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Differential forms on arithmetic jet spaces

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Abstract We study derivations and differential forms on the arithmetic jet spaces of smooth schemes, relative to several primes. As applications, we give a new interpretation of arithmetic Laplacians, and we discuss the de Rham cohomology of some specific arithmetic jet spaces, especially arithmetic jet spaces of linear tori, elliptic curves, and Kummer surfaces.

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1 Introduction

Arithmetic jet spaces with respect to a single prime p (also called p-jet spaces) were introduced in [7] and further studied in a series of papers; see [9] and the bibliography therein. A multiple-prime generalization of these spaces was introduced independently in [10] and [3,4]. Multiple-prime arithmetic jet spaces are, in some sense, assembled from single prime arithmetic jet spaces, but the way in which they are assembled is non-trivial and, in particular, primes interact in a non-trivial manner, within the multiple-prime spaces. As explained in [9,10], arithmetic jet spaces can be viewed as an arithmetic analog of usual jet spaces in differential geometry and classical mechanics; the role of the derivatives with respect to various directions is played, in the arithmetic setting, by Fermat quotient operators with respect to various primes.

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In particular, the functions on arithmetic jet spaces with respect to several primes can be viewed as arithmetic analogs of classical partial differential equations on manifolds. See [10], for instance, for an arithmetic analog of Laplacians. Alternatively, arithmetic jet spaces can be viewed as a realization, within classical algebraic geometry, of what one might call absolute geometry, or geometry over the field with one element. See [2–4] or the introduction to [9].

As explained in [9], pp. 88–100, in the case of one prime, the tangent (and cotangent) bundles of arithmetic jet spaces carry some remarkable structures that are analogous to structures appearing in classical mechanics. In this paper, we extend this to the case of several primes, and then, as applications, we give a new interpretation of the arithmetic Laplacians in [10], and we discuss the de Rham cohomology of certain specific arithmetic jet spaces, especially arithmetic jet spaces of linear tori, elliptic curves, and Kummer surfaces. Although we will not make conjectures about general varieties, our results in the above-mentioned special cases suggest that *the de Rham cohomology of jet spaces of varieties should encode subtle arithmetic information about the varieties in question*. The latter "conjectural principle" is one of the main motivations behind the de Rham computations in the present paper.

In order to explain our results in some detail recall that in the paper [10], one considers arithmetic jet spaces $Y = \mathcal{J}^r_{\mathcal{P}}(X)$ of order $r \in \mathbb{Z}^d_{\geq 0}$ attached to smooth schemes X over $\mathbb{Z}$, with respect to a finite set of primes $\mathcal{P} = \{p_1, \dots, p_d\}$, and in case X is a one dimensional group scheme, one constructs arithmetic analogs of Laplacians. These *arithmetic Laplacians* are constructed as families $(f_1, \dots, f_d)$ where each f_k is a formal function on the completion of Y along $Y_{p_k} := Y \otimes (\mathbb{Z}/p_k\mathbb{Z})$; these formal functions are required to be "analytically continued" along the zero section Z of X in the sense that there exists a formal function f_0 on the completion of Y along Z such that f_0 and f_k coincide on the completion of Y along $Z \cap Y_{p_k}$ for each k. In this paper, we want to revisit the idea of analytic continuation between different primes. Indeed, we will show that:

Theorem 1.1 *The arithmetic Laplacians $(f_1, \dots, f_d)$ have the property that the 1-forms $df_1, \dots, df_d$ extend to 1-forms defined on the whole of Y and these extended 1-forms all coincide.*

See Theorems 6.3 and 7.2 below, and the discussion preceding them, for the precise statement of the result and the precise definitions of arithmetic Laplacians. So, the arithmetic Laplacians appear as primitives, existing only on formal neighborhoods of certain divisors, of a 1-form that exists on the whole of the arithmetic jet space Y. This provides an alternative view on analytic continuation between different primes and has a global flavor, as opposed to the formally local flavor of the one in [10].

As a consequence of the above, we will show that for X a linear torus, an affine open set of an elliptic curve, or an affine open set of the Kummer surface attached to the twofold product of an elliptic curve and for $r = (1, \dots, 1)$ (in the case of the linear torus) respectively $r = (2, \dots, 2)$ (in the other cases) we have

Theorem 1.2 *The canonical volume form on the arithmetic jet space $Y = \mathcal{J}^r_{\mathcal{P}}(X)$ is not exact on Y, but it is exact on the completion of Y along each Y_{p_k}.*

See Corollaries 6.4, 7.3, and Theorem 8.1 below, and the discussion preceding them, for precise statements and for the definition of the *canonical volume form*.

Here is the plan of the paper. In section 2, we formulate this concept of analytic continuation based on globally defined differential forms; this makes sense on any scheme Y, not necessarily on arithmetic jet spaces. In section 3, we review and complement the main concepts in [3,4,10] related to arithmetic jet spaces. In Sects. 4 and 5, we extend the discussion in [9] on the tangent and cotangent bundles of arithmetic jet spaces from the case of one prime to the case of several primes. In Sects. 6, 7, and 8, we apply these concepts to the case when X is, respectively, the multiplicative group, an affine open set of an elliptic curve, or an affine open set of a K3 surface attached, via the Kummer construction, to a product of elliptic curves. We will perform "analytic continuation" on the arithmetic jet spaces of such schemes X, between various primes, based on globally defined differential forms, and as a by-product, we will derive various de Rham-style consequences for the arithmetic jet spaces in question. In particular, we prove Theorems 1.1 and 1.2. We have chosen to restrict ourselves to the case when X is affine because this simplifies the exposition and also captures the essential points of the theory. However, our theory can be extended to the case when X is not necessarily affine.

Remark 1.3 One should call the attention upon the fact that our arithmetic jet spaces are objects completely different from Vojta [14]; indeed, Vojta's jet spaces are constructed using Hasse-Schmidt derivations (which are morally "differentiations in the geometric direction") while our jet spaces are constructed using Fermat quotients (which are morally "differentiations in the arithmetic directions".)

2 Analytic continuation between primes via differential forms

For any ring B (respectively scheme Y), we denote by Ω_B (respectively Ω_Y) the B-module (respectively the sheaf on Y) of Kähler differentials of B (respectively Y) over $\mathbb{Z}$. We denote by

$$T_B = Hom_B(\Omega_B, B) = Der(B, B)$$

the dual of Ω_B; we denote by T_Y the dual sheaf of Ω_Y, which we refer to as the *tangent sheaf* of Y. Also for $i \geq 0$, we let $\Omega^i_B = \wedge^i \Omega_B$, $\Omega^i_Y = \wedge^i \Omega_Y$ denote the exterior powers of Ω_B, Ω_Y, respectively. Elements of $H^0(Y, \Omega^i_Y)$ are referred to as *i-forms* on Y. Also, we have at our disposal de Rham complexes (Ω^*_B, d) and (Ω^*_Y, d), respectively, and we have the usual notions of closed and exact forms. Recall that if B, Y are smooth over $\mathbb{Z}$ (or more generally over a ring of fractions of $\mathbb{Z}$) of relative dimension m then Ω^i_B, Ω^i_Y are locally free of rank $\binom{m}{i}$; by a *volume form* on Y, we will mean an m-form on Y that is invertible (i.e. is a basis of Ω^m_Y). Volume forms are, of course, closed.

If M is a module over a ring B and if I is an ideal in B, we denote by

$$M^{\widehat{I}} = \lim_{\leftarrow} M/I^n M$$

the I-adic completion of M. We say that M is *I-adically complete* if the map $M \to M^{\widehat{I}}$ is an isomorphism. For any Noetherian scheme Y and any closed subscheme $Z \subset Y$, we denote by $Y^{\widehat{Z}}$ the formal scheme obtained by completing Y along Z. If I is the ideal sheaf of Z in Y, we also write $Y^{\widehat{I}}$ in place of $Y^{\widehat{Z}}$. We let $Z_n \subset Y$ be the closed subscheme with ideal I^n. We let

$$\Omega^i_{Y^{\widehat{I}}} = \lim_{\leftarrow} \Omega^i_{Z_n}, \quad T_{Y^{\widehat{I}}} = \lim_{\leftarrow} T_{Z_n}.$$

Elements of $H^0(Y^{\widehat{I}}, \Omega^i_{Y^{\widehat{I}}})$ are referred to as *i-forms* on $Y^{\widehat{I}}$. Of course, $H^0(Y^{\widehat{I}}, \Omega^0_{Y^{\widehat{I}}})$ is just the space of formal functions $\mathcal{O}(Y^{\widehat{I}})$. Again, we have at our disposal an obvious deRham complex and a notion of closed and exact forms. If Y is smooth over $\mathbb{Z}$ (or a ring of fractions of $\mathbb{Z}$), we have a concept of volume form.

We will use the following basic terminology:

Definition 2.1 An $(i-1)$-form $\eta \in H^0(Y^{\widehat{I}}, \Omega^{i-1}_{Y^{\widehat{I}}})$ is an *I-adic primitive* of an i-form $\nu \in H^0(Y, \Omega^i_Y)$ if $\nu = d\eta$ in $H^0(Y^{\widehat{I}}, \Omega^i_{Y^{\widehat{I}}})$. An i-form $\nu \in H^0(Y, \Omega^i_Y)$ is called *I-adically exact* if it has a I-adic primitive.

In what follows, throughout the paper, we consider a finite set of primes $\mathcal{P} = \{p_1, \ldots, p_d\} \subset \mathbb{Z}$ and we will denote by $A_0 = S^{-1}\mathbb{Z}$ a ring of fractions of $\mathbb{Z}$ with respect to a multiplicative system S of integers coprime to the primes in $\mathcal{P}$.

Let Y be a smooth affine scheme of finite type over A_0. We will be interested in the following examples of subschemes $Z \subset Y$. As before we denote by I the ideal defining Z.

1) *Vertical case:* Z is defined by the ideal $I = (p_1 \ldots p_d)$. In this case

$$Y^{\widehat{p_1 \cdots p_d}} = \coprod_{k=1}^{d} Y^{\widehat{p_k}}$$

and hence

$$H^0(Y^{\widehat{p_1 \cdots p_d}}, \Omega^i_{Y^{\widehat{p_1 \cdots p_d}}}) = \prod_{k=1}^{d} H^0(Y^{\widehat{p_k}}, \Omega^i_{Y^{\widehat{p_k}}}).$$

In this case, we shall use the phrases *$\mathcal{P}$-adically exact* and *$\mathcal{P}$-adic primitive* in place of I-adically exact and I-adic primitive, respectively. If Y is smooth over A_0, then $\Omega^i_{Y^{\widehat{p_k}}}$ are locally free.

2) *Horizontal case:* Z is the image in Y of an A_0-point $P \in Y(A_0)$, i.e. of a section $P : \operatorname{Spec} A_0 \to Y$ of $Y \to \operatorname{Spec} A_0$. In this case, the ideal I is the kernel of $P^* : \mathcal{O}(Y) \to A_0$; by abuse of notation, we denote the ideal I, again, by P. If $\mathcal{S} \subset Y(A_0)$ is a set of points, we use the phrase *$\mathcal{S}$-addically exact* to mean P-adically exact for all $P \in \mathcal{S}$. Let now $P \in Y(A_0)$ be a *uniform point* (in the sense of [10], Definition 2.25); recall that this means that there exists a Zariski

open set $Y_1 \subset Y$ containing the image of P and possessing an étale morphism $Y_1 \to \mathbb{A}^N$ to an affine space $\mathbb{A}^N = \text{Spec } A_0[T]$, where $T = \{T_j\}$ is a tuple of variables. If this is the case, then one can choose Y_1 and T such that the ideal in $\mathcal{O}(Y_1)$ of the image of $P : \text{Spec } A_0 \to Y_1$, i.e. the kernel of $P^* : \mathcal{O}(Y_1) \to A_0$, is generated by T; we then say that T are *uniform* coordinates on Y_1. It then follows that the sheaves $\Omega^i_{Y^{\widehat{P}}}$ are free and

$$H^0(Y^{\widehat{P}}, \Omega^i_{Y^{\widehat{P}}}) = \bigoplus_{j_1<\ldots<j_i} A_0[[T]]dT_{j_1} \wedge \cdots \wedge dT_{j_i}.$$

3) *Case* $Z = Z' \cap Z''$ where the ideal of Z' is generated by p_k for some k, and Z'' is the image in Y of a point $P \in Y(A_0)$; hence, Z is defined by the ideal $I = (p_k, P)$. In this case

$$H^0(Y^{\widehat{(p_k,P)}}, \Omega^i_{Y^{\widehat{(p_k,P)}}}) = \bigoplus_{j_1<\cdots<j_i} \mathbb{Z}_{p_k}[[T]]dT_{j_1} \wedge \ldots \wedge dT_{j_i}.$$

Corresponding to examples 1) and 2) above, one can introduce more terminology as follows. Assume Y is a smooth affine scheme over A_0 with irreducible geometric fibers, let $P \in Y(A_0)$ be a point, and let ω be a 1-form on Y.

1) If ω is $\mathcal{P}$-adically exact, then there is a unique $\mathcal{P}$-adic primitive

$$f = (f_k)_k = (f_1, \ldots, f_d) \in \mathcal{O}(Y^{\widehat{p_1\cdots p_d}}) = \mathcal{O}(Y^{\widehat{p_1}}) \times \ldots \times \mathcal{O}(Y^{\widehat{p_d}}) \qquad (2.1)$$

of ω with $f_k(P) = 0$ for all $k = 1, \ldots, d$. (Here, $f_k(P)$ is the image of f_k via the map $\mathcal{O}(Y^{\widehat{p_k}}) \to \mathbb{Z}_{p_k}$ defined by P.) We shall refer to $f = (f_k)_k$ as the *$\mathcal{P}$-adic primitive of ω normalized along P*.

2) If ω is P-adically exact, then there is a unique P-adic primitive $f_0 \in \mathcal{O}(Y^{\widehat{P}})$ of ω such that $f_0(P) = 0$. (Here, $f_0(P)$ is the image of f_0 via the map $\mathcal{O}(Y^{\widehat{P}}) \to A_0$ defined by P.) We shall refer to f_0 as the *P-adic primitive of ω normalized along P*.

Remark 2.2 The concepts above are directly related to the concept of *analytic continuation between primes* considered in [10]. Indeed, assume Y is a smooth affine scheme over A_0 with irreducible geometric fibers, let $P \in Y(A_0)$ be a uniform point, and let ω be a 1-form on Y. Assume that ω is both $\mathcal{P}$-adically exact and P-adically exact. (This will be often the case in the main examples to be encountered later in the paper.) Let $f = (f_k)$ and f_0 be the $\mathcal{P}$-adic and the P-adic primitives of ω normalized along P, respectively. Since $df_k = df_0$ in $\bigoplus_j \mathbb{Z}_{p_k}[[T]]dT_j$ and $f_k(P) = f_0(P)$ in $\mathbb{Z}_{p_k}$, it follows that $f_k = f_0$ in $\mathbb{Z}_{p_k}[[T]]$ which shows that f is *analytically continued along* P and is *represented by* f_0 in the sense of [10], Definition 2.23.

Remark 2.3 The above concepts can be used to define a certain type of integration (and periods) of 1-forms in the arithmetic setting as follows. Indeed, assume again that Y is a smooth affine scheme over A_0 with irreducible geometric fibers and let $\mathcal{S} \subset Y(A_0)$

be a non-empty set of points. (In our applications, $\mathcal{S}$ will be the whole of $Y(A_0)$.) By an *elementary chain*, we will understand a pair (P_1, P_2) of points $P_1 \in Y(\mathbb{Z}_{p_{k_1}})$, $P_2 \in Y(\mathbb{Z}_{p_{k_2}})$ such that either 1) $k_1 = k_2 =: k$ or 2) $k_1 \neq k_2$ and there exists $P_{12} \in \mathcal{S}$ inducing both P_1 and P_2. In case 1), we say (P_1, P_2) is *vertical* and we let $\overline{P_1 P_2}$ denote the ideal (p_k) in $\mathcal{O}(Y)$. In case 2), we say (P_1, P_2) is *horizontal* and denote by $\overline{P_1 P_2}$ the ideal $P_{12} \subset \mathcal{O}(Y)$. By a *chain*, we understand a tuple $\Gamma := (P_1, \dots, P_N)$ where $(P_1, P_2), \dots, (P_{N-1}, P_N)$ are elementary chains. A chain as above is called a *cycle* if $P_1 = P_N$. We define the *group of abstract periods*

$$\Pi := \frac{\bigoplus_{k=1}^{d} \mathbb{Z}_{p_k}}{\{(a_1, \dots, a_d) \in A_0^d;\ \sum_{k=1}^{d} a_k = 0\}}.$$

For each k, the natural morphism $\mathbb{Z}_{p_k} \to \Pi$ is injective, and we shall view it as an inclusion. Now let ω be a 1-form on Y which is both $\mathcal{P}$-adically exact and $\mathcal{S}$-adically exact. Then, one can define the *integral of ω along a chain* $\Gamma = (P_1, \dots, P_N)$ by the formula

$$\int_{\Gamma} \omega := \sum_{j=1}^{N-1} (f_j(P_{j+1}) - f_j(P_j)) \in \Pi, \tag{2.2}$$

where f_j is any $\overline{P_j P_{j+1}}$-adic primitive of ω. (Here, if (P_j, P_{j+1}) is horizontal, then $f_j(P_j)$ and $f_j(P_{j+1})$ are defined as the corresponding images of f_j via the homomorphisms $\mathcal{O}(Y^{\widehat{P_{12}}}) \to A_0 \subset \mathbb{Z}_{p_{k_j}}$ and $\mathcal{O}(Y^{\widehat{P_{12}}}) \to A_0 \subset \mathbb{Z}_{p_{k_{j+1}}}$ defined by P_{12} so they are equal; hence, in the sum (2.2), the terms corresponding to horizontal elementary chains are equal to 0. Also, note that summation above gives a well-defined element of $\bigoplus_k \mathbb{Z}_{p_k}$, not only of Π.) If Γ is a cycle, we may refer to $\int_\Gamma \omega$ as a *period*. Note that if ω is exact on Y, then its periods vanish. (Indeed, for ω exact on Y, the summation (2.2) viewed as an element of $\bigoplus_k \mathbb{Z}_{p_k}$, although generally not zero, becomes zero in Π.)

In the main examples to be later encountered in the paper, the periods will be typically non-zero. In contrast with this phenomenon, in the projective (as opposed to the affine) case, and under certain Arakelov-style conditions at infinity, forms that are $\mathcal{P}$-adically and $\mathcal{S}$-adically exact should be expected to be exact; cf. [6], where arithmetic analogs of the Hironaka–Matsumura theorems [12] from formal geometry are proved.

3 Arithmetic jet spaces: review and complements

In this section, we review some of the concepts introduced in [3,4,7,9,10] and provide some complements to that theory.

Let $C_p(X, Y) \in \mathbb{Z}[X, Y]$ be the polynomial with integer coefficients

$$C_p(X, Y) := \frac{X^p + Y^p - (X + Y)^p}{p}.$$

A *p-derivation* from a ring A into an A-algebra $\varphi : A \to B$ is a set map $\delta : A \to B$ such that

$$\delta(1) = 0, \tag{3.1}$$

$$\delta(x + y) = \delta x + \delta y + C_p(x, y), \tag{3.2}$$

$$\delta(xy) = \varphi(x^p) \cdot \delta y + \varphi(y^p) \cdot \delta x + p \cdot \delta x \cdot \delta y, \tag{3.3}$$

for all $x, y \in A$. Clearly, the conditions above are equivalent to saying that the map from A to $B \times B$ given by $a \mapsto (\varphi(a), \delta a)$ is a ring homomorphism if we view $B \times B$ equipped with the Witt addition and multiplication. Given a p-derivation we always denote by $\phi : A \to B$ the map

$$\phi(x) = \varphi(x)^p + p\delta x; \tag{3.4}$$

then ϕ is a ring homomorphism. Conversely, if p is not a zero divisor in B and if $\phi : A \to B$ is a ring homomorphism satisfying the Frobenius lift property $\phi(x) \equiv \varphi(x)^p \mod pB$ for all $x \in A$, then there is a unique p-derivation $\delta : A \to B$ satisfying (3.4). In particular, $\mathbb{Z}$ has a unique p-derivation δ_p given by $\delta_p n = \frac{n-n^p}{p}$.

For any two distinct rational primes p_1, p_2, consider the polynomial C_{p_1,p_2} in the ring $\mathbb{Z}[X_0, X_1, X_2]$ defined by

$$C_{p_1,p_2}(X_0, X_1, X_2) := \frac{C_{p_2}(X_0^{p_1}, p_1X_1)}{p_1} - \frac{C_{p_1}(X_0^{p_2}, p_2X_2)}{p_2} - \frac{\delta_{p_1}p_2}{p_2}X_2^{p_1} + \frac{\delta_{p_2}p_1}{p_1}X_1^{p_2}. \tag{3.5}$$

Let $\mathcal{P} = \{p_1, \dots, p_d\}$ be a finite set of primes in $\mathbb{Z}$. A *$\delta_{\mathcal{P}}$-ring* is a ring A equipped with p_k-derivations $\delta_{p_k} : A \to A$, $k = 1, \dots, d$, such that

$$\delta_{p_k}\delta_{p_l}a - \delta_{p_l}\delta_{p_k}a = C_{p_k,p_l}(a, \delta_{p_k}a, \delta_{p_l}a) \tag{3.6}$$

for all $a \in A$, $k, l = 1, \dots, d$. A *homomorphism of $\delta_{\mathcal{P}}$-rings* A and B is a homomorphism of rings $\varphi : A \to B$ that commutes with the p_k-derivations in A and B, respectively. If $\phi_{p_k}(x) = x^{p_k} + p_k\delta_{p_k}x$ is the homomorphism associated to δ_{p_k}, condition (3.6) implies that

$$\phi_{p_k}\phi_{p_l}(a) = \phi_{p_l}\phi_{p_k}(a) \tag{3.7}$$

for all $a \in A$. Conversely, if the commutation relations (3.7) hold, and the numbers p_k are not zero divisors in A, then conditions (3.6) hold, and we have that $\phi_{p_k}\delta_{p_l}a = \delta_{p_l}\phi_{p_k}a$ for all $a \in A$. If A is a $\delta_{\mathcal{P}}$-ring then for all k, the p_k-adic completions $A^{\widehat{p_k}}$ are $\delta_{\mathcal{P}}$-rings in a natural way.

For a relation between these concepts and the theory of λ-rings, we refer to [3] and the references therein.

We let $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$, and let $\mathbb{Z}^d_{\geq 0}$ be given the product order. We let e_k be the element of $\mathbb{Z}^d_{\geq 0}$ all of whose components are zero except the k-th, which is 1. We set $e = \sum e_k = (1, \ldots, 1)$. For $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d_{\geq 0}$, we set $\mathcal{P}^i = p_1^{i_1} \ldots p_d^{i_d}$ and $\delta^i_{\mathcal{P}} = \delta^{i_1}_{p_1} \circ \cdots \circ \delta^{i_d}_{p_d}$ and $\phi^i_{\mathcal{P}} = \phi_{\mathcal{P}^i} = \phi^{i_1}_{p_1} \circ \cdots \circ \phi^{i_d}_{p_d}$. A $\delta_{\mathcal{P}}$*-prolongation system* $A^* = (A^r)$ is an inductive system of rings A^r indexed by $r \in \mathbb{Z}^d_{\geq 0}$, provided with transition maps $\varphi_{rr'} : A^r \to A^{r'}$ for any pair of indices r, r' such that $r \leq r'$, and equipped with p_k-derivations

$$\delta_{p_k} : A^r \to A^{r+e_k},$$

$k = 1, \ldots, d$, such that (3.6) holds for all k, l, and such that

$$\varphi_{r+e_k, r'+e_k} \circ \delta_{p_k} = \delta_{p_k} \circ \varphi_{rr'} : A^r \to A^{r'+e_k}$$

for all $r \leq r'$ and all k. A morphism of prolongation systems $A^* \to B^*$ is a system of ring homomorphisms $u^r : A^r \to B^r$ that commute with all the maps $\varphi_{rr'}$ and the δ_{p_k} of A^* and B^*.

Any $\delta_{\mathcal{P}}$-ring A induces a $\delta_{\mathcal{P}}$-prolongation system A^* where $A^r = A$ for all r and φ is the identity map. Conversely, if (A^r) is a $\delta_{\mathcal{P}}$-prolongation system, then the ring

$$A^\infty := \varinjlim A^r$$

has a natural structure of $\delta_{\mathcal{P}}$-ring.

Let us say that a $\delta_{\mathcal{P}}$-prolongation system (A^r) is *faithful* if the primes in $\mathcal{P}$ are non-zero divisors in all the rings A^r and if all the homomorphisms $\varphi_{rr'} : A^r \to A^{r'}$ are injective. When this is the case, we will usually view $\varphi_{rr'}$ as inclusions $A^r \subset A^{r'}$ and the primes in $\mathcal{P}$ are non-zero divisors in A^∞.

If $A^* = (A^r)$ is a $\delta_{\mathcal{P}}$-prolongation system, then for each $p_k \in \mathcal{P}$, the system of p_k-adic completions $((A^r)^{\widehat{p_k}})_r$ is easily seen to have a unique structure of $\delta_{\mathcal{P}}$-prolongation system with the property that the natural maps $A^r \to (A^r)^{\widehat{p_k}}$ define a morphism of prolongation systems. Further, the operators δ_{p_k} are continuous.

Let us say that a $\delta_{\mathcal{P}}$-ring A is $\delta_{\mathcal{P}}$*-generated* by a subring $A^0 \subset A$ if A is generated as an A^0-algebra by the set $\{\delta^s_{\mathcal{P}} a;\ a \in A^0,\ s \geq 0\}$.

For any affine scheme of finite type X over $A_0 = S^{-1}\mathbb{Z}$, one can define a system of schemes of finite type, $\mathcal{J}^r_{\mathcal{P}}(X)$ over A_0, called the $\delta_{\mathcal{P}}$*-jet spaces* of X (or $\mathcal{P}$*-typical jet spaces* in the language of [4]); if $X = \text{Spec}\, A^0$, with $A^0 = A_0[T]/(f)$, T a tuple of variables, and f a tuple of polynomials, then $\mathcal{J}^r_{\mathcal{P}}(X) = \text{Spec}\, A^r$, where

$$A^r = A_0[\delta^i_{\mathcal{P}} T; i \leq r]/(\delta^i_{\mathcal{P}} f; i \leq r). \tag{3.8}$$

Here, each $\delta^i_{\mathcal{P}} T$ is to be interpreted as a tuple of free variables, and each relation $\delta^i_{\mathcal{P}} f$ is interpreted as a polynomial expression in the variables $\delta^i_{\mathcal{P}} T$ by expanding it using the rules (3.1)–(3.3) and (3.6). These same rules define a structure of $\delta_{\mathcal{P}}$-prolongation system on the family $(A^r)_r$. In the case where $\mathcal{P}$ consists of a single prime p, formal

p-adic versions of these spaces were introduced in [7]. In the multiple-prime case considered here, they were introduced (independently) in [10] and [3,4]. In [4], the spaces $\mathcal{J}^r_{\mathcal{P}}(X)$ were denoted by $W_{r*}(X)$; the notation here follows [10].

The prolongation system (A^r), where A^r is still $\mathcal{O}\left(\mathcal{J}^r_{\mathcal{P}}(X)\right)$, has the following universal property: for any $\delta_{\mathcal{P}}$-prolongation system (B^r), each ring map $A^0 \to B^0$ extends uniquely to a map $(A^r) \to (B^r)$ of $\delta_{\mathcal{P}}$-prolongation systems. Let us now apply this in the case where B^r is the constant inductive system A_0, where each B^r is A_0 and each $\varphi_{rr'}$ the identity map. Since A_0 has a unique $\delta_{\mathcal{P}}$-ring structure, this system has a unique $\delta_{\mathcal{P}}$-prolongation structure. By the universal property, any ring map $A^0 \to A_0$ extends to a unique map $A^r \to B^r = A_0$ of $\delta_{\mathcal{P}}$-prolongation systems and hence defines a map $X(A_0) \to \mathcal{J}^r_{\mathcal{P}}(X)(A_0)$ for each r. We call the image of a point $P \in X(A_0)$ under this map its *canonical lift* (to $\mathcal{J}^r_{\mathcal{P}}(X)$). We will often denote it by P^r (because of its relation with the map (3.10) below).

For A_0-algebras C, there is a functorial isomorphism

$$\mathrm{Hom}_{A_0}(\mathrm{Spec}\, C, \mathcal{J}^r_{\mathcal{P}}(X)) = \mathrm{Hom}_{A_0}(\mathrm{Spec}\, W_r(C), X) \tag{3.9}$$

Here, $W_r(C)$ denotes the A_0-algebra of $\mathcal{P}$-typical Witt vectors of length r with entries in C. If $\mathcal{P}$ consists of a single prime p, then $W_r(C)$ is the usual ring of p-typical Witt vectors of length r (length $r+1$ in the traditionally more common numbering) with entries in C. In [4], the equation (3.9) is taken to be the definition of $\mathcal{J}^r_{\mathcal{P}}(X)$, or rather of its functor of points. This lets us define $\mathcal{J}^r_{\mathcal{P}}(X)$ for rather general X, such as algebraic spaces. We will not need this generality here, but we will make use of (3.9), as well as some results in [4] proved using this functorial point of view, such as the following:

Proposition 3.1 *Let $f : X' \to X$ be a smooth (respectively étale) morphism. Then, the induced morphism $g : \mathcal{J}^r_{\mathcal{P}}(X') \to \mathcal{J}^r_{\mathcal{P}}(X)$ is smooth (respectively étale). If f is also surjective, then so is g.*

The rings $A^r = \mathcal{O}(\mathcal{J}^r_{\mathcal{P}}(X))$ form a $\delta_{\mathcal{P}}$-prolongation system with A^∞ being $\delta_{\mathcal{P}}$-generated by A^0. If $Y \subset X$ is a principal open set of X, $\mathcal{O}(Y) = \mathcal{O}(X)_f$, then $\mathcal{O}(\mathcal{J}^r_{\mathcal{P}}(Y)) \simeq \mathcal{O}(\mathcal{J}^r_{\mathcal{P}}(X))_{f_r}$ where $f_r = \prod_{i \le r} \phi^i_{\mathcal{P}}(f)$. In particular, the induced morphism $\mathcal{J}^r_{\mathcal{P}}(Y) \to \mathcal{J}^r_{\mathcal{P}}(X)$ is an open immersion.

In corollary 3.6, we will show that if X/A_0 smooth and $\mathcal{J}^r_{\mathcal{P}}(X) = \mathrm{Spec}\, A^r$, then the prolongation system (A^r) is faithful.

Let us note that the homomorphisms

$$\varphi_{sr} \circ \phi^s_{\mathcal{P}} : A^0 \to A^r$$

for $s \le r$ induce morphisms $\mathcal{J}^r_{\mathcal{P}}(X) \to X$, and hence a natural morphism

$$\mathcal{J}^r_{\mathcal{P}}(X) \xrightarrow{\kappa_{\le r}} \prod_{s \le r} X. \tag{3.10}$$

This map becomes an isomorphism after tensoring with $\mathbb{Z}[1/p_1, \ldots, 1/p_k]$.

Proposition 3.2 *If X is a group scheme, then the maps $\mathcal{J}^s_{\mathcal{P}}(X) \to \mathcal{J}^r_{\mathcal{P}}(X)$ and $\mathcal{J}^s_{\mathcal{P}}(X) \to X$ induced by φ_{sr} and $\phi^s_{\mathcal{P}}$ are group homomorphisms.*

Proof It is enough to show that each map induces a group homomorphism on B-valued points for each A_0-algebra B. We can describe these maps simply using (3.9). For φ_{sr}, it can be identified with the map

$$X(W_s(B)) \to X(W_r(B))$$

induced by a map $W_s(B) \to W_r(B)$ (namely, the natural projection map). For $\phi^s_{\mathcal{P}}$, it can be identified with the map

$$X(W_s(B)) \to X(B)$$

induced by a map $W_s(B) \to B$ (namely, the s-th ghost component map). In particular, each map is a group homomorphism. □

Remark 3.3 Let X be a smooth scheme over A_0 and consider an affine open cover $X = \bigcup X_i$. Let p be a prime and let $r \geq 0$ be an integer. Then, one can glue the formal schemes $\mathcal{J}^r_{\{p\}}(X_i)\widehat{^p}$ along the formal schemes $\mathcal{J}^r_{\{p\}}(X_i \cap X_j)\widehat{^p}$ to construct a formal scheme that we denote by $J^r_p(X)$. This latter formal scheme does not depend on the covering we chose for X and will be referred to as the *p-jet space* of X of order r. After base change to $\hat{\mathbb{Z}}^{ur}_p$ (the completion of the maximum unramified extension of $\mathbb{Z}_p$), $J^r_p(X)$ becomes equal to the p-jet space of X considered in [7,9]. Note that taking formal completions here is needed for this approach to work. Otherwise, one would use the method of [4].

Proposition 3.4 *Let X be a smooth affine scheme over A_0, and let $r, s \in \mathbb{Z}^d_{\geq 0}$ be elements with $r \leq s$. Then, the map $\mathcal{J}^s_{\mathcal{P}}(X) \to \mathcal{J}^r_{\mathcal{P}}(X)$ induced by φ_{sr} is smooth and surjective.*

Proof First, let us reduce to the case where the number d of primes in $\mathcal{P}$ is 1. The case $d = 0$ is trivial. Now assume the theorem when $d = 1$, and suppose $d \geq 2$. Let $\mathcal{P}'$ denote the set $\{p_1, \dots, p_{d-1}\}$, let s' denote $(s_1, \dots, s_{d-1})$, and let r' denote $(r_1, \dots, r_{d-1})$. Then, we have the following factorization of the map in question:

$$\begin{array}{ccccc}
\mathcal{J}^s_{\mathcal{P}}(X) & \longrightarrow & & & \mathcal{J}^r_{\mathcal{P}}(X) \\
\downarrow \wr & & & & \downarrow \wr \\
\mathcal{J}^{s_d}_{\{p_d\}}\mathcal{J}^{s'}_{\mathcal{P}'}(X) & \xrightarrow{a} & \mathcal{J}^{s_d}_{\{p_d\}}\mathcal{J}^{r'}_{\mathcal{P}'}(X) & \xrightarrow{b} & \mathcal{J}^{r_d}_{\{p_d\}}\mathcal{J}^{r'}_{\mathcal{P}'}(X).
\end{array}$$

By Proposition 3.1, the space $\mathcal{J}^{r'}_{\mathcal{P}'}(X)$ is smooth over $\mathcal{J}^{r'}_{\mathcal{P}'}(\mathrm{Spec}\, A_0) = \mathrm{Spec}\, A_0$. Therefore by the case $d = 1$, the map b is smooth and surjective. And by induction, the map $\mathcal{J}^{s'}_{\mathcal{P}'}(X) \to \mathcal{J}^{r'}_{\mathcal{P}'}(X)$ is smooth and surjective. By [4] (11.1(c)), the map a is then smooth and surjective, and therefore $b \circ a$ is.

Thus, it suffices to assume $d = 1$. Let us write $p = p_1$, $\mathcal{J} = \mathcal{J}_{\{p\}}$, and so on. We can further assume $s = r + 1$ with $r \geq 0$, again by induction. Let φ denote the map $\mathcal{J}^{r+1}(X) \to \mathcal{J}^r(X)$.

To show φ is smooth, we need to show it is formally smooth and locally of finite presentation. Finite presentation is clear: by (3.8), both $\mathcal{J}^{r+1}(X)$ and $\mathcal{J}^r(X)$ are of finite type over A_0, which is Noetherian.

Let us now show that φ is formally smooth. Let $g : B \to A$ be a surjection of rings with square-zero kernel. By definition, we need to show that the evident map

$$\mathcal{J}^{r+1}(X)(B) \longrightarrow \mathcal{J}^{r+1}(X)(A) \times_{\mathcal{J}^r(X)(A)} \mathcal{J}^r(X)(B)$$

is a surjection. By (3.9), we can identify this map with the evident map

$$X(W_{r+1}(B)) \longrightarrow X(W_{r+1}(A)) \times_{X(W_r(A))} X(W_r(B)).$$

Since X is affine, this map can be identified with the evident map

$$X(W_{r+1}(B)) \longrightarrow X(W_{r+1}(A) \times_{W_r(A)} W_r(B)).$$

Thus, since X is formally smooth over $\mathbb{Z}$, we only need to show that the evident ring map

$$W_{r+1}(B) \longrightarrow W_{r+1}(A) \times_{W_r(A)} W_r(B) \tag{3.11}$$

is surjective with nilpotent kernel. Using the usual Witt components, we can identify this with the map

$$B^{r+2} \longrightarrow A^{r+2} \times_{A^{r+1}} B^{r+1},$$

defined by

$$(b_0, \ldots, b_{r+1}) \mapsto ((g(b_0), \ldots, g(b_{r+1})), (b_0, \ldots, b_r)).$$

This map can further be identified with the map

$$B^{r+2} \longrightarrow B^{r+1} \times A \tag{3.12}$$

defined by $(b_0, \ldots, b_{r+1}) \mapsto (b_0, \ldots, b_r, g(b_{r+1}))$. Since g is surjective, so is (3.12), and hence so is (3.11). On the other hand, the kernel of (3.12) is the set of elements $(0, \ldots, 0, b)$, where $b \in \ker(g)$. This kernel then has square zero, since we have

$$(0, \ldots, 0, b)(0, \ldots, b') = (0, \ldots, 0, p^{r+1}bb') = 0 \tag{3.13}$$

and $\ker(g)^2 = 0$. Therefore, (3.11) has square-zero kernel. It follows that φ is formally smooth.

Let us finally show that $\mathcal{J}^{r+1}(X) \to \mathcal{J}^r(X)$ is surjective. It suffices to show this after base change to $\mathbb{Z}[1/p] \times \mathbf{F}_p$. Over $\mathbb{Z}[1/p]$, the map can be identified with the projection $X^{r+2} \to X^{r+1}$, which is surjective.

For the base change to $\mathbf{F}_p$, we will show the stronger property that for any $\mathbf{F}_p$-algebra A, the map

$$\mathcal{J}^{r+1}(X)(A) \longrightarrow \mathcal{J}^r(X)(A)$$

is surjective. By (3.9), this can be identified with the map

$$X(W_{r+1}(A)) \longrightarrow X(W_r(A)).$$

But this map is surjective because X is formally smooth and because the map $W_{r+1}(A) \to W_r(A)$ is a surjection with nilpotent kernel, again by (3.13). □

Remark 3.5 Proposition 3.4 holds, more generally, for X an arbitrary smooth algebraic space over A_0 (See [4] for the definition of arithmetic jet spaces of algebraic spaces.) The proof is as follows: by general étale localization properties of $\mathcal{J}^r_{\mathcal{P}}$, it is enough to replace X with an affine étale cover, and in this case, the result was shown above. We will not need the non-affine version in this paper.

Corollary 3.6 *If* $\mathcal{J}^r_{\mathcal{P}}(X) = \operatorname{Spec} A^r$ *then the* $\delta_{\mathcal{P}}$*-prolongation systems* (A^r) *and* $((A^r)^{\widehat{p_k}})$ *are faithful.*

Proof By Proposition 3.4, the transition maps $A^s \to A^r$ are faithfully flat and hence injective. Also, since A^0 is flat over $\mathbb{Z}$ the primes in $\mathcal{P}$ are not zero divisors in any of the rings A^r. Hence (A^r) is faithful. Now the primes in $\mathcal{P}$ continue to be non-zero divisors in $(A^r)^{\widehat{p_k}}$. Moreover, the maps $A^s/p_kA^s \to A^r/p_kA^r$ are faithfully flat hence injective. It follows that the maps $(A^s)^{\widehat{p_k}} \to (A^r)^{\widehat{p_k}}$ are injective. So $((A^r)^{\widehat{p_k}})$ is faithful. □

We also have the following useful result:

Proposition 3.7 *Let* X *be a smooth affine scheme over* A_0 *and let* $X = \bigcup_i X_i$ *be a covering with principal affine open sets. Then, for each* r*, the closed set*

$$\mathcal{J}^r_{\mathcal{P}}(X) \backslash \bigcup_i \mathcal{J}^r_{\mathcal{P}}(X_i) \tag{3.14}$$

has codimension ≥ 2 *in* $\mathcal{J}^r_{\mathcal{P}}(X)$.

Proof Let $U = \bigcup_i \mathcal{J}^r_{\mathcal{P}}(X_i)$ and $Y = \mathcal{J}^r_{\mathcal{P}}(X)$. It is enough to show that for any prime p, the complement $(Y \otimes \mathbf{F}_p)\backslash(U \otimes \mathbf{F}_p)$ has codimension ≥ 2 in $Y \otimes \mathbf{F}_p$ and that $(Y \otimes \mathbb{Q})\backslash(U \otimes \mathbb{Q})$ has codimension ≥ 2 in $Y \otimes \mathbb{Q}$. This follows exactly as in the proof of Proposition 2.22 in [10]. □

Remark 3.8 We will later need the following consequence of [3] (3.4.2) and [4], Proposition 11.1 (cf. [10], Remark 2.27). Let X be a smooth affine scheme over A_0 with connected geometric fibers and let $P \in X(A_0)$ be a uniform point with uniform coordinates T in X. Let $Y = \mathcal{J}^r_{\mathcal{P}}(X)$ and let $P^r \in Y(A_0)$ be the canonical lift of P, as defined above. Then, P^r is a uniform point of Y with uniform coordinates $(\delta^i_{\mathcal{P}}T; i \leq r)$ in Y.

4 The tangent bundle of arithmetic jet spaces

In this section, we extend the theory in [9], pp. 88–100, from the case where $\mathcal{P}$ consists of one prime to the case where it consists of several primes.

Let A be a $\delta_{\mathcal{P}}$-ring in which the primes in $\mathcal{P}$ are non-zero divisors and let $A^0 \subset A$ be a subring. Let $\partial : A^0 \to A^0$ be a derivation and let $r \in \mathbb{Z}^d_{\geq 0}$. A derivation $\partial_r : A \to A$ will be called an *r-conjugate* of ∂ on A if for any $s \in \mathbb{Z}^d_{\geq 0}$ we have

$$\partial_r \circ \phi^s_{\mathcal{P}} = \delta_{rs} \cdot \mathcal{P}^r \cdot \phi^s_{\mathcal{P}} \circ \partial : A^0 \to A, \tag{4.1}$$

where δ_{rs} is the Kronecker symbol.

In other words, let $X = \operatorname{Spec} A^0$, let $\kappa : \mathcal{J}_{\mathcal{P}}(X) \longrightarrow \prod_r X$ denote the limit of the maps $\kappa_{\leq r}$ of (3.10), let ∂' denote the vector field $(\ldots, \mathcal{P}^r\partial, \ldots)_r$ on $\prod_r X$, and let ∂'' denote the vector field on $\prod_r X$ with values in A induced by ∂'. Then, an r-conjugate of ∂ is an extension of ∂'' to a vector field ∂_r on $X = \operatorname{Spec} A$.

A system of derivations $(\partial_r)_r$, $\partial_r : A \to A$, indexed by multi-indices $r \in \mathbb{Z}^d_{\geq 0}$, will be called a *complete system of conjugates* of ∂ on A if for any r the derivation ∂_r is an r-conjugate of ∂ on A.

Clearly, we have the following uniqueness result:

Proposition 4.1 *Assume A is a $\delta_{\mathcal{P}}$-ring in which the primes in $\mathcal{P}$ are non-zero divisors and assume A is $\delta_{\mathcal{P}}$-generated by a subring $A^0 \subset A$. Let $\partial : A^0 \to A^0$ be a derivation. Then, for each r, there is at most one r-conjugate ∂_r of ∂ in A. In addition, we have $\partial_r A^n = 0$ for $r \not\leq n$.*

On the other hand, we have the following existence result.

Proposition 4.2 *Let $X = \operatorname{Spec} A^0$ be an affine smooth scheme over A_0 and let $\mathcal{J}^r_{\mathcal{P}}(X) = \operatorname{Spec} A^r$ be its $\delta_{\mathcal{P}}$-jet spaces. Let $\partial : A^0 \to A^0$ be a derivation. Then, there exists a complete system of conjugates $(\partial_r)_r$ of ∂ in A^∞ (which is unique by Proposition 4.1). Moreover, if, by Corollary 3.6, we view each A^n as a subset of A^∞, then we have $\partial_r A^n \subset A^n$ for all r, n.*

We denote by $\partial_{r|n} : A^n \to A^n$ the restriction of ∂_r to A^n.

Proof Let $A^0 = A_0[T]/(f)$ where T is a tuple of indeterminates T_a and f is a tuple of polynomials f_b. Lift ∂ to a derivation $\partial : A_0[T] \to A_0[T]$ and, for any n define the derivation ∂_r on $\mathbb{Q}[\delta^s_{\mathcal{P}}T; s \geq 0] = \mathbb{Q}[\phi^s_{\mathcal{P}}T; s \geq 0]$ as the unique derivation satisfying

$$\partial_r(\phi^s_{\mathcal{P}}T_a) = \delta_{rs} \cdot \mathcal{P}^r \cdot \phi^s_{\mathcal{P}}(\partial T_a). \tag{4.2}$$

Clearly, (4.1) holds and ∂_r sends each $\mathbb{Q}[\delta^s_{\mathcal{P}}T; s \leq n]$ into itself. Set $\partial_r = 0$ for $r \in \mathbb{Z}^d \backslash \mathbb{Z}^d_{\geq 0}$.

Claim 1 $\partial_r \circ \phi_{p_k} = p_k \cdot \phi_{p_k} \circ \partial_{r-e_k}$ on $\mathbb{Q}[\delta^s_{\mathcal{P}} T; s \leq n]$.

Indeed, the difference between the left-hand side and the right-hand side of the above equality is a derivation that vanishes on the set of generators $\{\phi^s_{\mathcal{P}} T; s \leq n\}$ of $\mathbb{Q}[\delta^s_{\mathcal{P}} T; s \leq n]$; for, using (4.2), we have

$$\begin{aligned}
(\partial_r \phi_{p_k})(\phi^s_{\mathcal{P}} T_a) &= \partial_r \phi^{s+e_k}_{\mathcal{P}} T_a \\
&= \delta_{r,s+e_k} \cdot \mathcal{P}^r \cdot \phi^{s+e_k}_{\mathcal{P}} \partial T_a \\
&= \delta_{r-e_k,s} \cdot \mathcal{P}^r \cdot \phi_{p_k}(\phi^s_{\mathcal{P}}(\partial T_a)) \\
&= p_k \cdot \phi_{p_k}(\mathcal{P}^{r-e_k} \cdot \delta_{r-e_k,s} \cdot \phi^s_{\mathcal{P}}(\partial T_a)) \\
&= (p_k \cdot \phi_{p_k} \circ \partial_{r-e_k})(\phi^s_{\mathcal{P}} T_a).
\end{aligned}$$

Claim 2 ∂_r maps $A_0[\delta^s_{\mathcal{P}} T; s \leq n]$ into itself.

Indeed, it is enough to show that $\partial_r(\delta^s_{\mathcal{P}} T_a) \in A_0[\delta^s_{\mathcal{P}} T; s \leq n]$ for all $s \leq n$ and all a. We proceed by induction on the sum of the components of $r + s$. The statement is clearly true for $s = 0$ so for $r + s = 0$. Now assume $r + s$ arbitrary. We may assume $s \neq 0$, so we may assume there exists k such that $s_k \geq 1$. Then

$$\begin{aligned}
\partial_r(\delta^s_{\mathcal{P}} T_a) &= \partial_r(\delta_{p_k} \delta^{s-e_k}_{\mathcal{P}} T_a) \\
&= \partial_r \left(\frac{\phi_{p_k}(\delta^{s-e_k}_{\mathcal{P}} T_a) - (\delta^{s-e_k}_{\mathcal{P}} T_a)^{p_k}}{p_k} \right) \\
&= \phi_{p_k} \partial_{r-e_k}(\delta^{s-e_k}_{\mathcal{P}} T_a) - (\delta^{s-e_k}_{\mathcal{P}} T_a)^{p_k - 1} \partial_r(\delta^{s-e_k}_{\mathcal{P}} T_a),
\end{aligned}$$

by Claim 1. The latter belongs to $A_0[\delta^s_{\mathcal{P}} T; s \leq n]$ by the induction hypothesis.

Claim 3 ∂_r maps the ideal $(\delta^{rs}_{\mathcal{P}} f; s \leq n)$ of $A_0[\delta^s_{\mathcal{P}} T; s \leq n]$ into itself.

Indeed, one can repeat the argument in the proof of Claim 2 with T_a replaced by f_b.

Now Claims 2 and 3 imply that ∂_r induce derivations $A^\infty \to A^\infty$ satisfying (4.1) which ends our proof. □

Remark 4.3 The proof above shows that we have the commutation relations:

$$\partial_r \circ \phi^s_{\mathcal{P}} = \mathcal{P}^s \cdot \phi^s_{\mathcal{P}} \circ \partial_{r-s} : A^\infty \to A^\infty \tag{4.3}$$

for all r, s. (Here, as usual, $\partial_{r-s} = 0$ if $s \not\leq r$.)

Remark 4.4 Under the assumption of Proposition 4.1, fix k and consider the derivation (still denoted by) ∂ on $(A^0)^{\widehat{p_k}}$ induced by ∂. Then, the system of conjugates $(\partial_r)_r$ on A^∞ induces a complete system of conjugates of $\partial : (A^0)^{\widehat{p_k}} \to (A^0)^{\widehat{p_k}}$ on

$$A_k^\infty := \lim_{\to} (A^r)^{\widehat{p_k}}$$

which we continue to denote by $(\partial_r)_r$. Recall from Corollary 3.6 that $(A^n)^{\widehat{p_k}} \subset A_k^\infty$ for all n. Note that we have

$$\partial_r((A^n)^{\widehat{p_k}}) \subset (A^n)^{\widehat{p_k}}.$$

Remark 4.5 Under the assumptions of Proposition 4.2, we have the following formula:

$$(a \cdot \partial)_r = \phi_{\mathcal{P}}^r(a) \cdot \partial_r,$$

for any $a \in A$. This follows from the uniqueness in Proposition 4.1.

Proposition 4.6 *Let $X = \text{Spec}\, A^0$ be smooth over A_0 and assume $(\partial^a)_{1 \le a \le m}$ is a basis for the tangent sheaf T_X of X over A_0. Then, the family of conjugates $(\partial^a_{r|n})_{1 \le a \le m,\ 0 \le r \le n}$ is a basis for the tangent sheaf $T_{\mathcal{J}^n_{\mathcal{P}}(X)}$.*

Proof Assume first X has étale coordinates, i.e. there is an étale map $X \to \mathbb{A}^m = \text{Spec}\ A_0[T]$, where T is a family of indeterminates T_b. Note that, by (4.1),

$$\begin{aligned}
\mathcal{P}^r \cdot \phi_{\mathcal{P}}^r(\partial^a T_b) &= \partial_r^a(\phi_{\mathcal{P}}^r T_b) \\
&= \partial_r^a(\mathcal{P}^r \cdot \delta_{\mathcal{P}}^r T_b + (\text{polynomial in } \delta_{\mathcal{P}}^s T \text{ with } s \le r, s \ne r)) \\
&= \partial_r^a(\mathcal{P}^r \cdot \delta_{\mathcal{P}}^r T_b) \\
&= \mathcal{P}^r \cdot \partial_r^a(\delta_{\mathcal{P}}^r T_b)
\end{aligned}$$

Hence

$$\partial_r^a(\delta_{\mathcal{P}}^r T_b) = \phi_{\mathcal{P}}^r(\partial^a T_b). \tag{4.4}$$

Similarly, we get

$$\partial_r^a(\delta_{\mathcal{P}}^s T_b) = 0 \tag{4.5}$$

for $s \le r$, $s \ne r$. Since $\mathcal{J}^n_{\mathcal{P}}(X) \to \mathcal{J}^n_{\mathcal{P}}(\mathbb{A}^m)$ is étale we know that

$$\left(\frac{\partial}{\partial(\delta_{\mathcal{P}}^s T_b)}\right)_{1 \le b \le m,\ s \le n}$$

is a basis of $T_{\mathcal{J}^n_{\mathcal{P}}(X)}$. So we may write

$$\partial^a_r = \sum_{0 \le s \le n} \sum_{b=1}^{m} \alpha^{sb}_{ra} \cdot \frac{\partial}{\partial(\delta^s_{\mathcal{P}} T_b)},$$

with $\alpha^{sb}_{ra} \in A^n = \mathcal{O}(\mathcal{J}^n_{\mathcal{P}}(X))$. Now order the multiindices (r, a) by the lexicographic order. Then, by (4.4) and (4.5), the matrix $\alpha := (\alpha^{sb}_{ra})$ consists of $m \times m$ blocks such that all blocks under the diagonal are 0 and the diagonal blocks are of the form $\phi^r_{\mathcal{P}}\beta$, where $\beta = (\partial^a T_b)$. Therefore, the matrix α is invertible which ends our proof in case X has étale coordinates.

Assume now X does not necessarily have étale coordinates. Take a finite affine open cover $X = \cup X_i$ such that each X_i has étale coordinates and let $X' := \coprod X_i$ be the disjoint union. Then, X' has étale coordinates and is an étale cover of X. By [4], Propositions 11.1 and 11.4, the map $\pi^r : \mathcal{J}^r_{\mathcal{P}}(X') \to \mathcal{J}^r_{\mathcal{P}}(X)$ is an étale cover, in particular it is faithfully flat. Consider the homomorphism $u : \mathcal{O}^N_{\mathcal{J}^r_{\mathcal{P}}(X)} \to T_{\mathcal{J}^r_{\mathcal{P}}(X)}$ defined by the collection of sections ∂^a_r of $T_{\mathcal{J}^r_{\mathcal{P}}(X)}$ (where N is the cardinality of this collection). The pull-back of u to $\mathcal{J}^r_{\mathcal{P}}(X')$ coincides with the homomorphism $u' : \mathcal{O}^N_{\mathcal{J}^r_{\mathcal{P}}(X')} \to T_{\mathcal{J}^r_{\mathcal{P}}(X')}$ defined by the collection of corresponding sections (still denoted by) ∂^a_r of $T_{\mathcal{J}^r_{\mathcal{P}}(X')}$; this follows from the uniqueness in Proposition 4.1 plus the fact that π^r is étale. By the first part of the proof, u' is an isomorphism. Since π^r is faithfully flat, it follows that u itself is an isomorphism. This ends the proof. □

5 The cotangent bundle of arithmetic jet spaces

Lemma 5.1 *Let $X = \mathrm{Spec}\, A^0$ be smooth over A_0 and let $\omega \in \Omega^i_{A^0}$ be a i-form on X. Then $\phi^{r*}_{\mathcal{P}}\omega \in \mathcal{P}^{ir} \cdot \Omega^i_{A^r}$.*

Proof It is enough to show that $\phi^*_{p_k}\eta \in p_k \cdot \Omega_{A^{r+e_k}}$ for any $\eta \in \Omega_{A^r}$. But if $\eta = f dg$ with $f, g \in A^r$ then

$$\begin{aligned}\phi^*_{p_k}\eta &= (f^{p_k} + p_k\delta_{p_k} f)d(g^{p_k} + p_k\delta_{p_k} g)\\ &= (f^{p_k} + p_k\delta_{p_k} f)(p_k g^{p_k-1}dg + p_k d(\delta_{p_k} g)),\end{aligned}$$

and we are done. □

Since, for X/A_0 smooth, the primes in $\mathcal{P}$ are non-zero divisors in all the rings A^r (and hence in Ω_{A^r}), we may define, for any i-form $\omega \in \Omega^i_{A^0}$, the i-forms

$$\omega_r := \frac{\phi^{r*}_{\mathcal{P}}\omega}{\mathcal{P}^{ir}} \in \Omega^i_{A^r}. \tag{5.1}$$

Furthermore, for $n \geq r$, we consider the forms $\varphi_{rn}^* \omega_r \in \Omega^i_{A^n}$; when n is clear from the context, we simply denote these forms, again, by ω_r. Also, for $m := \mathcal{P}^r$ we write

$$\omega_{[m]} := \omega_r. \tag{5.2}$$

Note that if $\omega \in \Omega^i_{A^0}$ and $\eta \in \Omega^j_{A^0}$ then

$$(\omega \wedge \eta)_r = \omega_r \wedge \eta_r \tag{5.3}$$

for all r. Since the exterior derivative $d : \Omega^i_{A^r} \to \Omega^{i+1}_{A^r}$ commutes with $\phi^{r*}_{\mathcal{P}}$, we have

$$(d\omega)_r = \mathcal{P}^r d(\omega_r) \tag{5.4}$$

In particular, $\omega \in \Omega^i_{A^0}$ is closed then the forms $\omega_r \in \Omega^i_{A^r}$ are also closed for all r.

Finally, let us record the formula

$$\frac{\phi_p^*}{p}(xdy) = \phi_p(x)\left(y^{p-1}dy + d(\delta_p y)\right), \tag{5.5}$$

for $p \in \mathcal{P}$.

Remark 5.2 Of course, it is possible to take (5.5) as the definition of ϕ_p^*/p. We could then define $\phi^{r*}_{\mathcal{P}}/\mathcal{P}^r$ as compositions of the single prime operators $\phi^*_{p_k}/p_k$. This approach has the benefit that it works in general, without any smoothness assumptions on X. (See [5], 12.5 and 12.8, where ϕ^*/p is denoted θ.)

Remark 5.3 Also, these operators can be viewed as "universal lifts" of the inverse Cartier operator on the corresponding forms. To explain this, assume for simplicity that $\mathcal{P} = \{p\}$ and consider the case of 1-forms. We recall the definition of the inverse Cartier operator [11]. Assume B is a smooth $\mathbf{F}_p$-algebra. For any B-module M defined by $B \times M \to M$, $(b, m) \mapsto bm$ we denote by F_*M the additive group M viewed as a B-module under $B \times F_*M \to F_*M$, $(b, m) \mapsto b \cdot m := b^p m$. Then, the deRham complex

$$F_*\Omega^*_B = (F_*B \xrightarrow{d} F_*\Omega_B \xrightarrow{d} F_*\Omega^2_B \to \cdots)$$

is a complex of B-modules, in particular $H^1(F_*\Omega^*_B)$ is a B-module. The map $B \to H^1(F_*\Omega^*_B)$, $b \mapsto [b^{p-1}db]$ (where [ ] means class in H^1) is a derivation, so it induces a B-module homomorphism $C^{-1} : \Omega_B \to H^1(F_*\Omega^*_B)$, called the *inverse Cartier operator*; it satisfies

$$C^{-1}(adb) = a \cdot C^{-1}(db) = [a^p b^{p-1} db].$$

Now if $X = \mathrm{Spec}\, A^0$ and A^r are as in the discussion before this remark and $\omega \in \Omega_{A^0}$ then $\omega_1 \in \Omega_{A^1}$. Let $\phi_0 : (A^0)\widehat{^p} \to (A^0)\widehat{^p}$ be any lift of Frobenius. There is an induced section $s : X^{\widehat{p}} \to J^1_p(X)$ of the projection $J^1_p(X) \to X^{\widehat{p}}$, so we may consider the pull-back $s^*\omega_1 \in \Omega_{X\hat{p}}$ which is of course nothing but $s^*\omega_1 = \frac{\phi_0^*\omega}{p}$. Denoting by upper bars reduction mod p, a trivial computation shows that the class $[\overline{s^*\omega_1}]$ of $\overline{s^*\omega_1} \in \Omega_{A^0\otimes\mathbf{F}_p}$ in $H^1(F_*\Omega^*_{A^0\otimes\mathbf{F}_p})$ equals $C^{-1}(\overline{\omega})$, image of $\overline{\omega} \in \Omega_{A^0\otimes\mathbf{F}_p}$ under the inverse Cartier operator. (See also [11] for the relation between the Cartier operator and lifts of Frobenius.)

Proposition 5.4 *Let $X = \operatorname{Spec} A$ be smooth over A_0 of relative dimension m and assume $(\partial^a)_{1\le a\le m}$ is a basis for the tangent sheaf T_X of X over A_0. Let $(\omega^a)_{1\le a\le m}$ be the dual basis for Ω_X (i.e. $\langle\omega^a, \partial^b\rangle = \delta_{ab}$). Then, $(\omega^a_r)_{1\le a\le m,\ 0\le r\le n}$ is a basis for Ω_{A^n}, dual to the basis $(\partial^a_{r|n})_{1\le a\le m, 0\le r\le n}$.*

Proof As in the proof of Proposition 4.6, we may assume there is an étale map $X \to \mathbb{A}^m = \operatorname{Spec} A_0[T]$. So we may write $\partial^b = \sum_{b'} \alpha_{bb'} \frac{\partial}{\partial T_{b'}}$, with $\alpha_{bb'} \in A$. Hence $\omega^a = \sum_{a'} \beta_{aa'} dT_{a'}$ with (β_{ab}) the transposed of the inverse of (α_{ab}). We get, using Remark 4.5, that

$$\begin{aligned}
\left\langle\omega^a_r, \partial^b_s\right\rangle &= \left\langle \sum_{a'} \phi^r_{\mathcal{P}}(\beta_{aa'})\mathcal{P}^{-r}\phi^{r*}_{\mathcal{P}}(dT_{a'}), \sum_{b'} \phi^s_{\mathcal{P}}(\alpha_{bb'})\left(\frac{\partial}{\partial T_{b'}}\right)_s \right\rangle \\
&= \sum_{a'}\sum_{b'} \phi^r_{\mathcal{P}}(\beta_{aa'})\phi^s_{\mathcal{P}}(\alpha_{bb'})\left\langle \mathcal{P}^{-r} d(\phi^{r*}_{\mathcal{P}} T_{a'}), \mathcal{P}^s \frac{\partial}{\partial(\phi^s_{\mathcal{P}} T_{b'})}\right\rangle \\
&= \delta_{rs} \sum_{a'}\sum_{b'} \phi^r_{\mathcal{P}}(\beta_{aa'})\phi^s_{\mathcal{P}}(\alpha_{bb'})\delta_{a'b'} \\
&= \delta_{rs}\delta_{ab},
\end{aligned}$$

which ends the proof. □

Corollary 5.5 *Assume X is a smooth affine scheme over A_0 possessing a volume form ω. Then, the form $\bigwedge_{r\le n} \omega_r$ is a volume form on $\mathcal{J}^n_{\mathcal{P}}(X)$.*

Remark 5.6 The form $\bigwedge_{r\le n} \omega_r$ is clearly well defined up to a sign. In fact, if we interpret it in the following way, it is completely well defined. Let $\bigwedge_{r\le n} \Omega_X$ denote the module determined by the property that a linear map $\bigwedge_{r\le n} \Omega_X \to M$ is the same as an alternating linear map $\prod_{r\le n} \Omega_X \to M$, where $\prod_{r\le n} \Omega_X$ denotes the set of functions $\{r; r \le n\} \to \Omega_X$. Observe that we do not need to choose an ordering on $\{r; r \le n\}$. Then define $\bigwedge_{r\le n} \omega_r$ to be the image of the function $r \mapsto \omega_r$ under the universal alternating map $\prod_{r\le n} \Omega_X \to \bigwedge_{r\le n} \Omega_X$.

Proof of Corollary 5.5 By Corollary 3.6, $\mathcal{J}^n_{\mathcal{P}}(X)$ is smooth over A_0. Let N be its relative dimension over A_0. Then

$$\bigwedge^N \Omega_{\mathcal{J}^n_{\mathcal{P}}(X)}$$

is a locally free sheaf of rank one and the form $\bigwedge_{r\le n} \omega_r$ is a section of this sheaf. Let $X = \bigcup_i X_i$ be a covering with principal affine open sets such that Ω_{X_i} is free for each i. Then, by Proposition 5.4, $\bigwedge_{r\leq n}\omega_r$ is a volume form on each $\mathcal{J}^n_{\mathcal{P}}(X_i)$ and hence on the union $\bigcup_i \mathcal{J}^n_{\mathcal{P}}(X_i)$. But by Proposition 3.7, the complement of this union in $\mathcal{J}^n_{\mathcal{P}}(X)$ has codimension ≥ 2. This implies that $\bigwedge_{r\leq n}\omega_r$ is a volume form on the whole of $\mathcal{J}^n_{\mathcal{P}}(X)$. □

Remark 5.7 Let S be an abelian scheme or a K3 surface over A_0 (i.e. a smooth projective scheme of relative dimension 2 with $H^1(S,\mathcal{O})=0$ and trivial canonical bundle Ω^2_S) and fix a volume form ω on S. (So ω is well defined up to multiplication by a an element of $A_0^\times$.) Let $X\subset S$ be an affine open set. Then, one can consider on each $\mathcal{J}^n_{\mathcal{P}}(X)$ the volume form $\bigwedge_{r\leq n}\omega_r$. The latter will be referred to as the *canonical volume form* on $\mathcal{J}^n_{\mathcal{P}}(X)$ and (once ω has been fixed) is well defined up to sign.

Corollary 5.8 *Assume the hypotheses of Proposition 5.4. For any $f\in A^n=\mathcal{O}(\mathcal{J}^n_{\mathcal{P}}(X))$, we have the following formula in Ω_{A^n}:*

$$df=\sum_{1\leq a\leq m}\sum_{0\leq r\leq n}(\partial^a_r f)\omega^a_r. \tag{5.6}$$

Remark 5.9 By continuity, the formula (5.6) continues to hold in $\Omega_{(A^n)^{\widehat{p_k}}}$, for any $f\in(A^n)^{\widehat{p_k}}$.

Finally, we will need the following:

Proposition 5.10 *Let $Y\to X$ be an étale finite Galois cover of affine smooth schemes over A_0 with Galois group Γ, let p be a prime, and let $n,i\geq 0$ be integers. Let ω be an i-form on the p-jet space $J^n_p(Y)$ which is Γ-invariant. Then, ω is the pull-back of a unique i-form on the p-jet space $J^n_p(X)$.*

Remark 5.11 Proposition 5.10 fails to be true if the p-jet spaces $J^n_p(X)=\mathcal{J}^n_{\{p\}}(X)^{\widehat{p}}$ and $J^n_p(Y)=\mathcal{J}^n_{\{p\}}(Y)^{\widehat{p}}$ are replaced by the arithmetic jet spaces $\mathcal{J}^n_{\{p\}}(X)$ and $\mathcal{J}^n_{\{p\}}(Y)$, respectively. An example is given by $A_0=\mathbb{Z}$, $p\neq 2$, $Y=\operatorname{Spec}\mathbb{Z}[y,y^{-1}]$, $X=\operatorname{Spec}\mathbb{Z}[x,x^{-1}]$, $x\mapsto y^2$ and the 0-form $\omega:=yy'$, where $y'=\delta_p y$. Indeed, yy' is invariant under $y\mapsto -y$, but yy' does not belong to the image of

$$\mathcal{O}(\mathcal{J}^1_{\{p\}}(X))\to\mathcal{O}(\mathcal{J}^1_{\{p\}}(Y)),$$

as one can easily see by tensoring with $\mathbb{Q}$. On the other hand, yy' belongs to the image of

$$\mathcal{O}(J^1_p(X))\to\mathcal{O}(J^1_p(Y)),$$

as one can see directly from the formula

$$yy'=x^{(p+1)/2}\sum_{n\geq 2}\binom{1/2}{n}p^{n-1}\left(\frac{x'}{x^p}\right)^n\in\mathbb{Z}_p[x,x',x^{-1}]^{\widehat{p}}=\mathcal{O}(J^1_p(X)).$$

Also, Proposition 5.10 fails to be true if the p-jet spaces $J^n_p(X) = \mathcal{J}^n_{\{p\}}(X)^{\widehat{p}}$ and $J^n_p(Y) = \mathcal{J}^n_{\{p\}}(Y)^{\widehat{p}}$ are replaced by the spaces $\mathcal{J}^n_{\mathcal{P}}(X)^{\widehat{p}}$ and $\mathcal{J}^n_{\mathcal{P}}(Y)^{\widehat{p}}$, respectively, where $\mathcal{P}$ consists of at least 2 primes one of which is p.

Finally, note that Proposition 5.10 obviously fails "over $\mathbb{Q}$" since over $\mathbb{Q}$ the jet spaces of order n become $n+1$-fold products, and hence the projection between the jet spaces of Y and X becomes a Galois cover with group Γ^{n+1} (rather than Γ).

Proof of Proposition 5.10 For $i = 0$, this was proved in [9], Proposition 3.27. So we may assume $i \geq 1$. Let $Y = \bigcup_j Y_j$ be an affine open cover such that Ω_{Y_j} is free for each j, and let X_j be the preimage of Y_j in X. Since $J^n_p(X) = \bigcup_j J^n_p(X_j)$ it is sufficient to prove the Proposition for $X_j \to Y_j$ for each j. Replacing Y be Y_j we may assume Ω_Y is free. Let $\eta^1, \ldots, \eta^m$ be a basis of Ω_Y and $\omega^1, \ldots, \omega^m$ be the pull-back of this basis on X which is a basis of Ω_X. By Proposition 5.4, we may write

$$\omega = \sum_{\alpha_1 \ldots \alpha_i} a_{\alpha_1 \ldots \alpha_i} \omega^{\alpha_1} \ldots \omega^{\alpha_i},$$

for unique $a_{\alpha_1 \ldots \alpha_i} \in \mathcal{O}(J^n_p(Y))$. Since ω and $\omega^1, \ldots, \omega^m$ are Γ-invariant, it follows that $a_{\alpha_1 \ldots \alpha_i}$ are Γ-invariant hence (by the $i = 0$ case of the Proposition) they are pull-backs of unique functions in $\mathcal{O}(J^n_p(X))$, which ends our proof. □

6 The multiplicative group

In this section, we let $A_0 = S^{-1}\mathbb{Z}$ where S is the multiplicative system of all integers coprime to $p_1, \ldots, p_d$. Assume $X = \mathbf{G}_m = \text{Spec}\, A^0$, $A^0 := A_0[x, x^{-1}]$, is the multiplicative group scheme over A_0. Then, the origin is a uniform point of X with uniform coordinate $T = x - 1$ in X. Let $\omega = \frac{dx}{x} \in \Omega_{\mathbf{G}_m/A_0}$. Clearly, ω is closed but not exact on $\mathbf{G}_m$. Recall that we set $e = (1, \ldots, 1) \in \mathbb{Z}^d_{\geq 0}$ and define the 1-form

$$\omega^{(e)} = -\sum_{0 \leq r \leq e} (-1)^{|r|} \omega_r \in H^0(\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m), \Omega_{\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)}), \tag{6.1}$$

where $|r|$ is, as usual, the sum of the components of r and ω_r are defined as in (5.1). Note that

$$\omega^{(e)} = -\left(\prod_{k=1}^{d} \left(1 - \frac{\phi^*_{p_k}}{p_k}\right)\right)\omega = -\sum_{m | p_1 \ldots p_d} \mu(m) \omega_{[m]},$$

where μ is the Moebius function and $\omega_{[m]}$ are as in (5.2).

Consider now the elements

$$\psi^1_{p_k} \in (A^{e_k})^{\widehat{p_k}} = \mathbb{Z}_{p_k}[x, x^{-1}, \delta_{p_k} x]^{\widehat{p_k}},$$

$$\psi^1_{p_k} := \frac{1}{p_k} \log\left(1 + p\frac{\delta_{p_k} x}{x^{p_k}}\right) = \frac{\delta_{p_k} x}{x^{p_k}} - \frac{p_k}{2}\left(\frac{\delta_{p_k} x}{x^{p_k}}\right)^2 - \cdots .$$

Symbolically, one might write $\psi^1_{p_k} = p_k^{-1} \log \phi_{p_k}(x) - \log x$. We pass to the multiple-prime case by defining the elements

$$f_k := \left(\prod_{\substack{l=1 \\ l \neq k}}^{d} \left(1 - \frac{\phi_{p_l}}{p_l}\right) \right) \psi^1_{p_k} \in (A^e)^{\widehat{p_k}}. \tag{6.2}$$

Note that the vector

$$\psi^e_m := (f_1, \ldots, f_d) \tag{6.3}$$

is the *arithmetic Laplacian* of $\mathbf{G}_m$ introduced in [10]. Also consider the logarithm of the formal multiplicative group

$$l_{\mathbf{G}_m}(T) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{T^n}{n} \in \mathbb{Q}[[T]]$$

and the series

$$\psi^e_{m,0} := -\left(\prod_{l=1}^{d} \left(1 - \frac{\phi_{p_l}}{p_l}\right) \right) l_{\mathbf{G}_m}(T) \in \mathbb{Q}[[\delta^i_{\mathcal{P}} T ; i \leq e]]. \tag{6.4}$$

(In fact, we have $\psi^e_{m,0} \in A_0[[\delta^i_{\mathcal{P}} T ; i \leq e]]$. See the proof of Theorem 3.3 in [10].)

As discussed in the introduction, we can think of the f_k and $\psi^e_{m,0}$ as being series expansions of the formal expression $-\prod_{l=1}^{d}(1 - \frac{\phi_{p_l}}{p_l}) \cdot \log(x)$ in different regions of the A_0-scheme $\mathbf{G}_m$. In [10], it was shown that the expansions agree on what could be interpreted as the intersection these regions. In Theorem 6.3 below, we show that they are solutions to a common differential equation.

Lemma 6.1 $d\psi^1_{p_k} = -\left(1 - \frac{\phi^*_{p_k}}{p_k}\right) \omega.$

Essentially, this was given in [9], Proposition 7.26, but for the convenience of the reader, we will include a simple proof here.

Proof Let us abbreviate $p = p_k$. Then we have

$$\begin{aligned} p d\psi^1_p &= d \log(1 + p x^{-p} \delta_p x) \\ &= \frac{d(1 + p x^{-p} \delta_p x)}{1 + p x^{-p} \delta_p x} \\ &= \frac{d(1 + p x^{-p} \delta_p x)}{1 + p x^{-p} \delta_p x} + \frac{d(x^p)}{x^p} - p \frac{dx}{x} \\ &= \frac{d(x^p + p \delta_p x)}{x^p + p \delta_p x} - p \frac{dx}{x} \end{aligned}$$

$$= \frac{d\phi_p(x)}{\phi_p(x)} - p\frac{dx}{x}$$
$$= (\phi_p^* - p)\frac{dx}{x}.$$

Dividing by p then completes the proof. □

Let us say that a 1-form $\tilde{\omega}$ on some space $\mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m)$ is *invariant* if

$$\mu^*\tilde{\omega} = pr_1^*\tilde{\omega} + pr_2^*\tilde{\omega},$$

where

$$\mu, pr_1, pr_2 : \mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m) \times \mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m) \to \mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m)$$

are the multiplication and the 2 projections, respectively. (This agrees with the standard definition of invariance of 1-forms because we are in a situation when our group scheme is commutative.)

Lemma 6.2 *The A_0-module consisting of the invariant 1-forms on $\mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m)$ is free with basis $\{\omega_r \mid r \leq n\}$. Furthermore, any invariant 1-form on $\mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m)$ is closed.*

Proof Let us first show each ω_r is invariant. By definition, we have $\omega_r = \varphi_{nr}^* \frac{\phi^{r*}}{\mathcal{P}^r}(\omega)$. And by Proposition 3.2, both φ_{nr} and ϕ^{r*} are group homomorphisms. Since ω is an invariant differential on $\mathbf{G}_m$, we have that ω_r is an invariant differential on $\mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m)$.

By Proposition 5.4, the set $\{\omega_r \mid r \leq n\}$ is a basis of the $\mathcal{O}(\mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m))$-module of global 1-forms on $\mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m)$. In particular, it is A_0-linearly independent. It remains to show every invariant differential is in its A_0-linear span. Let $\tilde{\omega}$ be an invariant differential, and write $\tilde{\omega} = \sum_{r \leq n} a_r\omega_r$ with $a_r \in \mathcal{O}(\mathcal{J}^n_{\mathcal{P}}(\mathbf{G}_m))$. Since $\tilde{\omega}$ and the ω_r are invariant, each function a_r is invariant. Therefore $a_r \in A_0$.

Finally, since ω is closed, so is each ω_r, by (5.4). It then follows from the above that every invariant 1-form is closed. □

Let $P \in \mathbf{G}_m(A_0)$ be the origin and let $\mathcal{S} := \mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)(A_0)$. The canonical lift $P^e \in \mathcal{S}$ is the origin of $\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)$. Since P is uniform, P^e is also uniform. Since $\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)$ is a group all points in $\mathcal{S}$ are uniform. Here is our main result on $\mathbf{G}_m$:

Theorem 6.3

1) *$\omega^{(e)}$ is invariant and hence closed;*
2) *$\omega^{(e)}$ is not exact;*
3) *The arithmetic Laplacian ψ^e_m (of (6.3)) is the $\mathcal{P}$-adic primitive of $\omega^{(e)}$ normalized along P^e, and $\psi^e_{m,0}$ (of (6.4)) is the P^e-adic primitive of $\omega^{(e)}$ normalized along P^e. In particular, $\omega^{(e)}$ is $\mathcal{P}$-adically exact and $\mathcal{S}$-adically exact.*
4) *If $\tilde{\omega}$ is a 1-form on $\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)$ that is invariant and $\mathcal{P}$-adically exact, then $\tilde{\omega}$ is an A_0-multiple of $\omega^{(e)}$.*

Proof Assertion 1) follows from Lemma 6.2 and the Definition (6.1) of $\omega^{(e)}$. To prove assertion 2) consider the isomorphism

$$\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m) \otimes \mathbb{Q} \longrightarrow \prod_{s \le e} \mathbf{G}_{m/\mathbb{Q}}. \tag{6.5}$$

induced by $\kappa_{\le e}$ of (3.10). Under this map, ω_s corresponds to the differential $\mathcal{P}^{-s}\mathrm{pr}_s^*(\omega)$ on the right-hand side. In particular, if i denotes the inclusion of $\mathbf{G}_m$ into the factor $s = 0$ (and 1 on all other factors), then i^* applied to the differential corresponding to $\omega^{(e)}$ is ω, which is not exact. And so $\omega^{(e)}$ cannot be exact.

To prove the first assertion in 3), we must prove, first, that if f_k is as in (6.2), then we have $df_k = \omega^{(e)}$ for all $k = 1, \ldots, d$ and, second, that $d\psi^e_{m,0} = \omega^{(e)}$. By Lemma 6.1, we have

$$\begin{aligned}
df_k &= d\left(\left(\prod_{\substack{l=1\\ l\neq k}}^{d}\left(1 - \frac{\phi^*_{p_l}}{p_l}\right)\right)\psi^1_{p_k}\right)\\
&= \left(\prod_{\substack{l=1\\ l\neq k}}^{d}\left(1 - \frac{\phi^*_{p_l}}{p_l}\right)\right) d\psi^1_{p_k}\\
&= -\left(\prod_{\substack{l=1\\ l\neq k}}^{d}\left(1 - \frac{\phi^*_{p_l}}{p_l}\right)\right)\cdot\left(1 - \frac{\phi^*_{p_k}}{p_k}\right)\omega\\
&= -\left(\prod_{l=1}^{d}\left(1 - \frac{\phi^*_{p_l}}{p_l}\right)\right)\omega\\
&= \omega^{(e)}.
\end{aligned}$$

The statement about $\psi^e_{m,0}$ follows in the same way. To prove the second assertion in 3), we use the translation by points in $\mathcal{S}$ to reduce to the case of the origin.

To prove assertion 4) embed

$$A_k := \mathcal{O}(\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m))^{\widehat{p_k}}$$

into

$$B_k := \mathbb{Z}_{p_k}[[\delta^r_{\mathcal{P}}T; r \le e]]$$

for each k. By hypothesis $\tilde{\omega} = dg_k$, with $g_k \in A_k$ for all k. We may assume that the g_k, viewed as elements of B_k, have no constant term. Since $\tilde{\omega}$ is defined over A_0 it follows that each g_k belongs to $B_0 := \mathbb{Q}[[\delta^r_{\mathcal{P}}T; r \le e]]$ and that the g_k are equal to some $g_0 \in B_0$. So $g_0 \in A_0[[\delta_{\mathcal{P}}^r T; r \leq e]]$. Since $\mu^* \tilde{\omega} = pr_1^* \tilde{\omega} + pr_2^* \tilde{\omega}$ and g_0 has no constant term, it follows that

$$\mu^* g_0 = pr_1^* g_0 + pr_2^* g_0, \tag{6.6}$$

where

$$\mu^*, pr_1^*, pr_2^* : A_0[[\delta_{\mathcal{P}}^r T; r \leq e]] \to A_0[[\delta_{\mathcal{P}}^r T_1, \delta_{\mathcal{P}}^r T_2; r \leq e]]$$

in (6.6) are induced by the formal group law and the 2 projections respectively. We conclude that the tuple (g_k) is a $\delta_{\mathcal{P}}$-character on $\mathbf{G}_m$ in the sense of Definition 2.33 in [10]. Since the order of (g_k) is e, which is also the order of (f_k), Theorem 3.4 in [10] implies there exists $\rho \in A_0$ such that $g_k = \rho \cdot f_k$. Hence $\tilde{\omega} = \rho \cdot \omega^{(e)}$ and we are done. □

Consider next, for any multi-index n, the volume form $\bigwedge_{r \leq n} \omega_r$ on $\mathcal{J}_{\mathcal{P}}^n(\mathbf{G}_m)$. (See Corollary 5.5.) It will be referred to as the *canonical volume form* on $\mathcal{J}_{\mathcal{P}}^n(\mathbf{G}_m)$.

Corollary 6.4 *If $n \geq e$, the canonical volume form on $\mathcal{J}_{\mathcal{P}}^n(\mathbf{G}_m)$ is $\mathcal{P}$-adically exact and $\mathcal{S}$-adically exact but not exact.*

Proof Indeed, we have $\bigwedge_{r \leq n} \omega_r = \omega^{(e)} \wedge \left(\bigwedge_{0 \neq r \leq n} \omega_r\right)$, which is $\mathcal{P}$-adically exact and $\mathcal{S}$-adically exact because $\omega^{(e)}$ is $\mathcal{P}$-adically exact and $\mathcal{S}$-adically exact, by Theorem 6.3. Assume now that the canonical volume form on $\mathcal{J}_{\mathcal{P}}^n(\mathbf{G}_m)$ is exact and let us derive a contradiction. Using the isomorphism (6.5), we deduce that the form

$$\nu = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_N}{x_N}$$

on $\mathbf{G}_{m/\mathbb{Q}}^N = \operatorname{Spec} \mathbb{Q}[x_1^{\pm}, \ldots, x_N^{\pm}]$ (where N is the number of elements r that are $\leq n$) is exact. So $\nu = d\eta$,

$$\eta = \sum_{i=1}^N f_i \frac{dx_1}{x_1} \wedge \cdots \wedge \widehat{\frac{dx_i}{x_i}} \wedge \cdots \wedge \frac{dx_N}{x_N},$$

$f_i \in \mathbb{Q}[x_1^{\pm}, \ldots, x_N^{\pm}]$. Hence

$$\sum_{i=1}^N (-1)^i x_i \frac{\partial f_i}{\partial x_i} = 1.$$

But this is impossible because none of the Laurent polynomials $x_i \frac{\partial f_i}{\partial x_i}$ has a constant term. □

Remark 6.5 We expect that if $n \not\geq e$ then the canonical volume form on $\mathcal{J}_{\mathcal{P}}^n(\mathbf{G}_m)$ is not $\mathcal{P}$-adically exact. In any case, this form is not exact (cf. the proof above that applies to any N).

Corollary 6.6 *Consider the derivation* $\partial = x\frac{d}{dx} : \mathcal{O}(\mathbf{G}_m) \to \mathcal{O}(\mathbf{G}_m)$ *and let* ∂_r *be the* r*-conjugates of* ∂. *Then*

$$\partial_r f_k = (-1)^{|r|}$$

for all $k = 1, \ldots, d$ *and* $r \leq e$.

Proof By assertion 3) in Theorem 6.3 and Remark 5.9, we have

$$\sum_{r \leq n} (-1)^{|r|} \omega_r = \omega^{(e)} = df_k = \sum_{r \leq n} (\partial_r f_k)\omega_r.$$

By Proposition 5.4, the ω_r form a basis. Thus $\partial_r f_k = (-1)^{|r|}$. □

Remark 6.7 Using arguments from [10], it is easy to show that the 1-form $\omega^{(e)}$ on $\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)$ typically has non-zero periods. Indeed, consider the cycle

$$\Gamma = (P_1^e, P_2^e, P_3^e, P_4^e, P_1^e)$$

on $\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)$, where $P_j^e \in \mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)(A_0) \subset \mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)(\mathbb{Z}_{p_{k_j}})$ are the canonical lifts of $P_j \in \mathbf{G}_m(A_0)$, $k_1 = k_2 \neq k_3 = k_4$, P_2 and P_3 are induced by the identity section $P_{23} = 1 \in \mathbf{G}_m(A_0) = A_0^\times$, and P_1 and P_4 are induced by a section $P_{14} \in A_0^\times$. If $P_{14} = \pm 1$, then

$$\int_\Gamma \omega^{(e)} = 0.$$

Indeed, each of the p_{k_j}-adic primitives f_{k_j} for $\omega^{(e)}$ gives a group homomorphism $\mathcal{J}^e_{\mathcal{P}}(\mathbf{G}_m)(\mathbb{Z}_{p_{k_j}}) \to \mathbf{G}_a(\mathbb{Z}_{p_{k_j}})$. But the target is torsion free, and P_{14}^e is a torsion point. So we have $f_{k_j}(P_{14}^e) = 0$.

However, if $P_{14} \neq \pm 1$, we have that

$$\int_\Gamma \omega^{(e)} = f_{k_1}(P_{14}^e) - f_{k_4}(P_{14}^e) \neq 0.$$

This follows from [10], proof of Theorem 3.4. (The argument is that $f_{k_1}(P_{14}^e)$ is a non-zero rational number times the p_{k_1}-adic logarithm of an element in $1 + p_{k_1}\mathbb{Z}_{p_{k_1}} \backslash \{1\}$; but the latter logarithm is not in $\mathbb{Q}$ by Mahler's p-adic analog [1,13] of the Hermite–Lindeman theorem.)

7 Elliptic curves

Again, we let $A_0 = S^{-1}\mathbb{Z}$ where S is the multiplicative system of all integers coprime to $p_1, \ldots, p_d$. Assume all primes in $\mathcal{P}$ are ≥ 5. Consider an elliptic curve over A_0,

$$E := E_{a,b} := \text{Proj}\, A_0[x_0, x_1, x_2]/(x_0x_2^2 - x_1^3 - ax_1x_0^2 - bx_0^3),$$

with $a, b \in A_0$ and $-4a^3 - 27b^2 \in A_0^\times$. Let $\omega = \frac{dx}{y} \in H^0(E, \Omega_E)$, $x := \frac{x_1}{x_0}$, $y := \frac{x_2}{x_0}$, $T = \frac{x}{2y}$. Let $X \subset E$ be the affine open set where x_2 is invertible. The origin $P = [0, 0, 1]$ is uniform with uniform coordinate T. We continue to denote by ω the image of ω in $H^0(X, \Omega_X)$. Clearly, ω is closed but not exact on X. Let a_{p_k} be the trace of Frobenius on $E \otimes \mathbb{F}_{p_k}$. (So E has $1 - a_{p_k} + p_k$ points with coordinates in $\mathbb{F}_{p_k}$.) Also, we extend this definition by setting

$$a_m := a_{p_1}^{i_1} \dots a_{p_d}^{i_d}$$

for $m = p_1^{i_1} \dots p_d^{i_d} | p_1 \dots p_d$, $i_j \in \{0, 1\}$. Consider the form

$$\omega^{(2e)} := \left(\prod_{l=1}^{d} \left(1 - a_{p_l} \frac{\phi_{p_l}^*}{p_l} + p_l \frac{\phi_{p_l^2}^*}{p_l^2} \right) \right) \omega \in H^0 \left(\mathcal{J}_{\mathcal{P}}^{2e}(X), \Omega_{\mathcal{J}_{\mathcal{P}}^{2e}(X)} \right).$$

One can write

$$\omega^{(2e)} = \sum_{m | p_1^2 \dots p_d^2} \mu(m') m'' a_{m'} \omega_{[m]},$$

where $m = m'(m'')^2$ with m', m'' square free and coprime; μ is the Moebius function; and $\omega_{[m]}$ is as in (5.2). It follows from [9], Theorem 7.22 and Corollary 7.28, plus [8] Theorem 1.10, that there exist elements

$$\psi_{p_k}^2 \in \mathcal{O}(J_{p_k}^2(E))$$

(where $J_{p_k}^2(E)$ denotes the formal scheme defined in Remark 3.3) such that $\psi_{p_k}^2$ vanish at 0 and

$$d\psi_{p_k}^2 = \left(1 - a_{p_k} \frac{\phi_{p_k}^*}{p_k} + p_k \left(\frac{\phi_{p_k}^*}{p_k} \right)^2 \right) \omega. \tag{7.1}$$

Clearly, $\psi_{p_k}^2$ are unique with the above properties. We continue to denote by $\psi_{p_k}^2$ the image of this element in $\mathcal{O}(J_{p_k}^2(X)) = \mathcal{O}(\mathcal{J}_{\mathcal{P}}^{2e_k}(X)^{\widehat{p_k}})$. (N.B. The superscript 2 in $\psi_{p_k}^2$ is not an exponent; it merely indicates that the order of that element is 2.) Following [10] consider the elements

$$f_k := \left(\prod_{\substack{l=1 \\ l \neq k}}^{d} \left(1 - a_{p_l} \frac{\phi_{p_l}}{p_l} + p_l \frac{\phi_{p_l^2}}{p_l^2} \right) \right) \psi_{p_k}^2 \in \mathcal{O}(\mathcal{J}_{\mathcal{P}}^{2e}(X)^{\widehat{p_k}}).$$

The vector

$$\psi_E^{2e} := (f_1, \dots, f_d) \tag{7.2}$$

is the (restriction to X of the) *arithmetic Laplacian* of E introduced in [10]. Moreover, we may consider

$$\psi_{E,0}^{2e} := \left[\prod_{l=1}^{d}\left(1 - a_{p_l}\frac{\phi_{p_l}}{p_l} + p_l\left(\frac{\phi_{p_l}}{p_l}\right)^2\right)\right] l_E(T) \in \mathbb{Q}[[\delta_{\mathcal{P}}^i T; i \leq 2e]], \quad (7.3)$$

where $l_E(T) \in \mathbb{Q}[[T]]$ is the logarithm of the formal group of E. By [10], proof of Theorem 3.7, we have

$$\psi_{E,0}^{2e} \in A_0[[\delta_{\mathcal{P}}^i T; i \leq 2e]].$$

Let $\mu : E \times E \to E$ be the multiplication map and let

$$X^{(2)} := (X \times X) \cap \mu^{-1}(X) \subset E \times E.$$

Let us say that a 1-form $\tilde{\omega}$ on some space $\mathcal{J}_{\mathcal{P}}^n(X)$ is *invariant* if

$$\mu^*\tilde{\omega} = \pi_1^*\tilde{\omega} + \pi_2^*\tilde{\omega},$$

where $\mu, \pi_1, \pi_2 : \mathcal{J}_{\mathcal{P}}^n(X^{(2)}) \to \mathcal{J}_{\mathcal{P}}^n(X)$ come from the multiplication and the 2 projections. (Again, this agrees with the standard definition of invariance of 1-forms because we are in a commutative situation.)

Lemma 7.1 *The A_0-module consisting of the invariant 1-forms on $\mathcal{J}_{\mathcal{P}}^n(X)$ is free with basis $\{\omega_r \mid r \leq n\}$. Furthermore, any invariant 1-form on $\mathcal{J}_{\mathcal{P}}^n(X)$ is closed.*

Proof Exactly as in the case of $\mathbf{G}_m$. □

Since the origin $P = [0, 0, 1] \in X(A_0)$ is uniform so is its canonical lift $P^{2e} \in \mathcal{J}_{\mathcal{P}}^{2e}(X)(A_0)$; cf. Sect. 3. Let $\mathcal{S} = \mathcal{J}_{\mathcal{P}}^{2e}(X)(A_0)$. Since $\mathcal{J}_{\mathcal{P}}^{2e}(X)$ is a group all points in $\mathcal{S}$ are uniform.

Theorem 7.2

1) *$\omega^{(2e)}$ is invariant and hence closed;*
2) *$\omega^{(2e)}$ is not exact;*
3) *The arithmetic Laplacian ψ_E^{2e} (of (7.2)) is the $\mathcal{P}$-adic primitive of $\omega^{(2e)}$ normalized along P^{2e} and $\psi_{E,0}^{2e}$ (of (7.3)) is the P^{2e}-adic primitive of $\omega^{(2e)}$ normalized along P^{2e}. In particular, $\omega^{(2e)}$ is $\mathcal{P}$-adically exact and $\mathcal{S}$-adically exact.*
4) *Assume E has ordinary reduction at all the primes in $\mathcal{P}$ and $\tilde{\omega}$ is a 1-form on $\mathcal{J}_{\mathcal{P}}^{2e}(X)$ that is invariant and $\mathcal{P}$-adically exact. Then, $\tilde{\omega}$ is an A_0-multiple of $\omega^{(2e)}$.*

Proof Assertions 1) and 2) follow exactly as in Theorem 6.3. To prove assertion 3) we must prove, as in the case of $\mathbf{G}_m$, that $df_k = \omega^{(2e)}$ for all $k = 1, \ldots, d$ and

$d\psi^{2e}_{E,0} = \omega^{(2e)}$. By (7.1) we have

$$\begin{aligned}
df_k &= d\left(\left(\prod_{\substack{l=1\\ l\neq k}}^{d}\left(1 - a_{p_l}\frac{\phi_{p_l}}{p_l} + p_l\left(\frac{\phi_{p_l}}{p_l}\right)^2\right)\right)\psi^1_{p_k}\right)\\
&= \left(\prod_{\substack{l=1\\ l\neq k}}^{d}\left(1 - a_{p_l}\frac{\phi^*_{p_l}}{p_l} + p_l\left(\frac{\phi^*_{p_l}}{p_l}\right)^2\right)\right)d\psi^1_{p_k}\\
&= \left(\prod_{\substack{l=1\\ l\neq k}}^{d}\left(1 - a_{p_l}\frac{\phi^*_{p_l}}{p_l} + p_l\left(\frac{\phi^*_{p_l}}{p_l}\right)^2\right)\cdot\left(1 - a_{p_k}\frac{\phi^*_{p_k}}{p_k} + p_k\left(\frac{\phi^*_{p_k}}{p_k}\right)^2\right)\right)\omega\\
&= \left(\prod_{l=1}^{d}\left(1 - a_{p_l}\frac{\phi^*_{p_l}}{p_l} + p_l\left(\frac{\phi^*_{p_l}}{p_l}\right)^2\right)\right)\omega\\
&= \omega^{(2e)}.
\end{aligned}$$

A similar computation proves the statement about $\psi^{2e}_{E,0}$. To prove assertion 4), we proceed exactly as in the proof of assertion 4) in Theorem 6.3; instead of Theorem 3.4 in [10], we need to use Theorem 3.8 in [10] in conjunction with Lemma 7.33 in [9]. □

Consider next, for any multi-index n, the volume form $\bigwedge_{r\leq n}\omega_r$ on $\mathcal{J}^n_{\mathcal{P}}(X)$. (See Corollary 5.5.) It will be referred to as the *canonical volume form* on $\mathcal{J}^n_{\mathcal{P}}(X)$. Also let $\mathcal{S} = \mathcal{J}^n_{\mathcal{P}}(X)(A_0)$.

Corollary 7.3 *If $n \geq 2e$ the canonical volume form on $\mathcal{J}^n_{\mathcal{P}}(X)$ is $\mathcal{P}$-adically exact and $\mathcal{S}$-adically exact but not exact.*

Proof The same argument as for $\mathbf{G}_m$. To prove non-exactness it is enough to prove, as in the case of $\mathbf{G}_m$, that if we view E and X as schemes over $\mathbb{Q}$ and $\nu \in H^0(E^N, \Omega^N_{E^N}) \subset H^N_{DR}(E^N)$ is a volume form (where $N \geq 1$ is an integer) then the image of ν via the restriction map

$$H^N_{DR}(E^N) \to H^N_{DR}(X^N)$$

is non-zero. But ν lies in the $H^1_{DR}(E)\otimes\cdots\otimes H^1_{DR}(E)$ Künneth summand of $H^N_{DR}(E^N)$ so the image of ν in $H^N_{DR}(X^N)$ lies in the $H^1_{DR}(X)\otimes\cdots\otimes H^1_{DR}(X)$ Künneth summand of $H^N_{DR}(X^N)$. So we are reduced to show that the restriction map $H^1_{DR}(E) \to H^1_{DR}(X)$ is injective, which is true. □

Remark 7.4 We expect that if $n \not\geq 2e$ then no invariant volume form on $\mathcal{J}^n_{\mathcal{P}}(X)$ is $\mathcal{P}$-adically exact. In any case, no such form is exact (cf. the proof above that applies to any N).

Remark 7.5 Let S be an abelian scheme or a K3 surface over A_0 and let $X \subset S$ be an affine open set. It is natural to ask if the canonical volume forms (cf. Remark 5.7) on the various jet spaces $\mathcal{J}^n_{\mathcal{P}}(X)$ are $\mathcal{P}$-adically exact. We are not able to treat this question in general. Note however that by Theorem 7.2 the answer to this question is positive for abelian schemes of the form $S = E_1 \times \cdots \times E_m$ with E_j elliptic curves as in this section and $X = X_1 \times \cdots \times X_m$ with $X_j \subset E_j$ affine open sets. A special case of K3 surfaces will be treated in the next section.

Corollary 7.6 *Consider the derivation* $\partial = y\frac{\partial}{\partial x} : \mathcal{O}(X) \to \mathcal{O}(X)$ *and let* ∂_r *be the* r*-conjugates of* ∂. *Then*

$$\partial_r f_k = \mu(m')m'' a_{m'}$$

for all $k = 1, \ldots, d$ *and* $r \leq 2e$, $m = \mathcal{P}^r$.

Proof As in the Corollary 6.6, this follows from assertion 3) in Theorem 7.2 and Remark 5.9. □

Remark 7.7 As in the case of $\mathbf{G}_m$, using arguments from [10], it is easy to show that the 1-form $\omega^{(2e)}$ on $\mathcal{J}^{2e}_{\mathcal{P}}(X)$ typically has non-zero periods. Indeed, consider the cycle

$$\Gamma = (P_1^{2e}, P_2^{2e}, P_3^{2e}, P_4^{2e}, P_1^{2e})$$

on $\mathcal{J}^{2e}_{\mathcal{P}}(X)$, where $P_j^{2e} \in \mathcal{J}^{2e}_{\mathcal{P}}(X)(A_0) \subset \mathcal{J}^{2e}_{\mathcal{P}}(X)(\mathbb{Z}_{p_{k_j}})$ are the canonical lifts of $P_j \in X(A_0)$, $k_1 = k_2 \neq k_3 = k_4$, P_2 and P_3 are induced by the identity section of $E(A_0)$ and P_1 and P_4 are induced by a section $P_{14} \in X(A_0)$. As with $\mathbf{G}_m$, if P_{14} is torsion then

$$\int_\Gamma \omega^{(2e)} = 0.$$

However, if P_{14} is non-torsion, we have that

$$\int_\Gamma \omega^{(2e)} = f_{k_1}(P_{14}^{2e}) - f_{k_4}(P_{14}^{2e}) \neq 0.$$

This follows from the proof of Theorem 3.8 in [10]. (The argument is that $f_{k_1}(P_{14}^{2e})$ is a non-zero rational number times the p_{k_1}-adic elliptic logarithm of an element in $p_{k_1}\mathbb{Z}_{p_{k_1}} \backslash \{0\}$, but the latter logarithm is not in $\mathbb{Q}$ by Bertrand's p-adic analog [1] of the Hermite–Lindeman theorem.)

8 K3 surfaces

In this section, we consider our theory for a special class of K3 surfaces, namely for Kummer surfaces attached to products of two elliptic curves over $\mathbb{Q}$. For simplicity, we will only discuss $\mathcal{P}$-adic exactness; an analysis of $\mathcal{S}$-adic exactness can be done but requires a slight generalization of the discussion in the previous section (in which sections are replaced by multisections), so it will be omitted here.

Again, we let $A_0 = S^{-1}\mathbb{Z}$ where S is the multiplicative system of all integers coprime to $p_1, \dots, p_d$ and assume that all primes in $\mathcal{P}$ are ≥ 5. Consider two elliptic curves $E = E_{a,b}$ and $\tilde{E} = \tilde{E}_{\tilde{a},\tilde{b}}$ over A_0, as in the previous section, defined by cubics with coefficients $a, b \in A_0$ and $\tilde{a}, \tilde{b} \in A_0$, respectively. Denote by ω and $\tilde{\omega}$ the corresponding 1-forms on E and $\tilde{E}$ and denote by a_{p_k} and $\tilde{a}_{p_k}$ the corresponding traces of Frobenius. For each $k = 1, \dots, d$, we have at our disposal the functions $\psi^2_{p_k} \in \mathcal{O}(J^2_{p_k}(E))$ and $\tilde{\psi}^2_{p_k} \in \mathcal{O}(J^2_{p_k}(\tilde{E}))$, respectively. Consider the abelian scheme $A = E \times \tilde{E}$. We view ω and $\tilde{\omega}$ as 1-forms on A via pull-back. Then $\alpha := \omega \wedge \tilde{\omega}$ is a volume form on A. We also view $\psi^2_{p_k}$ and $\tilde{\psi}^2_{p_k}$ as elements of $\mathcal{O}(J^2_p(A))$ via pull-back.

Let $T \subset E$ and $\tilde{T} \subset \tilde{E}$ be the kernels of $[2]$ (the multiplication by 2) on E and $\tilde{E}$ respectively and assume that T and $\tilde{T}$ are unions of sections of our elliptic curves over A_0 (in other words, we assume that the cubics $x^3 + ax + b$ and $\tilde{x}^3 + \tilde{a}\tilde{x} + \tilde{b}$ have all their roots in A_0). Let $S = Km(A)$ be the Kummer surface attached to A, i.e., $S = B/i$ where B is the blow up of A at $T \times \tilde{T}$ and $i : B \to B$ is the involution lifting of the multiplication by -1, $[-1] : A \to A$. We have a diagram

$$A \stackrel{\epsilon}{\longleftarrow} B \stackrel{\pi}{\longrightarrow} S.$$

Recall that

$$\pi(\epsilon^{-1}(T \times \tilde{T})) = \bigcup_{j=1}^{16} L_j$$

where L_j are $\mathbf{P}^1$s on S with self-intersection -2. Let σ be the volume form on S normalized such that $\pi^*\sigma = \epsilon^*\alpha$ on B.

Let $U \subset A$ be an affine open set that is *symmetric* (i.e. $[-1]U = U$) and disjoint from $T \times \tilde{T}$. Moreover, let $Y = \epsilon^{-1}(U)$ and $X = Y/i$. So we have a diagram

$$U \stackrel{\epsilon}{\longleftarrow} Y \stackrel{\pi}{\longrightarrow} X$$

where the first map is an isomorphism and the second is a finite étale Galois cover with group $\langle i \rangle$.

For any multi-index $n \in \mathbb{Z}^d_{\geq 0}$, consider the *canonical volume form* $\bigwedge_{r \leq n} \sigma_r$ on $\mathcal{J}^n_{\mathcal{P}}(X)$.

Theorem 8.1 *Let $n \geq 2e$. Then, the canonical volume form on $\mathcal{J}^n_{\mathcal{P}}(X)$ is $\mathcal{P}$-adically exact. Moreover, if $U = (E \backslash T) \times (\tilde{E} \backslash \tilde{T})$, then the canonical volume form on $\mathcal{J}^n_{\mathcal{P}}(X)$ is not exact.*

In order to prove our Theorem, we need to introduce certain 1-forms. Indeed, for each $k = 1, \dots, d$ consider the 1-form

$$\eta_k := \psi^2_{p_k} \tilde{\omega}_0 = \psi^2_{p_k} \tilde{\omega} \qquad (8.1)$$

on $J^2_{p_k}(A)$ and hence on $J^2_{p_k}(U)$. Note that

$$d\eta_k = d\psi^2_{p_k} \wedge \tilde{\omega}_0 = \omega_0 \wedge \tilde{\omega}_0 - a_{p_k}\omega_{e_k} \wedge \tilde{\omega}_0 + p_k\omega_{2e_k} \wedge \tilde{\omega}_0. \tag{8.2}$$

Clearly $[-1]^*\psi^2_{p_k} = -\psi^2_{p_k}$ and $[-1]^*\tilde{\omega}_0 = -\tilde{\omega}_0$ on $J^2_{p_k}(A)$ so $[-1]^*\eta_k = \eta_k$ and hence $i^*(\epsilon^*\eta_k) = \epsilon^*\eta_k$ on $J^2_{p_k}(Y)$. By Proposition 5.10

$$\epsilon^*\eta_k = \pi^*\theta_k \tag{8.3}$$

on $J^2_{p_k}(Y)$ for some 1-form θ_k on $J^2_{p_k}(X)$.

Proof of Theorem 8.1 Let us still denote by θ_k the induced 1-form on $\mathcal{J}^n_{\mathcal{P}}(X)^{\widehat{p_k}}$. We claim that

$$\bigwedge_{r\le n} \sigma_r = d\theta_k \wedge \bigwedge_{0\ne r\le n} \sigma_r = d\left(\theta_k \wedge \bigwedge_{0\ne r\le n} \sigma_r\right) \tag{8.4}$$

on $\mathcal{J}^n_{\mathcal{P}}(X)^{\widehat{p_k}}$. This will prove that $\bigwedge_{r\le n}\sigma_r$ is $\mathcal{P}$-adically exact on $\mathcal{J}^n_{\mathcal{P}}(X)$. The second equality in (8.4) is clear because σ is closed and hence each σ_r is closed. To check the first equality in (8.4), it is enough to prove that the left-hand side and the right-hand side become equal when pulled back by π, since π is étale. Now on the one hand, we have

$$\pi^*\left(\bigwedge_{r\le n}\sigma_r\right) = \bigwedge_{r\le n}\pi^*\sigma_r = \bigwedge_{r\le n}(\pi^*\sigma)_r = \bigwedge_{r\le n}(\epsilon^*\alpha)_r = \epsilon^*\left(\bigwedge_{r\le n}\alpha_r\right).$$

Similarly

$$\pi^*\left(\bigwedge\nolimits_{0\ne r\le n}\sigma_r\right) = \epsilon^*\left(\bigwedge\nolimits_{0\ne r\le n}\alpha_r\right). \tag{8.5}$$

On the other hand, we have

$$\begin{aligned}
\pi^*\left(d\theta_k \wedge \left(\bigwedge\nolimits_{0\ne r\le n}\sigma_r\right)\right) &= d(\pi^*\theta_k) \wedge \pi^*\left(\bigwedge\nolimits_{0\ne r\le n}\sigma_r\right) \\
&= d(\epsilon^*\eta_k) \wedge \epsilon^*\left(\bigwedge\nolimits_{0\ne r\le n}\alpha_r\right) && \text{by (8.5)} \\
&= \epsilon^*\left(d\eta_k \wedge \left(\bigwedge\nolimits_{0\ne r\le n}(\omega_r \wedge \tilde{\omega}_r)\right)\right) \\
&= \epsilon^*\left(\omega_0 \wedge \tilde{\omega}_0 \wedge \left(\bigwedge\nolimits_{0\ne r\le n}(\omega_r \wedge \tilde{\omega}_r)\right)\right) && \text{by (8.2)} \\
&= \epsilon^*\left(\bigwedge\nolimits_{r\le n}\alpha_r\right),
\end{aligned}$$

which ends the proof of the $\mathcal{P}$-adic exactness assertion in the Theorem. To prove the second assertion of the Theorem, let us assume for a contradiction that $\bigwedge_{r\leq n}\sigma_r$ is exact on $\mathcal{J}^n_{\mathcal{P}}(X)$. Pulling back by π, we get that $\bigwedge_{r\leq n}\alpha_r$ is exact on $\mathcal{J}^n_{\mathcal{P}}(U)$. Tensoring with $\mathbb{Q}$, we get that the volume form on $E^N\times\tilde{E}^N$ over $\mathbb{Q}$ restricted to $(E\backslash T)^N\times(\tilde{E}\backslash\tilde{T})^N$ is exact for some N. But this cannot be the case, by the same argument as in the proof of Theorem 7.2. □

Of course the affine scheme $X\subset S$ we have been considering is disjoint from the union $\bigcup_{j=1}^{16}L_j$ of the 16 lines on S. On the other hand, we expect that if $X'\subset S$ is an affine open set which intersects $\bigcup_{j=1}^{16}L_j$, then the volume form $\bigwedge_{r\leq 2e}\sigma_r$ on $\mathcal{J}^{2e}_{\mathcal{P}}(X')$ is not $\mathcal{P}$-adically exact. We can only prove a partial result in this direction; see Theorem 8.2 below.

By the way each 1-form θ_k extends to a 1-form on

$$J^2_{p_k}\left(S\backslash\bigcup_{j=1}^{16}L_j\right)$$

simply because $S\backslash\bigcup_{j=1}^{16}L_j$ can be covered by open subsets X of the type we have been considering. Now if the 1-forms θ_k could be extended to 1-forms on the whole of $J^2_{p_k}(S)$, then it would follow (as in the proof of Theorem 8.1) that the canonical volume form on $\mathcal{J}^{2e}_{\mathcal{P}}(X')$ is $\mathcal{P}$-adically closed for any affine open set $X'\subset S$. As mentioned above, we do not expect this to be true. And indeed, the extension property for θ_k does not hold, as shown by the following Theorem. Let us fix k in $\{1,\ldots,d\}$.

Theorem 8.2 *The* 1*-form* θ_k *on* $J^2_{p_k}\left(S\backslash\bigcup_{j=1}^{16}L_j\right)$ *cannot be extended to a* 1*-form on the whole of* $J^2_{p_k}(S)$.

Proof Let us write write $p=p_k$, $\psi^2_p=\psi^2_{p_k}$, $\eta=\eta_k$, $\theta=\theta_k$ (cf. (8.3)), etc. Assume θ can be extended to a 1-form on $J^2_p(S)$. Let $z=x/y$ $\tilde{z}=\tilde{x}/\tilde{y}$. Then, the completion of A along the origin has ring (of global sections) $A_0[[z,\tilde{z}]]$. There is a point P on B, lying above the origin of A such that the completion of B along P has ring $A_0[[z,v]]$ where $\tilde{z}=zv$. Then, the completion of S along the image of P has ring $A_0[[u,v]]$, with $u=z^2$. Note that there is a natural identification

$$\mathbb{Q}_p[[z,\tilde{z},z',\tilde{z}',z'',\tilde{z}'']]=\mathbb{Q}_p[[z,\tilde{z},z^{\phi},\tilde{z}^{\phi},z^{\phi^2},\tilde{z}^{\phi^2}]],$$

where $z'=\delta_p z$, $z''=\delta_p^2 z$, etc., and $z^{\phi}=\phi_p(z)=z^p+pz'$, $z^{\phi^2}=\phi_p^2(z)$, etc. So we have natural inclusions

$$\begin{aligned}\mathbb{Q}_p[[z,\tilde{z},z^{\phi},\tilde{z}^{\phi},z^{\phi^2},\tilde{z}^{\phi^2}]]&\to\mathbb{Q}_p[[z,v,z^{\phi},v^{\phi},z^{\phi^2},v^{\phi^2}]]\\&\leftarrow\mathbb{Q}_p[[u,v,u^{\phi},v^{\phi},u^{\phi^2},v^{\phi^2}]].\end{aligned}$$

Let $\sum_{n\geq 1} c_n z^n$ be the logarithm of the formal group of E with respect to z and let $\sum_{n\geq 1} \tilde{c}_n \tilde{z}^n$ be defined similarly. (So $c_1 = \tilde{c}_1 = 1$.) Then

$$\psi_p^2 = \frac{1}{p}(\phi^2 - a_p\phi + p)(\sum c_n z^n)$$

in $\mathbb{Q}_p[[z, z^\phi, z^{\phi^2}]]$ and similarly for $\tilde{\psi}_p^2$. So we have

$$\eta = \left(\sum c_n z^n - \frac{a_p}{p}\sum c_n (z^\phi)^n + \frac{1}{p}\sum c_n (z^{\phi^2})^n\right) \times \left(\sum n\tilde{c}_n \tilde{z}^{n-1} d\tilde{z}\right). \quad (8.6)$$

Since we assumed that θ extends to $J_p^2(S)$, it follows that there exist series

$$f_0, f_1, f_2, g_0, g_1, g_2 \in \mathbb{Q}_p[[u, v, u^\phi, v^\phi, u^{\phi^2}, v^{\phi^2}]]$$

such that

$$\eta = f_0 du + f_1 du^\phi + f_2 du^{\phi^2} + g_0 dv + g_1 dv^\phi + g_2 dv^{\phi^2}. \quad (8.7)$$

Replacing $\tilde{z}$ by zv in (8.6) and u by z^2 in (8.7), expressing η as a linear combination of

$$dz, dz^\phi, dz^{\phi^2}, dv, dv^\phi, dv^{\phi^2},$$

and picking out the coefficient of dz we get

$$\left(\sum c_n z^n - \frac{a_p}{p}\sum c_n (z^\phi)^n + \frac{1}{p}\sum c_n (z^{\phi^2})^n\right) \times \left(\sum n\tilde{c}_n z^{n-1} v^n\right) = f_0 \times 2z$$

of series in $\mathbb{Q}_p[[z, v, z^\phi, v^\phi, z^{\phi^2}, v^{\phi^2}]]$. Picking out the coefficient of $z^{\phi^2} v$, we get $\frac{1}{p} = 0$, a contradiction. □

Remark 8.3 Note that the form θ_k is just one of six 1-forms

$$\theta_k, \theta_k', \theta_k'', \tilde{\theta}_k, \tilde{\theta}_k', \tilde{\theta}_k'' \quad (8.8)$$

on $J_{p_k}^2(X)$ that one could consider and that could be used in the proof of Theorem 8.1 in the same way in which θ_k was used. These six 1-forms are the unique 1-forms on $J_{p_k}^2(X)$ such that

$$\begin{aligned} \pi^*\theta_k &= \epsilon^*(\psi_{p_k}^2 \tilde{\omega}_0),\ \pi^*\theta_k' = \epsilon^*(\psi_{p_k}^2 \tilde{\omega}_{e_k}),\ \pi^*\theta_k'' = \epsilon^*(\psi_{p_k}^2 \tilde{\omega}_{2e_k}), \\ \pi^*\tilde{\theta}_k &= \epsilon^*(\tilde{\psi}_{p_k}^2 \omega_0),\ \pi^*\tilde{\theta}_k' = \epsilon^*(\tilde{\psi}_{p_k}^2 \omega_{e_k}),\ \pi^*\tilde{\theta}_k'' = \epsilon^*(\tilde{\psi}_{p_k}^2 \omega_{2e_k}). \end{aligned} \quad (8.9)$$

They can all be extended to $J^2_{p_k}(S\setminus\bigcup_{j=1}^{16} L_j)$. On the other hand, as before, none of them can be extended to $J^2_{p_k}(S)$.

It would be interesting to know if the spaces $\mathcal{J}^n_{\mathcal{P}}(X)$ (where X is as above) possess non-zero 2-forms that are $\mathcal{P}$-adically exact. For instance, one can ask the following question: *is there a non-trivial A_0-linear combination of the σ_rs (with $r \leq 2e$) on $\mathcal{J}^{2e}_{\mathcal{P}}(X)$ which is $\mathcal{P}$-adically exact?* The answer to the latter question is *yes* in case E and $\tilde{E}$ have supersingular reduction at all the primes in $\mathcal{P}$; see Theorem 8.4 below. But we expect that the answer to this question is *no* in case E and $\tilde{E}$ have ordinary reduction at the primes in $\mathcal{P}$. Again, see the comments below.

Indeed, it is a trivial exercise in linear algebra to show that if E and $\tilde{E}$ have ordinary reduction at p_k (hence $a_{p_k} \neq 0, \tilde{a}_{p_k} \neq 0$), then no non-zero 2-form on $J^{2e_k}_{\mathcal{P}}(X)$ that is a A_0-linear combination of $\sigma_0, \sigma_{e_k}, \sigma_{e_k}$ can be a $\mathbb{Z}_{p_k}$-linear combination of the forms

$$d\theta_k, d\theta_k', d\theta_k'', d\tilde{\theta}_k, d\tilde{\theta}_k', d\tilde{\theta}_k''$$

on $J^2_{p_k}(X)$.

On the other hand, if E and $\tilde{E}$ have supersingular reduction at p_k (hence $a_{p_k} = \tilde{a}_{p_k} = 0$), then

$$\sigma_0 - p_k^2\sigma_{2e_k} = \frac{1}{2}(d\theta_k - p_k d\theta_k'' - d\tilde{\theta}_k + p_k d\tilde{\theta}_k''). \tag{8.10}$$

This can be checked by a straight-forward computation using (7.1) and the defining formulas (8.9).

In the next Theorem, we consider the 2-form

$$\sigma^{(2e)} := \left(\prod_{k=1}^{d}\left(1 - \frac{\phi^{*2}_{p_k}}{p_k^2}\right)\right)\sigma = \sum_{m|p_1\ldots p_d} \mu(m)m^2\sigma_{[m]}$$

on $\mathcal{J}^{2e}_{\mathcal{P}}(X)$, where μ is the Moebius function and $\sigma_{[m]}$ are defined as in (5.2).

Theorem 8.4 *Assume E and $\tilde{E}$ have supersingular reduction at all primes in $\mathcal{P}$. Then, the 2-form $\sigma^{(2e)}$ on $\mathcal{J}^{2e}_{\mathcal{P}}(X)$ is $\mathcal{P}$-adically exact. Moreover, if $U = (E\setminus T) \times (\tilde{E}\setminus\tilde{T})$ this 2-form is not exact.*

Proof Exactly as in the proof of Theorems 6.3 and 7.2, using (8.10), one shows that $\sigma^{(2e)} = d\beta_l$ for each $l = 1, \ldots, d$, where β_l are the 1-forms

$$\beta_l := \frac{1}{2}\left(\prod_{\substack{k=1\\k\neq l}}^{d}\left(1 - \frac{\phi^{*2}_{p_k}}{p_k^2}\right)\right)(\theta_k - p_k\theta_k'' - \tilde{\theta}_k + p_k\tilde{\theta}_k'')$$

on $\mathcal{J}^{2e}_{\mathcal{P}}(X)^{\widehat{p_l}}$ and $\mathcal{P}$-adic exactness follows. Non-exactness follows as in Theorem 8.1. $\square$

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