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# Higher-order conservation laws for the nonlinear Poisson equation via characteristic cohomology

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**Abstract** We study higher-order conservation laws of the nonlinearizable elliptic Poisson equation as elements of the characteristic cohomology of the associated exterior differential system. The theory of characteristic cohomology determines a normal form for differentiated conservation laws by realizing them as elements of the kernel of a linear differential operator. We show that the $\mathbb{S}^1$-symmetry of the PDE leads to a normal form for the undifferentiated conservation laws. Zhiber and Shabat (in Sov Phys Dokl Akad 24(8):607–609, 1979) determine which potentials of nonlinearizable Poisson equations admit nontrivial Lie–Bäcklund transformations. In the case that such transformations exist, they introduce a pseudo-differential operator that can be used to generate infinitely many such transformations. We obtain similar results using the theory of characteristic cohomology: we show that for higher-order conservation laws to exist, it is necessary that the potential satisfies a linear second-order ODE. In this case, at most two new conservation laws in normal form appear at each even prolongation. By using a recursion motivated by Killing fields, we show that, for the simplest class of potentials, this upper bound is attained. The recursion circumvents the use of pseudo-differential operators. We relate higher-order conservation laws to generalized symmetries of the exterior differential system by identifying their generating functions. This Noether correspondence provides the connection between conservation laws and the canonical Jacobi fields of Pinkall and Sterling.

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## 1 Introduction

A select set of elliptic Poisson equations

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = -f(u), \tag{1}$$

where $u : \mathbb{C} \to \mathbb{R}$, is central in the study of submanifold geometries: When

$$f(u) = \begin{cases} -\frac{(\epsilon+\delta^2)}{4}\sinh(2u) & \text{if } \epsilon+\delta^2 > 0 \\ -\frac{(\epsilon+\delta^2)}{4}\cosh(2u) & \text{if } \epsilon+\delta^2 < 0 \\ e^{-2u} & \text{if } \epsilon+\delta^2 = 0 \end{cases} \tag{2}$$

then Eq. (1) arises as the Gauss equation for a surface of constant mean curvature $-2\delta$ in a three-dimensional space form of constant sectional curvature $\epsilon$. When

$$f(u) = e^{-2u} - e^{u},$$

Eq. (1) is the Gauss equation for a special Legendrian surface in $\mathbb{S}^5$. In all of these cases, the metric on the surface is locally given by $e^{2u}\mathrm{d}z \circ \mathrm{d}\bar{z}$. Once one has a solution $u(z, \bar{z})$ of Eq. (1), the map of the surface into the space form can be recovered by solving a system of ODE. It is also well known that the hyperbolic equation $u_{xt} = \sin(u)$, where $x, t$ are coordinates on $\mathbb{R}^2$, is the Gauss equation for surfaces in $\mathbb{R}^3$ with Gauss curvature equal to $-1$.

For all of the potentials $f(u)$ listed above, Eq. (1) is often referred to as a *soliton equation* or *integrable system* and is known to have many special properties, including a loop-group formulation, infinitely many conserved quantities, and a description of solutions using algebraic geometry (the spectral curve). The literature on soliton equations is fascinating but also sprawling and tangled.

Integrable systems of the form in Eq. (1) underlie the simplest cases of primitive maps from Riemann surfaces into $k$-symmetric spaces [10]. For this perspective, the reader might consult the articles by Uhlenbeck [39], Pinkall and Sterling [33], Hitchin [25], Bobenko [1], Burstall [11], Bolton et al. [2], Dai and Terng [15], McIntosh [29], and the references within. Hyperbolic equations of the form $u_{tx} = f(u)$ fit into the hierarchies developed by Terng and Uhlenbeck [37,38]. All of these references use the fact that soliton equations can be phrased in terms of flat connections on a Riemann surface.

A markedly different approach using recursion operators was initiated by Lenard (a private communication cited in [20]) and Olver [31,32] and later developed and formalized by, for example, Guthrie [23], Dorfman [17], and Sanders and Wang [34].

Yet another approach to investigating integrable systems is through the theory of characteristic cohomology developed by Bryant and Griffiths [4]. They cite Vinogradov (see the references in [4]) as their main influence, but the theory of characteristic cohomology, and in particular its formulation in the special case of Euler–Lagrange systems, is also closely related to the work of Shadwick using the Hamilton–Cartan formalism ([35] and the references within) and the work of Olver [32].

Thus far, the theory of characteristic cohomology has mostly been used as a method for classifying partial differential equations, or more generally, exterior differential systems (EDS). In [4,5,7,8,13,14,40], scalar parabolic and hyperbolic PDE for 2 and 3 independent variables are studied (using the method of equivalence and the characteristic cohomology) in terms of the dimension of the space of conservation laws. Bryant, Griffiths, and Hsu [4,7–9] make many interesting suggestions for other ways in which it might be used, including, for example, to study boundaries of integral manifolds and to study singularities. Motivated by this, the first author introduced an elementary approach to studying boundaries of integral manifolds using conservation laws in [19].

The references to the literature given above are by no means exhaustive or even representative. They were highlighted to give examples of other approaches that turn out to have close links to the theory of characteristic cohomology. It is not clear, for example, how the existence of hydrodynamic reductions (see [18] and the references within) relate to the existence of conservation laws. No doubt there are many more approaches and many more connections to be made between the various techniques in the literature.

In [6] (Proposition 4.6), it is shown that the nonlinear Poisson equation $\Delta u = f(u)$, where $u : \mathbb{R}^n \to \mathbb{R}$ and $\Delta$ is the Laplacian, admits no nonclassical conservation laws if $n \geq 3$ and $f_{uu} \neq 0$. On the other hand, the class of equations $\Delta u = f(u)$ with $n = 2$ and $u$ and $f(u)$ vector valued encompass the Toda equations, which are known to be integrable [2]. It does not appear to be known whether higher-order conservation laws exist for the nonlinear Poisson equation when $n \geq 3$ and $u$ and $f(u)$ are vector valued.

In this article, we study the (possibly infinite-dimensional) space of conservation laws of Eq. (1) using the characteristic cohomology. We show that for there to exist higher-order conservation laws, it is necessary that $f$ satisfies a linear second-order ODE. In [42], Zhiber and Shabat reach the same classification by looking for Lie–Bäcklund transformations. We give a complete and explicit description of the conservation laws in terms of the characteristic cohomology in the case that $f_{uu} = \beta f$ and $f$ does not satisfy a linear ODE. A forthcoming article by the first author completes the description for the case that $f_{uu} = \alpha f_u + 2\alpha^2 f$, which corresponds to the Tzitzeica equation. We find that in this case the conservation laws for Eq. (1) are equivalent to those studied by Olver in the hyperbolic case [31], though his characterization is not complete because he does not prove that the necessary recursion operator is always well defined, nor does he prove that the method would produce the complete set of conservation laws. The generating functions for the conservation laws are equivalent to the integrals in [42]. There is also some overlap with the work of Dodd and Bullough [16], though they also do not address the issue of completeness.

The conservation laws turn out to be equivalent to the canonical Jacobi fields of [33] and thus to the formal Killing fields of [12]. This is not surprising given the Noether correspondence between generalized symmetries and conservation laws (see Sect. 10). In a future article, we will describe how the characteristic cohomology can be used to recapture the notion of finite type solutions [12,33]. We will also elaborate on the relationship between conservation laws and formal/polynomial Killing fields for primitive map systems.

We conclude this section with a sketch of the remainder of the article. In Sect. 2, we reformulate Eq. (1) as an exterior differential system, Eq. (3), and present the structure equations for the $k^{th}$-prolongation $(M^{(k)}, \mathcal{I}^{(k)})$, allowing $k = \infty$. An $\mathbb{S}^1$-symmetry of the PDE leads to an $\mathbb{S}^1$-symmetry of $(M^{(k)}, \mathcal{I}^{(k)})$ and to the notion of weighted degree for functions and differential forms. We also introduce an almost complex structure $J$ on a codimension-one subbundle of $T^*M^{(k)}$ which leads to $\partial$ and $\overline{\partial}$ operators.

In Sect. 3, we present the basic definition of classical and higher-order conservation laws for an EDS, in both their differentiated and undifferentiated forms. The classical conservation laws for Eq. (3) are presented in Sect. 4. In Sect. 5, we use the general theory [4] to obtain the first approximation to the (differentiated) conservation laws. In Sect. 6, we refine the first approximation, obtaining an exact formula for differentiated conservation laws in normal form in terms of a generating function, which is a solution to an (overdetermined) system of linear PDE, Eqs. (13) and (14). Complicated calculations that would be necessary to directly verify that this formula does in fact convert solutions of Eqs. (13) and (14) into conservation laws (i.e. to show that the thus defined differential forms are closed) are circumvented by studying (weighted) homogeneous conservation laws in Sect. 7. The $\mathbb{S}^1$-symmetry of the EDS allows one to produce from the differentiated conservation laws a normal form for undifferentiated conservation laws—something that has not appeared in the general theory but is likely to be generally applicable to systems that have a gauge symmetry. See Sect. 11.

In Sect. 8, we use the normal form of undifferentiated conservation laws to show that any solution to Eqs. (13) and (14) defines a nontrivial conservation law. Furthermore, we show that there is an at most one-dimensional complex space of solutions of Eqs. (13) and (14) for each odd weighted degree, none of nonzero even weighted degree, and that these solutions are either 'holomorphic' or 'anti-holomorphic' polynomials in the derivatives $\frac{\partial^i u}{\partial z^i}$.

In Sect. 9, we investigate the space of solutions of Eqs. (13) and (14) under certain assumptions on $f$. We prove that if $f$ does not satisfy a linear second-order ODE, no higher-order conservation laws exist. When $f_{uu} = \beta f$ and $f$ does not satisfy any first-order ODE, we prove the existence of the maximal possible number of generating functions and hence determine the complete (infinite-dimensional) space of conservation laws. We also provide examples of higher-order conservation laws for the case when $f_{uu} = \alpha f_u + 2\alpha^2 f$. In this case, a coordinate change of Eq. (1) transforms it to the Tzitzeica equation $u_{z\bar{z}} = e^u - e^{-2u}$.

In Sect. 10, we show that generalized symmetries of $(M^{(\infty)}, \mathcal{I}^{(\infty)})$ are determined by generating functions that are solutions to Eq. (14), though they need not satisfy Eq. (13). This leads to a limited version of Noether's theorem that explains the relationship between conservation laws and the canonical Jacobi fields of Pinkall and Sterling [33]. Section 11 contains some concluding remarks.

## 2 The EDS and its prolongations

To begin, we encode the PDE as an exterior differential system (EDS) with independence condition. For a basic introduction to EDS, see [3] or [27]. Recall that an **exterior differential system** consists of a smooth manifold $M$ and a homogeneous differential ideal $\mathcal{I} \subset \bigoplus_p \Omega^p(M, \mathbb{C})$. An **integral manifold** of $(M, \mathcal{I})$ is an immersed submanifold $\iota : N \to M$ such that $\iota^*(\mathcal{I}) = 0$. If the ideal is generated by forms $\alpha_i$ (and their exterior derivatives since it is a *differential* ideal), we will write $\mathcal{I} = \langle \alpha_i \rangle$. For any set of 1-forms $\beta_i \in \Omega^1(M, \mathbb{C})$, we use $\{\beta_i\} \subset \Omega^1(M, \mathbb{C})$ to denote the subbundle they span. If $\alpha \in \mathcal{I}$ is a complex valued differential form, then by $\iota^*(\alpha) = 0$ we mean that both the real and imaginary parts pull back to the real manifold $N$ to be zero.

In order to encode Eq. (1) as an EDS, let $M = \mathbb{C} \times \mathbb{R} \times \mathbb{C}$ have coordinates $(z, u, u_0)$ and define the differential forms

$$\begin{aligned}
\zeta &= \mathrm{d}z \\
\omega_1 &= \mathrm{d}u_0 + f\bar{\zeta} \\
\eta_0 &= \mathrm{d}u - u_0\zeta - \bar{u}_0\bar{\zeta} \\
\psi &= \mathrm{Im}(\zeta \wedge \omega_1) = -\frac{\sqrt{-1}}{2}(\zeta \wedge \omega_1 - \bar{\zeta} \wedge \bar{\omega}_1).
\end{aligned}$$

The reader may recognize $M$ as the first jet space of maps $u : \mathbb{C} \to \mathbb{R}$. The desired differential ideal is $\mathcal{I} = \langle \eta_0, \psi \rangle$. We calculate that

$$\begin{aligned}
\mathrm{d}\eta_0 &= 2\,\mathrm{Re}(\zeta \wedge \omega_1) = \zeta \wedge \omega_1 + \bar{\zeta} \wedge \bar{\omega}_1 \\
\mathrm{d}\psi &= -\sqrt{-1}\, f_u \eta_0 \wedge \zeta \wedge \bar{\zeta}.
\end{aligned}$$

Thus, the differential ideal can be expressed as

$$\mathcal{I} = \langle \eta_0, \zeta \wedge \omega_1 \rangle.$$

One checks that a surface $\iota : \mathbb{C} \to M$ for which $\iota^*(\zeta \wedge \bar{\zeta}) \neq 0$ and $\iota^*(\eta_0) = 0$ is the 1-jet of a function $u : \mathbb{C} \to \mathbb{R}$ with $u_0 = \frac{\partial u}{\partial z}$ and $\bar{u}_0 = \frac{\partial u}{\partial \bar{z}}$. If in addition $\iota^*(\psi) = 0$, then the function $u(z, \bar{z})$ is a solution to Eq. (1). Thus, solutions to Eq. (1) correspond to integral surfaces $(N, \iota)$ such that $\iota^*(\zeta \wedge \bar{\zeta}) \neq 0$.

The goal of this article is to study the conservation laws of the EDS

$$(M, \mathcal{I}), \quad \text{where} \quad M = \mathbb{C}^2 \times \mathbb{R} \quad \text{and} \quad \mathcal{I} = \langle \eta_0, \psi \rangle, \tag{3}$$

*and* its prolongations. This EDS is involutive with Cartan characters $s_0 = 1$, $s_1 = 2$, $s_2 = 0$. Again, see [3,27] for the basics of EDS.

We outline the process of the first prolongation. For a general discussion of the prolongation process, see [3] or [27]. We comment that for the system under study, all of the integral elements are regular and so the process of prolongation is well defined. However, we will define all of the higher prolongations explicitly and so the reader need not be concerned about the general theory. We will restrict our attention to integral manifolds corresponding to solutions of Eq. (1). If $\iota : N \to M$ is such an integral manifold, then its tangent space at $\iota(n)$ is a real 2-plane $\iota_*(T_n N) \subset T_{\iota(n)} M$ on which the ideal pulls back to be zero and $\iota^*(\zeta \wedge \bar{\zeta}) \neq 0$. Any real 2-plane $E \subset T_{\iota(n)} M$ on which $\zeta \wedge \bar{\zeta} \neq 0$ is defined by relations

$$\eta_0|_E = c_1\zeta + \bar{c}_1\bar{\zeta}$$
$$\omega_1|_E = c_2\zeta + c_3\bar{\zeta}$$

for some complex numbers $c_1, c_2, c_3$. The ideal $\mathcal{I} = \langle \eta_0, \zeta \wedge \omega_1 \rangle$ vanishes on $E$ if and only if $c_1 = c_3 = 0$. Thus, the space of possible tangent planes to integral manifolds is parametrized by one complex number, which we will call $u_1$, via the conditions $\eta_0|_E = 0$ and $\omega_1|_E = u_1\zeta$.

Let $M^{(1)} = M \times \mathbb{C}$ and let $u_1$ be a holomorphic coordinate on $\mathbb{C}$. Define the complex one-form $\eta_1 = \omega_1 - u_1\zeta$ and the subbundle $\mathrm{I}^{(1)} = \{\eta_0, \eta_1, \bar{\eta}_1\} \subset \Omega^1(M^{(1)}, \mathbb{C})$, which generates a differential ideal $\mathcal{I}^{(1)}$. The new system $(M^{(1)}, \mathrm{I}^{(1)})$ is the first prolongation of $(M, \mathcal{I})$ with respect to the independence condition $\zeta \wedge \bar{\zeta} \neq 0$. Thus, we construct the prolongation by adjoining a new coordinate parametrizing the possible tangent spaces to integral manifolds and introducing tautological 1-forms that vanish on potential tangent planes to integral manifolds.

So what is the meaning of $u_1$? It contains the new second-order information of $u(z, \bar{z})$: The vanishing of $\eta_0 = \mathrm{d}u - u_0\zeta - \bar{u}_0\bar{\zeta}$ implies that $u_0 = \frac{\partial u}{\partial z}$. The vanishing of $\eta_1 = \mathrm{d}u_0 - u_1\zeta + f\bar{\zeta}$ implies that $u_1 = \frac{\partial u_0}{\partial z}$ and $-f = \frac{\partial u_0}{\partial \bar{z}}$. The first tells us that $u_1 = \frac{\partial^2 u}{\partial z^2}$, the new second derivative information on $u$, and the second of these recaptures the PDE condition that was encoded in the vanishing of $\psi$.

Now using the fact that $\mathrm{d}\eta_1$ must vanish on solutions of $(M^{(1)}, \mathcal{I}^{(1)})$, one can find the possible tangent planes of solutions of $(M^{(1)}, \mathcal{I}^{(1)})$ and in the same way as before construct the second prolongation. Let $M^{(k)}$ denote the $k^{th}$-prolongation. It is not hard to see that $M^{(k+1)} = M^{(k)} \times \mathbb{C}$ and we will always use $u_{k+1}$ for the new holomorphic coordinate on $M^{(k+1)}$. Furthermore, on a (real) two-dimensional integral manifold $\iota : N \to M$ for which $\iota^*(\zeta \wedge \bar{\zeta}) \neq 0$,

$$\iota^*(u_i) = \frac{\partial^{i+1} u}{\partial z^{i+1}}.$$

By calculating the first few prolongations, one is motivated to define complex functions and forms

$$\begin{aligned} & & \omega_{i+1} &= \mathrm{d}u_i + T^i\bar{\zeta} \\ T^0 &= f & \eta_0 &= \mathrm{d}u - u_0\zeta - \bar{u}_0\bar{\zeta} \\ T^{i+1} &= \textstyle\sum_{j=0}^{i} \binom{i}{j} u_{i-j} T_u^j & \eta_{i+1} &= \omega_{i+1} - u_{i+1}\zeta \\ & & \tau^i &= \textstyle\sum_{j=0}^{i} \binom{i}{j} T_u^j \eta_{i-j}. \end{aligned}$$

The real and imaginary parts of $\zeta, \eta_0, \ldots, \eta_k, \omega_{k+1}$ form a coframe of $M^{(k)}$ and

$$\mathrm{I}^{(k)} = \{\eta_0, \eta_1, \bar{\eta}_1, \ldots, \eta_k, \bar{\eta}_k\} \subset \Omega^1(M^{(k)}, \mathbb{C})$$

generates the ideal $\mathcal{I}^{(k)}$. The vector fields on $M^{(k)}$ dual to this coframe are

$$\begin{array}{lcl} e^k_{-1} = \frac{\partial}{\partial z} + u_0\frac{\partial}{\partial u} + \sum_{i=0}^{k-1} u_{i+1}\frac{\partial}{\partial u_i} - \sum_{i=0}^{k} \overline{T}^i \frac{\partial}{\partial \bar{u}_i} & & \zeta \\ e_0 = \frac{\partial}{\partial u} & \longleftrightarrow & \eta_0 \\ e_i = \frac{\partial}{\partial u_{i-1}} \quad i = 1\dots k+1 & & \eta_i \ (i = 1\dots k),\ \omega_{k+1} \end{array}$$

and their complex conjugates.[1]

To compute the structure equations, we will need

**Lemma 2.1** *For $i, j \geq 0$, we have*

1. $T^i = (e^k_{-1})^i f$ *for* $k \geq i$
2. $T^{i+j+1}_{u_j} = \binom{i+j+1}{i} T^i_u$.

*Proof* We use induction and the binomial identity $\binom{i-1}{j} + \binom{i-1}{j-1} = \binom{i}{j}$ for both formulas. We illustrate the calculation for the second formula only, making use of the first identity in the calculation. Suppose that the second formula holds for all $n < i+1$ and all $j$. Then,

$$\begin{aligned} T^{i+j+1}_{u_j} &= e_{j+1}e^k_{-1}T^{i+j} = (e^k_{-1}e_{j+1} + e_j)T^{i+j} \\ &= \binom{i+j}{i-1} e^k_{-1}T^{i-1}_u + \binom{i+j}{i} T^i_u = \binom{i+j+1}{i} T^i_u. \end{aligned}$$

In the last equality, we used the fact that $[e^k_{-1}, e_0]T^{i-1} = 0$ because $[e^k_{-1}, e_0]$ is in the span of the $e_{\bar{i}}$, which annihilate $T^r$ for all $r$. □

Thus, on an integral manifold $T^i = \frac{\partial^i f}{\partial z^i}$. Using Lemma 2.1, it is not hard to compute that

$$[e^k_{-1}, e^k_{\overline{-1}}] = \overline{T}^{k+1}\bar{e}_{k+1} - T^{k+1}e_{k+1}, \tag{4}$$

which will be needed later.

**Proposition 2.2** *For $1 \leq i \leq k$ the following structure equations are satisfied on $M^{(k)}$:*

$$\begin{aligned} \mathrm{d}T^i &\equiv T^{i+1}\zeta + \tau^i \mod \bar{\zeta} \\ \mathrm{d}\zeta &= 0 \\ \mathrm{d}\eta_0 &= \zeta \wedge \eta_1 + \bar{\zeta} \wedge \bar{\eta}_1 \\ \mathrm{d}\omega_{k+1} &= \tau^k \wedge \bar{\zeta} + T^{k+1}\zeta \wedge \bar{\zeta} \\ \mathrm{d}\eta_i &= -\eta_{i+1} \wedge \zeta + \tau^{i-1} \wedge \bar{\zeta} \end{aligned}$$

[1] Although the notation $e^k_0$ and $e^k_i$ would be more correct because, for example, $e^k_0$ and $e^{k'}_0$ are vector fields on different manifolds, we drop the indexing of the prolongation because the same formula holds and there are natural inclusions and surjections between $M^{(k)}$ and $M^{(k')}$ which identify the corresponding vector fields. We leave the superscript on $e^k_{-1}$ because this vector field does change from prolongation to prolongation.

*Proof* The second and the third equation follows directly from the definitions and the last two equations follow easily from the first. To prove the first equation, we calculate, using Lemma 2.1 to differentiate $T^i$:

$$\mathrm{d}T^i \equiv T^{i+1}\zeta + \sum_{j=0}^{i} T^i_{u_{j-1}}\eta_j \equiv T^{i+1}\zeta + \sum_{j=0}^{i} \binom{i}{j} T^{i-j}_u \eta_j \equiv T^{i+1}\zeta + \tau^i,$$

where all equivalences are modulo $\bar{\zeta}$. □

It will also be convenient to work on the infinite prolongation $M^{(\infty)}$ (see Section 4.3.1 of [6]). The infinite-dimensional space $M^{(\infty)}$ is the inverse limit of the sequence

$$\left\{\cdots \to M^{(k)} \overset{\pi_k}{\to} M^{(k-1)} \overset{\pi_{k-1}}{\to} \cdots \to M^{(1)} \overset{\pi_1}{\to} M^{(0)}\right\};$$

that is,

$$M^{(\infty)} = \left\{(p_0, p_1, \ldots) \in M^{(0)} \times M^{(1)} \times \cdots : \pi_k(p_k) = p_{k-1} \ \text{ for each } k \geq 1\right\}.$$

Let $\pi_{(k)} : M^{(\infty)} \to M^{(k)}$ be the natural surjections

$$\pi_{(k)}(p_0, p_1, \ldots) = (p_0, p_1, \ldots, p_k).$$

A smooth function or differential form on $M^{(\infty)}$ is given by the pullback via $\pi_{(k)}$ of a corresponding object on some finite prolongation. On $M^{(\infty)}$, the real and imaginary parts of $\zeta, \eta_0, \eta_1, \ldots$ form a coframe and the dual vector fields on $M^{(\infty)}$ are the real and imaginary parts of

$$\begin{array}{lll} e_{-1} = \frac{\partial}{\partial z} + u_0 \frac{\partial}{\partial u} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i} - \sum_{i=0}^{\infty} \overline{T}^i \frac{\partial}{\partial \bar{u}_i} & & \zeta \\ e_0 = \frac{\partial}{\partial u} & \leftrightarrow & \eta_0 \\ e_i = \frac{\partial}{\partial u_{i-1}} \quad i = 1 \ldots k+1 & & \eta_i \quad (i = 1, 2, \ldots) \end{array}$$

The ideal is generated by the (formally Frobenius) subbundle

$$\mathrm{I}^{(\infty)} = \{\eta_0, \eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2, \ldots\}.$$

Note that if $F \in \Omega^0(M^{(k)}, \mathbb{C})$, then $\pi^*_{(k)}(e^k_{-1}F) = e_{-1}(\pi^*_{(k)}(F))$.

Thus far, we have calculated a coframe, its dual frame, and the structure equations of an arbitrary prolongation of $(M, \mathcal{I})$. We now turn to some of the special structures on $(M^{(k)}, \mathcal{I}^{(k)})$ that arise due to the ellipticity and the $\mathbb{S}^1$-symmetry.

The PDE Eq. (1) is invariant under the $\mathbb{S}^1$-action $(u, z, \bar{z}) \to (u, \lambda z, \overline{\lambda z})$ (with $\lambda \in \mathbb{C}$ and $|\lambda| = 1$). This leads to a symmetry of $(M^{(k)}, \mathcal{I}^{(k)})$, which yields a decomposition of differential forms and thus conservation laws. To see this, let $F : \mathbb{S}^1 \times M^{(k)} \to M^{(k)}$ be defined as

$$F(\lambda, u, z, u_j) = (u, \lambda^{-1}z, \lambda^{j+1}u_j). \tag{5}$$

For $p \geq 0$ and $j \in \mathbb{Z}$, we define the spaces of differential forms of **homogeneous weighted degree** $j$ to be

$$\Omega^p_j(M^{(k)}) = \left\{\varphi \in \Omega^p(M^{(k)}, \mathbb{C}) \mid F^*\varphi = \lambda^j\varphi\right\}. \tag{6}$$

For an element $\varphi \in \Omega^p_j(M^{(k)}, \mathbb{C})$, we write $\mathrm{wd}(\varphi) = j$. Note that $\mathrm{wd}(\varphi) = -\mathrm{wd}(\overline{\varphi})$. This grading is preserved by exterior differentiation:

$$\mathrm{d} : \Omega^p_j(M^{(k)}) \to \Omega^{p+1}_j(M^{(k)}).$$

We will use the $\mathbb{S}^1$-symmetry and grading in Sections 7 and 8. For later reference, we note that

$$\begin{array}{ll}
\mathrm{wd}(z) = -1 & \mathrm{wd}(\bar{z}) = 1 \\
\mathrm{wd}(u_j) = +(j+1) & \mathrm{wd}(\bar{u}_j) = -(j+1) \\
\mathrm{wd}(u) = 0 & \mathrm{wd}(\eta_0) = 0 \\
\mathrm{wd}(\zeta) = -1 & \mathrm{wd}(\bar{\zeta}) = 1 \\
\mathrm{wd}(\omega_j) = +j & \mathrm{wd}(\bar{\omega}_j) = -j \\
\mathrm{wd}(\eta_j) = +j & \mathrm{wd}(\bar{\eta}_j) = -j.
\end{array}$$

The ellipticity of Eq. (1) leads us to the following

**Definition 2.3** Define the subspaces $\Omega^{(1,0)}(M^{(k)}) = \mathbb{C} \cdot \{\zeta, \eta_1, \ldots, \eta_k, \omega_{k+1}\}$ and $\Omega^{(0,1)}(M^{(k)}) = \mathbb{C} \cdot \{\bar{\zeta}, \bar{\eta}_1, \ldots, \bar{\eta}_k, \bar{\omega}_{k+1}\}$, and in the standard way also $\Omega^{(p,q)}(M^{(k)})$. We define the operators $\partial : C^\infty(M^{(k)}, \mathbb{C}) \to \Omega^{(1,0)}(M^{(k)})$ and $\overline{\partial} : C^\infty(M^{(k)}, \mathbb{C}) \to \Omega^{(0,1)}(M^{(k)})$ as

$$\partial A = e^k_{-1}(A)\zeta + \sum_{i=1}^{k} A_{u_{i-1}}\eta_i + A_{u_k}\omega_{k+1}$$

$$\overline{\partial} A = e^k_{\overline{-1}}(A)\bar{\zeta} + \sum_{i=1}^{k} A_{u_{i-1}}\bar{\eta}_i + A_{u_k}\bar{\omega}_{k+1},$$

allowing for $k = \infty$.

It will be convenient to use the following linear operator

$$J : \Omega^1(M^{(k)}) \to \Omega^1(M^{(k)}),$$

which acts by $\sqrt{-1}$ on $\Omega^{(1,0)}(M^{(k)})$, by $-\sqrt{-1}$ on $\Omega^{(0,1)}(M^{(k)})$, and as the identity on $\mathbb{R} \cdot \eta_0$. This is an almost complex structure on the annihilator of $e_0$.

## 3 Conservation laws as elements of the characteristic cohomology

Let $(M, \mathcal{I})$ be an involutive exterior differential system with maximal integral submanifolds of dimension $n$ and characteristic number $l$. The characteristic number is computed from $\mathcal{I}$ using linear algebra. See Section 4.2 of [4] for the definition. It is 1 for Eq. (3). Its characteristic cohomology is defined to be

$$H_0^p(M, \Omega/\mathcal{I}),$$

that is, the cohomology for the complex $\Omega/\mathcal{I}$ with differential $\bar{\mathrm{d}} : \Omega^p/(\mathcal{I} \cap \Omega^p) \to \Omega^{p+1}/(\mathcal{I} \cap \Omega^{p+1})$ induced by the standard exterior derivative. The subscript 0 indicates that we are working on the zeroth prolongation. We say that we are in the **local** case if $H^p_{dR}(M, \mathbb{R}) = 0$ for $p > 0$. In [4], it is shown that in the local involutive case $H_0^p(M, \Omega/\mathcal{I}) = 0$ for $p < n - l$. The first nontrivial group is of special interest.

**Definition 3.1** The space of **classical undifferentiated conservation laws** for $(M, \mathcal{I})$ is $H_0^{n-l}(M, \Omega/\mathcal{I})$.

*Remark* For the system Eq. (3), we have $n = 2$ and $l = 1$ so that the space of classical undifferentiated conservation laws is $H_0^1(M, \Omega/\mathcal{I})$.

In addition to the quotient complex $(\Omega/\mathcal{I}, \bar{\mathrm{d}})$, we also have the subcomplex $(\mathcal{I} \cap \Omega^p(M, \mathbb{R}), \mathrm{d})$ and its cohomology $H_0^p(M, \mathcal{I})$. To calculate the conservation laws in the local case, one uses the isomorphism

$$H_0^{n-l}(M, \Omega/\mathcal{I}) \cong H_0^{n-l+1}(M, \mathcal{I})$$

which follows from the long exact sequence in cohomology, which is induced by the short exact sequence

$$0 \to \mathcal{I} \to \Omega \to \Omega/\mathcal{I} \to 0$$

and the fact that $H^p(M, \mathbb{R}) = 0$ for $p > 0$.

**Definition 3.2** The space of **classical differentiated conservation laws** for $(M, \mathcal{I})$ is $H_0^{n-l+1}(M, \mathcal{I})$.

As stated above, for the case at hand, $l = 1$. An element of $H_0^{n-l+1}(M, \mathcal{I})$ is a closed $(n - l + 1)$-form in the ideal and we only care about it modulo d of $(n - l)$-forms in the ideal. The characteristic cohomology machinery developed in [4] identifies the space of conservation laws as the kernel of a linear differential operator (as opposed to elements of a quotient space), much as one finds harmonic representatives of de Rham classes in Hodge theory.

On $(M^{(\infty)}, \mathcal{I}^{(\infty)})$, one has the associated characteristic cohomology, which we abbreviate as $\bar{H}^p := H^p(M^{(\infty)}, \Omega/\mathcal{I})$. We continue to restrict to the local case, $H^p(M^{(k)}, \mathbb{R}) = 0$ for $k \geq 0$ and $p > 0$.

**Definition 3.3** The space of **higher-order undifferentiated conservation laws** for $(M, \mathcal{I})$ is

$$\bar{H}^{n-l} := H^{n-l}(M^{(\infty)}, \Omega/\mathcal{I}).$$

The space of **higher-order undifferentiated** *complex* **conservation laws** for $(M, \mathcal{I})$ is

$$\bar{H}^{n-l}_{\mathbb{C}} := H^{n-l}(M^{(\infty)}, \Omega_{\mathbb{C}}/\mathcal{I}_{\mathbb{C}}),$$

where the subscript $\mathbb{C}$ denotes complexification.

Any element of $\bar{H}^{n-l}$ is represented by an element of $\Omega^{n-l}(M^{(\infty)}, \mathbb{R})$ which, by definition, is the pullback of an element in $\Omega^{n-l}(M^{(k)}, \mathbb{R})$ under $\pi_{(k)} : M^{(\infty)} \to M^{(k)}$ for some $k$. Again, we can study the conservation laws via the isomorphism $H^{n-l}(M^{(k)}, \Omega/\mathcal{I}) \cong H^{n-l+1}(M^{(k)}, \mathcal{I}^{(k)})$ because $H^p(M^{(k)}, \mathbb{R}) = 0$ for $p > 0$.

**Definition 3.4** The space of **higher-order differentiated conservation laws** for $(M, \mathcal{I})$ is $H^{n-l+1}(M^{(\infty)}, \mathcal{I})$. The space of **higher-order differentiated** *complex* **conservation laws** for $(M, \mathcal{I})$ is $H^{n-l+1}(M^{(\infty)}, \mathcal{I}_{\mathbb{C}})$.

Exterior differentiation provides isomorphisms

$$\mathrm{d} : \bar{H}^{n-l} \stackrel{\cong}{\to} H^{n-l+1}(M^{(\infty)}, \mathcal{I})$$
$$\mathrm{d} : \bar{H}^{n-l}_{\mathbb{C}} \stackrel{\cong}{\to} H^{n-l+1}(M^{(\infty)}, \mathcal{I}_{\mathbb{C}})$$

in the local involutive case. We recall again that for the case at hand, $l = 1$.

## 4 Classical conservation laws

For the system in Eq. (3), the maximum integral manifolds are of dimension 2 and the characteristic number is 1. In the notation of the last section, $n = 2$ and $l = 1$. Thus, a classical differentiated conservation law is represented by a closed form in $\mathcal{I} \cap \Omega^2(M, \mathbb{R})$. Any 2-form in $\mathcal{I}$ can be written as

$$\Phi' = \rho \wedge \eta_0 + A\psi + B\mathrm{d}\eta_0$$

for some 1-form $\rho$ and functions $A$ and $B$. This can be rewritten as

$$\Phi' = (\rho - \mathrm{d}B) \wedge \eta_0 + A\psi + \mathrm{d}(B\eta_0).$$

As we are only interested in the class $[\Phi'] \in H^2(M, \mathcal{I})$, we need only concern ourselves with finding a 1-form $\rho$ and a function $A$ such that

$$\Phi = \rho \wedge \eta_0 + A\psi$$

is closed and $\Phi \neq \mathrm{d}\alpha$ for $\alpha \in \mathcal{I}$. It is easy to check that for any $\Phi$ of the form given, $\Phi \neq \mathrm{d}(g\eta_0)$ for any function $g$. Examining $\eta_0 \wedge \mathrm{d}\Phi = 0$ uncovers that $\rho \equiv -\frac{1}{2} J\mathrm{d}A$ mod $\eta_0$. Then, considering the terms in $\mathrm{d}\Phi = 0$ that involve $\eta_0$ uncovers the condition

$$\frac{1}{2}\mathrm{d}J\mathrm{d}A - \sqrt{-1} f_u A\zeta \wedge \bar{\zeta} + A_u\psi \equiv 0 \mod \eta_0.$$

By studying the coefficients of this vanishing 2-form, one can check that if one has $\log(f)_{uu} \neq 0$ and $f_{uu} \neq 0$, then the only conservation laws are given by setting

$$A = P + \overline{P}$$
$$P = au_0 + \sqrt{-1}bzu_0$$

where $a \in \mathbb{C}$ and $b \in \mathbb{R}$ are arbitrary constants.

In Sect. 7, we will introduce a systematic way of finding undifferentiated conservation laws from differentiated ones. It will fail only for the classical conservation laws with $a = 0$ and $b \neq 0$. For that reason, we present here the 1-form

$$\varphi_0 = G\eta_0 + E\zeta + \overline{E}\bar{\zeta} \tag{7}$$

with $G = -(zu_0 + \bar{z}\bar{u}_0)$ and $E = -\frac{1}{2}zu_0^2 + \bar{z}\int f$. It satisfies $\mathrm{d}\varphi_0 = \Phi$ when we take $P = \sqrt{-1}zu_0$ in the definition of $\Phi$. Notice that $\varphi_0$ is also an element of $\Omega^1_0(M, \mathbb{R})$, a fact used in Sect. 8.

In order to look for higher-order conservation laws—that is, conservation laws of the prolonged system—we will make use of some spectral sequence machinery which we now describe.

## 5 The first approximation of the characteristic cohomology

The material in this section is based on Sections 1.3, 2.1–2.4, and 5.1 of [4]. Let $(M^{(\infty)}, \mathcal{I}^{(\infty)})$ be the infinite prolongation of an involutive EDS $(M, \mathcal{I})$. Assume that $H^p_{dR}(M^{(k)}, \mathbb{R}) = 0$ for all $p > 0$ and for all $k \geq 0$ so that we are in the local involutive case. The system in Eq. (3) is in the local involutive case. Let $\Omega^p = \Omega^p(M^{(\infty)}, \mathbb{C})$ and let $\mathrm{I}^{(\infty)} = \mathcal{I}^{(\infty)}_{\mathbb{C}} \cap \Omega^1(M^{(\infty)}, \mathbb{C})$. Then, define

$$F^p\Omega^q = \mathrm{Im}\{\mathrm{I}^{(\infty)} \wedge \ldots \wedge \mathrm{I}^{(\infty)} \wedge \Omega^{q-p} \to \Omega^q\}$$
$$\overline{\Omega}^{p,q} = F^p\Omega^q / F^{p+1}\Omega^q.$$

Let $(E_r^{p,q}, \mathrm{d}_r)$ denote the spectral sequence associated with this filtration [22] whose first two terms are

$$E_0^{p,q} = \overline{\Omega}^{p,q}$$
$$E_1^{p,q} = H(E_0^{p,q}, \mathrm{d}_0) = \frac{\{\phi \in F^p\Omega^q : \mathrm{d}\phi \in F^{p+1}\Omega^{q+1}\}}{\mathrm{d}F^p\Omega^{q-1} \oplus F^{p+1}\Omega^q}.$$

Notice that

$$H^q(\Omega/\mathcal{I}^{(\infty)}, \bar{\mathrm{d}}) = E_1^{0,q}.$$

We study this space indirectly using the spectral sequence $(E_r^{p,q}, \mathrm{d}_r)$, which includes the complex

$$0 \longrightarrow E_1^{0,q} \xrightarrow{\mathrm{d}_1} E_1^{1,q+1} \xrightarrow{\mathrm{d}_1} E_1^{2,q+2} \cdots$$

From Equation (4) of Section 4.2 in [4], we know that this sequence is exact at $E_1^{0,q}$ and $E_1^{1,q+1}$ so that the characteristic cohomology $H^q(\Omega/\mathcal{I}^{(\infty)}, \bar{\mathrm{d}})$ is isomorphic to

$$\ker\{\mathrm{d}_1 : E_1^{1,q+1} \to E_1^{2,q+2}\}.$$

A second spectral sequence allows one to obtain a first approximation to the kernel of $\mathrm{d}_1$. For this, a weight filtration $wt$ is introduced. This weighting system will not be used after this section and is distinct from the notion of weighted degree (wd) defined in Sect. 2 and used throughout the paper. While we refer the reader to Section 2.4 of [4] for the definition of $wt$, we record here the properties needed to make the necessary computations for Eq. (3). Let $f \neq 0$ be a smooth function and $\varphi$ and $\psi$ smooth differential forms. Then,

$$\begin{aligned} wt(f\varphi) &= wt(\varphi) \\ wt(\varphi \wedge \psi) &\leq wt(\varphi) + wt(\psi) \\ wt(\varphi + \psi) &\leq \max(wt(\varphi), wt(\psi)). \end{aligned}$$

The second spectral sequence is obtained from the following filtration:

$$\overline{F}_k \overline{\Omega}^{p,q} = \{\phi \in \overline{\Omega}^{p,q} : wt(\phi) \leq k\} \subset \overline{\Omega}^{p,q},$$

We denote the associated graded spaces as $\overline{\Omega}_k^{p,q} := \overline{F}_k \overline{\Omega}^{p,q} / \overline{F}_{k-1} \overline{\Omega}^{p,q}$. This quotient complex has the associated cohomology groups

$$\mathcal{H}_k^{p,q} = H^q(\overline{\Omega}_k^{p,*}, \delta),$$

where $\delta$ is the differential induced by exterior differentiation. For each fixed $p > 0$, there is a spectral sequence $\{\bar{E}_r^{k,q}\}$ associated with the weight filtration that converges to $E_1^{p,q}$ and which satisfies

$$\bar{E}_1^{k,q} = \mathcal{H}_k^{p,q}.$$

Thus computing $\mathcal{H}_k^{p,q}$ gives us a first approximation of the form of a conservation law. The importance of this step is that $\delta$ is linear over functions and computing $\mathcal{H}_k^{p,q}$ is a purely algebraic process depending only on the principal symbol of the EDS.

Now suppose that $(M^{(\infty)}, \mathcal{I}^{(\infty)})$ is the infinite prolongation of Eq. (3). Then, using the definition of $wt$ in Section 2.4 of [4], we obtain the following weighting system:

$$wt(\zeta) = wt(\bar{\zeta}) = -1 \quad wt(\eta_0) = 1 \quad wt(\eta_j) = wt(\bar{\eta}_j) = j \quad for\ j > 0.$$

To compute the necessary cohomology groups, we need the spaces

$$
\begin{aligned}
&\overline{\Omega}^{1,1}_{-1} = 0 && \overline{\Omega}^{1,2}_{-2} = 0\\
&\overline{\Omega}^{1,1}_{0} = 0 && \overline{\Omega}^{1,2}_{-1} = 0\\
&\overline{\Omega}^{1,1}_{1} = \mathbb{C}\cdot\{\eta_0, \eta_1, \bar{\eta}_1\} && \overline{\Omega}^{1,2}_{0} = \mathbb{C}\cdot\{\eta_0\wedge\zeta, \eta_0\wedge\bar{\zeta}, \eta_1\wedge\zeta, \eta_1\wedge\bar{\zeta}, \bar{\eta}_1\wedge\zeta, \bar{\eta}_1\wedge\bar{\zeta}\}\\
&\overline{\Omega}^{1,1}_{j} = \mathbb{C}\cdot\{\eta_j, \bar{\eta}_j\} && \overline{\Omega}^{1,2}_{j} = \mathbb{C}\cdot\{\eta_{j+1}\wedge\zeta, \eta_{j+1}\wedge\bar{\zeta}, \bar{\eta}_{j+1}\wedge\zeta, \bar{\eta}_{j+1}\wedge\bar{\zeta}\} \ \text{ for } \ j > 1
\end{aligned}
$$

Easy calculation uncovers that the only nonzero cohomology group for the complexes $(\overline{\Omega}^{p,q}_k, \delta)$ is

$$
\mathcal{H}^{1,2}_0 = \mathbb{C}\cdot\{\eta_0\wedge\zeta, \eta_0\wedge\bar{\zeta}, \zeta\wedge\eta_1, \bar{\zeta}\wedge\bar{\eta}_1\}.
$$

This implies that a conservation law can be represented by a form

$$
\tilde{\Phi} \equiv R\eta_0\wedge\zeta + S\eta_0\wedge\bar{\zeta} + A''\,\mathrm{Re}(\zeta\wedge\eta_1) + B''\,\mathrm{Im}(\zeta\wedge\eta_1) \qquad \mathrm{mod} \quad F^2\Omega^2
$$

for some functions $R$, $S$, $A''$, $B''$. This can be rewritten as

$$
\begin{aligned}
\tilde{\Phi} &\equiv R\eta_0\wedge\zeta + S\eta_0\wedge\bar{\zeta} + A'\mathrm{d}\eta_0 + B'\psi\\
&\equiv R\eta_0\wedge\zeta + S\eta_0\wedge\bar{\zeta} + \eta_0\wedge\mathrm{d}A' + B'\psi + \mathrm{d}(A'\eta_0) \quad \mathrm{mod} \quad F^2\Omega^2
\end{aligned}
$$

or, as we will continue with, it can be written as

$$
\Phi \equiv \eta_0\wedge\rho + A\psi \qquad \mathrm{mod} \quad F^2\Omega^2 + \mathrm{d}(F^1\Omega^1)
$$

for some 1-form $\rho$ and function $A$, where we have left off the exact piece because that does not alter its class in $H^2(M^{(\infty)}, \mathcal{I}^{(\infty)})$. In the next section, we remove the congruence, finding how the closure of $\Phi$ determines $\rho$ and the other coefficients in terms of $A$, as well as equations that $A$ must satisfy. However, to prove the closure of $\Phi$ directly requires verifying some elaborate equations. We circumvent this in Sect. 8 by using the normal form for undifferentiated conservation laws found in Sect. 7.

## 6 The normal form of the differentiated conservation laws

We say that $f$ satisfies an $n$th-order autonomous linear ODE if it satisfies an equation of the form

$$
\frac{\mathrm{d}^n f}{\mathrm{d}u^n} = Z\left(f, \frac{\mathrm{d}f}{\mathrm{d}u}, \frac{\mathrm{d}^2 f}{\mathrm{d}u^2}, \ldots, \frac{\mathrm{d}^{n-1} f}{\mathrm{d}u^{n-1}}\right) \tag{8}
$$

where $Z$ is an $\mathbb{R}$-linear function of $n$ variables. From now on assume that $f$ does not satisfy any first-order linear autonomous ODE. We make the following definition.

**Definition 6.1** A representative $\Phi \in \mathcal{I}^{(\infty)} \cap \Omega^2(M^{(\infty)}, \mathbb{R})$ of a differentiated conservation law on $M^{(\infty)}$ is in **normal form** if

$$\begin{aligned}\Phi &= \eta_0 \wedge \rho + A\psi \\ &\quad + \sum_{1\le i<j\le k} \left( B^{ij}\eta_i \wedge \eta_j + \overline{B}^{ij}\bar{\eta}_i \wedge \bar{\eta}_j \right) + \sum_{1\le i\le j\le k} \left( D^{ij}\eta_i \wedge \bar{\eta}_j + \overline{D}^{ij}\bar{\eta}_i \wedge \eta_j \right).\end{aligned}$$

for some $k$ and some functions $A, B^{ij}, D^{ij} : M^{(\infty)} \to \mathbb{C}$ with $\overline{A} = A$.

There is an analogy between the role of conservation laws in normal form and harmonic representatives of de Rham cohomology classes which we now recall. The de Rham cohomology of a smooth closed manifold $X$ consists of the *quotient* groups

$$H^p_{dR}(X, \mathbb{R}) = \frac{\ker(\mathrm{d} : \Omega^p(X, \mathbb{R}) \to \Omega^{p+1}(X, \mathbb{R}))}{\mathrm{Im}(\mathrm{d} : \Omega^{p-1}(X, \mathbb{R}) \to \Omega^p(X, \mathbb{R}))}. \tag{9}$$

Hodge theory shows that if one has a Riemannian metric, then one can represent these quotient spaces as subspaces of $\Omega^p(X, \mathbb{R})$ in a natural way—each class in the quotient space has a unique harmonic representative. Analogously, *elements of the characteristic cohomology have unique representatives in normal form* [4].

**Definition 6.2** Let $\mathcal{C} \subset \Omega^2(M^{(\infty)}, \mathbb{R}) \cap \mathcal{I}^{(\infty)}$ denote the space of representatives of differentiated conservation laws in normal form. A conservation law on $M^{(\infty)}$ in normal form is said to have **level** $k$ if it is defined on $M^{(k)}$. Let $\mathcal{C}_{(k)}$ denote the space of representatives of conservation laws of level $k$ in normal form.

Pulling back with $\pi : M^{(k+1)} \to M^{(k)}$ induces the inclusion $\pi^*\mathcal{C}_{(k)} \subset \mathcal{C}_{(k+1)}$, allowing us to identify $\mathcal{C}_{(k)}$ as a subspace of $\mathcal{C}_{(k+1)}$. Using this identification, we have $\mathcal{C} = \bigcup_k \mathcal{C}_{(k)}$. In a series of lemmas, we will prove the following proposition, in which the normal form is further refined. In this section, we will not prove the existence of elements of $\mathcal{C}_{(k)}$; the proposition only tells us what the elements of $\mathcal{C}_{(k)}$ must look like if they do exist.

**Proposition 6.3** *Any element of* $\mathcal{C}_{(k)}$ *is of the form*

$$\Phi = \eta_0 \wedge \rho + A\psi + \sum_{1\le i<j\le k} \left( B^{ij}\eta_i \wedge \eta_j + \overline{B}^{ij}\bar{\eta}_i \wedge \bar{\eta}_j \right). \tag{10}$$

*The one-form* $\rho$ *and the function* $B$ *are determined by* $A$ *via the formulas*

$$\rho = -\frac{1}{2} J\mathrm{d}A \tag{11}$$

$$B^{ij} = \sqrt{-1} \sum_{m=0}^{k-j-i+1} (-1)^{m-i+1} \binom{m+i-1}{i-1} (e_{-1})^m A_{u_{m+j+i-1}} \tag{12}$$

*if we normalize $\rho$ so that $e_0 \lrcorner \rho = 0$. The function $A$ on $M^{(k)}$—which we henceforth refer to as the* **generating function** *of $\Phi$—satisfies*

$$A_{u_i,\overline{u_j}} = A_u = 0 \tag{13}$$

*and*

$$\mathcal{E}(A) := e_{\overline{-1}}e_{-1}A + f_u A = 0. \tag{14}$$

*Remark* The normal form of the differentiated conservation laws can be anticipated. From its definition, $\psi \in \Omega^{(2,0)}(M^{(k)}) \oplus \Omega^{(0,2)}(M^{(k)})$ so that modulo $\eta_0$, $\Phi \in \Omega^{(2,0)}(M^{(k)}) \oplus \Omega^{(0,2)}(M^{(k)})$. When $f = 0$ Eq. (1) is Laplace's equation, and then (3) is an integrable extension of the EDS for holomorphic curves in $\mathbb{C}^2$. Differentiated conservation laws of the EDS for holomorphic curves in $\mathbb{C}^n$ are closed forms in $\Omega^{(2,0)}(\mathbb{C}^n) \oplus \Omega^{(0,2)}(\mathbb{C}^n)$ [4].

For notational simplicity, extend the index set of $B^{ij}$ to infinity by setting $B^{ij} = 0$ unless $1 \le i < j \le k$.

Let $\Phi \in \mathcal{C}_{(k)}$. To prove the proposition, we unravel the consequences of $\mathrm{d}\Phi = 0$. First, we examine the weaker condition $\eta_0 \wedge \mathrm{d}\Phi = 0$.

**Lemma 6.4** *A conservation law $\Phi$ is of type $(2,0)+(0,2)$ modulo $\eta_0$; in other words, $D^{ij} = 0$ for all $1 \le i, j \le k$, and thus $\Phi$ is of the form in Eq.* (10).

*Proof* For $i = 1 \dots k$, the $\eta_0 \wedge \bar{\zeta} \wedge \eta_i \wedge \bar{\eta}_{k+1}$-coefficient of $\eta_0 \wedge \mathrm{d}\Phi$ is $D^{ik}$, so $D^{ik} = 0$. Now assume that $D^{i,k-r} = 0$ for $r = 0 \dots j$ and $i \le k - r$. We show that $D^{i,k-j-1}$ for $i \le k - j - 1$. The coefficient of $\eta_0 \wedge \bar{\zeta} \wedge \eta_i \wedge \bar{\eta}_{k-j}$ when $i < k - j - 1$ is $D^{i,k-j-1}$ plus terms that vanish by the induction hypothesis. When $i = k - j - 1$, the coefficient is $D^{k-j-1,k-j-1} - \overline{D}^{k-j-1,k-j-1}$. But $D^{l,l}$ is imaginary since $\Phi$ is real. □

**Lemma 6.5** *The following identities hold:*

$$\begin{aligned} \rho &\equiv -\frac{1}{2} J \mathrm{d}A \equiv -\frac{\sqrt{-1}}{2}\left(\partial A - \bar{\partial} A\right) && \mod \eta_0 \\ A_{u_{j-1}} &= -\sqrt{-1}(e_{-1}B^{1j} + B^{1,j-1}) && j = 2 \dots k \\ A_{u_k} &= -\sqrt{-1} B^{1k} \end{aligned}$$

*Proof* We express $\rho$ in the standard coframe as

$$\rho = \rho^0 \eta_0 + \rho^{-1}\zeta + \rho^{\overline{-1}}\bar{\zeta} + \sum_{i=1}^{k} (\rho^i \eta_i + \rho^{\bar{i}} \bar{\eta}_i).$$

Taking $\rho^0 = 0$, which we are free to do, we calculate

$$0 = e_1 \lrcorner e_{-1} \lrcorner \mathrm{d}\Phi \equiv \rho - \rho^{-1}\zeta - \rho^1\eta_1 - \frac{\sqrt{-1}}{2}(\mathrm{d}A - e_{-1}(A)\zeta - A_{u_0}\eta_1) + \sum_{j=2}^{k} e_{-1}B^{1j}\eta_j + \sum_{j=2}^{k} B^{1j}\eta_{j+1} \mod \eta_0. \tag{15}$$

This implies that

$$\rho^{(0,1)} = \frac{\sqrt{-1}}{2}\bar{\partial}A.$$

The first statement follows from the identities $\rho \equiv \rho^{(1,0)} + \rho^{(0,1)} \mod \eta_0$ and $\overline{\rho^{(0,1)}} = \rho^{(1,0)}$. The other two relations follow from the vanishing of the coefficients of $\eta_i$, $\eta_{k+1}$ in Eq. (15). □

**Lemma 6.6** *Equation* (13) *is true: we have* $A_u = 0$ *and* $A_{u_i,\overline{u_j}} = 0$ *for all* $i$, $j$.

*Proof* The coefficient of $\eta_0 \wedge \zeta \wedge \eta_1$ in $\mathrm{d}\Phi$ is $\frac{\sqrt{-1}}{2}(-e_1e_{-1}A + e_{-1}e_1A - A_u)$. Together with the commutator $[e_{-1}, e_1]A = -A_u$, this proves $A_u = 0$. The coefficient of $\eta_0 \wedge \eta_i \wedge \bar{\eta}_j$ is $\frac{\sqrt{-1}}{2}(-e_{\bar{j}}e_iA - e_ie_{\bar{j}}A)$. Combined with the commutator $[e_i, e_{\bar{j}}] = 0$, this proves the second claim. □

The vanishing $A_u = 0$ allows us to express $\rho$ exactly:

**Corollary 6.7** *Equation* (11) *is true: we have* $\rho = -\frac{1}{2}J\mathrm{d}A$.

**Lemma 6.8** *The* $B^{ij}$ *are given by Eq.* (12):

$$B^{ij} = \sqrt{-1} \sum_{m=0}^{k-j-i+1} (-1)^{m-i+1} \binom{m+i-1}{i-1} (e_{-1})^m A_{u_{m+j+i-1}}.$$

Therefore, if $A$ is weighted-homogeneous (cf. Eq. 6), then $B^{ij}$ is weighted-homogeneous and $\mathrm{wd}(B^{ij}) = \mathrm{wd}(A) - i - j$.

*Proof* First of all note that for $1 < i \leq j \leq k$, the $\eta_0 \wedge \zeta \wedge \eta_i \wedge \eta_{j+1}$-coefficient in $\eta_0 \wedge \mathrm{d}\Phi = 0$ is

$$B^{ij} + B^{i-1,j+1} + e_{-1}B^{i,j+1} = 0. \tag{16}$$

In particular, $B^{ik} = 0$ for $i > 1$, which is compatible with the formula to be proven.

We prove the lemma by induction on $i$. For $i = 1$, we have to show that for any $j = 2 \ldots k$

$$B^{1j} = \sqrt{-1} \sum_{m=0}^{k-j} (-1)^m (e_{-1})^m A_{u_{m+j}}, \tag{17}$$

which we prove by induction on $j$, going down from $k$ to 2. For $j = k$, the right-hand side is $\sqrt{-1}A_{u_k}$, which equals $B^{1k}$ by the third item of Lemma 6.5. Assume we have shown Eq. (17) for some $j$, then by the second item of Lemma 6.5,

$$\begin{aligned} B^{1,j-1} &= \sqrt{-1}A_{u_{j-1}} - e_{-1}B^{1j} \\ &= \sqrt{-1}A_{u_{j-1}} + \sqrt{-1}\sum_{m=0}^{k-j}(-1)^{m+1}(e_{-1})^{m+1}A_{u_{m+j}} \\ &= \sqrt{-1}\sum_{m=0}^{k-j+1}(-1)^m(e_{-1})^m A_{u_{m+j-1}}. \end{aligned}$$

Assume now that $i > 1$ is such that the formula for $B^{i-1,j}$ is true for all $j = 1 \dots k$. We will prove the formula for $B^{ij}$ by induction on $j$ as in the case $i = 1$. Above, we argued that it is correct for $j = k$, and assuming that $j$ is such that the formula for $B^{i,j+1}$ is correct we use Eq. (16) to prove the formula for $B^{ij}$:

$$\begin{aligned} \sqrt{-1}B^{ij} &= -\sqrt{-1}B^{i-1,j+1} - \sqrt{-1}e_{-1}B^{i,j+1} \\ &= (-1)^i A_{u_{j+i-1}} \\ &\quad + \sum_{m=1}^{k-j-i+1}(-1)^{m-i}\left[\binom{m+i-2}{i-2} + \binom{m+i-2}{i-1}\right](e_{-1})^m A_{u_{m+j+i-1}} \\ &= \sum_{m=0}^{k-j-i+1}(-1)^{m-i}\binom{m+i-1}{i-1}(e_{-1})^m A_{u_{m+j+i-1}}. \end{aligned}$$

□

The following unassuming corollary has important consequences that will be unraveled in Sect. 8.

**Corollary 6.9** *If $k$ is odd, then $B^{1k} = A_{k+1} = 0$.*

*Proof* For $i + j = k + 1$, Lemma 6.8 gives

$$B^{ij} = (-1)^{i+1}\sqrt{-1}A_{u_k}.$$

Writing $k + 1 = 2n$ and choosing $i = j = n$, $B^{n,n} = 0$ implies that $A_{u_k} = 0$ and consequently, taking $i = 1$ and $j = k$, that $B^{1k} = 0$. □

Finally, we deduce Eq. (14). The coefficient of $\eta_0 \wedge \zeta \wedge \bar{\zeta}$ in $\mathrm{d}\Phi = 0$ is

$$e_{-1}e_{\overline{-1}}A + e_{\overline{-1}}e_{-1}A + 2f_u A = 0.$$

Since we are working on $M^{(\infty)}$, we have $[e_{-1}, e_{\overline{-1}}]A = 0$, which allows us to rewrite this as

$$e_{\overline{-1}}e_{-1}A + f_u A = 0.$$

This completes the proof of Proposition 6.3.

## 7 Homogeneity and a normal form for undifferentiated conservation laws

Exterior differentiation commutes with the decomposition

$$\Omega^p(M^{(\infty)}, \mathbb{C}) = \bigoplus_{d\in\mathbb{Z}} \Omega^p_d(M^{(\infty)}),$$

where the space $\Omega^p_d(M^{(\infty)})$ of differential $p$-forms of homogeneous weighted degree $d$ was defined in Eq. (6). Let $\mathcal{C}_d$ be the image of the projection $\pi_d$ from $\mathcal{C} \subset \Omega^2(M^{(\infty)}, \mathbb{R}) \subset \Omega^2(M^{(\infty)}, \mathbb{C})$ to $\Omega^2_d(M^{(\infty)})$.

**Lemma 7.1** *The $\mathcal{C}_d$ are complex subspaces of $\Omega^2_d(M^{(\infty)})$ and $\mathcal{C} \otimes \mathbb{C} = \bigoplus_d \mathcal{C}_d$. Furthermore, if $\Phi_d \in \mathcal{C}_d$, then $\Phi_d + \overline{\Phi_d} \in \mathcal{C}$.*

*Proof* As mentioned above, any representative of a differentiated conservation law in normal form can be decomposed into weighted-homogeneous pieces, so we have $\mathcal{C} \subset \bigoplus_d \mathcal{C}_d$.

Let $\Phi_d \in \mathcal{C}_d$. By definition, $\Phi_d = \pi_d(\Phi)$ for some representative $\Phi \in \mathcal{C}$. Since $\Phi$ is a real-valued form and $\mathrm{wd}(\Psi) = -\mathrm{wd}(\overline{\Psi})$ for all weighted-homogeneous forms $\Psi$, we have $\pi_{-d}(\Phi) = \overline{\Phi_d}$. Since summands of different weighted degree cannot cancel, it follows that $\Phi_d + \overline{\Phi_d}$ is in normal form and hence an element of $\mathcal{C}$. But then, for any $b \in \mathbb{C}$, it follows that $b\Phi_d + \overline{b\Phi_d} \in \mathcal{C}$, so $b\Phi_d \in \mathcal{C}_d$, and $\mathcal{C}_d$ is a complex subspace.

In the last argument, taking $b = \sqrt{-1}$ implies that $\sqrt{-1}\Phi_d - \sqrt{-1}\overline{\Phi_d} \in \mathcal{C}$, so that $\Phi_d - \overline{\Phi_d} \in \mathcal{C} \otimes \mathbb{C}$. Therefore, $\Phi_d = \frac{1}{2}(\Phi_d + \Phi_d) + \frac{1}{2}(\Phi_d - \Phi_d) \in \mathcal{C} \otimes \mathbb{C}$ and $\bigoplus_d \mathcal{C}_d \subset \mathcal{C} \otimes \mathbb{C}$. From $\mathcal{C} \subset \bigoplus_d \mathcal{C}_d$, it follows that $\mathcal{C} \otimes \mathbb{C} \subset \bigoplus_d \mathcal{C}_d$ and so we can conclude that $\mathcal{C} \otimes \mathbb{C} = \bigoplus_d \mathcal{C}_d$. □

Given a conservation law $\Phi \in \mathcal{C}$ in normal form

$$\Phi = \eta_0 \wedge \rho + A\psi + B^{ij}\eta_i \wedge \eta_j + \overline{B}^{ij}\bar{\eta}_i \wedge \bar{\eta}_j$$

as in Proposition 6.3, and writing $A = \sum_{d\geq 0}(P_d + \overline{P_d})$ with $\mathrm{wd}(P_d) = d$ then

$$\Phi_{P_d} := \pi_d(\Phi) = \eta_0 \wedge \rho + P_d\psi + B^{ij}(P_d)\eta_i \wedge \eta_j + \overline{B^{ij}(\overline{P_d})}\bar{\eta}_i \wedge \bar{\eta}_j \in \mathcal{C}_d$$

with $\rho = -\frac{1}{2}J\mathrm{d}P_d$ and $B^{ij}(P_d)$ and $B^{ij}(\overline{P_d})$ being given by Eq. (12) using $P_d$, resp. $\overline{P_d}$, in place of $A$ in the formula. Using the weighted homogeneity of $\Phi_{P_d}$, we will produce a canonical representative of a class in $\bar{H}^1_{\mathbb{C}}$ from the normal form of a class in $\mathcal{C}_d$. To simplify notation, we drop the subscript $d$ on $P$ but continue to assume that $\mathrm{wd}(P) = d$.

Let $F$ be the $\mathbb{S}^1$-action defined in Eq. (5) and $v = \frac{dF}{dt}\big|_{t=0}$ where $\lambda = e^{it}$. One can calculate directly that

$$v = i\left(qe_0 + \bar{z}e_{\overline{-1}} - ze_{-1} + (e_{-1})^j(q)e_j + (e_{\overline{-1}})^j(q)e_{\bar{j}}\right),$$

where $q = zu_0 - \overline{zu_0}$. For $\mathrm{wd}(P) = d \neq 0$, define

$$\varphi_P = \frac{1}{d}\left(v \lrcorner \Phi_P\right). \tag{18}$$

**Lemma 7.2** *If* $\mathrm{wd}(P) = d \neq 0$ *and* $\mathrm{d}\Phi_P = 0$*, then* $\Phi_P = \mathrm{d}\varphi_P$*.*

*Proof* Suppose $\Phi_p$ is closed and homogeneous. Then,

$$d \cdot \Phi_P = \frac{\partial(F^*\Phi_P)}{\partial t}|_{t=0} = \mathcal{L}_v\Phi_P = \mathrm{d}(v \lrcorner \Phi_P).$$

□

The formulas for $\varphi_P$ and $v$ lead to

$$\varphi_P \equiv \frac{\sqrt{-1}}{2d} J\left(P\mathrm{d}q - q\mathrm{d}P\right) \mod \mathrm{I}^{(\infty)}, \tag{19}$$

which we use in Lemma 8.15.

*Remark* It is simple to check that, given any function $G$ on $M^{(\infty)}$ satisfying Eq. (14) (but not necessarily Eq. (13) ), then $[J(q\mathrm{d}G - G\mathrm{d}q)] \in \bar{H}^1$. It remains to show that it is a nontrivial element. Furthermore, one still obtains a conservation law if one replaces $q$ with any solution to Eq. (14). This structure is closely related to the Poisson bracket defined in Theorem 4 of [35] though we do not pursue this further here.

We can now define canonical representatives for elements of $\bar{H}^1$. For $d \neq 0$, let $\mathcal{H}^1_d$ be the image of the linear map

$$\begin{aligned} \mathcal{C}_d &\to \Omega^1_d(M^{(\infty)}, \mathbb{C}) \\ \Phi_P &\mapsto \varphi_P \end{aligned}$$

and let $\mathcal{H}^1_0 = \mathbb{R} \cdot \varphi_0$, where $\varphi_0$ is defined in Eq. (7). Then, let $\mathcal{H}^1_{\mathbb{C}} = \bigoplus_d \mathcal{H}^1_d$ and $\mathcal{H}^1 = \mathcal{H}^1_{\mathbb{C}} \cap \Omega^1(M^{(\infty)}, \mathbb{R})$.

**Definition 7.3** The **normal form** for an undifferentiated conservation law in $\bar{H}^1$ is the representative $\varphi \in \Omega^1(M^{(\infty)}, \mathbb{R})$ lying in $\mathcal{H}^1$.

*Remark* It would be interesting to find a definition of $\mathcal{H}^1$ that is independent of $\mathcal{C}$.

*Remark* The elements of $\mathcal{H}^1$ are not invariant under translations in a lattice in the $z$-plane, even if $u(z, \bar{z})$ is. One can prove that there are translation invariant representatives that therefore induce cohomology classes on the torus domains of doubly periodic solutions $u(z, \bar{z})$. We will report on this and its implications in a forthcoming article.

## 8 The space of conservation laws

So far we have seen that $u_0$ and $q = zu_0 - \bar{z}\bar{u}_0$ are solutions to Eqs. (13) and (14). These equations preserve weighted homogeneity and so to understand their solutions, it is enough to understand the weighted-homogeneous solutions.

**Definition 8.1** Let $V_d$ be the space of solutions to $\mathcal{E}(P) = 0$ (Eq. (14)) of weighted degree $d$ that also satisfy $P_{u_i,\bar{u}_j} = P_u = 0$.

*Example 8.2* It is easy to check that $u_0 \in V_1$, $\bar{u}_0 \in V_{-1}$, and $q \in V_0$.

In this section, we prove

**Theorem 8.3** *Suppose that $f$ does not satisfy a linear first-order ODE. Then,*

1. *$V_0$ is spanned by $q$. If $d$ is a nonzero even integer, then $V_d = 0$. If $d$ is odd, then* $\dim_{\mathbb{C}} V_d \leq 1$.
2. *For all $d$, we have isomorphisms*

$$\begin{aligned} V_d &\to \mathcal{H}^1_d \to \mathcal{C}_d \\ P &\mapsto \varphi_P \mapsto \Phi_P, \end{aligned}$$

*where $\varphi_P$ is defined as in Sect. 7, and the second map is just the exterior derivative.*

3. $\dim_{\mathbb{R}}(\mathcal{C}_{(2n+1)}/\mathcal{C}_{(2n)}) = 0$.
4. $\dim_{\mathbb{R}}(\mathcal{C}_{(2n+2)}/\mathcal{C}_{(2n)}) \leq 2$ *with equality if and only if* $\dim_{\mathbb{C}}(V_{2n+3}) = 1$.

We prove this theorem via the series of Lemmas 8.4–8.18. Let $P \in V_d$ and write $P = U(z, \bar{z}, u_j) + V(z, \bar{z}, \bar{u}_j) + R(z, \bar{z})$, such that neither $U$ nor $V$ have any terms that do not involve at least one $u_j$ or $\bar{u}_j$. We calculate that

$$\mathcal{E}(U) = f_u U + U_{z\bar{z}} + u_{j+1} U_{u_j,\bar{z}} - T^l \frac{\partial}{\partial u_l} \left( U_z + u_{j+1} U_{u_j} \right) \tag{20}$$

$$\mathcal{E}(V) = f_u V + V_{z\bar{z}} + \bar{u}_{j+1} V_{\bar{u}_j,z} - \overline{T}^l \frac{\partial}{\partial \bar{u}_l} \left( V_{\bar{z}} + \bar{u}_{j+1} V_{\bar{u}_j} \right) \tag{21}$$

**Lemma 8.4** $U_{\bar{z}} = V_z = R = 0$.

*Proof* The terms in $\mathcal{E}(P) = 0$ that do not involve $u$ imply that

$$R_{z\bar{z}} + U_{z\bar{z}} + u_{j+1} U_{u_j,\bar{z}} + V_{z\bar{z}} + \bar{u}_{j+1} V_{\bar{u}_j,z} = 0.$$

Let $u_k$ be the variable of highest weighted degree in $P$ that appears multiplied with a $\bar{z}$ and $\bar{u}_m$, the variable of lowest weighted degree appearing in $P$ with a $z$. Then, $u_{k+1}U_{u_k\bar{z}}$ produces a monomial which, because of the maximality of $u_k$, cannot be canceled by any of the other terms. Then by induction $U_{u_j\bar{z}} = 0$ for all $u_j$. A similar argument shows that $V_{\bar{u}_j z} = 0$ and because $U$ and $V$ do not have any terms without some $u_j$ or $\bar{u}_j$, we have that $U_{\bar{z}} = V_z = 0$. This then implies that $R_{z\bar{z}} = 0$.

Thus, the only remaining possibility for $R$, if it does not vanish, is that is consists of exactly one monomial of the appropriate degree. In the case $d > 0$, it follows that $R = c\bar{z}^d$ for some constant $c$. We have $\mathcal{E}(R) = cf_u\bar{z}^d$, so the negative of this term has to appear in $\mathcal{E}(U + V)$. But from Eqs. (20) and (21), we see that the only possibilities to get a summand in $\mathcal{E}(U + V)$ without any $u_j$ are the terms $-T^0\frac{\partial}{\partial u_0}U_z = -fU_{z,u_0}$ and $-fV_{\bar{z},\bar{u}_0}$. Since $f$ and $f_u$ are not linearly dependent, it follows $R = 0$. The same argument works for $d < 0$ and $d = 0$. □

We have $P = U(z, u_j) + V(\bar{z}, \bar{u}_j)$, where $U$ and $V$, expressed as power series in $z$ and $\bar{z}$, can be written as $U = \sum U^n z^n$ with $\mathrm{wd}(U^n) = d + n$ and $V = \sum V^n\bar{z}^n$ with $\mathrm{wd}(V^n) = d - n$. Each coefficient $U^n$ or $V^n$ is a polynomial in the $u_j$ or the $\bar{u}_j$, never constant. Now we can expand $\mathcal{E}(P) = 0$ in terms of $z$ and $\bar{z}$:

$$\mathcal{E}(U) = z^n\left[f_uU^n - T^l\frac{\partial}{\partial u_l}\left((n+1)U^{n+1} + u_{j+1}U^n_{u_j}\right)\right] \tag{22}$$

$$\mathcal{E}(V) = \bar{z}^n\left[f_uV^n - \overline{T}^l\frac{\partial}{\partial \bar{u}_l}\left((n+1)V^{n+1} + \bar{u}_{j+1}V^n_{\bar{u}_j}\right)\right] \tag{23}$$

The $n = 0$ coefficients only sum to be zero, but otherwise the coefficients of $z^n$ and $\bar{z}^n$ must vanish separately.

**Lemma 8.5** *If $u_k$ is the variable of highest weighted degree appearing in $U$, then $U_{u_k,u_j} = 0$ for all $u_j$. Similarly, if $\bar{u}_k$ is the variable of lowest weighted degree appearing in $V$, then $V_{\bar{u}_k,\bar{u}_j} = 0$ for all $\bar{u}_j$.*

*Proof* Let $u_k$ be the variable of highest weighted degree appearing in $U$. Suppose that there is a summand in $U^n$ where $u_k$ appears to a power higher than 1 or multiplied by some other $u_j$. Denote the monomial in $U^n$ of highest lexicographic ordering with this property by $u_{j_1}^{i_1}\cdot\dots\cdot u_{j_r}^{i_r}$, with $k = j_1 > \ldots > j_r$ and all exponents $\geq 1$. Our assumption says that either $r \geq 2$ or $r = 1$ and $i_1 \geq 2$.

Let us look at the case $r \geq 2$ first: By finding a nonvanishing summand in Eq. (22), we will derive a contradiction. Exactly for $j = j_1(= k)$, there appear summands involving $u_{k+1}$ in Eq. (22): exactly those of the form $-T^l\frac{\partial}{\partial u_l}\left(u_{k+1}U^n_{u_k}\right)$ with $l \neq k+1$. Our monomial above produces

$$-n_1T^l\frac{\partial}{\partial u_l}\left[u_{j_1+1}u_{j_1}^{i_1-1}\cdot u_{j_2}^{i_2}\cdot\dots\cdot u_{j_r}^{i_r}\right].$$

For $l = j_r$, we obtain

$$
\begin{aligned}
&-i_1 i_r T^{j_r} u_{j_1+1} u_{j_1}^{i_1-1} \cdot u_{j_2}^{i_2} \cdot \cdots \cdot u_{j_r}^{i_r-1} \\
&\quad = \begin{cases} -i_1 i_r f_u \left[ u_{j_1+1} u_{j_1}^{i_1-1} \cdot u_{j_2}^{i_2} \cdot \cdots \cdot u_{j_r}^{i_r-1} u_{j_r-1} \right] + \text{lower lex. ord.} & j_r > 0 \\ -i_1 i_r f \left[ u_{j_1+1} u_{j_1}^{i_1-1} \cdot u_{j_2}^{i_2} \cdot \cdots \cdot u_{j_r}^{i_r-1} \right] + \text{lower lex. ord.} & j_r = 0 \end{cases}
\end{aligned}
$$

and because the original monomial was the one of highest lexicographic ordering among those in $U^n$, this monomial cannot be canceled by any other that is produced from $U^n$ in Eq. (22). But it also cannot cancel with a summand coming from $U^{n+1}$ since that would contradict our assumption that $u_k$ is the variable of highest weighted degree appearing in all of $U$. Thus, $u_k$ cannot appear in a monomial with any other $u_j$.

Now suppose that $r = 1$ and $m = i_1 \geq 2$ so that the highest monomial is $u_k^m$. Taking $j = l = k$ gives a summand of Eq. (22) of the form

$$
\begin{aligned}
&-mT^k \frac{\partial}{\partial u_k} \left[ u_{k+1} u_k^{m-1} \right] \\
&= -m(m-1) T^k u_{k+1} u_k^{m-2} \\
&= \begin{matrix} -m(m-1) f_u \left[ u_{k+1} u_k^{m-2} u_{k-1} \right] + \text{lower lex. ord.} & k > 0 \\ -m(m-1) f \left[ u_{k+1} u_k^{m-2} \right] + \text{lower lex. ord.} & k = 0 \end{matrix}
\end{aligned}
$$

which again is the unique highest one. The same considerations as before lead to a contradiction. Thus, $u_k$ must appear linearly. When it does, the terms only involving $U^n$ allow it to cancel.

An analogous argument gives the corresponding result for $V$. □

**Corollary 8.6** *If $u_k$ is the highest variable appearing in $U$, then it only appears in $U^{k+1-d}$, so $d \leq k+1$. If $\bar{u}_m$ is the lowest variable appearing in $V$, then it only appears in $V^{m+1+d}$, so that $d \geq -(m+1)$.*

*Proof* This follows by considerations of weighted degree and Lemma 8.5. □

**Corollary 8.7** *If $u_k$ is the highest variable appearing in $U$, then $U^n = 0$ for $n > k+1-d$. If $\bar{u}_m$ is the lowest variable appearing in $V$, then $V^n = 0$ for $n > m+1+d$.*

*Proof* Let $u_l$ with $l < k$ be the highest variable appearing in those $U^n$ with $n > k+1-d$. If we regard such an $n$, the same argument as given in the proof of Lemma 8.5 implies that $u_l$ appears linearly and without any other $u_j$. This implies that $\mathrm{wd}(U^n) = l+1 < k+1$, but this contradicts $\mathrm{wd}(U^n) = n+d > k+1$. A similar argument can be made for $V$. □

**Corollary 8.8** *Both $U$ and $V$ are polynomials. In fact,*

$$
U = \sum_{n=0}^{k+1-d} U^n z^n \text{ with } U^{k+1-d} = b(u_k + \cdots) \text{ for some } b \in \mathbb{C}
$$

*and*

$$V = \sum_{n=0}^{m+1+d} V^n \bar{z}^n \textit{ with } V^{k+1+d} = c(\bar{u}_m + \cdots) \textit{ for some } \ c \in \mathbb{C}.$$

We will make use of the following lemma repeatedly.

**Lemma 8.9** *The operator $\overline{e_{-1}}$, acting on polynomials in $u_i$, has only the constants as kernel.*

*Proof* It suffices to prove that for a weighted-homogeneous polynomial $h$ of degree at least one, $\overline{e_{-1}}h = 0$ implies $h = 0$. Write

$$h = \sum_{|I|=k+1} h_I u^I,$$

where the sum runs over all multi-indices $I = (i_0, \ldots, i_k)$ of weighted degree $\sum_j (i_j + 1)$ equal to $k + 1$, and $u^I = u_0^{i_0} \cdot \cdots \cdot u_k^{i_k}$, and assume

$$0 = -\overline{e_{-1}}(h) = \sum_{j=0}^{k} \sum_{|I|=k+1} h_I T^j \frac{\partial}{\partial u_j} u^I. \tag{24}$$

Let $I_0 = (i_0, \ldots, i_k) \neq 0$ be the highest index such that $h_{I_0} \neq 0$. Let furthermore $l$ be the smallest number such that $i_l \neq 0$ and assume first that $l > 0$. In other words, $u^I = u_l^{i_l} u_{l+1}^{i_{l+1}} \cdot \cdots \cdot u_k^{i_k}$. In Eq. (24), we find $h_{I_0}$, for example, in the summand

$$h_{I_0} T^l i_l \cdot u_l^{i_l - 1} u_{l+1}^{i_{l+1}} \cdot \cdots \cdot u_k^{i_k}.$$

(This is the summand for $I = I_0$ and $j = l$.) Since $T^l$ reads $T^l = u_{l-1} f_u +$ terms with lower $u$'s, we have found a summand

$$h_{I_0} i_l f_u \cdot u_{l-1} u_l^{i_l - 1} u_{l+1}^{i_{l+1}} \cdot \cdots \cdot u_k^{i_k}.$$

Let us try to spot the full coefficient of this monomial $u_{l-1} u_l^{i_l-1} u_{l+1}^{i_{l+1}} \cdot \cdots \cdot u_k^{i_k}$ in Eq. (24). For which $I$ and $j$ can the summand $h_I T^j \frac{\partial}{\partial u_j} u^I$ contribute? If $j > 0$, some of the $u$'s in the monomial have to appear in $T^j$. But then, necessarily $I \geq I_0$, since $u^I$ is differentiated with respect to $u_j$, and $u_j$ is higher than all the $u$'s appearing in $T^j$. For $I = I_0$, we already have found the one contribution, so since we assumed that $I_0$ is the highest multi-index such that $h_{I_0} \neq 0$, the only further summands that can contribute are those with $j = 0$. Here, we only have a new contribution if $u^I = u_0 u_{l-1} u_l^{i_l-1} u_{l+1}^{i_{l+1}} \cdot \cdots \cdot u_k^{i_k}$. Denoting the corresponding multi-index by $I_1$, we have shown

$$0 = h_{I_0} i_j f_u + h_{I_1} c f,$$

where $c = 1$ or $c = 2$, depending on whether $l > 1$ or $l = 1$. Since $f$ and $f_u$ are linearly independent, $h_{I_0} = 0$ (and also $h_{I_1} = 0$), a contradiction.

In the case $l = 0$, the monomial in question is $u_l^{i_l - 1} u_{l+1}^{i_{l+1}} \cdot \cdots \cdot u_k^{i_k}$, and there is only the summand for $j = 0$ and $I = I_0$ contributing to this monomial; we also conclude $h_{I_0} = 0$. □

**Lemma 8.10** $\mathrm{wd}(U) \geq 0$ *and* $\mathrm{wd}(V) \leq 0$.

*Proof* Suppose that $\mathrm{wd}(U) = d < 0$. We know that $U = \sum_{n=0}^{k+1-d} U^n z^n$. Since $\mathrm{wd}(U^n) \geq 0$, we have $U^0 = 0$; let $m \geq 0$ be such that $U^0 = \ldots = U^m = 0$ and $U^{m+1} \neq 0$. If $m = 0$, i.e. $U^1 \neq 0$, it would follow that $d = -1$ and that $U^1$ is constant, which was ruled out in Lemma 8.4.

So we are in the case $m > 0$. From $\mathcal{E}(U + V) = 0$ and Eq. (22), we find that

$$\sum_l -z^m T^l \frac{\partial}{\partial u_l} \left( (m+1) U^{m+1} \right) = 0 \quad \text{(no sum on } m\text{)}$$

when $m > 0$. By Lemma 8.9, this implies that $U^{m+1} = 0$ and so by induction $U = 0$. A similar argument gives the result for $V$. □

**Corollary 8.11** *If* $\mathrm{wd}(P) > 0$, *then* $V = 0$. *If* $\mathrm{wd}(P) < 0$, *then* $U = 0$.

**Lemma 8.12** *If* $\mathrm{wd}(P) = 0$, *then* $P = b \cdot (z u_0 - \bar{z} \bar{u}_0)$ *for some constant* $b \in \mathbb{C}$.

*Proof* We have

$$U = b z u_0 + \sum_{n \geq 2}^{k+1} U^n z^n, \qquad V = c \bar{z} \bar{u}_0 + \sum_{n \geq 2}^{m+1} V^n \bar{z}^n$$

for some constants $b, c$. Then, the first terms in Eqs. (22) and (23) lead to

$$\mathcal{E}(U+V) = z^0 \left( -f U^1_{u_0} - f V^1_{\bar{u}_0} \right) + z^1 \left[ f_u U^1 - T^l \frac{\partial}{\partial u_l} \left( u_{j+1} U^1_{u_j} + 2U^2 \right) \right]$$
$$+ \bar{z}^1 \left[ f_u V^1 - \overline{T}^l \frac{\partial}{\partial \bar{u}_l} \left( \bar{u}_{j+1} V^1_{\bar{u}_j} + 2V^2 \right) \right] + \cdots$$

The $z^0$-term implies that $c = -b$. The terms from $U^1$ in the $z^1$-term cancel so that by Lemma 8.9 we have $U^2 = 0$. Similarly, the $\bar{z}^1$-term implies that $V^2 = 0$. Then, using induction, Eqs. (22) and (23) imply that $U^n = V^n = 0$ for $n > 1$. □

**Lemma 8.13** *If* $\mathrm{wd}(P) = d > 0$, *then* $P_z = 0$, *i.e.* $P$ *is a polynomial in the* $u_j$ *of the form* $P = b u_{d-1} + \ldots$ *with* $b \neq 0$. *If* $\mathrm{wd}(P) = d < 0$, *then* $P_{\bar{z}} = 0$, *i.e.* $P$ *is a polynomial in the* $\bar{u}_j$ *of the form* $P = c \bar{u}_{-d-1} + \ldots$ *with* $c \neq 0$.

*Proof* Suppose $d > 0$ so that $P = \sum_{n=0}^{k+1-d} U^n z^n$. Assuming that $k + 1 - d > 0$ will lead to a contradiction. We have

$$U^{k+1-d} = bu_k + b'u_{k-1}u_0 + \cdots$$

with $b \neq 0$, and

$$U^{k-d} = b''u_{k-1} + \cdots$$

The $z^{k-d}$-coefficient in Eq. (22) reads

$$f_u U^{k-d} - T^l \frac{\partial}{\partial u_l}\Big((k-d+1)U^{k-d+1} + u_{j+1}U^{k-d}_{u_j}\Big);$$

neglecting summands without $u_{k-1}$ and finding that the $b''$ terms cancel, we obtain

$$0 = -(k-d+1)(T^0 b' u_{k-1} + T^k b) = -(k-d+1)u_{k-1}(fb' + f_u b) + \cdots$$

Since $f$ and $f_u$ are linearly independent, $(k-d+1) > 0$ and $b \neq 0$, we have arrived at a contradiction. A similar argument works for $V$. □

**Corollary 8.14** *For all $d$, we have* $\dim_{\mathbb{C}}(V_d) \leq 1$.

*Proof* We know that any nonzero element in $V_d$ is of the form $bu_{d-1} + \cdots$ with $b \neq 0$ for $d > 0$, or $c\bar{u}_{d-1} + \cdots$ with $c \neq 0$ for $d < 0$ and $a \cdot (zu_0 - \bar{z}\bar{u}_0)$ with $a \neq 0$ for $d = 0$. The bound on dimension then follows because $\mathcal{E}(P) = 0$ is a linear equation. □

**Lemma 8.15** *For all $d$ the linear map*

$$\begin{aligned} V_d &\to \bar{H}^1_{\mathbb{C}} \\ P &\mapsto [\varphi_P] \end{aligned}$$

*is injective.*

*Proof* Assume that $d > 0$. The $d < 0$ case follows by complex conjugation. Let $P \in V_d$ be nonzero and normalized, i.e. $P$ is a polynomial of the form

$$P = u_{d-1} + \cdots .$$

The 1-form $\varphi_P$ was defined in Eq. (18), and a more explicit form was given in Eq. (19). It will be convenient to modify $\varphi_P$ by an exact form as follows: If we define $\tilde{E}' = qe_{-1}(P)$, $\tilde{E}'' = e_{\overline{-1}}(q)P$ and

$$\tilde{\varphi}_P = \tilde{E}'\zeta + \tilde{E}''\bar{\zeta},$$

then

$$\tilde{\varphi}_P \equiv d \cdot \varphi_P + \mathrm{d}(qP) \quad \mod \mathrm{I}^{(d)}.$$

First, we have to show that $\tilde{\varphi}_P$ defines a cohomology class in $\bar{H}^1_{\mathbb{C}}$. The only obstacle that could arise is that $\mathrm{d}\tilde{\varphi}_P$ could have a $\zeta\wedge\bar{\zeta}$ term. However, the corresponding coefficient is

$$\begin{aligned}-e_{\overline{-1}}\tilde{E}' + e_{-1}\tilde{E}'' &= -e_{\overline{-1}}(q)e_{-1}P - qe_{\overline{-1}}e_{-1}P + e_{\overline{-1}}e_{-1}(q)P + e_{\overline{-1}}(q)e_{-1}P\\ &= qf_u P - f_u qP = 0\end{aligned}$$

because $P \in V_d$ and $q \in V_0$.

Thus, it remains to show that $[\tilde{\varphi}_P] = d\cdot[\varphi_P]$ is a nontrivial class. The one-form $\tilde{\varphi}_P$ represents $0 \in \bar{H}^1_{\mathbb{C}}$ if and only if

$$\mathrm{d}\tilde{\varphi}_P = \mathrm{d}\alpha \tag{25}$$

for some $\alpha \in \mathrm{I}^{(l)}$. Assuming that $\alpha = \sum_{j=0}^{l}(a^j\eta_j + b^j\bar{\eta}_j)$ (with $b^0 = 0$) satisfies Eq. (25) will lead to a contradiction.

For $j > 1$, the $\zeta\wedge\eta_j$-coefficient of Eq. (25) implies

$$-qe_je_{-1}P = -\tilde{E}'_{u_{j-1}} = e_{-1}a^j + a^{j-1},$$

from which we can determine the $a^j$ recursively: We have $e_{-1}P = u_d + \cdots$, so $a^j$ vanishes for $j > d$. The first two nonvanishing coefficients are

$$a^d = -qe_{d+1}e_{-1}P = -q$$

and

$$a^{d-1} = -e_{-1}a^d - qe_de_{-1}P = e_{-1}q - qe_de_{-1}P. \tag{26}$$

We obtain

$$\begin{aligned}a^1 &= \sum_{j=2}^{d+1}(-1)^{j-1}(e_{-1})^{j-2}(qe_je_{-1}P)\\ &= (-1)^d\left[(e_{-1})^{d-1}(q) - (e_{-1})^{d-2}(qe_de_{-1}P) \pm\cdots + (-1)^{d-1}qe_2e_{-1}P\right]\end{aligned} \tag{27}$$

The condition on the $\eta_1\wedge\eta_{d-1}$-coefficients of Eq. (25) is

$$a^{d-1}_{u_0} = a^1_{u_{d-2}},$$

which will provide a contradiction. One finds that

$$(e_{-1})^j q = ju_{j-1} + zu_j + \bar{z}T^{j-1}.$$

Using Eq. (26), we compute

$$a^{d-1}_{u_0} = 1 - z e_d e_{-1} P - q e_1 e_d e_{-1} P,$$

which has a constant term 1. On the other hand, the only way to obtain a constant term from differentiating Equation (27) with respect to $u_{d-2}$ is via the summand $(-1)^d (e_{-1})^{d-1}(q)$:

$$a^1_{d-1} = (-1)^d d + \text{non-constant terms}.$$

But $(-1)^d d \neq 1$ for all $d > 0$. □

**Lemma 8.16** *If* $P \in V_d$ *then* $\mathrm{d}\varphi_P = \Phi_P$.

*Proof* The $d = 0$ case was done explicitly in Sect. 4, so assume $d \neq 0$. If $P \in V_d$ is nonzero, then $[\varphi_P]$ is a nontrivial class and so $[\mathrm{d}\varphi_P] = [\Phi_{P'}]$ for some other solution $P'$ to Eqs. (13) and (14). Weighted degree is preserved by exterior differentiation, so $\mathrm{wd}(P') = \mathrm{wd}(P)$ and hence $P' \in V_d$, which implies that $P' = c \cdot P$ for some constant $c$, by Corollary 8.14. Since $\Phi_{P'}$ is closed, this implies that $\Phi_P$ is closed. Then by Lemma 7.2, we reach the desired conclusion. □

**Corollary 8.17** *If* $P \in V_{k+1}$, *then* $\Phi_P$ *is closed and* $\Phi_P + \overline{\Phi_P}$ *is a real element of* $\mathcal{C}_{(k)}$.

**Lemma 8.18** *For even degree* $d \neq 0$, $V_d = 0$.

*Proof* For $d \neq 0$, complex conjugation gives an isomorphism $V_d \to V_{-d}$. Thus, it suffices to prove the lemma for positive $d$.

Suppose that $d = 2n > 0$ and $P = u_{d-1} + \cdots \in V_d$ is a normalized solution. Then, by Lemmas 8.15 and 8.16, the two-form

$$\Phi_P = \eta_0 \wedge \rho_P + P\psi + \sum_{i,j} B^{ij}(P)\eta_i \wedge \eta_j \in \Omega^2_d(M^{(d-1)})$$

defines a weighted-homogeneous differentiated conservation law. By Corollary 6.9, we can conclude that $B^{1,d-1}(P) = 0$, which contradicts the third item of Lemma 6.5 due to the fact that, if such a $P$ exists, then $P = u_{d-1} + \cdots$. □

*Proof of Thm 8.3* The statements about $V_d$ are exactly Lemma 8.12, Corollary 8.14, and Lemma 8.18. By Corollary 8.17, the map $P \to \Phi_P$ is an isomorphism from $V_d$ to $\mathcal{C}_d$. By definition, the map $\mathcal{C}_d \to \mathcal{H}^1_d$ is an isomorphism and Lemma 8.16 implies that its inverse is given by exterior differentiation. The last two items are immediate consequences of the second item of the theorem. □

## 9 Potentials satisfying linear second-order ODEs

So far, the only assumption on $f$ was that it does not satisfy a linear first-order ODE, i.e. that $f$ and $f_u$ are linearly independent.[2] The following theorem shows that $f$ has to satisfy a linear second-order ODE for higher-order conservation laws to exist.

**Theorem 9.1** *Assume that $f$ does not satisfy any linear second-order ODE, i.e. that $f$, $f_u$ and $f_{uu}$ are linearly independent over $\mathbb{R}$. Then, $V_d = 0$ for $|d| \geq 2$, i.e. no higher-order conservation laws occur.*

*Proof* It suffices to prove $V_d = 0$ for $d \geq 2$. Assume that $V_d \neq 0$ for some $d \geq 2$, and let $P \in V_d$ be a nonzero element. By Lemma 8.13, $P$ is a polynomial in the variables $u_j$ and can be normalized such that

$$P = u_{d-1} + cu_{d-2}u_0 + \cdots$$

The polynomial $P$ satisfies Eq. (14)

$$\sum_{j=0}^{d} \sum_{i=0}^{d-1} T^j \frac{\partial}{\partial u_j} \left(u_{i+1} P_{u_i}\right) = f_u P. \tag{28}$$

Recall from Lemma 2.1 that $T^j = (e_{-1})^j f$. It follows that $T^0 = f, T^1 = u_0 f_u$, $T^2 = u_1 f_u + u_0^2 f_{uu}$ and for $j \geq 3$,

$$T^j = u_{j-1} f_u + j u_{j-2} u_0 f_{uu} + \text{terms without } u_{j-1} \text{ and } u_{j-2}.$$

Therefore, the summands on the left-hand side of Eq. (28) that involve $u_{d-2}$ are

$$\begin{array}{lll} j = 0, & i = d-3, \text{ if } d \geq 3 & f u_{d-2} P_{u_{d-3},u_0} \\ j = 1, & i = d-3, \text{ if } d \geq 4 & f_u u_{d-2} u_0 P_{u_{d-3},u_1} \\ j = 1, & i = 0 & f_u c u_{d-2} u_0 \\ j = d-1, & i = d-2, \text{ if } d \geq 3 & f_u c u_{d-2} u_0 \\ j = d, & i = d-1 & f_{uu} d u_{d-2} u_0. \end{array}$$

It follows that the vanishing of the $u_{d-2}u_0$-coefficient of Eq. (28) contradicts the assumption that $f_{uu}$ is linearly independent from $f$ and $f_u$. Therefore, $V_d = 0$. The statement about conservation laws follows from Proposition 6.3. □

On the other hand, in certain cases, the upper bound for the dimensions of the spaces of higher-order conservation laws given in Theorem 8.3 is sharp.

**Lemma 9.2** *Suppose that $f_{uu} = \beta f$ with $\beta \neq 0$ and that $f$ does not satisfy any first-order ODE. Then,* $\dim_{\mathbb{C}}(V_{2n+1}) = 1$ *for all $n \in \mathbb{Z}$.*

---

[2] If $f_u = \beta f$ for some constant $\beta$, then Eq. (1) is the Liouville equation. It is not hard to check that it has infinitely many classical conservation laws. It is well known that the Liouville equation is linearizable. With respect to its role as the Gauss equation for constant mean curvature surfaces with $\epsilon + \delta^2 = 0$ (see Eq. (2)), the linearizability is equivalent to the existence of the Weierstrass representation.

*Proof* By the results of Sect. 8, we only need to demonstrate the existence of a non-zero element in each $V_{2n+1}$ for $n \geq 0$. Let $P^0 = u_0 \in V_1$. We will define the other solutions recursively. Suppose that for any solution $P^n \in V_{2n+1}$, we can find a weighted-homogeneous polynomial $Q^n$ only involving the $u_j$ that satisfies

$$\begin{aligned} e_{-1} Q^n &= 2u_0 e_{-1} P^n \\ e_{\overline{-1}} Q^n &= -2 f P^n. \end{aligned} \tag{29}$$

Given such a $Q^n$, define

$$P^{n+1} = e_{-1} e_{-1} P^n - \frac{\beta}{2} u_0 Q^n. \tag{30}$$

Using Eq. (29) and the fact that $P^n \in V_{2n+1}$, one can check that

$$e_{\overline{-1}} e_{-1} P^{n+1} = -f_u P^{n+1}. \tag{31}$$

Given that $Q^n$ is a polynomial only involving the $u_j$, Eqs. (30) and (31) imply that $P^{n+1} \in V_{2n+3}$. We now prove the existence of such a $Q^n$.

For any solution $P^n \in V_{2n+1}$, define $\alpha^n = 2u_0 P^n_{-1} \zeta - 2 f P^n \bar{\zeta}$. It is readily checked that $\mathrm{d}\alpha^n \equiv 0 \mod \mathrm{I}^{(\infty)}$. Thus, $[\alpha^n] \in \bar{H}^1$. However, $\mathrm{wd}(\alpha^n) = 2n + 2$ and by the results of Sect. 8, this implies that $[\alpha^n] = 0 \in \bar{H}^1$. Thus, there exists a function $Q^n$ on $M^{(2n)}$ and $\beta^n \in \mathrm{I}^{(2n)}$ such that

$$\mathrm{d} Q^n = \alpha^n + \beta^n. \tag{32}$$

The $\zeta$ and $\bar{\zeta}$ terms of Eq. (32) imply Eq. (29) and the other terms lead to

$$\beta^n = Q^n_u \eta_0 + Q^n_{u_{j-1}} \eta_j + Q^n_{\bar{u}_{j-1}} \bar{\eta}_j.$$

We are finished once we show that $Q^n$ is independent of $u, z, \bar{z}$ and the $\bar{u}_j$. First, we show that $Q^n_z = Q^n_{\bar{z}} = 0$. Differentiating Eq. (32) with respect to $z$ results in

$$\mathrm{d}\left(\frac{\partial Q^n}{\partial z}\right) \in \mathrm{I}^{(2n)}$$

$\frac{\partial \eta_i}{\partial z} = \frac{\partial \eta_i}{\partial \bar{z}} = 0$. But there are no exact forms in the ideal and so $\frac{\partial Q^n}{\partial z}$ is constant. Together with a similar argument involving $\bar{z}$, this implies that $Q^n$ is at most linear in $z$ or $\bar{z}$ so we can write $Q^n = \hat{Q}^n + az + b\bar{z}$ where $\hat{Q}^n$ is independent of $z$ and $\bar{z}$. However, because the right-hand side of Eq. (32) does not contain any terms of the form $a\mathrm{d}z$ or $b\mathrm{d}\bar{z}$, we must have $a = b = 0$. Thus, $Q^n$ is independent of $z$ and $\bar{z}$.

Now write $Q^n = \sum_{a=0}^{\infty} Q^{n,a} \bar{u}^a_{2n}$ where $Q^{n,a}$ is independent of $\bar{u}_{2n}$. The second of the two equations in Eq. (29) can be written as

$$\sum_{a=0}^{\infty} \left( Q^{n,a} a \bar{u}^{a-1}_{2n} \bar{u}_{2n+1} + Q^{n,a}_{\overline{-1}} \bar{u}^a_{2n} \right) = -2 f P^n.$$

The highest variable appearing in $Q^n$ is $\bar{u}_{2n}$, and it does not appear in $P^n$. By induction, we find that $Q^n$ is independent of $\bar{u}_{2n}$. Then using induction again, we find that $Q^n$ is independent of $\bar{u}_j$ for all $j \geq 0$. Once this is done, expanding the same equation in a power series in $u$ implies that $Q^n$ is independent of $u$. The appearance of $f(u)$ on the right-hand side does not spoil the argument because it is the image of the $\bar{u}_0 \frac{\partial}{\partial u}$ term in $e_{\overline{-1}}$ that leads to the vanishing. Thus, $Q^n$ is a function only of the $u_j$. This completes the proof. □

The process used to prove Lemma 9.2, and in particular the definition of the one-form $\alpha^n$, is derived from the recursive equations satisfied by the coefficients of a formal Killing field associated with the system for harmonic maps into $\mathrm{SU}(2)/\mathrm{SO}(2)$. A more complicated recursion involving a formal Killing field for the system of primitive maps into $\mathrm{SU}(3)/\mathrm{SO}(2)$ should result in an existence theorem for a basis of $V_{2n+1}$ for the Tzitzeica equation.[3] In fact, many of the results in this article will hold for any Toda field equation. We expect such results to be analogous to the approach of Terng and Uhlenbeck in [38]. We will report on this in a future article.

Proposition 3.1 of [33] provides an independent proof of the existence of the $Q^n$ by writing down a highly nonlinear but explicit recursive formula. It is unknown whether that amazing formula will generalize to other systems of PDE.

In [42], they determine the equations of the form $u_{xt} = f(u)$ that admit a nontrivial Lie–Bäcklund transformation group. This is essentially the same classification that we have found because Noether's theorem relates conservation laws and symmetries (see Sect. 10). Our work is much closer to the approach in [16] where they look for the existence of polynomial conserved quantities, though we do not rely on either [16] or [42].

The recursion used in the proof of Lemma 9.2 fits into the general theory of recursion operators introduced by Guthrie [23]. However, it appears that Theorem 1 of [23] does not actually guarantee existence. In the example above, Guthrie's theorem does not seem to guarantee the existence of the $Q^n$ because his integrability condition is only that $d\alpha^n \equiv 0 \mod \mathrm{I}^{(\infty)}$, not that $[\alpha^n] = 0 \in \bar{H}^1$. Though it may be that we do not fully understand his results.

*Example 9.3* Suppose that $f_{uu} = \beta f$ with $\beta \neq 0$ and that $f$ does not satisfy a first-order ODE. Then, $\dim V_d = 1$ for all odd integers $d$. If the generators $P^n$ of $V_{2n+1}$ are normalized so that $P^n = u_{2n} + \cdots$ then the first four of them are

$$
\begin{aligned}
P^0 &= u_0 \\
P^1 &= u_2 - \frac{1}{2}\beta u_0^3 \\
P^2 &= u_4 - \frac{5}{2}\beta u_2 u_0^2 - \frac{5}{2}\beta u_1^2 u_0 + \frac{3}{8}\beta^2 u_0^5
\end{aligned}
$$

[3] In [16], Dodd and Bullough claim that there are only finitely many polynomial conserved quantities for the Tzitzeica equation. This would be strange since both families of potentials discussed in this section are obtained from primitive maps into $k$-symmetric spaces and thus should have similar theories.

$$P^3 = u_6 - \frac{7}{2}\beta u_4 u_0^2 - 14\beta u_3 u_1 u_0 - \frac{21}{2}\beta u_2^2 u_0 - \frac{35}{2}\beta u_2^2 u_1^2 + \frac{35}{8}\beta^2 u_2 u_0^4 + \frac{35}{4}\beta^2 u_1^2 u_0^3 - \frac{5}{16}\beta^3 u_0^7.$$

*Example 9.4* In the case that $f_{uu} = \alpha f_u + 2\alpha^2 f$ with $\alpha \neq 0$, a coordinate change transforms Eq. (1) into the Tzitzeica equation $u_{z\bar{z}} = e^u - e^{-2u}$. For $f(u) = e^u - e^{-2u}$, the first four spaces $V_{2i+1}$ are as follows:

$$\begin{aligned}
&\dim V_1 = 1, \quad u_0 \in V_1\\
&\dim V_3 = 0\\
&\dim V_5 = 1, \quad u_4 + 5u_2u_1 - 5u_2u_0^2 - 5u_1^2u_0 + u_0^5 \in V_5\\
&\dim V_7 = 1, \quad u_6 + 7u_4u_1 - 7u_4u_0^2 + 14u_3u_2 - 28u_3u_1u_0 - 21u_2^2u_0 - 28u_2u_1^2\\
&\qquad -14u_2u_1u_0^2 + 14u_2u_0^4 - \frac{28}{3}u_1^3u_0 + 28u_1^2u_0^3 - \frac{4}{3}u_0^7 \in V_7.
\end{aligned}$$

*Remark* We believe that $f$ must satisfy either $f_{uu} = \beta f$ or $f_{uu} = \alpha f_u + 2\alpha^2 f$ if *higher-order* conservation laws exist at *any* level, but we do not have a proof of this. It is not hard to show that for there to exist new conservation laws in normal form at the second prolongation, then $f$ must satisfy $f_{uu} = \beta f$, and for there to exist new conservation laws in normal form at the fourth prolongation, then $f$ must satisfy either $f_{uu} = \beta f$ or $f_{uu} = \alpha f_u + 2\alpha^2 f$.

## 10 Generalized symmetries

Noether's theorem can be formulated as an isomorphism between the space of proper conservation laws (viewed as elements of the characteristic cohomology) and the space of proper generalized symmetries [6]. See, for example, [32,35] for related formulations of Noether's theorem. In order to discuss this for the system at hand, we begin by introducing the appropriate class of generalized symmetries. In Lemma 10.3, we prove a weaker version of Noether's theorem that has appeared previously using other machinery—for example, see [36]. We end by discussing how generalized symmetries relate to the Jacobi fields of Pinkall and Sterling [33].

It is most convenient to study symmetries on $M^{(\infty)}$. There we have the following

**Definition 10.1** A real vector field $v$ on $M^{(\infty)}$ is a **generalized symmetry of order** $r$ for $(M^{(\infty)}, \mathcal{I}^{(\infty)})$ if $\mathcal{L}_v(\mathcal{I}^{(l)}) \subset \mathcal{I}^{(l+r)}$ for all $l \geq 0$. A **trivial generalized symmetry** for $(M^{(\infty)}, \mathcal{I}^{(\infty)})$ is a generalized symmetry $v$ that satisfies $v \lrcorner \mathrm{I}^{(\infty)} = 0$.

A natural candidate for a trivial generalized symmetry is $\mathrm{Re}(e_{-1})$ or $\mathrm{Im}(e_{-1})$ since $e_{-1} \lrcorner \mathrm{I}^{(\infty)} = 0$. We calculate that

$$\begin{aligned}
\mathcal{L}_{e_{-1}}\eta_l &= \eta_{l+1}\\
\mathcal{L}_{e_{-1}}\eta_l &= -\tau^{l-1},
\end{aligned}$$

showing that $e_{-1}$ is an order 1 generalized symmetry of $(M^{(\infty)}, \mathcal{I}^{(\infty)})$. By the observation above it is trivial. In fact, the same is true for $\mathrm{Re}(Qe_{-1})$ for any complex valued function $Q$ on $M^{(\infty)}$.

**Definition 10.2** A **proper** generalized symmetry is a generalized symmetry $v$ also satisfying $v \lrcorner \zeta = 0$.

In [6] the proper generalized symmetries are realized as a quotient of the space of all generalized symmetries. Using our specified coframe allows us to recognize them as a subspace rather than a quotient. The following lemma has been proven previously in many other contexts.

**Lemma 10.3** *Let $v$ be a (real) vector field on $M^{(\infty)}$ such that $\zeta(v) = 0$ and let $g = \eta_0(v)$. Then, $v$ is a generalized symmetry of order one of $(M^{(\infty)}, \mathcal{I}^{(\infty)})$ if and only if $\eta_i(v) = (e_{-1})^i(g)$ and $g$ is a solution to Eq. (14).*

*Proof* The $\zeta$-coefficient of $\mathcal{L}_v(\eta_0)$ is $v^1 - e_{-1}(g)$. Thus, the condition that $\mathcal{L}_v(\eta_0) \in \mathrm{I}^{(2)}$ implies that $v^1 = e_{-1}(g)$. In general, we find that the $\zeta$-coefficient of $\mathcal{L}_v(\eta_l)$ is $v^{l+1} - e_{-1}^{l+1}(g)$. This implies that $\eta_i(v) = (e_{-1})^i(g)$ for all $i \geq 0$.

Using this, the $\bar{\zeta}$-coefficient of $\mathcal{L}_v(\eta_0)$ is $e_{\overline{-1}}e_{-1}g + f_u g$. In general, the $\bar{\zeta}$-coefficient of $\mathcal{L}_v(\eta_i)$ is

$$e_{-1}v^i + \sum_{j=0}^{i-1} \binom{i-1}{j} T_u^{i-1-j} v^j.$$

Using $v^j = e_{-1}^j(g)$ and $T^i = (e_{-1})^i(f)$, this becomes $(e_{-1})^{i-1}(e_{\overline{-1}}e_{-1}g + f_u g)$. □

To state Noether's theorem in this context, we need to recall a standard definition and introduce a refinement of the space of generalized symmetries.

**Definition 10.4** If $v$ is a generalized symmetry, then $\eta_0(v)$ is its **generating function**.

**Definition 10.5** Let $\hat{\mathcal{S}} \subset \mathcal{S}$ be the subspace of proper generalized symmetries $v$ whose generating functions satisfy Eq. (13).

**Proposition 10.6** (Noether's Theorem) There is an isomorphism between $\hat{\mathcal{S}}$ and $\mathcal{C}$ given by sending the generating function of a generalized symmetry to the generating function of a conservation law.

*Proof* This follows immediately from the definitions, Lemma 10.3, and Theorem 8.3. □

The central equation to solve in order to produce either generalized symmetries or conservation laws is Eq. (14). This equation restricts to any integral manifold of $(M, \mathcal{I})$ defined by a solution $u(z, \bar{z})$ of Eq. (1) to be the linearization of Eq. (1):

$$A_{z\bar{z}} = -f_u A. \tag{33}$$

The Jacobi fields studied in [33] are defined to be solutions to Eq. (33) and thus are generating functions for both proper generalized symmetries of $(M^{(\infty)}, \mathcal{I}^{(\infty)})$ and conservation laws. This explains the appearance of the canonical Jacobi fields of [33] as generating functions for conservation laws in Lemma 9.2. The generating functions for conservation laws/generalized symmetries are produced by Olver [31] using recursion operators. However, his treatment is not complete because, in the case of the sin-Gordon equation, he does not prove that the recursion operator can be applied indefinitely to generate the full infinite sequence. More rigorous treatments have since been given by Guthrie [23], Dorfman [17], and Sanders and Wang [34]. The methods in [17] and [34] are specifically for evolution equations of the form $u_t = K(u)$, where $K$ depends on $u$ and its derivatives with respect to the other independent variables. The treatment in [23] is more general. One can also obtain the conservation laws for the hyperbolic case with $f(u) = -\frac{1}{4}\sin(u)$ using the minus one flow in the work of Terng and Uhlenbeck [38]. Presumably, the conservation laws studied in the present article are equivalent to those derived by Ward [26], but this is not clear to us.

## 11 Concluding remarks

We end this article with a number of observations. We begin with some issues internal to the theory of characteristic cohomology.

The spectral sequence machinery used in Sect. 5 to get a first approximation to the space of conservation laws is extremely useful. Without it, one has a bewildering freedom in the choice of a representative, which will not be easy to deal with. However, as we found in what is probably the simplest nontrivial class of elliptic equations, the machinery of Sect. 5 and the calculations of Sect. 6 still leave one with some very difficult equations to verify, even once the generating functions are found. This suggests that the use of the gauge symmetry (Sect. 7 and in particular Eq. (19)) in order to produce a direct relationship between solutions to the linearized equation and undifferentiated conservation laws may prove extremely useful, if not essential, in proving the existence of conservation laws for more complicated EDS.

We have begun exploring this for more complicated systems such as special Lagrangian 3-folds in $\mathbb{C}^3$, special Legendrian 3-folds in $\mathbb{S}^7$, and the EDS for constant mean curvature surfaces in three-dimensional space forms. In each of these cases, a gauge symmetry allows one to find a direct relationship between solutions of the linearized system and undifferentiated conservation laws. However, one is still left with the formidable challenge of finding solutions to the linearized equation. For surface geometries, recursion relations for Killing fields prove useful, but for higher-dimensional submanifold geometries, there is no theory of formal/polynomial Killing fields. It is unclear how to produce canonical solutions to the linearized equation for these higher dimensional systems. Adapting the theory of recursion operators [17,23] or Killing fields [12] to this context seems essential to developing a complete theory of exterior differential systems with infinitely many higher-order conservation laws.

A characteristic property of integrable equations is that they belong to a hierarchy of higher commuting flows [38,41]. These higher commuting flows can be understood as a canonical sequence of solutions to the linearization of the original equation. As described by Mukai-Hidano and Ohnita [30], the Killing fields of harmonic (or primitive) map systems are solutions to the linearization of the harmonic map equation. The framework of characteristic cohomology and the work in [33] suggest that these Killing fields are canonically defined objects on an appropriate jet space that one may restrict to any solution. It would be particularly interesting to develop an approach that could work for integral manifolds with any topology. Integrable system approaches to harmonic maps with higher genus domains have begun to appear [21,30]. Though at present the only approach to higher genus surface geometries (that do not have a Weierstrass representation) that has beared fruit has been through gluing constructions using geometric analysis [24,28].

Pinkall and Sterling [33] use the canonical Jacobi fields to define a notion of finite type solution. In the context of harmonic or primitive maps into homogeneous spaces, this has been generalized using the notion of formal and polynomial Killing fields [12]. One can use conservation laws to define a notion of finite type solutions which, in the case at hand, recovers the notion defined by Pinkall and Sterling. We will expand upon this and the relationship between formal/polynomial Killing fields and conservation laws in a forthcoming article.

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## References

1. Bobenko, A.I.: All constant mean curvature tori in $\mathbf{R}^3$, $S^3$, $H^3$ in terms of theta-functions. Math. Ann. **290**(2), 209–245 (1991)
2. Bolton, J., Pedit, F., Woodward, L.: Minimal surfaces and the affine Toda field model. J. Reine Angew. Math. **459**, 119–150 (1995)
3. Bryant, R.L., Chern, S.S., Gardner, R.B., Goldschmidt, H.L., Griffiths, P.A.: Exterior Differential Systems, Mathematical Sciences Research Institute Publications, vol. 18. Springer, New York (1991)
4. Bryant, R.L., Griffiths, P.A.: Characteristic cohomology of differential systems. I. General theory. J. Am. Math. Soc. **8**(3), 507–596 (1995)
5. Bryant, R.L., Griffiths, P.A.: Characteristic cohomology of differential systems. II. Conservation laws for a class of parabolic equations. Duke Math. J. **78**(3), 531–676 (1995)
6. Bryant, R.L., Griffiths, P.A., Grossman, D.: Exterior Differential Systems and Euler-Lagrange Partial Differential Equations, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL (2003)
7. Bryant, R.L., Griffiths, P.A., Hsu, L.: Hyperbolic exterior differential systems and their conservation laws. I. Selecta Math. (N.S.) **1**(1), 21–112 (1995)
8. Bryant, R.L., Griffiths, P.A., Hsu, L.: Hyperbolic exterior differential systems and their conservation laws. II. Selecta Math. (N.S.) **1**(2), 265–323 (1995)
9. Bryant, R.L., Griffiths, P.A., Hsu, L.: Toward a Geometry of Differential Equations, Geometry, Topology, and Physics. Conf. Proc. Lecture Notes Geom Topology, IV, Int. Press Cambridge, MA pp. 1–76
10. Burstall, F.E., Pedit, F.: Dressing orbits of harmonic maps. Duke Math. J. **80**(2), 353–382 (1995)
11. Burstall, F.E.: Harmonic maps and soliton theory. Mat. Contemp. **2**:1–18, (1992) Workshop on the Geometry and Topology of Gauge Fields (Campinas, 1991)
12. Burstall, F.E., Ferus, D., Pedit, F., Pinkall, U.: Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras. Ann. Math. (2) **138**(1), 173–212 (1993)

13. Clelland, J.N.: Geometry of conservation laws for a class of parabolic partial differential equations. Selecta Math. (N.S.) **3**(1), 1–77 (1997)
14. Clelland, J.N.: Geometry of conservation laws for a class of parabolic PDE's. II. Normal forms for equations with conservation laws. Selecta Math. (N.S.) **3**(4), 497–515 (1997)
15. Dai, B., Terng, C.-L.: Bäcklund transformations, ward solitons, and unitons. J. Differ. Geom. **75**(1), 57–108 (2007)
16. Dodd, R.K., Bullough, R.K.: Polynomial conserved densities for the sine-Gordon equations. Proc. Roy. Soc. Lond. Ser. A **352**(1671), 481–503 (1977)
17. Dorfman, I.: Dirac Structures and Integrability of Nonlinear Evolution Equations, Nonlinear Science: Theory and Applications. John Wiley & Sons Ltd., Chichester (1993)
18. Ferapontov, E.V., Khusnutdinova, K.R., Tsarev, S.P.: On a class of three-dimensional integrable Lagrangians. Commun. Math. Phys. **261**(1), 225–243 (2006)
19. Fox, D.: Boundaries of graphs of harmonic functions. SIGMA **5**(068), 8 (2009)
20. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Korteweg-deVries equation and generalization. VI. Methods for exact solution. Commun. Pure Appl. Math. **27**, 97–133 (1974)
21. Gerding, A., Heller, S., Pedit, F., Schmitt, N.: Global aspects of integrable surface geometry. In: Proceedings of Integrable Systems and Quantum Field Theory at Peyresq, Fifth Meeting (To appear)
22. Griffiths, P.A., Harris, J.D.: Principles of Algebraic Geometry. Wiley Classics Library, John Wiley & Sons Inc., New York (1994). Reprint of the 1978 original
23. Guthrie, G.A.: Recursion operators and non-local symmetries. Proc. Roy. Soc. Lond. Ser. A **446**(1926), 107–114 (1994)
24. Haskins, M., Kapouleas, N.: Special Lagrangian cones with higher genus links. Invent. Math. **167**(2), 223–294 (2007)
25. Hitchin, N.J.: Harmonic maps from a 2-torus to the 3-sphere. J. Differ. Geom. **31**(3), 627–710 (1990)
26. Ioannidou, T., Ward, R.S.: Conserved quantities for integrable chiral equations in $2 + 1$ dimensions. Phys. Lett. A **208**(3), 209–213 (1995)
27. Ivey, T.A., Landsberg, J.M.: Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems, Graduate Studies in Mathematics, vol. 61. American Mathematical Society, Providence (2003)
28. Kapouleas, N.: Complete constant mean curvature surfaces in Euclidean three-space. Ann. Math (2) **131**(2), 239–330 (1990)
29. McIntosh, I.: Harmonic tori and generalised Jacobi varieties. Commun. Anal. Geom. **9**(2), 423–449 (2001)
30. Mukai-Hidano, M., Ohnita, Y.: Gauge-theoretic approach to harmonic maps and subspaces in moduli spaces, integrable systems, geometry, and topology, AMS/IP Stud. Adv. Math., vol. 36, Am. Math. Soc., Providence, RI, pp. 191–234 (2006)
31. Olver, P.J.: Evolution equations possessing infinitely many symmetries. J. Math. Phys. **18**(6), 1212–1215 (1977)
32. Olver, P.J.: Applications of Lie groups to differential equations, 2nd edn., Graduate Texts in Mathematics, vol. 107. Springer, New York (1993)
33. Pinkall, U., Sterling, I.: On the classification of constant mean curvature tori. Ann. Math. (2) **130**(2), 407–451 (1989)
34. Sanders, J.A., Wang, J.P.: Integrable systems and their recursion operators. In: Proceedings of the Third World Congress of Nonlinear Analysts, Part 8 (Catania, 2000), vol. 47, pp. 5213–5240 (2001)
35. Shadwick, W.F.: The Hamilton-Cartan formalism for higher order conserved currents. I. Regular first order Lagrangians. Rep. Math. Phys. **18**(2), 243–256 (1983)
36. Shadwick, W.F.: Noether's theorem and Steudel's conserved currents for the sine-Gordon equation. Lett. Math. Phys. **4**(3), 241–248 (1980)
37. Terng, C.-L.: A higher dimension generalization of the sine-Gordon equation and its soliton theory. Ann. of Math. (2) **111**(3), 491–510 (1980)
38. Terng, C.-L., Uhlenbeck, K.: Bäcklund transformations and loop group actions. Commun. Pure Appl. Math. **53**(1), 1–75 (2000)
39. Uhlenbeck, K.: Harmonic maps into Lie groups: classical solutions of the chiral model. J. Differ. Geom. **30**(1), 1–50 (1989)
40. Wang, S.H.: Conservation laws for a class of third order evolutionary differential systems. Trans. Am. Math. Soc. **356**(10), 4055–4073 (2004)

41. Wilson, G.: Commuting flows and conservation laws for Lax equations. Math. Proc. Cambridge Philos. Soc. **86**(1), 131–143 (1979)
42. Zhiber, A.V., Shabat, A.B.: Klein-Gordon equations with a nontrivial group. Sov. Phys. Dokl. Akad. **24**(8), 607–609 (1979)