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Characters of unipotent groups over finite fields

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Abstract Let G be a connected unipotent group over a finite field $\mathbb{F}_q$. In this article, we propose a definition of L-packets of complex irreducible representations of the finite group $G(\mathbb{F}_q)$ and give an explicit description of L-packets in terms of the so-called admissible pairs for G. We then apply our results to show that if the centralizer of every geometric point of G is connected, then the dimension of every complex irreducible representation of $G(\mathbb{F}_q)$ is a power of q, confirming a conjecture of Drinfeld. This paper is the first in a series of three papers exploring the relationship between representations of a group of the form $G(\mathbb{F}_q)$ (where G is a unipotent algebraic group over $\mathbb{F}_q$), the geometry of G, and the theory of character sheaves.

Keywords Unipotent group · Finite field · Geometric character theory

Mathematics Subject Classification (2010) Primary 20C15; Secondary 20G40

1 Introduction

In 1960, Higman asked[1] [24, p. 29] whether it is true that if q is a prime power and $n \in \mathbb{N}$, then the dimension of every complex irreducible representation of $UL_n(\mathbb{F}_q)$ is a power of q. Here, $\mathbb{F}_q$ is a finite field with q elements and $UL_n(\mathbb{F}_q)$ denotes the group of unipotent upper-triangular matrices of size n over $\mathbb{F}_q$. This question was later advertised and popularized by Thompson and Kirillov, among others. The answer is affirmative, and a generalization of this fact (to the so-called algebra groups over finite fields) was proved by Isaacs in [25]. It is natural to ask whether Isaacs's result can be

[1] I am grateful to Jon Alperin for providing this reference.

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further generalized to groups of the form $G(\mathbb{F}_q)$, where G is a connected unipotent group over $\mathbb{F}_q$.

If G is as above, it is not always the case that the dimension of every complex irreducible representation of $G(\mathbb{F}_q)$ is a power of q. This interesting phenomenon was first observed by Lusztig, who showed in [31] that if U is a maximal unipotent subgroup of the symplectic group Sp_4 over a finite field $\mathbb{F}_{2^r}$, where $r \in \mathbb{N}$, then $U(\mathbb{F}_{2^r})$ always has irreducible representations of dimension 2^{r-1}. The fake Heisenberg groups introduced in [10] (see also Sect. 2.10) provide similar counterexamples in every characteristic $p > 2$.

In 2005, Drinfeld conjectured that if a unipotent group G has the property that every geometric point of G is contained in the neutral connected component of its centralizer, then the dimension of every irreducible representation of $G(\mathbb{F}_q)$ is a power of q. One of the main goals of our paper is to prove this conjecture (see Theorem 2.5). From the viewpoint of character theory for finite groups, this is the most appealing result of the paper. However, we must point out that the proof we present (which is the only one known to us) heavily relies on geometric techniques, and may appear to be somewhat indirect. In particular, it is based on the notion of an *L-packet* of irreducible representations of $G(\mathbb{F}_q)$, which we introduce for an *arbitrary* connected unipotent group G over $\mathbb{F}_q$, and on our second main result, which provides a description of L-packets in more concrete terms.

The idea of using geometry to study representations of groups of the form $G(\mathbb{F}_q)$, where G is an algebraic group over $\mathbb{F}_q$, is not new. For *reductive* G, one has the theory of Deligne and Lusztig, which constructs many virtual representations of $G(\mathbb{F}_q)$ in the ℓ-adic cohomology of certain varieties X over $\mathbb{F}_q$ with a G-action, as well as Lusztig's theory of character sheaves, which expresses the irreducible characters of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$ as linear combinations of the "trace of Frobenius functions" of certain irreducible perverse ℓ-adic sheaves on G.

The case of *unipotent* G was also originally considered by Lusztig. In [31], he predicted the existence of an interesting theory of character sheaves for unipotent groups in positive characteristic[2] and defined the character sheaves in an ad hoc manner for the maximal unipotent subgroup[3] U of the symplectic group Sp_4 over a field of characteristic 2. Lusztig proved, moreover, that if the ground field is a finite field $\mathbb{F}_{2^r}$, then the trace functions associated with the character sheaves on U form a basis of the space of class functions on $U(\mathbb{F}_{2^r})$, and the relationship between these functions and the irreducible characters of $U(\mathbb{F}_{2^r})$ is similar to the one that exists in the theory of character sheaves for reductive groups over finite fields.

Lusztig's work led Drinfeld to formulate a series of definitions and conjectures that should form a basis of a general theory of character sheaves for unipotent groups in

[2] In characteristic zero, a unipotent group is "the same" as a finite dimensional nilpotent Lie algebra, and in this case the theory of character sheaves is essentially equivalent to Kirillov's orbit method [27]. In particular, character sheaves themselves are simply the Fourier transforms of the constant rank 1 local systems on the coadjoint orbits for the group.

[3] This is the first interesting example where the orbit method does not apply, cf. [10].

positive characteristic. We refer the reader to [10] for an overview. At present, many of these conjectures are already known [8,9].

The approach taken in the present article is somewhat different, although the methods we use are closely related to those proposed by Drinfeld and Lusztig, and they form a basis for [8,9]. Character sheaves do not appear in this paper, but our goal is still to study irreducible representations of $G(\mathbb{F}_q)$ by relating them to constructible ℓ-adic complexes on G, or, more precisely, to objects of the equivariant derived category $\mathscr{D}_G(G)$.

Our work can be thought of as an attempt to geometrize two classical and well-known results of character theory for finite groups, which we now state. If Γ is a finite group, let $\mathrm{Fun}(\Gamma)^{\Gamma}$ denote the space of conjugation-invariant functions $\Gamma \longrightarrow \mathbb{C}$. It is a commutative algebra under convolution of functions. The first result is that there is a natural bijection between complex irreducible characters of Γ and the minimal (in other terminology: "indecomposable" or "primitive") idempotents in $\mathrm{Fun}(\Gamma)^{\Gamma}$, given by $\chi \longleftrightarrow |\Gamma|^{-1}\chi(1)\cdot\chi$ (see e.g., [10]). The second result is that if Γ is nilpotent, then every complex irreducible representation of Γ is induced from a 1-dimensional representation of a subgroup of Γ.

In this paper, the word "geometrization" refers to replacing finite groups with algebraic groups G over finite fields $\mathbb{F}_q$ and studying representations of groups of the form $G(\mathbb{F}_q)$ by relating them to the geometry of G. The ground field for the representations is taken to be $\overline{\mathbb{Q}}_\ell$ rather than $\mathbb{C}$. Geometrization also involves replacing functions on finite groups with (complexes of) constructible ℓ-adic sheaves on algebraic groups and using Grothendieck's sheaves-to-functions correspondence.

The geometric analogue of the bijection between irreducible characters of a finite group Γ and minimal idempotents in $\mathrm{Fun}(\Gamma)^{\Gamma}$ is not a result at all, but rather a definition. More precisely, for a connected unipotent group G over $\mathbb{F}_q$, we propose a definition of L-packets of irreducible representations of $G(\mathbb{F}_q)$ based on the notion of a "weak idempotent" in the equivariant derived category $\mathscr{D}_G(G)$ (Definition 2.7). For groups G of this type, the result on representations of finite nilpotent groups mentioned above has a geometric analogue, which is more subtle: it becomes an *explicit description of L-packets* in terms of the so-called admissible pairs for G (Theorem 2.14). This is the second main result of our work.

We tried to keep the amount of geometry involved in our proofs to a minimum. In particular, not all the structures present on $\mathscr{D}_G(G)$ have been explored. Notably, we avoided using the *braided* monoidal structure on this category: only the *square* of the braiding appears in the proof of Theorem 2.5, and only does so implicitly.

2 Main definitions and results

In this section, we state the two main results of our work (Theorems 2.5 and 2.14), explaining most of the relevant definitions (although some technical details are postponed until later sections) and giving some historical background. In Sect. 2.10, we illustrate our theory by describing the L-packets of irreducible representations of $G(\mathbb{F}_q)$, where G is a so-called fake Heisenberg group over $\mathbb{F}_q$. The strategy we use to prove our main results is outlined in Sect. 3.

2.1 Conventions

If k is any field, an *algebraic group* over k is defined as a smooth group scheme of finite type over k. We recall that "smooth" is equivalent to "geometrically reduced" in this situation, and if k is perfect, the word "geometrically" can be omitted. By a *unipotent group over* k, we will mean a unipotent algebraic group (in particular, smooth) over k. We denote by $\overline{\mathbb{Q}}_\ell$ a fixed algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, and whenever the notation $\overline{\mathbb{Q}}_\ell$ is used, we invariably assume that ℓ is a prime different from the characteristic of the base field k.

Our conventions regarding finite fields are as follows. Let p be a prime number, fixed once and for all, and let $\mathbb{F}$ be a fixed algebraic closure of the finite field $\mathbb{F}_p$ with p elements. If $q = p^r$ for some $r \in \mathbb{N}$, we write $\mathbb{F}_q$ for the unique subfield of $\mathbb{F}$ consisting of q elements. All representations of finite groups that we consider are assumed to be defined over $\overline{\mathbb{Q}}_\ell$, where $\ell \neq p$. (This restriction only becomes relevant when we use geometric methods coming from ℓ-adic cohomology [21]. Much of our theory can be developed over an arbitrary algebraically closed field of characteristic 0, but for consistency we will work over $\overline{\mathbb{Q}}_\ell$ throughout this article.)

2.2 Easy unipotent groups

Let us recall a definition from [10].

Definition 2.1 Let k be a field and $\overline{k}$ an algebraic closure of k. An algebraic group G over k is said to be *easy* if every $g \in G(\overline{k})$ is contained in the neutral connected component, $Z(g)^\circ$, of its centralizer, $Z(g)$, in $G \otimes_k \overline{k}$.

It is clear that an easy algebraic group G over k has to be connected (if not, then applying the definition to any element $g \in G(\overline{k})$ that does not belong to the neutral connected component $(G \otimes_k \overline{k})^\circ$ leads to a contradiction).

The group GL_n is easy. A connected reductive group in characteristic 0 is easy if and only if its derived group is simply connected and its center is connected. From this point on, all easy groups discussed in this article will be unipotent.

Remark 2.2 We know of no examples of easy unipotent groups G that do not satisfy the stronger condition that the centralizer of every geometric point of G is *connected*. It appears plausible that there are no such examples.

We observe that if k has characteristic zero, then every unipotent group over k is connected, and since closed subgroups of unipotent groups are unipotent, it follows that every unipotent group over k is easy. Therefore, from now on we will only be interested in the case $\operatorname{char} k > 0$.

The first obvious example of an easy unipotent group in positive characteristic is provided by UL_n, the so-called *unipotent linear group*, defined as the group of unipotent upper-triangular matrices of size n. More generally, if G is any reductive group over k and U is a maximal connected unipotent subgroup of G, then U is easy provided the characteristic of k is large enough (depending on the types of the simple constituents of $G \otimes_k \bar{k}$). For instance, if G is the symplectic group Sp_{2n}, where $n \geq 2$, then U is easy if and only if char $k > 2$.

Another type of generalizations of the group UL_n comes from the so-called algebra groups. If A is a finite dimensional associative unital k-algebra, let J be the Jacobson radical[4] of A, and let $G(A)$ denote the algebraic group over k defined as follows. For any commutative k-algebra R, we let $G(A)(R)$ denote the multiplicative group of all elements of $R \otimes_k A$ of the form $1 + x$, where $x \in R \otimes_k J$. Then, $G(A)$ is an easy unipotent group over k because the centralizers of geometric points of $G(A)$ can be identified with linear subspaces of $\bar{k} \otimes_k J$. We call $G(A)$ the *unipotent algebra group* associated with A. Observe that if A is the algebra of all upper-triangular matrices of size n over k, then $G(A) \cong UL_n$.

The example of a maximal unipotent subgroup U of Sp_4 over the finite field $\mathbb{F}_{2^r}$, where $r \in \mathbb{N}$, was originally considered by Lusztig. This group is not easy. Lusztig computed the character table of $U(\mathbb{F}_{2^r})$ in Sect. 7 of [31] and found that this group has irreducible representations of dimension 2^{r-1}. The fake Heisenberg groups defined in [10] (see also Sect. 2.10) are also not easy (Lemma 2.16).

2.3 Representations of algebra groups over finite fields

It is known that the dimension of every irreducible representation of $UL_n(\mathbb{F}_q)$ is a power of q, which yields an affirmative answer to a question of Higman [24]. A stronger and more general result is provided by

Theorem 2.3 (Halasi) *If A is a finite dimensional algebra over $\mathbb{F}_q$, then every irreducible representation of $G(A)(\mathbb{F}_q)$ is induced from a 1-dimensional representation of a subgroup of the form $G(B)(\mathbb{F}_q)$, where $B \subset A$ is an $\mathbb{F}_q$-subalgebra.*

Corollary 2.4 (Isaacs) *In the situation of Theorem 2.3, the dimension of every irreducible representation of $G(A)(\mathbb{F}_q)$ is a power of q.*

Theorem 2.3 was first stated by Gutkin in [22]; however, Gutkin's proof of it was incomplete. Isaacs proved Corollary 2.4 in [25]. Later, Halasi proved Theorem 2.3 in [23]; it is worth noting that his proof uses Corollary 2.4 in an essential way. A more direct proof of Theorem 2.3, based on Halasi's methods, was given in [6], and an improved version later appeared in [7].

One of the main goals of this article is to extend Corollary 2.4 to all easy unipotent groups over finite fields. The result (Theorem 2.5) is stated below.

2.4 Character degrees of easy unipotent groups

One of the main results of this paper is the following theorem, proved in Sect. 9.4.

Theorem 2.5 (Main Theorem 1) *If G is an easy unipotent group over $\mathbb{F}_q$, the dimension of every irreducible representation of $G(\mathbb{F}_q)$ is a power of q.*

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[4] Since A is clearly Artinian as a ring, J is also the maximal two-sided nilpotent ideal of A.

This result was conjectured by Drinfeld in 2005. In our opinion, it explains the "geometry behind [the positive answer to] Higman's question."

After the first version of this article was written, Drinfeld informed us that the following extension of Theorem 2.3 (which gives a weaker result when applied to unipotent algebra groups) can be proved.

Theorem 2.6 *If G is an easy unipotent group over $\mathbb{F}_q$, then every irreducible representation of $G(\mathbb{F}_q)$ is induced from a 1-dimensional representation of a subgroup of the form $P(\mathbb{F}_q)$, where $P \subset G$ is a closed connected subgroup.*

We note that this result implies Theorem 2.5, because the index of $P(\mathbb{F}_q)$ in $G(\mathbb{F}_q)$ equals $q^{\dim G - \dim P}$. However, the proof of Theorem 2.6 relies on many of the key ingredients needed for our proof of Theorem 2.5. With his kind permission, we reproduce Drinfeld's proof of Theorem 2.6 in Appendix 9.6.

2.5 From easy to arbitrary connected unipotent groups

It turns out that in order to prove Theorem 2.5 one has to formulate and prove a more general statement about irreducible characters of $G(\mathbb{F}_q)$ for an *arbitrary* connected unipotent group G over $\mathbb{F}_q$. The reason is that all approaches to representation theory for unipotent groups known to us are based on induction on $\dim G$ in one way or another, reducing the questions one is interested in to similar questions for subgroups of G of smaller dimension. For instance, the proof of Theorem 2.3 ultimately relies on the possibility of constructing many nontrivial multiplicatively closed subspaces inside the Jacobson radical $J(A)$ of a finite dimensional algebra A. However, if G is an arbitrary easy unipotent group over $\mathbb{F}_q$, it is not known to us how to construct sufficiently many *easy* subgroups of G to make it possible to give an inductive proof of Theorem 2.5. On the other hand, G has lots of *connected* closed subgroups, and most of our paper is devoted to the study of arbitrary connected unipotent groups over finite fields.

2.6 Definition of L-indistinguishability

In the remainder of this section, we will freely use the language of ℓ-adic cohomology [1,14,21]. A brief review of the terminology appears in the first half of Sect. 4.

Let G be a connected unipotent group over $\mathbb{F}_q$, and let $\mu : G \times G \longrightarrow G$ be the multiplication morphism. The definition of the equivariant derived category $\mathscr{D}_G(G)$, together with the functor of convolution with compact supports,

$$\mathscr{D}_G(G) \times \mathscr{D}_G(G) \longrightarrow \mathscr{D}_G(G), \qquad (M, N) \longmapsto M * N = R\mu_!(M \boxtimes N),$$

is recalled in Sect. 4.5. An object $e \in \mathscr{D}_G(G)$ is said to be a *weak idempotent* if $e * e \cong e$. If this holds, it is clear that the associated trace function $t_e : G(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell$ is a central idempotent with respect to the usual convolution on the space of $\overline{\mathbb{Q}}_\ell$-valued functions on $G(\mathbb{F}_q)$. In particular, t_e acts either as zero or as the identity in every irreducible representation of $G(\mathbb{F}_q)$.

Definition 2.7 Two irreducible representations, ρ_1 and ρ_2, of $G(\mathbb{F}_q)$ are said to be *L-indistinguishable* if for every weak idempotent $e \in \mathscr{D}_G(G)$, the function t_e acts in the same way in ρ_1 and in ρ_2. The equivalence classes with respect to the relation of L-indistinguishability are called *L-packets* of irreducible representations.

Remark 2.8 (Drinfeld) The conjectural notion of an L-packet in representation theory of reductive groups over *local* fields was introduced by Langlands in [29]. It is hard to compare it with the notion introduced above because technically the two definitions are given in quite different terms. However, the philosophical ideas behind them are the same.

2.7 Multiplicative local systems

If G is an arbitrary connected unipotent group over $\mathbb{F}_q$, it is not at all clear how to describe all weak idempotents in the category $\mathscr{D}_G(G)$. For instance, it is not even obvious that there are any apart from the zero object and the unit object $\mathbb{1}$. In Sect. 2.9, we will state our second main result (Theorem 2.14), which yields a description of L-packets of irreducible representations of $G(\mathbb{F}_q)$ in terms of more concrete objects, the so-called admissible pairs (Sect. 2.8) for G. This description is one of the key ingredients in our proof of Theorem 2.5. In Sect. 2.10, we show how it can be used to describe all L-packets of irreducible representations of $G(\mathbb{F}_q)$ when G is a fake Heisenberg group over $\mathbb{F}_q$. We begin by introducing

Definition 2.9 If k is a field and ℓ is a prime different from char k, a *nonzero* $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on a connected algebraic group H over k is said to be *multiplicative* if $\mu^*(\mathcal{L}) \cong \mathcal{L} \boxtimes \mathcal{L}$, where $\mu : H \times_k H \longrightarrow H$ denotes the multiplication morphism.

Remark 2.10 If $k = \mathbb{F}_q$ and $\mathcal{L}$ is a multiplicative $\overline{\mathbb{Q}}_\ell$-local system on H, it is clear that the "trace function" (Sect. 4.2) $t_{\mathcal{L}}$ defined by $\mathcal{L}$ is a homomorphism $H(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. Moreover, $\mathcal{L}$ can be recovered from $t_{\mathcal{L}}$ up to isomorphism. If H is *commutative*, every homomorphism $H(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ arises in this way (cf. Proposition A.18). For noncommutative H, this statement fails in general, even if H is unipotent (cf. the example of the fake Heisenberg groups discussed in [10] and in Sect. 2.10).

2.8 Admissible pairs

Let $(H, \mathcal{L})$ denote a pair consisting of a closed connected subgroup $H \subset G$ and a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on H. The notion of what it means for this pair to be *admissible* is introduced in Sect. 7.3. The precise definition is somewhat technical, so here we will only remark that admissibility is a certain geometric non-degeneracy condition. (In this context, the word "geometric" refers to the fact that this property depends only on the triple $(G \otimes_{\mathbb{F}_q} \mathbb{F}, H \otimes_{\mathbb{F}_q} \mathbb{F}, \mathcal{L} \otimes_{\mathbb{F}_q} \mathbb{F})$ obtained from $(G, H, \mathcal{L})$ by base change to $\mathbb{F}$, an algebraic closure of $\mathbb{F}_q$.) It should be thought of as a geometrization of the following purely algebraic version.

Definition 2.11 Let Γ be a finite group and consider a pair (H, χ) consisting of a subgroup $H \subset \Gamma$ and a homomorphism $\chi : H \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. Let Γ' be the stabilizer of the pair (H, χ) for the conjugation action of Γ. We say that the pair (H, χ) is admissible if the following three conditions are satisfied:

(1) Γ'/H is commutative;
(2) the bi-additive map $B_\chi : (\Gamma'/H) \times (\Gamma'/H) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ induced by

$$(\gamma_1, \gamma_2) \longmapsto \chi\left(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\right)$$

(which by (1) is well defined) is a perfect pairing of finite abelian groups, i.e., induces an isomorphism $\Gamma'/H \xrightarrow{\simeq} \operatorname{Hom}(\Gamma'/H, \overline{\mathbb{Q}}_\ell^\times)$; and
(3) for every $g \in \Gamma$, $g \notin \Gamma'$, we have $\chi\,|_{H\cap H^g} \neq \chi^g\,|_{H\cap H^g}$, where $H^g = g^{-1}Hg$ and $\chi^g : H^g \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ is obtained from χ by transport of structure.

Remark 2.12 Conditions (1) and (2) in the algebraic definition of admissibility imply that the group Γ' has a unique irreducible representation π_χ over $\overline{\mathbb{Q}}_\ell$ which acts on H by the homomorphism χ. Condition (3) further implies that the induced representation $\operatorname{Ind}_{\Gamma'}^{\Gamma} \pi_\chi$ is irreducible (in view of Mackey's irreducibility criterion). The geometric notion of admissibility serves a somewhat similar purpose.

2.9 Explicit description of L-packets

We now return to the geometric setting. Let G be a connected unipotent group over $\mathbb{F}_q$, and $(H_1, \mathcal{L}_1)$, $(H_2, \mathcal{L}_2)$ two pairs consisting of closed connected subgroups $H_1, H_2 \subset G$ and multiplicative local systems $\mathcal{L}_j$ on H_j $(j = 1, 2)$. We say that these pairs are *geometrically conjugate* if there exists $g \in G(\mathbb{F})$ which conjugates one of them into the other. Note that, in general, geometric conjugacy is weaker than conjugacy by an element of $G(\mathbb{F}_q)$.

Definition 2.13 Let $\mathscr{C}$ be a geometric conjugacy class of admissible pairs $(H, \mathcal{L})$ as above for G. We define a set $L(\mathscr{C})$ of (isomorphism classes of) irreducible representation of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$ as follows. We say that $\rho \in L(\mathscr{C})$ if there exists $(H, \mathcal{L}) \in \mathscr{C}$ such that ρ is an irreducible summand of $\operatorname{Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)} t_{\mathcal{L}}$.

It is immediate that each of the sets $L(\mathscr{C})$ is nonempty. The second main result of our paper claims that the sets $L(\mathscr{C})$ are precisely the L-packets of irreducible representations of $G(\mathbb{F}_q)$. We prove it in Sect. 9.2. This result, along with the definition of admissible pairs, was also formulated by Drinfeld.

Theorem 2.14 (Main Theorem 2) *Let G be a connected unipotent group over $\mathbb{F}_q$. For every geometric conjugacy class $\mathscr{C}$ of admissible pairs for G, the set $L(\mathscr{C})$ is an L-packet. Conversely, every L-packets of irreducible representations of $G(\mathbb{F}_q)$ is of the form $L(\mathscr{C})$ for some geometric conjugacy class $\mathscr{C}$ of admissible pairs.*

Corollary 2.15 *With the notation above,*

(a) *every irreducible representation of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$ lies in $L(\mathscr{C})$ for some geometric conjugacy class $\mathscr{C}$ of admissible pairs for G; and*
(b) *if $\mathscr{C}_1$ and $\mathscr{C}_2$ are two geometric conjugacy classes of admissible pairs for G, the sets $L(\mathscr{C}_1)$ and $L(\mathscr{C}_2)$ are either equal or disjoint.*

Note that, a priori, neither of the statements of this corollary is obvious.

2.10 Example: L-packets for the fake Heisenberg groups

We conclude this overview with an example which to some extent motivated the notion of an admissible pair. If k is a field of characteristic $p > 2$, we define a *fake Heisenberg group* over k to be a connected noncommutative unipotent algebraic group G over k of exponent p and dimension 2 (hence the word "fake"). The reason for imposing the restriction $p > 2$ is that every 2-dimensional unipotent group in characteristic 0 is commutative (which follows from the corresponding statement for Lie algebras), and that every group of exponent 2 is commutative.[5]

However, if $p > 2$, there are plenty of examples of fake Heisenberg groups over $\mathbb{F}_q$: see [10]. Here we will describe the L-packets of irreducible representations for such groups. We begin with a simple auxiliary result.

Lemma 2.16 *If G is a noncommutative connected unipotent group of dimension 2 over a field k, then the only nontrivial proper closed connected subgroup of G is its commutator, $[G, G]$. Moreover, such a group G is not easy (Definition 2.1).*

Proof The assumptions imply that $[G, G]$ is connected and $\dim[G, G] = 1$, so that $\dim G^{ab} = 1$ as well, where $G^{ab} = G/[G, G]$ is the abelianization of G. Moreover, $[G, G]$ is contained in the center of G. Now let $H \subset G$ be a proper closed connected subgroup, and suppose $H \not\subset [G, G]$. Then, H projects epimorphically onto G^{ab}, which implies that $G = H \cdot [G, G]$. Hence $[G, G] \not\subset H$ as well. Therefore $[H, H] \subset H \cap [G, G]$ is connected and 0-dimensional, whence trivial. Thus, H is commutative. This implies that G is commutative, which is a contradiction.

For the second claim, note that if g is a geometric point of G which does not lie in the center of G, then $Z(g) \neq G \otimes_k \overline{k}$, whereas $[G, G] \otimes_k \overline{k} \subset Z(g)$, whence $Z(g)^\circ = [G, G] \otimes_k \overline{k}$, which implies that $g \notin Z(g)^\circ(\overline{k})$. □

Let G be a fake Heisenberg group over $\mathbb{F}_q$ and consider a pair $(H, \mathcal{L})$ consisting of a closed connected subgroup $H \subset G$ and a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on H. If $H = G$, this pair is trivially admissible; its geometric conjugacy class $\mathscr{C}$ reduces to the single pair $(H, \mathcal{L})$; and the corresponding L-packet $L(\mathscr{C})$ consists of the single 1-dimensional representation $t_{\mathcal{L}} : G(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$.

[5] In characteristic 2, there also exist connected noncommutative 2-dimensional unipotent groups, but they all have exponent 4.

Usually, however, not every L-packet of irreducible representations of $G(\mathbb{F}_q)$ is of this form. For instance, if $G(\mathbb{F}_q)$ is noncommutative, it has irreducible representations of dimension > 1. On the other hand, in many cases where $G(\mathbb{F}_q)$ *is* commutative, not every 1-dimensional representation of $G(\mathbb{F}_q)$ comes from a multiplicative local system on G.

To find other L-packets of irreducible representations of $G(\mathbb{F}_q)$, we must allow $H \neq G$. It is clear that H cannot be trivial, so by Lemma 2.16, the only remaining possibility is $H = [G, G]$. In this case, H is central in G, so every $\overline{\mathbb{Q}}_\ell$-local system on H is automatically G-invariant. It is easy to check that if $\mathcal{L}$ is a multiplicative $\overline{\mathbb{Q}}_\ell$-local system on H, the pair $(H, \mathcal{L})$ is admissible for G if and only if $\mathcal{L}$ is nontrivial. Moreover, in this case, the geometric conjugacy class $\mathscr{C}$ of $(H, \mathcal{L})$ also reduces to the single pair $(H, \mathcal{L})$, and the corresponding L-packet $L(\mathscr{C})$ consists of all irreducible representations of $G(\mathbb{F}_q)$ that act by the scalar $t_{\mathcal{L}}$ on $[G, G](\mathbb{F}_q)$.

3 The structure of the proofs

In this section, we describe the methods we used to prove Theorems 2.5 and 2.14, stating several other results that are interesting in their own right along the way.

The main technical tools used in our proofs are:

- the equivariant derived category $\mathscr{D}_G(G)$ for a unipotent group G, along with the bifunctor $(M, N) \longmapsto M * N$ of convolution with compact supports and the collection of "twists" $\theta_M : M \xrightarrow{\simeq} M$ defined for all $M \in \mathscr{D}_G(G)$;
- the functor of induction with compact supports $\mathrm{ind}_{G'}^{G} : \mathscr{D}_{G'}(G') \longrightarrow \mathscr{D}_G(G)$, defined for any closed subgroup $G' \subset G$; and
- the notion of an admissible pair for G, along with an extension of Serre duality [34] to noncommutative connected unipotent groups.

The first two of these are introduced in Sects. 4 and 5, respectively, where we also establish some auxiliary results involving these technical tools. The extension of Serre duality to the noncommutative setting, along with some new results on the classical Serre duality and bi-extensions of connected commutative unipotent groups by $\mathbb{Q}_p/\mathbb{Z}_p$, appears in the (rather extensive) "Appendix". Admissible pairs are defined in Sect. 7.3. The proofs of Theorems 2.5 and 2.14 occupy Sects. 7–9; together, they can be split into the following sequence of steps.

3.1 Step 1

Let G be a connected unipotent group over $\mathbb{F}_q$. We begin by proving the one result which explicitly relates representations of $G(\mathbb{F}_q)$ with the geometry of G: namely, that for every irreducible representation ρ of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$, there exists a geometric conjugacy class $\mathscr{C}$ of admissible pairs for G such that ρ lies in $L(\mathscr{C})$ (cf. Definition 2.13). Theorem 7.1 gives a slightly more precise statement.

One of the ingredients in this step should be useful in other situations. Namely, in Proposition 7.7 we formulate a condition under which a multiplicative $\overline{\mathbb{Q}}_\ell$-local system on a closed connected subgroup H of a connected unipotent group G can be extended to a multiplicative $\overline{\mathbb{Q}}_\ell$-local system on all of G.

3.2 Step 2

Next we relate admissible pairs to L-packets. If $(H, \mathcal{L})$ is an admissible pair for a unipotent group G over an arbitrary field k, we consider the object $e_{\mathcal{L}} = \mathbb{K}_H \otimes \mathcal{L} \in \mathscr{D}_H(H)$, where $\mathbb{K}_H \in \mathscr{D}_H(H)$ is the dualizing complex of H. One checks easily that $e_{\mathcal{L}} * e_{\mathcal{L}} \cong e_{\mathcal{L}}$, i.e., $e_{\mathcal{L}}$ is a weak idempotent. If G' is the stabilizer of $(H, \mathcal{L})$ for the conjugation action of G, it is easy to see that the extension of $e_{\mathcal{L}}$ by zero to all of G' defines an object $e'_{\mathcal{L}} \in \mathscr{D}_{G'}(G')$. Of course, $e'_{\mathcal{L}}$ is also a weak idempotent. Finally, we apply the functor of induction with compact supports, and we show that $e_{H,\mathcal{L}} := \operatorname{ind}_{G'}^{G} e'_{\mathcal{L}}$ is a *minimal weak idempotent* in $\mathscr{D}_G(G)$, i.e., a nonzero weak idempotent such that if $e \in \mathscr{D}_G(G)$ is any weak idempotent, then either $e_{H,\mathcal{L}} * e = 0$, or $e_{H,\mathcal{L}} * e \cong e_{H,\mathcal{L}}$. All these results are proved in Sects. 8 and 9.

One of the ingredients here is a more general result, proved in Sect. 5.8, which gives a condition on a given weak idempotent $e \in \mathscr{D}_{G'}(G')$ under which $f = \operatorname{ind}_{G'}^{G}(e)$ is a weak idempotent in $\mathscr{D}_G(G)$ and the functor $\operatorname{ind}_{G'}^{G}$ restricts to an *equivalence of semigroupal categories* $e * \mathscr{D}_{G'}(G') \xrightarrow{\sim} f * \mathscr{D}_G(G)$. The condition is reminiscent of Mackey's criterion for the irreducibility of an induced representation.

3.3 Step 3

We explore the relationship between the functor $\operatorname{ind}_{G'}^{G}$ and the operation of induction of class functions studied in Sect. 6.7 to prove that in the situation of Step 2, if G is connected, $k = \mathbb{F}_q$, and $\mathscr{C}$ is the geometric conjugacy class of the admissible pair $(H, \mathcal{L})$, then the set $L(\mathscr{C})$ of irreducible $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{F}_q)$ introduced in Definition 2.13 coincides with the set of irreducible $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{F}_q)$ on which the trace function $t_{e_{H,\mathcal{L}}} : G(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell$ acts as the identity.

Remark 3.1 The functor $\operatorname{ind}_{G'}^{G}$ is often not compatible with induction of class functions on the nose (unless G' is connected), which is why we must work with geometric conjugacy classes of admissible pairs, rather than $G(\mathbb{F}_q)$-conjugacy classes.

After we put the previous steps together, proving Theorem 2.14 becomes very easy. Namely, let $e \in \mathscr{D}_G(G)$ be any weak idempotent. If $t_e \equiv 0$, then we can discard e while trying to describe L-packets. Otherwise there exists an irreducible $\overline{\mathbb{Q}}_\ell$-representation ρ of $G(\mathbb{F}_q)$ on which t_e acts nontrivially. By Step 1, there exists a geometric conjugacy class $\mathscr{C}$ of admissible pairs for G such that $\rho \in L(\mathscr{C})$. By Step 3, if $(H, \mathcal{L}) \in \mathscr{C}$, then $t_{e_{H,\mathcal{L}}}$ acts nontrivially on ρ. This implies that $t_e * t_{e_{H,\mathcal{L}}} \not\equiv 0$. A fortiori, $e * e_{H,\mathcal{L}} \neq 0$ (as convolution of functions is clearly compatible with the convolution with compact supports of ℓ-adic complexes). By Step 2, this implies that $e * e_{H,\mathcal{L}} \cong e_{H,\mathcal{L}}$, and therefore, applying Step 3 again, we see that e acts as the identity on every irreducible representation of $G(\mathbb{F}_q)$ appearing in $L(\mathscr{C})$. This result, together with the statement proved in Step 1, implies Theorem 2.14.

3.4 Step 4

Now let G be an easy unipotent group over $\mathbb{F}_q$. In the proof of Theorem 2.5, we use the result of Step 1 above (but not the results of Steps 2 and 3). Thus, let ρ be an irreducible $\overline{\mathbb{Q}}_\ell$-representation of $G(\mathbb{F}_q)$, and choose an admissible pair $(H, \mathcal{L})$ for G such that ρ is an irreducible summand of $\mathrm{Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)} t_{\mathcal{L}}$.

We employ the compatibility of the functor $\mathrm{ind}_{G'}^{G}$ with twists (Proposition 5.17) and the triviality of twists in $\mathscr{D}_G(G)$ (Lemma 4.16) to prove that if G' is the stabilizer of $(H, \mathcal{L})$ for the conjugation action of G, then G' is necessarily connected and the homomorphism $(G'/H)_{perf} \longrightarrow (G'/H)^*_{perf}$ appearing in the definition of an admissible pair (see Sect. 7.3) is an isomorphism (not merely an isogeny). Here, $(G'/H)_{perf}$ denotes the *perfectization* of the group G'/H (see Sect. 9.6), and $(G'/H)^*_{perf}$ is its *Serre dual* (see Sects. 3.6 and 9.6).

From this, we deduce that $G'(\mathbb{F}_q)$ has a unique irreducible $\overline{\mathbb{Q}}_\ell$-representation ρ' which acts by the scalar $t_{\mathcal{L}}$ on $H(\mathbb{F}_q)$. Mackey's criterion implies that $\mathrm{Ind}_{G'(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho'$ is irreducible, and Frobenius reciprocity forces $\rho \cong \mathrm{Ind}_{G'(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho'$. In particular, $\dim \rho = q^{\dim G - \dim G'} \cdot \dim \rho'$ (because G' is connected).

3.5 Step 5

To complete the proof of Theorem 2.5, we must demonstrate that, in the situation of the previous step, the dimension of ρ' is a power of q. Since $\dim \rho' = [G'(\mathbb{F}_q) : H(\mathbb{F}_q)]^{1/2}$, this is the same as showing that $\dim(G'/H)$ is even. In view of the fact that the canonical map $(G'/H)_{perf} \longrightarrow (G'/H)^*_{perf}$ is an isomorphism, this follows from a more general result, Proposition A.28, proved in Sect. 9.6.

Remark 3.2 As we already mentioned in the Introduction, in this paper we do not define or use the braided monoidal structure on the category $\mathscr{D}_G(G)$, without which the significance of the "twists" in $\mathscr{D}_G(G)$ cannot be fully appreciated (see [10]). However, we believe that the full power of the geometric techniques should be reserved for the theory of character sheaves.

The reader who is only interested in understanding the general ideas behind our arguments does not have to read any further. The missing details of the proofs sketched above are filled in the remaining sections, which are more technical.

3.6 On Serre duality

We end with a comment on the notion of a multiplicative $\overline{\mathbb{Q}}_\ell$-local system used in the main body of the paper and the Serre duality studied in the "Appendix". In the proofs of our main results, Serre duality serves mostly as a tool, and if G is a connected unipotent group over a perfect field k of characteristic $p > 0$, we think of the Serre dual G^* of G as the "moduli space of multiplicative $\overline{\mathbb{Q}}_\ell$-local systems on G". However, if one wishes to prove foundational results about Serre duality, the most natural framework (which, in particular, is independent of ℓ) is that of *central extensions* by the discrete group $\mathbb{Q}_p/\mathbb{Z}_p$. It would have been inconvenient for us to choose one of these viewpoints once and for all, and to completely discard the other one. The relationship between them is described in Sect. 7.2.

4 The category $\mathscr{D}_G(G)$ and L-packets

4.1 Derived categories of constructible ℓ-adic complexes

Fix an arbitrary field k and a prime $\ell \neq \operatorname{char} k$ (in Sects. 4.2 and 4.8, we take k to be finite). Throughout this section, we will work with schemes of finite type over k. If X is such a scheme, one defines the bounded derived category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ of constructible complexes of $\overline{\mathbb{Q}}_\ell$-sheaves on X. We will denote this category simply by $\mathscr{D}(X)$, with the understanding that ℓ is fixed once and for all. It is a triangulated $\overline{\mathbb{Q}}_\ell$-linear category. For perfect k, the definition of $\mathscr{D}(X)$ appears in [17], and in general we define $\mathscr{D}(X) = \mathscr{D}(X \otimes_k k^{perf})$, where k^{perf} is the perfect closure of k.

Remark 4.1 For the purposes of this work, it would be enough to consider the case where k is finite or algebraically closed. Here the definition of $\mathscr{D}(X)$ is more classical [1,13,14]. However, with future applications in mind, we consider the more general case in this section.

We will often use Grothendieck's "formalism of the six functors" for the categories $\mathscr{D}(X)$ (as well as their equivariant versions, defined in Sect. 4.3). For a morphism $f : X \longrightarrow Y$ of k-schemes of finite type one has the pullback functor $f^* : \mathscr{D}(Y) \longrightarrow \mathscr{D}(X)$, the pushforward functor $f_* : \mathscr{D}(X) \longrightarrow \mathscr{D}(Y)$, the functor $f_! : \mathscr{D}(X) \longrightarrow \mathscr{D}(Y)$ (pushforward with compact supports), and the functor $f^! : \mathscr{D}(Y) \longrightarrow \mathscr{D}(X)$. *We always omit the letters "L" and "R" from our notation for the six functors; thus, $f_!$ stands for $Rf_!$ and $\otimes$ stands for $\overset{L}{\otimes}_{\overline{\mathbb{Q}}_\ell}$, etc.*

Remark 4.2 In [14], the functor $f_!$ is defined for separated morphisms f when k is finite or algebraically closed. This case would suffice for the purposes of the present work. However, the formalism we need was extended to arbitrary fields k in [17], and the assumption that f is separated is unnecessary [30].

The most important result we will need is the proper base change theorem; see Exp. XII and XVII in [1] and Exp. IV in [14] for the case where k is finite or algebraically closed; and Theorem 6.3(iii) in [17] for the general case.

Theorem 4.3 (Proper base change) *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{\;g'\;} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\;g\;} & Y \end{array}$$

of k-schemes of finite type. There is a natural isomorphism of functors

$$(g^* \circ f_!) \cong (f'_! \circ g'^*) : \mathscr{D}(X) \longrightarrow \mathscr{D}(Y'). \tag{4.1}$$

4.2 Reminder on the sheaves-to-functions correspondence

In this subsection, we assume that the base field is finite: $k = \mathbb{F}_q$. Let X be a scheme of finite type over $\mathbb{F}_q$. Given an object $M \in \mathscr{D}(X)$, one can define the corresponding function $t_M : X(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell$. Namely, a point $x \in X(\mathbb{F}_q)$ can be thought of as an $\mathbb{F}_q$-morphism $x : \operatorname{Spec} \mathbb{F}_q \longrightarrow X$. Then, $x^*M \in \mathscr{D}(\operatorname{Spec} \mathbb{F}_q)$, and the cohomology sheaves $\mathcal{H}^i(x^*M) \cong x^*\mathcal{H}^i(M)$ are constructible ℓ-adic sheaves on $\operatorname{Spec} \mathbb{F}_q$, i.e., continuous finite dimensional representations of the absolute Galois group $\operatorname{Gal}(\mathbb{F}/\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$. Let $F_q \in \operatorname{Gal}(\mathbb{F}/\mathbb{F}_q)$ be the geometric Frobenius, defined as the inverse of the Frobenius substitution $a \longmapsto a^q$. Then, one defines

$$t_M(x) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot \operatorname{tr}\left(F_q; \mathcal{H}^i(x^*M)\right).$$

The main properties of the map $M \longmapsto t_M$ are summarized in

Lemma 4.4 *Let X and Y be schemes of finite type over $\mathbb{F}_q$, and let $f : X \longrightarrow Y$ be an $\mathbb{F}_q$-morphism.*

(1) *If $N \in \mathscr{D}(Y)$, then $t_{f^*N} = f^*t_N \overset{def}{:=} t_N \circ f$.*
(2) *If $M, K \in \mathscr{D}(X)$, then $t_{M \otimes K} = t_M \cdot t_K$ (pointwise product).*
(3) *Assume that f is separated. If $M \in \mathscr{D}(X)$, then $t_{f_!M} = f_!t_M$, where, by abuse of notation, we also write f for the induced map of sets $X(\mathbb{F}_q) \longrightarrow Y(\mathbb{F}_q)$, and $(f_!t_M)(y) = \sum_{x \in f^{-1}(y)} t_M(x)$.*

Of these, (1) and (2) follow rather easily from the definitions, while (3) is more subtle. It follows from the proper base change theorem and the special case of (3) where $Y = \operatorname{Spec} \mathbb{F}_q$, which is known as the *Lefschetz-Grothendieck trace formula*; see Theorem 3.2 of "Rapport sur la formule des traces" in [14].

4.3 Equivariant derived categories

We return to the situation where the base field k is arbitrary. Let G be an algebraic group over k, let X be a scheme of finite type over k, and suppose that we are given a regular left action of G on X. We would like to define the "equivariant derived category" $\mathscr{D}_G(X)$.

In general, to get the correct definition one must either adopt the approach of Bernstein and Lunts [5] (when G is affine), or use the definition of ℓ-adic derived categories for Artin stacks due to Laszlo and Olsson [30] and define $\mathscr{D}_G(X) = \mathscr{D}(G \backslash X)$, where $G \backslash X$ is the quotient stack of X by G.

From now on, we assume that G is unipotent. In this case, one knows that the naive definition of $\mathscr{D}_G(X)$ (taken from [10]), given below, already gives the correct answer.

Roughly speaking, this definition amounts to looking at the "category of G-equivariant objects in $\mathscr{D}(X)$".

Let us write $\alpha : G \times X \to X$ for the action morphism and $\pi : G \times X \to X$ for the projection. Let $\mu : G \times G \to G$ be the product in G. Let $\pi_{23} : G \times G \times X \to G \times X$ be the projection along the first factor G. The category $\mathscr{D}_G(X)$ is defined as follows.

Definition 4.5 An *object* of the category $\mathscr{D}_G(X)$ is a pair (M, ϕ), where $M \in \mathscr{D}(X)$ and $\phi : \alpha^* M \xrightarrow{\simeq} \pi^* M$ is an isomorphism in $\mathscr{D}(G \times_k X)$ such that

$$\pi_{23}^*(\phi) \circ (\mathrm{id}_G \times \alpha)^*(\phi) = (\mu \times \mathrm{id}_X)^*(\phi), \tag{4.2}$$

i.e., the composition of the natural isomorphisms

$$(\mathrm{id}_G \times \alpha)^* \alpha^* M \cong (\mu \times \mathrm{id}_X)^* \alpha^* M \xrightarrow{(\mu \times \mathrm{id}_X)^*(\phi)} (\mu \times \mathrm{id}_X)^* \pi^* M \cong \pi_{23}^* \pi^* M$$

equals the composition

$$(\mathrm{id}_G \times \alpha)^* \alpha^* M \xrightarrow{(\mathrm{id}_G \times \alpha)^*(\phi)} (\mathrm{id}_G \times \alpha)^* \pi^* M \cong \pi_{23}^* \alpha^* M \xrightarrow{\pi_{23}^*(\phi)} \pi_{23}^* \pi^* M.$$

A *morphism* $(M, \phi) \longrightarrow (N, \psi)$ in $\mathscr{D}_G(X)$ is a morphism $\nu : M \longrightarrow N$ in $\mathscr{D}(X)$ satisfying $\psi \circ \alpha^*(\nu) = \pi^*(\nu) \circ \phi$. The *composition* of morphisms in $\mathscr{D}_G(X)$ is defined to be equal to their composition in $\mathscr{D}(X)$.

Remark 4.6 If G is a connected unipotent group, the forgetful functor $\mathscr{D}_G(X) \longrightarrow \mathscr{D}(X)$ is fully faithful.

4.4 Functors between equivariant derived categories

In the situation of Sect. 4.3, let us assume that H is another unipotent group over k acting on a scheme Y of finite type over k. Suppose we are given a homomorphism $i : G \longrightarrow H$ of k-groups and a morphism $f : X \longrightarrow Y$ of k-schemes which is G-equivariant with respect to the G-action on Y induced by i. Then, the functor $f^* : \mathscr{D}(Y) \longrightarrow \mathscr{D}(X)$ naturally lifts to a functor $f^* : \mathscr{D}_H(Y) \longrightarrow \mathscr{D}_G(X)$.

In the special case where $H = G$ and i is the identity, we can also define a functor $f_! : \mathscr{D}_G(X) \longrightarrow \mathscr{D}_G(Y)$. Indeed, we have cartesian diagrams

$$\begin{array}{ccc} G \times X & \xrightarrow{\mathrm{id}_G \times f} & G \times Y \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ X & \xrightarrow{f} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} G \times X & \xrightarrow{\mathrm{id}_G \times f} & G \times Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

where α_X, α_Y are the action morphisms and π_X, π_Y are the projections, so Theorem 4.3 implies that $f_! : \mathscr{D}(X) \longrightarrow \mathscr{D}(Y)$ lifts to a functor $f_! : \mathscr{D}_G(X) \longrightarrow \mathscr{D}_G(Y)$.

From now on, we assume that if f is an equivariant morphism between k-schemes of finite type equipped with a G-action, then f^* and $f_!$ are understood as functors between the corresponding equivariant derived categories.

4.5 Convolution in $\mathscr{D}(G)$ and $\mathscr{D}_G(G)$

Let G be an algebraic group over an arbitrary field k, and $\mu : G \times_k G \longrightarrow G$ the multiplication morphism. The bifunctor

$$\mathscr{D}(G) \times \mathscr{D}(G) \longrightarrow \mathscr{D}(G), \qquad (M, N) \longmapsto M * N = \mu_!(M \boxtimes N), \tag{4.3}$$

is called the *convolution with compact supports*. Replacing $\mu_!$ with μ_* in the above definition would yield the "usual" convolution bifunctor; however, convolution with compact supports is the only one that will be used in this article and will be referred to simply as "convolution" of constructible ℓ-adic complexes on G.

It is easy to construct an associativity constraint for the bifunctor $*$, and check that it makes $\mathscr{D}(G)$ a monoidal category, where the unit object $\mathbb{1}$ is the delta-sheaf at the identity element of G, i.e., $\mathbb{1} = 1_*\overline{\mathbb{Q}}_\ell = 1_!\overline{\mathbb{Q}}_\ell$.

Lemma 4.4 implies that the bifunctor $*$ is compatible with convolution of functions via the sheaves-to-functions correspondence. Namely, for a finite group Γ, let us define the convolution of two functions $f_1, f_2 : \Gamma \longrightarrow \overline{\mathbb{Q}}_\ell$ by the formula $(f_1 * f_2)(g) = \sum_{\gamma\in\Gamma} f_1(\gamma) f_2(\gamma^{-1}g)$. Then, for any algebraic group G over $\mathbb{F}_q$ and any $M, N \in \mathscr{D}(G)$, we have $t_{M*N} = t_M * t_N$ as functions on $G(\mathbb{F}_q)$.

Next, suppose that G is a unipotent algebraic group over k. Unless otherwise explicitly stated, whenever we consider a G-action on itself, we will always mean the *conjugation* action. We also have the induced action of G on $G \times_k G$ (by simultaneous conjugation), and the multiplication morphism $\mu : G \times_k G \longrightarrow G$ is G-equivariant. It follows (see Sect. 4.4) that (4.3) can be upgraded to a bifunctor

$$\mathscr{D}_G(G) \times \mathscr{D}_G(G) \longrightarrow \mathscr{D}_G(G), \qquad (M, N) \longmapsto M * N = \mu_!(M \boxtimes N), \tag{4.4}$$

which we also call convolution with compact supports.

Just as in the non-equivariant case, (4.4) can be upgraded to a monoidal structure on the category $\mathscr{D}_G(G)$. Moreover, this category has a natural braiding, defined explicitly in [10]. We will only need a weaker assertion:

Lemma 4.7 *There exist functorial isomorphisms* $\beta_{M,N} : M * N \xrightarrow{\simeq} N * M$ *for all* $M, N \in \mathscr{D}_G(G)$.

Proof Consider the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \xi\ } & G \times G \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\mu} \\ G \times G & \xrightarrow{\ \mu\ } & G, \end{array}$$

where $\tau(g,h) := (h,g)$ and $\xi(g,h) := (g, g^{-1}hg)$. We have $M * N = \mu_!(M \boxtimes N)$, and the above diagram shows that $N * M = (\mu\tau)_!(M \boxtimes N) = \mu_!\xi_!(M \boxtimes N)$. We define $\beta_{M,N} : \mu_!(M \boxtimes N) \xrightarrow{\sim} \mu_!\xi_!(M \boxtimes N)$ by $\beta_{M,N} := \mu_!(f)$, where $f : M \boxtimes N \xrightarrow{\sim} \xi_!(M \boxtimes N)$ comes from the G-equivariant structure on N. □

4.6 Semigroupal categories

The notion of a *semigroupal* category is obtained from that of a monoidal category by discarding all the axioms that involve the unit object. Thus, a semigroupal category is a triple $(\mathcal{M}, \otimes, \alpha)$, where $\mathcal{M}$ is a category, $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ is a bifunctor, and α is an associativity constraint for $\otimes$, i.e., a collection of trifunctorial isomorphisms $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)$ for all triples of objects $X, Y, Z \in \mathcal{M}$ satisfying a standard coherence condition.

The reason we need this notion is that even though the categories $\mathscr{D}(G)$ and $\mathscr{D}_G(G)$ introduced in Sect. 4.5 are monoidal, we will have the occasion to consider certain semigroupal subcategories of $\mathscr{D}_G(G)$ that, at least a priori, may not be monoidal (cf. Remark 4.8).

The notion of a (weak, strong or strict) *semigroupal functor* between semigroupal categories is also obtained from the notion of a monoidal functor in the obvious way. Thus, if $(\mathcal{M}, \otimes, \alpha)$ and $(\mathcal{N}, \otimes', \alpha')$ are semigroupal categories, a functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ is said to be *strict semigroupal* if F commutes with the semigroupal structures "on the nose," i.e., $F(X \otimes Y) = F(X) \otimes' F(Y)$ for every pair of objects X, Y of $\mathcal{M}$; $F(f \otimes g) = F(f) \otimes' F(g)$ for every pair of morphisms f, g in $\mathcal{M}$; and $F(\alpha_{X,Y,Z}) = \alpha'_{F(X),F(Y),F(Z)}$ for every triple of objects X, Y, Z of $\mathcal{M}$.

On the other hand, a *weak semigroupal structure* on a functor $\Phi : \mathcal{M} \longrightarrow \mathcal{N}$ is a collection of bifunctorial morphisms $\varphi_{X,Y} : \Phi(X) \otimes' \Phi(Y) \longrightarrow \Phi(X \otimes Y)$ for all $X, Y \in \mathcal{M}$, which are compatible with the associativity constraints in the obvious sense. The structure is said to be *strong* if every $\varphi_{X,Y}$ is an isomorphism.

An *additive semigroupal category* is a semigroupal category $(\mathcal{M}, \otimes, \alpha)$ such that $\mathcal{M}$ is an additive category, and the bifunctor $\otimes$ is bi-additive.

4.7 Weak idempotents

Let $\mathcal{M} = (\mathcal{M}, \otimes, \alpha)$ be a semigroupal category. An object $e \in \mathcal{M}$ is said to be a *weak idempotent* if $e \otimes e \cong e$. Observe that this notion depends only on the bifunctor $\otimes$ and not on the associativity constraint α.

If $e \in \mathcal{M}$ is a weak idempotent, the corresponding *Hecke subcategory* is defined as the full subcategory $e\mathcal{M}e \subset \mathcal{M}$ consisting of all objects $N \in \mathcal{M}$ such that $N \cong e \otimes N \otimes e$. Equivalently, $e\mathcal{M}e$ can be described as the essential image of the functor $\mathcal{M} \longrightarrow \mathcal{M}$ given by $M \longmapsto (e \otimes M) \otimes e$, which explains the notation.

The Hecke subcategory $e\mathcal{M}e \subset \mathcal{M}$ is stable under $\otimes$, so it becomes a semigroupal category in its own right. If $\mathcal{M}$ is additive, so is $e\mathcal{M}e$.

Remark 4.8 Even if $\mathcal{M}$ is a *monoidal* category, one cannot expect $e\mathcal{M}e$ to be a monoidal category in general. Indeed, even though $e \otimes N \cong N \cong N \otimes e$ for all $N \in e\mathcal{M}e$, there is no guarantee that the functor $N \longmapsto e \otimes N$ is an auto-equivalence of $e\mathcal{M}e$; if it is not, then $e\mathcal{M}e$ cannot have a unit object.[6]

Definition 4.9 A semigroupal category $\mathcal{M}$ is *weakly symmetric* if $M \otimes N \cong N \otimes M$ for all pairs of objects $M, N \in \mathcal{M}$.

For example, if a semigroupal category admits a braiding, then it is weakly symmetric. The category $(\mathscr{D}_G(G), *)$ is weakly symmetric by virtue of Lemma 4.7. Observe that if $\mathcal{M}$ is a weakly symmetric semigroupal category, then for any weak idempotent $e \in \mathcal{M}$, we have $e\mathcal{M}e = e\mathcal{M} = \mathcal{M}e$.

Definition 4.10 If $\mathcal{M}$ is an additive weakly symmetric semigroupal category, a weak idempotent $e \in \mathcal{M}$ is said to be *minimal* if $e \neq 0$ and, for any weak idempotent $e' \in \mathcal{M}$, we have either $e \otimes e' = 0$, or $e \otimes e' \cong e$.

Remark 4.11 If $\mathcal{M}$ is an additive weakly symmetric semigroupal category, a weak idempotent $e \in \mathcal{M}$ is minimal if and only if the Hecke subcategory $e\mathcal{M}$ contains exactly two weak idempotents (up to isomorphism), namely, 0 and e.

4.8 Idempotents and L-packets

In this subsection, we again assume that the base field is finite: $k = \mathbb{F}_q$. Let us fix a connected unipotent group G over $\mathbb{F}_q$. Recall from Definition 2.7 that two irreducible representations, ρ_1 and ρ_2, of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$, are said to be *$L$-indistinguishable* if for every weak idempotent $e \in \mathscr{D}_G(G)$, the function t_e acts in the same way in ρ_1 and ρ_2. In this subsection, we will explain that in this definition one can restrict attention to a special class of weak idempotents.

Definition 4.12 If k is any field and U is a unipotent group over k, a weak idempotent $e \in \mathscr{D}_U(U)$ is said to be *geometrically minimal* if for every algebraic extension k' of k, the induced weak idempotent $e' = e \otimes_k k'$ in $\mathscr{D}_{U'}(U')$, where $U' = U \otimes_k k'$, is minimal in the sense of Definition 4.10.

Every geometrically minimal weak idempotent in $\mathscr{D}_U(U)$ is minimal, but the converse need not be true. The next result is proved in Sect. 9.3.

Proposition 4.13 *Let G be a connected unipotent group over $\mathbb{F}_q$, and let ρ_1, ρ_2 be two irreducible representations of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$. The following are equivalent.*

(i) *The representations ρ_1 and ρ_2 are L-indistinguishable.*
(ii) *For every minimal weak idempotent $e \in \mathscr{D}_G(G)$, the function t_e acts in the same way in ρ_1 and in ρ_2.*
(iii) *For every geometrically minimal weak idempotent $e \in \mathscr{D}_G(G)$, the function t_e acts in the same way in ρ_1 and in ρ_2.*

[6] This is one of the reasons why we chose the term "*weak* idempotent". The notion of a *closed idempotent* in a monoidal category, defined in [10], is much more rigid; in particular, if e is any closed idempotent in a monoidal category $\mathcal{M}$, then $e\mathcal{M}e$ is monoidal as well.

Remarks 4.14 (1) Note that if $e_1, e_2 \in \mathscr{D}_G(G)$ are non-isomorphic minimal weak idempotents, then $e_1 * e_2 = 0$, so $t_{e_1} * t_{e_2} = 0$. Now if $e \in \mathscr{D}_G(G)$ is a geometrically minimal weak idempotent such that $t_e \not\equiv 0$, we can define $L(e)$ as the set of irreducible representations of $G(\mathbb{F}_q)$ on which t_e acts as the identity, and it follows (from Definition 2.7 and the last observation) that $L(e)$ is an L-packet. Proposition 4.13 implies that, conversely, *every* L-packet of irreducible representations of $G(\mathbb{F}_q)$ is of this form.

(2) It is shown in [8] that if $e \in \mathscr{D}_G(G)$ is any geometrically minimal weak idempotent, then $t_e \not\equiv 0$. In particular, one obtains a bijection between L-packets of irreducible representations of $G(\mathbb{F}_q)$ and isomorphism classes of geometrically minimal weak idempotents in $\mathscr{D}_G(G)$. However, the proofs of these facts use some of the methods developed in [9] (as well as additional techniques) and are beyond the scope of the present article.

(3) On the other hand, Proposition 4.13 easily implies that if $e \in \mathscr{D}_G(G)$ is a minimal weak idempotent which is *not geometrically minimal*, then $t_e \equiv 0$.

4.9 Twists in the category $\mathscr{D}_G(G)$

A structure on the equivariant derived category $\mathscr{D}_G(G)$ that plays an important role in the proof of Theorem 2.5 is a canonical automorphism of the identity functor, whose construction we now recall. Fix a unipotent group G over k, let $c : G \times G \to G$ be the conjugation action morphism $c(g, h) = ghg^{-1}$, let $p_2 : G \times G \to G$ denote the second projection, and write $\Delta : G \to G \times G$ for the diagonal. Then, $c \circ \Delta = \mathrm{id}_G = p_2 \circ \Delta$. For each $M \in \mathscr{D}_G(G)$, the G-equivariant structure on M yields an isomorphism $c^*M \xrightarrow{\simeq} p_2^*M$. Pulling it back by Δ, we obtain an isomorphism $\theta_M : M = \Delta^*c^*M \xrightarrow{\simeq} \Delta^*p_2^*M = M$.

Definition 4.15 One calls θ_M the *twist automorphism* of M, or the *balancing isomorphism*. The collection $\{\theta_M \mid M \in \mathscr{D}_G(G)\}$ defines an automorphism of the identity functor on $\mathscr{D}_G(G)$, which we simply denote by θ if no confusion can arise.

The following fact will be used in our proof of Theorem 2.5.

Lemma 4.16 *Let G be an easy unipotent group over a field k of characteristic $p > 0$. For every object $M \in \mathscr{D}_G(G)$, the twist automorphism θ_M of M is trivial.*

Proof (cf. [10]) We may and do assume that k is algebraically closed. Consider the usual (non-perverse) t-structure on $\mathscr{D}(G)$ whose heart is the category $Sh_c(G, \overline{\mathbb{Q}}_\ell)$ of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on G. If $M \in \mathscr{D}(G)$, we will write $\mathcal{H}^i(M)$ for the cohomology sheaves of M with respect to this t-structure ($i \in \mathbb{Z}$).

Fix $M \in \mathscr{D}_G(G)$ and $i \in \mathbb{Z}$. For every $x \in G(k)$, we have the induced action of the centralizer $Z_G(x)$ of x in G on the stalk $\mathcal{H}^i(M)_x$, and by continuity, the neutral component $Z_G(x)^\circ \subset Z_G(x)$ acts trivially on $\mathcal{H}^i(M)_x$. In particular, since G is easy, x acts trivially on $\mathcal{H}^i(M)_x$, which shows that $\theta_{\mathcal{H}^i(M)} = \mathrm{id}_{\mathcal{H}^i(M)}$. This implies that θ_M is a *unipotent* automorphism of M. On the other hand, since G is unipotent, it has exponent p^n for some $n \in \mathbb{N}$, and it follows that $(\theta_M)^{p^n} = \mathrm{id}_M$. Finally, we conclude that $\theta_M = \mathrm{id}_M$, as desired. □

5 Induction with compact supports

5.1 Setup

Throughout this section, G is a unipotent algebraic group over a field k, $G' \subset G$ is a closed subgroup, and ℓ is a prime different from char k. Both G and G' are allowed to be disconnected. We will use the notation introduced in Sect. 4 above. Our goal is to define the functor of "induction with compact supports"

$$\mathrm{ind}_{G'}^{G} : \mathscr{D}_{G'}(G') \longrightarrow \mathscr{D}_G(G) \tag{5.1}$$

and establish its basic properties, such as the existence of a weak semigroupal structure (cf. Sect. 4.6) on this functor and its compatibility with twists (Sect. 5.9). In Sect. 5.8, we study the restriction of the functor (5.1) to a functor between suitable Hecke subcategories of $\mathscr{D}_{G'}(G')$ and of $\mathscr{D}_G(G)$.

5.2 Definition of $\mathrm{ind}_{G'}^{G}$

In this subsection, we will define the functor (5.1).

5.2.1 Motivation

To motivate the definition of (5.1), we first rewrite the formula for the induced character (in the setting of representations of finite groups) in a suggestive way. Let Γ be a finite group, and $\Gamma' \subset \Gamma$ a subgroup. Consider the (free) right action of Γ' on the product $\Gamma \times \Gamma'$ given by $(g, g')\cdot\gamma = (g\gamma, \gamma^{-1}g'\gamma)$, and let $\widetilde{\Gamma} = (\Gamma \times \Gamma')/\Gamma'$ be the set of orbits for this action. The left Γ-action on $\Gamma \times \Gamma'$ given by $\gamma : (g, g') \longmapsto (\gamma g, g')$ descends to a left Γ-action on $\widetilde{\Gamma}$. We have a natural Γ'-equivariant injection $i : \Gamma' \hookrightarrow \widetilde{\Gamma}$ induced by $g' \longmapsto (1, g')$, and a natural Γ-equivariant map $\pi : \widetilde{\Gamma} \longrightarrow \Gamma$ induced by $(g, g') \longmapsto gg'g^{-1}$ (as always, Γ' and Γ act on themselves by conjugation).

We will use the following notation. If X is any set, $\mathrm{Fun}(X)$ denotes the vector space of all functions $X \longrightarrow \overline{\mathbb{Q}}_\ell$. If H is an (abstract) group acting on X, we write $\mathrm{Fun}(X)^H \subset \mathrm{Fun}(X)$ for the subspace of H-invariant functions. If $\phi : X \longrightarrow Y$ is a map of sets, we have the pullback map $\phi^* : \mathrm{Fun}(Y) \longrightarrow \mathrm{Fun}(X)$ given by $\phi^*(f) = f \circ \phi$. Finally, if ϕ has finite fibers (in particular, if X itself is finite), we can also define a linear map $\phi_! : \mathrm{Fun}(X) \longrightarrow \mathrm{Fun}(Y)$ by the formula

$$(\phi_! f)(y) = \sum_{x \in \phi^{-1}(y)} f(x).$$

With this notation, one can easily verify the following statements.

(a) The map $i^* : \mathrm{Fun}(\widetilde{\Gamma})^{\Gamma} \longrightarrow \mathrm{Fun}(\Gamma')^{\Gamma'}$ is an isomorphism.

(b) Let $\mathrm{ind}_{\Gamma'}^{\Gamma} : \mathrm{Fun}(\Gamma')^{\Gamma'} \longrightarrow \mathrm{Fun}(\Gamma)^{\Gamma}$ be defined by $\mathrm{ind}_{\Gamma'}^{\Gamma} = \pi_! \circ (i^*)^{-1}$. If ρ is any finite dimensional representation of Γ' over $\overline{\mathbb{Q}}_\ell$ and $\chi \in \mathrm{Fun}(\Gamma')^{\Gamma'}$ is its character, then the character of the representation $\mathrm{Ind}_{\Gamma'}^{\Gamma}\rho$ of Γ equals $\mathrm{ind}_{\Gamma'}^{\Gamma}(\chi)$.

5.2.2 Auxiliary constructions

We will define the functor $\mathrm{ind}_{G'}^{G}$ by imitating the formula presented in Sect.5.2.1. We have a free right action of G' on $G \times G'$, given by $(g, g') \cdot \gamma = (g\gamma, \gamma^{-1} g' \gamma)$, so we can form the quotient $\widetilde{G} = (G \times G')/G'$ (it exists as a scheme, for instance, because $G \times G'$ is affine), and G acts on $\widetilde{G}$ on the left. Similarly, we can consider the G'-equivariant injection $i : G' \hookrightarrow \widetilde{G}$ induced by $g' \longmapsto (1, g')$, and the G-equivariant morphism $\pi : \widetilde{G} \longrightarrow G$ induced by $(g, g') \longmapsto g g' g^{-1}$, where G and G' act on themselves by conjugation. Applying the constructions of Sect. 4.4, we obtain functors between the equivariant derived categories

$$\mathscr{D}_{G'}(G') \xleftarrow{\ i^*\ } \mathscr{D}_{G}(\widetilde{G}) \xrightarrow{\ \pi_!\ } \mathscr{D}_{G}(G).$$

The geometric analogue of statement (a) in Sect. 5.2.1 is the following

Lemma 5.1 *The functor $i^* : \mathscr{D}_G(\widetilde{G}) \longrightarrow \mathscr{D}_{G'}(G')$ is an equivalence of categories.*

This result is proved in Sect. 5.2.4.

5.2.3 The main definition

In the situation of Sect. 5.2.2, let us choose a quasi-inverse to the functor i^*, and denote it by $(i^*)^{-1}$, by a slight abuse of notation.

Definition 5.2 The functor $\mathrm{ind}_{G'}^{G} : \mathscr{D}_{G'}(G') \longrightarrow \mathscr{D}_G(G)$ of *induction with compact supports* is defined as the composition

$$\mathscr{D}_{G'}(G') \xrightarrow{\ (i^*)^{-1}\ } \mathscr{D}_{G}(\widetilde{G}) \xrightarrow{\ \pi_!\ } \mathscr{D}_{G}(G).$$

Remarks 5.3 (1) Strictly speaking, the definition we gave depends on the choice of $(i^*)^{-1}$. However, different choices lead to isomorphic functors $\mathrm{ind}_{G'}^{G}$, and since we only use induction as a technical tool, we prefer to ignore this issue.

(2) Along with the functor $\mathrm{ind}_{G'}^{G}$, one can introduce an induction functor

$$\mathrm{Ind}_{G'}^{G} = \pi_* \circ (i^*)^{-1} \ : \ \mathscr{D}_{G'}(G') \longrightarrow \mathscr{D}_G(G).$$

We will only need this functor in the proof of Proposition 5.17.

5.2.4 Proof of Lemma 5.1

The result would have been more or less obvious, had we used the definition of equivariant derived categories in terms of quotient stacks (cf. Sect. 4.3). Indeed, the map $g' \longmapsto (1, g')$ induces an isomorphism $G' \xrightarrow{\ \simeq\ } G\backslash(G \times G')$, and hence an isomorphism $G'/(\mathrm{Ad}\, G') \xrightarrow{\ \simeq\ } G\backslash(G \times G')/G' = G\backslash\widetilde{G}$. However, since we used an ad hoc definition of the equivariant derived category, we will give a proof of Lemma 5.1 that only uses that definition. The argument is based on two results:

Lemma 5.4 *Let U be a unipotent group over k, let $N \subset U$ be a closed normal subgroup, let X be a scheme of finite type over k with a left U-action, and let $\phi : X \longrightarrow Y$ be a morphism of k-schemes which makes X an N-torsor over Y (a fortiori, the induced action of N on X is free). The pullback functor ϕ^* can be upgraded to an equivalence of categories*

$$\phi^* : \mathscr{D}_{U/N}(Y) \longrightarrow \mathscr{D}_U(X).$$

Lemma 5.5 *With the notation of Lemma 5.4, assume, in addition, that N admits a complement in U (i.e., a closed subgroup $H \subset U$ which maps isomorphically onto U/N), and that ϕ admits an H-equivariant section $\sigma : Y \longrightarrow X$. Then, the functor*

$$\sigma^* : \mathscr{D}_U(X) \longrightarrow \mathscr{D}_{U/N}(Y) = \mathscr{D}_H(Y),$$

understood as the composition of the forgetful functor $\mathscr{D}_U(X) \longrightarrow \mathscr{D}_H(X)$ and the pullback via σ, is a quasi-inverse to the functor $\phi^* : \mathscr{D}_{U/N}(Y) \longrightarrow \mathscr{D}_U(X)$.

Let us now prove Lemma 5.1. Recall that $\widetilde{G}$ is defined as the quotient of $G \times G'$ by the right G'-action defined by $(g, g') \cdot \gamma = (g\gamma, \gamma^{-1}g'\gamma)$. Since we will also need to consider left G-actions, we prefer to turn this action into a left G'-action as well, given by $\gamma : (g, g') \longmapsto (g\gamma^{-1}, \gamma g'\gamma^{-1})$ (merely for notational convenience).

Let us write $\mathfrak{q} : G \times G' \longrightarrow \widetilde{G}$ for the quotient morphism. We define a left action of $G \times G'$ on $G \times G'$ by $(h, \gamma) : (g, g') \longmapsto (hg\gamma^{-1}, \gamma g'\gamma^{-1})$. Applying Lemma 5.4 to $U = G \times G'$ and $N = \{1\} \times G' \subset U$, we obtain an equivalence

$$\mathfrak{q}^* : \mathscr{D}_G(\widetilde{G}) \xrightarrow{\sim} \mathscr{D}_{G\times G'}(G \times G').$$

On the other hand, let $\mathfrak{p}' : G \times G' \longrightarrow G'$ denote the second projection. It can be viewed as a quotient map for the induced action of G on $G \times G'$, where we embed $G \hookrightarrow G \times G'$ via $g \longmapsto (g, 1)$. Of course, the quotient group $(G \times G')/(G \times \{1\})$ is naturally identified with G'. Let $\Delta : G' \hookrightarrow G \times G'$ denote the diagonal embedding. Then, $\Delta(G')$ is a complement to $G \times \{1\}$ in $G \times G'$. Moreover, the map $j : G' \longrightarrow G \times G'$ defined by $g' \longmapsto (1, g')$ is a $\Delta(G')$-equivariant section of $\mathfrak{p}'$.

Applying Lemma 5.5, we see that the functors

$$\mathfrak{p}'^* : \mathscr{D}_{G'}(G') \longrightarrow \mathscr{D}_{G\times G'}(G \times G') \quad \text{and} \quad j^* : \mathscr{D}_{G\times G'}(G \times G') \longrightarrow \mathscr{D}_{G'}(G')$$

are equivalences of categories that are quasi-inverse to each other.

Finally, since the composition of $\Delta : G' \hookrightarrow G \times G'$ and the natural projection $G \times G' \longrightarrow G$ is equal to the inclusion map $G' \longrightarrow G$, and since $i = \mathfrak{q} \circ j$ by definition, we see that the functor $i^* : \mathscr{D}_G(\widetilde{G}) \longrightarrow \mathscr{D}_{G'}(G')$ is isomorphic to the composition

$$\mathscr{D}_G(\widetilde{G}) \xrightarrow{\mathfrak{q}^*} \mathscr{D}_{G\times G'}(G \times G') \xrightarrow{j^*} \mathscr{D}_{G'}(G').$$

We just showed that $\mathfrak{q}^*$ and j^* are equivalences, whence so is i^*.

5.3 An alternative viewpoint on induction functors

Suppose Y is a scheme of finite type over k equipped with a *transitive* left G-action. We let G act on itself by conjugation, as usual, and consider the induced diagonal action of G on $G \times Y$. Write $Z = \{(g, y) \mid g \cdot y = y\} \subset G \times Y$ and observe that Z is G-stable. Given $y \in Y(k)$, we let G^y denote the stabilizer of y in G and consider the inclusion morphism $j_y : G^y \hookrightarrow Z$ given by $g \mapsto (g, y)$.

Proposition 5.6 (a) *For every $y \in Y(k)$, the pullback $j_y^* : \mathscr{D}_G(Z) \longrightarrow \mathscr{D}_{G^y}(G^y)$ is an equivalence of categories.*
(b) *If $\mathrm{pr}_1 : Z \longrightarrow G$ is the first projection, the functors*

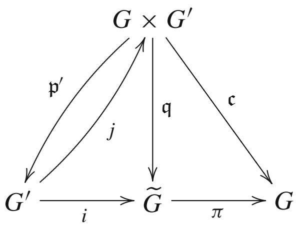
$$\mathrm{pr}_{1*} \circ (j_y^*)^{-1}, \quad \mathrm{pr}_{1!} \circ (j_y^*)^{-1} : \mathscr{D}_{G^y}(G^y) \longrightarrow \mathscr{D}_G(G)$$

are isomorphic to the induction functors $\mathrm{Ind}_{G^y}^G$ and $\mathrm{ind}_{G^y}^G$, respectively.

Proof Fix $y \in Y(k)$, write $G' = G^y$, and let $\widetilde{G} = (G \times G')/G'$ be defined as before. The morphism $G \times G' \longrightarrow G \times Y$ given by $(g, g') \mapsto (gg'g^{-1}, g \cdot y)$ has image in Z and induces a G-equivariant isomorphism $\widetilde{G} \xrightarrow{\simeq} Z$, which identifies j_y with $i : G' \hookrightarrow \widetilde{G}$ and pr_1 with $\pi : \widetilde{G} \longrightarrow G$. The proposition follows. □

5.4 Useful notation

In this subsection, we collect some of the notation introduced in Sect. 5.2.2 and in the proof of Lemma 5.1, given in Sect. 5.2.4 above. It is convenient to put all the maps we defined together into the following diagram:



Here, $\mathfrak{q}$ is the quotient map for the G'-action, $\mathfrak{p}'$ is the second projection, j is the natural inclusion given by $j(g') = (1, g')$, and $i = \mathfrak{q} \circ j$. Also, $\mathfrak{c}$ is the conjugation map $(g, g') \longmapsto gg'g^{-1}$, and π is the unique morphism satisfying $\mathfrak{c} = \pi \circ \mathfrak{q}$. Finally, let us agree, from now on, to denote the chosen quasi-inverse of the functor i^* by

$$\mathscr{D}_{G'}(G') \ni M \longmapsto \widetilde{M} \in \mathscr{D}_G(\widetilde{G}).$$

5.5 Weak semigroupal structure on $\mathrm{ind}_{G'}^{G}$

In this subsection, we will define functorial morphisms

$$\varphi_{M,N} : \mathrm{ind}_{G'}^{G}(M) * \mathrm{ind}_{G'}^{G}(N) \longrightarrow \mathrm{ind}_{G'}^{G}(M * N) \tag{5.2}$$

for all $M, N \in \mathscr{D}_{G'}(G')$, where the convolution on the left (respectively, on the right) is computed on G (respectively, on G'). In Sect. 5.7, we prove that under a suitable condition on M and N, the arrow $\varphi_{M,N}$ is an isomorphism. Of course, there is no reason for it to be an isomorphism in general (in the setting of finite groups, induction of class functions usually does not commute with convolution).

5.5.1 Preparations

We keep the notation of Sect. 5.4. We have an obvious morphism $\widetilde{G} \longrightarrow G/G'$ induced by the first projection $G \times G' \longrightarrow G$. Form the fiber product

$$Z = \widetilde{G} \underset{G/G'}{\times} \widetilde{G}.$$

Thus, Z is a closed subscheme of $\widetilde{G} \times \widetilde{G}$, and the morphism $i \times i : G' \times G' \longrightarrow \widetilde{G} \times \widetilde{G}$ factors through Z. Further, let $\mu : G \times G \longrightarrow G$ and $\mu' : G' \times G' \longrightarrow G'$ denote the respective multiplication morphisms. The next result is straightforward.

Lemma 5.7 *There exists a morphism $\widetilde{\mu} : Z \longrightarrow \widetilde{G}$ such that*

$$\widetilde{\mu}(y_1, y_2) = \left[(g_1, h_1 \cdot g_1^{-1} g_2 \cdot h_2 \cdot g_2^{-1} g_1)\right] \qquad \forall\, (y_1, y_2) \in Z \subset \widetilde{G} \times \widetilde{G},$$

where $(g_j, h_j) \in G \times G'$ are representatives of the G'-orbits $y_j \in \widetilde{G}$ ($j = 1, 2$), and $[(g, h)]$ denotes the G'-orbit of a point $(g, h) \in G \times G'$. Furthermore, the square

$$\begin{array}{ccc} G' \times G' & \xhookrightarrow{\ i \times i\ } & Z \\ \Big\downarrow{\scriptstyle \mu'} & & \Big\downarrow{\scriptstyle \widetilde{\mu}} \\ G' & \xhookrightarrow{\ \ i\ \ } & \widetilde{G} \end{array}$$

commutes and is cartesian, and the square

$$\begin{array}{ccc} Z & \xrightarrow{\ \pi \times \pi\ } & G \times G \\ \Big\downarrow{\scriptstyle \widetilde{\mu}} & & \Big\downarrow{\scriptstyle \mu} \\ \widetilde{G} & \xrightarrow{\ \ \pi\ \ } & G \end{array}$$

commutes. Also, Z is stable under the diagonal action of G, and $\widetilde{\mu}$ is G-equivariant.

5.5.2 Definition of the weak semigroupal structure

Let us choose $M, N \in \mathscr{D}_{G'}(G')$. By definition, $(i^* \times i^*)(\widetilde{M} \boxtimes \widetilde{N}) \cong M \boxtimes N$, whence, by the proper base change theorem, we have functorial isomorphisms

$$i^* \left(\widetilde{\mu}_!(\widetilde{M} \boxtimes \widetilde{N})\,|_Z\right) \cong \mu'_!(M \boxtimes N) \cong M * N,$$

and thus we have functorial isomorphisms

$$\operatorname{ind}_{G'}^{G}(M * N) \cong \pi_! \widetilde{\mu}_!(\widetilde{M} \boxtimes \widetilde{N})\,|_Z \cong \mu_!(\pi \times \pi)_!(\widetilde{M} \boxtimes \widetilde{N})\,|_Z\,.$$

If $f : Z \hookrightarrow \widetilde{G} \times \widetilde{G}$ denotes the inclusion morphism, we see that the adjunction morphism $\widetilde{M} \boxtimes \widetilde{N} \longrightarrow f_! f^*(\widetilde{M} \boxtimes \widetilde{N})$ induces a natural morphism

$$(\operatorname{ind}_{G'}^{G} M) * (\operatorname{ind}_{G'}^{G} N) \cong \mu_!(\pi \times \pi)_!(M \boxtimes N) \longrightarrow \operatorname{ind}_{G'}^{G}(M * N).$$

This is the desired morphism (5.2).

5.6 Some auxiliary results

The following facts will be used several times in the rest of the section.

Lemma 5.8 *Let X be a scheme of finite type over k, let $U \subset X$ be an open subset, let $Z = X \backslash U$ be equipped with the reduced induced subscheme structure, and let*

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$$

be the natural inclusions. For every $\mathcal{F} \in \mathscr{D}(X)$, there is a distinguished triangle

$$j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow j_! j^! \mathcal{F}[1],$$

functorial in $\mathcal{F}$, where the morphisms $j_! j^! \mathcal{F} \longrightarrow \mathcal{F}$ and $\mathcal{F} \longrightarrow i_* i^* \mathcal{F}$ are induced via adjunction by the identity morphisms $j^! \mathcal{F} \longrightarrow j^! \mathcal{F}$ and $i^* \mathcal{F} \longrightarrow i^* \mathcal{F}$.

Let us write, as usual, $\mathcal{F}\,|_U = j^* \mathcal{F}$ and $\mathcal{F}\,|_Z = i^* \mathcal{F}$. Since $j^! = j^*$ and $i_* = i_!$, the distinguished triangle of Lemma 5.8 can also be rewritten as

$$j_!\,(\mathcal{F}\,|_U) \longrightarrow \mathcal{F} \longrightarrow i_!\,(\mathcal{F}\,|_Z) \longrightarrow j_!\,(\mathcal{F}\,|_U)\,[1]. \tag{5.3}$$

Lemma 5.9 *Let H be a possibly disconnected unipotent group over a field k, and let $h : X \longrightarrow Y$ be an H-torsor, where Y is a scheme of finite type over k. For every $M \in \mathscr{D}(Y)$, consider the canonical adjunction morphism $\epsilon_M : h_! h^! M \longrightarrow M$.*

(a) *If H is connected, then ϵ_M is an isomorphism for all $M \in \mathscr{D}(Y)$.*
(b) *In general, ϵ_M has a natural splitting, i.e., there exist functorial morphisms $s_M : M \longrightarrow h_! h^! M$ for all $M \in \mathscr{D}(Y)$, such that $\epsilon_M \circ s_M = \operatorname{id}_M$.*

Proof (a) First, we may clearly assume that k is algebraically closed (using base change to an algebraic closure of k). Second, using base change by the smooth surjective morphism $X \xrightarrow{h} Y$, we may assume that X is a trivial H-torsor over Y. Since k is algebraically closed and H is a connected unipotent group over k, it follows that H has a filtration by normal connected subgroups with successive subquotients isomorphic to the additive group $\mathbb{G}_a$ over k. Thus, we may also assume that $H = \mathbb{G}_a$. In this case, the result reduces to the standard computation of the cohomology with compact supports for an affine line over k.

(b) In view of (a), we may assume that the neutral connected component of H is trivial, i.e., that H is a finite étale group scheme over k. In this case $h_* = h_!$, $h^! = h^*$, so that $h^!$ is also left adjoint to $h_!$, and we obtain a canonical adjunction morphism $\eta_M : M \longrightarrow h_! h^! M$. Let $\overline{k}$ be an algebraic closure of k, and let $n = |H(\overline{k})|$. We claim that the composition $\epsilon_M \circ \eta_M$ equals the multiplication by n; assuming this claim, we can define $s_M = n^{-1} \cdot \eta_M$, because $\overline{\mathbb{Q}}_\ell$ has characteristic 0, and then s_M is the desired splitting. To prove the last claim, we again extend the base field to $\overline{k}$ and thus assume that $k = \overline{k}$. Then, H is a discrete group of order n, and the claim becomes trivial. □

In practice, we will apply the following corollary of the lemma. Note that h is a smooth morphism of relative dimension $d = \dim H$, so that there is a natural isomorphism of functors $h^! \cong h^*[2d](d)$. Thus, the next result is immediate.

Corollary 5.10 *In the situation of Lemma 5.9, let $d = \dim H$. If H is connected (respectively, in general), every $M \in \mathscr{D}(Y)$ is naturally isomorphic to (respectively, is naturally isomorphic to a direct summand of) $h_! h^* M[2d](d)$.*

5.7 A case where (5.2) is an isomorphism

In this subsection, we establish a sufficient condition for (5.2) to be an isomorphism. First we introduce more notation. If M is an object of $\mathscr{D}(G')$ or $\mathscr{D}_{G'}(G')$, we will denote by $\overline{M}$ the object of $\mathscr{D}(G)$ obtained from M by extension by zero. It is clear that if $M, N \in \mathscr{D}(G')$, then

$$\overline{M} * \overline{N} \cong \overline{M * N}, \qquad \text{functorially in } M, N. \tag{5.4}$$

Next, we choose an algebraic closure, $\overline{k}$, of k. We can consider the algebraic group $G \otimes_k \overline{k}$ over $\overline{k}$. By a slight abuse of notation, given an object of $\mathscr{D}(G)$, we will denote the corresponding object of $\mathscr{D}(G \otimes_k \overline{k})$ by the same letter. If $x \in G(\overline{k})$, we denote by δ_x the corresponding delta-sheaf on $G \otimes_k \overline{k}$.

Proposition 5.11 *If $M, N \in \mathscr{D}_{G'}(G')$ are such that $\overline{M} * \delta_x * \overline{N} = 0$, as objects of $\mathscr{D}(G \otimes_k \overline{k})$, for all $x \in G(\overline{k}) \setminus G'(\overline{k})$, then (5.2) is an isomorphism.*

Proof Using base change from k to $\overline{k}$, we may and do assume that k is algebraically closed. Let $U = (\widetilde{G} \times \widetilde{G}) \setminus Z$, an open subset of $\widetilde{G} \times \widetilde{G}$. We will use the following shorthand notation: $(\widetilde{M} \boxtimes \widetilde{N})_U$ is the extension of $(\widetilde{M} \boxtimes \widetilde{N})|_U$ to $\widetilde{G} \times \widetilde{G}$ by zero outside of U. Applying the distinguished triangle (5.3) to our situation (where $X = \widetilde{G} \times \widetilde{G}$), we see that it is enough to check that $\mu_!(\pi \times \pi)_!(\widetilde{M} \boxtimes \widetilde{N})_U = 0$.

We still keep the notation of Sect. 5.4. Consider the morphism

$$\mathfrak{q} \times \mathfrak{q} : G \times G' \times G \times G' \longrightarrow \widetilde{G} \times \widetilde{G},$$

which is a torsor under $G' \times G'$. According to Corollary 5.10, if $d' = \dim G'$, then $(\widetilde{M} \boxtimes \widetilde{N})_U$ is a direct summand of $(\mathfrak{q} \times \mathfrak{q})_!(\mathfrak{q} \times \mathfrak{q})^*(\widetilde{M} \boxtimes \widetilde{N})_U[4d'](2d')$, so to complete the proof of the proposition, it will suffice to show that

$$\mu_!(\pi \times \pi)_!(\mathfrak{q} \times \mathfrak{q})_!(\mathfrak{q} \times \mathfrak{q})^*(\widetilde{M} \boxtimes \widetilde{N})_U = 0.$$

We have $(\mathfrak{q} \times \mathfrak{q})^*(\widetilde{M} \boxtimes \widetilde{N})_U = (\mathfrak{q}^*\widetilde{M} \boxtimes \mathfrak{q}^*\widetilde{N})_{(\mathfrak{q}\times\mathfrak{q})^{-1}(U)}$, where the meaning of the subscript $(\mathfrak{q} \times \mathfrak{q})^{-1}(U)$ is similar to that of the subscript U. According to the proof of Lemma 5.1 given in Sect. 5.2.4, we have $\mathfrak{q}^*\widetilde{M} \cong \mathfrak{p}'^*M$, where $\mathfrak{p}' : G \times G' \longrightarrow G'$ is the projection onto the second factor. Similarly, $\mathfrak{q}^*\widetilde{N} \cong \mathfrak{p}'^*N$. Thus, we are reduced to showing that

$$[\mu \circ (\pi \times \pi) \circ (\mathfrak{q} \times \mathfrak{q})]_! \left(\mathfrak{p}'^*M \boxtimes \mathfrak{p}'^*N\right)_{(\mathfrak{q}\times\mathfrak{q})^{-1}(U)} = 0. \tag{5.5}$$

In fact, we will prove a stronger statement. Namely, consider the morphism

$$\Phi = [\mu \circ (\pi \times \pi) \circ (\mathfrak{q} \times \mathfrak{q})] \times \mathrm{pr}_1 \times \xi \ : \ G \times G' \times G \times G' \longrightarrow G \times G \times G,$$

where $\mathrm{pr}_1 : G \times G' \times G \times G' \longrightarrow G$ is the first projection and $\xi : G \times G' \times G \times G' \longrightarrow G$ is given by $(g_1, g_1', g_2, g_2') \longmapsto g_1^{-1}g_2$. We will prove that

$$\Phi_! \left(\mathfrak{p}'^*M \boxtimes \mathfrak{p}'^*N\right)_{(\mathfrak{q}\times\mathfrak{q})^{-1}(U)} = 0, \tag{5.6}$$

which will of course imply (5.5).

By definition, it is easy to check that $(\mathfrak{q} \times \mathfrak{q})^{-1}(U) = \Phi^{-1}\left(G \times G \times (G \backslash G')\right)$. Hence, it suffices to prove the vanishing of the stalk of $\Phi_!(\mathfrak{p}'^*M \boxtimes \mathfrak{p}'^*N)$ at any given point $(g, g_1, x) \in G(k) \times G(k) \times \left(G(k) \backslash G'(k)\right)$. As usual, we apply the proper base change theorem. The fiber $\Phi^{-1}(g, g_1, x)$ is naturally identified with the closed subset

$$\left\{(h_1, h_2) \in G' \times G' \,\middle|\, h_1 x h_2 = g_1^{-1} g g_1 x\right\} \subset G' \times G'$$

via the morphism $(h_1, h_2) \longmapsto (g_1, h_1, g_1 x, h_2)$. (This is simply because the equation $g_1 h_1 g_1^{-1} \cdot (g_1 x) h_2 (g_1 x)^{-1} = g$ is equivalent to $h_1 x h_2 = g_1^{-1} g g_1 x$.) Hence, the stalk of $\Phi_!(\mathfrak{p}'^*M \boxtimes \mathfrak{p}'^*N)$ at (g, g_1, x) is quasi-isomorphic to the stalk of the convolution $\overline{M} * \delta_x * \overline{N}$ at $g_1^{-1} g g_1 x$. Since $x \in G(k) \backslash G'(k)$, we have $\overline{M} * \delta_x * \overline{N} = 0$ by assumption, which implies (5.6) and completes the proof of Proposition 5.11. □

5.8 Induction of weak idempotents

In this subsection, we keep all the notation introduced earlier (notably, at the beginning of Sect. 5.7).

5.8.1 Statement of the main theorem

Let $e \in \mathscr{D}_{G'}(G')$ be a weak idempotent, and recall that the corresponding Hecke subcategory of $\mathscr{D}_{G'}(G')$ is denoted by $e\mathscr{D}_{G'}(G')$, because $\mathscr{D}_{G'}(G')$ is weakly symmetric by Lemma 4.7 (see Sect. 4.7 for the all relevant terminology).

Assume that $\overline{e} * \delta_x * \overline{e} = 0$ for all $x \in G(\overline{k}) \setminus G'(\overline{k})$. It follows from Proposition 5.11 that $f := \operatorname{ind}_{G'}^{G}(e)$ is a weak idempotent in $\mathscr{D}_G(G)$. Moreover, if $M \in e\mathscr{D}_{G'}(G')$, then $M \cong e * M$, which implies that $\overline{e} * \delta_x * \overline{M} = 0$ for all $x \in G(\overline{k}) \setminus G'(\overline{k})$ (in view of (5.4)), and Proposition 5.11 shows that $\operatorname{ind}_{G'}^{G}(M) \in f\mathscr{D}_G(G)$.

Theorem 5.12 (a) *In this situation, the functor*

$$\operatorname{ind}_{G'}^{G}\big|_{e\mathscr{D}_{G'}(G')} \;:\; e\mathscr{D}_{G'}(G') \longrightarrow f\mathscr{D}_G(G) \tag{5.7}$$

is strong semigroupal (with respect to the semigroupal structure introduced in Sect. 5.5) and induces a bijection on isomorphism classes of objects.

(b) *If the functor $M \longmapsto e * M$ is isomorphic to the identity functor on $e\mathscr{D}_{G'}(G')$, the functor (5.7) is faithful.*

(c) *If the functors $M \longmapsto e * M$ and $N \longmapsto f * N$ are isomorphic to the identity functors on $e\mathscr{D}_{G'}(G')$ and $f\mathscr{D}_G(G)$, respectively, then (5.7) is an equivalence of categories, a quasi-inverse to which is provided by the functor*

$$f\mathscr{D}_G(G) \longrightarrow e\mathscr{D}_{G'}(G'), \qquad N \longmapsto e * (N|_{G'}).$$

5.8.2 An immediate consequence

Let us note at once the following

Corollary 5.13 *If e is a minimal (resp., geometrically minimal) weak idempotent in $\mathscr{D}_{G'}(G')$ and the other assumptions are in force, then $f = \operatorname{ind}_{G'}^{G}(e)$ is a minimal (resp., geometrically minimal) weak idempotent in $\mathscr{D}_G(G)$.*

It suffices to check that if e is minimal, then so is f, as all the hypotheses of the corollary are obviously invariant under base change to algebraic extensions of k. To this end, observe that a weak idempotent $f \in \mathscr{D}_G(G)$ is minimal if and only if the semigroupal category $f\mathscr{D}_G(G)$ contains no weak idempotents other than 0 and f. By Proposition 5.11 and Theorem 5.12, the functor (5.7) induces a bijection between the set of isomorphism classes of weak idempotents in $e\mathscr{D}_{G'}(G')$ and the set of isomorphism classes of weak idempotents in $f\mathscr{D}_G(G)$, whence the claim.

5.8.3 Reduction of Theorem 5.12 to two auxiliary propositions

From now on, we fix a weak idempotent $e \in \mathscr{D}_{G'}(G')$ satisfying $\overline{e} * \delta_x * \overline{e} = 0$ for all $x \in G(\overline{k}) \backslash G'(\overline{k})$. If $M, N \in e\mathscr{D}_{G'}(G')$, then $M \cong M*e$ and $N \cong e*N$, which implies that $\overline{M} * \delta_x * \overline{N}$ for all $x \in G(\overline{k}) \backslash G'(\overline{k})$. Thus, the first assertion of Theorem 5.12(a) results from Proposition 5.11.

Put $f = \operatorname{ind}_{G'}^{G}(e) \in \mathscr{D}_G(G)$. The following two propositions are proved below.

Proposition 5.14 *For each $N \in \mathscr{D}_G(G)$, there is an isomorphism*

$$f * N \xrightarrow{\simeq} \operatorname{ind}_{G'}^{G}(e * (N|_{G'})),$$

functorial with respect to N.

Proposition 5.15 *For each $M \in e\mathscr{D}_{G'}(G')$, there is an isomorphism*

$$e * \left((\operatorname{ind}_{G'}^{G} M)|_{G'}\right) \xrightarrow{\simeq} e * M,$$

functorial with respect to M.

These propositions clearly imply part (c) and the second assertion of part (a) of Theorem 5.12 (by restricting attention to objects $N \in f\mathscr{D}_G(G)$). To see that they also imply part (b) of the theorem, observe that by Proposition 5.15, the functor $M \longmapsto e * M$ on $e\mathscr{D}_{G'}(G')$ is isomorphic to the composition

$$e\mathscr{D}_{G'}(G') \xrightarrow{\operatorname{ind}_{G'}^{G}} \mathscr{D}_G(G) \xrightarrow{\text{restriction}} \mathscr{D}_{G'}(G') \xrightarrow{e*-} e\mathscr{D}_{G'}(G').$$

If the composition is isomorphic to the identity functor on $e\mathscr{D}_{G'}(G')$, then the first term in the composition, $\operatorname{ind}_{G'}^{G}|_{e\mathscr{D}_{G'}(G')}$, has to be faithful.

5.8.4 Proof of Proposition 5.14

The argument follows a pattern similar to the one used in the proof of Proposition 5.11. By definition, we have

$$f * N = (\operatorname{ind}_{G'}^{G} e) * N = \mu_!(\pi \times \mathrm{id})_!(\widetilde{e} \boxtimes N), \tag{5.8}$$

where we are using the morphisms

$$\widetilde{G} \times G \xrightarrow{\pi \times \mathrm{id}} G \times G \xrightarrow{\mu} G.$$

Consider the closed subset

$$Z' = \left\{([(g,h)], \gamma) \in \widetilde{G} \times G \;\middle|\; g^{-1}\gamma g \in G'\right\} \subset \widetilde{G} \times G.$$

It is easy to check that Z' is well defined and is stable under the diagonal action of G on $\widetilde{G} \times G$ (where, as always, G acts on itself by conjugation). Moreover, let us recall the closed subset $Z \subset \widetilde{G} \times \widetilde{G}$ introduced in Sect. 5.5.1:

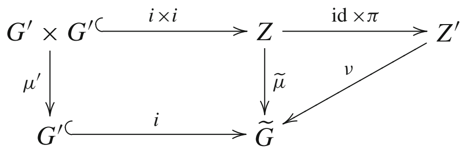
$$Z = \left\{ ([(g_1, h_1)], [(g_2, h_2)]) \in \widetilde{G} \times \widetilde{G} \mid g_1^{-1} g_2 \in G' \right\}.$$

It is clear that the morphism $\mathrm{id} \times \pi : \widetilde{G} \times \widetilde{G} \longrightarrow \widetilde{G} \times G$ takes Z into Z'.

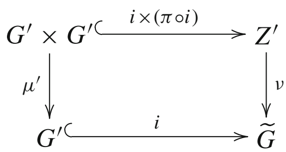
Now consider the (clearly G-equivariant) morphism

$$\nu : Z' \longrightarrow \widetilde{G}, \qquad ([(g, h)], \gamma) \longmapsto \left[(g, hg^{-1}\gamma g) \right].$$

It is easy to check that ν is well defined, and we obtain a commutative diagram



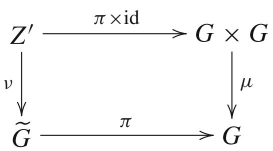
Furthermore, it is also easy to check that the following square is cartesian:



Now we use the same argument as before. Put $U' = (\widetilde{G} \times G) \backslash Z'$, write $(\widetilde{e} \boxtimes N)_{Z'}$ for the extension of $(\widetilde{e} \boxtimes N)\,|_{Z'}$ to $\widetilde{G} \times G$ by zero outside of Z', and define $(\widetilde{e} \boxtimes N)_{U'}$ similarly. Applying the proper base change theorem to the cartesian square above, we obtain functorial isomorphisms

$$i^* \nu_! \left((\widetilde{e} \boxtimes N)\,|_{Z'} \right) \cong \mu'_! \left(e \boxtimes (N\,|_{G'}) \right) \cong e * (N\,|_{G'}),$$

whence $\mathrm{ind}_{G'}^{G} (e * (N\,|_{G'})) \cong \pi_! \nu_! \left((\widetilde{e} \boxtimes N)\,|_{Z'} \right)$. We also have a commutative diagram



which implies that

$$\mathrm{ind}_{G'}^{G} (e * (N\,|_{G'})) \cong \mu_! (\pi \times \mathrm{id})_! \left((\widetilde{e} \boxtimes N)_{Z'} \right).$$

In view of (5.8), we see that the adjunction morphism $(\widetilde{e}\boxtimes N)\longrightarrow(\widetilde{e}\boxtimes N)_{Z'}$ yields a morphism $f*N\longrightarrow \operatorname{ind}_{G'}^{G}(e*(N|_{G'}))$, functorial with respect to N. As before, to complete the proof of Proposition 5.14, it is enough to show that if $\overline{e}*\delta_x*\overline{e}=0$ for all $x\in G(\overline{k})\backslash G'(\overline{k})$, then $\mu_!(\pi\times \mathrm{id})_!((\widetilde{e}\boxtimes N)_{U'})=0$.

Henceforth, we may and do assume that k is algebraically closed. By Corollary 5.10, it is enough to prove that $\mu_!(\pi\times\mathrm{id})_!(\mathfrak{q}\times\mathrm{id})_!\left[(\mathfrak{q}\times\mathrm{id})^*(\widetilde{e}\boxtimes N)_{U'}\right]=0$, which is equivalent to

$$\mu_!(\pi\times\mathrm{id})_!(\mathfrak{q}\times\mathrm{id})_!\left[\left((\mathfrak{p}'^*e)\boxtimes N\right)_{(\mathfrak{q}\times\mathrm{id})^{-1}(U')}\right]=0. \tag{5.9}$$

(As in Sect. 5.4, we write $\mathfrak{q}:G\times G'\longrightarrow\widetilde{G}$ for the quotient map and $\mathfrak{p}':G\times G'\longrightarrow G'$ for the projection onto the second factor; and we are using the proof of Lemma 5.1 given in Sect. 5.2.4 to conclude that $\mathfrak{q}^*\widetilde{e}\cong\mathfrak{p}'^*e$.)

Note that the morphism

$$\mu\circ(\pi\times\mathrm{id})\circ(\mathfrak{q}\times\mathrm{id}):G\times G'\times G\longrightarrow G$$

is given by $(g,h,\gamma)\longmapsto ghg^{-1}\gamma$. Let us consider the morphism

$$\Phi':G\times G'\times G\longrightarrow G\times G,\qquad (g,h,\gamma)\longmapsto(ghg^{-1}\gamma,g).$$

To establish (5.9), it suffices to prove that

$$\Phi'_!\left[\left((\mathfrak{p}'^*e)\boxtimes N\right)_{(\mathfrak{q}\times\mathrm{id})^{-1}(U')}\right]=0. \tag{5.10}$$

We also observe that

$$(\mathfrak{q}\times\mathrm{id})^{-1}(U')=\left\{(g,h,\gamma)\in G\times G'\times G\,\middle|\,g^{-1}\gamma g\in G\backslash G'\right\}.$$

We will use the proper base change theorem to compute the stalk of the left hand side of (5.10) at a point $(x,g)\in G(k)\times G(k)$. The fiber $\Phi'^{-1}(x,g)$ is identified with the closed subset $\left\{(h,\gamma)\in G'\times G\,\middle|\,ghg^{-1}\gamma=x\right\}\subset G'\times G$. The equation $ghg^{-1}\gamma=x$ can be rewritten as $h\cdot g^{-1}\gamma g=g^{-1}xg$, so we see that $\Phi'^{-1}(x,g)\cap(\mathfrak{q}\times\mathrm{id})^{-1}(U)$ can be identified with the closed subset $W=\left\{(h,\gamma')\in G'\times(G\backslash G')\,\middle|\,h\gamma'=g^{-1}xg\right\}$ via the morphism $w:W\longrightarrow G\times G'\times G$ given by $(h,\gamma')\longmapsto(g,h,g\gamma'g^{-1})$.

Since N is G-equivariant, the pullback $\lambda^*((\mathfrak{p}'^*e)\boxtimes N)$ is naturally identified with $(e\boxtimes N)|_W$, and thus, by the proper base change theorem, the complex $R\Gamma_c\left(W,\lambda^*((\mathfrak{p}'^*e)\boxtimes N)\right)$ is naturally identified with the stalk at $g^{-1}xg$ of the convolution $\overline{e}*N_{G/G'}$, where the meaning of the subscript $G\backslash G'$ is as before: $N_{G\backslash G'}$ is the extension of $N\left|_{G\backslash G'}\right.$ to G by zero outside of $G\backslash G'$. Hence, we are reduced to the following

Lemma 5.16 *Under the assumptions of Proposition 5.14, we have*

$$\overline{e}*N_{G\backslash G'}=0\qquad \textit{for all } N\in\mathscr{D}_G(G).$$

To prove Lemma 5.16, we use the distinguished triangle (5.3) with $M = N$, $X = G$ and $Z = G'$. We see that it suffices to show that the natural morphism $\overline{e} * N \longrightarrow \overline{e} * N_{G'}$ is an isomorphism. But $\overline{e} \cong \overline{e} * \overline{e}$, and since $N \in \mathscr{D}_G(G)$, we see that $\overline{e} * N \cong \overline{e} * N * \overline{e}$, functorially in N (see Lemma 4.7). On the other hand, it is clear that $\overline{e} * N_{G'} = \overline{e * (N|_{G'})}$, and since e is a weak idempotent in $\mathscr{D}_{G'}(G')$, we also have $e * M \cong e * M * e$ functorially with respect to $M \in \mathscr{D}(G')$ (applying Lemma 4.7 to G' in place of G). Thus, we are reduced to showing that the morphism

$$\overline{e} * N * \overline{e} \longrightarrow \overline{e} * N_{G'} * \overline{e},$$

induced by the adjunction morphism $N \longrightarrow N_{G'}$, is an isomorphism. Applying the distinguished triangle (5.3) once again, we see that it is enough to show that

$$\overline{e} * N_{G \setminus G'} * \overline{e} = 0 \qquad \text{for all } N \in \mathscr{D}_G(G). \tag{5.11}$$

Finally, to prove (5.11), note that $\overline{e} * N_{G\setminus G'} * \overline{e} = \mu'_{3!}(e \boxtimes N\big|_{G\setminus G'} \boxtimes e)$, where $\mu'_3 : G' \times (G\setminus G') \times G' \longrightarrow G$ is given by $(g_1, g_2, g_3) \longmapsto g_1 g_2 g_3$. Consider the map

$$\lambda : G' \times (G\setminus G') \times G' \longrightarrow G \times (G\setminus G'), \qquad (g_1, g_2, g_3) \longmapsto (g_1 g_2 g_3, g_2).$$

By the proper base change theorem, the stalk of $\lambda_! \left(e \boxtimes N\big|_{G\setminus G'} \boxtimes e\right)$ at a point $(g, x) \in G(k) \times \left(G(k)\setminus G'(k)\right)$ is isomorphic to $N_x \otimes (\overline{e} * \delta_x * \overline{e})_g$, where N_x is the stalk of N at x. But $\overline{e} * \delta_x * \overline{e} = 0$ by assumption, so $\lambda_!(e \boxtimes N\big|_{G\setminus G'} \boxtimes e) = 0$. This forces (5.11), completing the proof of Lemma 5.16 and of Proposition 5.14.

5.8.5 Proof of Proposition 5.15

Once again, the argument is very similar to the ones used in the proofs of Propositions 5.11 and 5.14. The morphism $i : G' \longrightarrow \widetilde{G}$ is a closed immersion; let $U'' \subset \widetilde{G}$ denote the complement of its image. As usual, we have an exact triangle $\widetilde{M}_{U''} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M}_{i(G')} \longrightarrow \widetilde{M}_{U''}[1]$, where the meaning of the subscripts U'' and $i(G')$ is as before. In addition, we have $\widetilde{M}_{i(G')} \cong i_! M$ by definition, and therefore $\pi_! \widetilde{M}_{i(G')} \cong \overline{M}$. Thus, we obtain a natural morphism

$$\overline{e} * \operatorname{ind}_{G'}^G(M) = \mu_!(\overline{e} \boxtimes \pi_! \widetilde{M}) \longrightarrow \mu_!(\overline{e} \boxtimes \pi_! \widetilde{M}_{i(G')}) \cong \overline{e} * \overline{M} \cong \overline{e * M}.$$

Restricting it to G' yields a morphism

$$e * \left(\left(\operatorname{ind}_{G'}^G M\right)\Big|_{G'}\right) \longrightarrow e * M,$$

functorial in $M \in e\mathscr{D}_{G'}(G')$, and we would like to show that it is an isomorphism.

As before, it is enough to prove that

$$\overline{e} * \left(\pi_! \widetilde{M}_{U''}\right) = 0. \tag{5.12}$$

In turn, to establish this equality, it is enough (by Corollary 5.10) to check that

$$\overline{e} * \pi_! \mathfrak{q}_! \left(\mathfrak{q}^*(\widetilde{M}_{U''})\right) = 0. \tag{5.13}$$

where $\mathfrak{q} : G \times G' \longrightarrow \widetilde{G}$ is the quotient map.

As always, we may and do assume that $k = \overline{k}$. Now $\mathfrak{q}^{-1}(U'') = (G\backslash G') \times G'$, and $\mathfrak{q}^*(\widetilde{M}_{U''}) = (\mathfrak{p}'^* M)_{((G\backslash G')\times G')}$, where $\mathfrak{p}' : G \times G' \longrightarrow G'$ is the projection onto the second factor. Thus, the left hand side of (5.13) can be rewritten as $\Psi_!(e \boxtimes \overline{\mathbb{Q}}_\ell \boxtimes M)$, where we define

$$\Psi : G' \times (G\backslash G') \times G' \longrightarrow G \qquad \text{by} \qquad (h_1, g, h_2) \longmapsto h_1 \cdot g h_2 g^{-1},$$

and $\overline{\mathbb{Q}}_\ell$ denotes the constant rank 1 local system on $G\backslash G'$.

Let us consider instead the morphism

$$\Phi'' : G' \times (G\backslash G') \times G' \longrightarrow G \times (G\backslash G'), \qquad (h_1, g, h_2) \longmapsto (h_1 g h_2 g^{-1}, g).$$

The fiber of Φ'' over $(x, g) \in G(k) \times (G(k)\backslash G'(k))$ is naturally identified with

$$W' = \left\{(h_1, h_2) \in G' \times G' \,\middle|\, h_1 g h_2 = xg\right\}$$

via the morphism

$$w' : W' \longrightarrow G' \times (G\backslash G') \times G', \qquad (h_1, h_2) \longmapsto (h_1, g, h_2).$$

By the proper base change theorem, we have

$$\Phi''_!(e \boxtimes \overline{\mathbb{Q}}_\ell \boxtimes M)_{(x,g)} \cong R\Gamma_c\left(W', w'^*(e \boxtimes \overline{\mathbb{Q}}_\ell \boxtimes M)\right) \cong (\overline{e} * \delta_g * \overline{M})_{xg}.$$

The latter stalk is zero because $g \in G(k)\backslash G'(k)$ and $\overline{M} \cong \overline{e} * \overline{M}$. Thus, we have proved that $\Phi''_!(e \boxtimes \overline{\mathbb{Q}}_\ell \boxtimes M) = 0$. A fortiori, $\Psi_!(e \boxtimes \overline{\mathbb{Q}}_\ell \boxtimes M) = 0$, which is equivalent to (5.13), which in turn implies (5.12) and completes the proof of Proposition 5.15.

5.9 Compatibility of induction with twists

Our final goal in this section is the following

Proposition 5.17 *For every $M \in \mathscr{D}_{G'}(G')$, we have $\operatorname{ind}_{G'}^{G}(\theta'_M) = \theta_{\operatorname{ind}_{G'}^{G}(M)}$ as automorphisms of $\operatorname{ind}_{G'}^{G}(M)$, where θ' and θ are the twists in the categories $\mathscr{D}_{G'}(G')$ and $\mathscr{D}_G(G)$, respectively, introduced in Definition 4.15.*

Lemma 5.18 *Let $j : G' \hookrightarrow G$ denote the inclusion map and $d = \dim(G/G')$. Then, $\operatorname{Ind}_{G'}^{G}$ is right adjoint to the restriction functor $j^* : \mathscr{D}_G(G) \longrightarrow \mathscr{D}_{G'}(G')$, and $\operatorname{ind}_{G'}^{G}$ is left adjoint to the functor $j^![2d](d) : \mathscr{D}_G(G) \longrightarrow \mathscr{D}_{G'}(G')$.*

Proof Observe that $j = \pi \circ i$ with the notation of Sect. 5.2.2. Next apply the definitions of $\mathrm{Ind}_{G'}^{G}$ and $\mathrm{ind}_{G'}^{G}$ (see Definition 5.2 and Remark 5.3(2)) and use the fact that π_* and $\pi_!$ are, respectively, right and left adjoint to π^* and $\pi^!$. ◻

To formulate the next lemma, we let $\mathbb{D}_G : \mathscr{D}_G(G) \longrightarrow \mathscr{D}_G(G)$ denote the Verdier duality functor and write $\mathbb{D}_G^- = \iota^* \circ \mathbb{D}_G = \mathbb{D}_G \circ \iota^* : \mathscr{D}_G(G) \longrightarrow \mathscr{D}_G(G)$ (following [10]), where $\iota : G \longrightarrow G$ is given by $g \mapsto g^{-1}$.

Lemma 5.19 *The functors* $\mathbb{D}_G^- \circ \mathrm{Ind}_{G'}^{G} \circ \mathbb{D}_{G'}^-$ *and* $\mathrm{ind}_{G'}^{G}[2d](d)$ *are isomorphic.*

Proof This follows from Lemma 5.18 using the fact that $\mathbb{D}_G^-$ and $\mathbb{D}_{G'}^-$ are anti-auto-equivalences together with the isomorphism $\mathbb{D}_{G'}^- \circ (j^![2d](d)) \circ \mathbb{D}_G^- \cong j^*$. ◻

Proof Proof of Proposition 5.17 By definition, $\theta'_{j^*N} = j^*(\theta_N)$ for all $N \in \mathscr{D}_G(G)$. Lemma 5.18 formally implies that $\mathrm{Ind}_{G'}^{G}(\theta'_M) = \theta_{\mathrm{Ind}_{G'}^{G}(M)}$ for all $M \in \mathscr{D}_{G'}(G')$. It is shown in [15, Prop. 7.2] that $\mathbb{D}_G^-(\theta_N) = \theta_{\mathbb{D}_G^- N}$ for all $N \in \mathscr{D}_G(G)$ (and a similar statement holds for $\mathbb{D}_{G'}^-$). Now Lemma 5.19 finishes the proof. ◻

6 Inner forms of algebraic groups and G-schemes

The material of this section will be used to study the relationship between the induction functor introduced in Sect. 5 above and the operation of induction of class functions on finite groups (see Sect. 6.7). It is also a necessary ingredient in the formulation of the relationship between character sheaves on a *disconnected* unipotent group G over $\mathbb{F}_q$ and irreducible representations of $G(\mathbb{F}_q)$; cf. [8].

6.1 Notation

We fix an algebraic closure $\mathbb{F}$ of a finite field of characteristic $p > 0$. If q is a power of p, we write $\mathbb{F}_q$ for the unique subfield of $\mathbb{F}$ consisting of q elements. Given a scheme X over $\mathbb{F}_q$, we write Fr_q for the Frobenius endomorphism of $X \otimes_{\mathbb{F}_q} \mathbb{F}$ (it is obtained by extension of scalars from the absolute Frobenius $\Phi_q : X \longrightarrow X$).

Suppose Γ is an abstract group and $\varphi : \Gamma \xrightarrow{\simeq} \Gamma$ is an automorphism. We can use φ to define an action of $\mathbb{Z}$ on Γ, and hence obtain the pointed set $H^1(\mathbb{Z}, \Gamma)$. Concretely, $H^1(\mathbb{Z}, \Gamma)$ can be identified with the set of *φ-conjugacy classes* in Γ, the latter being the orbits of the Γ-action on itself defined by $\gamma : g \mapsto \varphi(\gamma) g \gamma^{-1}$.

6.2 Galois cohomology and torsors

If G is an algebraic group over $\mathbb{F}_q$, the first Galois cohomology $H^1(\mathbb{F}_q, G)$ is the pointed set of isomorphism classes of right G-torsors. We can consider the action of $\mathbb{Z}$ on $G(\mathbb{F})$ such that $1 \in \mathbb{Z}$ acts via Fr_q and form the pointed set $H^1(\mathbb{Z}, G(\mathbb{F}))$ as above. The following result is standard (part (b) is due to Serge Lang [28]).

Lemma 6.1 (a) *Let P be a right G-torsor, choose $p \in P(\mathbb{F})$, and let $g \in G(\mathbb{F})$ be the unique element such that $p = \mathrm{Fr}_q(p) \cdot g$. Then, the Fr_q-conjugacy class of g in $G(\mathbb{F})$ is independent of p, and the map $[P] \longmapsto [g]$ gives a bijection*

$$H^1(\mathbb{F}_q, G) \xrightarrow{\simeq} H^1(\mathbb{Z}, G(\mathbb{F})).$$

(b) *Let G° denote the neutral connected component of G, and let $\Pi = G/G^\circ$. The natural map $H^1(\mathbb{Z}, G(\mathbb{F})) \longrightarrow H^1(\mathbb{Z}, \Pi(\mathbb{F}))$ is a bijection.*

(c) *Suppose that G is a closed subgroup of an algebraic group U over $\mathbb{F}_q$, form the quotient $Y = U/G$, and let $\pi : U \longrightarrow Y$ denote the quotient morphism. The map $y \longmapsto \pi^{-1}(y)$ induces a bijection between the set of $U(\mathbb{F}_q)$-orbits in $Y(\mathbb{F}_q)$ and the kernel of the natural map $H^1(\mathbb{F}_q, G) \longrightarrow H^1(\mathbb{F}_q, U)$.*

Remarks 6.2 (1) We recall that the kernel of a pointed map between pointed sets $(S_1, s_1) \xrightarrow{f} (S_2, s_2)$ is defined as the subset $f^{-1}(s_2) \subset S_1$.

(2) The action of U on Y is by left translations.

(3) If $y \in Y(\mathbb{F}_q)$, then $\pi^{-1}(y)$ is a closed subvariety of U defined over $\mathbb{F}_q$, and the action of G on U by right multiplication makes $\pi^{-1}(y)$ a right G-torsor.

6.3 Inner forms of algebraic groups

We continue working in the setup of Sect. 6.2. Given $\alpha \in H^1(\mathbb{F}_q, G)$, we would like to define an inner form G^α of G determined by α. Let P be a right G-torsor whose isomorphism class equals α. Briefly, G^α is the group of automorphisms of P that commute with the right G-action. To define G^α more formally, we consider a functor, which we denote by G^P, from the category of $\mathbb{F}_q$-schemes to the category of groups, constructed as follows.

Let S be any $\mathbb{F}_q$-scheme. We can view $P \times S$ as an S-scheme, and we have a right action of G on $P \times S$ by S-scheme automorphisms. Then, $G^P(S)$ is defined as the group of S-scheme automorphisms of $P \times S$ that commute with the G-action.

Lemma 6.3 *The functor G^P is representable by an algebraic group over $\mathbb{F}_q$. Moreover, $G^P \otimes_{\mathbb{F}_q} \mathbb{F} \cong G \otimes_{\mathbb{F}_q} \mathbb{F}$ as algebraic groups over $\mathbb{F}$.*

Proof In the case where P is a trivial torsor (i.e., $P(\mathbb{F}_q) \neq \varnothing$), one checks that G^P is representable by G itself. In general, we have $P(\mathbb{F}_{q^n}) \neq \varnothing$ for some $n \geq 1$. Thus, G^P is representable by $G \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ after base change to $\mathbb{F}_{q^n}$, and Galois descent implies that G^P is representable over $\mathbb{F}_q$. □

Remark 6.4 If P' is another right G-torsor that is isomorphic to P, then a choice of an isomorphism $P \xrightarrow{\simeq} P'$ induces an isomorphism $G^P \xrightarrow{\simeq} G^{P'}$. Moreover, by definition, any two isomorphisms $P \longrightarrow P'$ differ by an element of $G^P(\mathbb{F}_q)$. Consequently, we have an isomorphism between G^P and $G^{P'}$ that is unique up to inner automorphisms.

Definition 6.5 Let G be an algebraic group over $\mathbb{F}_q$, and let $\alpha \in H^1(\mathbb{F}_q, G)$. For a representative P of the isomorphism class α, we will write, somewhat imprecisely, $G^\alpha = G^P$. We call G^α the *inner form of G defined by* α.

Remark 6.6 In view of the previous remark, we see that the set of conjugacy classes in the group $G^\alpha(\mathbb{F}_q)$ is determined *canonically* by α. Similarly, we can speak of the set of irreducible characters of the group $G^\alpha(\mathbb{F}_q)$.

Remark 6.7 The reader may prefer the following more concrete description of the inner form G^P. Fix $p \in P(\mathbb{F})$, and let $g \in G(\mathbb{F})$ be the unique element such that $p = \mathrm{Fr}_q(p) \cdot g$ (so the Fr_q-conjugacy class of g in $G(\mathbb{F})$ is the element of $H^1(\mathbb{Z}, G(\mathbb{F}))$ corresponding to the isomorphism class of P). Then, there exists an isomorphism $G^P \otimes_{\mathbb{F}_q} \mathbb{F} \xrightarrow{\simeq} G \otimes_{\mathbb{F}_q} \mathbb{F}$ such that the Frobenius endomorphism for G^P becomes identified with the endomorphism $x \mapsto g^{-1} \mathrm{Fr}_q(x) g$ of G.

6.4 Inner forms of G-schemes

We remain in the setup of Sect. 6.2. Given $\alpha \in H^1(\mathbb{F}_q, G)$ and an $\mathbb{F}_q$-scheme of finite type X equipped with a left G-action, we would like to define an inner form X^α of X so that the corresponding inner form G^α of G acts on X^α. Once again, let P be a representative of α, and define G^P as above. Note that by construction, G^P acts on P on the left; in fact, P is a left G^P-torsor. We also consider the free left action of G on the product $P \times X$ given by $g \cdot (p, x) = (p \cdot g^{-1}, g \cdot x)$, and we form the quotient $X^P := G \backslash (P \times X)$. The actions of G and G^P on $P \times X$ commute (here, G^P acts on X trivially), so we obtain an induced action of G^P on X^P.

Definition 6.8 We write $X^\alpha = X^P$ (somewhat imprecisely), and we call X^α (together with the left action of G^α constructed above) the *inner form of the G-scheme X defined by the cohomology class* α.

The next fact follows directly from the definitions.

Lemma 6.9 *Let G be an algebraic group over* $\mathbb{F}_q$*, let P be a right G-torsor, and let* $X = G$ *equipped with the conjugation action of G. Then,* X^P *is naturally isomorphic to* G^P*, also equipped with the conjugation action of* G^P.

6.5 Transport of equivariant complexes

In this section, we assume that G is a unipotent[7] algebraic group over $\mathbb{F}_q$. Given $\alpha \in H^1(\mathbb{F}_q, G)$ and a scheme X of finite type over $\mathbb{F}_q$ equipped with a left G-action, our goal is to define a canonical "transport functor" (in fact, an equivalence of categories) $\mathscr{D}_G(X) \xrightarrow{\sim} \mathscr{D}_{G^\alpha}(X^\alpha)$.

[7] This assumption is imposed only because we decided to work with the "naive" definition of an equivariant derived category.

As usual, we choose a representative P of X. Let $\mathrm{pr}_2 : P \times X \longrightarrow X$ denote the second projection, and let $q : P \times X \longrightarrow X^P$ denote the quotient morphism for the free left G-action defined in Sect. 6.4. As we already remarked, the product $G \times G^P$ acts on $P \times X$ on the left; moreover, pr_2 is the quotient map for the action of G^P, which is also free. Thus, both pullback functors

$$\mathscr{D}_G(X) \xrightarrow{\mathrm{pr}_2^*} \mathscr{D}_{G\times G^P}(P \times X) \xleftarrow{q^*} \mathscr{D}_{G^P}(X^P)$$

are equivalences of categories.

Definition 6.10 The composition $q^* \circ (\mathrm{pr}_2^*)^{-1} : \mathscr{D}_G(X) \xrightarrow{\sim} \mathscr{D}_{G^\alpha}(X^\alpha)$ is called the functor of *transport of equivariant complexes* and is denoted by $M \longmapsto M^\alpha$.

Remark 6.11 If an object $M \in \mathscr{D}_G(X)$ comes from a G-equivariant local system on X, then M^α is also a G^α-equivariant local system on X^α.

As a corollary of Lemma 6.9, we now also have the construction of a transport functor $\mathscr{D}_G(G) \xrightarrow{\sim} \mathscr{D}_{G^\alpha}(G^\alpha)$, which is again denoted by $M \longmapsto M^\alpha$.

6.6 Alternative descriptions

In this subsection, we present a slightly different viewpoint on the constructions introduced in Sects. 6.3–6.5. It has the advantage of being somewhat more concrete, although it is less evident that the constructions appearing in this subsection are independent of the choices involved in them.

Proposition 6.12 *Let G be a closed subgroup of an algebraic group U over $\mathbb{F}_q$. Define $Y = U/G$, equipped with the left U-action by translations, let $\pi : U \longrightarrow Y$ denote the quotient map, write $\overline{1} = \pi(1)$, and put $Z = \{(u, y) \mid u \cdot y = y\} \subset U \times Y$. We consider the diagonal action of U on $U \times Y$, where the action on the first factor is by conjugation, and remark that Z is stable under this action.*

Finally, for each $y \in Y(\mathbb{F}_q)$, let $\alpha(y) \in H^1(\mathbb{F}_q, G)$ denote the isomorphism class of the right G-torsor $\pi^{-1}(y)$ (cf. Lemma 6.1(c)).

(a) *For every $y \in Y(\mathbb{F}_q)$, the stabilizer, U^y, of y in U is isomorphic to the inner form $G^{\alpha(y)}$ of G defined by the cohomology class $\alpha(y)$.*
(b) *Let X be a scheme of finite type over $\mathbb{F}_q$ equipped with a left G-action, and let $\widetilde{X} = (U \times X)/G$, where the right G-action on $U \times X$ is given by $(u, x) \cdot g = (ug, g^{-1} \cdot x)$. Write $p : \widetilde{X} \longrightarrow Y$ for the induced morphism. For every $y \in Y(\mathbb{F}_q)$, the fiber $p^{-1}(y)$ is isomorphic to*[8] *the inner form $X^{\alpha(y)}$ in a way compatible with the isomorphism of part (a).*
(c) *Assume that U is unipotent. For every $y \in Y(\mathbb{F}_q)$, the inclusion $j_y : U^y \hookrightarrow Z$, given by $g \mapsto (g, y)$, induces an equivalence $j_y^* : \mathscr{D}_U(Z) \xrightarrow{\sim} \mathscr{D}_{U^y}(U^y)$ (as*

[8] Observe that $p^{-1}(y)$ is stable under $U^y \subset U$; thus we have a left action of U^y on $p^{-1}(y)$.

usual, U^y acts on itself by conjugation). Furthermore, the composition

$$j_y^* \circ (j_1^*)^{-1} : \mathscr{D}_G(G) \xrightarrow{\sim} \mathscr{D}_{U^y}(U^y) \simeq \mathscr{D}_{G^{\alpha(y)}}(G^{\alpha(y)})$$

is isomorphic to the transport functor introduced in Sect. 6.5.

(d) *Again, assume that U is unipotent, and let the notation be as in part (b). Given $y \in Y(\mathbb{F}_q)$, write $\widetilde{X}^y = p^{-1}(y)$, and let $i_y : \widetilde{X}^y \hookrightarrow \widetilde{X}$ denote the inclusion. Then, $i_y^* : \mathscr{D}_U(\widetilde{X}) \longrightarrow \mathscr{D}_{U^y}(\widetilde{X}^y)$ is an equivalence, and the composition*

$$i_y^* \circ (i_1^*)^{-1} : \mathscr{D}_G(X) \xrightarrow{\sim} \mathscr{D}_{U^y}(\widetilde{X}^y) \simeq \mathscr{D}_{G^{\alpha(y)}}(X^{\alpha(y)})$$

is isomorphic to the transport functor of Definition 6.10.

6.7 Relation between $\mathrm{ind}_{G'}^G$ and induction of class functions

Proposition 6.13 *Let G be a unipotent group over $\mathbb{F}_q$, let $G' \subset G$ be a closed subgroup, and let $M \in \mathscr{D}_{G'}(G')$. Then*

$$t_{\mathrm{ind}_{G'}^G M} = \sum_{\alpha \in \mathrm{Ker}(H^1(\mathbb{F}_q, G') \longrightarrow H^1(\mathbb{F}_q, G))} \mathrm{ind}_{G'^{\alpha}(\mathbb{F}_q)}^{G(\mathbb{F}_q)} t_{M^{\alpha}}. \tag{6.1}$$

Remarks 6.14 (1) For every $\alpha \in \mathrm{Ker}(H^1(\mathbb{F}_q, G') \longrightarrow H^1(\mathbb{F}_q, G))$, we realize G'^{α} as a subgroup of G using Proposition 6.12(a).

(2) The notation on the right hand side of (6.1) is as follows: if $\Gamma' \subset \Gamma$ are finite groups, then $\mathrm{ind}_{\Gamma'}^{\Gamma} : \mathrm{Fun}(\Gamma')^{\Gamma'} \longrightarrow \mathrm{Fun}(\Gamma)^{\Gamma}$ denotes the usual induction map from class functions on Γ' to class functions on Γ (cf. Sect. 5.2.1).

(3) As a special case of Proposition 6.13, we observe that if G' is connected, then $H^1(\mathbb{F}_q, G')$ is trivial, so the sum in (6.1) reduces to $\mathrm{ind}_{G'(\mathbb{F}_q)}^{G(\mathbb{F}_q)} t_M$. Hence for connected G', the proposition states that $\mathrm{ind}_{G'}^G$ is compatible with induction of class functions (via the sheaves-to-functions correspondence) "on the nose."

Proof of Proposition 6.13 Form the quotient $\widetilde{G} = (G \times G')/G'$, where the right action of G' on $G \times G'$ is given by $(g, g') \cdot \gamma = (g\gamma, \gamma^{-1} g' \gamma)$, and equip it with the G-action induced by the left translation action of G on the first factor in $G \times G'$.

The conjugation map $G \times G' \longrightarrow G$ (given by $(g, g') \mapsto g g' g^{-1}$) induces a G-equivariant morphism $\pi : \widetilde{G} \longrightarrow G$, and the map $i : G' \hookrightarrow G \times G'$ given by $g' \mapsto (1, g')$ induces a G'-equivariant morphism $i : G' \longrightarrow \widetilde{G}$.

Fix $M \in \mathscr{D}_{G'}(G')$. By Lemma 5.1, there is a (unique up to isomorphism) object $\widetilde{M} \in \mathscr{D}_G(\widetilde{G})$ such that $i^* \widetilde{M} \cong M$. We put $N = \pi_!(\widetilde{M}) \in \mathscr{D}_G(G)$, so that, by definition, $N \cong \mathrm{ind}_{G'}^G M$. By Lemma 4.4(c), $t_N = \pi_!(t_{\widetilde{M}})$, so to prove (6.1) we need to calculate the function $t_{\widetilde{M}} : \widetilde{G}(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell$.

To this end, define $Y = G/G'$ and equip it with the translation action of G. The first projection $G \times G' \longrightarrow G$ induces a G-equivariant morphism $p : \widetilde{G} \longrightarrow Y$. Then, $\widetilde{G}(\mathbb{F}_q)$ is the disjoint union of the sets of $\mathbb{F}_q$-points of the fibers $p^{-1}(y)$, where

y ranges over $Y(\mathbb{F}_q)$. For each $y \in Y(\mathbb{F}_q)$, write $G^y \subset G$ for the stabilizer of y in G and observe that the fiber $p^{-1}(y) \subset \widetilde{G}$ is G^y-stable.

The next result is straightforward.

Lemma 6.15 *Let $y \in Y(\mathbb{F}_q)$ and choose $g \in G(\mathbb{F})$ that maps onto y. Then*

(a) $G^y = gG'g^{-1}$;
(b) *the map $G^y \longrightarrow G \times G'$ given by $\gamma \mapsto (g, g^{-1}\gamma g)$ induces a G^y-equivariant inclusion $i_y : G^y \hookrightarrow \widetilde{G}$ (where G^y acts on itself by conjugation);*
(c) *i_y induces an isomorphism $G^y \xrightarrow{\simeq} p^{-1}(y)$, which is independent of the choice of g; and*
(d) *the composition $\pi \circ i_y : G^y \longrightarrow G$ is equal to the natural inclusion $G^y \hookrightarrow G$.*

We can now complete the proof of (6.1). For each $y \in Y(\mathbb{F}_q)$, write $\alpha(y) \in H^1(\mathbb{F}_q, G')$ for the isomorphism class of the right G'-torsor $q^{-1}(y) \subset G$, where $q : G \longrightarrow Y$ is the quotient map. If $M^y = i_y^*\widetilde{M} \in \mathscr{D}_{G^y}(G^y)$, then by Lemma 6.15(c) and Proposition 6.12(d), we can identify M^y with $M^{\alpha(y)} \in \mathscr{D}_{G'^{\alpha(y)}}^{(}G'^{\alpha(y)})$, where $G'^{\alpha(y)}$ is identified with G^y using Proposition 6.12(a). Since $t_N = \pi_!(t_{\widetilde{M}})$, Lemma 6.15(d) shows that

$$t_{\mathrm{ind}_{G'}^G M} = t_N = \sum_{y \in Y(\mathbb{F}_q)} \overline{t_{M^y}}, \tag{6.2}$$

where $\overline{t_{M^y}}$ denotes the function on $G(\mathbb{F}_q)$ obtained from $t_{M^y} : G^y(\mathbb{F}_q) \longrightarrow G(\mathbb{F}_q)$ via extension by zero. Now suppose $\mathcal{O} \subset Y(\mathbb{F}_q)$ is a single $G(\mathbb{F}_q)$-orbit and set $\alpha(\mathcal{O}) = \alpha(y)$ for any $y \in \mathcal{O}$ (note that $\alpha(y)$ does not depend on the choice of $y \in \mathcal{O}$). It is then easy to see that

$$\sum_{y \in \mathcal{O}} \overline{t_{M^y}} = \mathrm{ind}_{G'^{\alpha}(\mathbb{F}_q)}^{G(\mathbb{F}_q)} t_{M^\alpha}, \qquad \text{where } \alpha = \alpha(\mathcal{O}). \tag{6.3}$$

As $\mathcal{O}$ ranges over all $G(\mathbb{F}_q)$-orbits in $Y(\mathbb{F}_q)$, the corresponding cohomology class $\alpha(\mathcal{O}) \in H^1(\mathbb{F}_q, G')$ ranges over $\mathrm{Ker}(H^1(\mathbb{F}_q, G') \longrightarrow H^1(\mathbb{F}_q, G))$ by Lemma 6.1(c). Combining this observation with (6.2)–(6.3) yields (6.1).

7 Geometric reduction process

7.1 Overview

The main goal of this section is to prove the following result:

Theorem 7.1 *Let G be a (possibly disconnected) unipotent group over $\mathbb{F}_q$, let ρ be a nonzero representation of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$, and let $(A, \mathcal{N})$ be a pair consisting of a normal connected subgroup $A \subset G$ and a G-invariant multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{N}$ on A such that the restriction of ρ to $A(\mathbb{F}_q)$ is scalar, given by the 1-dimensional* *character* $t_{\mathcal{N}} : A(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. *Then, there exists an admissible pair* $(H, \mathcal{L})$ *for* G *such that* $A \subset H$, $\mathcal{N} \cong \mathcal{L}|_A$, *and the restriction of* ρ *to* $H(\mathbb{F}_q)$ *has as a direct summand the* 1*-dimensional representation defined by* $t_{\mathcal{L}} : H(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$.

We note that, in particular, we can take G to be connected, ρ to be irreducible, and the pair $(A, \mathcal{N})$ to be trivial. In this case, in view of the Frobenius reciprocity, the proposition implies the existence of an admissible pair $(H, \mathcal{L})$ for G such that ρ is a direct summand of $\operatorname{Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)} t_{\mathcal{L}}$, which proves the claim of Sect. 3.1 above.

The reason for stating Theorem 7.1 the way we did is that this approach makes it easier to give an inductive proof of the proposition. The title of this section ("Geometric reduction process") is motivated by an analogy between (the proof of) Theorem 7.1 and the (algebraic) reduction process described in one of the appendices to [10], where it is proved that every irreducible representation of a finite nilpotent group Γ can be *canonically* realized as a representation induced from a "Heisenberg representation" (op. cit.) of a subgroup of Γ.

The notion of an admissible pair (for a unipotent group over an arbitrary field of characteristic $p > 0$) is defined in Sect. 7.3. However, it is more convenient to formulate this definition in the framework of Serre duality developed in the "Appendix", rather than in the language of multiplicative local systems. Therefore we first explain the relationship between these two approaches to Serre duality in Sect. 7.2. We then state an auxiliary result in Sect. 7.4 (it is equivalent to one of the results proved in the "Appendix") and use it to prove Theorem 7.1 in Sects. 7.5–7.6.

7.2 Two approaches to Serre duality

Let us first fix an arbitrary field k and a connected algebraic group G over k. If A is an abstract abelian group, we will view A as a discrete group scheme over k. Thus, we have the notion of a central extension of G by A, as well as the notion of a *multiplicative A-torsor* on G (defined by an obvious analogy with Definition 2.9). Let us define a *rigidification* of an A torsor $\mathcal{E}$ on G to be a trivialization of the pullback $1^*\mathcal{E}$, where $1 : \operatorname{Spec} k \longrightarrow G$ is the unit morphism, and let us define a *rigidified* A-torsor on G to be an A-torsor on G equipped with a chosen rigidification. Since G is connected and A is discrete, it is easy to see that rigidified A-torsors on G form a discrete groupoid (i.e., a category with no non-identity morphisms). On the other hand, plain A-torsors on G form a groupoid where the group of automorphisms of every object is isomorphic to A.

The following result is proved in [26].

Lemma 7.2 (a) *Every multiplicative A-torsor on G admits a rigidification, and the forgetful functor induces a bijection between the set of isomorphism classes of rigidified multiplicative A-torsors on G and that of plain ones.*

(b) *The natural forgetful functor from the groupoid of central extensions of G by A to the groupoid of rigidified multiplicative A-torsors on G is an equivalence.*

Now we assume that k has characteristic $p > 0$ and G is a connected *unipotent* group over k. Fix a prime $\ell \neq p$. Our goal is to relate multiplicative $\overline{\mathbb{Q}}_\ell$-local systems on G to central extensions of G by the discrete group $\mathbb{Q}_p/\mathbb{Z}_p$.

Let us fix a homomorphism $\psi : (\mathbb{Q}_p, +) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ with kernel equal to $\mathbb{Z}_p$. Then, ψ identifies $\mathbb{Q}_p/\mathbb{Z}_p$ with the group $\mu_{p^\infty}(\overline{\mathbb{Q}}_\ell)$ of roots of unity in $\overline{\mathbb{Q}}_\ell^\times$ whose order is a power of p. Given a central extension $1 \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$, we can view $\widetilde{G}$ as a multiplicative $\mathbb{Q}_p/\mathbb{Z}_p$-torsor on G, and using the homomorphism ψ, we obtain the induced multiplicative $\overline{\mathbb{Q}}_\ell$-local system on G, which we denote $\widetilde{G}_\psi$.

Lemma 7.3 *The map $\widetilde{G} \mapsto \widetilde{G}_\psi$ constructed above is an isomorphism between the group $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ of isomorphism classes of central extensions of G by $\mathbb{Q}_p/\mathbb{Z}_p$ and the group of isomorphism classes of multiplicative $\overline{\mathbb{Q}}_\ell$-local systems on G.*

Proof In view of Lemma 7.2, it is enough to show that if $\mathcal{L}$ is an arbitrary multiplicative $\overline{\mathbb{Q}}_\ell$-local system on G, then $\mathcal{L}$ is induced from a $\mathbb{Q}_p/\mathbb{Z}_p$-torsor on G via ψ. To this end, we will show that if $f : \pi_1(G) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ is the homomorphism corresponding to $\mathcal{L}$, then the image of f is finite and is contained in the subgroup $\mu_{p^\infty}(\overline{\mathbb{Q}}_\ell)$, where $\pi_1(G)$ is the algebraic fundamental group of G, see [20].

Choose an algebraic closure $\bar{k}$ of k, write $\overline{G} = G \otimes_k \bar{k}$, let $1 : \operatorname{Spec} k \longrightarrow G$ denote the unit morphism as before, and let $\bar{1}$ denote the corresponding $\bar{k}$-point of either G or $\overline{G}$. By [20, Thm. IX.6.1], we have a short exact sequence of groups

$$1 \longrightarrow \pi_1(\overline{G}, \bar{1}) \longrightarrow \pi_1(G, \bar{1}) \longrightarrow \operatorname{Gal}(\bar{k}/k) \longrightarrow 1,$$

which is split by the homomorphism $\operatorname{Gal}(\bar{k}/k) \longrightarrow \pi_1(G, \bar{1})$ induced by the morphism $1 : \operatorname{Spec} k \longrightarrow G$. By the definition of multiplicativity, the $\overline{\mathbb{Q}}_\ell$-local system $1^*\mathcal{L}$ on $\operatorname{Spec} k$ satisfies $(1^*\mathcal{L}) \otimes (1^*\mathcal{L}) \cong 1^*\mathcal{L}$, whence $1^*\mathcal{L}$ is trivial. In other words, the composition $\operatorname{Gal}(\bar{k}/k) \xrightarrow{1_*} \pi_1(G, \bar{1}) \xrightarrow{f} \overline{\mathbb{Q}}_\ell^\times$ is trivial. Thus, f is determined by its restriction to $\pi_1(\overline{G}, \bar{1})$, which we will also denote by f.

By definition, f factors through a continuous homomorphism $\pi_1(\overline{G}, \bar{1}) \to K^\times$, where K is a finite extension of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$. Moreover, by compactness, the image of f must lie in $\mathcal{O}_K^\times$, where $\mathcal{O}_K \subset K$ is the ring of integers of K. The structure of $\mathcal{O}_K^\times$ is known; in particular, if $\mathfrak{m}_K$ denotes the unique maximal ideal of $\mathcal{O}_K$, then $(\mathcal{O}_K^\times)/(1 + \mathfrak{m}_K) \cong (\mathcal{O}_K/\mathfrak{m}_K)^\times$ is finite, and $1 + \mathfrak{m}_K$ has a descending filtration by closed subgroup with successive quotients isomorphic to the additive group of the residue $\mathcal{O}_K/\mathfrak{m}_K$, which is a finite field of characteristic ℓ. It follows that the subgroup $\mu_{p^\infty}(K) = K \cap \mu_{p^\infty}(\overline{\mathbb{Q}}_\ell) \subset \mathcal{O}_K^\times$ is finite, and the quotient $\mathcal{O}_K^\times/\mu_{p^\infty}(K)$ is a profinite group whose order is relatively prime to p.

On the other hand, since G is connected, $\overline{G}$ is isomorphic to an affine space over $\bar{k}$, so its algebraic fundamental group has no nontrivial quotients of order prime to p. Thus, the image of f lies in $\mu_{p^\infty}(K)$, completing the proof. □

7.3 Definition of an admissible pair

The notion of an admissible pair is a *geometric* one; thus we will first formulate it for an algebraically closed base field, and then for an arbitrary one. Moreover, it is more convenient to begin by working in the framework of Serre duality developed in the "Appendix".

Normalizer of a pair (H, χ)

Let us fix a perfect field k of characteristic $p > 0$ and a unipotent[9] algebraic group G over k. From now on, by default, all subgroups of group schemes are assumed to be closed. Consider a pair (H, χ) consisting of a *connected* subgroup $H \subset G$ and a central extension $1 \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \widetilde{H} \xrightarrow{\chi} H \longrightarrow 1$. We can view χ as a k-point of the Serre dual H^* of H, see Sect. 9.6, which is a perfect (possibly disconnected) commutative unipotent group over k by Proposition A.30. Let $N_G(H)$ denote the normalizer of H in G. Since the Serre dual H^* is defined by a universal property, we obtain an induced regular action of the perfectization $N_G(H)_{perf}$ on H^* by k-group scheme automorphisms. We let $G'_{perf} \subset N_G(H)_{perf}$ denote the stabilizer of χ under this action. The notation is explained by the fact that G'_{perf} is the perfectization of a uniquely determined closed subgroup of G, which we denote by G' and (unambiguously) call the "normalizer of (H, χ) in G". We remark that G' may be disconnected even if G is connected.

Definition 7.4 Let k be an algebraically closed field of characteristic $p > 0$, let G be a unipotent algebraic group (or perfect unipotent group) over k, and let (H, χ) be a pair consisting of a connected subgroup $H \subset G$ and an element $\chi \in H^*(k)$. We say that this pair is *admissible for G* if the following three conditions are satisfied.

(1) Let G' be the normalizer of (H, χ) in G, defined in the previous paragraph. Then, the quotient group G'°/H is commutative, i.e., $[G'^{\circ}, G'^{\circ}] \subset H$.
(2) The homomorphism $(G'^{\circ}/H)_{perf} \longrightarrow (G'^{\circ}/H)^*_{perf}$ constructed in Sect. 9.6, which is well defined in our situation in view of condition (1), is an *isogeny*.
(3) Given $g \in G(k)$, write $H^g = g^{-1}Hg$, and let $\chi^g \in (H^g)^*(k)$ be obtained from χ by transport of structure via $H^g \xrightarrow{\simeq} H, h \longmapsto ghg^{-1}$. If $g \notin G'(k)$, then

$$\chi \big|_{(H\cap H^g)^{\circ}} \not\cong \chi^g \big|_{(H\cap H^g)^{\circ}}$$

Definition 7.5 Let k be an arbitrary field of characteristic $p > 0$, let G be a unipotent algebraic group over k, let $H \subset G$ be a connected subgroup, and let χ be a central extension of H by $\mathbb{Q}_p/\mathbb{Z}_p$. The pair (H, χ) is said to be *admissible for G* if the pair $(H \otimes_k \overline{k}, \chi \otimes_k \overline{k})$ obtained from (H, χ) by base change to an algebraic closure $\overline{k}$ of k is admissible for $G \otimes_k \overline{k}$.

Admissible pairs in the context of multiplicative local systems

Now let k be a field of characteristic $p > 0$, let ℓ be a prime different from p, and choose a homomorphism $\psi : (\mathbb{Q}_p, +) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ with kernel $\mathbb{Z}_p$, as in Sect. 7.2. Fix a unipotent algebraic group G over k, a connected subgroup $H \subset G$, and a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on H (see Definition 2.9). By Lemma 7.3, $\mathcal{L}$ is asso-

[9] The definition applies equally well (with obvious simplifications) in the case where G is a *perfect* unipotent group over k (see Sect. 9.6).

ciated with a unique (up to isomorphism) central extension χ of H by $\mathbb{Q}_p/\mathbb{Z}_p$ via the homomorphism ψ.

Definition 7.6 We say that the pair $(H, \mathcal{L})$ is *admissible for* G if the pair (H, χ) is admissible for G in the sense of Definition 7.5. In the case where k is perfect, the *normalizer* of the pair $(H, \mathcal{L})$ in G is defined as the normalizer of (H, χ) in G.

It is not hard to see that this definition does not depend on the choice of ψ.

7.4 Extension of multiplicative local systems

The next result is used in the proof of Theorem 7.1. However, we also find it to be interesting in its own right. It is a natural geometrization of the fact that if Γ is a group, $H \subset \Gamma$ is a subgroup such that $[\Gamma, \Gamma] \subset H$, and $\chi : H \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ is a homomorphism, then χ extends to a homomorphism $\Gamma \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ if and only if $\chi\big|_{[\Gamma,\Gamma]} \equiv 1$ (a simple exercise).

Proposition 7.7 *Let G be a connected unipotent group over an arbitrary field k of characteristic $p > 0$, let ℓ be a prime different from p, let $H \subset G$ be a closed connected subgroup such that $[G, G] \subset H$, and let $\mathcal{L}$ be a multiplicative $\overline{\mathbb{Q}}_\ell$-local system on H. Then, there exists a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}'$ on G with $\mathcal{L}'|_H \cong \mathcal{L}$ if and only if the pullback* $\mathrm{com}^* \mathcal{L}$ *is a trivial $\overline{\mathbb{Q}}_\ell$-local system on $G \times G$, where* $\mathrm{com} : G \times G \longrightarrow H$ *is the commutator morphism,* $\mathrm{com}(g_1, g_2) = g_1 g_2 g_1^{-1} g_2^{-1}$.

In view of Lemma 7.3, this result is essentially equivalent to Proposition A.35, proved in Sect. 9.6 of the “Appendix”.

Remark 7.8 Naively, one might have replaced the condition that $\mathrm{com}^* \mathcal{L}$ is trivial by the stronger requirement that $\mathcal{L}\big|_{[G,G]}$ is a trivial $\overline{\mathbb{Q}}_\ell$-local system on $[G, G]$. However, the latter condition is *not* necessary. Indeed, as explained in [26], there are examples of connected unipotent groups G for which there is a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on G with $\mathcal{L}\big|_{[G,G]}$ being nontrivial. (We leave it to the reader to check that the fake Heisenberg groups defined in Sect. 2.10 are among such examples.)

7.5 A special case of Theorem 7.1

In this subsection, we prove Theorem 7.1 in the special case where ρ is irreducible and $[G, G^\circ] \subset A$. Using the construction explained in Sect. 9.6, we see that $\mathcal{N}$ induces a homomorphism of perfect $\mathbb{F}_q$-groups $\phi_{\mathcal{N}} : (G/A)_{perf} \longrightarrow (G^\circ/A)^*_{perf}$. Let H be a maximal (with respect to inclusion) connected subgroup of G with the property that $A \subset H$ and the composition

$$(H/A)_{perf} \hookrightarrow (G/A)_{perf} \xrightarrow{\phi_{\mathcal{N}}} (G^\circ/A)^*_{perf} \twoheadrightarrow (H/A)^*_{perf}$$

is trivial (the subscript “perf” is defined in Sect. 9.6). By the definition of $\phi_{\mathcal{N}}$, this implies that the pullback of $\mathcal{N}$ by the commutator map $H \times H \longrightarrow A$ is trivial. By Proposition 7.7, there is a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on H with $\mathcal{N} \cong \mathcal{L}|_A$.

We first claim that $\mathcal{L}$ can be chosen so that $t_{\mathcal{L}} : H(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ is a direct summand of the restriction of ρ to $H(\mathbb{F}_q)$. Indeed, since the homomorphism

$$(H/A)(\mathbb{F}_q) = (H/A)_{perf}(\mathbb{F}_q) \longrightarrow (H/A)^*_{perf}(\mathbb{F}_q) = (H/A)^*(\mathbb{F}_q)$$

induced by $\mathcal{N}$ is trivial, we see, in particular, that $t_{\mathcal{N}} : A(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ is trivial on $[H(\mathbb{F}_q), H(\mathbb{F}_q)]$. Now let V denote the representation space of ρ, so that $A(\mathbb{F}_q)$ acts on V through the character $t_{\mathcal{N}}$. Then, we see that $\rho(H(\mathbb{F}_q)) \subset \mathrm{Aut}(V)$ is a commutative subgroup; in particular, by Schur's lemma, there exists a character $\nu : H(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ which is contained in the restriction of ρ to $H(\mathbb{F}_q)$. *A fortiori*, ν and $t_{\mathcal{L}}$ agree on $A(\mathbb{F}_q)$, and hence $\nu \cdot t_{\mathcal{L}}^{-1}$ comes from a character $(H/A)(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. But H/A is a connected commutative algebraic group over $\mathbb{F}_q$, so, as we mentioned in Remark 2.10, there exists a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{E}$ on H/A such that $\nu \cdot t_{\mathcal{L}}^{-1} = t_{\widetilde{\mathcal{E}}}$, where $\widetilde{\mathcal{E}}$ is the pullback of $\mathcal{E}$ to H. In other words, $\nu = t_{\mathcal{L}\otimes\widetilde{\mathcal{E}}}$. But the restriction of $\widetilde{\mathcal{E}}$ to A is trivial by construction, so we may replace $\mathcal{L}$ with $\mathcal{L} \otimes \widetilde{\mathcal{E}}$ without loss of generality.

It remains to verify that $(H, \mathcal{L})$ is admissible. Since $[G, H] \subset [G, G^\circ] \subset A \subset H$, we see that H is normal in G, so condition (3) in the definition of admissibility is automatic. Condition (1) holds because $H \supset A$ and G°/A is central in G/A. Finally, condition (2) holds by the maximality requirement in the choice of H.

7.6 Proof of Theorem 7.1

Let us complete the proof of Theorem 7.1 in general. We will use simultaneous induction on $\dim G$ and on the length of the étale group $\pi_0(G) = G/G^\circ$ (i.e., the number of elements of $\pi_0(G)(\mathbb{F})$). By the result of Sect. 7.5, we may assume that $[G, G^\circ] \not\subset A$. We may also assume that ρ is irreducible.

Let $Z \subset G^\circ$ denote the preimage in G of the neutral component of the center of G/A. By assumption, $Z \neq G^\circ$, so Z is a proper connected subgroup of G°. As in Sect. 7.5, $\mathcal{N}$ induces a k-group morphism $\phi_{\mathcal{N}} : (G/A)_{perf} \longrightarrow (Z/A)^*_{perf}$. Since $\dim Z < \dim G$, the restriction of this morphism to $(Z/A)_{perf}$ has positive dimensional kernel.[10] Hence, there is a connected subgroup $B \subset Z$ such that $A \subsetneq B$ and the composition

$$(B/A)_{perf} \hookrightarrow (G/A)_{perf} \to (Z/A)^*_{perf} \twoheadrightarrow (B/A)^*_{perf}$$

is trivial. As in Sect. 7.5, we see that there exists a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{N}'$ on B such that $\mathcal{N} \cong \mathcal{N}'|_A$ and $t_{\mathcal{N}'}$ is a summand of $\rho\big|_{B(\mathbb{F}_q)}$.

On the other hand, B is normal in G since $A \subset B \subset Z$, and hence $\mathcal{N}'$ is not G-invariant by the maximality of $(A, \mathcal{N})$. Let G_1 denote the normalizer of $\mathcal{N}'$ in G. Then, G_1 is a proper subgroup of G, so either $\dim G_1 < \dim G$, or $|\pi_0(G_1)(\mathbb{F})| <$

[10] If K is the neutral connected component of the kernel of $\phi_{\mathcal{N}}$, then K is normal in G/A and we obtain a decreasing sequence $K, [G/A, K], [G/A, [G/A, K]], \ldots,$ of normal connected subgroups of G/A. The last nontrivial term of this sequence is contained in Z/A by the definition of Z.

$|\pi_0(G)(\mathbb{F})|$. In either case, if we let ρ_1 be the restriction of ρ to $G_1(\mathbb{F}_q)$, we may assume that Theorem 7.1 holds for ρ_1 and the pair $(B, \mathcal{N}')$.

Let $(H, \mathcal{L})$ denote a pair consisting of a connected subgroup $H \subset G_1$ and a multiplicative $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on H, which satisfies the conclusion of Theorem 7.1 for the 4-tuple $(G_1, \rho_1, B, \mathcal{N}')$. We assert that it also satisfies the conclusion of this theorem for $(G, \rho, A, \mathcal{N})$. To prove this assertion, we only need to check that $(H, \mathcal{L})$ is admissible with respect to G.

Let G' denote the normalizer of the pair $(H, \mathcal{L})$ in G. We have $G' \subset G_1$. Indeed, if $g \in G(\mathbb{F})$, $g \notin G_1(\mathbb{F})$, then, by construction, g does not fix $\mathcal{N}'$, and hence, a fortiori, it cannot fix $(H, \mathcal{L})$ (because B is normal in G and $\mathcal{L}|_B \cong \mathcal{N}'$).

Since $G' \subset G_1$, conditions (1) and (2) in the definition of admissibility for $(H, \mathcal{L})$ hold with respect to G because they hold with respect to G_1. To verify condition (3), let $g \in G(\mathbb{F})$, $g \notin G'(\mathbb{F})$. If $g \in G_1(\mathbb{F})$, there is nothing to do because $(H, \mathcal{L})$ is admissible with respect to G_1. If $g \notin G_1(\mathbb{F})$, then, since $B \subset (H \cap H^g)^\circ$ and g does not fix $\mathcal{N}'$, it follows that the restrictions of $\mathcal{L}$ and $\mathcal{L}^g$ to $(H \cap H^g)^\circ$ cannot be isomorphic, completing the induction step in the proof of Theorem 7.1.

8 Analysis of Heisenberg idempotents

In this section, we study a certain special type of geometrically minimal weak idempotents (cf. Definition 4.12) in the equivariant derived categories of unipotent algebraic groups. The main result of the section is Proposition 8.1.

8.1 Setup

Throughout this section, we fix a field k of characteristic $p > 0$, let U be a possibly disconnected unipotent group over k, and let $(N, \mathcal{L})$ be an admissible pair for U in the sense of Definition 7.6, such that its normalizer in U is all of U (so that the third condition in the definition of admissibility is vacuous). In particular, N is a normal closed connected subgroup of U, and $\mathcal{L}$ is a multiplicative $\overline{\mathbb{Q}}_\ell$-local system on N which is invariant under the conjugation action of U. In this subsection, we construct certain objects associated with the data $U, N, \mathcal{L}$.

8.1.1 Construction of $e_{\mathcal{L}}$ and $e'_{\mathcal{L}}$

It follows from our assumptions that $\mathcal{L}$ has a natural U-equivariant structure (because N is connected). Let $\mathbb{K}_N$ denote the dualizing complex of N; then $\mathbb{K}_N$ has a natural U-equivariant structure as well, since N is normal in U. It follows that we can define $e_{\mathcal{L}} = \mathcal{L} \otimes \mathbb{K}_N$ as an object of $\mathscr{D}_U(N)$. Let $e'_{\mathcal{L}}$ denote the object of $\mathscr{D}_U(U)$ obtained from $e_{\mathcal{L}}$ via extension by zero.

8.1.2 Construction of a morphism $\mathbb{1} \longrightarrow e'_{\mathcal{L}}$

Let $1 : \operatorname{Spec} k \longrightarrow U$ be the unit morphism, and let $\mathbb{1} = 1_!\overline{\mathbb{Q}}_\ell$ be the delta-sheaf at 1, equipped with the "trivial" U-equivariant structure. Recall that $\mathbb{1}$ is a unit object in the monoidal category $\mathscr{D}_U(U)$ under convolution. Of course, we can equally well think of $\mathbb{1}$ as an object of $\mathscr{D}_N(N)$. If $p : N \longrightarrow \operatorname{Spec} k$ is the structure morphism, then $\mathbb{K}_N = p^!\overline{\mathbb{Q}}_\ell$, so we get a canonical identification $\overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} 1^!\mathbb{K}_N$. By adjunction, we get a canonical morphism $\mathbb{1} \longrightarrow \mathbb{K}_N$. On the other hand, since the stalk $\mathcal{L}_1$ of $\mathcal{L}$ at 1 has a natural trivialization, we obtain an isomorphism $\mathbb{1} \otimes \mathcal{L}^\vee \xrightarrow{\simeq} \mathbb{1}$, where $\mathcal{L}^\vee = \operatorname{hom}(\mathcal{L}, \overline{\mathbb{Q}}_\ell)$ is the dual local system on $\mathcal{L}$. Composing the two morphisms we just constructed, we obtain a natural morphism $\mathbb{1} \otimes \mathcal{L}^\vee \longrightarrow \mathbb{K}_N$, which induces a morphism $\mathbb{1} \longrightarrow e_\mathcal{L}$ in $\mathscr{D}_U(N)$, and hence a morphism $\mathbb{1} \longrightarrow e'_\mathcal{L}$ in $\mathscr{D}_U(U)$.

8.1.3 Construction of a homomorphism $\varphi_\mathcal{L} : (U^\circ/N)_{perf} \longrightarrow (U^\circ/N)^*_{perf}$

Before stating the main result of the section, we need one last construction. Let us fix a homomorphism $\psi : (\mathbb{Q}_p, +) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ with kernel $\mathbb{Z}_p$. By Lemma 7.3, the multiplicative local system $\mathcal{L}$ on N is induced from a central extension $\widetilde{N}$ of N by $\mathbb{Q}_p/\mathbb{Z}_p$ via ψ. Let k_{perf} be the perfect closure of k (see [19]); it is the maximal purely inseparable algebraic extension of k, and hence is determined up to a unique k-isomorphism. As recalled in Sect. 9.6, for every k-scheme X, we can construct its *perfectization*, X_{perf}, which is a scheme over k_{perf}. In particular, we obtain the induced central extension $\widetilde{N}_{perf}$ of N_{perf} by $\mathbb{Q}_p/\mathbb{Z}_p$, which is U_{perf}-invariant. Using the construction explained in Sect. 9.6 with $Z = U^\circ_{perf}$, the neutral connected component of U (recall that U°/N is commutative, so this choice of Z is allowed), we obtain a homomorphism of perfect unipotent groups $U^\circ_{perf}/N_{perf} \longrightarrow \left(U^\circ_{perf}/N_{perf}\right)^*$ over k_{perf}, i.e., a homomorphism

$$\varphi_\mathcal{L} : (U^\circ/N)_{perf} \longrightarrow (U^\circ/N)^*_{perf}.$$

By the definition of admissibility, $\varphi_\mathcal{L}$ is an isogeny.

8.2 Statement of the main result

Proposition 8.1 *Let U, N, $\mathcal{L}$, $e_\mathcal{L}$, $e'_\mathcal{L}$ and $\varphi_\mathcal{L}$ be as above.*

(a) *The morphism $\mathbb{1} \longrightarrow e'_\mathcal{L}$ constructed in Sect. 8.1.2 becomes an isomorphism after convolving with $e'_\mathcal{L}$. A fortiori, $e'_\mathcal{L}$ is a weak idempotent in $\mathscr{D}_U(U)$.*
(b) *In fact, $e'_\mathcal{L}$ is a geometrically minimal weak idempotent in $\mathscr{D}_U(U)$ (see Definition 4.12).*
(c) *Let θ denote the canonical automorphism of the identity functor on $\mathscr{D}_{U\otimes_k\bar{k}}(U \otimes_k \bar{k})$, introduced in Sect. 4.9. If the restriction of θ to the Hecke subcategory*

$$e'_\mathcal{L}\mathscr{D}_{U\otimes_k\bar{k}}(U \otimes_k \bar{k}) \subset \mathscr{D}_{U\otimes_k\bar{k}}(U \otimes_k \bar{k})$$

is trivial, then U is connected and $\varphi_\mathcal{L}$ is an isomorphism.

This proposition is the last ingredient in the proofs of the main results of our work, stated in Sect. 2. The rest of the section is devoted to its proof. If U is as above, a weak idempotent in $\mathscr{D}_U(U)$ isomorphic to one of the form $e'_{\mathcal{L}}$ will be called a *Heisenberg idempotent*,[11] which explains the title of the section.

Remark 8.2 One can show that the converse of Proposition 8.1(c) holds as well (see [15]), but we do not need this fact.

8.3 Proof of Proposition 8.1(a)

It is enough to show that the morphism $\mathbb{1} \longrightarrow e_{\mathcal{L}}$ in $\mathscr{D}_N(N)$ becomes an isomorphism after convolving with $e_{\mathcal{L}}$. Without loss of generality, we may and do assume that k is algebraically closed. Then, $e_{\mathcal{L}} \cong \mathcal{L}[2\dim N]$, so it suffices to prove that $\mathbb{1} \longrightarrow e_{\mathcal{L}}$ becomes an isomorphism after convolving with $\mathcal{L}$.

Fix $g \in N(k)$, and let $\rho_g : N \longrightarrow N$ be defined by $n \longmapsto n^{-1}g$. By the proper base change theorem, the induced morphism on the stalks, $(\mathbb{1} * \mathcal{L})_g \longrightarrow (e_{\mathcal{L}} * \mathcal{L})_g$, is the same as the morphism obtained by applying the functor $R\Gamma_c(N, -)$ to the induced morphism $\mathbb{1} \otimes (\rho_g^*\mathcal{L}) \longrightarrow e_{\mathcal{L}} \otimes (\rho_g^*\mathcal{L})$. However, $\rho_g^*\mathcal{L}$ is naturally isomorphic to $\mathcal{L}^\vee$, because $\mathcal{L}$ is multiplicative, and the morphism $\mathbb{1} \otimes (\rho_g^*\mathcal{L}) \longrightarrow e_{\mathcal{L}} \otimes (\rho_g^*\mathcal{L})$ becomes the canonical morphism $\mathbb{1} \longrightarrow \mathbb{K}_N$. Applying the functor $R\Gamma_c(N, -)$, we recover the adjunction morphism $\overline{\mathbb{Q}}_\ell \longrightarrow p_! p^! \overline{\mathbb{Q}}_\ell$ (where $p : N \longrightarrow \operatorname{Spec} k$ is the structure morphism), which is an isomorphism because N is a connected unipotent group over k, and hence is isomorphic to an affine space over k.

8.4 Proof of Proposition 8.1(b)

Without loss of generality, we may and do assume that k is algebraically closed. Then, we must prove that $e'_{\mathcal{L}}$ is a minimal weak idempotent in $\mathscr{D}_U(U)$, which is equivalent to showing that the Hecke subcategory $e'_{\mathcal{L}}\mathscr{D}_U(U)$ contains no weak idempotents apart from 0 and $e'_{\mathcal{L}}$.

The category $e'_{\mathcal{L}}\mathscr{D}_U(U)$ is studied in[12] [15]. Let us recall the results of op. cit. that will be used in the current proof.

Theorem 8.3 [15, Theorem 1.5]

Let $\mathcal{M} \subset e'_{\mathcal{L}}\mathscr{D}_U(U)$ be the full subcategory consisting of objects M for which $M[-\dim N]$ is a perverse[13] sheaf on U.

(a) *The natural functor $D^b(\mathcal{M}) \longrightarrow e'_{\mathcal{L}}\mathscr{D}_U(U)$ is an equivalence of categories.*
(b) *The subcategory $\mathcal{M}$ is closed under convolution and is a (semisimple) fusion category with unit object $e'_{\mathcal{L}}$.*
(c) *There exists a ribbon structure on the fusion category $\mathcal{M}$, which makes $\mathcal{M}$ a modular category and is such that the corresponding twist (in other terminology,*

[11] The notion of a Heisenberg idempotent is the geometric analogue of the notion of a Heisenberg representation of a finite group, introduced in [10].

[12] The results of [15] use the construction of the arrow $\mathbb{1} \longrightarrow e'_{\mathcal{L}}$ presented in Sect. 8.1.2, but are otherwise independent of the current section.

[13] See [4]; we only consider the *middle perversity* in this article.

"balancing") is equal to the canonical automorphism θ of the identity functor introduced in Sect. 4.9.

Remarks 8.4 (1) The word "semisimple" in the formulation of part (b) is only added for emphasis. Our use of the term "fusion category" agrees with that of [18]. Thus, part (b) means that $\mathcal{M}$ is a semisimple $\overline{\mathbb{Q}}_\ell$-linear monoidal category over $\overline{\mathbb{Q}}_\ell$, which is rigid, has finitely many simple objects and finite dimensional Hom-spaces and is such that the unit object is simple.

(2) We do not require the precise definitions of the terms "ribbon structure" or "modular category"; see e.g., [2]. All we need is the fact that a modular fusion category where the twist is trivial has only one simple object (which is necessarily the unit object).

We see that Proposition 8.1(b) follows from the more general

Lemma 8.5 *Let $\mathcal{M}$ be a weakly symmetric fusion category. The bounded derived category $D^b(\mathcal{M})$ (equipped with the induced tensor product) has no weak idempotents other than* 0 *and the unit object.*

Proof Let $\otimes$ and $\mathbb{1}$ denote the monoidal functor and the unit object of $\mathcal{M}$. In the proof, we will repeatedly use the observation that if $X, Y \in \mathcal{M}$ are nonzero, then $X \otimes Y$ is also nonzero. The reason is that if X^* is a right dual[14] of X, then $X^* \otimes X$ contains $\mathbb{1}$ as a direct summand, and hence $X^* \otimes X \otimes Y$ contains Y as a direct summand.

First let us check that every weak idempotent in $D^b(\mathcal{M})$ comes from a simple object of $\mathcal{M}$. Write $\mathcal{M}^{gr}$ for the category of bounded graded objects of $\mathcal{M}$; in other words, $\mathcal{M}^{gr}$ is the category of bounded complexes over $\mathcal{M}$ in which all the differentials are equal to 0. The cohomology functor $H^\bullet : D^b(\mathcal{M}) \longrightarrow \mathcal{M}^{gr}$ is an equivalence of categories because $\mathcal{M}$ is semisimple, and it is moreover a monoidal equivalence by the Künneth formula. The comment in the previous paragraph implies that the length function $\ell : \mathcal{M}^{gr} \longrightarrow \mathbb{Z}_{\geq 0}$, which assigns to an object $X \in \mathcal{M}^{gr}$ the sum of lengths of all the components of X, satisfies $\ell(X \otimes Y) \geq \ell(X) \cdot \ell(Y)$. It follows that every weak idempotent in $\mathcal{M}^{gr}$ has length 1, and hence it must be a simple object concentrated in a single degree k. Now we are forced to have $k + k = k$, and hence $k = 0$, as desired.

Now let $X \in \mathcal{M}$ be a nonzero weak idempotent. Then, the right dual X^* is also a weak idempotent, and hence so is $X^* \otimes X$ (since $\mathcal{M}$ is weakly symmetric). We already saw that $X^* \otimes X$ must be simple, and thus $X^* \otimes X \cong \mathbb{1}$. So X is invertible, and since $X \otimes X \cong X$, we see that $X \cong \mathbb{1}$, proving the lemma. □

8.5 Proof of Proposition 8.1(c)

By Remark 8.4(2), the hypothesis of Proposition 8.1(c) implies that the category $\mathcal{M} \subset e'_{\mathcal{L}} \mathscr{D}_U(U)$ defined in Theorem 8.3 has only one simple object, namely, $e'_{\mathcal{L}}$ itself. Write $\Gamma = U/U^\circ$, where $U^\circ \subset U$ is the neutral connected component, and let $\mathcal{M}_0 \subset e'_{\mathcal{L}} \mathscr{D}_{U^\circ}(U)$ be the full subcategory consisting of objects M for which $M[-\dim N]$ is a perverse sheaf on U. The natural action of Γ on $\mathscr{D}_{U^\circ}(U)$ induces an

[14] We are using the terminology of [2, 18].

action of Γ on $\mathcal{M}_0$, and by [15, Lemma 1.4], the Γ-equivariantization $\mathcal{M}_0^\Gamma$ is equivalent to $\mathcal{M}$. By [15, Theorem 1.3], $\mathcal{M}_0$ is also a fusion category, so we see that $e'_\mathcal{L}$ is the only simple object of $\mathcal{M}_0$ (indeed, every simple object of $\mathcal{M}_0$ can be realized as a direct summand of a simple object of $\mathcal{M}_0^\Gamma$). On the other hand, all simple objects of $\mathcal{M}_0$ are described in [15, §4.1], and that description implies that U is connected and that $\varphi_\mathcal{L}$ is an isomorphism.

9 The proofs of the main results

In this section, we put together all the preliminary results obtained in Sects. 4–8 to prove Theorem 2.5, Theorem 2.14 and Proposition 4.13.

9.1 The key result

Throughout this section, we work with a fixed connected unipotent group G over a field k of characteristic $p > 0$. For the most part, we will take $k = \mathbb{F}_q$, but it is convenient to formulate one of the results in the more general setting. Below we will state a result (Proposition 9.1) to which all the other results to be proved in this section are easily reduced.

It is convenient to introduce the following notation. If $k = \mathbb{F}_q$ and $e \in \mathscr{D}_G(G)$ is any weak idempotent, let us write $L(e)$ for the set of isomorphism classes of irreducible representations ρ of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$ in which t_e acts by the identity operator. If e is minimal (Definition 4.10), it follows from Definition 2.7 and Remark 4.14(1) that $L(e)$ is either empty (when $t_e \equiv 0$) or an L-packet (when $t_e \not\equiv 0$).

Let $(H, \mathcal{L})$ be an admissible pair for G, and let G' be its normalizer in G (see Definition 7.6). Let $e'_\mathcal{L} \in \mathscr{D}_{G'}(G')$ be the object obtained by applying the construction of Sect. 8.1.1 with G' and H in place of U and N, and let $e_{H,\mathcal{L}} = \operatorname{ind}_{G'}^G e'_\mathcal{L}$.

Proposition 9.1 *With this notation,*

(a) *$e_{H,\mathcal{L}}$ is a geometrically minimal weak idempotent in $\mathscr{D}_G(G)$; and*
(b) *if $\mathscr{C}$ denotes the geometric conjugacy class of $(H, \mathcal{L})$ (see Sect. 2.9) and $k = \mathbb{F}_q$, then*

$$L(\mathscr{C}) = L(e_{H,\mathcal{L}}),$$

where $L(\mathscr{C})$ is introduced in Definition 2.13.

This proposition is proved in Sects. 9.5 and 9.6. First we explain how it implies all the other results to be proved in this section. In Sects. 9.2 and 9.4, we assume that $k = \mathbb{F}_q$.

9.2 Proof of Theorem 2.14

Proposition 9.1 implies that if $\mathscr{C}$ is a geometric conjugacy class of admissible pairs for G, then $L(\mathscr{C})$ is an L-packet of irreducible representations of $G(\mathbb{F}_q)$.

Conversely, consider an L-packet $\mathscr{P}$ of irreducible representations of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$, and choose $\rho \in \mathscr{P}$. By Theorem 7.1 (and Frobenius reciprocity), there exists a geometric conjugacy class $\mathscr{C}$ of admissible pairs for G such that $\rho \in L(\mathscr{C})$. Then, $L(\mathscr{C}) \cap \mathscr{P} \neq \varnothing$, and since $L(\mathscr{C})$ is also an L-packet by the previous paragraph, we see that $L(\mathscr{C}) = \mathscr{P}$, proving Theorem 2.14. □

9.3 Proof of Proposition 4.13

Let us fix two irreducible representations, ρ_1 and ρ_2, of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$. We tautologically have $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ in the statement of Proposition 4.13, so we only need to show that $(iii) \Longrightarrow (i)$. Assume that (iii) holds. By the arguments above, there exists an admissible pair $(H, \mathcal{L})$ for G such that $\rho_1 \in L(e_{H,\mathcal{L}})$. By Proposition 9.1, $e_{H,\mathcal{L}}$ is a geometrically minimal weak idempotent in $\mathscr{D}_G(G)$. Now $t_{e_{H,\mathcal{L}}}$ acts as the identity in ρ_1, and hence, by assumption, it also acts as the identity in ρ_2. This means that ρ_1 and ρ_2 both lie in $L(e_{H,\mathcal{L}})$, which is a single L-packet, and the proof is complete. □

9.4 Proof of Theorem 2.5

In this subsection, we assume that G is an *easy* unipotent group over $\mathbb{F}_q$. Let ρ be an irreducible representation of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$. We must prove that the dimension of ρ is a power of q.

9.4.1 Step 1

By the arguments above, there exists an admissible pair $(H, \mathcal{L})$ for G such that ρ is a direct summand of $\mathrm{Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)} t_{\mathcal{L}}$. Let G' be the normalizer of $(H, \mathcal{L})$ in G (Definition 7.6). We first show, using Proposition 8.1, that G' is connected and the dimension of G'/H is even.

Let $e'_{\mathcal{L}} \in \mathscr{D}_{G'}(G')$ be the extension of $\mathcal{L} \otimes \mathbb{K}_H$ by zero to G', as before. From Proposition 8.1(a) and Lemma 9.2, it follows that the functor $M \longmapsto e'_{\mathcal{L}} * M$ is isomorphic to the identity functor on the Hecke subcategory $e'_{\mathcal{L}}\mathscr{D}_{G'}(G') \subset \mathscr{D}_{G'}(G')$. Now Theorem 5.12(b) implies that the restriction

$$\mathrm{ind}_{G'}^{G}\Big|_{e'_{\mathcal{L}}\mathscr{D}_{G'}(G')} : e'_{\mathcal{L}}\mathscr{D}_{G'}(G') \longrightarrow \mathscr{D}_G(G)$$

is a *faithful* functor. By Lemma 4.16, the twist automorphism of the identity functor on $\mathscr{D}_G(G)$ is trivial, because G is easy. As $\mathrm{ind}_{G'}^{G}$ is compatible with twists (Proposition 5.17), the restriction of the twist on $\mathscr{D}_{G'}(G')$ to the Hecke subcategory $e'_{\mathcal{L}}\mathscr{D}_{G'}(G')$ is trivial as well. These statements continue to hold after base change from $\mathbb{F}_q$ to $\mathbb{F}$. By Proposition 8.1(c), G' is connected, and the homomorphism $\varphi_{\mathcal{L}} : (G'/H)_{perf} \longrightarrow (G'/H)^*_{perf}$ induced by $\mathcal{L}$ is an isomorphism. Since $\varphi_{\mathcal{L}}$ obviously arises from a skew-symmetric bi-extension of G'/H by $\mathbb{Q}_p/\mathbb{Z}_p$ (cf. Remark A.34; see also Sect. 9.6 for the construction of $\varphi_{\mathcal{L}}$, and Sect. 9.6 for the terminology), it follows from Proposition A.28(b) that G'/H is even-dimensional.

We now pause to state and prove the lemma used in the previous paragraph.

Lemma 9.2 *Let $\mathcal{M}$ be a monoidal category with monoidal bifunctor $\otimes$ and unit object $\mathbb{1}$ and consider an arrow $\mathbb{1} \longrightarrow e$ in $\mathcal{M}$ that becomes an isomorphism after tensoring with e on the right. Then, the functor $X \longmapsto e \otimes X$ is isomorphic to the identity functor on the subcategory $e\mathcal{M} \subset \mathcal{M}$.*

Remark 9.3 Here, the notation is similar to that used in Sect. 4.7, namely, $e\mathcal{M}$ is the essential image of the functor $\mathcal{M} \longrightarrow \mathcal{M}$ given by $X \longmapsto e \otimes X$. With the assumption of the lemma, it is obvious that e is a weak idempotent in $\mathcal{M}$ in the sense of Sect. 4.7. However, the existence of an arrow $\mathbb{1} \longrightarrow e$ satisfying the assumption of Lemma 9.2 is a much stronger condition than merely requiring e to be a weak idempotent. (In [10], arrows $\mathbb{1} \longrightarrow e$ that become isomorphisms after tensoring with e on either side are called *closed idempotents*.) In particular, we do not expect the conclusion of Lemma 9.2 to hold for an arbitrary weak idempotent $e \in \mathcal{M}$.

Proof of Lemma 9.2 We use the fact that $\otimes$ is equipped with an associativity constraint. If $X \in e\mathcal{M}$, then $X \cong e \otimes X$, because $e \cong e \otimes e$. Hence for any $X \in e\mathcal{M}$, the arrow $\mathbb{1} \longrightarrow e$ becomes an isomorphism after we apply the functor $Y \longmapsto Y \otimes X$ to it. This (together with the unit constraint for $\otimes$) gives us a functorial collection of isomorphisms $X \xrightarrow{\simeq} e \otimes X$ for all $X \in e\mathcal{M}$, as desired.

9.4.2 Step 2

Now we complete the proof of Theorem 2.5. Consider the commutator morphism $\mathrm{com} : G' \times G' \longrightarrow H$, $(g_1, g_2) \longmapsto g_1 g_2 g_1^{-1} g_2^{-1}$, and form the pullback local system $\mathcal{L}' = \mathrm{com}^* \mathcal{L}$ on $G' \times G'$. Since the map $\varphi_{\mathcal{L}} : (G'/H)_{perf} \longrightarrow (G'/H)^*_{perf}$ induced by $\mathcal{L}$ is an isomorphism, it is easy to deduce from Proposition A.18 that the trace function $t_{\mathcal{L}'} : G'(\mathbb{F}_q) \times G'(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ descends to a perfect pairing

$$B_{\mathcal{L}} : (G'/H)(\mathbb{F}_q) \times (G'/H)(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times,$$

i.e., a bimultiplicative map that induces an isomorphism

$$(G'/H)(\mathbb{F}_q) \xrightarrow{\simeq} \mathrm{Hom}\Big((G'/H)(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times\Big).$$

Next, the definition of $\varphi_{\mathcal{L}}$ implies that $B_{\mathcal{L}}$ is equal to the map induced by the commutator pairing defined by the character $t_{\mathcal{L}} : H(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell$, namely,

$$G'(\mathbb{F}_q) \times G'(\mathbb{F}_q) \xrightarrow{\mathrm{com}} H(\mathbb{F}_q) \xrightarrow{t_{\mathcal{L}}} \overline{\mathbb{Q}}_\ell^\times.$$

It is well known (see e.g., the appendix on Heisenberg representations in [10]) that the non-degeneracy of $B_{\mathcal{L}}$ implies that $G'(\mathbb{F}_q)$ has a unique irreducible representation, call it ρ', which acts on $H(\mathbb{F}_q)$ by the scalar $t_{\mathcal{L}}$. Moreover, ρ' has dimension

$[G'(\mathbb{F}_q) : H(\mathbb{F}_q)]^{1/2}$, which is a power of q by the first step of the proof. Furthermore, by the Frobenius reciprocity, the irreducible representation of $G(\mathbb{F}_q)$ with which we started, ρ, is a direct summand of $\operatorname{Ind}_{G'(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho'$. However, the definition of an admissible pair, together with Mackey's irreducibility criterion, imply that $\operatorname{Ind}_{G'(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho'$ is irreducible. Thus, $\rho \cong \operatorname{Ind}_{G'(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho'$, whence (as G' is connected)

$$\dim \rho = [G(\mathbb{F}_q) : G'(\mathbb{F}_q)] \cdot \dim \rho = q^{\dim G - \dim G'} \cdot [G'(\mathbb{F}_q) : H(\mathbb{F}_q)]^{1/2},$$

which is a power of q, completing the proof of Theorem 2.5.

9.5 Proof of Proposition 9.1(a)

In this subsection, k is allowed to be an arbitrary field of characteristic $p > 0$. We will use the notation introduced at the beginning of Sect. 5.7. In view of Corollary 5.13, it suffices to check that for every $g \in G(\overline{k})\backslash G'(\overline{k})$, we have $\overline{e'_{\mathcal{L}}} * \delta_g * \overline{e'_{\mathcal{L}}} = 0$. Since the notion of an admissible pair is stable under base change from k to $\overline{k}$, we may as well *assume that k is algebraically closed*. Let us fix $g \in G(k)\backslash G'(k)$.

Consider the morphism

$$m_g : H \times H \longrightarrow G, \qquad (h_1, h_2) \longmapsto h_1 g h_2.$$

By definition, $\overline{e'_{\mathcal{L}}} * \delta_g * \overline{e'_{\mathcal{L}}} = m_{g!}(e_{\mathcal{L}} \boxtimes e_{\mathcal{L}})$. To complete the proof, it suffices to show that for every $x \in G(k)$, the stalk $m_{g!}(e_{\mathcal{L}} \boxtimes e_{\mathcal{L}})_x = 0$. Up to cohomological shift,[15] this is the same as proving that $m_{g!}(\mathcal{L} \boxtimes \mathcal{L})_x = 0$. By the proper base change theorem, this is equivalent to $R\Gamma_c(m_g^{-1}(x), \mathcal{L} \boxtimes \mathcal{L}) = 0$.

Let us fix $x \in G(k)$. If $m_g^{-1}(x) = \varnothing$, there is nothing to check. Otherwise, fix a k-point (h_1, h_2) of $m_g^{-1}(x)$. Then, $m_g^{-1}(x)$ can be identified with $H \cap gHg^{-1}$ via the map $w : H \cap gHg^{-1} \longrightarrow H \times H$ given by $w(h) = (h_1 h, g^{-1}h^{-1}gh_2)$.

The (isomorphism class of the) local system $\mathcal{L}$ on H is invariant under left and right translations (this follows from the multiplicativity of $\mathcal{L}$). Thus

$$w^*(\mathcal{L} \boxtimes \mathcal{L}) \cong \mathcal{L}\big|_{H\cap gHg^{-1}} \otimes {}^g\mathcal{L}^\vee\big|_{H\cap gHg^{-1}},$$

where ${}^g\mathcal{L}$ denotes the multiplicative $\overline{\mathbb{Q}}_\ell$-local system on gHg^{-1} obtained from $\mathcal{L}$ by transport of structure via $h \longmapsto ghg^{-1}$, and ${}^g\mathcal{L}^\vee$ is its dual local system.

By the definition of admissibility, we are reduced to the following well known.

Lemma 9.4 *Let A be an algebraic group over a field k, and let $\mathcal{L}$ be a multiplicative $\overline{\mathbb{Q}}_\ell$-local system on A such that $\mathcal{L}|_{A^\circ}$ is nontrivial. Then, $R\Gamma_c(A, \mathcal{L}) = 0$.*

[15] Recall that $\mathbb{K}_H \cong \overline{\mathbb{Q}}_\ell[2\dim H](\dim H)$, and Tate twists are trivial since $k = \overline{k}$.

Proof It suffices to show that $f_! \mathcal{L} = 0$, where $f : A \longrightarrow \pi_0(A)$ is the natural quotient morphism and $\pi_0(A) = A/A^\circ$. Since $\mathcal{L}$ is multiplicative, it in turn suffices to show that $R\Gamma_c(A^\circ, \mathcal{L}|_{A^\circ}) = 0$. Thus, we may assume, without loss of generality, that A is connected and $\mathcal{L}$ is as before.

The following diagram is clearly cartesian:

$$\begin{array}{ccc} A \times A & \xrightarrow{\ \mu\ } & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\ \pi\ } & \operatorname{Spec} k \end{array}$$

where $\pi : A \longrightarrow \operatorname{Spec} k$ is the structure morphism and $\mathrm{pr}_1 : A \times A \longrightarrow A$ is the projection onto the first factor. By the proper base change theorem,

$$\pi^* R\Gamma_c(A, \mathcal{L}) \cong \mathrm{pr}_{1!} \mu^* \mathcal{L} \cong \mathrm{pr}_{1!}(\mathcal{L} \boxtimes \mathcal{L}) \cong R\Gamma_c(A, \mathcal{L}) \otimes \mathcal{L}.$$

Since $\mathcal{L}$ is a nontrivial local system on A, this clearly forces $R\Gamma_c(A, \mathcal{L}) = 0$. □

9.6 Proof of Proposition 9.1(b)

We now take $k = \mathbb{F}_q$ and recall that $G' \subset G$ denotes the normalizer of the given admissible pair $(H, \mathcal{L})$ and $\mathscr{C}$ denotes the geometric conjugacy class of $(H, \mathcal{L})$. We can identify $\mathscr{C}$ with $(G/G')(\mathbb{F}_q)$. Note also that $H^1(\mathbb{F}_q, G)$ is trivial because G is connected, so there are representatives $\{(H_\alpha, \mathcal{L}_\alpha)\}_{\alpha \in H^1(\mathbb{F}_q, G')}$ of the $G(\mathbb{F}_q)$-orbits in $\mathscr{C}$ such that the normalizers of $(H_\alpha, \mathcal{L}_\alpha)$ are the inner forms $G'^\alpha \subset G$ of G' (see Definition 6.5 and Proposition 6.12).

By Definition 2.13, the set $L(\mathscr{C})$ consists of all irreducible representations ρ of $G(\mathbb{F}_q)$ such that the function[16] $t_{\mathcal{L}_\alpha}$ acts nontrivially in the representation $\rho\big|_{G'^\alpha(\mathbb{F}_q)}$ for some $\alpha \in H^1(\mathbb{F}_q, G')$. On the other hand, Proposition 6.13 shows that $t_{e_{H,\mathcal{L}}} = t_{\mathrm{ind}_{G'}^G e'_{\mathcal{L}}}$ is equal to the character of the representation $\bigoplus_{\alpha \in H^1(\mathbb{F}_q, G')} \mathrm{Ind}_{G'^\alpha(\mathbb{F}_q)}^{G(\mathbb{F}_q)} t_{\mathcal{L}_\alpha}$ of $G(\mathbb{F}_q)$. In view of Frobenius reciprocity and the definition of $L(e_{H,\mathcal{L}})$, this implies that $L(\mathscr{C}) = L(e_{H,\mathcal{L}})$ and completes the proof of Proposition 9.1(b).

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[16] Here we view $t_{\mathcal{L}_\alpha}$ as a conjugation-invariant function on the finite group $G'^\alpha(\mathbb{F}_q)$.

written. Last but not least, I am grateful to the anonymous referee for suggesting multiple improvements and corrections.

Appendix A: Serre duality and bi-extensions

In this appendix, which can (for the most part) be read independently of the rest of the paper, we recall the classical Serre duality theory [3,34] for connected commutative unipotent groups, explain how to extend this theory to the case where the commutativity assumption is dropped (following a suggestion of Drinfeld), and establish a number of technical results on Serre duality and skewsymmetric bi-extensions that are used in the main body of the text. Our presentation closely follows [16], and we verify a few of the statements conjectured there.

A.1 Prologue

If G is an algebraic group over a field k and ℓ is a prime different from char k, we recall that a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on G is said to be *multiplicative* if $\mu^*(\mathcal{L}) \cong \mathcal{L} \boxtimes \mathcal{L}$, where $\mu : G \times_k G \longrightarrow G$ is the multiplication morphism. This notion is a natural geometrization of the notion of a homomorphism $\Gamma \longrightarrow \overline{\mathbb{Q}}_\ell^\times$, where Γ is an abstract group. In the purely algebraic setting, the set of all such homomorphisms is itself an abelian group, and this observation is useful in the character theory of finite groups. It is natural to ask whether this statement has a geometric analogue.

In particular, we would like to construct a "moduli space" of multiplicative $\overline{\mathbb{Q}}_\ell$-local systems on G. We assume that G is connected: otherwise local systems on G have nontrivial automorphisms, and there is no convenient way to "rigidify" them. Moreover, if we want this moduli space to be something resembling an algebraic group as well, it is not hard to see [10] that G must be unipotent. Next, if G is a unipotent group over a field of characteristic 0, then every local system on G is constant, so we will assume that char $k = p > 0$.

In this case, fix an injection of groups $\psi : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \overline{\mathbb{Q}}_\ell^\times$. It identifies $\mathbb{Q}_p/\mathbb{Z}_p$ with the group of roots of unity in $\overline{\mathbb{Q}}_\ell^\times$ whose order is a power of p, and one easily checks (see Lemma 7.3) that every multiplicative $\overline{\mathbb{Q}}_\ell$-local system on G comes from a multiplicative $\mathbb{Q}_p/\mathbb{Z}_p$-torsor on G (defined in an obvious manner). This observation allows us to have a more natural theory which is independent of ℓ.

Next, even for connected G, multiplicative $\mathbb{Q}_p/\mathbb{Z}_p$-torsors on G are still not rigid, because being multiplicative is only a property. To rigidify the situation, we must look at multiplicative $\mathbb{Q}_p/\mathbb{Z}_p$-torsors $\mathcal{E}$ on G equipped with a trivialization of the pullback $1^*\mathcal{E}$, where $1 : \operatorname{Spec} k \longrightarrow G$ is the multiplicative identity. Giving such data is equivalent to giving a central extension of G by the discrete group $\mathbb{Q}_p/\mathbb{Z}_p$ in the category of group schemes over k. This is proved in [26]. We find it more natural, and technically much more convenient, to work with central extensions of group schemes rather than multiplicative local systems or torsors. Therefore the results of this appendix will usually be phrased in the language of (bi-)extensions.

Finally, we recall (see Remark A.10) that the "moduli space" of central extensions of G by $\mathbb{Q}_p/\mathbb{Z}_p$ can only be canonically defined as a *perfect* scheme.

A.2 Organization

The first half of the appendix is devoted to the Serre duality[17] for connected *commutative* perfect unipotent groups, the idea of which goes back to [34]. In Sect. 9.6, we provide some background on perfect schemes, perfect group schemes, and the perfectization functor, following [16,19]. In Sect. 9.6, we define perfect quasi-algebraic groups and perfect unipotent groups. In Sect. 9.6, we recall the main statement of the classical Serre duality theory following [3]. In Sect. 9.6, we recall Mumford's notion [33] of a bi-extension, and in Sect. 9.6 we relate it to the notion of a "bimultiplicative torsor". In Sect. 9.6, we relate Serre duality for connected commutative unipotent groups over finite fields to Pontryagin duality for finite p-groups. In Sect. 9.6, we prove a result on bi-extensions of commutative connected unipotent groups which is used in the study of admissible pairs. Finally, in Sect. 9.6 we prove the existence of "almost Lagrangian" subgroups with respect to a skewsymmetric bi-extension (defined in Sect. 9.6) of a commutative unipotent group by $\mathbb{Q}_p/\mathbb{Z}_p$, under suitable additional assumptions, cf. [16].

The second half of the appendix discusses Serre duality for *noncommutative* groups. In Sect. 9.6, we define the Serre dual of any connected perfect unipotent group, and in Sects. 9.6–9.6 we establish the geometric analogues of certain standard constructions and results on 1-dimensional characters of abstract groups.

A.3 Perfect schemes and group schemes

Fix a prime p. Let us recall that a scheme S in characteristic p, i.e., such that p annihilates the structure sheaf $\mathcal{O}_S$ of S, is said to be *perfect* if the morphism $\mathcal{O}_S \longrightarrow \mathcal{O}_S$, given by $f \longmapsto f^p$ on the local sections of $\mathcal{O}_S$, is an isomorphism of sheaves. In particular, a commutative ring A of characteristic p is perfect [19] if and only if Spec A is a perfect scheme.

Let $\mathfrak{Sch}_p$ denote the category of all $\mathbb{F}_p$-schemes, and let $\mathfrak{Perf}_p$ be the full subcategory of $\mathfrak{Sch}_p$ formed by perfect schemes. The inclusion functor $\mathfrak{Perf}_p \hookrightarrow \mathfrak{Sch}_p$ has a right adjoint which we will call the *perfectization functor* and will denote by $X \longmapsto X_{perf}$. We note that this functor was constructed by Greenberg in [19], who denotes it by $X \longmapsto X^{1/p^\infty}$, and calls X^{1/p^∞} the *perfect closure* of X.

Next let k be a perfect field of characteristic p, let $\mathfrak{Sch}_k$ be the category of k-schemes and $\mathfrak{Perf}_k$ the full subcategory consisting of perfect schemes. The natural morphism $(\mathrm{Spec}\, k)_{perf} \longrightarrow \mathrm{Spec}\, k$ is an isomorphism, so for any $X \in \mathfrak{Sch}_k$, the perfectization X_{perf} is automatically a scheme over k (if k is not perfect, then X_{perf} is a scheme over the perfect closure of k). Hence, $X \longmapsto X_{perf}$ can be upgraded to a functor $\mathfrak{Sch}_k \longrightarrow \mathfrak{Perf}_k$, which is also right adjoint to the natural inclusion.

Remark A.5 If A and B are perfect k-algebras, so is their tensor product $A \otimes_k B$. Indeed the p-th power homomorphism $A \otimes_k B \longrightarrow A \otimes_k B$, $x \longmapsto x^p$, is the tensor product of the corresponding homomorphisms $A \longrightarrow A$ and $B \longrightarrow B$. It follows

[17] Not to be confused with Serre duality in the theory of cohomology of coherent sheaves.

that the product of two perfect schemes over k is perfect, so the inclusion functor $\mathfrak{Perf}_k \hookrightarrow \mathfrak{Sch}_k$ preserves products. Hence, a group object in the category $\mathfrak{Perf}_k$ is automatically a group scheme over k in the usual sense, which is perfect as a scheme. In particular, the term "perfect group scheme over k" is unambiguous.

Remark A.6 On the other hand, the perfectization functor $\mathfrak{Sch}_k \longrightarrow \mathfrak{Perf}_k$ preserves limits by abstract nonsense (because it has a left adjoint). In particular, if G is a group scheme over k, then G_{perf} becomes a perfect group scheme over k.

A.4 Perfect unipotent groups

Let us fix a perfect field k of characteristic $p > 0$. A perfect scheme Y over k is said to be *of quasi-finite type over* k if it is isomorphic to X_{perf} for a scheme X of finite type over k. We define a *quasi-algebraic group* over k to be a perfect group scheme such that the underlying scheme is of quasi-finite type over k.

The next result is not strictly necessary for what follows, but we find it to be at least psychologically helpful.

Lemma A.7 *If G is an affine quasi-algebraic group over k, then G is isomorphic to the perfectization of an affine algebraic group over k.*

Proof In view of Remark A.5, we have $G = \operatorname{Spec} A$, where A is a commutative Hopf algebra over k which is perfect as a ring. By assumption, there exists a finitely generated k-subalgebra $B \subset A$ such that A is the perfect closure [19] of B. Every coalgebra over a field is the filtered union of its finite dimensional sub-coalgebras, so there is a finitely generated Hopf subalgebra $B' \subset A$ such that $B \subset B'$. Then, A is the perfect closure of B' as well, and $G' = \operatorname{Spec} B'$ is an affine algebraic group over k (because A is reduced), and $G \cong G'_{perf}$, as desired. □

Definition A.8 A *perfect unipotent group* over k is a perfect group scheme over k which is isomorphic to the perfectization of a unipotent algebraic group over k.

The two basic examples of perfect unipotent groups over k are the discrete group $\mathbb{Z}/p\mathbb{Z}$ and the perfectization $\mathbb{G}_{a,\,perf}$ of the additive group $\mathbb{G}_a$. If $k = \overline{k}$, then every connected perfect unipotent group over k has a finite filtration by closed normal subgroups with successive subquotients isomorphic to $\mathbb{G}_{a,\,perf}$.

We denote by $\mathfrak{cpu}_k$ the category of all commutative perfect unipotent groups over k, and by $\mathfrak{cpu}_k^\circ \subset \mathfrak{cpu}_k$ the full subcategory formed by connected group schemes. It is not hard to see that $\mathfrak{cpu}_k$ is an abelian category; in particular, for a morphism $f : G \longrightarrow H$ in $\mathfrak{cpu}_k$, we can talk about the kernel, $\operatorname{Ker} f$, of f, and we have the notion of an exact sequence in $\mathfrak{cpu}_k$. Moreover, $\mathfrak{cpu}_k^\circ$ is an exact subcategory of $\mathfrak{cpu}_k$.

A.5 Classical Serre duality

We continue to work over a fixed perfect field k of characteristic $p > 0$. If $G \in \mathfrak{cpu}_k^\circ$, we define a contravariant functor

$$G^* : \mathfrak{Sch}_k \longrightarrow \{\text{abelian groups}\}, \qquad S \longmapsto \operatorname{Ext}^1_S(G \times_k S, \mathbb{Q}_p/\mathbb{Z}_p), \qquad \text{(A.1)}$$

where Ext^1 denotes the first Ext group computed in the category of commutative group schemes over S, and $\mathbb{Q}_p/\mathbb{Z}_p$ is viewed as a discrete group scheme over S. We call this functor the *Serre dual* of the group G.

The idea of this construction goes back to Serre's article [34]. However, in the form needed for our purposes, the duality theory appears to be due to Begueri:

Theorem A.9 [3] *The restriction of the functor G^* to the subcategory $\mathfrak{Perf}_k$ is representable by an object of $\mathfrak{cpu}_k^\circ$, which is also denoted by G^*. Moreover, the functor $G \longmapsto G^*$ is an exact anti-auto-equivalence of the category $\mathfrak{cpu}_k^\circ$.*

Remark A.10 If G is a connected commutative unipotent group over k in the usual sense, the natural morphism $G^* \longrightarrow (G_{perf})^*$ is an isomorphism of functors on $\mathfrak{Sch}_k$. On the other hand, as explained, e.g., in [10], the functor G^* is not representable[18] on the whole category $\mathfrak{Sch}_k$ already for $G = \mathbb{G}_a$. This is the reason for working with perfect group schemes in the context of Serre duality.

A.6 Bi-extensions

The notion of a bi-extension of group schemes was discovered by Mumford in [33], and later generalized by Grothendieck in SGA 7-1. It can be formulated in several equivalent ways; the following approach will be convenient for us. Let G_1, G_2 be group schemes over a field k, and let A be a commutative group scheme over k. A *bi-extension* of (G_1, G_2) by A is a scheme E over k, equipped with an action of A and a morphism $\pi : E \longrightarrow G_1 \times_k G_2$ which makes E an A-torsor over $G_1 \times_k G_2$, together with the following additional structures.

(a) Choices of sections of π along $\{1\} \times G_2$ and $G_1 \times \{1\}$, by means of which the "slices" $\pi^{-1}(\{1\} \times G_2)$ and $\pi^{-1}(G_1 \times \{1\})$ will be identified with $A \times_k G_2$ and $G_1 \times_k A$, respectively, where 1 denotes the unit in G_1 or G_2.
(b) A morphism $\bullet_1 : E \times_{G_2} E \longrightarrow E$ which makes E a group scheme over G_2 and makes π a central extension of $G_1 \times_k G_2$, viewed as a group scheme over G_2, by $A \times_k G_2$, in a way compatible with the identification $A \times_k G_2 \cong \pi^{-1}(\{1\} \times G_2)$.
(c) A morphism $\bullet_2 : E \times_{G_1} E \longrightarrow E$ which makes E a group scheme over G_1 and makes π a central extension of $G_1 \times_k G_2$, viewed as a group scheme over G_1, by $G_1 \times_k A$, in a way compatible with the identification $G_1 \times_k A \cong \pi^{-1}(G_1 \times \{1\})$.

These data are required to satisfy the following compatibility condition: if T is any k-scheme and $e_{11}, e_{12}, e_{21}, e_{22} \in E(T) = \mathrm{Hom}_{k-schemes}(T, E)$, then

$$(e_{11} \bullet_2 e_{12}) \bullet_1 (e_{21} \bullet_2 e_{22}) = (e_{11} \bullet_1 e_{21}) \bullet_2 (e_{12} \bullet_1 e_{22})$$

whenever both sides of this equality are defined, i.e., whenever

$$\pi(e_{11}) = (g_1, g_2), \quad \pi(e_{12}) = (g_1, g_2'), \quad \pi(e_{21}) = (g_1', g_2), \quad \pi(e_{22}) = (g_1', g_2')$$

for some $g_1, g_1' \in G_1(T)$ and $g_2, g_2' \in G_2(T)$.

[18] However, it is ind-representable: see the "Appendix" on Serre duality in [10].

Definition A.11 The notion of an *isomorphism* of bi-extensions is defined in the obvious way, and bi-extensions of (G_1, G_2) by A form a groupoid which we denote by $\underline{\text{Bi-ext}}(G_1, G_2; A)$. (It is even a strictly commutative Picard groupoid, see [16], but we will not use this fact.) A *trivial* bi-extension is one which is isomorphic to $A \times_k G_1 \times_k G_2$ equipped with the obvious A-action, the natural projection $A \times_k G_1 \times_k G_2 \longrightarrow G_1 \times_k G_2$, and the obvious partial group laws coming from the group law on A. Equivalently, a bi-extension E as above is trivial if it has a *trivialization*, i.e., a bimultiplicative section $\sigma : G_1 \times_k G_2 \longrightarrow E$ of π, which means that for any k-scheme T and any choice of $g_1, g_1' \in G_1(T)$ and $g_2, g_2' \in G_2(T)$, we have $\sigma(g_1 g_1', g_2) = \sigma(g_1, g_2) \bullet_1 \sigma(g_1', g_2)$ and $\sigma(g_1, g_2 g_2') = \sigma(g_1, g_2) \bullet_2 \sigma(g_1, g_2')$.

Remark A.12 [16] Bi-extensions are to central extensions as bimultiplicative maps are to homomorphisms, where, for abstract groups Γ_1, Γ_2, A, we say that a map $\beta : \Gamma_1 \times \Gamma_2 \longrightarrow A$ is bimultiplicative if $\beta(\gamma_1, -)$ is a homomorphism for every fixed $\gamma_1 \in \Gamma_1$, and $\beta(-, \gamma_2)$ is a homomorphism for every fixed $\gamma_2 \in \Gamma_2$. This analogy manifests itself in many different ways.

For instance, if G is a group scheme over k and A is a commutative group scheme over k, then the group of automorphisms of any central extension of G by A is naturally isomorphic to the group $\text{Hom}(G, A)$ of morphisms of k-group schemes $G \longrightarrow A$. Consequently, for any trivial central extension of G by A, its trivializations form a torsor under $\text{Hom}(G, A)$. Similarly, if G_1 and G_2 are group schemes over k, then the group of automorphisms of any bi-extension of (G_1, G_2) by A is naturally isomorphic to the group of bi-multiplicative morphisms of k-schemes $G_1 \times_k G_2 \longrightarrow A$, and hence trivializations of any trivial bi-extension of (G_1, G_2) by A form a torsor under the latter group.

Corollary A.13 *If A is a discrete commutative group and G_1, G_2 are group schemes over k, at least one of which is a connected algebraic or quasi-algebraic group, then bi-extensions of (G_1, G_2) by A have no non-trivial automorphisms. In particular, every such bi-extension has at most one trivialization.*

Definition A.14 A bi-extension E of (G_1, G_2) by A is said to be *commutative* if the two partial group laws on E are commutative. Of course, this can only happen if G_1 and G_2 are commutative group schemes. In [33], Mumford imposes the commutativity requirement in the very definition of a bi-extension.

A.7 Bi-extensions and bimultiplicative torsors

Some of the data and the compatibility conditions that appear in the definition of a bi-extension can often be ignored. To explain this comment, let us introduce the notion of a bimultiplicative torsor, by analogy with the notion of a multiplicative torsor.

Definition A.15 In the situation above, let

$$\mu_1 : G_1 \times_k G_1 \longrightarrow G_1 \qquad \text{and} \qquad \mu_2 : G_2 \times_k G_2 \longrightarrow G_2$$

be the multiplication morphisms, and let

$$\mathrm{pr}_{13}, \mathrm{pr}_{23} : G_1 \times_k G_1 \times_k G_2 \longrightarrow G_1 \times_k G_2$$

and

$$\mathrm{pr}_{12}, \mathrm{pr}_{13} : G_1 \times_k G_2 \times_k G_2 \longrightarrow G_1 \times_k G_2$$

be the projections. An A-torsor $\mathcal{E}$ on $G_1 \times_k G_2$ is said to be *bimultiplicative* if

$$(\mu_1 \times \mathrm{id}_{G_2})^*(\mathcal{E}) \cong \mathrm{pr}_{13}^*(\mathcal{E}) \otimes \mathrm{pr}_{23}^*(\mathcal{E}) \qquad \text{as } A - \text{torsors on } G_1 \times_k G_1 \times_k G_2$$

and

$$(\mathrm{id}_{G_1} \times \mu_2)^*(\mathcal{E}) \cong \mathrm{pr}_{12}^*(\mathcal{E}) \otimes \mathrm{pr}_{13}^*(\mathcal{E}) \qquad \text{as } A - \text{torsors on } G_1 \times_k G_2 \times_k G_2.$$

It is clear that a bi-extension E of (G_1, G_2) by A determines a bimultiplicative A-torsor on $G_1 \times_k G_2$ by forgetting the partial group laws on E. The proof of the following analogue of Lemma 7.2 is straightforward and is therefore omitted.

Lemma A.16 *Let G_1 and G_2 be connected algebraic or quasi-algebraic groups over a field k, and let A be an abstract commutative group, viewed as a discrete group scheme over k. Consider the groupoid $\mathscr{G}$ of bimultiplicative A-torsors $\mathcal{E}$ on $G_1 \times_k G_2$ equipped with a trivialization of $(1 \times 1)^*\mathcal{E}$. Then, the forgetful functor $\underline{\text{Bi-ext}}(G_1, G_2; A) \longrightarrow \mathscr{G}$ is an equivalence of categories, and both groupoids are discrete. Furthermore, if G_1 and G_2 are commutative, then every bi-extension of (G_1, G_2) by A is automatically commutative as well.*

A.8 Serre duality and Pontryagin duality

In the remainder of the appendix, the only bi-extensions that we consider will be bi-extensions of connected unipotent groups by $\mathbb{Q}_p/\mathbb{Z}_p$. Let us fix a perfect field k of characteristic $p > 0$ and an object $G \in \mathfrak{cpu}_k^\circ$. If $G^* \in \mathfrak{cpu}_k^\circ$ is the Serre dual (Sect. 9.6) of G, then, by definition, we have a central extension $\mathcal{U}$ of $G \times_k G^*$ by $\mathbb{Q}_p/\mathbb{Z}_p$ in the category of group schemes over G^*, which is universal in the obvious sense. The following claim is readily verified:

Lemma A.17 *Let A be a perfect group scheme over k, and let $f : A \longrightarrow G^*$ be an arbitrary morphism of k-schemes. Then, f is a morphism of group schemes if and only if $(\mathrm{id}_G \times f)^*\mathcal{U}$ has a structure of a bi-extension of (G, A) by $\mathbb{Q}_p/\mathbb{Z}_p$, compatible with its structure of a central extension of $G \times_k A$ by $\mathbb{Q}_p/\mathbb{Z}_p$ as an A-group scheme.*

In particular, we see that $\mathcal{U}$ itself comes from a (unique) bi-extension of (G, G^*) by $\mathbb{Q}_p/\mathbb{Z}_p$, which we will also denote by $\mathcal{U}$. Furthermore, morphisms of k-group schemes $f : A \longrightarrow G^*$ correspond bijectively with bi-extensions of (G, A) by $\mathbb{Q}_p/\mathbb{Z}_p$.

In this subsection, we assume that $k = \mathbb{F}_q$ is finite. Then, $G(\mathbb{F}_q)$ is a finite abelian p-group, and we can consider its Pontryagin dual, which can be canonically defined as $G(\mathbb{F}_q)^* = \mathrm{Hom}(G(\mathbb{F}_q), \mathbb{Q}_p/\mathbb{Z}_p)$. It is natural to ask about the relationship between $G^*(\mathbb{F}_q)$ and $G(\mathbb{F}_q)^*$. As we will see shortly, these two groups are canonically isomorphic. First, note that we have an analogue of the sheaves-to-functions correspondence (Sect. 4.2) in the context of $\mathbb{Q}_p/\mathbb{Z}_p$-torsors. Namely, isomorphism classes of $\mathbb{Q}_p/\mathbb{Z}_p$-torsors on $\operatorname{Spec} \mathbb{F}_q$ are in natural bijection with continuous homomorphisms $\phi : \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$. In turn, such homomorphisms are in bijection with elements of $\mathbb{Q}_p/\mathbb{Z}_p$, via $\phi \longmapsto \phi(F_q)$, where $F_q \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ is the geometric Frobenius. Now if X is an arbitrary scheme over $\mathbb{F}_q$ and $\mathcal{E}$ is a $\mathbb{Q}_p/\mathbb{Z}_p$-torsor over X, we obtain a functor $t_{\mathcal{E}} : X(\mathbb{F}_q) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$, defined by sending $x \in X(\mathbb{F}_q)$ to the element of $\mathbb{Q}_p/\mathbb{Z}_p$ corresponding to the $\mathbb{Q}_p/\mathbb{Z}_p$-torsor $x^*\mathcal{E}$ over $\operatorname{Spec} \mathbb{F}_q$.

Proposition A.18 *If G is a perfect connected commutative unipotent group over $\mathbb{F}_q$, G^* is its Serre dual, and $\mathcal{U}$ is the universal bi-extension of (G, G^*) by $\mathbb{Q}_p/\mathbb{Z}_p$, then the map*

$$G(\mathbb{F}_q) \times G^*(\mathbb{F}_q) = (G \times G^*)(\mathbb{F}_q) \xrightarrow{t_{\mathcal{U}}} \mathbb{Q}_p/\mathbb{Z}_p$$

is a perfect pairing, i.e., it is bi-additive and identifies $G^*(\mathbb{F}_q)$ with $G(\mathbb{F}_q)^*$.

Proof The only nontrivial part is to show that every $\chi \in G(\mathbb{F}_q)^*$ can be represented as $x \longmapsto t_{\mathcal{U}}(x, y)$ for some $y \in G^*(\mathbb{F}_q)$. By Lang's theorem [28], we have an exact sequence of perfect commutative unipotent groups over $\mathbb{F}_q$,

$$0 \longrightarrow G(\mathbb{F}_q) \longrightarrow G \xrightarrow{L} G \longrightarrow 0, \tag{A.2}$$

where $L : G \longrightarrow G, g \longmapsto \Phi(g)g^{-1}$ is the Lang isogeny.[19] Choose a homomorphism $\chi : G(\mathbb{F}_q) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$. The pushforward of (A.2) by χ is an extension

$$0 \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \widetilde{G}_\chi \longrightarrow G \longrightarrow 0,$$

which defines an element $y \in G^*(\mathbb{F}_q)$. One checks easily ([14], *Sommes. trig.*) that the function $t_{\widetilde{G}_\chi} : G(\mathbb{F}_q) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$, defined by $\widetilde{G}_\chi$ (viewed as a $\mathbb{Q}_p/\mathbb{Z}_p$-torsor over G), is equal to χ. Hence, $\chi(x) = t_{\mathcal{U}}(x, y)$ for all $x \in G(\mathbb{F}_q)$. □

A.9 Canonical pairing associated with a bi-extension

Let us fix two objects $G_1, G_2 \in \mathfrak{cpu}_k^\circ$ and a morphism $f : G_1 \longrightarrow G_2$. Since $(G_2^*)^*$ is canonically identified with G_2 by Theorem A.9, we see from Lemma A.17 that f corresponds to a bi-extension of (G_2^*, G_1) by $\mathbb{Q}_p/\mathbb{Z}_p$. Consider the dual morphism $f^* : G_2^* \longrightarrow G_1^*$. The kernels $\operatorname{Ker} f$ and $\operatorname{Ker} f^*$ are possibly disconnected objects of $\mathfrak{cpu}_k$.

Until the end of the subsection we assume that k is an algebraically closed field of characteristic $p > 0$. Then, the groups of connected components $\pi_0(\operatorname{Ker} f)$ and

[19] Here, $\Phi : G \longrightarrow G$ is the absolute Frobenius morphism, defined as the identity map on the underlying set of G, and the map $f \longmapsto f^q$ on local sections of the structure sheaf $\mathscr{O}_G$ of G.

$\pi_0(\text{Ker } f^*)$ are finite discrete abelian p-groups. Our goal is to define, following [16], a canonical nondegenerate pairing of abelian p-groups

$$B_f : \pi_0(\text{Ker } f^*) \times \pi_0(\text{Ker } f) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p, \tag{A.3}$$

i.e., a bi-additive map inducing an isomorphism of abelian groups

$$\pi_0(\text{Ker } f^*) \longrightarrow \text{Hom}(\pi_0(\text{Ker } f), \mathbb{Q}_p/\mathbb{Z}_p). \tag{A.4}$$

To define (A.3), note that since $f\,\big|_{\text{Ker } f} = 0$, the restriction $E\big|_{G_2^* \times (\text{Ker } f)}$ is a trivial bi-extension. Since G_2^* is connected, there is only one trivialization of $E\big|_{G_2^* \times (\text{Ker } f)}$ by Corollary A.13. Similarly, the bi-extension $E\big|_{(\text{Ker } f^*) \times G_1}$ has a unique trivialization. Thus, we obtain *two* trivializations of $E\,\big|_{(\text{Ker } f^*) \times (\text{Ker } f)}$, which, by Remark A.12, must "differ" by a bi-additive morphism $(\text{Ker } f^*) \times (\text{Ker } f) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete, the latter factors through a bi-additive map (A.3).

Proposition A.19 *The pairing (A.3) we just defined is nondegenerate.*

To prove this proposition, it suffices to show the injectivity of the induced homomorphism (A.4), for then we can apply the same argument replacing f with f^*. Thus, let $g \in (\text{Ker } f^*)(k)$ be such that $B_f(\overline{g}, x) = 0$ for all $x \in \pi_0(\text{Ker } f)$, where $\overline{g}$ denotes the image of g in $\pi_0(\text{Ker } f^*)$. We must show that $g \in (\text{Ker } f^*)^\circ$. To this end, we need to obtain a more concrete description of $(\text{Ker } f^*)^\circ$.

Consider an arbitrary extension of commutative group schemes over k,

$$0 \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \widetilde{G_2} \xrightarrow{\eta} G_2 \longrightarrow 0.$$

Let $t \in G_2^*(k)$ denote the corresponding element, and assume that $t \in \text{Ker } f^*$, which means that there exists a morphism $\widetilde{f} : G_1 \longrightarrow \widetilde{G_2}$ such that $\eta \circ \widetilde{f} = f$. Since G_1 is connected and $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete, $\widetilde{f}$ is unique. Moreover, it takes $\text{Ker } f$ to $\mathbb{Q}_p/\mathbb{Z}_p$, so we obtain an induced homomorphism $t' : \pi_0(\text{Ker } f) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$.

The following claim is simply a matter of unwinding the definitions:

Lemma A.20 *We have $t'(x) = B_f(\bar{t}, x)$ for all $x \in \pi_0(\text{Ker } f)$.*

In view of this result, it is clear that Proposition A.19 follows from

Lemma A.21 *In the situation above, the following are equivalent:*

(i) *t belongs to the neutral connected component* $(\text{Ker } f^*)^\circ$;
(ii) *the restriction of t to the image of f is trivial*;
(iii) *the homomorphism t' is identically zero.*

Proof The equivalence between conditions (ii) and (iii) is obvious. Indeed, since we are already assuming that $f^*(t) = 0$, it is clear that the restriction of t to the image of f is trivial if and only if the homomorphism $\widetilde{f}$ defined above vanishes on $\text{Ker } f$, which is in turn equivalent to the vanishing of t'.

For the equivalence between (i) and (ii), note first that the composition of f with the quotient map $G_2 \twoheadrightarrow G_2/f(G_1)$ equals zero, hence the composition

$$(G_2/f(G_1))^* \hookrightarrow G_2^* \xrightarrow{f^*} G_1^*$$

also equals zero. Since $(G_2/f(G_1))^*$ is connected, we see that (ii) implies (i). The argument of the previous paragraph shows that we have an exact sequence

$$0 \longrightarrow (G_2/f(G_1))^*(k) \longrightarrow (\mathrm{Ker}\, f^*)(k) \longrightarrow \mathrm{Hom}(\pi_0(\mathrm{Ker}\, f), \mathbb{Q}_p/\mathbb{Z}_p),$$

and the last group is finite. Thus, $(G_2/f(G_1))^*$ maps isomorphically onto $(\mathrm{Ker}\, f^*)^\circ$, whence (i) also implies (ii) and the proof of the lemma is complete. □

A.10 Symmetric and skewsymmetric bi-extensions

Let k be a perfect field of characteristic $p > 0$, let $G \in \mathfrak{cpu}_k^\circ$, and let $E \xrightarrow{\pi} G \times_k G$ be a bi-extension of (G, G) by $\mathbb{Q}_p/\mathbb{Z}_p$. As explained above, E determines (and is determined by) a homomorphism $f : G \longrightarrow G^*$. We will think of its dual, f^*, as a homomorphism $G \longrightarrow G^*$ as well, using the canonical identification $(G^*)^* \cong G$ (see Theorem A.9).

Definition A.22 The bi-extension E is said to be *symmetric* if $\tau^*(E) \cong E$, where $\tau : G \times_k G \longrightarrow G \times_k G$ is the transposition of the two factors. Equivalently, E is symmetric if $f^* = f$. We say that E is *skewsymmetric*[20] if the restriction of E to the diagonal in $G \times_k G$ is a trivial $\mathbb{Q}_p/\mathbb{Z}_p$-torsor.

Remark A.23 In general, skewsymmetry is not a property of a bi-extension; rather, one needs to define an extra structure, called a *skewsymmetry constraint* in [16]. However, since G is connected and $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete, being skewsymmetric becomes a property in our situation (similarly to Lemma A.16).

Remark A.24 If the bi-extension E considered above is skewsymmetric, it is easy to check that $f = -f^*$. The converse statement holds if (and only if) $p > 2$.

Lemma A.25 ("Parity change") *In the situation of this subsection, assume that k is algebraically closed and consider the canonical pairing (A.3) defined in Sect. 9.6. If E is symmetric (respectively, skewsymmetric), so that, in particular,* $\mathrm{Ker}\, f = \mathrm{Ker}\, f^*$, *then B_f is alternating*[21] *(respectively, symmetric).*

This lemma is proved at the end of the subsection. In the skewsymmetric case, we can in fact prove a more precise result (see Lemma A.26). To state it, we introduce some notation. Because we are working in the commutative situation, we will denote the partial group laws on E by $+_1$ and $+_2$, as opposed to $\bullet_1$ and $\bullet_2$.

[20] The term "alternating" might be more appropriate, but we decided to be consistent with [16].

[21] In other words, $B_f(x, x) = 0$ for all $x \in \pi_0(\mathrm{Ker}\, f) = \pi_0(\mathrm{Ker}\, f^*)$.

The group laws on G and A will also be written additively. Since E is either symmetric or skewsymmetric, we have $\operatorname{Ker} f = \operatorname{Ker} f^*$, and thus we have unique trivializations $\lambda : (\operatorname{Ker} f) \times_k G \longrightarrow E$ and $\rho : G \times_k (\operatorname{Ker} f) \longrightarrow E$ (cf. Definition A.11), as in Sect. 9.6.

With this notation, the bi-additive map (A.3) is explicitly defined by

$$\lambda(x, y) = B_f(x, y) + \rho(x, y) \qquad \forall\, x, y \in \operatorname{Ker} f.$$

If, in addition, E is skewsymmetric, then, by Definition A.22, we have a unique morphism $\widetilde{\Delta} : G \longrightarrow E$ such that $\widetilde{\Delta}(0) = 0$ and $\pi \circ \widetilde{\Delta}$ is the diagonal morphism $\Delta : G \longrightarrow G \times_k G$. In this case, we define a morphism $q : \operatorname{Ker} f \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ by

$$\lambda(x, x) = q(x) + \widetilde{\Delta}(x) \qquad \forall\, x \in \operatorname{Ker} f.$$

Lemma A.26 *Assume that k is algebraically closed, and that E is skewsymmetric. Then, q descends to a map $q : \pi_0(\operatorname{Ker} f) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ satisfying the following identities:*

(1) $q(nx) = n^2 \cdot q(x)$ *for all* $n \in \mathbb{Z}$ *and all* $x \in \pi_0(\operatorname{Ker} f)$*; and*
(2) $B_f(x, y) = q(x + y) - q(x) - q(y)$ *for all* $x, y \in \pi_0(\operatorname{Ker} f)$.

Hence, q is a "nondegenerate $\mathbb{Q}_p/\mathbb{Z}_p$-valued quadratic form" on $\pi_0(\operatorname{Ker} f)$.

Proof (1) Given $n \in \mathbb{Z}$, let $e \longmapsto n *_1 e$ and $e \longmapsto n *_2 e$ denote the "multiplication by n" action on E obtained from the first and the second partial group laws, respectively. Consider the identity $\widetilde{\Delta}(n \cdot x) = n *_1 (n *_2 \widetilde{\Delta}(x))$ for $x \in G$. It holds for $x = 0$ by construction, and becomes true after we apply π to both sides. Hence, this identity holds for all x, by continuity. On the other hand, since λ is bi-additive, we have $\lambda(n \cdot x, n \cdot x) = n *_1 (n *_2 \lambda(x, x))$ for all $x \in \operatorname{Ker} f$, proving (1).

(2) We take $x, y \in (\operatorname{Ker} f)(k)$. By construction, we have

$$\lambda(x + y, x + y) = q(x + y) + \widetilde{\Delta}(x + y). \tag{A.5}$$

On the other hand, using the bi-additivity of λ, we find

$$\begin{aligned}\lambda(x + y, x + y) &= (\lambda(x, x) +_2 \lambda(x, y)) +_1 (\lambda(y, x) +_2 \lambda(y, y)) \\ &= q(x) + q(y) + \big(\widetilde{\Delta}(x) +_2 \lambda(x, y)\big) +_1 \big(\lambda(y, x) +_2 \widetilde{\Delta}(y)\big)\end{aligned} \tag{A.6}$$

Comparing (A.5) and (A.6), we see that (2) reduces to verifying the identity

$$\widetilde{\Delta}(x + y) = \big(\widetilde{\Delta}(x) +_2 \rho(x, y)\big) +_1 \big(\lambda(y, x) +_2 \widetilde{\Delta}(y)\big).$$

However, the last identity is meaningful for all $(x, y) \in G(k) \times (\operatorname{Ker} f)(k)$. Moreover, it is satisfied by continuity, because it holds whenever $x = 0$, because G is connected, and because it becomes true after we apply π to both sides. □

Proof of Lemma A.25 If E is skewsymmetric, the symmetry of B_f follows from Lemma A.26(2). Now assume that E is symmetric and use the notation introduced above. Let $\widetilde{\tau} : E \xrightarrow{\simeq} E$ be the isomorphism of k-schemes which induces an isomorphism of bi-extensions between $\tau^*(E)$ and E. In other words, $\pi \circ \widetilde{\tau} = \tau \circ \pi$, and $\widetilde{\tau}$ interchanges the two partial group laws on E. Uniqueness of trivializations implies that $\lambda(x, y) = \widetilde{\tau}(\rho(y, x))$ for all $(x, y) \in (\operatorname{Ker} f)(k) \times G(k)$. On the other hand, by continuity, $\widetilde{\tau}$ is the identity above the diagonal in $G \times_k G$. Thus, $\lambda(x, x) = \rho(x, x)$ for all $x \in (\operatorname{Ker} f)(k)$, which means that $B_f(x, x) = 0$, i.e., B_f is alternating.

A.11 Lagrangian subgroups

Every finite dimensional vector space V equipped with an alternating bilinear form ω admits a Lagrangian subspace, i.e., a subspace $L \subset V$ which coincides with its annihilator with respect to ω. In particular, the rank of ω is even.

Let us study the geometric analogue of this statement. Fix $G \in \mathfrak{cpu}_k^\circ$ and a skew-symmetric bi-extension E of (G, G) by $\mathbb{Q}_p/\mathbb{Z}_p$. Let $f : G \longrightarrow G^*$ denote the corresponding morphism. The *rank* of E is defined to be $\operatorname{rk} E = \dim f(G)$.

Definition A.27 An *almost Lagrangian* subgroup of G with respect to E is a closed connected subgroup $L \subset G$ such that $L \subset f^{-1}(\operatorname{Ann}(L))$, and L has finite index in $f^{-1}(\operatorname{Ann}(L))$, where $\operatorname{Ann}(L) = \operatorname{Ker}(G^* \twoheadrightarrow L^*) \subset G^*$ is the annihilator of L in G^*. We say that L is simply *Lagrangian* if $f^{-1}(\operatorname{Ann}(L)) = L$.

Note that if G has an almost Lagrangian subgroup with respect to E, then $\operatorname{rk} E$ must be even. However, $\operatorname{rk} E$ is not always even in the geometric setup: for instance, there exist nontrivial skewsymmetric bi-extensions of $(\mathbb{G}_{a,\,perf}, \mathbb{G}_{a,\,perf})$ by $\mathbb{Q}_p/\mathbb{Z}_p$. This is the first obstruction to the existence of almost Lagrangian subgroups. If the base field k is algebraically closed, this is the only obstruction. The non-algebraically closed case appears to be more intricate. Part (a) of the following result was conjectured in [16].

Proposition A.28 (a) *If k is algebraically closed and* $\operatorname{rk} E$ *is even, G has an almost Lagrangian subgroup with respect to E.*

(b) *Allow k to be merely perfect, and suppose that E is nondegenerate in the strong*[22] *sense, i.e., $f : G \longrightarrow G^*$ is an isomorphism. Then,* $\dim G$ *is even and every almost Lagrangian subgroup of G with respect to E is Lagrangian.*

Before proving this proposition we need to understand Serre duality more explicitly in a special case. It is well known that the objects of $\mathfrak{cpu}_k^\circ$ that are annihilated by p are isomorphic to direct sums of copies of $\mathbb{G}_{a,\,perf}$. On the other hand, we have

$$\operatorname{End}_{\mathfrak{cpu}_k^\circ}(\mathbb{G}_{a,\,perf}) \cong R := k\{\tau, \tau^{-1}\}, \tag{A.7}$$

the ring of twisted Laurent polynomials, determined by $\tau \cdot c = c^p \cdot \tau$ for $c \in k$. The isomorphism (A.7) is obtained by letting elements of k act on $\mathbb{G}_{a,\,perf}$ by dilations, and letting τ act by $x \longmapsto x^p$.

[22] The "weak" non-degeneracy condition is that f is an isogeny.

To describe the Serre duality functor on the subcategory of p-torsion objects of $\mathfrak{cpu}_k^\circ$, we only need to explain how it acts on $\mathbb{G}_{a,\,perf}$ and on endomorphisms of $\mathbb{G}_{a,\,perf}$. This was done in [10]. One can identify $\mathbb{G}_{a,\,perf}^*$ with $\mathbb{G}_{a,\,perf}$ so that the action of the Serre duality functor on endomorphisms of $\mathbb{G}_{a,\,perf}$ becomes the anti-involution $f \longmapsto f^*$ of the ring R determined by $c^* = c$ for $c \in k$, and $\tau^* = \tau^{-1}$.

Below we will prove the following purely algebraic result:

Lemma A.29 *Let k be algebraically closed, and let $f : \mathbb{G}_{a,\,perf}^2 \longrightarrow (\mathbb{G}_{a,\,perf}^2)^*$ be a morphism of k-group schemes such that $f^* = -f$. Then, there exists a nonzero morphism $\alpha : \mathbb{G}_{a,\,perf} \longrightarrow \mathbb{G}_{a,\,perf}^2$ such that $\alpha^* \circ f \circ \alpha = 0$.*

Let us explain why this lemma implies Proposition A.28.

For part (a) of the proposition, we use induction on $\dim G$. What allows us to reduce $\dim G$ is the construction of subquotients of E. Namely, if $H \subset G$ is a closed connected *isotropic* subgroup, i.e., such that $H \subset H^\perp := f^{-1}(\mathrm{Ann}(H))$, then E induces a skewsymmetric bi-extension E' of $G' := (H^\perp)^\circ/H$ by $\mathbb{Q}_p/\mathbb{Z}_p$, and if L' is an almost Lagrangian subgroup of G' with respect to E', then the preimage of L' in $H^\perp$ is an almost Lagrangian subgroup of G with respect to E. Furthermore, $\mathrm{rk}\, E'$ has the same parity as $\mathrm{rk}\, E$, and $\dim G' < \dim G$ so long as H is nontrivial.

First we reduce to the case $p \cdot G = 0$. If $p \cdot G \neq 0$, let $n \geq 2$ be the smallest integer for which $p^n \cdot G = 0$ (it exists because G is a commutative perfect unipotent group). Then, $p^{n-1} \cdot G$ is a nontrivial connected isotropic subgroup of G with respect to E, so we are done by induction on $\dim G$, as explained above.

Now assume that $p \cdot G = 0$, so that $G \cong \mathbb{G}_{a,\,perf}^d$ for some $d \in \mathbb{N}$. If $d = 1$, then $E = 0$ because $\mathrm{rk}\, E$ is even by assumption, so G is almost Lagrangian in itself. Otherwise, let $G'' \subset G$ be a closed subgroup isomorphic to $\mathbb{G}_{a,\,perf}^2$. By Lemma A.29, G'' has a nontrivial connected isotropic subgroup H with respect to $E\big|_{G'' \times_k G''}$. Then, H is also isotropic in G, and we are again done by induction.

Next we prove part (b) of the proposition. The second statement is obvious: if $L \subset G$ is any connected subgroup, then $\mathrm{Ann}(L) \subset G^*$ is also connected, whence $L^\perp = f^{-1}(\mathrm{Ann}(L))$ is connected because f is an isomorphism. Thus, L is Lagrangian if and only if it is almost Lagrangian. For the first statement, use the base change from k to an algebraic closure of k (this changes neither the non-degeneracy property of E nor $\dim G$). Now let us try to repeat the same inductive argument as above to prove that G has an almost Lagrangian subgroup with respect to E, which will imply that $\dim G$ is even. Note that if $H \subset G$ is a closed connected subgroup which is isotropic with respect to E, then the induced skewsymmetric bi-extension E' of $H^\perp/H$ by $\mathbb{Q}_p/\mathbb{Z}_p$ is also nondegenerate. Thus, the only place where we cannot repeat the same argument as above is when $G = \mathbb{G}_{a,\,perf}$.

However, we claim that $\mathbb{G}_{a,\,perf}$ cannot have nondegenerate skewsymmetric bi-extensions by $\mathbb{Q}_p/\mathbb{Z}_p$. Indeed, consider a morphism $f : \mathbb{G}_{a,\,perf} \longrightarrow \mathbb{G}_{a,\,perf}^*$ defining a skewsymmetric bi-extension. We identify $\mathbb{G}_{a,\,perf}^*$ with $\mathbb{G}_{a,\,perf}$ as explained above, so that f becomes an element of the ring $R = k\{\tau, \tau^{-1}\}$. It is easy to check that being skewsymmetric becomes the property that f can be written as a sum of elements of the form $\tau^j \cdot c - c \cdot \tau^{-j}$, where $j \in \mathbb{N}$ and $c \in k$. However, no such element is invertible in R, since all units of R are of the form $c \cdot \tau^i$, for $c \in k^\times$ and $i \in \mathbb{Z}$.

Thus, we have proved both parts of Proposition A.28.

Proof of Lemma A.29 Our argument uses dimension counting, which is why we need to assume that k is algebraically closed. Using the identification of $\mathbb{G}^*_{a,\,perf}$ with $\mathbb{G}_{a,\,perf}$, we can represent the morphism f by a two-by-two matrix

$$\begin{pmatrix} a & b \\ -b^* & d \end{pmatrix}, \qquad \text{where } a, b, d \in R \text{ and } a = -a^*,\ d = -d^*.$$

An arbitrary morphism $\alpha : \mathbb{G}_{a,\,perf} \longrightarrow \mathbb{G}^2_{a,\,perf}$ can be represented by a vector $(x, y) \in R^2$, and then the element $\alpha^* \circ f \circ \alpha \in \mathrm{End}_{\mathfrak{cpu}^\circ_k}(\mathbb{G}_{a,\,perf})$ equals

$$F(x, y) := x^* a x - y^* b^* x + x^* b y + y^* d y. \tag{A.8}$$

We must show that in this situation there exist $x, y \in R$, not both zero, such that $F(x, y) = 0$. Note that we have $F(x, y)^* = -F(x, y)$ for all $x, y \in R$.

In what follows, we will view R as a vector space over k with respect to the action of k on R by left multiplication. For each $N \in \mathbb{N}$, let R_N be the subspace of R spanned over k by $\{\tau^{-N}, \tau^{-N+1}, \dots, \tau^N\}$. It has dimension $2N + 1$.

Let R_N^{skew} denote the subset of R_N consisting of the elements z satisfying $z^* = -z$. It is not a k-subspace of R_N. However, we can identify it with a suitable k-vector space. Namely, let R_N^+ be the subspace of R spanned by $\{\tau, \tau^2, \dots, \tau^N\}$ if $p > 2$, and by $\{1, \tau, \dots, \tau^N\}$ if $p = 2$. There is a unique additive bijection $\phi : R_N^+ \xrightarrow{\simeq} R_N^{skew}$ given by $c\tau^j \longmapsto c\tau^j - \tau^{-j}c$ for $1 \le j \le N$, $c \in k$, and $c \longmapsto c$ if $p = 2$. We have $\dim R_N^+ = N$ if $p > 2$, and $\dim R_N^+ = N + 1$ if $p = 2$.

Choose $m \in \mathbb{N}$ such that $a, b, d \in R_m$. Then, the map F defined by (A.8) takes R_N^2 to R_{2N+m}^{skew}. Hence, $F' := \phi^{-1} \circ \left(F\big|_{R_N}\right)$ is a map $R_N^2 \longrightarrow R_{2N+m}^+$. This map is not quite polynomial with respect to the obvious coordinates on the k-vector spaces R_N^2 and R_{2N+m}^+, because of the equation $\tau^{-1} \cdot c = c^{1/p} \cdot \tau^{-1}$ for $c \in k$. However, for any $s \in \mathbb{N}$, we have a bijection $k^{4N+2} \xrightarrow{\simeq} R_N^2$ given by

$$(x_{-N}, \dots, x_N, y_{-N}, \dots, y_N) \longmapsto \left(\sum_i x_i^{p^s} \cdot \tau^i, \sum_j y_j^{p^s} \cdot \tau^j \right),$$

and the resulting composition $k^{4N+2} \longrightarrow R_{2N+m}^+$ is given by a polynomial map with coefficients in k if s is large enough. Since $4N + 2 > \dim_k R_{2N+m}^+$ whenever $N \ge m$, and since k is assumed to be algebraically closed, the standard theorem of the dimension of fibers of an algebraic map implies that the equation $F(x, y) = 0$ has a nonzero solution $(x, y) \in R_N^2$ for $N \ge m$, which proves the lemma. □

A.12 Noncommutative Serre duality

Let us once again fix a perfect field k of characteristic $p > 0$. Let G be a connected perfect unipotent group over k. This time we do not assume that G is commutative. The Serre dual of G can be defined, as a functor, in the same way as in Sect. 9.6, by formula (A.1), except that the right hand side has to be interpreted as the group of isomorphism classes of central extensions of $G \times_k S$ by the discrete group scheme $\mathbb{Q}_p/\mathbb{Z}_p$. The following result was conjectured by Drinfeld in [16]; the key idea of the proof is also due to him.

Proposition A.30 *If G is a connected perfect unipotent group over k, the restriction of the functor G^* to $\mathfrak{Perf}_k$ is represented by an object of $\mathfrak{cpu}_k$, which we also denote by G^*. Moreover, the natural homomorphism $(G^{ab})^* \longrightarrow G^*$, induced by the quotient map $G \longrightarrow G^{ab} = G/[G, G]$, identifies $(G^{ab})^*$ with $(G^*)^\circ$.*

Let us first explain why the problem is nontrivial. Naively one might expect that every central extension of G by $\mathbb{Q}_p/\mathbb{Z}_p$ restricts to a trivial extension of $[G, G]$, so that $G^* = (G^{ab})^*$ as functors. However, this is not so, as was already observed in [16]; see [26] for more details. Fortunately, as we will see below, this naive expectation only "fails by a finite amount," which allows us to get a handle on G^*.

We use the following result of [26] in a crucial way:

Theorem A.31 [26] *There exists a (unique) central extension*

$$1 \longrightarrow \Pi \longrightarrow [G, G]_{true} \longrightarrow [G, G] \longrightarrow 1, \tag{A.9}$$

where Π is a finite unipotent k-group,[23] *characterized by the properties that*

- *$[G, G]_{\text{true}}$ is connected,*
- *every central extension of G by a finite unipotent k-group splits after pullback to $[G, G]_{\text{true}}$, and*
- *the commutator morphism $G \times_k G \longrightarrow G$ lifts to $[G, G]_{\text{true}}$.*

Moreover, there exists a central extension $\widetilde{G} \xrightarrow{\pi} G$ of G by a finite unipotent k-group such that (A.9) is isomorphic to $\pi^{-1}([G, G])^\circ \longrightarrow [G, G]$. Finally, the formation of (A.9) commutes with base change to algebraic extensions of k.

The group $[G, G]_{\text{true}}$, together with the homomorphism $[G, G]_{\text{true}} \longrightarrow G$, is called the *true (étale) commutator* [16] of G, for the obvious reason.

To prove Proposition A.30, we use induction of $\dim G$. If $\dim G = 1$, then G is commutative and we can apply Theorem A.9. Assume that $\dim G > 1$ and the result holds for all connected perfect unipotent groups H over k such that $\dim H < \dim G$.

If S is any perfect scheme over k and A is an abstract (discrete) abelian group, we will write $H^2(G \times_k S, A)$ for the abelian group of isomorphism classes of central extensions of $G \times_k S$ by A in the category of group schemes over S.

[23] In other words, a finite étale unipotent group scheme over k.

Lemma A.32 *If S is a perfect scheme over k, the natural homomorphism*

$$H^2(G \times_k S, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^2([G, G]_{true} \times_k S, \mathbb{Q}_p/\mathbb{Z}_p)$$

equals zero.

Proof We may assume that S is affine. There exists $n \in \mathbb{N}$ such that $g^{p^n} = 1$ for all $g \in G$ or $[G, G]_{\text{true}}$. Hence, for any S as above, the natural homomorphisms

$$H^2(G \times_k S, \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow H^2(G \times_k S, \mathbb{Q}_p/\mathbb{Z}_p)$$

and

$$H^2([G, G]_{\text{true}} \times_k S, \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow H^2([G, G]_{\text{true}} \times_k S, \mathbb{Q}_p/\mathbb{Z}_p)$$

are isomorphisms. On the other hand, it is easy to see that since S is affine,

$$\varinjlim H^2(G \times_k S', \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} H^2(G \times_k S, \mathbb{Z}/p^n\mathbb{Z}),$$

where the inductive limit is taken over all morphisms $S \longrightarrow S'$, where S' is a perfect affine k-scheme of quasi-finite type (Sect. 9.6) over k. Hence, it suffices to prove the lemma in the case where S is of quasi-finite type over k.

Now, by the induction assumption, the functor $[G, G]^*_{\text{true}}$ is representable by an object of $\mathfrak{cpu}_k$. Given an element of $H^2(G \times_k S, \mathbb{Q}_p/\mathbb{Z}_p)$, its pullback to $[G, G]_{\text{true}} \times_k S$ defines a morphism of k-schemes $S \longrightarrow [G, G]^*_{\text{true}}$. By Theorem A.31, the induced map $S(\overline{k}) \longrightarrow [G, G]^*_{true}(\overline{k})$ is identically 0, where $\overline{k}$ is an algebraic closure of k. Since S is of quasi-finite type over k, this implies that $S \longrightarrow [G, G]^*_{\text{true}}$ is constant. □

We can now complete the proof of Proposition A.30. Consider the sequence (A.9) defined in Theorem A.31. In view of Lemma A.32, we obtain an exact sequence of functors from $\mathfrak{Perf}_k$ to the category of abelian groups,

$$0 \longrightarrow (G^{ab})^* \longrightarrow G^* \longrightarrow \text{Hom}(\Pi, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 0,$$

where $\text{Hom}(\Pi, \mathbb{Q}_p/\mathbb{Z}_p)$ is viewed as a finite unipotent k-group in the natural way.[24] Since $(G^{ab})^*$ is representable by a connected commutative perfect unipotent group over k by Theorem A.9, both statements of Proposition A.30 follow.

[24] For instance, Π is nothing but a finite abelian p-group equipped with a continuous action of $\text{Gal}(\overline{k}/k)$, where $\overline{k}$ is an algebraic closure of k. Then, $\text{Hom}(\Pi, \mathbb{Q}_p/\mathbb{Z}_p)$ can be equipped with the contragredient action of $\text{Gal}(\overline{k}/k)$.

A.13 An auxiliary construction

In this subsection, we describe a construction used in the definition of an admissible pair for a unipotent group in characteristic $p > 0$ (see Sect. 7.3). The construction is a geometric counterpart of the following simple observation. If Γ is a finite group, $N \subset \Gamma$ is a normal subgroup, $\chi : N \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ is a homomorphism, which is invariant under the conjugation action of Γ, and $Z \subset \Gamma$ is a subgroup such that $N \subset Z$ and $[\Gamma, Z] \subset N$, then χ induces a homomorphism $\Gamma/N \longrightarrow \mathrm{Hom}(Z/N, \mathbb{Q}_p/\mathbb{Z}_p)$ given by $\gamma \longmapsto \big(z \mapsto \chi(\gamma z \gamma^{-1} z^{-1})\big)$.

We fix a perfect field k of characteristic $p > 0$, let U be a (possibly disconnected) perfect unipotent group over k and let $N \subset U$ be a normal connected subgroup. By Proposition A.30, we can speak about the Serre dual, $N^* \in \mathfrak{cpu}_k$, of N, and since N^* is defined by a universal property (in the category of perfect k-schemes), it is clear that U acts on N^* regularly by k-group scheme automorphisms.

Let $\nu \in N^*(k)$ be a U-invariant element, and let $Z \subset U$ be a connected subgroup such that $N \subset Z$ and $[U, Z] \subset N$. (Without loss of generality, one can take Z to be the preimage in U of the neutral connected component of $Z(U/N)$.)

We claim that ν defines a k-group morphism $\varphi_\nu : U/N \longrightarrow (Z/N)^*$.

In the proof of this claim, we will use the standard correspondence between central extensions (or bi-extensions) of connected (quasi-)algebraic groups by $\mathbb{Q}_p/\mathbb{Z}_p$ and (bi)multiplicative $\mathbb{Q}_p/\mathbb{Z}_p$-torsors (see [26] and Lemma A.16). Thus, we will also denote by ν the multiplicative $\mathbb{Q}_p/\mathbb{Z}_p$-torsor on N defined by ν.

First we will define a morphism of k-schemes $U \longrightarrow Z^*$. By definition, this is the same as constructing a central extension of $U \times_k Z$, viewed as a group scheme over U, by $\mathbb{Q}_p/\mathbb{Z}_p$. We define a $\mathbb{Q}_p/\mathbb{Z}_p$-torsor on $U \times_k Z$ by $\mathcal{E} = c^*\nu$, where $c : U \times_k Z \longrightarrow N$, $c(u, z) = [u, z] := uzu^{-1}z^{-1}$. Now we apply the following result.

Lemma A.33 *The restrictions of $\mathcal{E}$ to $N \times_k Z$ and $U \times_k N$ are trivial torsors. Moreover, let $\mu_U : U \times_k U \longrightarrow U$, $\mu_Z : Z \times_k Z \longrightarrow Z$ denote the multiplication morphisms, and let $p_1, p_2 : U \times_k U \longrightarrow U$ and $q_1, q_2 : Z \times_k Z \longrightarrow Z$ be the natural projections. Then*

$$(\mathrm{id}_U \times \mu_Z)^*\mathcal{E} \cong (\mathrm{id}_U \times q_1)^*\mathcal{E} \otimes (\mathrm{id}_U \times q_2)^*\mathcal{E} \tag{A.10}$$

as $\mathbb{Q}_p/\mathbb{Z}_p$-torsors on $U \times_k Z \times_k Z$, and

$$(\mu_U \times \mathrm{id}_Z)^*\mathcal{E} \cong (p_1 \times \mathrm{id}_Z)^*\mathcal{E} \otimes (p_2 \times \mathrm{id}_Z)^*\mathcal{E} \tag{A.11}$$

as $\mathbb{Q}_p/\mathbb{Z}_p$-torsors on $U \times_k U \times_k Z$.

The proof of Lemma A.33 is given below. Note that formula (A.10) implies that $\mathcal{E}$ corresponds to a central extension of $U \times_k Z$, viewed as a group scheme over U, by $\mathbb{Q}_p/\mathbb{Z}_p$, and hence defines a morphism $U \longrightarrow Z^*$ of k-schemes. Formula (A.11) implies that, moreover, this morphism is a group homomorphism. Finally, the first sentence of the lemma means that this homomorphism factors through a homomorphism $\varphi_\nu : U/N \longrightarrow (Z/N)^*$, as claimed.

Remark A.34 Note that $Z/N \in \mathfrak{cpu}_k^\circ$, because $[U, Z] \subset N$, and hence, *a fortiori*, $[Z, Z] \subset N$. The restriction of φ_ν to Z/N is a bi-extension of $(Z/N, Z/N)$ by $\mathbb{Q}_p/\mathbb{Z}_p$. In fact, this extension is *skewsymmetric* (Definition A.22) by the very construction of φ_ν (indeed, observe that the restriction of c to the diagonal in $Z \times_k Z$ is constant).

Proof of Lemma A.33 The following observation will be used several times in the proof. Let $\alpha : U \times_k N \longrightarrow N$ denote the conjugation action map: $(u, n) \longmapsto unu^{-1}$. Then, $\alpha^*\nu$ is a $\mathbb{Q}_p/\mathbb{Z}_p$-torsor over $U \times_k N$, and it is clear that if we view $U \times_k N$ as a group scheme over U, then $\alpha^*\nu$ becomes a *multiplicative* $\mathbb{Q}_p/\mathbb{Z}_p$-torsor, in the sense that

$$(\mathrm{id}_U \times \mu_N)^*\alpha^*\nu \cong (\mathrm{id}_U \times r_1)^*\alpha^*\nu \otimes (\mathrm{id}_U \times r_2)^*\alpha^*\nu$$

as $\mathbb{Q}_p/\mathbb{Z}_p$-torsors on $U \times_k N \times_k N$, where $\mu_N : N \times_k N \longrightarrow N$ is the multiplication morphism and $r_1, r_2 : N \times_k N \longrightarrow N$ are the two projections. Hence, $\alpha^*\nu$ defines a morphism of k-schemes $U \longrightarrow N^*$, which is nothing but the orbit map for the U-action on $\nu \in N^*(k)$. By assumption, ν is U-invariant, whence

$$\alpha^*\nu \cong (\mathbb{Q}_p/\mathbb{Z}_p)_U \boxtimes \nu, \tag{A.12}$$

where $(\mathbb{Q}_p/\mathbb{Z}_p)_U$ denotes the trivial $\mathbb{Q}_p/\mathbb{Z}_p$-torsor on U.

Let us prove that, in the notation of Lemma A.33, the torsor $\mathcal{E}\big|_{N\times_k Z}$ is trivial. The following composition clearly equals $c\big|_{N\times_k Z}$:

$$N \times_k Z \xrightarrow{\iota} N \times_k Z \times_k N \xrightarrow{\mathrm{id}\times(\alpha|_{Z\times_k N})} N \times_k N \xrightarrow{\mu_N} N,$$

where $\iota(n, z) = (n, z, n^{-1})$. Therefore

$$\begin{aligned}\mathcal{E}\big|_{N\times_k Z} &\cong \left(c\big|_{N\times_k Z}\right)^* \nu \\ &\cong \iota^* \left[\mathrm{id}\times(\alpha\big|_{Z\times_k N})\right]^* (\mu_N^*\nu) \\ &\cong \iota^* \left[\mathrm{id}\times(\alpha\big|_{Z\times_k N})\right]^* (\nu \boxtimes \nu) \\ &\cong \iota^* \left(\nu \boxtimes (\mathbb{Q}_p/\mathbb{Z}_p)_U \boxtimes \nu\right) \cong (\mathbb{Q}_p/\mathbb{Z}_p)_{N\times_k Z},\end{aligned}$$

as claimed, where the isomorphism before the last one uses (A.12).

The triviality of $\mathcal{E}\big|_{U\times_k N}$ is proved by a completely analogous argument.

Let us prove (A.10). It is straightforward to verify the identity

$$c(u, z_1z_2) = c(u, z_1)\cdot c(u, z_2)\cdot \alpha(z_1, c(u, z_2)) \qquad \forall\, u \in U,\ z_1, z_2 \in Z.$$

It translates into the commutativity of the following diagram:

$$\begin{array}{ccc} U \times_k Z \times_k Z & \xrightarrow{\beta} & N \times_k N \times_k N \\ {\scriptstyle \mathrm{id}_U \times \mu_Z}\downarrow & & \downarrow{\scriptstyle \mu_3} \\ U \times_k Z & \xrightarrow{c} & N \end{array}$$

where $\mu_3(n_1, n_2, n_3) = n_1 n_2 n_3$ and

$$\beta(u, z_1, z_2) = (c(u, z_1), c(u, z_2), \alpha(z_1, c(u, z_2)))\,.$$

Therefore, using (A.12), we find that

$$\begin{aligned}(\mathrm{id}_U \times \mu_Z)^* \mathcal{E} &\cong \beta^*(\nu \boxtimes \nu \boxtimes \nu) \\ &\cong (\mathrm{id}_U \times q_1)^* c^* \nu \otimes (\mathrm{id}_U \times q_2)^* c^* \nu \otimes (\mathbb{Q}_p/\mathbb{Z}_p)_{U \times_k Z \times_k Z} \\ &\cong (\mathrm{id}_U \times q_1)^* \mathcal{E} \otimes (\mathrm{id}_U \times q_2)^* \mathcal{E},\end{aligned}$$

which proves (A.10).

Finally, the proof of (A.11) is very similar, so we omit the details. The argument uses the easily verifiable identity

$$c(u_1 u_2, z) = c(u_1, z) \cdot c(u_2, z) \cdot c\left(z u_1 z^{-1}, c(u_2, z)^{-1}\right)^{-1}$$

together with the multiplicativity property of ν (i.e., $\mu_3^* \nu \cong \nu \boxtimes \nu \boxtimes \nu$) and the fact that $\mathcal{E}\,\big|_{U \times_k N}$ is trivial. This completes the proof of Lemma A.33. □

A.14 Lifting central extensions

In this subsection, we prove a result on lifting central extensions of connected unipotent groups by $\mathbb{Q}_p/\mathbb{Z}_p$ that is essentially equivalent to Proposition 7.7 used in the main body of the text.

Proposition A.35 *Let k be (as usual) a perfect field of characteristic $p > 0$, let G be a connected unipotent group over k, let $H \subset G$ be a connected subgroup such that $[G, G] \subset H$ and consider a central extension*

$$0 \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \widetilde{H} \xrightarrow{\pi} H \longrightarrow 0. \qquad \text{(A.13)}$$

Assume that the commutator morphism com : $G \times G \longrightarrow H$ *lifts to a morphism $G \times G \longrightarrow \widetilde{H}$. Then, (A.13) lifts to a central extension of G by $\mathbb{Q}_p/\mathbb{Z}_p$.*

Remark A.36 The reader may observe that in this result we departed from our tradition of working with perfect unipotent groups. However, the difference between algebraic and quasi-algebraic groups is absolutely irrelevant in Proposition A.35 (which we could have equally well stated for perfect unipotent groups). Indeed, if G is any algebraic group over k, the natural pullback map $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^2(G_{perf}, \mathbb{Q}_p/\mathbb{Z}_p)$ is easily seen to be an isomorphism, where, as in the proof of Proposition A.30, we write $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ for the abelian group of isomorphism classes of central extensions of G by $\mathbb{Q}_p/\mathbb{Z}_p$ in the category of k-group schemes.

Proof of Proposition A.35 It is clear that we can choose a lift $c : G \times G \longrightarrow \widetilde{H}$ of the commutator morphism which satisfies $c(1, 1) = 1$. We assert that c makes the homomorphism $\delta : \widetilde{H} \longrightarrow G$ obtained by composing π with the inclusion $H \hookrightarrow G$ a *strictly stable crossed module* in the terminology of [12].

Let us briefly recall what this means. Define a morphism $G \times \widetilde{H} \longrightarrow \widetilde{H}$ by $(g, h) \longmapsto {}^g h := c(g, \delta(h)) \cdot h$. This morphism is a regular left action of G on $\widetilde{H}$ by algebraic group automorphisms, satisfying

$$\delta({}^g h) = g\delta(h)g^{-1} \quad \text{and} \quad {}^{\delta(h)}h' = hh'h^{-1} \quad \forall\, g \in G,\ h, h' \in \widetilde{H}.$$

(This is a slight abuse of notation since we should not be using "elements" of G and $\widetilde{H}$, but it is easy to rewrite the identities above purely in terms of morphisms of schemes.) In addition, c satisfies $c(g, g) = 1$, $c(g, h)c(h, g) = 1$, and a few other conditions (coming from the axioms for a braided monoidal category), all of which are carefully formulated in [12] and in [26].

All the equations for the morphism c mentioned in the previous paragraph are automatically satisfied in our situation because G is connected (and therefore geometrically integral), $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete, and $c(1, 1) = 1$ by assumption. The full proof is left as a simple exercise for the reader. The details can also be found in [26].

According to [12], c induces the structure of a *strictly commutative Picard stack* on the quotient stack $G/\widetilde{H}$; see [12] or [1, Exp. XVIII, Sect. 1.4] for the definition of a strictly commutative Picard stack. According to Proposition 1.4.15 in loc. cit., if $\mathcal{A}$ and $\mathcal{B}$ are sheaves of abelian groups on any site, then strictly commutative Picard stacks $\mathcal{P}$ with $\pi_0(\mathcal{P}) = \mathcal{A}$ and $\pi_1(\mathcal{P}) = \mathcal{B}$ are classified up to equivalence by the group $\mathrm{Ext}^2(\mathcal{A}, \mathcal{B})$. In our situation, $\pi_0(G/\widetilde{H}) = G/H$ and $\pi_1(G/\widetilde{H}) = \mathbb{Q}_p/\mathbb{Z}_p$.

We claim that $\mathrm{Ext}^2(G/H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Indeed, G/H is a connected commutative unipotent group over k, so since k is perfect, G/H has a filtration by connected subgroups with all the successive subquotients isomorphic to $\mathbb{G}_a$. By induction on $\dim(G/H)$, we are reduced to the following lemma, which is proved in Sect. 9.6.

Lemma A.37 *If k is any perfect field of characteristic $p > 0$, then*

$$\mathrm{Ext}^r(\mathbb{G}_a, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \qquad \textit{for all } r \geq 2,$$

where $\mathbb{Q}_p/\mathbb{Z}_p$ is viewed as a discrete group scheme over k and the group Ext^2 *is computed in the category of sheaves of abelian groups on the fppf site of* $\mathrm{Spec}\, k$.

We see that the Picard stack $G/\widetilde{H}$ is equivalent to the "trivial one", defined as the product $(G/H) \times (\mathbb{Q}_p/\mathbb{Z}_p - \underline{\text{tors}})$, where G/H is the discrete (i.e., having no nontrivial morphisms) strictly commutative Picard stack defined by the commutative algebraic group G/H and $\mathbb{Q}_p/\mathbb{Z}_p - \underline{\text{tors}}$ is the Picard stack of $\mathbb{Q}_p/\mathbb{Z}_p$-torsors. This implies that the 1-morphism $H \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p - \underline{\text{tors}}$ of gr-stacks (cf. [12] or [26]) obtained from the central extension (A.13) extends to a 1-morphism $G \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p - \underline{\text{tors}}$, which in turn implies that χ lifts to a central extension of G by $\mathbb{Q}_p/\mathbb{Z}_p$.

A.15 Proof of Lemma A.37

The argument, which we present, is based on an idea borrowed from Sect. III.0 of [32], and uses a result of Breen [11]. Breen also has independently suggested another proof of Lemma A.37 (in private communication).

The *perfect site*, k_{pf}, of Spec k is defined as the category of perfect k-schemes of quasi-finite type (see Sect. 9.6) over k, with the Grothendieck topology for which the covering families are the surjective families of étale morphisms.

For brevity, we will introduce the following (non-standard) notation. Let us write $\mathscr{A}$ for the category of sheaves of abelian groups on the fppf site k_{fppf} of Spec k, and let us write $\mathscr{A}(p)$ for the category of sheaves of $\mathbb{F}_p$-vector spaces on the site k_{pf}. We claim that there is a natural (quasi-)isomorphism

$$R\operatorname{Hom}_{\mathscr{A}(p)}(\mathbb{G}_a, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\simeq} R\operatorname{Hom}_{\mathscr{A}}(\mathbb{G}_a, \mathbb{Q}_p/\mathbb{Z}_p). \tag{A.14}$$

Indeed, following the proof of Lemma III.0.13(a) of [32], let us choose an injective resolution $\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathscr{I}^\bullet$ in the category $\mathscr{A}$. Injective abelian sheaves are divisible, so if we let $\mathscr{I}_p^j$ be the kernel of the multiplication by p map $\mathscr{I}^j \longrightarrow \mathscr{I}^j$ for every $j \geq 0$, then the complex $\mathscr{I}_p^\bullet$ is a resolution of $\mathbb{Z}/p\mathbb{Z}$, which restricts to an *injective* resolution of $\mathbb{Z}/p\mathbb{Z}$ in the category $\mathscr{A}(p)$. Moreover, since any morphism $\mathbb{G}_a \longrightarrow \mathscr{I}^j$ automatically factors through $\mathscr{I}_p^j$ (because $\mathbb{G}_a$ is annihilated by p), we obtain (A.14).

To complete the proof of Lemma A.37, we use the Artin–Schreier sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{x \longmapsto x^p - x} \mathbb{G}_a \longrightarrow 0 \tag{A.15}$$

In [11], it is proved that $\operatorname{Ext}^j_{\mathscr{A}(p)}(\mathbb{G}_a, \mathbb{G}_a) = 0$ for all $j \geq 1$. Applying the functor $\operatorname{Hom}_{\mathscr{A}(p)}(\mathbb{G}_a, -)$ to (A.15) and using the associated long exact sequence of the Ext groups, we find that $\operatorname{Ext}^r_{\mathscr{A}(p)}(\mathbb{G}_a, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $r \geq 2$. By (A.14), we are done.

Appendix B: Proof of Theorem 2.6

In this appendix, we present a proof of Theorem 2.6, which is due to Drinfeld.

B.1 Setup

We fix an easy unipotent group G over $\mathbb{F}_q$ and an irreducible representation ρ of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$. Let us list a few facts, which were established in the course of proving Theorem 2.5 in the main body of the text.

(1) There exists an admissible pair $(H, \mathcal{L})$ for G (where H is a connected subgroup of G defined over $\mathbb{F}_q$ and $\mathcal{L}$ is a multiplicative $\overline{\mathbb{Q}}_\ell$-local system on H) such that the restriction of ρ to $H(\mathbb{F}_q)$ contains $t_{\mathcal{L}}$ as a direct summand (Theorem 7.1).
(2) Let G' denote the normalizer of $(H, \mathcal{L})$ in G. Then, G' is connected and the dimension of G'/H is even (Sect. 9.4.1). Also, G'/H is commutative, which results from the connectedness of G' and the definition of an admissible pair.
(3) Let $\varphi_{\mathcal{L}} : (G'/H)_{perf} \longrightarrow (G'/H)^*_{perf}$ denote the homomorphism induced by as in Sect. 9.6. Then, $\varphi_{\mathcal{L}}$ is an isomorphism (see Sect. 9.4.1).
(4) The skewsymmetric bi-additive map $B_{\mathcal{L}} : (G'/H)(\mathbb{F}_q) \times (G'/H)(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ induced by $\varphi_{\mathcal{L}}$ is nondegenerate (see Sect. 9.4.2).
(5) Let $L \subset (G'/H)(\mathbb{F}_q)$ be any Lagrangian subgroup with respect to $B_{\mathcal{L}}$, and let $\widetilde{L}$ denote the preimage of L in $G'(\mathbb{F}_q)$. Then, $t_{\mathcal{L}} : H(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ can be extended to a homomorphism $\chi : \widetilde{L} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$, and for any such extension, we have $\rho \cong \mathrm{Ind}_{\widetilde{L}}^{G(\mathbb{F}_q)} \chi$ (see Sect. 9.4.2).

B.2 Existence of Lagrangian subgroups

Next we will formulate a result that implies Theorem 2.6 in view of the facts we listed above. Let A be a perfect connected commutative unipotent group over $\mathbb{F}_q$, and let $\varphi : A \longrightarrow A^*$ denote an isomorphism that induces a skewsymmetric bi-extension (Definition A.22) of (A, A) by $\mathbb{Q}_p/\mathbb{Z}_p$ (in particular, $\varphi^* = -\varphi$). Let $B_\varphi : A(\mathbb{F}_q) \times A(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ denote the skewsymmetric bi-additive map induced by φ; it is nondegenerate by Proposition A.18.

Proposition B.1 *With the notation above, there is a Lagrangian subgroup $L \subset A(\mathbb{F}_q)$ with respect to B_φ such that $L = \underline{L}(\mathbb{F}_q)$ for some connected subgroup $\underline{L} \subset A$.*

The proof of the proposition is given below. It is clear that Theorem 2.6 follows from the proposition, because the latter implies that the Lagrangian subgroup L mentioned in Sect. 9.6(5) can be chosen to have the form $\underline{L}(\mathbb{F}_q)$ for a connected subgroup $\underline{L} \subset G'/H$. Letting P be the preimage of $\underline{L}$ in G', we have $\widetilde{L} = P(\mathbb{F}_q)$, and the proof of the theorem is complete.

B.3 Strategy of the proof

The proof of Proposition B.1 has two steps. First we will reduce it to the case where A is annihilated by p. In this case, a stronger statement holds:

Lemma B.2 *Let A be a connected commutative unipotent group*[25] *over $\mathbb{F}_q$ such that $p \cdot A = 0$, and let $L \subset A(\mathbb{F}_q)$ be any subgroup whose order is a power of q. Then, there exists a connected subgroup $\underline{L} \subset A$ such that $L = \underline{L}(\mathbb{F}_q)$.*

[25] We can take A to be either an algebraic group in the usual sense, or a perfect algebraic group. The distinction between the two classes of groups is irrelevant here.

Note that when $p \cdot A = 0$, Lemma B.2 implies Proposition B.1, because in the setting of the proposition, the order of any Lagrangian subgroup $L \subset A(\mathbb{F}_q)$ with respect to B_φ equals $A(\mathbb{F}_q)^{1/2} = q^{(\dim A)/2}$, which is a power of q because $\dim A$ is even by Proposition A.28(b). The lemma is proved in Sect. 9.6.

B.4 Reduction of Proposition B.1 to the case $p \cdot A = 0$

Let us suppose that A is not annihilated by p, and let $n \geq 2$ be the smallest integer such that $p^n \cdot A = 0$. Let $A_0 = p^{n-1} \cdot A$. Then, A_0 is a nontrivial connected subgroup of A and is isotropic with respect to the skewsymmetric bi-extension induced by φ, because $p^{n-1} \cdot p^{n-1} \geq p^n$. Let $A_0^\perp$ be the orthogonal complement to A_0 in A, i.e.,

$$A_0^\perp = \varphi^{-1}\left(\mathrm{Ker}(A^* \longrightarrow A_0^*)\right) \subset A.$$

Then, $A_0 \subset A_0^\perp$, and $A_1 := A_0^\perp / A_0$ is also connected. Moreover, φ induces a strongly nondegenerate skewsymmetric bi-extension of (A_1, A_1) by $\mathbb{Q}_p/\mathbb{Z}_p$. Since $\dim A_1 < \dim A$, we may assume that Proposition B.1 holds for this bi-extension. Since A_0 is a connected algebraic subgroup of A, the reduction to the case $p \cdot A = 0$ is complete.

B.5 Proof of Lemma B.2

Throughout the proof, q is assumed to be fixed, and $\mathbb{G}_a$ denotes the additive group over the field $\mathbb{F}_q$. Let $W = A(\mathbb{F}_q)$; it is canonically an $\mathbb{F}_p$-vector space. Since A is annihilated by p, it is isomorphic to a direct sum of copies of $\mathbb{G}_a$, whence we may assume that $A = \mathbb{G}_a^n$. Let $L \subset W$ be any subgroup of order q^k, where $0 \leq k \leq n$. Let $A' \subset A$ be the direct sum of the first k copies of $\mathbb{G}_a$. Then, $A'(\mathbb{F}_q)$ and L are $\mathbb{F}_p$-subspaces of W and have the same dimension over $\mathbb{F}_p$. Hence, there exists an $\mathbb{F}_p$-linear map $f : W \longrightarrow W$ such that $L = f(A'(\mathbb{F}_q))$.

The ring $\mathrm{End}(\mathbb{G}_a)$ of endomorphisms of $\mathbb{G}_a$ as an algebraic group over $\mathbb{F}_q$ contains all the elements of $\mathbb{F}_q$ (acting by dilations), as well as the Frobenius map $x \mapsto x^p$. It follows from Lemma B.3 that the natural ring homomorphism $\mathrm{End}(\mathbb{G}_a) \longrightarrow \mathrm{End}_{\mathbb{F}_p}(\mathbb{F}_q)$ is surjective. This easily implies that the natural homomorphism $\mathrm{End}(A) \longrightarrow \mathrm{End}_{\mathbb{F}_p}(W)$ is surjective as well. In particular, the linear map f in the previous paragraph is induced by an endomorphism of A, which, by abuse of notation, we will also denote by $f : A \longrightarrow A$. Then, $f(A')$ is a connected subgroup of A. Moreover, $L = f(A'(\mathbb{F}_q)) \subset f(A')(\mathbb{F}_q)$, while $\dim f(A') \leq \dim A' = k$, which implies that $|f(A')(\mathbb{F}_q)| \leq q^k = |L|$. Thus, $L = f(A')(\mathbb{F}_q)$. □

B.6 An auxiliary result

Let us recall a construction. If R is a ring and Γ is a group acting on R by ring automorphisms, the smash product $R\#\Gamma$ is defined to be the ring whose elements are formal sums $\sum_{\gamma\in\Gamma} a_\gamma \gamma$, where $a_\gamma \in R$ and all but finitely many a_γ are 0; the addition in $R\#\Gamma$ is defined in the obvious way; and the multiplication is determined by $(a_1\gamma_1) \cdot (a_2\gamma_2) = (a_1 \cdot \gamma_1(a_2)) \cdot (\gamma_1\gamma_2)$.

Lemma B.3 *Let $K \subset L$ be a finite Galois extension of fields, and let $\Gamma = \mathrm{Gal}(L/K)$. The natural homomorphism*[26] $L\#\Gamma \longrightarrow \mathrm{End}_K(L)$ *is an isomorphism of K-algebras.*

Proof One can easily show that $L\#\Gamma$ is a simple ring.[27] Furthermore, L is simple as a module over itself, and hence, a fortiori, as a module over $L\#\Gamma$. Since $\mathrm{End}_L(L) = L$, it follows that $\mathrm{End}_{L\#\Gamma}(L) = L^\Gamma = K$. Using Wedderburn theory, we conclude that $L\#\Gamma \longrightarrow \mathrm{End}_K(L)$ is an isomorphism. □

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[26] Induced by the action of L on itself by left multiplication, and by the tautological action of Γ on L; we denote by $\mathrm{End}_K(L)$ the algebra of endomorphisms of L as a vector space over K.

[27] Let $I \subset L\#\Gamma$ be a nonzero two-sided ideal, let $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ be a nonzero element in I such that the number of nonzero coefficients a_γ is as small as possible, etc.

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