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Pseudo-potentials, nonlocal symmetries and integrability of some shallow water equations

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Abstract. Zero curvature formulations, pseudo-potentials, modified versions, "Miura transformations", conservation laws, and nonlocal symmetries of the Korteweg–de Vries, Camassa–Holm and Hunter–Saxton equations are investigated from a unified point of view: these three equations belong to a two-parameter family of equations describing pseudo-spherical surfaces, and therefore their basic integrability properties can be studied by geometrical means.

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1. Introduction

The goal of this work is to present a unified account of some integrability properties of three important shallow water models, the Korteweg–de Vries (KdV), Camassa–Holm (CH) and Hunter–Saxton (HS) equations. The main motivation behind this research comes from the papers [4, 5, 30]: in [4, 5] R. Beals, D. Sattinger and J. Szmigielski considered the scattering/inverse scattering analysis of these three equations from a unified perspective, and in [30] B. Khesin and G. Misiołek gave an integrated account of their bi-hamiltonian formulations and showed that these three dynamical systems can be understood as geodesic equations associated to different right-invariant metrics on (appropriate homogeneous spaces of) the Virasoro group. In this article it is pointed out that the existence of zero curvature formulations, quadratic pseudo-potentials, modified versions, Miura transformations, conservation laws and nonlocal symmetries for them follows from some developments linking differential geometry of surfaces and integrability of nonlinear partial differential equations [13, 44, 45, 48, 52].

Two reasons why these observations may be of importance—besides the fact that they show the usefulness of the geometric approach to integrability advocated in the works just mentioned above—are that the construction of nonlocal symmetries carried out in this paper can be considered as a geometric implementation of the "algebraic method" used by M. Leo, R. A. Leo, G. Soliani, and P. Tempesta [35, 36] to find nonlocal symmetries of nonlinear equations, and also as a nontrivial instance of the geometric approach to nonlocalities developed by I. Krasil'shchik, A. Vinogradov and their coworkers (the theory of coverings and more generally differential varieties or "diffieties"; see [33, 34, 54] and references therein). On the other hand, this geometric approach is silent on analytic properties such as well-posedness of the Cauchy problems for the KdV, CH and HS equations, although *it is* relevant for analysis since it provides one with zero curvature representations and conservation laws. For analytic studies of the equations considered here the reader is referred to [4, 5, 10, 14, 15, 16, 37].

Recall that if $u_t = F$ is a scalar partial differential evolution equation in two independent variables x and t, a (generalized) symmetry of $u_t = F$ is a smooth function G depending on x, t, u, and a finite number of derivatives of u such that for any solution $u(x,t)$ of $u_t = F$, the deformed function $u(x,t) + \tau G(x,t)$ is also a solution to first order in τ. At least at a formal level [39, Chapter 5] a generalized symmetry G allows one to generate new solutions from old ones and, if G depends at most on x, t, u, u_x, one can indeed show the existence of a (local) one-parameter group of transformations on the space of first order jets of the trivial bundle $(x,t,u) \mapsto (x,t)$ which "sends solutions to solutions" (see [34, 38, 39]).

It appears that A. Vinogradov and I. Krasil'shchik [55] were the first researchers who studied *nonlocal* symmetries of partial differential equations rigorously and also the first who pointed out some of their applications. By a nonlocal symmetry of $u_t = F$ one means (see Section 3 for a rigorous definition) a function G which depends on x, t, u, a finite number of x-derivatives of u *and for example* integrals of u, such that for any solution $u(x,t)$ of $u_t = F$, the function $u(x,t) + \tau G(u(x,t))$ is also a solution to first order in τ. That these symmetries are both important and natural to consider has been increasingly acknowledged since Vinogradov and Krasil'shchik's paper [55]. A few highlights are the following:

In 1982, Kaptsov [29] considered an evolution equation $u_t = F$ and solved the "recursion operator equation" $R_t = [F_*, R]$, in which F_* is the formal linearization of F (see [38, 39] and Section 3 below) finding that his solution R induced sequences of nonlocal symmetries of $u_t = F$. In the late 1980's, Bluman, Kumei and Reid [7] and Bluman and Kumei [6] used nonlocal symmetries to find linearizing transformations for nonlinear equations. In 1991 V. E. Adler [2] introduced Lie algebras of nonlocal symmetries associated to equations integrable by the scattering/inverse scattering method, generalizing the classical construction of integrable hierarchies due to M. Adler [1], Reiman and Semenov-Tyan-Shanski [42], and others (see Faddeev and Takhtajan [18] for details and historical notes). Lastly, Galas [25], Leo *et al.* [35, 36], and Schiff [49] have recently obtained nonlocal symmetries of some well-known integrable equations, found their flows, and used them to construct special solutions for the equations at hand.

An interesting characteristic of the papers by Galas, Leo *et al.*, and Schiff [25, 35, 36, 49] is that the nonlocalities appearing in their symmetries are more involved than simply integrals of smooth functions of x, t, u and a finite number of derivatives of u. In their examples, the nonlocal symmetries depend on *pseudo-potentials* of the equations they consider. These symmetries were anticipated and studied by Krasil'shchik and Vinogradov in the 1980's using their theory of coverings of differential equations (see [33, 34] and references therein) and several examples were given by Kiso [31] about that time. However, it appears that it is only in [25, 35, 36, 49] that they have been used to find explicit solutions. It is then of interest to further the work carried out in these papers, and to show novel applications of the theory. This is what the geometric constructs of this article allow one to do:

The notion of a scalar equation describing pseudo-spherical surfaces (or "of pseudo-spherical type") is introduced in Section 2. Equations in this class are of interest because they share with the classical sine-Gordon equation the property that their (suitably generic) solutions determine two-dimensional surfaces equipped with Riemannian metrics of constant Gaussian curvature -1, and also because equations possessing this structure are naturally the integrability condition of $\mathrm{sl}(2,\mathbb{R})$-valued linear problems. Section 3 is on (local/nonlocal) symmetries and pseudo-potentials for equations describing pseudo-spherical surfaces. A short introduction to the theory of coverings is also included there. From the point of view of this paper, the pseudo-potentials of the scalar equations considered in [25, 36, 49] determine geodesics of the pseudo-spherical structures described by the relevant equations, and the nonlocal symmetries G for them are obtained by studying infinitesimal deformations $u \mapsto u + \tau u$ of the dependent variable u which preserve geodesics to first order in the deformation parameter τ.

Section 4 contains the application of the work carried out in Sections 2 and 3 to the Korteweg–de Vries [32], Camassa–Holm [11] and Hunter–Saxton [26, 27] equations. Their zero curvature representations, quadratic pseudo-potentials, "Miura transformations" and modified versions are introduced (and consequently, a method for finding sequences of conservation laws is pointed out), and then nonlocal symmetries of "pseudo-potential type" are constructed for them. Furthermore, it is shown that these symmetries can be integrated, and that consideration of their flows yields smooth local existence theorems for solutions. Examples of solutions are also included here.

Special cases of some of the results appearing in this paper have been announced in [46, 47].

2. Equations of pseudo-spherical type

Equations of pseudo-spherical type were introduced by S. S. Chern and K. Tenenblat in 1986 [13], motivated by the fact that generic solutions of equations integrable by the Ablowitz, Kaup, Newell and Segur (AKNS) inverse scattering scheme determine—whenever their associated linear problems are real—pseudo-spherical surfaces, that is, Riemannian surfaces of constant Gaussian curvature equal to -1 [48].

Henceforth, partial derivatives $\partial^{p+q}u/\partial x^p \partial t^q$ are denoted by $u_{x^p t^q}$.

Definition 2.1. A scalar differential equation $\Xi(x, t, u, u_x, \ldots, u_{x^n t^m}) = 0$ in two independent variables x, t is *of pseudo-spherical type* (or, *describes pseudo-spherical surfaces*) if there exist one-forms $\omega^i \neq 0$, $i = 1, 2, 3$,

$$\omega^i = f_{i1}(x, t, u, \ldots, u_{x^r t^p})dx + f_{i2}(x, t, u, \ldots, u_{x^s t^q})dt, \tag{2.1}$$

whose coefficients f_{ij} are differential functions, such that the one-forms $\overline{\omega}^i = \omega^i(u(x,t))$ satisfy the structure equations

$$d\overline{\omega}^1 = \overline{\omega}^3 \wedge \overline{\omega}^2, \qquad d\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}^3, \qquad d\overline{\omega}^3 = \overline{\omega}^1 \wedge \overline{\omega}^2, \tag{2.2}$$

whenever $u = u(x,t)$ is a solution to $\Xi = 0$.

Recall that a *differential function* is a smooth function depending on the independent variables x, t, the dependent variable u, and a finite number of derivatives of u (see [39]), that is, a smooth function on some finite order jet bundle of the (trivial) fiber bundle $(x, t, u) \mapsto (x, t)$. The case when all the functions f_{ij} depend only on x and t appears to be irrelevant for differential equations and therefore it is excluded from the considerations below.

Example. The equation

$$-f_t + \frac{\partial}{\partial x}[g_x + fg] = 0, \tag{2.3}$$

in which f and g are arbitrary differential functions, is of pseudo-spherical type with associated one-forms

$$\omega^1 = f dx + (g_x + fg)dt, \qquad \omega^2 = \lambda dx + \lambda g dt, \qquad \omega^3 = -\lambda dx - \lambda g dt.$$

The well-known Burgers equation $u_t = u_{xx} + uu_x$ is a special case of (2.3) with $f = g = (1/2)u$.

The expression "PSS equation" is sometimes used in this paper instead of "equation of pseudo-spherical type". The geometric interpretation of Definition 2.1 is based on the following genericity notions ([45] and references therein):

Definition 2.2. Let $\Xi = 0$ be a PSS equation with associated one-forms ω^i, $i = 1, 2, 3$. A solution $u(x,t)$ of $\Xi = 0$ is *I-generic* if $(\omega^3 \wedge \omega^2)(u(x,t)) \neq 0$, *II-generic* if $(\omega^1 \wedge \omega^3)(u(x,t)) \neq 0$, and *III-generic* if $(\omega^1 \wedge \omega^2)(u(x,t)) \neq 0$.

Proposition 2.3. *Let $\Xi = 0$ be a PSS equation with associated one-forms ω^i, let $u(x,t)$ be a local solution to $\Xi = 0$ and set $\overline{\omega}^i = \omega^i(u(x,t))$.*

(a) *If $u(x,t)$ is a I-generic solution, $\overline{\omega}^2$ and $\overline{\omega}^3$ determine a Lorentzian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with metric connection one-form $\overline{\omega}^1$.*

(b) *If $u(x,t)$ is a II-generic solution, $\overline{\omega}^1$ and $-\overline{\omega}^3$ determine a Lorentzian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with metric connection one-form $\overline{\omega}^2$.*

(c) *If $u(x,t)$ is a III-generic solution, $\overline{\omega}^1$ and $\overline{\omega}^2$ determine a Riemannian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with metric connection one-form $\overline{\omega}^3$.*

Proposition 2.3 follows from the structure equations of a (pseudo) Riemannian manifold, which appear, for example, in [52]. The notion of integrability introduced below is implicit in [13].

Definition 2.4. An equation is *geometrically integrable* if it describes a nontrivial one-parameter family of pseudo-spherical surfaces.

Proposition 2.5. *A geometrically integrable equation $\Xi = 0$ with associated one-forms ω^i, $i = 1,2,3$, is the integrability condition of a one-parameter family of* $\mathrm{sl}(2,\mathbf{R})$*-valued linear problems.*

Proof. The linear problem $d\psi = \Omega\psi$, in which

$$\Omega = X dx + T dt = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}, \tag{2.4}$$

is integrable whenever $u(x,t)$ is a solution of $\Xi = 0$. □

An important idea in integrable systems [18] is that an equation $\Xi = 0$ which is integrable via scattering/inverse scattering is not just the integrability condition of a linear problem $\psi_x = X\psi$, $\psi_t = T\psi$, but it is in fact *equivalent* to the zero curvature equation $X_t - T_x + [X,T] = 0$. For evolutionary PSS equations $u_t = F(x,t,u,\ldots,u_{x^n})$ one formalizes this remark thus [28, 45]:

Consider the differential ideal I_F generated by the two-forms

$$du \wedge dx + F(x,t,u,\ldots,u_{x^n}) dx \wedge dt, \quad du_{x^l} \wedge dt - u_{x^{l+1}} dx \wedge dt, \quad 1 \le l \le n-1,$$

on a manifold J with coordinates $x, t, u, u_x, \ldots, u_{x^n}$.

Definition 2.6. An evolution equation $u_t = F(x,t,u,\ldots,u_{x^n})$ is *strictly pseudo-spherical* if there exist one-forms $\omega^i = f_{i1}dx + f_{i2}dt$, $i = 1,2,3$, whose coefficients f_{ij} are smooth functions on J, such that the two-forms

$$\Omega_1 = d\omega^1 - \omega^3 \wedge \omega^2, \quad \Omega_2 = d\omega^2 - \omega^1 \wedge \omega^3, \quad \Omega_3 = d\omega^3 - \omega^1 \wedge \omega^2 \tag{2.5}$$

generate I_F.

Local solutions of $u_t = F$ correspond to integral submanifolds of the exterior differential system $\{I_F, dx \wedge dt\}$. Thus, if $u_t = F$ is strictly pseudo-spherical, it is necessary and sufficient for the structure equations $\Omega_\alpha = 0$ to hold. The following lemma [44, 45] is used in Section 3 below.

Lemma 2.7. *Necessary and sufficient conditions for an n^{th} order equation $u_t = F$ to be strictly pseudo-spherical are the conjunction of:*

(a) *The functions f_{ij} satisfy $f_{i1,u_{x^a}} = 0$, $a \geq 1$; $f_{i2,u_{x^n}} = 0$, $i = 1, 2, 3$; and*

$$f_{11,u}^2 + f_{21,u}^2 + f_{31,u}^2 \neq 0. \tag{2.6}$$

(b) *F and f_{ij} satisfy the identities*

$$-f_{i1,u}F + \sum_{p=0}^{n-1} u_{x^{p+1}} f_{i2,u_{x^p}} + f_{j1}f_{k2} - f_{k1}f_{j2} + f_{i2,x} - f_{i1,t} = 0 \tag{2.7}$$

in which $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 2, 1)\}$.

Remark 2.8. In the interesting papers [21, 22], A. Fokas, I. Gelfand and their coworkers consider an *extrinsic* approach in which (systems of) nonlinear partial differential equations admitting zero curvature representations determine immersed surfaces in Lie groups and Lie algebras. This is different from the point of view taken here, in which the pseudo-spherical surfaces determined by (generic) solutions to equations of pseudo-spherical type are defined *intrinsically* by the one-forms ω^i, $i = 1, 2, 3$.

3. Symmetries and pseudo-potentials for PSS equations

3.1. Pseudo-potentials

The following geometrical result appears in [13, 52]:

Proposition 3.1. *Given an orthogonal coframe $\{\overline{\omega}^1, \overline{\omega}^2\}$ and corresponding metric connection one-form $\overline{\omega}^3$ on a Riemannian surface M with metric $\overline{\omega}^1 \otimes \overline{\omega}^1 + \overline{\omega}^2 \otimes \overline{\omega}^2$, there exists a new orthogonal coframe $\{\overline{\theta}^1, \overline{\theta}^2\}$ and new metric connection one-form $\overline{\theta}^3$ on M satisfying*

$$d\overline{\theta}^1 = 0, \qquad d\overline{\theta}^2 = \overline{\theta}^2 \wedge \overline{\theta}^1, \qquad \overline{\theta}^3 + \overline{\theta}^2 = 0, \tag{3.1}$$

if and only if the surface M is pseudo-spherical.

Proof. Assume that the local orthonormal frames dual to the coframes $\{\overline{\omega}^1, \overline{\omega}^2\}$ and $\{\overline{\theta}^1, \overline{\theta}^2\}$ have the same orientation. The one-forms $\overline{\omega}^\alpha$ and $\overline{\theta}^\alpha$ are then connected by means of

$$\overline{\theta}^1 = \overline{\omega}^1 \cos\rho + \overline{\omega}^2 \sin\rho, \qquad \overline{\theta}^2 = -\overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho, \qquad \overline{\theta}^3 = \overline{\omega}^3 + d\rho. \tag{3.2}$$

It follows that one-forms $\overline{\theta}^1, \overline{\theta}^2, \overline{\theta}^3$ satisfying (3.1) exist if and only if the Pfaffian system

$$\overline{\omega}^3 + d\rho - \overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho = 0 \tag{3.3}$$

on the space of coordinates (x, t, ρ) is completely integrable for $\rho(x, t)$, and it is easy to see that this happens if and only if M is pseudo-spherical. □

Equations (3.1) and (3.3) determine geodesic coordinates on M [13, 52]. If an equation $\Xi = 0$ describes pseudo-spherical surfaces with associated one-forms $\omega^i = f_{i1}dx + f_{i2}dt$, equations (3.1) and (3.3) imply that the Pfaffian system

$$\omega^3(u(x,t)) + d\rho - \omega^1(u(x,t))\sin\rho + \omega^2(u(x,t))\cos\rho = 0 \tag{3.4}$$

is completely integrable for $\rho(x,t)$ whenever $u(x,t)$ is a local solution of $\Xi = 0$. Moreover, equations (3.1) and (3.2) imply that for each solution $u(x,t)$ and corresponding solution $\rho(x,t)$ of (3.4), the one-form

$$\theta^1(u(x,t)) = \omega^1(u(x,t))\cos\rho + \omega^2(u(x,t))\sin\rho \tag{3.5}$$

is closed. Since one-forms which are closed on solutions of $\Xi = 0$ determine conservation laws [33, 34, 39] it follows that if the functions f_{ij} (and therefore ρ and θ^1) can be expanded as power series in a parameter λ, the PSS equation $\Xi = 0$ will possess, in principle, an infinite number of conservation laws, which may well be nonlocal. The reader is referred to [13, 43, 44, 46, 47, 52] for further discussions.

Lemma 3.2. *Let $\Xi = 0$ be a PSS equation with associated one-forms ω^i. Under the changes of variables $\Gamma = \tan(\rho/2)$ and $\hat{\Gamma} = \cot(\rho/2)$, the Pfaffian system* (3.4) *and the one-form* (3.5) *become, respectively,*

$$-2d\Gamma = (\overline{\omega}^3 + \overline{\omega}^2) - 2\Gamma\overline{\omega}^1 + \Gamma^2(\overline{\omega}^3 - \overline{\omega}^2), \tag{3.6}$$

$$\Theta = \overline{\omega}^1 - \Gamma(\overline{\omega}^3 - \overline{\omega}^2) \quad (\textit{up to an exact differential form}), \tag{3.7}$$

and

$$2d\hat{\Gamma} = (\overline{\omega}^3 - \overline{\omega}^2) - 2\hat{\Gamma}\overline{\omega}^1 + \hat{\Gamma}^2(\overline{\omega}^3 + \overline{\omega}^2), \tag{3.8}$$

$$\hat{\Theta} = -\overline{\omega}^1 + \hat{\Gamma}(\overline{\omega}^3 + \overline{\omega}^2) \quad (\textit{up to an exact differential form}), \tag{3.9}$$

where $\overline{\omega}^i = \omega^i(u(x,t))$, $i = 1,2,3$.

Pseudo-potentials are a generalization of conservation laws:

Definition 3.3. A real-valued function Γ is a *pseudo-potential* of a differential equation $\Xi(x,t,u,\ldots,u_{x^m t^n}) = 0$ if there exist smooth functions f, g depending on Γ, x, t, u, and a finite number of derivatives of u, such that the one-form

$$\Omega_\Gamma = d\Gamma - (f dx + g dt)$$

satisfies

$$d\Omega_\Gamma = 0 \bmod \Omega_\Gamma$$

whenever $u(x,t)$ is a solution to $\Xi = 0$.

One says that the one-form Ω_Γ is *associated* to the pseudo-potential Γ. Note that if the functions f and g appearing in the definition do not depend on Γ, this function is a potential for the *bona fide* conservation law $f dx + g dt$ of the equation $\Xi = 0$.

Pseudo-potentials were introduced by Wahlquist and Eastbrook [56]. They can be understood geometrically in the framework of covering theory (see [33, 34] and references therein). *Quadratic* pseudo-potentials, that is, pseudo-potentials Γ such that the functions f and g appearing in the associated one-form Ω_Γ are quadratic polynomials in Γ, possess a very appealing geometrical interpretation within the framework of PSS equations.

Proposition 3.4. *A differential equation* $\Xi = 0$ *is of pseudo-spherical type if and only if it admits a quadratic pseudo-potential.*

Proof. Equations (3.6) and (3.8) say that Γ and $\hat{\Gamma}$ are quadratic pseudo-potentials for the PSS equation $\Xi = 0$. On the other hand, if $\Xi = 0$ admits a pseudo-potential Γ with associated one-form $\Omega_\Gamma = d\Gamma - (f dx + g dt)$, in which $f = a + b\Gamma + c\Gamma^2$ and $g = a' + b'\Gamma + c'\Gamma^2$, then the system

$$\Gamma_x = a + b\Gamma + c\Gamma^2, \qquad \Gamma_t = a' + b'\Gamma + c'\Gamma^2 \tag{3.10}$$

is completely integrable on solutions of $\Xi = 0$, and therefore the equations

$$a_t + ba' = a'_x + b'a, \qquad b_t + 2ca' = b'_x + 2c'a, \qquad c_t + cb' = c'_x + c'b$$

are satisfied on solutions of $\Xi = 0$. It follows that $\Xi = 0$ is a PSS equation with associated one-forms ω^i, $i = 1, 2, 3$, given by

$$\omega^1 = bdx + b'dt, \quad \omega^2 = (-a+c)dx + (-a'+c')dt, \quad \omega^3 = -(a+c)dx - (a'+c')dt. \tag{3.11}$$

□

The quadratic pseudo-potential (3.6) induced by the one-forms (3.11) is, of course, (3.10), and therefore if a differential equation $\Xi = 0$ admits a quadratic pseudo-potential Γ, the function Γ determines the geodesics of the pseudo-spherical structures described by $\Xi = 0$.

3.2. Symmetries

As stated in Section 1, a differential function G is a generalized symmetry of $u_t = F$ if and only if $u(x,t) + \tau G(u(x,t))$ is—to first order in τ—a solution of $u_t = F$ whenever $u(x,t)$ is a solution of $u_t = F$. In other words, G is a generalized symmetry of $u_t = F$ if and only if the equation $D_t G = F_* G$, in which F_* denotes the formal linearization of F,

$$F_* = \sum_{i=0}^{k} \frac{\partial F}{\partial u_{x^i}} D_x^i, \tag{3.12}$$

and D_x, D_t are the total derivative operators with respect to x and t respectively [38, 39], holds identically once all the derivatives with respect to t appearing in it have been replaced by means of $u_t = F$. This definition extends straightforwardly to (systems of) equations not necessarily of evolutionary type (see [39, 34, 54]).

Now let $u_t = F$ be an n^{th} order strictly pseudo-spherical evolution equation with associated one-forms ω^i. Let $u(x,t)$ be a local solution of $u_t = F$, and set $\overline{G} = G(u(x,t))$, where G is an arbitrary differential function. Expand $\omega^i(u(x,t) + \tau\overline{G})$ about $\tau = 0$, thereby obtaining an infinitesimal deformation $\overline{\omega}^i + \tau\overline{\Lambda}_i$, $\overline{\Lambda}_i = \overline{g}_{i1}dx + \overline{g}_{i2}dt$, of the one-forms $\overline{\omega}^i = \omega^i(u(x,t))$. Lemma 1 implies that $\overline{g}_{i1} = f_{i1,u}(u(x,t))\overline{G}$ and $\overline{g}_{i2} = \sum_{p=0}^{n-1} f_{i2,u_{x^p}}(u(x,t))(\partial^p\overline{G}/\partial x^p)$, $i = 1, 2, 3$. One then has [44]:

Theorem 3.5. *Suppose that* $u_t = F(x,t,u,\dots,u_{x^n})$ *is strictly pseudo-spherical with associated one-forms* $\omega^i = f_{i1}dx + f_{i2}dt$, $i = 1,2,3$, *and let* G *be a differential function. The deformed one-forms* $\overline{\omega}^i + \tau\overline{\Lambda}_i$ *satisfy the structure equations of a pseudo-spherical surface up to terms of order* τ^2 *if and only if* G *is a generalized symmetry of* $u_t = F$.

Thus, generalized symmetries of strictly pseudo-spherical equations $u_t = F$ are identified with infinitesimal deformations of the pseudo-spherical structures determined by $u_t = F$ which preserve Gaussian curvature to first order in the deformation parameter. Theorem 3.5 has been used in [44, 45] to show existence of (generalized, nonlocal) symmetries of strictly PSS equations.

The symmetry concept will now be extended to encompass nonlocal data following [33, 34, 54]. In order to do this, one needs some notions from the formal geometry approach to differential equations [33, 34, 39, 54] (a short summary of this theory also appears in [43]) which are now recalled:

Fix E to be a (trivial) bundle given locally by $(x,t,u) \mapsto (x,t)$, and let $J^\infty E$ be the corresponding infinite jet bundle of E. Then:

(a) A scalar differential equation $\Xi = 0$ in two independent variables x,t is identified with a subbundle S^∞ of $J^\infty E$ called the *equation manifold* of $\Xi = 0$.
(b) The fiber bundles S^∞ and $J^\infty E$ come equipped with completely integrable distributions—the *Cartan distributions* of S^∞ and $J^\infty E$—denoted by $\mathbf{C}(S^\infty)$ and $\mathbf{C}$ respectively. They satisfy the condition $\mathbf{C}(S^\infty)_\theta = T_\theta S^\infty \cap \mathbf{C}_\theta$ for all $\theta \in S^\infty$.
(c) The Cartan distributions $\mathbf{C}$ and $\mathbf{C}(S^\infty)$ are (locally) generated by the vector fields D_x and D_t. It is understood that in the case of S^∞ the symbols D_x, D_t stand for the pullbacks $\iota^* D_x, \iota^* D_t$, in which $\iota : S^\infty \to J^\infty E$ is the inclusion map.

Definition 3.6. Let $\Xi = 0$ be a differential equation with equation manifold S^∞, and let $\pi : \overline{S} \to S^\infty$ be a fiber bundle over S^∞. The bundle π determines a *covering* structure (or, $\overline{S}$ is a *covering* of S^∞) if and only if

(a) There exists a completely integrable distribution $\overline{\mathbf{C}}$ on the bundle $\pi^\infty_M \circ \pi : \overline{S} \to M$.
(b) The distribution $\overline{\mathbf{C}}$ agrees with the Cartan distribution $\mathbf{C}(S^\infty)$ on S^∞, that is, for any vector field X on M, $\pi_*(\overline{X}) = \mathrm{pr}^\infty(X)$, where the vector field $\overline{X}$ on $\overline{S}$ is the horizontal lift of X induced by $\overline{\mathbf{C}}$, and $\mathrm{pr}^\infty(X)$ is the horizontal lift of the vector field X with respect to the Cartan distribution of S^∞.

Fiber bundles over equation manifolds S^∞ are rigorously defined in [33, 34]. In this paper only the following local description will be needed: Consider local coordinates $(x,t,u,\dots,w^1,\dots,w^N)$, $1 \le N \le \infty$, on a covering $\pi : \overline{S} \to S^\infty$ of S^∞ such that $(x,t,u,\dots)$ are canonical coordinates on S^∞ and $(w^1,\dots,w^N)$ are fiber coordinates on $\overline{S}$, and let D_x and D_t be the total derivative operators on S^∞. Definition 3.6 implies that the covering $\overline{S}$ is locally determined by the data $(\overline{S}, \overline{D}_x, \overline{D}_t, \pi)$, in which $\overline{D}_x$, $\overline{D}_t$ are differential operators on $\overline{S}$ satisfying:

(a) $\overline{D}_x$, $\overline{D}_t$ are of the form

$$\overline{D}_x = D_x + X_1 \qquad \text{and } \overline{D}_t = D_t + X_2, \tag{3.13}$$

where X_i, $i = 1, 2$, are vertical vector fields on $\overline{S}$, $X_i = \sum_{\beta=1}^{N} X^{\beta}_{i}\partial/\partial w^{\beta}$.

(b) $\overline{D}_x$ and $\overline{D}_t$ satisfy the integrability condition

$$[\overline{D}_x, \overline{D}_t] := D_x(X_2) - D_t(X_1) + [X_1, X_2] = 0. \tag{3.14}$$

The operators $\overline{D}_x$ and $\overline{D}_t$ are the *total derivative operators* on $\overline{S}$. As in the case of total derivatives on equation manifolds, $\overline{D}_x$ and $\overline{D}_t$ span the horizontal distribution of $\overline{S}$. The fiber coordinates w^i, $1 \leq i \leq N$, are called *nonlocal variables* with respect to S^∞, and N is the *dimension* of the covering $\pi : \overline{S} \to S^\infty$.

Example. Probably the most elementary example of a covering is the following [34]: Assume that the one-form $\kappa = f dx + g dt$, in which f and g are differential functions, satisfies $D_t f = D_x g$ on solutions of $u_t = F$. Then κ determines a covering $(\overline{S}, \overline{D}_x, \overline{D}_t, \pi)$ of the equation $u_t = F$: $\overline{S}$ is locally defined by $\overline{S} = \{(x, t, u, \dots, u_{x^m}, \dots, w)\}$, where $(x, t, u, u_x, \dots)$ are coordinates on the equation manifold S^∞ of $u_t = F$, and

$$\overline{D}_x = D_x + f\frac{\partial}{\partial w}, \qquad \overline{D}_t = D_t + g\frac{\partial}{\partial w}. \tag{3.15}$$

It is trivial to check that $D_t f = D_x g$ implies that the integrability condition (3.14) for $\overline{D}_x$ and $\overline{D}_t$ holds.

Example. Let S^∞ be the equation manifold of a "trivial" scalar equation in two independent variables, that is, $S^\infty = J^\infty E$. Set $u_{k_1,k_2} = D_x^{k_1} D_t^{k_2} u$, for $k_1, k_2 \in \mathbb{Z}$, and $u_{0,0} = u$. Introduce the manifold $\overline{S}$ locally by $\overline{S} = \{(x, t, , u, \dots, u_{k_1,k_2}, \dots)\}$ and define the projection map $\pi : \overline{S} \to S^\infty$ in the obvious way. For any pair $(k_1, k_2) \in \mathbb{Z}^2$, let π_{k_1,k_2} be the function

$$\pi_{k_1,k_2} = \begin{cases} \dfrac{x^{-k_1} t^{-k_2}}{(-k_1)!(-k_2)!}, & k_1, k_2 \leq 0, \\ 0, & \text{otherwise}. \end{cases}$$

The *ghost vector fields* γ_{k_1,k_2}, where $(k_1, k_2) \in \mathbb{Z}^2$, are vector fields on $\overline{S}$ defined by the rules

$$\gamma_{k_1,k_2}(u_{m,n}) = \pi_{k_1+m,k_2+n}, \qquad \gamma_{k_1,k_2}(x^i t^j) = 0.$$

Ghost vector fields have been introduced by Olver, Sanders, and Wang [41] and further considered by Olver [40], as a way to extend the Lie bracket of evolutionary vector fields to the nonlocal domain. Now set $X_1 = \sum \gamma_{k,0}$ and $X_2 = \sum \gamma_{0,k}$. Then $(\overline{S}, \overline{D}_x, \overline{D}_t, \pi)$ with $\overline{D}_x = D_x + X_1$ and $\overline{D}_t = D_t + X_2$ is an infinite-dimensional covering of S^∞. That the integrability condition (3.14) holds follows from the fact that ghost vector fields commute with each other (see [40, 41]).

Nonlocal symmetries are defined thus:

Definition 3.7. Let $(\overline{S}, \overline{D}_x, \overline{D}_t, \pi)$ be an N-dimensional covering of the evolution equation $u_t = F$. Assume that $\overline{S}$ is equipped with coordinates $(x, t, u, u_x, \ldots, w^\beta)$, $1 \leq \beta \leq N$, and that $\overline{D}_x$ and $\overline{D}_t$ are given by

$$\overline{D}_x = D_x + \sum_{\beta=1}^{N} X_1^\beta \frac{\partial}{\partial w^\beta} \quad \text{and} \quad \overline{D}_t = D_t + \sum_{\beta=1}^{N} X_2^\beta \frac{\partial}{\partial w^\beta}. \tag{3.16}$$

A *nonlocal symmetry of type* π of $u_t = F$ is a vector field $\overline{D}_\tau$ of the form

$$\overline{D}_\tau = \sum_{i=0}^{\infty} \overline{D}_x^i(G) \frac{\partial}{\partial u_{x^i}} + \sum_\beta I_\beta \frac{\partial}{\partial w^\beta}, \tag{3.17}$$

in which G and I_β, $1 \leq \beta \leq N$, are smooth functions on $\overline{S}$ such that

$$\overline{D}_t G = \sum_{i=0}^{n} \frac{\partial F}{\partial u_{x^i}} \overline{D}_x^i(G), \quad \overline{D}_x(I_\beta) = \overline{D}_\tau(X_1^\beta), \quad \overline{D}_t(I_\beta) = \overline{D}_\tau(X_2^\beta). \tag{3.18}$$

More generally, if $\Xi = 0$ is a scalar differential equation in two independent variables x, t, not necessarily of evolutionary type, a *nonlocal symmetry of type* π of $\Xi = 0$ is a vector field

$$\overline{D}_\tau = \sum \overline{D}_x^i \overline{D}_t^j(G) \frac{\partial}{\partial u_{x^i t^j}} + \sum_\beta I_\beta \frac{\partial}{\partial w^\beta}, \tag{3.19}$$

where $u_{x^i t^j}$ denote intrinsic coordinates on the equation manifold of $\Xi = 0$, such that

$$\overline{\Xi}_*(G) = 0, \quad \overline{D}_x(I_\beta) = \overline{D}_\tau(X_1^\beta), \quad \overline{D}_t(I_\beta) = \overline{D}_\tau(X_2^\beta), \tag{3.20}$$

where $\overline{\Xi}_*$ is the lift of the formal linearization of Ξ to the covering π,

$$\overline{\Xi}_* = \sum \frac{\partial \Xi}{\partial u_{x^i t^j}} \overline{D}_x^i \overline{D}_t^j. \tag{3.21}$$

This definition can be adapted straightforwardly to systems of equations [33, 34], and this extension will be used in what follows without further ado.

Note that the first equations of (3.18) and (3.20) depend only on G and the equation at hand. The vector field $G\partial/\partial u$, or, in the case of an evolution equation,

$$\sum_{i=0}^{\infty} \overline{D}_x^i(G) \frac{\partial}{\partial u_{x^i}}, \tag{3.22}$$

can be interpreted as a vector field on $\overline{S}$ *along* S^∞. This vector field, or simply G, is called the *shadow* of the nonlocal symmetry $\overline{D}_\tau$. In general, vector fields on $\overline{S}$ along S^∞ which satisfy the first equation of (3.18) are called π-*shadows*. An important question is whether one can extend π-shadows to *bona fide* nonlocal symmetries. General theorems along these lines have been proven by Nina Khor'kova in 1988 (see [33, 34]) and by Kiso [31]. An explicit example of such an extension appears in Subsection 4.3 below.

Now one would like to characterize nonlocal symmetries of strictly pseudo-spherical evolution equations. Let $u_t = F$ be an n^{th} order strictly pseudo-spherical equation with associated one-forms ω^i, $i = 1,2,3$, and equation manifold S^∞, and consider a covering $(\overline{S}, \overline{D}_x, \overline{D}_t, \pi)$ of S^∞. One first extends the "horizontal" exterior derivative operator from S^∞ to $\overline{S}$ thus [34]:

If $\omega = \sum a_{i_1 \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is a horizontal differential form on $\overline{S}$, in which $x_1 = x$, $x_2 = t$, then

$$\overline{d}_H \omega = \sum (\overline{D}_x a_{i_1 \dots i_k} dx + \overline{D}_t a_{i_1 \dots i_k} dt) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Next, let G be a function on $\overline{S}$. In analogy with the generalized symmetry case, one studies the one-forms $\omega^i + \tau \Lambda_i$ on $\overline{S}$, in which $\Lambda_i = g_{i1} dx + g_{i2} dt$ and

$$g_{i1} = f_{i1,u} G, \qquad g_{i2} = \sum_{k=0}^{n-1} f_{i2,u_{x^k}} \overline{D}_x^k G, \qquad i = 1,2,3. \tag{3.23}$$

Theorem 3.8. *Let $u_t = F(x,t,u,\dots,u_{x^n})$ be strictly pseudo-spherical with associated one-forms $\omega^i = f_{i1} dx + f_{i2} dt$, $i = 1,2,3$. Let G be a smooth function on a covering $(\overline{S}, \overline{D}_x, \overline{D}_t, \pi)$ of the equation manifold S^∞, and consider the deformed one-forms $\omega^\alpha + \tau \Lambda_\alpha$ defined above. They satisfy the structure equations*

$$\overline{d}_H \sigma^1 = \sigma^3 \wedge \sigma^2, \qquad \overline{d}_H \sigma^2 = \sigma^1 \wedge \sigma^3, \qquad \overline{d}_H \sigma^3 = \sigma^1 \wedge \sigma^2, \tag{3.24}$$

up to terms of order τ^2 if and only if G is a π-shadow of the equation $u_t = F$.

Proof. The one-forms $\omega^\alpha + \tau \Lambda_\alpha$ satisfy (3.24) up to terms of order τ^2 if and only if

$$-\overline{D}_t g_{11} + \overline{D}_x g_{12} = f_{31} g_{22} - f_{32} g_{21} + f_{22} g_{31} - f_{21} g_{32}, \tag{3.25}$$
$$-\overline{D}_t g_{21} + \overline{D}_x g_{22} = f_{11} g_{32} - f_{12} g_{31} + f_{32} g_{11} - f_{31} g_{12}, \tag{3.26}$$
$$-\overline{D}_t g_{31} + \overline{D}_x g_{32} = f_{11} g_{22} - f_{12} g_{21} + f_{22} g_{11} - f_{21} g_{12}. \tag{3.27}$$

Since $u_t = F$ is strictly pseudo-spherical, equations (2.7) of Lemma 2.7 are identities. Take Lie derivatives with respect to the vector field L_τ defined in (3.22), and substitute into (3.25)–(3.27). One finds that these equations are satisfied if and only if

$$-f_{\alpha 1,u} \overline{D}_t(G) + f_{\alpha 1,u} \sum_{i=0}^{n} \frac{\partial \overline{F}}{\partial u_{x^i}} \overline{D}_x^i(G) = 0, \qquad i = 1,2,3. \tag{3.28}$$

Since the constraint (2.6) holds, one concludes that equations (3.25)–(3.27) are satisfied if and only if G is a π-shadow of the equation $u_t = F$. $\square$

Theorem 3.8 appeared for the first time in [45]; it is included here for ease of reference.

4. Shallow water equations

In this section the equations due to Korteweg and de Vries [32],

$$u_t = u_{xxx} + 6uu_x, \tag{4.1}$$

Camassa and Holm [11],

$$m = u_{xx} - u, \qquad m_t = -m_x u - 2mu_x, \tag{4.2}$$

and Hunter and Saxton [26],

$$m = u_{xx}, \qquad m_t = -m_x u - 2mu_x, \tag{4.3}$$

are studied by taking advantage of the fact that they are members of a two-parameter family of equations of pseudo-spherical type.

Of course, the KdV equation has been subject of an impressive body of research since [32], and in fact, Peter Olver has pointed out that (4.1) was derived already in the 1870's by J. Boussinesq, who also found its first three conservation laws, and its one-soliton and periodic traveling wave solutions (see [8, 9]).

Cocerning the integrability properties of the important Camassa–Holm [11] and Hunter–Saxton equations [26] the following (at least) is known: their analysis by scattering/inverse scattering has been carried out (Beals, Sattinger and Szmigielski [4, 5]; Constantin and McKean [16]), their bi-hamiltonian character has been discussed (in the CH case this was first observed by B. Fuchssteiner and A. S. Fokas; see [24, 23, 20] and references therein. The bi-hamiltonian formulation of CH appears also in [11] together with a discussion on its Lax pair formulation; the bi-hamiltonian structure of the HS equation has been discussed by J. K. Hunter and Y. X. Zheng [27]), and moreover, it has been proven that the Korteweg–de Vries, Camassa–Holm and Hunter–Saxton equations exhaust, in a precise sense, the bi-hamiltonian equations which can be modeled as geodesic flows on (homogeneous spaces related to) the Virasoro group (Khesin and Misiołek [30]).

4.1. Pseudo-spherical structures

That the KdV equation describes pseudo-spherical surfaces was observed by Sasaki [48] and also by Chern and Tenenblat, who obtained this result from some general classification theorems proven by them in [13].

Example. The KdV equation $u_t = u_{xxx} + 6uu_x$ describes pseudo-spherical surfaces [48, 13] with associated one-forms $\omega^i = f_{i1}dx + f_{i2}dt$, in which

$$\omega^1 = (1-u)dx + (-u_{xx} + \lambda u_x - \lambda^2 u - 2u^2 + \lambda^2 + 2u)dt, \tag{4.4}$$

$$\omega^2 = \lambda dx + (\lambda^3 + 2\lambda u - 2u_x)dt, \tag{4.5}$$

$$\omega^3 = (-1-u)dx + (-u_{xx} + \lambda u_x - \lambda^2 u - 2u^2 - \lambda^2 - 2u)dt, \tag{4.6}$$

and λ is an arbitrary parameter.

The analysis carried out in Section 3 allows one to obtain the standard quadratic pseudo-potential for KdV found by Wahlquist and Eastbrook in [56]:

Example. Consider the KdV equation $u_t = u_{xxx} + 6uu_x$, and the associated one-forms ω^i given by (4.4)–(4.6). Rotate the coframe $\{\omega^1, \omega^2\}$ determined by (4.4)–(4.6) through $\pi/2$, and change Γ to $-\Gamma$. One can then write the Pfaffian system (3.6) as

$$\text{(a) } \Gamma_x = -u - \lambda\Gamma - \Gamma^2, \qquad \text{(b) } \Gamma_t = (\Gamma_{xx} - 3\Gamma^2\lambda - 2\Gamma^3)_x.$$

Since the KdV equation is strictly pseudo-spherical, $u(x,t)$ as determined by (a) solves KdV if $\Gamma(x,t)$ solves (b). One has thus recovered the Miura transformation and the modified KdV equation from geometrical considerations.

Henceforth ϵ will denote a real parameter. Theorem 4.1 below first appeared in [47]. It is reproduced here since its proof will be used momentarily.

Theorem 4.1. *The Camassa–Holm and Hunter–Saxton equations,* (4.2) *and* (4.3) *respectively, describe pseudo-spherical surfaces.*

Proof. Consider one-forms ω^i, $i = 1, 2, 3$, given by

$$\begin{aligned} \omega^1 &= (m - \beta + \epsilon\alpha^{-2}(\beta - 1))dx \\ &\quad + (-u_x\beta\alpha^{-1} - \beta\alpha^{-2} - um - 1 + u\beta + u_x\alpha^{-1} + \alpha^{-2})dt, && (4.7) \\ \omega^2 &= \alpha dx + (-\beta\alpha^{-1} - \alpha u + \alpha^{-1} + u_x)dt, && (4.8) \\ \omega^3 &= (m+1)dx + \left(\epsilon u\frac{\beta - 1}{\alpha^2} - um + \frac{1}{\alpha^2} + \frac{u_x}{\alpha} - u - \frac{\beta}{\alpha^2} - \frac{u_x\beta}{\alpha}\right)dt, && (4.9) \end{aligned}$$

where $m = u_{xx} - \epsilon u$ and the parameters α and β are constrained by the relation

$$\alpha^2 + \beta^2 - 1 = \epsilon\left[\frac{\beta - 1}{\alpha}\right]^2. \tag{4.10}$$

It is not hard to check that the structure equations (2.2) are satisfied whenever the equation

$$-2u_xu_{xx} + 3u_x\epsilon u - uu_{xxx} + \epsilon u_t - u_{xxt} = 0 \tag{4.11}$$

holds, and equation (4.11) becomes the Camassa–Holm equation (4.2) if $\epsilon = 1$, and the Hunter–Saxton equation (4.3) if $\epsilon = 0$. □

In order to include the KdV equation into the picture, one applies the Galilei transformation $\mathbf{T} : (X, T, U) \mapsto (x, t, u)$ given by

$$x = \frac{X}{\nu} + \frac{T}{\nu^3} - \frac{T}{\sqrt{\nu}}, \tag{4.12}$$

$$t = \frac{T}{\sqrt{\nu}}, \tag{4.13}$$

$$u = \frac{U}{\sqrt{\nu}} + \frac{1}{3}\frac{1}{\nu^{5/2}} - \frac{1}{3}, \tag{4.14}$$

to equation (4.11) and to the one-forms (4.7)–(4.9):

Corollary 4.2. *The nonlinear equation*

$$-2\nu^2 U_X U_{XX} + 3\epsilon U_X U - \nu^2 U_{XXX} U + \tfrac{2}{3}(1-\nu^{5/2})U_{XXX} + \epsilon U_T - \nu^2 U_{XXT} = 0 \tag{4.15}$$

describes pseudo-spherical surfaces with associated one-forms $\mathbf{T}^*\omega^i$*, in which* ω^i*,* $i = 1, 2, 3,$ *are given by* (4.7)–(4.9)*.*

Equation (4.15) does contain the KdV, CH, and HS equations as special cases, but the one-forms $\mathbf{T}^*\omega^i$ are singular in the KdV limit $\nu \to 0$. For example, $\mathbf{T}^*\omega^2$ is

$$\mathbf{T}^*\omega^2 = \frac{\alpha}{\nu}dX + \left(-\frac{\beta}{\sqrt{\nu}\alpha} - \frac{\alpha U}{\nu} + \frac{2}{3}\frac{\alpha}{\nu^3} - \frac{2}{3}\frac{\alpha}{\sqrt{\nu}} + \frac{1}{\sqrt{\nu}\alpha} + U_X\right)dT.$$

This difficulty is dealt with in the following subsection.

Remark 4.3. Equation (4.15) with $\epsilon = 1$ and $1 - \nu^{5/2} = \gamma$ has been derived as a shallow water equation by Dullin, Gottwald, and Holm [17] via an asymptotic expansion of the Euler equations. The well-posedness of the Cauchy problem for the Dullin–Gottwald–Holm equation and its analysis via scattering/inverse scattering have been discussed by L. Tian, G. Gui and Y. Liu [53].

Remark 4.4. Equation (4.15) can be interpreted as a geodesic equation on the Virasoro group. In fact, (4.15) is in the class of equations studied by Khesin and Misiołek in [30]: it is their equation (3.9) with $\beta = \nu^2$, $\alpha = \epsilon$, and $b = (2/3)(1 - \nu^{5/2})$.

Remark 4.5. Equation (4.15) is bi-hamiltonian. Indeed (this fact was pointed out by a referee) it is equation (4) in A. Fokas' paper [20] if one eliminates the term u_x in the latter equation via an appropriate translation $u \mapsto u+k$, and then replaces t by T/ϵ, chooses $\alpha = 3\epsilon$, $\nu = -N^2/\epsilon$, and $\beta = 2/3 - (2/3)N^{5/2}$, and finally changes N to ν.

Now one dispenses with the constraint (4.10) by using a parametrization of the family of curves $\alpha^2 + \beta^2 - 1 = \epsilon[(\beta - 1)/\alpha]^2$. For example one can take

$$\alpha = \sqrt{\epsilon + 1 - s^2}, \qquad \beta = \frac{\epsilon}{s-1} - s. \tag{4.16}$$

After rotating by $\pi/2$ and using (4.16), the one-forms $\mathbf{T}^*\omega^i$ associated with equation (4.15) become:

$$\begin{aligned}\mathbf{T}^*\omega^1 &= \frac{\sqrt{\epsilon+1-s^2}}{\nu}dX \\ &\quad + \left(\sqrt{\epsilon+1-s^2}\left(\frac{2}{3\nu^3} - \frac{2s+1}{3\sqrt{\nu}(s-1)} - \frac{U}{\nu}\right) + U_X\right)dT,\end{aligned} \tag{4.17}$$

$$\mathbf{T}^*\omega^2 = -\frac{1}{3}\frac{-3\epsilon U\nu^2 - \epsilon + \epsilon\nu^{5/2} + 3s\nu^{5/2} + 3\nu^4 U_{XX}}{\nu^{7/2}}dX$$
$$+ \frac{1}{9}\frac{1}{\nu^{11/2}}\Bigg(6\nu^4(\nu^{5/2}-1)U_{XX} - 9\epsilon\nu^4U^2 + 9\nu^6 U_{XX}U$$
$$- 9U_X\frac{\sqrt{\epsilon+1-s^2}}{1-s}\nu^{11/2} + 3\nu^2\left[\epsilon + \nu^{5/2}\left(3s+\epsilon\frac{s+2}{1-s}\right)\right]U$$
$$- \nu^{5/2}\left(6s - \epsilon\frac{4s-1}{1-s}\right) + 2\epsilon - \frac{1+2s}{1-s}(3s+\epsilon)\nu^5\Bigg)dT, \tag{4.18}$$
$$\mathbf{T}^*\omega^3 = \left(U_{XX}\sqrt{\nu} - \frac{\epsilon U}{\nu^{3/2}} - \frac{\epsilon}{3\nu^{7/2}} + \frac{\epsilon}{3\nu} + \frac{1}{\nu}\right)dX + \frac{1}{s-1}\Bigg[\sqrt{\nu}(1-s)U_{XX}U$$
$$+ \frac{1}{3}\left[(1-s)\left(\frac{\epsilon}{\nu^{7/2}} + \frac{3}{\nu}\right) + (s+2)\frac{\epsilon}{\nu}\right]U - \frac{\epsilon}{\nu^{3/2}}(1-s)U^2 - U_X\sqrt{\epsilon+1-s^2}$$
$$+ \frac{2}{3}(1-s)\left(\frac{\epsilon}{3\nu^{11/2}} - \frac{1}{\nu^3}\right) + \frac{\epsilon}{9\nu^3}(4s-1) - \frac{1}{3\sqrt{\nu}}\left(\frac{\epsilon}{3}+1\right)(1+2s)$$
$$+ \frac{2}{3}\left(\nu - \frac{1}{\nu^{3/2}}\right)(1-s)U_{XX}\Bigg]dT. \tag{4.19}$$

Corollary 4.6. *The nonlinear equation* (4.15) *is geometrically integrable.*

4.2. Pseudo-potentials

The one-forms (4.17)–(4.19) can now be used to compute the quadratic pseudo-potential (3.8) associated with Equation (4.15). The resulting formulae are very involved, but they can be simplified as follows. After writing down (3.8) with the help of the one-forms $\mathbf{T}^*\omega^i$, one applies the transformation

$$\hat{\Gamma} \mapsto \hat{\gamma}\sqrt{\nu} + \frac{\sqrt{\epsilon+1-s^2}}{1-s},$$

and changes the parameter s by setting

$$s - 1 = \sqrt{\nu}/\lambda.$$

Theorem 4.7. *Equation* (4.15) *admits a quadratic pseudo-potential* $\hat{\gamma}(X,T)$ *determined by the compatible system*

$$\frac{\partial}{\partial X}\hat{\gamma} = -\frac{1}{2}\frac{\hat{\gamma}^2}{\lambda} - \frac{\epsilon U}{\nu^2} + \frac{1}{2}\frac{\lambda\epsilon}{\nu^2} + \frac{1}{3}\frac{\epsilon}{\nu^{3/2}} + U_{XX} - \frac{1}{3}\frac{\epsilon}{\nu^4} \tag{4.20}$$

and

$$\frac{\partial}{\partial T}\hat{\gamma} = \left(\frac{1}{2}\frac{U}{\lambda} - \frac{1}{3}\frac{1}{\nu^2\lambda} + \frac{1}{2} + \frac{1}{3}\frac{\sqrt{\nu}}{\lambda}\right)\hat{\gamma}^2 - U_X\hat{\gamma} - \frac{2}{3}U_{XX}\sqrt{\nu}$$
$$- \frac{2}{9}\frac{\epsilon}{\nu^6} - U_{XX}U + \frac{\epsilon U^2}{\nu^2} - \frac{2}{3}\frac{\lambda\epsilon}{\nu^{3/2}} + \frac{2}{3}\frac{\lambda\epsilon}{\nu^4} + \frac{1}{2}\frac{\lambda\epsilon U}{\nu^2} - \frac{1}{2}\frac{\lambda^2\epsilon}{\nu^2}$$
$$- \frac{2}{9}\frac{\epsilon}{\nu} - \frac{1}{3}\frac{\epsilon U}{\nu^4} + \frac{4}{9}\frac{\epsilon}{\nu^{7/2}} + \frac{1}{3}\frac{\epsilon U}{\nu^{3/2}} + \frac{2}{3}\frac{U_{XX}}{\nu^2}, \tag{4.21}$$

in which $\lambda \neq 0$ *is a real parameter.*

Using the pseudo-potential $\hat{\gamma}$ one can simplify the linear problem associated with equation (4.15) which follows from (2.4) and the one-forms $\mathbf{T}^*\omega^i$. Applying Propositions 2.5 and 3.4 one finds the following result:

Proposition 4.8. *The family of equations* (4.15) *is the integrability condition of the one-parameter family of linear problems* $d\psi = (Xdx + Tdt)\psi$, *in which the matrices* X *and* T *are given by*

$$X = \begin{bmatrix} 0 & -\frac{1}{2}\lambda^{-1} \\ \frac{1}{3}\frac{\epsilon}{\nu^4} - \frac{1}{2}\frac{\lambda\epsilon}{\nu^2} - \frac{1}{3}\frac{\epsilon}{\nu^{3/2}} - U_{XX} + \frac{\epsilon U}{\nu^2} & 0 \end{bmatrix} \tag{4.22}$$

and

$$T = \begin{bmatrix} \frac{1}{2}U_X & \frac{U}{2\lambda} - \frac{1}{3\nu^2\lambda} + \frac{1}{2} + \frac{\sqrt{\nu}}{3\lambda} \\ \begin{array}{c} \frac{2}{3}U_{XX}\sqrt{\nu} - \frac{1}{2}\frac{\lambda\epsilon U}{\nu^2} - \frac{4}{9}\frac{\epsilon}{\nu^{7/2}} + \frac{2}{9}\frac{\epsilon}{\nu} - \frac{1}{3}\frac{\epsilon U}{\nu^{3/2}} \\ + \frac{2}{3}\frac{\lambda\epsilon}{\nu^{3/2}} - \frac{2}{3}\frac{\lambda\epsilon}{\nu^4} + \frac{2}{9}\frac{\epsilon}{\nu^6} \\ + \frac{1}{2}\frac{\lambda^2\epsilon}{\nu^2} - \frac{2}{3}\frac{U_{XX}}{\nu^2} + U_{XX}U - \frac{\epsilon U^2}{\nu^2} + \frac{1}{3}\frac{\epsilon U}{\nu^4} \end{array} & -\frac{1}{2}U_X \end{bmatrix}. \tag{4.23}$$

The linear problem $d\psi = (Xdx + Tdt)\psi$ with X, T given by (4.22) and (4.23) can be used to find a linear problem associated to (4.15) which is not singular at the KdV limit $\nu \to 0$: Applying the gauge transformation

$$X_A = AXA^{-1} + A_xA^{-1}, \quad T_A = ATA^{-1} + A_tA^{-1},$$

in which

$$A = \begin{bmatrix} 0 & \nu \\ -\nu^{-1} & 0 \end{bmatrix},$$

and changing the parameter λ to ζ by means of

$$\lambda = \tfrac{2}{3}\nu^{-2} + \tfrac{2}{3}\zeta, \tag{4.24}$$

one finds that equation (4.15) is the integrability condition of the linear problem $d\psi = (X_Adx + T_Adt)\psi$ with

$$X_A = \begin{bmatrix} 0 & \frac{1}{3}\epsilon\zeta + \frac{1}{3}\sqrt{\nu}\epsilon + \nu^2U_{XX} - \epsilon U \\ \frac{3}{4}(1 + \zeta\nu^2)^{-1} & 0 \end{bmatrix} \tag{4.25}$$

and

$$T_A = \begin{bmatrix} -\frac{1}{2}U_X & -\frac{2}{3}\nu^{5/2}U_{XX} + \frac{1}{3}\epsilon U\zeta - \frac{2}{9}\nu\epsilon \\ & +\frac{1}{3}\sqrt{\nu}\epsilon U - \frac{4}{9}\sqrt{\nu}\epsilon\zeta - \frac{2}{9}\epsilon\zeta^2 \\ & +\frac{2}{3}U_{XX} - \nu^2 U_{XX}U + \epsilon U^2 \\ -\dfrac{\frac{3}{4}U + \frac{1}{2}\zeta + \frac{1}{2}\sqrt{\nu}}{1+\zeta\nu^2} & \frac{1}{2}U_X \end{bmatrix}. \tag{4.26}$$

Example. If $\nu = 0$ and $\epsilon = 1$ the matrices X_A and T_A read

$$X_A = \begin{bmatrix} 0 & \frac{1}{3}\zeta - U \\ \frac{3}{4} & 0 \end{bmatrix}, \quad T_A = \begin{bmatrix} -\frac{1}{2}U_X & \frac{1}{3}U\zeta - \frac{2}{9}\zeta^2 + \frac{2}{3}U_{XX} + U^2 \\ -\frac{3}{4}U - \frac{1}{2}\zeta & \frac{1}{2}U_X \end{bmatrix}, \tag{4.27}$$

which is the usual linear problem associated to the KdV equation [48]. On the other hand, if $\nu = 1$, equation (4.24) implies that the matrices X_A and T_A become

$$X_A = \frac{1}{2}\begin{bmatrix} 0 & \epsilon\lambda + 2m \\ \lambda^{-1} & 0 \end{bmatrix}, T_A = \frac{1}{2}\begin{bmatrix} -U_X & -2Um + \epsilon\lambda U - \epsilon\lambda^2 \\ -1 - U\lambda^{-1} & U_X \end{bmatrix}, \tag{4.28}$$

where $m = U_{XX} - \epsilon U$, and one recovers the associated linear problems for the CH and HS equations derived in [47].

Corollary 4.9. (a) *The nonlinear equation*

$$-2\nu^2 U_X U_{XX} + 3\epsilon U_X U - \nu^2 U_{XXX}U + \tfrac{2}{3}(1-\nu^{5/2})U_{XXX} + \epsilon U_T - \nu^2 U_{XXT} = 0 \tag{4.29}$$

describes pseudo-spherical surfaces with associated one-forms $\omega^i = f_{i1}dx + f_{i2}dt$ *in which*

$$f_{11} = \tfrac{1}{3}\epsilon\zeta + \tfrac{1}{3}\sqrt{\nu}\epsilon + \nu^2 U_{XX} - \epsilon U + \tfrac{3}{4}(1+\zeta\nu^2)^{-1}, \tag{4.30}$$

$$f_{12} = \tfrac{2}{3}(1-\nu^{5/2})U_{XX} + \tfrac{1}{3}\epsilon(\zeta + \sqrt{\nu})U + \epsilon U^2 - \nu^2 U U_{XX} - \tfrac{2}{9}\epsilon(\zeta^2 + 2\sqrt{\nu}\zeta + \nu) - \frac{1}{2(1+\zeta\nu^2)}(\tfrac{3}{2}U + \zeta + \sqrt{\nu}), \tag{4.31}$$

$$f_{21} = 0, \tag{4.32}$$

$$f_{22} = -U_X, \tag{4.33}$$

$$f_{31} = -\tfrac{1}{3}\epsilon\zeta - \tfrac{1}{3}\sqrt{\nu}\epsilon - \nu^2 U_{XX} + \epsilon U + \tfrac{3}{4}(1+\zeta\nu^2)^{-1}, \tag{4.34}$$

$$f_{32} = -\tfrac{2}{3}(1-\nu^{5/2})U_{XX} - \tfrac{1}{3}\epsilon(\zeta + \sqrt{\nu})U - \epsilon U^2 + \nu^2 U U_{XX} + \tfrac{2}{9}\epsilon(\zeta^2 + 2\sqrt{\nu}\zeta + \nu) - \frac{1}{2(1+\zeta\nu^2)}(\tfrac{3}{2}U + \zeta + \sqrt{\nu}), \tag{4.35}$$

and ζ *is an arbitrary parameter.*

(b) *Equation* (4.29) *admits the quadratic pseudo-potential* $\gamma(X,T)$ *determined by the Pfaffian system*

$$-\gamma_X = \frac{3}{4(1+\zeta\nu^2)}\gamma^2 - (\nu^2 U_{XX} - \epsilon U + \tfrac{1}{3}\epsilon(\zeta+\sqrt{\nu})), \tag{4.36}$$

$$\begin{aligned} -\gamma_T = {} & \frac{1}{2(1+\zeta\nu^2)}(-\tfrac{3}{2}U - \zeta - \sqrt{\nu})\gamma^2 + U_X\gamma \\ & + (U(\nu^2 U_{XX} - \epsilon U) + \tfrac{2}{3}(\nu^{5/2}-1)U_{XX} - \tfrac{1}{3}\epsilon(\zeta+\sqrt{\nu})U \\ & + \tfrac{2}{9}\epsilon(\zeta^2 + 2\sqrt{\nu}\zeta + \nu)). \end{aligned} \tag{4.37}$$

Example. Taking $\nu = 0$ and $\epsilon = 1$ in (4.36) gives

$$-\gamma_X = \tfrac{3}{4}\gamma^2 + U - \tfrac{1}{3}\zeta,$$

and one recovers the usual Miura transformation for the KdV equation. On the other hand, taking $\nu = 1$ and $\lambda = \frac{2}{3}(1+\zeta)$ in (4.36) and (4.37) yields the system

$$U_{XX} - \epsilon U = \gamma_X + \tfrac{\gamma^2}{2\lambda} - \tfrac{\epsilon}{2}\lambda, \tag{4.38}$$

$$-\gamma_T = -\tfrac{1}{2}(U/\lambda + 1)\gamma^2 + U_X\gamma + (U(U_{XX} - \epsilon U) - \tfrac{1}{2}\epsilon U\lambda + \tfrac{1}{2}\epsilon\lambda^2). \tag{4.39}$$

Substitution of (4.38) into (4.39) implies that the Camassa–Holm equation (4.2) and the Hunter–Saxton equation (4.3) possess the parameter-dependent conservation law

$$\gamma_T = \lambda\left(U_X - \gamma - \frac{1}{\lambda}U\gamma\right)_X. \tag{4.40}$$

As in the KdV case, one can use (4.38) and (4.40) to construct conservation laws for the CH and HS equations [11, 19, 27, 46, 47]. Setting $\gamma = \sum_{n=1}^{\infty}\gamma_n\lambda^{n/2}$ yields the conserved densities

$$\begin{aligned} \gamma_1 &= \sqrt{2}\sqrt{m}, \\ \gamma_2 &= -\tfrac{1}{2}\ln(m)_X, \\ \gamma_3 &= \frac{1}{2\sqrt{2}\sqrt{m}}\left[\epsilon - \frac{m_X^2}{4m^2} + \ln(m)_{XX}\right], \\ \gamma_{n+1} &= -\frac{1}{\gamma_1}\gamma_{n,X} - \frac{1}{2\gamma_1}\sum_{j=2}^{n}\gamma_j\gamma_{n+2-j}, \qquad n \geq 3, \end{aligned}$$

in which $m = U_{XX} - \epsilon U$, while the expansion $\gamma = \epsilon\lambda + \sum_{n=0}^{\infty}\gamma_n\lambda^{-n}$ implies

$$\gamma_{0,X} + \epsilon\gamma_0 = m, \qquad \gamma_{n,X} + \epsilon\gamma_n = -\frac{1}{2}\sum_{j=0}^{n-1}\gamma_j\gamma_{n-1-j}, \qquad n \geq 1. \tag{4.41}$$

In the CH case, (4.41) allows one to find the local conserved densities U, $U_X^2 + U^2$, and $UU_X^2 + U^3$ appearing in [11], and a sequence of nonlocal conservation laws.

In view of the foregoing example, it is natural to postulate equation (4.38) as the analog of the Miura transformation for the CH and HS equations, and (4.40) as the corresponding "modified" equation. Note that, in contradistinction to the KdV case, the modified CH and HS equations are nonlocal equations for γ.

Remark 4.10. The question whether there exists a modified CH equation has been asked by J. Schiff in [50]. Earlier contributions to this problem have been made by Fokas [20], Fuchssteiner [23], Schiff [51], and Camassa and Zenchuk [12]. The reader is referred to [46] for a discussion on the relation between these works and the modified equation proposed here.

4.3. Nonlocal symmetries

In this subsection it is shown that one can find a nonlocal symmetry of equation (4.29) starting from the pseudo-potential $\gamma(X,T)$ given by (4.36), (4.37). First of all, note that substitution of (4.36) into (4.37) yields the conservation law

$$\gamma_T = [\tfrac{2}{3}(\zeta\nu^2+1)U_X - \tfrac{2}{3}(\zeta+\sqrt{\nu})\gamma - \gamma U]_X. \tag{4.42}$$

Then, in analogy with the Camassa–Holm case [46], one obtains:

Theorem 4.11. *Set $m = \nu^2 U_{XX} - \epsilon U$, define γ by the equations* (4.36) *and* (4.37)*, and let δ be a potential for the conservation law* (4.42)*. Then the nonlocal vector field*

$$V = \gamma \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial U} \tag{4.43}$$

determines a shadow of a nonlocal symmetry for the nonlinear equation (4.29).

Proof. Equation (4.42) implies that the potential δ satisfies the equations

$$\delta_X = \gamma, \qquad \delta_T = \tfrac{2}{3}(\zeta\nu^2+1)U_X - \tfrac{2}{3}(\zeta+\sqrt{\nu})\gamma - U\gamma, \tag{4.44}$$

which are compatible on solutions of (4.29). One can define a covering $\overline{S}$ of the equation manifold of (4.29) as follows. Locally, $\overline{S}$ is equipped with coordinates $(X,T,U,\ldots,U_{X^pT^q},\ldots,\gamma,\delta)$ where $p = 1,2,3,\ldots$ and $q = 0,1,2,\ldots$, and the total derivatives $\overline{D}_X$ and $\overline{D}_T$ are given by

$$\begin{aligned}
\overline{D}_X &= D_X + \left(\frac{-3}{4(1+\zeta\nu^2)}\gamma^2 + m + \tfrac{1}{3}\epsilon(\zeta+\sqrt{\nu})\right)\frac{\partial}{\partial\gamma} + \gamma\frac{\partial}{\partial\delta},\\
\overline{D}_T &= D_T + \Bigg(\frac{1}{2(1+\zeta\nu^2)}(\tfrac{3}{2}U+\zeta+\sqrt{\nu})\gamma^2 - U_X\gamma - U(\nu^2U_{XX}-\epsilon U)\\
&\quad - \tfrac{2}{3}(\nu^{5/2}-1)U_{XX} + \tfrac{1}{3}\epsilon(\zeta+\sqrt{\nu})U - \tfrac{2}{9}\epsilon(\zeta^2+2\sqrt{\nu}\zeta+\nu)\Bigg)\frac{\partial}{\partial\gamma}\\
&\quad + (\tfrac{2}{3}(\zeta\nu^2+1)U_X - \tfrac{2}{3}(\zeta+\sqrt{\nu})\gamma - U\gamma)\frac{\partial}{\partial\delta}.
\end{aligned}$$

Now, the vector field V determines the shadow of a nonlocal symmetry if the function

$$G = \gamma \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right) \tag{4.45}$$

satisfies the equation

$$\overline{D}_T G = \sum_{i=0}^{3} \frac{\partial F}{\partial u_{X^i}} \overline{D}^i_X(G) + \frac{\partial F}{\partial u_{XXT}} \overline{D}^2_X \overline{D}_T(G)$$

identically, in which F is the right hand side of (4.29) when written as $U_T = F$. Checking that this is so is a long but straightforward computation. It can be done using the MAPLE package VESSIOT developed by I. Anderson and his coworkers (see [3]). □

The next problem is to extend the shadow (4.43) to a *bona fide* nonlocal symmetry. For this, one needs to study the variations of the functions γ and δ induced by the infinitesimal deformation $U \mapsto U + \tau G$, in which G is given by (4.45).

Note that (4.29) can be written as a system of equations for two variables, m and U, as follows:

$$m = \nu^2 U_{XX} - \epsilon U, \qquad m_T = -m_X U - 2mU_X + \tfrac{2}{3}(1 - \nu^{5/2})U_{XXX}. \tag{4.46}$$

Theorem 4.12. *Let γ, δ and β be defined by the equations*

$$\gamma_X = -\frac{3}{4(1+\zeta\nu^2)}\gamma^2 + (m + \tfrac{1}{3}\epsilon(\zeta + \sqrt{\nu})), \tag{4.47}$$

$$\gamma_T = [\tfrac{2}{3}(\zeta\nu^2 + 1)U_X - \tfrac{2}{3}(\zeta + \sqrt{\nu})\gamma - \gamma U]_X, \tag{4.48}$$

$$\delta_X = \gamma, \tag{4.49}$$

$$\delta_T = \tfrac{2}{3}(\zeta\nu^2 + 1)U_X - \tfrac{2}{3}(\zeta + \sqrt{\nu})\gamma - U\gamma, \tag{4.50}$$

$$\beta_X = [\nu^2 m + \tfrac{1}{3}\epsilon(\nu^{5/2} - 1)] \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right), \tag{4.51}$$

$$\begin{aligned}\beta_T = {} & [-\tfrac{1}{3}(\nu^{5/2} - 1)(2m + \epsilon U) - \tfrac{1}{2}\gamma^2 + \tfrac{2}{9}\epsilon(2\zeta + \zeta^2\nu^2 - \nu^3 + 2\sqrt{\nu}) \\ & - \nu^2 U m] \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right),\end{aligned} \tag{4.52}$$

which are compatible on solutions of (4.46). *The system of equations* (4.46)–(4.52) *possesses the symmetry*

$$\begin{aligned} W = {} & \gamma \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial U} \\ & + \left[\nu^2 m_X + \frac{3\nu^2\gamma}{1+\zeta\nu^2} m + \gamma\epsilon\frac{\nu^{5/2} - 1}{1+\zeta\nu^2}\right] \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial m} \\ & + [\nu^2 m + \tfrac{1}{3}\epsilon(\nu^{5/2} - 1)] \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right)\frac{\partial}{\partial \gamma} \\ & + \beta\frac{\partial}{\partial\delta} \\ & + \left(\nu^2[\nu^2 m + \tfrac{1}{3}\epsilon(\nu^{5/2} - 1)]\exp\left(\frac{3\delta}{1+\zeta\nu^2}\right) + \frac{3}{4(1+\zeta\nu^2)}\beta^2\right)\frac{\partial}{\partial\beta}. \end{aligned} \tag{4.53}$$

As with Theorem 4.11, Theorem 4.12 can be verified using the MAPLE package VESSIOT [3]. In terms of the theory of coverings, one has

Corollary 4.13. *The vector field* (4.53) *determines a nonlocal symmetry of the system of equations* (4.46)*.*

Proof. Define a covering $\overline{S}$ of the equation manifold of (4.46): locally, $\overline{S}$ is equipped with coordinates $(X, T, U, U_X, U_T, \dots, U_{X^{2p+1}T^q}, \dots, m, \dots, m_{X^r}, \dots, \gamma, \delta, \beta)$, and the total derivatives $\overline{D}_X$ and $\overline{D}_T$ are given by the formulae

$$\begin{aligned}
\overline{D}_X &= D_X + \left(\frac{-3}{4(1+\zeta\nu^2)}\gamma^2 + m + \tfrac{1}{3}\epsilon(\zeta + \sqrt{\nu}) \right) \frac{\partial}{\partial \gamma} \\
&\quad + \gamma \frac{\partial}{\partial \delta} + [\nu^2 m + \tfrac{1}{3}\epsilon(\nu^{5/2} - 1)] \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2} \right) \frac{\partial}{\partial \beta}, \\
\overline{D}_T &= D_T + \left(\frac{1}{2(1+\zeta\nu^2)} (\tfrac{3}{2}U + \zeta + \sqrt{\nu})\gamma^2 - U_X\gamma - U(\nu^2 U_{XX} - \epsilon U) \right. \\
&\quad \left. - \tfrac{2}{3}(\nu^{5/2} - 1)U_{XX} + \tfrac{1}{3}\epsilon(\zeta + \sqrt{\nu})U - \tfrac{2}{9}\epsilon(\zeta^2 + 2\sqrt{\nu}\zeta + \nu) \right) \frac{\partial}{\partial \gamma} \\
&\quad + (\tfrac{2}{3}(\zeta\nu^2 + 1)U_X - \tfrac{2}{3}(\zeta + \sqrt{\nu})\gamma - U\gamma) \frac{\partial}{\partial \delta} \\
&\quad + [-\tfrac{1}{3}(\nu^{5/2} - 1)(2m + \epsilon U) - \tfrac{1}{2}\gamma^2 + \tfrac{2}{9}\epsilon(2\zeta + \zeta^2\nu^2 - \nu^3 + 2\sqrt{\nu}) \\
&\quad - \nu^2 U m] \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2} \right) \frac{\partial}{\partial \beta}.
\end{aligned}$$

One now defines the functions

$$\begin{aligned}
G_1 &= \gamma \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2} \right), \\
G_2 &= \left[\nu^2 m_X + \frac{3\nu^2\gamma}{1+\zeta\nu^2} m + \gamma\epsilon \frac{\nu^{5/2} - 1}{1+\zeta\nu^2} \right] \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2} \right) \\
I_\gamma &= [\nu^2 m + \tfrac{1}{3}\epsilon(\nu^{5/2} - 1)] \exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2} \right), \\
I_\delta &= \beta, \\
I_\beta &= \nu^2[\nu^2 m + \tfrac{1}{3}\epsilon(\nu^{5/2} - 1)] \exp\left(\frac{3\delta}{1+\zeta\nu^2} \right) + \frac{3}{4(1+\zeta\nu^2)}\beta^2.
\end{aligned}$$

Then the vector field

$$\overline{D}_\tau = \sum_{p,q} \overline{D}_X^p \overline{D}_T^q (G_1) \frac{\partial}{\partial U_{X^pT^q}} + \sum_{i \ge 0} \overline{D}_X^i (G_2) \frac{\partial}{\partial m_{X^i}} + I_\gamma \frac{\partial}{\partial \gamma} + I_\delta \frac{\partial}{\partial \delta} + I_\beta \frac{\partial}{\partial \beta}$$

is a nonlocal symmetry of the system of equations (4.46). In fact, the first equation of (3.20) becomes

$$\begin{pmatrix} -\nu^2 \overline{D}_X^2 + \epsilon & 1 \\ \frac{2}{3}(1 - \nu^{5/2})\overline{D}_X^3 + 2m\overline{D}_X + m_X & \overline{D}_T + U\overline{D}_X + 2U_X \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and (4.47)–(4.52) imply that this equation is equivalent to the fact that G_1 and G_2 satisfy the linearization of (4.46). The other equations of (3.20) hold because the functions I_γ, I_δ and I_β satisfy the linearizations of equations (4.47)–(4.52). □

It is clear from Theorem 4.12 and Corollary 4.13 that the results of Galas [25] and Leo *et al.* [35, 36], mentioned in Section 1, can also be interpreted in terms of coverings and provide further examples of nonlocal symmetries.

The advantage of the vector field W given by (4.53) over the shadow V defined in (4.43) is that one can find the flow of W simply by integrating a first order system of partial differential equations, and therefore one can obtain a (local) existence theorem for solutions of the nonlinear equation (4.46). Consider the following first order system in independent variables ξ and η:

$$\frac{\partial x}{\partial \xi} = -\nu^2 e^{D(\xi,\eta)}, \tag{4.54}$$

$$\frac{\partial m}{\partial \xi} = \frac{1}{1+\zeta\nu^2}(3\nu^2 m(\xi,\eta) + \epsilon(\nu^{5/2}-1))\gamma(\xi,\eta)e^{D(\xi,\eta)}, \tag{4.55}$$

$$\frac{\partial \gamma}{\partial \xi} = \left(\frac{3\nu^2}{4(1+\zeta\nu^2)}\gamma(\xi,\eta)^2 - \tfrac{1}{3}\epsilon(1+\zeta\nu^2)\right)e^{D(\xi,\eta)}, \tag{4.56}$$

$$\frac{\partial \delta}{\partial \xi} = \beta(\xi,\eta) - \nu^2\gamma(\xi,\eta)e^{D(\xi,\eta)}, \tag{4.57}$$

$$\frac{\partial \beta}{\partial \xi} = \frac{3}{4(1+\zeta\nu^2)}\beta(\xi,\eta)^2, \tag{4.58}$$

where

$$D(\xi,\eta) = \frac{3\delta(\xi,\eta)}{2(1+\zeta\nu^2)}. \tag{4.59}$$

Proposition 4.14. *The system of equations* (4.54)–(4.58) *with initial conditions* $\beta_0 = \beta(0,\eta)$, $\gamma_0 = \gamma(0,\eta)$, $\delta_0 = \delta(0,\eta)$, $m_0 = m(0,\eta)$, *and* $X_0 = X(0,\eta) = \eta$, *has the solution*

$$X(\xi,\eta) = -\nu^2\int_0^\xi e^{D(z,\eta)}dz + \eta, \tag{4.60}$$

$$\ln\left|\frac{3\nu^2 m(\xi,\eta) + \epsilon(\nu^{5/2}-1)}{3\nu^2 m_0 + \epsilon(\nu^{5/2}-1)}\right| = \frac{3\nu^2}{(1+\zeta\nu^2)}\int_0^\xi \gamma(z,\eta)e^{D(z,\eta)}dz, \tag{4.61}$$

$$\begin{aligned}\gamma(\xi,\eta) = {} & \frac{1}{9}\left(\frac{-4(1+\zeta\nu^2)}{(-3\xi\beta_0+4+4\zeta\nu^2)\beta_0} + \frac{1}{\beta_0}\right)(4\epsilon(1+\zeta\nu^2)^2 - 9\nu^2{\gamma_0}^2)e^{D(0,\eta)} \\ & + \gamma_0,\end{aligned} \tag{4.62}$$

$$\delta(\xi,\eta) = \tfrac{2}{3}(1+\zeta\nu^2)\ln\left|\frac{4(1+\zeta\nu^2)\frac{\partial\gamma}{\partial\xi}(\xi,\eta)}{3\nu^2\gamma(\xi,\eta)^2 - \frac{4}{3}\epsilon(1+\zeta\nu^2)^2}\right|, \tag{4.63}$$

$$\beta(\xi,\eta) = 4\frac{(1+\zeta\nu^2)\beta_0}{-3\xi\beta_0+4+4\zeta\nu^2}, \tag{4.64}$$

where the functions $D(0,\eta)$ and $D(z,\eta)$ are determined by (4.59)*, the initial condition $\delta_0 = \delta(0,\eta)$, and equations* (4.62) *and* (4.63)*.*

Now, equation (4.60) determines a transformation $(\xi,\eta) \mapsto (X,\tau)$, in which τ is a parameter along the flow of W, given by, say,

$$\tau = \xi, \qquad X = h(\xi,\eta). \tag{4.65}$$

Applying this change of variables to equations (4.55)–(4.58), and using equations (4.54), (4.47), (4.49), and (4.52), one sees that formulae (4.61)–(4.64) provide solutions for the flow equations

$$\frac{\partial m}{\partial \tau} = \left[\nu^2 m_X + \frac{3\nu^2\gamma}{1+\zeta\nu^2} m + \gamma\epsilon\frac{\nu^{5/2}-1}{1+\zeta\nu^2}\right]\exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right), \tag{4.66}$$

$$\frac{\partial \gamma}{\partial \tau} = [\nu^2 m + \tfrac{1}{3}\epsilon(\nu^{5/2}-1)]\exp\left(\frac{\frac{3}{2}\delta}{1+\zeta\nu^2}\right), \tag{4.67}$$

$$\frac{\partial \delta}{\partial \tau} = \beta, \tag{4.68}$$

$$\frac{\partial \beta}{\partial \tau} = \nu^2[\nu^2 m + \tfrac{1}{3}\epsilon(\nu^{5/2}-1)]\exp\left(\frac{3\delta}{1+\zeta\nu^2}\right) + \frac{3}{4(1+\zeta\nu^2)}\beta^2, \tag{4.69}$$

which one obtains from the formula for W in Theorem 4.12. Thus, finding a two-parameter (the “flow” parameter τ and the “spectral” parameter ζ) family of solutions to the nonlinear equation

$$-2\nu^2 U_X U_{XX} + 3\epsilon U_X U - \nu^2 U_{XXX} U + \tfrac{2}{3}(1-\nu^{5/2})U_{XXX} + \epsilon U_T - \nu^2 U_{XXT} = 0 \tag{4.70}$$

amounts to solving *one* simple equation. More exactly, one has

Corollary 4.15. *Let $U(X,T)$ be a solution of equation* (4.70)*. Then the solution $U(X,T,\tau)$ to the initial value problem*

$$\frac{\partial U}{\partial \tau} = \gamma(X,T,\tau)\exp\left(\frac{\frac{3}{2}\delta(X,T,\tau)}{1+\zeta\nu^2}\right), \tag{4.71}$$

$$U(X,T,0) = U(X,T), \tag{4.72}$$

in which $\gamma(X,T,\tau)$ and $\delta(X,T,\tau)$ are determined by (4.62)*,* (4.63)*, and* (4.65)*, is a two-parameter family of solutions to the family of equations* (4.70)*.*

This paper ends with two elementary examples.

Example. In the Camassa–Holm case, $\nu = 1$, $\epsilon = 1$, $\lambda = \frac{2}{3}(1+\zeta)$, the first order system (4.54)–(4.58) becomes

$$\frac{\partial X}{\partial \xi} = -e^{\delta/\lambda}, \qquad \frac{\partial m}{\partial \xi} = \frac{2}{\lambda}\gamma e^{\delta/\lambda} m, \tag{4.73}$$

$$\frac{\partial \gamma}{\partial \xi} = -\frac{1}{2\lambda}e^{\delta/\lambda}(\lambda^2-\gamma^2), \qquad \frac{\partial \delta}{\partial \xi} = \beta - \gamma e^{\delta/\lambda}, \qquad \frac{\partial \beta}{\partial \xi} = \frac{1}{2\lambda}\beta^2, \tag{4.74}$$

and the solutions (4.60)–(4.64) now read

$$X = \eta + \ln\left|\frac{-\xi\beta_0 + 2\lambda + (\gamma_0 - \lambda)\xi e^{\delta_0/\lambda}}{-\xi\beta_0 + 2\lambda + (\gamma_0 + \lambda)\xi e^{\delta_0/\lambda}}\right|, \tag{4.75}$$

$$m = \frac{m_0}{(-\xi\beta_0 + 2\lambda)^4} \times (-\xi\beta_0 + 2\lambda + (\gamma_0 - \lambda)\xi e^{\delta_0/\lambda})^2(-\xi\beta_0 + 2\lambda + (\gamma_0 + \lambda)\xi e^{\delta_0/\lambda})^2, \tag{4.76}$$

$$\gamma = \gamma_0 + \frac{\xi(\gamma_0^2 - \lambda^2)}{-\xi\beta_0 + 2\lambda} e^{\delta_0/\lambda}, \tag{4.77}$$

$$\delta = \lambda \ln\left|\frac{4\lambda^2 e^{\delta_0/\lambda}}{(-\xi\beta_0 + 2\lambda + (\gamma_0 + \lambda)\xi e^{\delta_0/\lambda})(-\xi\beta_0 + 2\lambda + (\gamma_0 - \lambda)\xi e^{\delta_0/\lambda})}\right| \tag{4.78}$$

$$\beta = 2\frac{\lambda\beta_0}{-\xi\beta_0 + 2\lambda}. \tag{4.79}$$

These formulae first appeared in [46]. Now consider the Camassa–Holm equation in the form

$$2U_\eta U_{\eta\eta} + UU_{\eta\eta\eta} = U_T - U_{\eta\eta T} + 3U_\eta U, \tag{4.80}$$

so that the "old" space variable is η, and choose an obvious solution of (4.80), say $U_0(\eta, T) = e^\eta$. The corresponding (pseudo)potentials γ_0, δ_0 and β_0, computed by means of (4.47)–(4.52), are given by

$$\gamma_0 = \lambda, \qquad \beta_0 = c, \qquad \delta_0 = \lambda\eta - \lambda^2 T.$$

Use these values as initial conditions for the system (4.73), (4.74), that is, take

$$U_0(\eta, T) = e^\eta, \qquad m_0 = 0, \qquad \gamma_0 = \lambda, \qquad \delta_0 = \lambda\eta - \lambda^2 T, \qquad \beta_0 = c.$$

The new space variable X is then given by equation (4.75). One finds

$$X(\xi, \eta, T) = \eta + \ln\left|\frac{-\xi c + 2\lambda}{-\xi c + 2\lambda + \xi e^{\eta - \lambda T} 2\lambda}\right|, \tag{4.81}$$

$$m(\xi, \eta, T) = 0, \tag{4.82}$$

and the (pseudo)potentials γ, δ and β become

$$\gamma(\xi, \eta, T) = \lambda, \tag{4.83}$$

$$\delta(\xi, \eta, T) = \lambda \ln\left|\frac{4\lambda^2 e^{\eta - \lambda T}}{(-\xi c + 2\lambda + 2\lambda\xi e^{\eta - \lambda T})(-\xi c + 2\lambda)}\right|, \tag{4.84}$$

$$\beta(\xi, \eta, T) = \frac{2\lambda c}{-\xi c + 2\lambda}. \tag{4.85}$$

Now invert equation (4.81) to find a change of variables $\eta = h(X, \tau)$, $\xi = \tau$. Taking $\beta_0 = c = 0$, one obtains

$$\eta = X - \ln|1 - \tau e^{X - \lambda T}|, \qquad \xi = \tau,$$

and therefore one can write the (pseudo)potentials γ, δ and β as functions of X, T, and τ:

$$\gamma(X, T, \tau) = \lambda, \qquad \delta(X, T, \tau) = \lambda(X - \lambda T), \qquad \beta(X, T, \tau) = 0. \tag{4.86}$$

Corollary 4.15 then implies that a two-parameter family of solutions of the Camassa–Holm equation

$$m = U_{XX} - U, \qquad m_T = -m_X U - 2mU_X,$$

is determined by the initial value problem

$$\frac{\partial U}{\partial \tau} = \gamma(X,T,\tau)e^{(1/\lambda)\delta(X,T,\tau)}, \qquad U(X,T,0) = e^X,$$

since at $\tau = 0$ the independent variables X and η coincide. One finds

$$u(X,T,\tau) = \lambda\tau e^{X-\lambda T} + e^X.$$

Example. Consider the nonlinear equation (4.46),

$$m = \nu^2 U_{\eta\eta} - \epsilon U, \qquad m_T = -m_\eta U - 2mU_\eta + \tfrac{2}{3}(1-\nu^{5/2})U_{\eta\eta\eta}, \tag{4.87}$$

and the trivial solution $U(\eta,T) = 0$. The corresponding (pseudo)potentials γ_0, δ_0 and β_0 which one obtains from (4.47)–(4.52) can be chosen to be

$$\gamma_0 = \tfrac{2}{3}\sqrt{1+\zeta\nu^2}\sqrt{\epsilon(\zeta+\sqrt{\nu})}, \tag{4.88}$$

$$\delta_0 = \tfrac{2}{3}\sqrt{1+\zeta\nu^2}\sqrt{\epsilon}\sqrt{\zeta+\sqrt{\nu}}\,(\eta - \tfrac{2}{3}(\zeta+\sqrt{\nu})T), \tag{4.89}$$

$$\begin{aligned}\beta_0 &= \tfrac{1}{3}(\nu^{5/2}-1)\sqrt{\epsilon}\frac{\sqrt{1+\zeta\nu^2}}{\sqrt{\zeta+\sqrt{\nu}}} \\ &\quad\times \exp\left(\frac{(\eta - \frac{2}{3}(\zeta+\sqrt{\nu})T)\sqrt{\epsilon}\sqrt{\zeta+\sqrt{\nu}}}{\sqrt{1+\zeta\nu^2}}\right).\end{aligned} \tag{4.90}$$

Proposition 4.14 yields expressions for the (pseudo)potentials γ, δ and β as functions of η, ξ and T, and then equation (4.60) and Corollary 4.15 allow one to find a two-parameter family of solutions to the system (4.87).

Consider, for instance, the $\nu = 0$ case. Equation (4.60) implies that in this case the "old" and "new" independent variables η and X agree. One then finds $\gamma(X,T,\tau)$, $\delta(X,T,\tau)$ and $\beta(X,T,\tau)$ to be

$$\gamma = -\frac{2}{3}\frac{\sqrt{\zeta}(\xi e^{-\frac{1}{3}\sqrt{\zeta}(-3X+2T\zeta)} - 4\sqrt{\zeta})}{\xi e^{-\frac{1}{3}\sqrt{\zeta}(-3X+2T\zeta)} + 4\sqrt{\zeta}}, \tag{4.91}$$

$$\delta = \frac{8}{3}\ln(2) + \frac{2}{3}\ln\left|\frac{e^{-\frac{1}{3}\sqrt{\zeta}(-3X+2T\zeta)}\zeta}{(\xi e^{-\frac{1}{3}\sqrt{\zeta}(-3X+2T\zeta)} + 4\sqrt{\zeta})^2}\right|, \tag{4.92}$$

$$\beta = -\frac{4}{3}\frac{e^{-\frac{1}{3}\sqrt{\zeta}(-3X+2T\zeta)}}{\xi e^{-\frac{1}{3}\sqrt{\zeta}(-3X+2T\zeta)} + 4\sqrt{\zeta}}, \tag{4.93}$$

and it follows from Corollary 4.15 that the function

$$U(X,T,\tau) = \frac{32}{3}\frac{\zeta^{3/2}e^{-\frac{1}{3}\sqrt{\zeta}(-3X+2T\zeta)}\xi}{(\xi e^{-\frac{1}{3}\sqrt{\zeta}(-3X+2T\zeta)} + 4\sqrt{\zeta})^2} \tag{4.94}$$

solves the KdV equation

$$3\left(\frac{\partial}{\partial X}U(X,T)\right)U(X,T)+\frac{2}{3}\frac{\partial^3}{\partial X^3}U(X,T)+\frac{\partial}{\partial T}U(X,T)=0.$$

This is a traveling wave solution if $\xi > 0$, and a singular solution for some negative values of the parameter ξ.

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