{C})$ is closed in the prounipotent topology. Define coordinates on $R_{ell}(\mathbb{C})$ as follows: $R_{ell}(\mathbb{C}) \ni \theta \leftrightarrow (c_b, d_b)_b$, where $b$ runs over a homogeneous basis of $\mathfrak{f}_2$ (generated by $\xi = \log X$, $\eta = \log Y$), e.g., $\{b\} = \{\xi, \eta, [\xi, \eta], \ldots\}$, and $\log\theta(e^\xi) = \sum_b c_b b$, $\log\theta(e^\eta) = \sum_b d_b b$. Then $G(\mathbb{C})$ is a finite intersection of sets of the form $\{\theta | P(c_\xi, c_\eta, \ldots, d_\xi, d_\eta, \ldots) = 0\}$, where $P$ is a polynomial in $(c_b, d_b)_b$, vanishing at the origin. Such a $G(\mathbb{C})$ necessarily contains $R_{ell}(\mathbb{C}) \cap \mathrm{Aut}^n(F_2(\mathbb{C}))$ for a large enough $n$. □

*Sequel of proof of Proposition 7.3* It follows that $i_e(\langle B_3\rangle(\mathbb{C})) \subset \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C})$ is closed in the prounipotent topology of $R_{ell}^{gr}(\mathbb{C})$ (as in the case of $R_{ell}(\mathbb{C})$, and it is defined by the inclusion in $\mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbb{C}})$), so $\mathrm{Lie}\, i_e(\langle B_3\rangle(\mathbb{C})) \subset \hat{\mathfrak{b}}_3^{\mathbb{C}}$ is closed.

Recall that $\log\psi_+ = -(2\pi\,\mathrm{i})^{-1} \cdot (e_- + \sum_{k\geq 1} a_{2k}\delta_{2k}) + \frac{2\pi\,\mathrm{i}}{12}\delta_0 \in \mathrm{Lie}\, i_e(\langle B_3\rangle(\mathbb{C}))$, where $a_{2k} := 2(2k+1)\zeta(2k+2) \neq 0$, while $\log\psi_- \in 2\pi\,\mathrm{i} \cdot (e_+ + \hat{\mathfrak{b}}_3^{+,\mathbb{C}})$. □

**Lemma 7.5** *Let $\mathfrak{g} \subset \hat{\mathfrak{b}}_3^{\mathbb{C}}$ be a closed (for the total degree topology) Lie subalgebra, such that $\mathfrak{g} \ni \tilde{e}_\pm$, where: $\tilde{e}_+ = e_+ +$ terms of degree $> 0$, $\tilde{e}_- = e_- + \sum_{k>0} a_{2k}\delta_{2k} - \frac{1}{12}\delta_0 + \sum_{p\geq 1, q>1}$ degree $(p,q)$, where $a_{2k} \neq 0$. Then $\mathfrak{g} = \hat{\mathfrak{b}}_3^{\mathbb{C}}$.*

*Proof* Set $\mathcal{G} = \oplus_{k\geq 0}\mathcal{G}_{2k} := \mathfrak{b}_3^{\mathbb{C}}$ (decomposition w.r.t. the total degree), $\hat{\mathcal{G}} := \hat{\mathfrak{b}}_3^{\mathbb{C}}$. Set $\hat{\mathcal{G}}_{\geq 2k} := \prod_{k'\geq k} \mathcal{G}_{2k'}$, then $\hat{\mathcal{G}} = \hat{\mathcal{G}}_{\geq 0} \supset \hat{\mathcal{G}}_{\geq 2} \supset \ldots$ is a complete descending Lie algebra filtration of $\hat{\mathcal{G}}$, with associated graded Lie algebra $\mathcal{G}$. Set $\mathfrak{g}_{\geq 2k} := \mathfrak{g} \cap \mathcal{G}_{2k}$, then $\mathfrak{g} = \mathfrak{g}_{\geq 0} \supset \mathfrak{g}_{\geq 2} \supset \ldots$ is a complete descending filtration of $\mathfrak{g}$. Let $\mathrm{gr}(\mathfrak{g}) := \oplus_{k\geq 0}\mathfrak{g}_{\geq 2k}$, where $\mathrm{gr}(\mathfrak{g}) := \mathfrak{g}_{\geq 2k}/\mathfrak{g}_{\geq 2(k+2)}$. We then have an inclusion $\mathrm{gr}(\mathfrak{g}) \subset \mathcal{G}$ of graded Lie algebras. We now prove that $\mathrm{gr}(\mathfrak{g}) = \mathcal{G}$.

As $\tilde{e}_\pm \in \mathfrak{g} = \mathfrak{g}_0$, $\mathrm{gr}(\mathfrak{g}_0)$ contains $e_\pm$. Set $h := [e_+, e_-]$. Then $[\tilde{e}_+, \tilde{e}_-] = h + \sum_{p,q\geq 1}$ terms of degree $(p,q)$, and $[h, e_-] = -2e_-$, $[h, \delta_{2n}] = 2n\delta_{2n}$. Then $\mathfrak{g} \ni P(\mathrm{ad}[\tilde{e}_+, \tilde{e}_-])(\tilde{e}_-) = P(-2)\Delta_0 + \sum_{n\geq 0} a_{2n}P(2n)\delta_{2n} + \sum_{p\geq 1, q>1}$ terms of degree $(p,q)$ (with $a_0 = -\frac{1}{12}$). Taking $P$ such that $P(-2) = P(0) = \ldots = P(2k-2) = 0$ and $a_{2k}P(2k) = 1$, we see that $\mathfrak{g}$ contains an element of the form $\delta_{2k} + \sum_{p\geq 1, q>1}$ terms of degree $(p,q)$. Applying $(\mathrm{ad}\,\tilde{e}_-)^{2k}$ to this element, and using the fact that $(\mathrm{ad}\,\tilde{e}_-)^{2k}(x) = 0$ for $x \in \mathcal{G}$ of total degree $\leq 2(k-1)$, we see that $\mathfrak{g}$ contains an element of the form $(\mathrm{ad}\,e_-)^{2k}(\delta_{2k}) + \sum$(terms of total degree $\geq 2(k+2)$). As the latter sum belongs to $\mathfrak{g}_{\geq 2(k+1)}$, we obtain that $(\mathrm{ad}\,e_-)^{2k}(\delta_{2k}) \in \mathrm{gr}(\mathfrak{g})_{2(k+1)}$. The Lie subalgebra $\mathrm{gr}(\mathfrak{g}) \subset \mathcal{G}$ then contains $e_\pm$ and $(\mathrm{ad}\,e_-)^{2k}(\delta_{2k})$, $k \geq 0$. As $(\mathrm{ad}\,e_-)^{2k+1}(\delta_{2k}) = 0$, $(\mathrm{ad}\,e_+)^{2k}(\mathrm{ad}\,e_-)^{2k}(\delta_{2k})$ is a nonzero multiple of $\delta_{2k}$. So $\mathrm{gr}(\mathfrak{g}) = \mathcal{G}$. It follows that $\mathfrak{g} = \mathcal{G}$. □

*End of proof of Proposition 7.3* Applying Lemma 7.5 with $\tilde{e}_+ = 2\pi\,\mathrm{i}\log\psi_-$, $\tilde{e}_- = -(2\pi\,\mathrm{i})^{-1}\log\psi$, we get $i_e(\mathrm{Lie}\langle B_3\rangle(\mathbb{C})) = \hat{\mathfrak{b}}_3^{\mathbb{C}}$, as wanted. □

The last part of Theorem 7.1 is a consequence of the following statement, applied to a torsor of isimorphisms of Lie algebras. It was communicated to the author by P. Etingof; it is inspired by the results of [9].

**Proposition 7.6** *Let $U = \lim_{\leftarrow} U_i$ be a prounipotent $\mathbb{Q}$-group scheme (where $U_0 = 1$) and let $T := \lim_{\leftarrow} T_i$, where $T_i$ are a compatible system of torsors under $U_i$, defined over $\mathbb{Q}$. If $T(\mathbb{C}) \neq \emptyset$, then $T(\mathbb{Q}) \neq \emptyset$.*

*Proof* Let $\tilde{U}_i := \mathrm{im}(U \to U_i)$, then $U = \lim_{\leftarrow} \tilde{U}_i$, where $\cdots \to \tilde{U}_2 \to \tilde{U}_1 \to \tilde{U}_0 = 1$ is a sequence of epimorphisms of unipotent groups. We set $K_i := \mathrm{Ker}(U \to \tilde{U}_i)$; then $K_i \lhd U$. If we set $\tilde{T}_i := \mathrm{im}(T \to T_i)$, then $\tilde{T}_i \simeq T/K_i$ is a torsor over $\tilde{U}_i$; $T$ is the inverse limit of $\cdots \to \tilde{T}_2 \to \tilde{T}_1$, where the morphisms are onto. □

We may therefore assume w.l.o.g. that the morphisms $U_{i+1} \to U_i$, $T_{i+1} \to T_i$ are onto; if $K_i := \mathrm{Ker}(U \to U_i)$, then $T_i = T/K_i$.

We now show that the projective systems $\cdots \to T_2 \to T_1$, $\cdots \to U_2 \to U_1$ may be completed so that for any $i$, $\mathrm{Ker}(U_{i+1} \to U_i) \simeq \mathbb{G}_a$. Indeed, for $U_{i+1} \to U' \to U_i$ a sequence of epimorphisms, we set $K' := \mathrm{Ker}(U \to U')$ and $T' := T/K'$. Then $T_{i+1} \to T' \to T_i$ is a sequence of epimorphisms, compatible with $U_{i+1} \to U' \to U_i$.

Let $t \in T(\mathbb{C})$. We construct a sequence $(k_i)_{i\geq 0}$, where $k_i \in K_i(\mathbb{C})$, such that $\mathrm{im}(k_i \cdots k_0 t \in T(\mathbb{C}) \to T_i(\mathbb{C})) \in T_i(\mathbb{Q})$. Then $k := \lim_i (k_i \cdots k_0) \in U(\mathbb{C})$ is such that $kt \in T(\mathbb{Q})$.

We first construct $k_0$. $T_1(\mathbb{C})$ is nonempty as it contains $t_1 := \mathrm{im}(t \in T(\mathbb{C}) \to T_1(\mathbb{C}))$, hence by Hilbert's Nullstellensatz $T_1(\bar{\mathbb{Q}})$ is nonempty. Using then $H^1(G_{\mathbb{Q}}, \bar{\mathbb{Q}}) = 0$ ([30]), we obtain that $T_1(\mathbb{Q})$ is nonempty; let $t_1' \in T_1(\mathbb{Q})$. Let $u_1 \in U_1(\mathbb{C})$ be such that $t_1' = u_1 t_1$. Let $k_0 \in U(\mathbb{C}) = K_0(\mathbb{C})$ be a preimage of $u_1$, then $\mathrm{im}(k_0 t \in T(\mathbb{C}) \to T_1(\mathbb{C})) \in T_1(\mathbb{Q})$.

Assume that $k_0, \ldots, k_{i-1}$ have been constructed and let us construct $k_i$. Let $\tilde{t} := k_{i-1} \cdots k_0 t$, then $t_{i-1} := \mathrm{im}(\tilde{t} \in T(\mathbb{C}) \to T_{i-1}(\mathbb{C})) \in T_{i-1}(\mathbb{Q})$. Then, $T_i(\mathbb{C}) \times_{T_{i-1}(\mathbb{C})} \{t_{i-1}\}$ is nonempty as it contains $\tau := \mathrm{im}(\tilde{t} \in T(\mathbb{C}) \to T_i(\mathbb{C}))$. As $t_{i-1} \in T_{i-1}(\mathbb{Q})$, we define a functor $\{\mathbb{Q}\text{-rings}\} \to \{\text{sets}\}$, $\mathbf{k} \mapsto X(\mathbf{k}) := T_i(\mathbf{k}) \times_{T_{i-1}(\mathbf{k})} \{t_{i-1}\}$; it is a $\mathbb{Q}$-scheme and a torsor under $K_i/K_{i+1} = \mathbb{G}_a$. We have seen that $X(\mathbb{C}) \neq \emptyset$, from which we derive as above that $X(\mathbb{Q}) \neq \emptyset$. Let $\tau' \in X(\mathbb{Q})$ and let $k_i \in K_i(\mathbb{C})$ be such that $\bar{k}_i \tau = \tau'$, where $\bar{k}_i := \mathrm{im}(k_i \in K_i(\mathbb{C}) \to K_i/K_{i-1}(\mathbb{C}))$; then $\mathrm{im}(k_i \cdots k_0 t \in T(\mathbb{C}) \to T_i(\mathbb{C})) = \mathrm{im}(k_i \tilde{t} \in T(\mathbb{C}) \to T_i(\mathbb{C})) = \bar{k}_i \tau = \tau' \in T_i(\mathbb{Q})$.

## 7.4 Proof of Theorem 7.2

The morphism $B_3 \to \underline{\mathrm{Aut}\ P_{1,n}}(\mathbb{Q})$ factors as $B_3 \to R_{ell}(\mathbb{Q}) \stackrel{\mu_{ell}}{\to} \underline{\mathrm{Aut}\ P_{1,n}}(\mathbb{Q})$.

The elliptic associator $e_{KZ}$ transports the morphism $R_{ell}(-) \to \underline{\mathrm{Aut}\ P_{1,n}}(-)$ to the morphism $R^{gr}_{ell}(-) \to \underline{\mathrm{Aut}\ P^{gr}_{1,n}}(-)$, whose Lie algebra morphism is

$$\mathfrak{r}^{gr}_{ell} \to \mathrm{Der}(\mathfrak{t}_{1,n}), \quad (\alpha_+, \alpha_-) \mapsto (x_i^{\pm} \mapsto \alpha_{\pm}^{i,1\ldots\check{i}\ldots n}).$$

The morphism $\mathfrak{t}_{1,n} \to \mathfrak{t}_{1,2}$, $x_i^{\pm} \mapsto x_i^{\pm}$ if $i = 1, 2$, $x_i^{\pm} \mapsto 0$ if $i \in \{3, \ldots, n\}$ can then be used to prove that this Lie algebra morphism is injective. It follows that that the group morphism $R_{ell}(-) \to \underline{\mathrm{Aut}\ P_{1,n}}(-)$ is injective.

One has $\langle B_3 \rangle_n = \cap_{H | H \subset \underline{\mathrm{Aut}\ P_{1,n}}(-), H(\mathbb{Q}) \supset \mathrm{im}(B_3)} H$, therefore

$$\langle B_3 \rangle_n \cap R_{ell}(-) = \cap_{H | H \subset \underline{\mathrm{Aut}\ P_{1,n}}(-), H(\mathbb{Q}) \supset \mathrm{im}(B_3)} (H \cap R_{ell}(-)).$$

The map

$$\{H | H \text{ algebraic subgroup of } \underline{\mathrm{Aut}\ P_{1,n}}(-), \text{ s.t. } H(\mathbb{Q}) \supset \mathrm{im}(B_3)\}$$
$$\to \{G | G \text{ algebraic subgroup of } R_{ell}(-), \text{ s.t. } G(\mathbb{Q}) \supset \mathrm{im}(B_3)\},$$

given by $H \mapsto G := H \cap R_{ell}(-)$, is surjective (a preimage of $G$ is $G$ itself). Therefore

$$\langle B_3 \rangle_n \cap R_{ell}(-) = \cap_{G | G \subset R_{ell}(-), G(\mathbb{Q}) \supset \mathrm{im}(B_3)} G = \langle B_3 \rangle.$$

The Zariski closure of $\mathrm{im}(B_3 \to \underline{\mathrm{Aut}\ P_{1,n}}(\mathbb{Q}))$ is contained in the Zariski closure of $\mathrm{im}(R_{ell}(\mathbb{Q}) \to \underline{\mathrm{Aut}\ P_{1,n}}(\mathbb{Q}))$, which is $R_{ell}(-)$ as the morphism $R_{ell}(-) \to \underline{\mathrm{Aut}\ P_{1,n}}(-)$ is injective. So

$$\langle B_3 \rangle_n \subset R_{ell}(-) \subset \underline{\mathrm{Aut}\ P_{1,n}}(-).$$

All this implies that $\langle B_3 \rangle \to \langle B_3 \rangle_n$ is an isomorphism.

## 8 Iterated integrals of Eisenstein series and MZVs

In this section, we define regularized iterated integrals of modular forms. This construction generalizes both that of iterated integrals of cusp forms ([Ma]) and the definition of the Mellin transform of Eisenstein series ([Za]): it is based on a truncation procedure and the use of modular properties. We study the relations between these numbers arising from modular invariance. We show that the relations (26)-(27) from [7], obtained by the study of a monodromy morphism, can be recovered from formula (60) for the isomorphism $i_{e_{KZ}}$. The study of these relations leads to a family of algebraic relations between the iterated integrals of Eisenstein series and the MZVs.

### 8.1 Iterated Mellin transforms of modular forms

Iterated Mellin transforms of cusp modular forms were studied in [20]. On the other hand, Mellin transforms of noncusp (e.g. Eisenstein) modular forms were studied in [33]. In this section, we study iterated Mellin transforms of general (i.e. nonnecessarily cusp) modular forms.

**Proposition 8.1** *Let* $\mathcal{E} := \{f : \mathrm{i}\,\mathbb{R}_+^\times \to \mathbb{C} | f$ *is smooth and* $f(\mathrm{i}\,t) = a + O(e^{-2\pi t})$ *as* $t \to \infty$ *for some* $a \in \mathbb{C}\}$. *Set*

$$F_{t_0}^{f_1,\ldots,f_n}(s_1,\ldots,s_n) := \int\limits_{t_0 \le t_1 \le \ldots \le t_n \le \infty} f_1(\mathrm{i}\,t_1) t_1^{s_1-1} dt_1 \cdots f_n(\mathrm{i}\,t_n) t_n^{s_n-1} dt_n, 0$$

*where* $f_1,\ldots,f_n \in \mathcal{E}$ *and* $t_0 \in \mathbb{R}_+^\times$. *This function is analytic for* $\Re(s_i) \ll 0$ *and admits a meromorphic prolongation to* $\mathbb{C}^n$, *where the only singularities are simple poles at the hyperplanes* $s_i + \cdots + s_j = 0$ $(1 \le i \le j \le n)$.

*Proof* Set $\mathcal{E}_0 := \{f \in \mathcal{E} | a = 0\}$. Then $\mathcal{E} = \mathcal{E}_0 \oplus \mathbb{C}1$. When $f_1,\ldots,f_n \in \mathcal{E}_0$, $F_{t_0}^{f_1,\ldots,f_n}$ is analytic on $\mathbb{C}^n$. Let now $f_1,\ldots,f_n \in \mathcal{E}$, and set $f_i = \bar{f}_i + a_i$, with $\bar{f}_i \in \mathcal{E}_0$. Using

$$\int\limits_{t \le t_1 \le \ldots \le t_n \le t'} t_1^{s_1-1} dt_1 \cdots t_n^{s_n-1} dt_n = \sum_{k=0}^{n} (-1)^k \frac{(t')^{s_{k+1}+\cdots+s_n}}{s_{k+1}(s_{k+1}+s_{k+2})\cdots(s_{k+1}+\cdots+s_n)} \frac{t^{s_1+\cdots+s_k}}{s_k(s_k+s_{k-1})\cdots(s_k+\cdots+s_1)},$$

we get

$$F_{t_0}^{f_1,\ldots,f_n}(s_1,\ldots,s_n) = \sum_{k=1}^{n} \sum_{1 \le i_1 < \ldots < i_k \le n} \left( \prod_{j \in \{1,\ldots,n\} - \{i_1,\ldots,i_k\}} a_j \right) \sum_{\substack{j_1 \in \{1,\ldots,j_1-1\}, \\ \ldots, \\ j_k \in \{i_{k-1},\ldots,i_k-1\}}} \frac{(-1)^{|A_1|+|A_2|+\ldots+|A_{k+1}|}}{\prod_{i=1}^{k+1} \tilde{s}_{A_i} \prod_{i=1}^{k} \tilde{\tilde{s}}_{B_i}} t_0^{s_{A_1}} F_{t_0}^{\bar{f}_{i_1},\ldots,\bar{f}_{i_k}}(s_{i_1}+s_{B_1}+s_{A_1},\ldots,s_{i_k}+s_{B_k}+s_{A_k}),$$

where $A_l := \{i_{l-1}+1, ..., j_l\}$, $B_l := \{j_l+1, ..., i_l-1\}$ for $l = 1, ..., k$, and $A_{k+1} := \{i_k+1, ..., n\}$, $s_A := \sum_{\alpha\in A} s_\alpha$, $\tilde{s}_A := s_b(s_b+s_{b-1})...(s_a+...+s_b)$, $\tilde{\tilde{s}}_A := s_a(s_a+s_{a+1})...(s_a+...+s_b)$, for $A = \{a, a+1, ..., b\}$. This implies the result in general. □

Note that $F_{t_0}^{f_1,...,f_n} = \frac{(-1)^n a_1...a_n}{(s_1+\cdots+s_n)...s_n} t_0^{s_1+\cdots+s_n} + O(t_0^\sigma e^{-2\pi t_0})$ as $t_0 \to \infty$, where $\sigma$ depends on the $\Re(s_i)$.

Let now $\tilde{\mathcal{E}} := \{f \in \mathcal{E} | \exists N \geq 0, f(\mathrm{i}t) = O(t^{-N})$ as $t \to 0^+\}$. Set

$$G_{t_0}^{f_1,...,f_n}(s_1, ..., s_n) := \int_{0\leq t_1\leq\cdots\leq t_n\leq t_0} f_1(\mathrm{i}t_1)t_1^{s_1-1}dt_1...f_n(\mathrm{i}t_n)t_n^{s_n-1}dt_n$$

for $f_1, ..., f_n \in \tilde{\mathcal{E}}$. This function is analytic for $\Re(s_i) \gg 0$.

**Proposition 8.2** *For $f_1, ..., f_n \in \tilde{\mathcal{E}}$, the function*

$$(s_1, ..., s_n) \mapsto \sum_{k=0}^{n} G_{t_0}^{f_1,...,f_k}(s_1, ..., s_k)F_{t_0}^{f_{k+1},...,f_n}(s_{k+1}, ..., s_n)$$

*is analytic for $\Re(s_i) \gg 0$ and independent of $t_0$. We denote it $L^*_{f_1,...,f_n}(s_1, ..., s_n)$.*

*Proof* The analyticity follows from the fact that $F_{t_0}^{f_{k+1},...,f_n}(s_{k+1}, ..., s_n)$ may be viewed as an analytic function for $\Re(s_i) \gg 0$. The independence of $t_0$ follows from

$$\partial_{t_0} G_{t_0}^{f_1,...,f_k}(s_1, ..., s_k) = f_k(\mathrm{i}t_0)t_0^{s_k-1}G_{t_0}^{f_1,...,f_{k-1}}(s_1, ..., s_{k-1}),$$
$$\partial_{t_0} F_{t_0}^{f_k,...,f_n}(s_k, ..., s_n) = -f_k(\mathrm{i}t_0)t_0^{s_k-1}F_{t_0}^{f_{k+1},...,f_n}(s_{k+1}, ..., s_n),$$

where the former identity in valid in the domain $\Re(s_i) \gg 0$, and the latter is analytically extended from the domain $\Re(s_i) \ll 0$ to $\Re(s_i) \gg 0$. □

Recall that if $f(\tau)$ is a modular form of weight $k$, then $(t \mapsto f(\mathrm{i}t)) \in \tilde{\mathcal{E}}$, $f(\tau+1) = f(\tau)$ and $f(\frac{-1}{\tau}) = \tau^k f(\tau)$.

*Proposition-Definition 8.3* Let $f_i$ be modular forms of weight $k_i$ ($i = 1, ..., n$), then the function $L^*_{f_1,...,f_n}(s_1, ..., s_n)$ extends to a meromorphic function on $\mathbb{C}^n$, whose only possible singularities are simple poles at the hyperplanes $s_i + ... + s_j = 0$ and $s_i + ... + s_j = k_i + \cdots + k_j$ (where $1 \leq i \leq j \leq n$). We call it the iterated Mellin transform of $f_1, \ldots, f_n$. □

*Proof* By modularity,

$$G_{t_0}^{f_1,...,f_l}(s_1, ..., s_l) = (-1)^{(k_1+\cdots+k_l)/2} F_{1/t_0}^{f_l,...,f_1}(k_l - s_l, ..., k_1 - s_1).$$

Plugging this equality in the definition of $L^*_{f_1,...,f_n}(s_1, ..., s_n)$ and using the poles structures of the functions $F_{1/t_0}^{f_l,...,f_1}$, $F_{t_0}^{f_{l+1},...,f_n}$, we obtain the result. □

When $n = 1$, we now relate $L^*_f(s)$ with the Mellin transform $L^*(f, s)$ defined in [33]. Let $f$ be a modular form with $f(\tau) \to a$ as $\tau \to \mathrm{i}\infty$. Then $L^*(f, s)$ is defined for $\Re(s) \gg 0$ by $L^*(f, s) = \int_0^\infty (f(\mathrm{i}t) - a)t^{s-1}dt$. Then:

**Proposition 8.4** $L^*(f, s) = L^*_f(s)$.

*Proof* $L^*(f, s) = \int_0^{t_0} f(\mathrm{i}t)t^{s-1}dt - a\frac{t_0^s}{s} + \int_{t_0}^\infty (f(\mathrm{i}t) - a)t^{s-1}dt$ for $\Re(s) \gg 0$.

On the other hand, $G^f_{t_0}(s) = \int_0^{t_0} f(\mathrm{i}t)t^{s-1}dt$ for $\Re(s) \gg 0$, while $F^f_{t_0}(s) = \int_{t_0}^\infty f(\mathrm{i}t)t^{s-1}dt = \int_{t_0}^\infty (f(\mathrm{i}t)-a)t^{s-1}dt - a\frac{t_0^s}{s}$ for $\Re(s) \ll 0$. The second expression of $F^f_{t_0}(s)$ is meromorphic on $\mathbb{C}$ with as its only possible singularity, a simple pole at $s = 0$; in particular, this expression coincides with $F^f_{t_0}(s)$ for $\Re(s) \gg 0$. Then for $\Re(s) \gg 0$, $L^*_f(s) = G^f_{t_0}(s) + F^f_{t_0}(s) = L^*(f, s)$. □

For $s_1, \dots, s_n \in \mathbb{Z}$, one sets

$$L^\sharp_{f_1,\dots,f_n}(s_1, \dots, s_n) := \mathrm{i}^{s_1+\cdots+s_n} L^*_{f_1,\dots,f_n}(s_1, \dots, s_n).$$

According to Proposition-Definition 8.3, the numbers

$$L^\sharp_{k_1,\dots,k_n}(l_1, \dots, l_n) := L^\sharp_{G_{k_1},\dots,G_{k_n}}(l_1, \dots, l_n), \tag{61}$$

for $k_1, \dots, k_n$ even integers $\geq 4$, $l_i \in \{1, \dots, k_i - 1\}$, are well-defined. One can prove that $L^\sharp_{k_1,\dots,k_n}(b_1, \dots, b_n) \in \mathrm{i}^{l_1+\cdots+l_n}\mathbb{R}$.

## 8.2 Monodromy relations and the isomorphism $i_{e_{KZ}}$

(60) defines a morphism

$$i_{e_{KZ}} : B_3 \to \exp(\hat{\mathfrak{b}}_3^+) \rtimes \mathrm{SL}_2(\mathbb{C})(\subset \mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbb{C}})^{op}),$$

such that

$$\forall \tilde{g}_3 \in B_3, \quad \tilde{g} * e_{KZ} = e_{KZ} * i_{e_{KZ}}(\tilde{g}) = (i_{e_{KZ}}(\tilde{g})(A_{KZ}), i_{e_{KZ}}(\tilde{g})(B_{KZ})), \tag{62}$$

where $e_{KZ} = (A_{KZ}, B_{KZ})$.

Specializing to $\tilde{g} = \Psi_+$, this gives

$$i_{e_{KZ}}(\Psi_+) : A_{KZ} \mapsto A_{KZ}, \quad B_{KZ} \mapsto B_{KZ}A_{KZ},$$

and for $\tilde{g} = \Theta$, this gives

$$i_{e_{KZ}}(\Theta) : A_{KZ} \mapsto B_{KZ}^{-1}, \quad B_{KZ} \mapsto B_{KZ}A_{KZ}B_{KZ}^{-1}.$$

In Calaque et al. [7], we introduced $\tilde{A}, \tilde{B} \in \exp(\hat{\mathfrak{t}}_{1,2})$ related to $A_{KZ}, B_{KZ}$ by

$$\tilde{A} = (2\pi/\mathrm{i})^t A_{KZ}(2\pi/\mathrm{i})^{-t}, \quad \tilde{B} = (2\pi/\mathrm{i})^t B_{KZ}(2\pi/\mathrm{i})^{-t},$$

and elements $[\Psi], [\Theta] \in \exp(\hat{\mathfrak{b}}_3^+) \rtimes \mathrm{SL}_2(\mathbb{C})$, and studying a monodromy morphism, showed relations (numbered (26), (27) in Calaque et al. [7])

$$[\Psi]e^{\mathrm{i}\frac{\pi}{6}\operatorname{ad}t} : A_{KZ} \mapsto A_{KZ}, \quad B_{KZ} \mapsto B_{KZ}A_{KZ},$$
$$[\Theta]e^{\mathrm{i}\frac{\pi}{2}\operatorname{ad}t} : A_{KZ} \mapsto B_{KZ}^{-1}, \quad B_{KZ} \mapsto B_{KZ}A_{KZ}B_{KZ}^{-1}.$$

One checks that $[\Psi]e^{\mathrm{i}\frac{\pi}{6}\operatorname{Ad}t} = i_{e_{KZ}}(\Psi)$, $[\Theta]e^{\mathrm{i}\frac{\pi}{2}\operatorname{ad}t} = i_{e_{KZ}}(\Theta)$. So (60) allows to recover relations (26), (27) from [7].

## 8.3 Relations between iterated Mellin transforms and MZVs

Another consequence of (62) is the behaviour of the automorphism $i_{e_{KZ}}(\Psi_-)$, namely

$$i_{e_{KZ}}(\Psi_-) : A_{KZ} \mapsto A_{KZ}B_{KZ}^{-1}, \quad B_{KZ} \mapsto B_{KZ}. \tag{63}$$

Notice that $\Psi_- = \Theta\Psi_+\Theta^{-1}$ and that $\log i_{e_{KZ}}(\Psi_+)$ is a well-defined derivation of $\hat{\mathfrak{f}}_2^{\mathbb{C}}$. Set

$$x_{KZ} \coloneqq \log A_{KZ|x\mapsto(2\pi\,\mathrm{i})^{-1}x,\, y\mapsto 2\pi\,\mathrm{i}\,y} \in \hat{\mathfrak{f}}_2^{\mathbb{C}}, \quad y_{KZ} \coloneqq \log B_{KZ|x\mapsto(2\pi\,\mathrm{i})^{-1}x,\, y\mapsto 2\pi\,\mathrm{i}\,y} \in \hat{\mathfrak{f}}_2^{\mathbb{C}},$$

so $\sigma(\Phi_{KZ}) = (e^{x_{KZ}}, e^{y_{KZ}})$. Then, (63) is equivalent to the statement that the derivation

$$D := \operatorname{Ad}(\operatorname{diag}((2\pi\,\mathrm{i})^{-1}, 2\pi\,\mathrm{i}) \circ i_{e_{KZ}}(\Theta))(\log i_{e_{KZ}}(\Psi_+)) \in \operatorname{Der}(\hat{\mathfrak{f}}_2^{\mathbb{C}})$$

acts as follows

$$D : x_{KZ} \mapsto -\frac{\operatorname{ad} x_{KZ}}{1 - e^{-\operatorname{ad} x_{KZ}}}(y_{KZ}), \quad y_{KZ} \mapsto 0,$$

where $t(2\pi\,\mathrm{i}) = \operatorname{diag}((2\pi\,\mathrm{i})^{-1}, 2\pi\,\mathrm{i}) \in \mathrm{SL}_2(\mathbb{C})$ is viewed as an automorphism of $\hat{\mathfrak{f}}_2^{\mathbb{C}}$ (see Proposition 6.4).

There is a decomposition $\operatorname{Der}(\hat{\mathfrak{f}}_2^{\mathbb{C}}) = \prod_{k,l\in\mathbb{Z}} \operatorname{Der}(\mathfrak{f}_2^{\mathbb{C}})[k,l]$, where the bracket indicates the bidegree in $x, y$. Let $D = \sum_{k,l} D[k,l]$ be the corresponding decomposition of $D$. One has $\operatorname{Der}(\mathfrak{f}_2^{\mathbb{C}})[k,l] = \operatorname{Der}(\mathfrak{f}_2^{\mathbb{Q}})[k,l] \otimes \mathbb{C}$.

Set $\mathcal{Z}_0 := \mathbb{Q}$ and for $l \geq 1$, set

$$\mathcal{Z}_l := \operatorname{Span}_{\mathbb{Q}}\{\zeta(l_1, \dots, l_s) | s \geq 1, l_1 \geq 1, \dots, l_{s-1} \geq 1, l_s \geq 2, l_1 + \cdots + l_s = l\} \subset \mathbb{C},$$

where

$$\zeta(l_1, \dots, l_s) = \sum_{1 \leq k_1 \leq \dots \leq k_s} k_1^{-l_1} \cdots k_s^{-l_s}.$$

For $V \subset \mathbb{C}$ a $\mathbb{Q}$-vector subspace and $k \in \mathbb{Z}$, set $V(k) := (2\pi\,\mathrm{i})^k V$.

**Proposition 8.5** *$D[k,l]$ has the following properties:*

- *it lies in $\mathfrak{b}_3^{\mathbb{Q}}[k,l] \otimes \mathbb{Q}(l)$ if $k=1,\ l \geq -1$;*
- *it lies in $\mathfrak{b}_3^{\mathbb{Q}}[k,l] \otimes (\mathcal{Z}_l(0) + \mathcal{Z}_{l+1}(-1))$ if $k \geq 2,\ l \geq 1$;*
- *it is equal to zero in all the other cases.*

*Proof* $\log i_{e_{KZ}}(\Psi_+) \in \sum_{k \geq -1} \mathbb{Q}(k) \otimes \mathfrak{b}_3^{\mathbb{Q}}[k,1]$, and $\mathrm{diag}((2\pi\,\mathrm{i})^{-1}, 2\pi\,\mathrm{i}) \circ i_{e_{KZ}}(\Theta) =$ (element of $\exp(\hat{\mathfrak{b}}_3^+)) \times (x \mapsto y, y \mapsto x)$, and the support of $\hat{\mathfrak{b}}_3^+$ is contained in $\{1,2,\ldots\}^2$. All this implies that

$$D \in \prod_{l \geq -1} \mathbb{Q}(l) \otimes \mathfrak{b}_3^{\mathbb{Q}}[1,l] \oplus \prod_{k \geq 2, l \geq 0} \mathfrak{b}_3^{\mathbb{C}}[k,l].$$

Since $D$ lies in $\hat{\mathfrak{b}}_3^{\mathbb{C}}$, whose support is contained in $\{(1,-1),(0,0),(-1,1)\} \cup \{1,2,\ldots\}^2$, this statement can be improved by changing the second product into $\prod_{k \geq 2, l \geq 1} \mathfrak{b}_3^{\mathbb{C}}[k,l]$. This implies the first and the last statement of the proposition.

Recall that

$$x_{KZ} = \mathrm{Ad}\Big(\Phi_{KZ}(-\frac{\mathrm{ad}\,x}{e^{\mathrm{ad}\,x}-1}(y), t)\Big)\Big(2\pi\,\mathrm{i}\,\frac{-\mathrm{ad}\,x}{e^{\mathrm{ad}\,x}-1}(y)\Big),$$
$$y_{KZ} = \mathrm{i}\pi t * \log \Phi_{KZ}(-\frac{\mathrm{ad}\,x}{e^{\mathrm{ad}\,x}-1}(y), t) * x * \log \Phi_{KZ}(\frac{\mathrm{ad}\,x}{e^{\mathrm{ad}\,x}-1}(y) + t, t),$$

where $*$ is the CBH product $a * b := \log e^a e^b$.

There exists a unique derivation $\tilde{D}$ of $\hat{\mathfrak{f}}_2^{\mathbb{C}}$, such that

$$\tilde{D} : x \mapsto 0, \quad y \mapsto -\frac{1}{2\pi\,\mathrm{i}} \frac{e^{\mathrm{ad}\,x}-1}{\mathrm{ad}\,x} \varphi\Big(\mathrm{ad}\Big(-2\pi\,\mathrm{i}\,\frac{\mathrm{ad}\,x}{e^{\mathrm{ad}\,x}-1}(y)\Big)\Big)(x),$$

where $\varphi(t) = (-t)/(1-e^{-t})$, and a unique automorphism $\theta$ of the same Lie algebra, such that

$$\theta : x \mapsto y_{KZ}, \quad y \mapsto -\frac{1}{2\pi\,\mathrm{i}} \frac{e^{\mathrm{ad}\,y_{KZ}}-1}{\mathrm{ad}\,y_{KZ}}(x_{KZ});$$

then $D = \theta \tilde{D} \theta^{-1}$. One has

$$\tilde{D} \in \mathbb{Q}(-1) \otimes \mathrm{Der}(\mathfrak{f}_2)[1,-1] + \prod_{k \geq 1, l \geq 0} \mathbb{Q}(l) \otimes \mathrm{Der}(\mathfrak{f}_2)[k,l]. \tag{64}$$

One computes

$$\log \Phi_{KZ}(-\frac{\mathrm{ad}\,x}{e^{\mathrm{ad}\,x}-1}(y), t), \log \Phi_{KZ}(\frac{\mathrm{ad}\,x}{e^{\mathrm{ad}\,x}-1}(y)+t, t) \in \prod_{k \geq 1, l \geq 2} \mathcal{Z}_l \otimes \mathfrak{f}_2^{\mathbb{Q}}[k,l],$$
$$\mathrm{i}\pi t \in \mathbb{Q}(1) \otimes \mathfrak{f}_2[1,1],$$

which implies

$$y_{KZ} \in x + \prod_{k\geq 1, l\geq 1} (\mathcal{Z}_l + \mathcal{Z}_{l-1}(1)) \otimes \mathfrak{f}_2^{\mathbb{Q}}[k,l]. \tag{65}$$

It also implies

$$-\frac{1}{2\pi \mathrm{i}} x_{KZ} = y + \prod_{k,l\geq 1} \mathcal{Z}_{l-1} \otimes \mathrm{Der}\, \mathfrak{f}_2^{\mathbb{Q}}[k,l],$$

which then implies

$$-\frac{1}{2\pi \mathrm{i}} \frac{e^{\mathrm{ad}\, y_{KZ}} - 1}{\mathrm{ad}\, y_{KZ}}(x_{KZ}) \in y + \prod_{k,l\geq 1} (\mathcal{Z}_l + \mathcal{Z}_{l-1}(1)) \otimes \mathfrak{f}_2^{\mathbb{Q}}[k,l]. \tag{66}$$

(65) and (66) imply that $\theta - \mathrm{id} \in \prod_{k\geq 0, l\geq 1}(\mathcal{Z}_l + \mathcal{Z}_{l-1}(1)) \otimes \mathrm{End}(\mathfrak{f}_2^{\mathbb{Q}})[k,l]$, so that $\log\theta$ belongs to the same space. Together with the estimate on $\tilde{D}$, this implies that for any $k \geq 1$,

$$\mathrm{ad}(\log\theta)^k(\tilde{D}) \in \prod_{k\geq 1, l\geq 0} \mathrm{Der}(\mathfrak{f}_2^{\mathbb{Q}})[k,l] \otimes \big(\mathcal{Z}_l + \mathcal{Z}_{l+1}(-1)\big).$$

Combining this with the estimate on $\tilde{D}$, one obtains that $D = \theta\tilde{D}\theta^{-1}$ belongs to the direct sum of $\mathrm{Der}(\mathfrak{f}_2^{\mathbb{Q}})[1,-1] \otimes \mathbb{Q}(-1)$ with this space, which together with the first and third statements of the proposition, and the fact that $D \in \hat{\mathfrak{b}}_3^{\mathbb{C}}$ implies the second statement of the proposition. □

For $\lambda \in \mathbb{C}^\times$, set $w(\lambda) := \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$.

**Lemma 8.6**

$$i_{e_{KZ}}(\Theta) \equiv w(2\pi \mathrm{i}) \cdot \sum_{n\geq 0} \sum_{\substack{k_1,\ldots,k_n \geq 1 \\ l_i \in \{0,\ldots,2k_i\}}} \left(\frac{-1}{2\pi \mathrm{i}}\right)^{l_1+1} \cdots \left(\frac{-1}{2\pi \mathrm{i}}\right)^{l_n+1} L^{\sharp}_{2k_1+2,\ldots,2k_n+2}(l_1+1,\ldots,l_n+1) \frac{2k_1+1}{l_1!} \mathrm{ad}(e_-)^{l_1}(\delta_{2k_1}) \cdots \frac{2k_n+1}{l_n!} \mathrm{ad}(e_-)^{l_n}(\delta_{2k_n})$$

*in* $\exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C})$*, up to multiplication by an element of* $\exp(\mathbb{C}\delta_0)$.

*Proof* $i_{e_{KZ}}(\Theta) = \tilde{F}(\frac{-1}{\tau})^{-1} w(2\pi \mathrm{i})\tilde{F}(\tau)e^{\log(\frac{-1}{\tau})\delta_0}$, where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\tilde{F}(\tau) := n_-(\frac{\tau}{2\pi \mathrm{i}})^{-1} F(\tau)$. As $\tilde{F}(\tau)$ satisfies

$$\partial_\tau \tilde{F}(\tau) = -\Big(\sum_{k\geq 0}\sum_{l=0}^{2k}\tau^l G_{2k+2}(\tau)(\frac{-1}{2\pi\,\mathrm{i}})^{l+1}\frac{2k+1}{l!}\,\mathrm{ad}(e_-)^l(\delta_{2k})\Big)\tilde{F}(\tau),$$

and taking into account the behaviour of $\tilde{F}(\tau)$ at $\tau \to \mathrm{i}\,\infty$, one obtains

$$\tilde{F}(\tau) \equiv \sum_{n\geq 0}\sum_{k_1,\ldots,k_n\geq 1}\sum_{l_i\in\{0,\ldots,2k_i\}} \phi_\tau^{G_{2k_1+2},\ldots,G_{2k_n+2}}(l_1+1,\ldots,l_n+1)\left(\frac{-1}{2\pi\,\mathrm{i}}\right)^{l_1+1} \cdots\left(\frac{-1}{2\pi\,\mathrm{i}}\right)^{l_n+1}\frac{2k_1+1}{l_1!}\,\mathrm{ad}(e_-)^{l_1}(\delta_{2k_1})\cdots\frac{2k_n+1}{l_n!}\,\mathrm{ad}(e_-)^{l_n}(\delta_{2k_n}),$$

where $\phi_{\mathrm{i}\,t_0}^{f_1,\ldots,f_n}(s_1,\ldots,s_n) := \mathrm{i}^{s_1+\cdots+s_n}\,F_{t_0}^{f_1,\ldots,f_n}(s_1,\ldots,s_n)$. Combining this with the similar formula for $\tilde{F}(\frac{-1}{\tau})^{-1}$, one obtains the result. □

Set $w := w(1) \in \mathrm{SL}_2(\mathbb{C})$.

**Lemma 8.7**

$$\mathrm{Ad}(w^{-1})\circ D = -\frac{1}{2\pi\,\mathrm{i}}e_- + \frac{2\pi\,\mathrm{i}}{12}\delta_0 + \sum_{\substack{n>0\\ k_1,\ldots,k_n\geq 1\\ l_i\in\{0,\ldots,2k_i\}}}\left(\frac{-1}{2\pi\,\mathrm{i}}\right)^{l_1+1}\cdots\left(\frac{-1}{2\pi\,\mathrm{i}}\right)^{l_n+1} \times\left\{\begin{matrix} -L^\sharp_{2k_1+2,\ldots,2k_n+2}(l_1+1,\ldots,l_n+1)\cdot l_n & \mathit{if}\, l_n\neq 0\\ L^\sharp_{2k_1+2,\ldots,2k_{n-1}+2}(l_1+1,\ldots,l_{n-1}+1)\cdot 2\zeta(2k_n+2) & \mathit{if}\, l_n=0\end{matrix}\right\} \times\left[\frac{2k_1+1}{l_1!}(\mathrm{ad}\,e_-)^{l_1}(\delta_{2k_1}),\ldots,\frac{2k_n+1}{l_n!}(\mathrm{ad}\,e_-)^{l_n}(\delta_{2k_n})\right],$$

*where* $L^\sharp_\emptyset(\emptyset) = 1$ *by convention, and* $[a_1,\ldots,a_n] := \mathrm{ad}\,a_1\circ\cdots\circ\mathrm{ad}\,a_{n-1}(a_n)$.

*Proof* One has $t(2\pi\,\mathrm{i})\circ w(2\pi\,\mathrm{i}) = w(1) = w$ in $\mathrm{Aut}(\hat{\mathfrak{f}}_2)$. One also has

$$\log i_{e_{KZ}}(\Psi_+) = \frac{2\pi\,\mathrm{i}}{12}\delta_0 - \frac{1}{2\pi\,\mathrm{i}}(e_- + \sum_{k>0}2(2k+1)\zeta(2k+2)\delta_{2k}).$$

The result then follows from the expansion of $i_{e_{KZ}}(\Theta)$ and from the identity $\mathrm{Ad}(g)(y) = \sum_{n\geq 0}a_{i_1,\ldots,i_n}[x_{i_1},\ldots,x_{i_n},y]$ for $g = \sum_{n\geq 0}\sum_{i_1,\ldots,i_n\in I}a_{i_1,\ldots,i_n}x_{i_1}\cdots x_{i_n}$ a group-like element of $U\mathfrak{g}$ and $y\in\mathfrak{g}$, where $\mathfrak{g}$ is a topological Lie algebra and $x_i,\ i\in I$ are positive degree elements of $\mathfrak{g}$. □

Combining Proposition 8.5 and Lemma 8.7, one obtains the following family of relations between iterated integrals of Eisenstein series and MZVs:

**Proposition 8.8** *Let* $I := \{(a,b)|a,b\geq 1, a+b$ *is even*$\}$. *For* $(a,b)\in I$, *let*

$$e_{a,b} := \frac{a+b-1}{(b-1)!}(\mathrm{ad}\,e_-)^{b-1}(\delta_{a+b-2}) \in \mathfrak{b}_3^{\mathbb{Q}}[a,b].$$

*Let* $A \geq 2$, $B \geq 1$. *Any* $\xi \in \mathfrak{b}_3^{\mathbb{Q}}[A,B]^*$ *gives rise to a relation*

$$\sum_{n>0} \sum_{\substack{(a_1,b_1),\dots,(a_n,b_n)| \\ (a_1,b_1)+\cdots+(a_n,b_n)=(A,B)}} \langle \xi, [e_{a_1,b_1},\dots,e_{a_n,b_n}]\rangle \times$$
$$\times \left\{ \begin{matrix} -L^{\sharp}_{a_1+b_1,\dots,a_n+b_n}(b_1,\dots,b_n)\cdot(b_n-1) \text{ if } b_n \neq 1 \\ L^{\sharp}_{a_1+b_1,\dots,a_{n-1}+b_{n-1}}(b_1,\dots,b_{n-1})\cdot 2\zeta(a_n+b_n) \text{ if } b_n = 1 \end{matrix} \right\}$$
$$\in \mathcal{Z}_A(B) + \mathcal{Z}_{A+1}(B-1).$$

## 8.4 Modular and shuffle relations

The numbers $L^{\sharp}_{k_1,\dots,k_n}(b_1,\dots,b_n)$ are subject to other relations:

(a) the shuffle relations

$$L^{\sharp}_{k_1,\dots,k_n}(b_1,\dots,b_n)L^{\sharp}_{k_{n+1},\dots,k_{n+m}}(b_{n+1},\dots,b_{n+m}) = \sum_{\sigma\in S_{n,m}} L^{\sharp}_{k_{\sigma(1)},\dots,k_{\sigma(n+m)}}(b_{\sigma(1)},\dots,b_{\sigma(n+m)}), \tag{67}$$

where $S_{n,m} = \{\sigma \in S_{n+m} | \sigma(i) < \sigma(j) \text{ if } i < j \leq n \text{ or } n+1 \leq i < j\}$, which can be reexpressed as the following statement: let $\mathfrak{M} := \oplus_{k\geq 4}\mathbb{C}G_k \otimes \mathbb{C}[t]_{\leq k-2}$, then the linear map $I : T(\mathfrak{M}) \to \mathbb{C}$ such that $I(G_{k_1}(t)t^{b_1},\dots,G_{k_n}(t)t^{b_n}) := L^{\sharp}_{k_1,\dots,k_n}(b_1+1,\dots,b_n+1)$ is an algebra morphism, $T(\mathfrak{M})$ being equipped with the shuffle algebra product ;

(b) the modular relations

$$I^{\otimes 2} \circ (\mathrm{id} \otimes S) \circ \Delta = J^{\otimes 3} \circ (\mathrm{id} \otimes U \otimes U^2) \circ \Delta^{(2)} = \varepsilon \tag{68}$$

(equalities in $\mathrm{Hom}_{alg}(T(\mathfrak{M}),\mathbb{C})$, $T(\mathfrak{M})$ being equipped with the shuffle product), where:

- $J : T(\mathfrak{M}) \to \mathbb{C}$ is defined by $J := (I \otimes \psi) \circ \Delta$, $\psi : T(\mathfrak{M}) \to \mathbb{C}$ being defined by

$$\psi(G_{k_1}t^{b_1-1} \otimes \cdots \otimes G_{k_n}t^{b_n-1}) := \frac{2\zeta(k_1)\cdots 2\zeta(k_n)}{b_1(b_1+b_2)\cdots(b_1+\cdots+b_n)};$$

- $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ act on $\mathfrak{M}$ by $S \cdot t^{b-1}G_k := t^{k-2}(\frac{-1}{t})^{b-1}G_k$, $U \cdot t^{b-1}G_k := t^{k-2}(1-\frac{1}{t})^{b-1}G_k$;
- $\varepsilon : T(\mathfrak{M}) \to \mathbb{C}$ is the augmentation morphism, $\Delta : T(\mathfrak{M}) \to T(\mathfrak{M})^{\otimes 2}$ is the shuffle coproduct morphism $x_1 \otimes \cdots \otimes x_n \mapsto \sum_{k=0}^n (x_1 \otimes \cdots \otimes x_k) \otimes (x_{k+1} \otimes \cdots \otimes x_n)$, $\Delta^{(2)} := (\Delta \otimes \mathrm{id}) \circ \Delta$.

The relations (68) are proved as follows. Let $\tilde{\mathfrak{b}}_3 := \mathrm{Lie}(\oplus_{a,b\geq 1,a+b \text{ even}}\mathbb{C}\tilde{e}_{a,b}) \rtimes \mathfrak{sl}_2$, where Lie(−) means the free Lie algebra generated by a vector space, and $\oplus_{a,b\geq 1,a+b \text{ even}}\mathbb{C}\tilde{e}_{a,b} = \oplus_{k \text{ even}}(\oplus_{a,b\geq 1,a+b=k}\mathbb{C}\tilde{e}_{a,b})$ is the direct sum of all the odd-dimensional simple $\mathfrak{sl}_2$-modules, the action being normalized by $e_-\cdot\tilde{e}_{a,b} = b\tilde{e}_{a-1,b+1}$. There is a unique morphism $\tilde{\mathfrak{b}}_3 \to \mathfrak{b}_3$, such that it induces the identity on $\mathfrak{sl}_2$ and such that $\tilde{e}_{a,b} \mapsto e_{a,b}$. The $\mathrm{SL}_2(\mathbb{Z})$-equivariant connection on $\mathfrak{H}$ with values in the trivial principal bundle with group $\exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C})$ defined by (56) admits a lift to a similar connection, where this group is replaced by its analogue with $\tilde{\mathfrak{b}}_3$ replacing $\mathfrak{b}_3$. This connection therefore gives rise to a morphism $\mathrm{SL}_2(\mathbb{Z}) \to \exp(\hat{\tilde{\mathfrak{b}}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C})$ to this group. The relations (68) express the fact that the relations between the usual generators of $\mathrm{SL}_2(\mathbb{Z})$ are satisfied by their images.

*Remark 8.9* Relations (68) are generalizations of the modular relations satisfied by the period polynomials of Eisenstein series ([33], Proposition p. 453). The contribution of $\psi$ to $J$ is the analogue of the contributions of the values at cusps to the period polynomials of Eisenstein series as defined in [33], (9).

*Remark 8.10* Let $\mathfrak{Z} = \oplus_{k\geq 0}\mathfrak{Z}_k$ be the $\mathbb{Q}$-algebra of formal MZVs, i.e., the $\mathbb{Q}$-algebra generated by formal versions of $2\pi$ i and of the $\zeta(k_1, \ldots, k_s)$, subject to the associator relations. Define $\mathfrak{Z}^*$ as the $\mathfrak{Z}$-algebra generated by formal analogues of the $L^\sharp_{k_1,\ldots,k_n}(b_1, \ldots, b_n)$, $k_1, \ldots, k_n$ even $\geq 4$, $b_i \in \{1, \ldots, k_i - 1\}$, modulo the shuffle relations (67), the modular relations (68), and the relations from Proposition 8.8, in which the right-hand side is replaced by any lift in $\mathfrak{Z}_A(B) + \mathfrak{Z}_{A+1}(B - 1)$. Then $\mathfrak{Z}^*$ is $\mathbb{N}$-graded, with the degree of $L^\sharp_{k_1,\ldots,k_n}(b_1, \ldots, b_n)$ being equal to $k_1 + \cdots + k_n$.

## 8.5 Computation of some regularized iterated integrals

Denote by $\mathrm{Sh}(\mathfrak{M})$ the vector space $T(\mathfrak{M})$, equipped with its (commutative) shuffle algebra structure. Let $\mathrm{Lie}(\mathfrak{M}) \subset T(\mathfrak{M})$ be the (free) Lie subalgebra of $T(\mathfrak{M})$ generated by $\mathfrak{M}$, $T(\mathfrak{M})$ being equipped with its tensor algebra structure. This inclusion gives rise to a commutative algebra morphism $S(\mathrm{Lie}(\mathfrak{M})) \to \mathrm{Sh}(\mathfrak{M})$, which can be shown to be an isomorphism. As $I : \mathrm{Sh}(\mathfrak{M}) \to \mathbb{C}$ is an algebra morphism, it is uniquely determined by its restriction

$$I : \mathrm{Lie}(\mathfrak{M}) \to \mathbb{C}.$$

$\mathrm{Lie}(\mathfrak{M})$ decomposes as $\mathfrak{M} \oplus \mathrm{Lie}_2(\mathfrak{M}) \oplus \cdots$. The restriction of $I$ to $\mathfrak{M}$ has been determined in [33]: for $k$ even $\geq 4$,

$$\begin{aligned} I(t^{k-2}G_k) &= -I(G_k) = \frac{2\pi \text{ i}}{k-1}\zeta(k-1), \\ I(t^a G_k) &= \frac{(-1)^{a+1}}{(k-1)!}\frac{B_{a+1}}{a+1}\frac{B_{k-a-1}}{k-a-1}(2\pi \text{ i})^k \quad \text{for } a = 1, \ldots, k-3. \end{aligned} \tag{69}$$

The grading $\mathfrak{M} = \oplus_{k\geq 4}\mathfrak{M}_k$, where $\mathfrak{M}_k = \mathbb{C}G_k \otimes \mathrm{Span}_{\mathbb{C}}(1, t, \dots, t^{k-2})$, induces a grading $\mathrm{Lie}(\mathfrak{M}) = \oplus_{k\geq 4, k \text{ even}} \mathrm{Lie}(\mathfrak{M})_k$. The restriction of $I$ to $\mathrm{Lie}(\mathfrak{M})_k$ for the first values of $k$ can be carried out as follows.

- $k = 4, 6$. In these cases, $\mathrm{Lie}(\mathfrak{M})_k = \mathfrak{M}_k$, so (69) determines the restriction of $I$ to $\mathrm{Lie}(\mathfrak{M})_k$.
- $k = 8$. $\mathrm{Lie}(\mathfrak{M})_8 = \mathfrak{M}_8 \oplus \mathrm{Lie}_2(\mathfrak{M}_4)$. (69) determines the restriction of $I$ to $\mathfrak{M}_8$, so it remains to compute its restriction to $\mathrm{Lie}_2(\mathfrak{M}_4) = \mathrm{Span}_{\mathbb{C}}([G_4, tG_4], [G_4, t^2G_4], [tG_4, t^2G_4])$. The modular relations imply that

$$I([G_4, t^2G_4]) = -\left(\frac{2\pi\,\mathrm{i}}{3}\zeta(3)\right)^2 - \frac{418}{45}\zeta(4)^2,$$

  and that $I([G_4, tG_4]) + I([tG_4, t^2G_4]) = 0$. Proposition 8.8 for $(A, B) = (4, 4)$, together with the fact that the restriction of the morphism $\tilde{\mathfrak{b}}_3 \to \mathfrak{b}_3$ to degree 8 is an isomorphism, then implies that $I([G_4, tG_4]) \in \mathcal{Z}_5(3) + \mathcal{Z}_4(4) = \mathrm{Span}_{\mathbb{Q}}((2\pi\,\mathrm{i})^3\zeta(5), (2\pi\,\mathrm{i})^5\zeta(3), (2\pi\,\mathrm{i})^8)$. As $I([G_4, tG_4])$ is pure imaginary, one even has

$$I([G_4, tG_4]) = -I([tG_4, t^2G_4]) \in \mathbb{Q}(2\pi\,\mathrm{i})^3\zeta(5) + \mathbb{Q}(2\pi\,\mathrm{i})^5\zeta(3)$$

  (the rational coefficients can be determined from the expression of the components of the derivation $D$ in a generating family of MZVs).
- $k = 10$. $\mathrm{Lie}(\mathfrak{M})_{10} = \mathfrak{M}_{10} \oplus \mathfrak{M}_4 \otimes \mathfrak{M}_6$, and as a $\mathrm{SL}_2(\mathbb{C})$-module, $\mathfrak{M}_4 \otimes \mathfrak{M}_6$ decomposes as a direct sum $V_7 \oplus V_5 \oplus V_3$ of irreducible modules of the indicated dimensions, generated by the highest weight vectors

$$[G_4, G_6], \quad [tG_4, G_6] - [G_4, tG_6], \quad [t^2G_4, G_6] - 2[tG_4, tG_6] + [G_4, t^2G_6]. \tag{70}$$

  The modular relations determine the restriction of $I$ to 1-codimensional subspaces of $V_i$ $(i = 3, 5, 7)$, for which the highest weight vectors (70) span supplementary subspaces.
  On the other hand, the expansion of $\log(w(2\pi\,\mathrm{i})^{-1}i_{e_{KZ}}(\Theta))$ up to degree 10 yields the identity

$$\begin{aligned}
&\mathrm{Ad}(w^{-1})(D)\\
&= \mathrm{Ad}\exp\Bigg(\sum_{a,b}\left(\frac{-1}{2\pi\,\mathrm{i}}\right)^b I(t^{b-1}G_{a+b})e_{a,b}\\
&\quad + \frac{1}{4}\sum_{a,b,a',b'}\left(\frac{-1}{2\pi\,\mathrm{i}}\right)^{b+b'} I([t^{b-1}G_{a+b}, t^{b'-1}G_{a'+b'}])[e_{a,b}, e_{a',b'}]\Bigg)\cdot\\
&\quad \cdot\Bigg(\frac{-1}{2\pi\,\mathrm{i}}(e_- + \sum_{k\geq 1} 2\zeta(2k+2)\delta_{2k})\Bigg)
\end{aligned}$$

modulo degree $\geq 12$, from where one derives the expression in terms of MZVs of

$$\left[ e_-, \sum_{a,b,a',b'|a+b+a'+b'=10} \left(\frac{-1}{2\pi\,\mathrm{i}}\right)^{b+b'} I([t^{b-1}G_{a+b}, t^{b'-1}G_{a'+b'}])[e_{a,b}, e_{a',b'}] \right]. \tag{71}$$

On the other hand, let $\tilde{V}_k := \mathrm{Span}_{\mathbb{C}}(\tilde{e}_{2k,1}, \ldots, \tilde{e}_{1,2k}) \subset \tilde{\mathfrak{b}}_3$; the degree 10 part of $\tilde{\mathfrak{b}}_3$ decomposes as $(\tilde{\mathfrak{b}}_3)_{10} = \tilde{V}_{10} \oplus \tilde{V}_4 \otimes \tilde{V}_6$. This $\mathrm{SL}_2(\mathbb{C})$-module is dual to $\mathrm{Lie}(\mathfrak{M})_{10}$, in particular

$$\sum_{a,b,a',b'|a+b+a'+b'=10} \left(\frac{-1}{2\pi\,\mathrm{i}}\right)^{b+b'} [t^{b-1}G_{a+b}, t^{b'-1}G_{a'+b'}] \otimes [\tilde{e}_{ab}, \tilde{e}_{a'b'}] \tag{72}$$

is the canonical element of $(\mathfrak{M}_4 \otimes \mathfrak{M}_6) \otimes (\tilde{V}_4 \otimes \tilde{V}_6)$. Decompose $\tilde{V}_4 \otimes \tilde{V}_6$ as a direct sum $\tilde{W}_7 \oplus \tilde{W}_5 \oplus \tilde{W}_3$ of irreducible $\mathrm{SL}_2(\mathbb{C})$-modules of the indicated dimensions, then (72) is the sum of the canonical elements in each summand of $(V_7 \otimes \tilde{W}_7) \oplus (V_5 \otimes \tilde{W}_5) \oplus (V_3 \otimes \tilde{W}_3)$; these canonical elements have the form

$$\begin{aligned} &[G_4, G_6] \otimes (\text{lowest weight vector of } \tilde{W}_7) \\ &\quad + \text{a sum of tensors of different weights}, \\ &([tG_4, G_6] - [G_4, tG_6]) \otimes (\text{lowest weight vector of } \tilde{W}_5) \\ &\quad + \text{a sum of tensors of different weights}, \\ &\quad ([t^2G_4, G_6] - 2[tG_4, tG_6] + [G_4, t^2G_6]) \otimes (\text{l.w.v. of } \tilde{W}_3) \\ &\quad + \text{tensors of different weights}. \end{aligned}$$

**Lemma 8.11** *The composite maps* $\tilde{W}_7 \subset (\tilde{\mathfrak{b}}_3)_{10} \to (\mathfrak{b}_3)_{10}$, $\tilde{W}_5 \subset (\tilde{\mathfrak{b}}_3)_{10} \to (\mathfrak{b}_3)_{10}$ *and* $\tilde{W}_3 \to (\tilde{\mathfrak{b}}_3)_{10} \to (\mathfrak{b}_3)_{10}$ *are injective.*

*Proof* The images of the highest weight vectors of $\tilde{W}_7$, $\tilde{W}_5$ in $\mathfrak{b}_3 \subset \mathrm{Der}_t(\mathfrak{f}_2)$ can be partially computed (here $t = -[x, y]$ and $\mathrm{Der}_t$ means the derivations taking $t$ to zero) as follows. The commutator of derivations induces a map $\mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2')^{\otimes 2} \to \mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2'')$ (where $\mathfrak{f}_2' = [\mathfrak{f}_2, \mathfrak{f}_2]$, $\mathfrak{f}_2'' = [\mathfrak{f}_2', \mathfrak{f}_2']$), which in its turn induces a map $D_1^{\otimes 2} \to D_2$, where $D_1 := \mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2')/\mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2'')$, $D_2 := \mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2'')/\mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2''')$ (where $\mathfrak{f}_2''' := [\mathfrak{f}_2', \mathfrak{f}_2'']$). There is a natural map $D_1 \to \mathfrak{f}_2'/\mathfrak{f}_2''$ induced by $\mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2') \to \mathfrak{f}_2'/\mathfrak{f}_2''$, $D \mapsto$ (the class of an element $a \in \mathfrak{f}_2'$ such that $D - \mathrm{ad}\, a \in \mathrm{Der}(\mathfrak{f}_2, \mathfrak{f}_2'')$) and $D_2 \to \mathfrak{f}_2''/\mathfrak{f}_2'''$ defined similarly. There are isomorphisms $\mathbb{C}[u, v] \simeq \mathfrak{f}_2'/\mathfrak{f}_2''$, defined by $u^n v^m \mapsto$ (the class of $(\mathrm{ad}\, x)^n (\mathrm{ad}\, y)^m [x, y]$), and $\bigwedge^2 \mathbb{C}[u, v] \simeq \mathfrak{f}_2''/\mathfrak{f}_2'''$, induced by the Lie bracket $\bigwedge^2 \mathfrak{f}_2' \to \mathfrak{f}_2''$. The map $D_1^{\otimes 2} \to D_2$ is then compatible with an explicit map $\mathbb{C}[u, v]^{\otimes 2} \to \bigwedge^2 \mathbb{C}[u, v]$. The images in $\mathfrak{b}_3$ of the highest weight vectors of $\tilde{W}_7$, $\tilde{W}_5$ in fact lie in $\mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2'')$, and their images in $\bigwedge^2 \mathbb{C}[u, v]$ can be computed using the above map $\mathbb{C}[u, v]^{\otimes 2} \to \bigwedge^2 \mathbb{C}[u, v]$ and shown to be nonzero. On the other hand, the image in $\bigwedge^2 \mathbb{C}[u, v]$ of the highest weight vector of $\tilde{W}_3$ is zero, so the image of this highest weight vector in $\mathfrak{b}_3$ lies in $\mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2''')$. This derivation can be computed explicitly (by computer) and shown to be nonzero (this can also be derived from [28], Thm. 3, where $\mathrm{Ker}(D_1^{\otimes 2} \to D_2)$ is computed). Note that $[\mathrm{Der}_t(\mathfrak{f}_2, \mathfrak{f}_2'')_{10} : \underline{3}] = 1$, where $\underline{3}$ is the irreducible 3-dimension representation of $\mathrm{SL}_2(\mathbb{C})$, so this multiplicity space is spanned by the image of the highest weight vector of $\tilde{W}_3$.

The expression of (71) in terms of MZVs therefore allows one to express $I([G_4, G_6])$, $I([tG_4, G_6]-[G_4, tG_6])$ and $I([t^2G_4, G_6]-2[tG_4, tG_6]+[G_4, t^2G_6])$ in terms of MZVs, thereby completing the computation of the restriction of $I$ to $V_7$, $V_5$ and $V_3$. To summarize, the results of Sects. 8.3, 8.4 allow one to determine the restriction of $I$ to $\mathrm{Lie}(\mathfrak{M})_{10}$ in terms of MZVs of weight 10. □

- $k = 14$. It has been shown in [28] that $[\delta_2, \delta_8] = 3[\delta_4, \delta_6]$. Using the same techniques as for $k = 10$, one can prove that $81 \cdot I([G_4, G_{10}]) + 35 \cdot I([G_6, G_8])$ is a MZV of weight 14. These techniques do not give any information on the individual values of $I([G_4, G_{10}])$ and $I([G_6, G_8])$.

## 9 Galois aspects

In this section, we recall the links between $G_{\mathbb{Q}}$, $\widehat{\mathrm{GT}}$ and the Teichmüller groupoids in genus zero. We then establish the analogous results in genus one: they relate the arithmetic fundamental group $\pi_1(M_{1,1}^{\mathbb{Q}})$, $\widehat{\mathrm{GT}}_{ell}$ and the Teichmüller groupoid in genus one.

### 9.1 Galois groups and Teichmüller groupoids in genus zero

#### *9.1.1 Profinite Galois representations*

Let $n \geq 3$ and $M_{0,n}^{\mathbb{Q}}$ be the moduli stack over $\mathbb{Q}$ of genus zero smooth projective curves with $n$ marked points and $\overline{M}_{0,n}^{\mathbb{Q}}$ its Deligne-Mumford compactification. Maximally degenerate curves are rational points of this stack and correspond bijectively to planar unrooted trivalent trees with leaves indexed bijectively by $\{1, \ldots, n\}$, modulo 'mirror' symmetry. For $T$ such a tree, let $X_T^0$ the corresponding curve. The formal neighbourhood of $X_T^0$ is a fibration $X_T \to \mathrm{Spec}\,\mathbb{Q}[[q_e, e \text{ inner edge of } T]]$. Then the pull-back $X_T \otimes_{\mathbb{Q}[[\{q_e\}_e]]} \mathbb{Q}[[q]]$ corresponding to the morphism given by $q_e \mapsto q$ is a rational tangential base point of $M_{0,n}^{\mathbb{Q}}$ (recall that a rational tangential base point of a scheme $X$ is a morphism $X \to \mathrm{Spec}\,\mathbb{Q}((q))$; see [15,22].

Let $S$ be this set of rational tangential base points. The fundamental groupoid $\widehat{T}_{0,n} := \pi_1^{geom}(M_{0,n}^{\mathbb{Q}}, S)$ relative to this base set is the profinite completion of the groupoid $T_{0,n}$ described in [29]. There is a split exact sequence

$$1 \to \widehat{T}_{0,n} \to \pi_1(M_{0,n}^{\mathbb{Q}}, S) \overset{\curvearrowleft}{\to} G_{\mathbb{Q}} \to 1$$

with section induced by $S$. It results in a group morphism $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\widehat{T}_{0,n})$ (see [9,10]).

**Theorem 9.1** ([9,29]) *This morphism factors as* $G_{\mathbb{Q}} \to \widehat{\mathrm{GT}} \to \mathrm{Aut}(\widehat{T}_{0,n})$.

*9.1.2 Pro-l and prounipotent completions*

Let $\pi$ be a finitely generated group, and let $\pi_{\mathbb{Q}}(-)$ denote its $\mathbb{Q}$-prounipotent completion. It has the following properties: $\pi_{\mathbb{Q}}(-)$ is a prounipotent $\mathbb{Q}$-group scheme; there is a group morphism $\pi \to \pi_{\mathbb{Q}}(\mathbb{Q})$; any morphism $\pi \to U(\mathbb{Q})$, where $U(-)$ is a unipotent $\mathbb{Q}$-group scheme, induces a $\mathbb{Q}$-group scheme morphism $\pi_{\mathbb{Q}}(-) \to U(-)$, such that $(\pi \to U(\mathbb{Q})) = (\pi \to \pi_{\mathbb{Q}}(\mathbb{Q}) \to U(\mathbb{Q}))$.

If $\mathbf{k}$ is a $\mathbb{Q}$-ring, then $\pi_{\mathbf{k}}(-) := \pi_{\mathbb{Q}}(-) \otimes \mathbf{k}$ is a prounipotent $\mathbf{k}$-group scheme (it is the functor $\{\mathbf{k}\text{-rings}\} \to \{\text{groups}\}$, $K \mapsto \pi_{\mathbb{Q}}(K)$). There is a morphism $(\pi \to \pi_{\mathbf{k}}(\mathbf{k})) := (\pi \to \pi_{\mathbb{Q}}(\mathbb{Q}) \to \pi_{\mathbb{Q}}(\mathbf{k}) = \pi_{\mathbf{k}}(\mathbf{k}))$. Any morphism $\pi \to U(\mathbf{k})$, where $U(-)$ is a prounipotent $\mathbf{k}$-group scheme, gives rise to a morphism $\pi_{\mathbf{k}}(-) \to U(-)$, such that $(\pi \to U(\mathbf{k})) = (\pi \to \pi_{\mathbf{k}}(\mathbf{k}) \to U(\mathbf{k}))$ ([12], Sect. 4).

Let $l$ be a prime number, and let $\pi_l$ be the pro-$l$ completion of $\pi$. According to [13], Lemma A.7, there exists a morphism $\pi_l \to \pi_{\mathbb{Q}}(\mathbb{Q}_l)$, compatible with the maps from $\pi$.

If $\pi, \pi'$ are finitely generated groups, then a continuous morphism $\pi_l \to \pi'_l$ gives rise to the morphism $\pi \to \pi_l \to \pi'_l \to \pi'_{\mathbb{Q}}(\mathbb{Q}_l)$, and hence to a $\mathbb{Q}_l$-group scheme morphism $\pi_{\mathbb{Q}_l}(-) \to \pi'_{\mathbb{Q}_l}(-)$, such that $(\pi \to \pi'_{\mathbb{Q}}(\mathbb{Q}_l)) = (\pi \to \pi_{\mathbb{Q}_l}(\mathbb{Q}_l) \to \pi'_{\mathbb{Q}_l}(\mathbb{Q}_l))$. The resulting map $\mathrm{Hom}(\pi_l, \pi'_l) \to \mathrm{Hom}_{\substack{\mathbb{Q}_l\text{-group}\\ \text{schemes}}}(\pi_{\mathbb{Q}_l}, \pi'_{\mathbb{Q}_l})$ is compatible with compositions and hence gives rise to a group morphism

$$\mathrm{Aut}(\pi_l) \to \mathrm{Aut}_{\substack{\mathbb{Q}_l\text{-group}\\ \text{schemes}}}(\pi_{\mathbb{Q}_l}). \tag{73}$$

Let $U(-)$ be a prounipotent $\mathbb{Q}$-group scheme. Let $\underline{\mathrm{Aut}\,U}$ be the $\mathbb{Q}$-group scheme defined as the functor $\{\mathbb{Q}\text{-rings}\} \to \{\text{groups}\}$, $\mathbf{k} \mapsto \underline{\mathrm{Aut}\,U}(\mathbf{k}) := \mathrm{Aut}_{\substack{\mathbf{k}\text{-group}\\ \text{schemes}}}(U \otimes \mathbf{k}) = \mathrm{Aut}_{\substack{\mathbf{k}\text{-Lie}\\ \text{algebras}}}(\mathfrak{u} \otimes \mathbf{k})$, where $\mathfrak{u} = \mathrm{Lie}\,U$. Then, $\underline{\mathrm{Aut}\,U}$ is an extension of a group $\mathbb{Q}$-subscheme $G \subset \mathrm{GL}(\mathfrak{u}^{ab})$ by a prounipotent $\mathbb{Q}$-group scheme, explicitly

$$1 \to \underline{\mathrm{Aut}^+ U} \to \underline{\mathrm{Aut}\,U} \to G \to 1.$$

Namely, $G$ is the intersection of the decreasing sequence of group schemes $\mathrm{Im}(\underline{\mathrm{Aut}\,U/U^{(n)}} \to \mathrm{GL}(\mathfrak{u}^{ab}))$, which is stationary.

The morphism (73) may therefore be interpreted as a morphism

$$\mathrm{Aut}(\pi_l) \to \underline{\mathrm{Aut}\,\pi}(\mathbb{Q}_l).$$

Let $\mathcal{G} \rightrightarrows B$ be a groupoid where for any $b \in B$, $\mathcal{G}_b := \mathcal{G}_{bb}$ is finitely generated. We denote by $\mathcal{G}_l \rightrightarrows B$, $\mathcal{G}_{\mathbb{Q}}(-) \rightrightarrows B$ its pro-$l$ and $\mathbb{Q}$-prounipotent completions, given by $(\mathcal{G}_l)_{bc} := (\mathcal{G}_b)_l \times_{\mathcal{G}_b} \mathcal{G}_{bc}$ and $\mathcal{G}_{\mathbb{Q}}(\mathbf{k})_{bc} := \mathcal{G}_b(\mathbf{k}) \times_{\mathcal{G}_b} \mathcal{G}_{bc}$.

Assume that $\mathcal{G}$ is connected (i.e. for any $b, c \in B$, $\mathcal{G}_{bc} \neq \emptyset$). Define the group scheme $\underline{\mathrm{Aut}\mathcal{G}}$ by $\underline{\mathrm{Aut}\mathcal{G}}(\mathbf{k}) := \mathrm{Aut}(\mathcal{G}(\mathbf{k}))$. If $\mathcal{G}_{ab} \times \mathcal{G}_{bc} \to \mathcal{G}_{ac}$, $(g_{ab}, g_{bc}) \mapsto g_{bc}g_{ab}$ is the composition of $\mathcal{G}$, then $\mathrm{Aut}(\mathcal{G}(\mathbf{k})) = \{\theta_{ab} : \mathcal{G}_{ab} \to \mathcal{G}_{ab}(\mathbf{k}) | \forall a, b, c, \forall g_{ab}, g_{bc}, \theta_{ac}(g_{bc}g_{ab}) = \theta_{bc}(g_{bc})\theta_{ab}(g_{ab})\}$. The choice of $b \in B$ and of particular elements $g^0_{ab} \in \mathcal{G}_{ab}$ for any $a \in B - \{b\}$ gives rise to an isomorphism $\mathrm{Aut}(\mathcal{G}(\mathbf{k})) \simeq \mathcal{G}_b(\mathbf{k})^{B-\{b\}} \rtimes \underline{\mathrm{Aut}\mathcal{G}_b}(\mathbf{k})$, the inverse isomorphism taking $((X_a)_a, \theta)$ to the automorphism such that $\mathcal{G}_b(\mathbf{k}) \ni g_b \mapsto \theta(g_b) \in \mathcal{G}_b(\mathbf{k})$, and $\mathcal{G}_{ab} \ni g^0_{ab} \mapsto X_a g^0_{ab} \in \mathcal{G}_{ab}(\mathbf{k})$. The morphisms $\pi_l \to \pi_{\mathbb{Q}}(\mathbb{Q}_l)$ and $\mathrm{Aut}(\pi_l) \to \underline{\mathrm{Aut}\,\pi}(\mathbb{Q}_l)$, where $\pi = \mathcal{G}_b$, give rise to a morphism

$$\mathrm{Aut}(\mathcal{G}_l) \to \underline{\mathrm{Aut}\,\mathcal{G}}(\mathbb{Q}_l).$$

### *9.1.3 Pro-l Galois representations*

The following statement can be derived from [9,29].

**Proposition 9.2** *There exist morphisms* $\mathrm{GT}_l \to \mathrm{Aut}(T^l_{0,n})$, $\mathrm{GT}(-) \to \underline{\mathrm{Aut}\,T_{0,n}}(-)$, *such that the squares in the following diagram commute*

$$\begin{array}{ccccc} \widehat{\mathrm{GT}} & \longrightarrow & \mathrm{GT}_l & \longrightarrow & \mathrm{GT}(\mathbb{Q}_l) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Aut}(\widehat{T}_{0,n}) & \longrightarrow & \mathrm{Aut}(T^l_{0,n}) & \longrightarrow & \underline{\mathrm{Aut}\,T_{0,n}}(\mathbb{Q}_l) \end{array}$$

## 9.2 Arithmetic fundamental groups and Teichmüller groupoids in genus one

The Galois theoretic counterpart of the theory of elliptic associators is the action of the arithmetic fundamental group $\pi_1(M^{\mathbb{Q}}_{1,\vec{1}})$ on the completions of elliptic braid groups, based on the fibration $M^{\mathbb{Q}}_{1,n} \to M^{\mathbb{Q}}_{1,1}$, as studied in [11,27]. We first recall the main points of this study.

### *9.2.1 Arithmetic fundamental groups of moduli spaces*

Let $M^{\mathbb{Q}}_{1,1}$ (resp., $M^{\mathbb{Q}}_{1,\vec{1}}$, $\tilde{M}^{\mathbb{Q}}_{1,1}$) be the moduli space of elliptic curves with one puncture (respectively, with one puncture and a nonzero tangent vector at the puncture, with one puncture and a formal coordinate at the puncture).

A rational tangential base point $\xi$ of $M^{\mathbb{Q}}_{1,1}$ is defined as follows. The Deligne-Mumford compactification $\overline{M}^{\mathbb{Q}}_{1,1}$ of $M^{\mathbb{Q}}_{1,1}$ contains a unique curve $X^0$, which corresponds to the tadpole graph. A formal neighbourhood of $X^0$ in $\overline{M}^{\mathbb{Q}}_{1,1}$ is a curve $X \to \mathrm{Spec}\,\mathbb{Q}[[q]]$, whose generic fibre is the Tate elliptic curve $\mathbb{G}_m/q^{\mathbb{Z}}$ with marked point $[1] = q^{\mathbb{Z}}$. This may be viewed as a morphism $\mathrm{Spec}\,\mathbb{Q}[[q]] \to \overline{M}^{\mathbb{Q}}_{1,1}$, which restricts to $\xi : \mathrm{Spec}\,\mathbb{Q}((q)) \to M^{\mathbb{Q}}_{1,1}$.

A lift $\tilde{\xi}$ of $\xi$ to $\tilde{M}^{\mathbb{Q}}_{1,1}$ is defined by choosing the local coordinate $\log z$ at $[1] = q^{\mathbb{Z}}$, $z$ being the canonical coordinate on $\mathbb{G}_m$ (such that the function ring is $\mathbb{Q}[z, z^{-1}]$). Let $\vec{\xi}$ be the lift of $\xi$ to $M_{1,\vec{1}}$ given by the expansion of the local coordinate of $\tilde{\xi}$ at order one.

The isomorphism $\pi_1^{geom}(M^{\mathbb{Q}}_{1,\vec{1}}, \vec{\xi}) \simeq \widehat{B}_3$ gives rise to a split exact sequence

$$1 \to \widehat{B}_3 \to \pi_1(M^{\mathbb{Q}}_{1,\vec{1}}, \vec{\xi}) \overset{\curvearrowleft}{\to} G_{\mathbb{Q}} \to 1, \tag{74}$$

where the section is provided by the base point $\vec{\xi}$; the induced morphism $G_{\mathbb{Q}} \to \mathrm{Aut}(\widehat{B}_3)$ has been computed explicitly in [24], Cor. 4.15 (it is recalled in Sect. 9.3).

The result of [24] can be complemented as follows.

**Proposition 9.3** *There is a morphism from (74) to the split exact sequence*

$$1 \to \mathrm{SL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\widehat{\mathbb{Z}}) \overset{\curvearrowleft}{\to} \mathbb{Z}^{\times} \to 1, \tag{75}$$

*where the second morphism is the determinant* det *and the section is the morphism* $\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. *The rightmost morphism in (74)→ (75) is the cyclotomic character* $G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times}$.

The proof will be carried out in Sect. 9.3.

*9.2.2 Profinite representations*

Let $\tilde{M}^{\mathbb{Q}}_{1,n}$ be the moduli space of elliptic curves with $n$ punctures. There is a fibration $M^{\mathbb{Q}}_{1,n} \to M^{\mathbb{Q}}_{1,1}$ defined by forgetting all the punctures except the first one. One sets $\tilde{M}^{\mathbb{Q}}_{1,n} := \tilde{M}^{\mathbb{Q}}_{1,1} \times_{M^{\mathbb{Q}}_{1,1}} M^{\mathbb{Q}}_{1,n}$.

A *tangential section* of a morphism $X \to Y$ of $\mathbb{Q}$-schemes is defined to be a morphism $Y \times \mathrm{Spec}\,\mathbb{Q}((t)) \to X$, such that its composition with $X \to Y$ is the canonical projection.

An *n-tree* $T$ is defined to be a rooted trivalent planar tree, equipped with a bijection $i_T : \{\text{leaves}\} \to \{1, \dots, n\}$ (the root is not a leaf), such that the leftmost leaf is labelled 1. Such a tree gives rise to the assignment, to each $i \in \{1, \dots, n\}$, of a pair $(d_i, s_i)$, where $d_i$ is an integer $\geq 1$ (the distance between the leaf labelled $i$ and the root), and of a map $s_i \in \{l, r\}^{d_i}$ describing the path from the root to the leaf labelled $i$ ($s_i(k) = l$ or $r$ according to whether the $k$th interval of the path is a left or right descendant). It also gives rise to a permutation $s_T \in S_n$ such that $s_T(1) = 1$: $s_T$ is the composite map $\{1, \dots, n\} \to \{\text{leaves}\} \overset{i_T}{\to} \{1, \dots, n\}$, where the first map is the inverse of the lexicographic (according to the order left $<$ right) indexation of the leaves.

A tangential section $\sigma_T$ of the morphism $\tilde{M}^{\mathbb{Q}}_{1,n} \to \tilde{M}^{\mathbb{Q}}_{1,1}$ may be associated with each $n$-tree $T$ as follows: $\sigma_T$ is the morphism $\tilde{M}^{\mathbb{Q}}_{1,1} \times \mathrm{Spec}\,\mathbb{Q}((t)) \to \tilde{M}_{1,n}$, taking a pair $((E, p, z), t)$ to $(E, p_1, \dots, p_n, z)$, where $p_i := z^{-1}(\sum_{k \in s_i^{-1}(r)} t^k)$.

Let $\mathcal{F}_\xi$ be the fibre over $\xi$ of $M_{1,n}^{\mathbb{Q}} \to M_{1,1}^{\mathbb{Q}}$. There is a split exact sequence of groupoids

$$1 \to \pi_1^{geom}(\mathcal{F}_\xi, \{\sigma_T(\xi)\}) \to \pi_1(\tilde{M}_{1,n}^{\mathbb{Q}}, \{\sigma_T(\tilde{\xi})\}) \overset{\curvearrowleft}{\to} \pi_1(\tilde{M}_{1,1}^{\mathbb{Q}}, \tilde{\xi}) \to 1.$$

(see [11,27] and also [24], Sect. 5.1), which gives rise to a morphism

$$\pi_1(\tilde{M}_{1,1}^{\mathbb{Q}}, \tilde{\xi}) \to \mathrm{Aut}(\pi_1^{geom}(\mathcal{F}_\xi, \{\sigma_T(\xi)\})). \tag{76}$$

The fibre at $(E, p)$ of $M_{1,n}^{\mathbb{Q}} \to M_{1,1}^{\mathbb{Q}}$ is $(E - \{p\})^{n-1}$ − (diagonals), whose geometric fundamental group is the profinite completion of $\overline{P}_{1,n}$ (the quotient of the elliptic braid group with $n$ strands $P_{1,n}$ by the central $\mathbb{Z}^2$). The geometric fundamental groupoid $\pi_1^{geom}(\mathcal{F}_\xi, \{\sigma_T(\xi)\})$ is the profinite completion of the groupoid $T_{ell,n}$ where objects are $n$-trees and the set of morphisms from $T$ to $T'$ is $\overline{P}_{1,n} \times_{S_n} \{s_{T'} s_T^{-1}\}$, equipped with the composition of morphisms induced from the product in $\overline{P}_{1,n}$. On the other hand, there is an isomorphism $\pi_1(\tilde{M}_{1,1}^{\mathbb{Q}}, \tilde{\xi}) \simeq \pi_1(M_{1,\vec{1}}^{\mathbb{Q}}, \vec{\xi})$. (76) therefore gives rise to a morphism

$$\pi_1(M_{1,\vec{1}}^{\mathbb{Q}}, \vec{\xi}) \to \mathrm{Aut}(\widehat{T}_{ell,n}). \tag{77}$$

**Theorem 9.4** *There exists a morphism $\pi_1(M_{1,\vec{1}}^{\mathbb{Q}}, \vec{\xi}) \to \widehat{\mathrm{GT}}_{ell}$ and an action of $\widehat{\mathrm{GT}}_{ell}$ on $\widehat{T}_{ell,n}$, such that:*

*(a) the morphism (77) factors as $\pi_1(M_{1,\vec{1}}^{\mathbb{Q}}, \vec{\xi}) \to \widehat{\mathrm{GT}}_{ell} \to \mathrm{Aut}(\widehat{T}_{ell,n})$;*

*(b) the morphism of split morphisms induced by (74)→ (75) factors as $(\pi_1(M_{1,\vec{1}}^{\mathbb{Q}}, \vec{\xi}) \overset{\curvearrowleft}{\to} G_{\mathbb{Q}}) \to (\widehat{\mathrm{GT}}_{ell} \overset{\curvearrowleft}{\to} \widehat{\mathrm{GT}}) \to (\mathrm{GL}_2(\widehat{\mathbb{Z}}) \overset{\curvearrowleft}{\to} \widehat{\mathbb{Z}}^\times)$.*

The proof will be carried out in Sect. 9.4.

### *9.2.3 Pro-l representations*

**Proposition 9.5** *There exist morphisms $\mathrm{GT}_{ell}^l \to \mathrm{Aut}\, T_{ell,n}^l$, $\mathrm{GT}_{ell}(-) \to \underline{\mathrm{Aut}\, T_{ell,n}}(-)$, such that the squares in the following diagram commute*

$$\begin{array}{ccccc}
\widehat{\mathrm{GT}}_{ell} & \longrightarrow & \mathrm{GT}_{ell}^l & \longrightarrow & \mathrm{GT}_{ell}(\mathbb{Q}_l) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Aut}(\widehat{T}_{ell,n}) & \longrightarrow & \mathrm{Aut}(T_{ell,n}^l) & \longrightarrow & \underline{\mathrm{Aut}\, T_{ell,n}}(\mathbb{Q}_l)
\end{array}$$

9.3 Proof of Proposition 9.3

As in [24], let $\mathfrak{s}_0 : G_{\mathbb{Q}} \to \pi_1(M^{\mathbb{Q}}_{1,\vec{1}}, \vec{\xi})$ be the section induced by $\vec{\xi}$. The diagram $\pi_1(M^{\mathbb{Q}}_{1,\vec{1}}, \vec{\xi}) \overset{\curvearrowleft}{\to} G_{\mathbb{Q}}$ gives rise to the semidirect product decomposition $\pi_1(M^{\mathbb{Q}}_{1,\vec{1}}, \vec{\xi}) \simeq \widehat{B}_3 \rtimes G_{\mathbb{Q}}$, where the action $G_{\mathbb{Q}} \to \mathrm{Aut}(\widehat{B}_3)$ is $g * x := \mathfrak{s}_0(g)x\mathfrak{s}_0(g)^{-1}$. On the other hand, the diagram $\mathrm{GL}_2(\mathbb{Z}) \overset{\curvearrowleft}{\to} \widehat{\mathbb{Z}}^\times$ gives rise to the semidirect product decomposition $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \simeq \mathrm{SL}_2(\mathbb{Z}) \rtimes \widehat{\mathbb{Z}}^\times$, where the action $\widehat{\mathbb{Z}}^\times \to \mathrm{Aut}(\mathrm{SL}_2(\widehat{\mathbb{Z}}))$ is $\lambda \bullet m := \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} m \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1}$.

Let $\sigma_1, \sigma_2$ be the Artin generators of $B_3$ (denoted $\bar{a}_1, \bar{a}_2$ in [24]). As $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ is profinite, there is a unique morphism

$$\widehat{B}_3 \to \mathrm{SL}_2(\widehat{\mathbb{Z}}), \quad x \mapsto \overline{x} \tag{78}$$

extending the quotient morphism $B_3 \to B_3/Z(B_3) \simeq \mathrm{SL}_2(\mathbb{Z})$, $\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

The action of $G_{\mathbb{Q}}$ on $\widehat{B}_3$ can be made explicit as follows. Denote the map $G_{\mathbb{Q}} \to \widehat{\mathrm{GT}} \subset \widehat{\mathbb{Z}}^\times \times \widehat{F}_2$ by $g \mapsto (\chi(g), f_g)$. Using the formula $\beta_0(g) = \sigma_1^{8\rho_2(g)}\mathfrak{s}_0(g)$ in [24] before Proposition 4.12, and Corollary 4.15 in the same paper, one obtains

$$g * \sigma_1 = \sigma_1^{\chi(g)}, \quad g * \sigma_2 = \mathrm{Ad}_{\sigma_1^{-8\rho_2(g)} f_g(\sigma_1^2, \sigma_2^2)^{-1}}(\sigma_2^{\chi(g)})$$

(here $\rho_2 : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}$ is the Kummer cocycle related to the roots of 2).

Then

$$\overline{g * \sigma_1} = \overline{\sigma_1^{\chi(g)}} = \begin{pmatrix} 1 & \chi(g) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi(g) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi(g) & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \chi(g) \bullet \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \chi(g) \bullet \overline{\sigma}_1;$$

on the other hand, Corollary 4.15 in [24] says that

$$f_g\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\right) = \pm \begin{pmatrix} 1 & 0 \\ -8\rho_2(g) & 1 \end{pmatrix} \begin{pmatrix} \chi(g)^{-1} & 0 \\ 0 & \chi(g) \end{pmatrix} \begin{pmatrix} 1 & -8\rho_2(g) \\ 0 & 1 \end{pmatrix}$$

(identity in $\mathrm{SL}_2(\widehat{\mathbb{Z}})$), therefore

$$\begin{aligned} \overline{g * \sigma_2} &= \overline{\mathrm{Ad}_{\sigma_1^{-8\rho_2(g)} f_g(\sigma_1^2, \sigma_2^2)^{-1}} \sigma_2^{\chi(g)}} = \mathrm{Ad}_{\pm \begin{pmatrix} \chi(g) & 0 \\ 0 & \chi(g)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 8\rho_2(g) & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ -\chi(g) & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\chi(g)^{-1} & 1 \end{pmatrix} = \begin{pmatrix} \chi(g) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \chi(g) & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \chi(g) \bullet \overline{\sigma}_2. \end{aligned}$$

It follows that (78) intertwines the actions of $G_{\mathbb{Q}}$ and $\widehat{\mathbb{Z}}^\times$, which proves Proposition 9.3.

### 9.4 Proof of Theorem 9.4

Theorem 9.4 states the existence of a morphism $\pi_1(M_{1,\vec{1}}^{\mathbb{Q}}, \vec{\xi}) \to \widehat{\mathrm{GT}}_{ell}$, which will now be constructed.

**Proposition 9.6** *Set* $\widehat{R}_{ell} := \mathrm{Ker}(\widehat{\mathrm{GT}}_{ell} \to \widehat{\mathrm{GT}})$.

*(a) There is a unique morphism* $\widehat{B}_3 \to \widehat{R}_{ell}$*, extending the canonical morphism* $B_3 \simeq R_{ell} \to \widehat{R}_{ell}(\subset \mathrm{Aut}(\widehat{F}_2))$.

*(b) There is a unique morphism* $\pi_1(M_{1,\vec{1}}^{\mathbb{Q}}, \vec{\xi}) \to \widehat{\mathrm{GT}}_{ell}$*, such that the diagram*

$$\begin{array}{ccccccccc} 1 \to & \widehat{B}_3 & \to & \pi_1(M_{1,\vec{1}}^{\mathbb{Q}}, \vec{\xi}) & \overset{\mathfrak{s}_0}{\curvearrowleft}\!\!\!\to & G_{\mathbb{Q}} & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & \widehat{R}_{ell} & \to & \widehat{\mathrm{GT}}_{ell} & \overset{\sigma}{\curvearrowleft}\!\!\!\to & \widehat{\mathrm{GT}} & \to 1 \end{array}$$

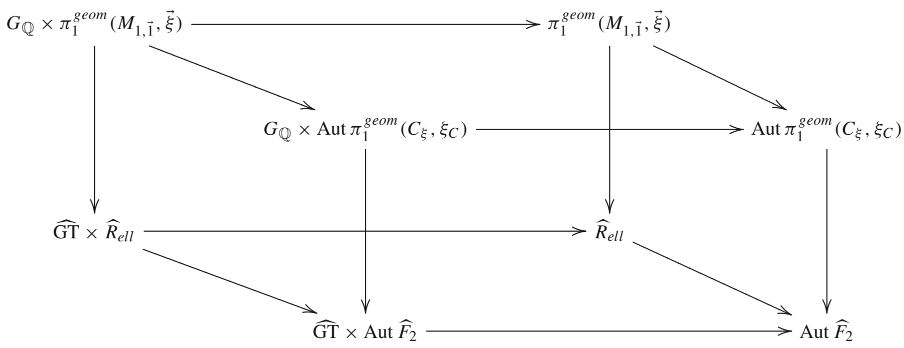
*commutes.*

*Proof* (a) Recall that $\widehat{R}_{ell}$ is a subgroup of $\mathrm{Aut}(\widehat{F}_2)$. $\mathrm{Aut}(\widehat{F}_2)$ is profinite ([8], Thm. 5.3), and the map $\mathrm{Aut}(\widehat{F}_2) \to \widehat{F}_2^2$, $\theta \mapsto (\theta(X), \theta(Y))$ is continuous (loc. cit., Ex. 2, p. 96). As $\widehat{R}_{ell}$ is the preimage of 1 by a continuous map $(\widehat{F}_2)^2 \to (\widehat{B}_{1,3})^2 \times \widehat{F}_2$, it is closed, so $\widehat{R}_{ell}$ is a closed subgroup of $\mathrm{Aut}(\widehat{F}_2)$, hence is profinite. The morphism $B_3 \to \widehat{R}_{ell}$ therefore extends to a morphism $\widehat{B}_3 \to \widehat{R}_{ell}$.

Statement (b) is equivalent of the compatibility of the morphism $\widehat{B}_3 \to \widehat{R}_{ell}$ with the actions of $G_{\mathbb{Q}}$ and $\widehat{\mathrm{GT}}$ on both sides via $\mathfrak{s}_0$ and $\sigma$ and the morphism $G_{\mathbb{Q}} \to \widehat{\mathrm{GT}}$, i.e. to the commutativity of

$$\begin{array}{ccc} G_{\mathbb{Q}} \times \widehat{B}_3 & \to & \widehat{B}_3 \\ \downarrow & & \downarrow \\ \widehat{\mathrm{GT}} \times \widehat{R}_{ell} & \to & \widehat{R}_{ell} \end{array} \tag{79}$$

Consider the following cubic diagram



where $C_\xi$ is the fibre of $M^{\mathbb{Q}}_{1,2} \to M^{\mathbb{Q}}_{1,1}$ at $\xi$ (which identifies with the fibre of $M^{\mathbb{Q}}_{1,\vec{2}} \to M^{\mathbb{Q}}_{1,\vec{1}}$, where $M^{\mathbb{Q}}_{1,\vec{2}} = M^{\mathbb{Q}}_{1,\vec{1}} \times_{M^{\mathbb{Q}}_{1,1}} M^{\mathbb{Q}}_{1,2}$), $\xi_C$ is a tangential base point of $C_\xi$ supported at the marked point, and the maps are defined as follows :

• the upper horizontal maps are the Galois actions; the map $\widehat{\mathrm{GT}} \times \widehat{R}_{ell} \to \widehat{R}_{ell}$ is the action induced by the section of $\widehat{\mathrm{GT}}_{ell} \to \widehat{\mathrm{GT}}$ defined in Proposition 3.21; the map $\widehat{\mathrm{GT}} \times \mathrm{Aut}\,\widehat{F}_2 \to \mathrm{Aut}\,\widehat{F}_2$ is induced by the composite map $\widehat{\mathrm{GT}} \to \mathrm{Aut}\,\widehat{F}_2 \to \mathrm{Aut}(\mathrm{Aut}\,\widehat{F}_2)$, where the second map is the inner action of $\mathrm{Aut}\,\widehat{F}_2$ on itself, and the first map is the composite morphism $\widehat{\mathrm{GT}} \to \widehat{\mathrm{GT}}_{ell} \subset \widehat{\mathrm{GT}} \times \mathrm{Aut}\,\widehat{F}_2 \to \mathrm{Aut}\,\widehat{F}_2$, where $\widehat{\mathrm{GT}}_{ell} \to \widehat{\mathrm{GT}}$ is the same morphism as above and $\widehat{\mathrm{GT}} \times \mathrm{Aut}\,\widehat{F}_2 \to \mathrm{Aut}\,\widehat{F}_2$ is the second projection;

- the vertical maps are induced by the morphisms $G_{\mathbb{Q}} \to \widehat{\mathrm{GT}}$, $\pi_1^{geom}(M^{\mathbb{Q}}_{1,\vec{1}}, \vec{\xi}) \simeq \widehat{B}_3 \to \widehat{R}_{ell}$, $\pi_1^{geom}(C_\xi, \xi_C) \xrightarrow{\sim} \widehat{F}_2$;
- the diagonal maps are induced by the canonical inclusion $\widehat{R}_{ell} \to \mathrm{Aut}\,\widehat{F}_2$, and by the action of $\pi_1^{geom}(M_{1,\vec{1}}, \vec{\xi})$ on $\pi_1^{geom}(C_\xi, \xi_C)$ induced by the fibration $M^{\mathbb{Q}}_{1,\vec{2}} \to M^{\mathbb{Q}}_{1,\vec{1}}$.

The square corresponding to the upper face of the cube commutes because the action of $\pi_1^{geom}(M_{1,\vec{1}}, \vec{\xi})$ on $\pi_1^{geom}(C_\xi, \xi_C)$ is compatible with the Galois action.

The square corresponding to the sides of the cube commute because this action identifies with the profinite completion of the action of $B_3$ on $F_2$.

The square corresponding to the lower face of the cube commutes by construction of the map $\widehat{\mathrm{GT}} \times \mathrm{Aut}\,\widehat{F}_2 \to \mathrm{Aut}\,\widehat{F}_2$.

The square corresponding to the lower front face commutes for the following reason. According to [24], Corollary 4.5, the action of $G_{\mathbb{Q}}$ on $\pi_1^{geom}(C_\xi, \xi_C)$ may be described as follows. $\pi_1^{geom}(C_\xi, \xi_C)$ is topologically free, generated by $x_1, x_2$. The action of $\gamma \in G_{\mathbb{Q}}$ on this group is

$$\gamma^*(x_1) = f_\gamma(x_1, z_1) x_1^{\chi(\gamma)} f_\gamma(x_1, z_1)^{-1}, \tag{80}$$

$$\gamma^*(x_2) = f_\gamma(x_1, z_1) x_1^{1-\chi(\gamma)} f_\gamma^{\vec{\infty}1}(x_1, x_1^{-1} z_1 x_1)^{-1} x_2 x_1^{\chi(\gamma)-1} f_\gamma(x_1, z_1)^{-1}, \tag{81}$$

where $z_1 = (x_2, x_1) = x_2 x_1 x_2^{-1} x_1^{-1}$, $\gamma \mapsto (\chi(\gamma), f_\gamma)$ is the map $G_{\mathbb{Q}} \to \widehat{\mathrm{GT}}$, and $f_\gamma^{\vec{\infty}1}(a, b) = f_\gamma(b, c) b^{(\chi(\gamma)-1)/2} f_\gamma(a, b)$ for $abc = 1$.

Under the identification $x_1 \mapsto X$, $x_2 \mapsto Y$, formula (80) corresponds to the expression of $g_+$ in Proposition 3.21. It follows from the hexagon and duality identities that any $(\lambda, f) \in \widehat{\mathrm{GT}}$ satisfies the octagon identity

$$f(X^{-1}Z^{-1}, Z)(ZX)^{-\lambda} f(Z, X^{-1}Z^{-1}) Z^{(\lambda+1)/2} f(X, Z) X^\lambda f(Z, X) Z^{(\lambda-1)/2} = 1,$$

where $Z := (Y, X)$. This identity implies

$$f(X, (Y, X)) X^{1-\lambda} f^{\vec{\infty}1}(X, (X^{-1}, Y))^{-1} Y X^{\lambda-1} f(X, (Y, X))^{-1}$$
$$= Z^{(\lambda-1)/2} f(X^{-1}Z^{-1}, Z) Y f(X, (Y, X))^{-1}$$

so that (81) corresponds to $g_-$ in Proposition 3.21. All this implies the commutativity of

$$\begin{array}{ccc} G_{\mathbb{Q}} & \to & \mathrm{Aut}\,\pi_1^{geom}(C_\xi, \xi_C) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{GT}}_{ell} & \to & \mathrm{Aut}\,\widehat{F}_2 \end{array}$$

Composing this square with the commutative square

$$\begin{array}{ccc} \mathrm{Aut}\,\pi_1^{geom}(C_\xi, \xi_C) & \to & \mathrm{Aut}(\mathrm{Aut}\,\pi_1^{geom}(C_\xi, \xi_C)) \\ \downarrow & & \downarrow \\ \mathrm{Aut}\,\widehat{F}_2 & \to & \mathrm{Aut}(\mathrm{Aut}\,\widehat{F}_2) \end{array}$$

where the horizontal maps are inner action morphisms, one obtains the commutativity of the square corresponding to the lower front face.

The commutativity of all these squares implies that the two composite maps

$$G_{\mathbb{Q}} \times \pi_1^{geom}(M_{1,\vec{1}}, \vec{\xi}) \to \widehat{R}_{ell} \to \mathrm{Aut}\,\widehat{F}_2$$

coincide, where the maps $G_{\mathbb{Q}} \times \pi_1^{geom}(M_{1,\vec{1}}, \vec{\xi}) \to \widehat{R}_{ell}$ are the two composite maps which can be obtained from the upper front face. As $\widehat{R}_{ell} \to \mathrm{Aut}\,\widehat{F}_2$ is injective, this implies the commutativity of the square corresponding to the upper front face, and therefore of (79). □

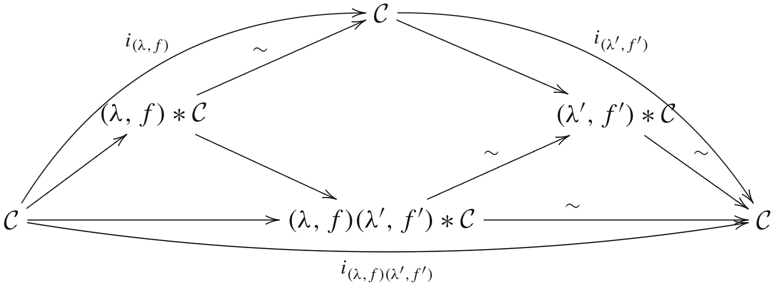
The next statement of Theorem 9.4 is the existence of an action of $\widehat{\mathrm{GT}}_{ell}$ on $\widehat{T}_{ell,n}$, which will now be constructed (Definition 9.8).

If $\mathcal{C}$ is a category, let $\mathrm{Aut}(\mathcal{C})$ be its group of automorphisms (as a category, even if $\mathcal{C}$ has a braided monoidal structure).

For $(\lambda, f) \in \underline{\widehat{\mathrm{GT}}}$, let $i_{\lambda,f}$ be the composite functor $\widehat{\mathbf{PaB}} \xrightarrow{\alpha_{(\lambda,f)}} (\lambda, f) * \widehat{\mathbf{PaB}} \xrightarrow{\sim} \widehat{\mathbf{PaB}}$, where the first functor is the unique tensor functor which induces the identity on objects, and the second functor is the identity functor (which is not tensor). $i_{\lambda,f}$ is then an endofunctor of $\widehat{\mathbf{PaB}}$.

**Lemma 9.7** $(\lambda, f) \mapsto i_{\lambda,f}^{-1}$ *is a morphism* $\widehat{\mathrm{GT}} \to \mathrm{Aut}(\widehat{\mathbf{PaB}})$.

*Proof* The identity $i_{(\lambda',f')} i_{(\lambda,f)} = i_{(\lambda,f)(\lambda',f')}$ follows from the commutativity of the diagram



in which the commutativity of the central square follows from that of

$$\begin{array}{ccc} (\lambda, f) * \mathcal{C} & \overset{(\lambda,f)*\varphi}{\to} & (\lambda, f) * \mathcal{D} \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{C} & \overset{\varphi}{\to} & \mathcal{D} \end{array}$$

for any braided monoidal categories $\mathcal{C}, \mathcal{D}$ and any tensor functor $\varphi : \mathcal{C} \to \mathcal{D}$. □

One constructs in the same way a morphism

$$\widehat{\mathrm{GT}}_{ell} \to \mathrm{Aut}(\widehat{\mathbf{PaB}}_{ell}). \tag{82}$$

If $\mathcal{C}_0$ is a braided monoidal category, then $\mathrm{Ob}\,\mathcal{C}_0$ is a magma (i.e. a set equipped with a composition map and a unit). Let $\phi : M \to \mathrm{Ob}\,\mathcal{C}_0$ be a magma morphism, then a braided monoidal category $\phi * \mathcal{C}_0$ can be constructed by $\mathrm{Ob}\,\phi^*\mathcal{C}_0 = M$, $\phi^*\mathcal{C}_0(m, m') := \mathcal{C}_0(\phi(m), \phi(m'))$ and by the condition that the obvious functor $\phi^*\mathcal{C}_0 \to \mathcal{C}_0$ is tensor. If $\mathcal{C}_0 \to \mathcal{C}$ is an elliptic structure over $\mathcal{C}_0$, then one defines an elliptic structure $\phi^*\mathcal{C}_0 \to \phi^*\mathcal{C}$ over $\phi^*\mathcal{C}_0$ in the same way. Then, there are natural group morphisms

$$\mathrm{Aut}\,\mathcal{C}_0 \to \mathrm{Aut}\,\phi^*\mathcal{C}_0, \quad \mathrm{Aut}\,\mathcal{C} \to \mathrm{Aut}\,\phi^*\mathcal{C}. \tag{83}$$

Let $\mu(S)$ be the free magma generated by a set $S$. The unique map $S \to \{\bullet\}$ induces a magma morphism $\phi : \mu(S) \to \mu(\{\bullet\}) \simeq \mathrm{Ob}\,\widehat{\mathbf{PaB}}$. Set $\widehat{\mathbf{PaB}}_S := \phi^*\widehat{\mathbf{PaB}}$, $\widehat{\mathbf{PaB}}_{ell,S} := \phi^*\widehat{\mathbf{PaB}}_{ell}$. The morphisms (84) then specialize to morphisms

$$\mathrm{Aut}(\widehat{\mathbf{PaB}}) \to \mathrm{Aut}(\widehat{\mathbf{PaB}}_S), \quad \mathrm{Aut}(\widehat{\mathbf{PaB}}_{ell}) \to \mathrm{Aut}(\widehat{\mathbf{PaB}}_{ell,S}). \tag{84}$$

$\widehat{T}_{ell,n}$ may be viewed as the full subcategory of $\widehat{\mathbf{PaB}}_{ell,[n]}$, where the objects are the preimages of $1\dot{+}\cdots\dot{+}n$ under the map $\mu([n]) \overset{\psi}{\to} \mathbb{N}[n]$, extending the identity on $[n]$, where $\mathbb{N}[n]$ is the free abelian semigroup generated by $[n] = \{1, \ldots, n\}$ (in which the addition is denoted $\dot{+}$). If $\mathcal{C}$ is any category and $\mathcal{C}'$ is any full subcategory, then there is a natural morphism $\mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(\mathcal{C}')$. It specializes to a group morphism

$$\mathrm{Aut}(\widehat{\mathbf{PaB}}_{ell,[n]}) \to \mathrm{Aut}(\widehat{T}_{ell,n}). \tag{85}$$

**Definition 9.8** The action of $\widehat{\mathrm{GT}}_{ell}$ on $\widehat{T}_{ell,n}$ is given by the composite morphism

$$\widehat{\mathrm{GT}}_{ell} \to \mathrm{Aut}\,\widehat{\mathbf{PaB}}_{ell} \to \mathrm{Aut}\,\widehat{\mathbf{PaB}}_{ell,[n]} \to \mathrm{Aut}(\widehat{T}_{ell,n}).$$

obtained from (82), (84) and (85).

Theorem 9.4 next states the compatibility of the 'arithmetic' action

$$\pi_1(\tilde{M}_{1,1}^{\mathbb{Q}}, \tilde{\xi}) \to \mathrm{Aut}(\pi_1^{geom}(\mathcal{F}_\xi, \{\sigma_T(\xi)\}))$$

(see (76)) with its 'algebraic model' $\widehat{\mathrm{GT}}_{ell} \to \mathrm{Aut}(\widehat{T}_{ell,n})$ (see Definition 9.8), namely the commutativity of

$$\begin{array}{ccc} \pi_1(\tilde{M}^{\mathbb{Q}}_{1,1}, \tilde{\xi}) & \longrightarrow & \mathrm{Aut}(\pi_1^{geom}(\mathcal{F}_\xi, \{\sigma_T(\xi)\})) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{GT}}_{ell} & \longrightarrow & \mathrm{Aut}(\widehat{T}_{ell,n}) \end{array} \tag{86}$$

The commutativity of the restriction of (86) to $\widehat{B}_3 \subset \pi_1(\tilde{M}^{\mathbb{Q}}_{1,1}, \tilde{\xi})$ can be proved as follows. Let $\widehat{\mathbf{B}}_{ell}$ be the category with $\mathrm{Ob}\,\widehat{\mathbf{B}}_{ell} = \mathbb{N}$, $\widehat{\mathbf{B}}_{ell}(n,m) = \emptyset$ if $m \neq n$, and $\widehat{\mathbf{B}}_{ell}(n,n) = \widehat{B}_{1,n}$. There is a natural functor $\widehat{\mathbf{PaB}}_{ell} \to \widehat{\mathbf{B}}_{ell}$, defined as the length map $l : \mu(\{\bullet\}) \to \mathbb{N}$ at the level of objects and as the identity at the level of morphisms; actually $\widehat{\mathbf{PaB}}_{ell} \simeq l^*\widehat{\mathbf{B}}_{ell}$. As $\widehat{R}_{ell} \subset \widehat{\mathrm{GT}}_{ell}$ acts trivially on the images of the associativity constraints under $\widehat{\mathbf{PaB}} \to \widehat{\mathbf{PaB}}_{ell}$, its action on $\widehat{\mathbf{PaB}}_{ell}$ is the lift of an action of $\widehat{R}_{ell}$ on $\widehat{\mathbf{B}}_{ell}$. One checks explicitly that the composition of this action with the morphism $\widehat{B}_3 \to \widehat{R}_{ell}$ coincides with the action of $\widehat{B}_3\widehat{\mathbf{B}}_{ell}$, which arises from its geometric action on the various groups $\widehat{B}_{1,n}$.

The commutativity of the composition of (86) with $G_{\mathbb{Q}} \overset{\sigma}{\to} \pi_1(M_{1,\vec{1}}, \vec{\xi})$ can be shown as follows. As the diagram

$$\begin{array}{ccc} G_{\mathbb{Q}} & \longrightarrow & \pi_1(M^{\mathbb{Q}}_{1,\vec{1}}, \vec{\xi}) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{GT}} & \longrightarrow & \widehat{\mathrm{GT}}_{ell} \end{array}$$

commutes, it suffices to proves that its composition with (86) commutes, i.e. that

$$\begin{array}{ccc} G_{\mathbb{Q}} & \longrightarrow & \mathrm{Aut}(\mathcal{F}_\xi, \{\sigma_T(\xi)\}) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{GT}} & \longrightarrow & \mathrm{Aut}(\widehat{T}_{ell,n}) \end{array}$$

commutes. According to [21], the morphism $G_{\mathbb{Q}} \to \mathrm{Aut}(\mathcal{F}_\xi, \{\sigma_T(\xi)\})$ can be derived explicitly from the actions of $G_{\mathbb{Q}}$ on $\pi_1(C_\xi, \xi_C)$ and on the profinite braid groups in genus zero. The former action has been computed in [24], Cor. 4.5. The resulting formulas for the action of $G_{\mathbb{Q}}$ can be shown to match those for the action of $\widehat{\mathrm{GT}}$ on $\widehat{T}_{ell,n}$.

The last statement of Theorem 9.4 says that the morphism $(\pi_1(M^{\mathbb{Q}}_{1,\vec{1}}, \vec{\xi}) \overset{\curvearrowleft}{\to} G_{\mathbb{Q}}) \to (G_{\mathbb{Q}} \overset{\curvearrowleft}{\to} \widehat{\mathbb{Z}}^\times)$ factors through $(\widehat{\mathrm{GT}}_{ell} \overset{\curvearrowleft}{\to} \widehat{\mathrm{GT}})$. This can be proved as follows. Firstly, one checks that the morphism $\widehat{B}_3 \to \mathrm{SL}_2(\widehat{\mathbb{Z}})$ factors through $\widehat{R}_{ell}$. The three morphisms between $\widehat{B}_3$, $\widehat{R}_{ell}$, and $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ are compatible with the actions of $G_{\mathbb{Q}}$, $\widehat{\mathrm{GT}}$, and $\widehat{\mathbb{Z}}^\times$; and the morphism $G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^\times$ factors through $\widehat{\mathrm{GT}}$. This ends the proof of Theorem 9.4.

### 9.5 Proof of Proposition 9.5

This statement follows from the form taken by the action of $\widehat{\mathrm{GT}}_{ell}$ on $\widehat{T}_{ell,n}$.

## 10 A question

In this section, we ask whether $\mathfrak{r}_{ell}$ is generated by the elements $\delta_{2n}$ arising from [7]. This question is analogous to the problem of whether $\mathfrak{grt}_1$ is generated by its Drinfeld generators, which is also open. We give an indication in favour of a positive answer: such an answer would imply a statement which is also implied by a transcendence conjecture about MZVs; this last conjecture would follow from Grothendieck's transcendence conjecture for the category of mixed Tate motives and the equality of the motivic Galois group $G_{MTM}(-)$ with $\mathrm{GT}(-)$ (see [2]). We record that in contrast with the fact that the Drinfeld generators of $\mathfrak{grt}_1$ generate a free Lie algebra (Brown), and several families of relations between the $\delta_{2n}$ have been found (see [28]).

### 10.1 A generation conjecture (GC)

The Drinfeld generators of $\mathfrak{grt}_1$ are obtained from the homogeneous decomposition of the logarithm of $\mathrm{im}(-1 \in \underline{\mathrm{GT}} \overset{j_{\Phi_{KZ}}}{\to} \mathrm{GRT}(\mathbb{C})) \cdot \mathrm{can}(-1)$, where $\mathrm{can} : \mathbb{C}^\times \to \mathrm{GRT}(\mathbb{C})$ is the canonical morphism. The analogue of the conjecture that these elements generate $\mathfrak{grt}_1$ is then:

**Conjecture 10.1** (Generation Conjecture) $\mathfrak{b}_3 \subset \mathfrak{r}_{ell}^{gr}$ *is an equality, i.e.,* $\mathfrak{r}_{ell}^{gr}$ *is generated by* $\mathfrak{sl}_2$ *and the* $\delta_{2n}$, $n \geq 0$.

This conjecture is equivalent to the inclusion $\exp(\hat{\mathfrak{b}}_3^{+,\mathbf{k}}) \rtimes \mathrm{SL}_2(\mathbf{k}) \subset R_{ell}^{gr}(\mathbf{k})$ being an equality.

**Proposition 10.2** *GC is equivalent to the Zariski density of* $B_3 \subset R_{ell}(-)$*, i.e.,* $\langle B_3 \rangle = R_{ell}(-)$.

*Proof* According to Lemma 3.19, $\langle B_3 \rangle$ is uniquely determined by its Lie algebra. This fact and Proposition 3.18 immediately imply that $\langle B_3 \rangle = R_{ell}(-)$ iff the inclusion $\mathrm{Lie}(u_+, u_-) = \langle \log \psi_+, \log \psi_- \rangle \subset \mathrm{Lie}\, R_{ell}(-)$ is actually an equality. Tensoring with $\mathbb{C}$, this holds iff

$$\langle \log \psi_+, \log \psi_- \rangle^{\mathbb{C}} \subset \hat{\mathfrak{r}}_{ell}^{\mathbb{C}}$$

is an equality. On the other hand, GC holds iff $\hat{\mathfrak{b}}_3^{+,\mathbb{C}} \subset \hat{\mathfrak{r}}_{ell}^{gr,\mathbb{C}}$ is an equality. Now $i_{e_{KZ}}$ sets up a diagram

$$\begin{array}{ccc} \langle \log \psi_+, \log \psi_- \rangle^{\mathbb{C}} & \hookrightarrow & \mathfrak{r}_{ell}^{\mathbb{C}} \\ \simeq\downarrow & & \downarrow\simeq \\ \hat{\mathfrak{b}}_3^{+,\mathbb{C}} & \hookrightarrow & \hat{\mathfrak{r}}_{ell}^{gr,\mathbb{C}} \end{array}$$

It follows that the upper inclusion is an equality iff the lower is. □

## 10.2 Relation with a transcendence conjecture

We first present the transcendence conjecture.

### *10.2.1 The coordinate ring of associators*

The functors $\{\mathbb{Q}\text{-rings}\} \to \{\text{sets}\}$, $\mathbf{k} \mapsto \underline{M}(\mathbf{k}), M(\mathbf{k})$ may be represented as follows. Let $\text{pent}_{\mathbf{k}} : \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}}) \to \exp(\hat{\mathfrak{t}}_3^{\mathbf{k}})$, $\text{hex} : \mathbf{k} \times \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}}) \to \exp(\hat{\mathfrak{t}}_3^{\mathbf{k}})$, $\text{dual} : \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}}) \to \exp(\hat{\mathfrak{t}}_3^{\mathbf{k}})$ be the maps $\text{pent}(\Phi) := \text{lhs}((24))^{-1}\,\text{rhs}((24))$, $\text{hex}(\mu, \Phi) := \text{lhs}((23))\,\text{rhs}((23))^{-1}$, $\text{dual}(\Phi) := \Phi\Phi^{3,2,1}$.

Let $B, B', C$ be homogeneous bases of $\mathfrak{f}_2, \mathfrak{t}_3, \mathfrak{t}_4$. Let $\mu, \phi_b$ $(b \in B)$ be free commutative variables and set $\mathbf{k}_0 := \mathbb{Q}[\varphi_b, b \in B]$, $\mathbf{k}_1 := \mathbb{Q}[\mu, \varphi_b, b \in B]$. Then $\mathbf{k}_0 \subset \mathbf{k}_1$. Let $\Phi := \exp(\sum_{b\in B} \varphi_b b) \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}_0}) \subset \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}_1})$. For $b' \in B'$, $c \in C$, define $\text{pent}_c, \text{dual}_{b'} \in \mathbf{k}_0 \subset \mathbf{k}_1$ by $\sum_{c\in C} \text{pent}_c\, c = \log(\text{pent}(\Phi))$, $\sum_{b'\in B'} \text{dual}_{b'}\, b' = \log(\text{dual}(\Phi))$, $\sum_{b'\in B'} \text{hex}_{b'}\, b' = \log(\text{hex}(\mu, \Phi))$.

We then set $\mathbb{Q}[\underline{M}] := \mathbf{k}_1/(\text{pent}_c, \text{dual}_{b'}, \text{hex}_{b'}, b' \in B', c \in C)$ and $\mathbb{Q}[M] := \mathbb{Q}[\underline{M}][\mu^{-1}] = \mathbb{Q}[\mu^{\pm 1}, \varphi_b, b \in B]/\{\text{ideal with the same generators}\}$. Then, for any $\mathbb{Q}$-ring $\mathbf{k}$, we have (functorial in $\mathbf{k}$) bijections
$$\begin{array}{ccc} M(\mathbf{k}) & \simeq & \text{Hom}_{\mathbb{Q}\text{-rings}}(\mathbb{Q}[M], \mathbf{k}) \\ \subset\downarrow & & \downarrow\subset \\ \underline{M}(\mathbf{k}) & \simeq & \text{Hom}_{\mathbb{Q}\text{-rings}}(\mathbb{Q}[\underline{M}], \mathbf{k}) \end{array}$$

### *10.2.2 The transcendence conjecture*

The KZ associator $(2\pi\,\mathrm{i}, \Phi_{KZ}) \in M(\mathbb{C})$ gives to a morphism $\varphi_{KZ} : \mathbb{Q}[M] \to \mathbb{C}$.

**Conjecture 10.3** *(Transcendence Conjecture) $\varphi_{KZ}$ is injective.*

Let $\mathbf{k}_{MZV} := \text{im}(\mathbb{Q}[M] \stackrel{\varphi_{KZ}}{\to} \mathbb{C})$. This is a subring of $\mathbb{C}$ (according to [18], this is the subring generated by $(2\pi\,\mathrm{i})^{\pm 1}$ and the MZVs).

## 10.3 Consequences of GC

**Proposition 10.4** *The inclusion*

$$i_e(B_3) \subset \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \text{SL}_2(\mathbb{C}) \tag{87}$$

*holds:*

*(a) for any* $e \in Ell(\mathbb{C}) \times_{M(\mathbb{C})} \{\Phi_{KZ}\}$ *iff* $\mathfrak{b}_3 \lhd \mathfrak{r}_{ell}^{gr}$*;*
*(b) for any* $e \in \sigma(M(\mathbb{C}))$ *iff* $[\sigma(\mathfrak{grt}), \mathfrak{b}_3] \subset \hat{\mathfrak{b}}_3$*;*
*(c) for any* $e \in Ell(\mathbb{C})$ *iff* $\mathfrak{b}_3 \lhd \mathfrak{grt}_{ell}$*, i.e., iff the two above-mentioned conditions are realized.*

*Moreover, GC implies that (87) holds for any* $e \in Ell(\mathbb{C})$*.*

We do not know whether the Lie algebraic statements in (a), (b), (c) hold, so they may be viewed as conjectures implied by GC.

*Proof* Note first that for any $e \in Ell(\mathbb{C})$, and by Zariski density, (87) $\Leftrightarrow$ $(i_e(\langle B_3\rangle(\mathbb{C})) = \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C}))$.

(a) is proved as follows. $e \in Ell(\mathbb{C}) \times_{M(\mathbb{C})} \{\Phi_{KZ}\}$ iff $e = \sigma(\Phi_{KZ}) * g$ for some $g \in R_{ell}^{gr}(\mathbb{C})$. So

$$\begin{aligned}
&((87) \text{ holds for any } e \in Ell(\mathbb{C}) \times_{M(\mathbb{C})} \{\Phi_{KZ}\})\\
&\Leftrightarrow (g(i_{\sigma(\Phi_{KZ})}(\langle B_3\rangle(\mathbb{C}))) = \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C}) \text{ for any } g \in R_{ell}^{gr}(\mathbb{C}))\\
&\Leftrightarrow (g(\Gamma) = \Gamma \text{ for any } g \in R_{ell}^{gr}(\mathbb{C}), \text{ where } \Gamma = \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C}))\\
&\Leftrightarrow (\mathfrak{b}_3 \lhd \mathfrak{r}_{ell}^{gr}).
\end{aligned}$$

Here, the second equivalence follows from Proposition 7.3.

(b), (c) are then proved in the same way, using

$$(e \in \sigma(M(\mathbb{C}))) \Leftrightarrow (e = \sigma(\Phi_{KZ}) * \sigma(g) \text{ for some } g \in \mathrm{GRT}(\mathbb{C})),$$
$$(e \in Ell(\mathbb{C})) \Leftrightarrow (e = \sigma(\Phi_{KZ}) * g \text{ for some } g \in \mathrm{GRT}_{ell}(\mathbb{C})).$$

The equivalence (c) $\Leftrightarrow$ ((a) and (b)) follows from $\mathfrak{grt}_{ell} = \sigma(\mathfrak{grt}) \oplus \mathfrak{r}_{ell}^{gr}$. Finally, GC means that $\langle \mathfrak{sl}_2, \delta_{2k}\rangle = \mathfrak{r}_{ell}^{gr}$, which immediately implies (a), (b), and (c) as $\mathfrak{r}_{ell}^{gr} \lhd \mathfrak{grt}_{ell}$. □

## 10.4 Consequences of the transcendence conjecture (TC)

**Proposition 10.5** *If TC holds, then for any* $\mathbb{Q}$*-ring* $\mathbf{k}$ *and any* $\Phi \in M(\mathbf{k})$, $i_{\sigma(\Phi)}(B_3) \subset \exp(\hat{\mathfrak{b}}_3^{+,\mathbf{k}}) \rtimes \mathrm{SL}_2(\mathbf{k})$.

*Proof* Recall that $\langle B_3\rangle(-) \hookrightarrow R_{ell}(-)$, $\exp(\hat{\mathfrak{b}}_3^{+}) \rtimes \mathrm{SL}_2(-) \hookrightarrow R_{ell}^{gr}(-)$ are inclusions of $\mathbb{Q}$-group schemes, and $Ell \to M$, $M \xrightarrow{\sigma} M$ are morphisms of $\mathbb{Q}$-group schemes.

In the notation of Definition 4.10, any $x \in X(\mathbf{k})$ gives rise to a morphism $i_x : G(\mathbf{k}) \to H(\mathbf{k})$, defined by $g * x = x * i_x(g)$ for any $g \in G(\mathbf{k})$. The assignment $x \mapsto i_x$ is functorial in the following sense: if $\mathbf{k} \to \mathbf{k}'$ is a morphism of $\mathbb{Q}$-rings and $x' := \mathrm{im}(x \in X(\mathbf{k}) \to X(\mathbf{k}'))$, then

$$\begin{array}{ccc} G(\mathbf{k}) & \stackrel{i_x}{\to} & H(\mathbf{k}) \\ \downarrow & & \downarrow \\ G(\mathbf{k}') & \stackrel{i_{x'}}{\to} & H(\mathbf{k}') \end{array}$$

commutes.

For any $\mathbb{Q}$-scheme $X$ and any $\mathbb{Q}$-ring $\mathbf{k}$, let $X \otimes \mathbf{k}$ be the $\mathbf{k}$-scheme $(X \otimes \mathbf{k})(\mathbf{k}') := X(\mathbf{k}')$ for any $\mathbf{k}' \in \{\mathbf{k}\text{-rings}\}$. Again with the notation of Definition 4.10, a torsor even gives rise to an assignment $X(\mathbf{k}) \ni x \mapsto (G \otimes \mathbf{k} \stackrel{i_x^{\mathbf{k}}}{\to} H \otimes \mathbf{k})$, where $i_x^{\mathbf{k}}$ is a morphism of $\mathbf{k}$-group schemes, defined by: $\forall \mathbf{k}' \in \{\mathbf{k}\text{-rings}\}$, $g * \bar{x} = \bar{x} * i_x^{\mathbf{k}}(g)$ for any $g \in (G \otimes \mathbf{k})(\mathbf{k}') = G(\mathbf{k}')$, where $\bar{x} := \mathrm{im}(x \in X(\mathbf{k}) \to X(\mathbf{k}'))$.

In particular, $\Phi_{KZ} \in M(\mathbf{k}_{MZV})$ gives rise to an isomorphism $i_{\sigma(\Phi_{KZ})} : R_{ell}(-) \otimes \mathbf{k}_{MZV} \stackrel{\sim}{\to} R_{ell}^{gr}(-) \otimes \mathbf{k}_{MZV}$, and therefore to a Lie algebra isomorphism Lie $i_{\sigma(\Phi_{KZ})} : \mathfrak{t}_{ell} \otimes \mathbf{k}_{MZV} \stackrel{\sim}{\to} (\mathfrak{t}_{ell}^{gr} \otimes \mathbf{k}_{MZV})^\wedge$, whose $\otimes_{\mathbf{k}_{MZV}} \mathbb{C}$ is the infinitesimal of the isomorphism of Proposition 7.3.

The group scheme inclusions $\langle B_3 \rangle(-) \subset R_{ell}(-)$ and $\exp(\hat{\mathfrak{b}}_3^+) \rtimes \mathrm{SL}_2 \subset R_{ell}^{gr}(-)$ give rise to Lie algebra inclusions $\mathrm{Lie}\langle B_3 \rangle(-) \subset \mathfrak{t}_{ell}$ and $\hat{\mathfrak{b}}_3^+ \subset \hat{\mathfrak{t}}_{ell}^{gr}$, and Proposition 7.3 implies that Lie $i_{\sigma(\Phi_{KZ})} \otimes_{\mathbf{k}_{MZV}} \mathbb{C}$ restricts to an isomorphism $\mathrm{Lie}\langle B_3 \rangle(-) \otimes_{\mathbb{Q}} \mathbb{C} \to (\mathfrak{b}_3 \otimes_{\mathbb{Q}} \mathbb{C})^\wedge$. This implies that Lie $i_{\sigma(\Phi_{KZ})}$ restricts to a Lie algebra isomorphism

$$\mathrm{Lie}\langle B_3 \rangle(-) \otimes_{\mathbb{Q}} \mathbf{k}_{MZV} \to (\mathfrak{b}_3 \otimes_{\mathbb{Q}} \mathbf{k}_{MZV})^\wedge.$$

There are Lie subalgebras $\mathbb{Q} \log \psi_\pm \otimes \mathbf{k}_{MZV}$ in the l.h.s., mapping to ($\mathbb{Q}e_\pm +$ terms of degree $> 0) \otimes \mathbf{k}_{MZV}$ in the r.h.s. (where $e_+ = e$, $e_- = f$, $\psi_+ = \psi$). This induces a diagram

$$\begin{array}{ccc} \langle B_3 \rangle(-) \otimes \mathbf{k}_{MZV} & \stackrel{i_{\sigma(\Phi_{KZ})}}{\to} & (\exp(\hat{\mathfrak{b}}_3^+) \rtimes \mathrm{SL}_2) \otimes \mathbf{k}_{MZV} \\ \uparrow & & \uparrow \\ \mathbb{G}_a \otimes \mathbf{k}_{MZV} & = & \mathbb{G}_a \otimes \mathbf{k}_{MZV} \end{array}$$

If now $\Phi \in M(\mathbf{k})$, the transcendence conjecture says that there exists a $\mathbb{Q}$-ring morphism $\mathbf{k}_{MZV} \stackrel{\varphi}{\to} \mathbf{k}$, such that $\Phi = \varphi_*(\Phi_{KZ})$. Applying this morphism to the above diagram, one gets

$$\begin{array}{ccc} \langle B_3 \rangle(-) \otimes \mathbf{k} & \stackrel{i_{\sigma(\Phi)}}{\to} & (\exp(\hat{\mathfrak{b}}_3^+) \rtimes \mathrm{SL}_2) \otimes \mathbf{k} \\ \uparrow & & \uparrow \\ \mathbb{G}_a \otimes \mathbf{k} & = & \mathbb{G}_a \otimes \mathbf{k} \end{array}$$

Taking $\mathbf{k}$-points, one obtains a commutative diagram

$$\begin{array}{ccc} \langle B_3 \rangle(\mathbf{k}) & \stackrel{i_{\sigma(\Phi)}}{\to} & \exp(\hat{\mathfrak{b}}_3^{+,\mathbf{k}}) \rtimes \mathrm{SL}_2(\mathbf{k}) \\ \uparrow & & \uparrow \\ \mathbf{k} & = & \mathbf{k} \end{array}$$

The image of $1 \in \mathbf{k}$ is $\Psi_\pm \subset \langle B_3\rangle(\mathbf{k})$; then $i_{\sigma(\Phi)}(\Psi_\pm) \in \exp(\hat{\mathfrak{b}}_3^{+,\mathbf{k}}) \rtimes \mathrm{SL}_2(\mathbf{k}) \subset \exp(\hat{\mathfrak{b}}_3^{+,\mathbb{C}}) \rtimes \mathrm{SL}_2(\mathbb{C})$. □

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