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A geometric proof of the definability of Hausdorff limits

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Abstract. We give a geometric proof of the following well-established theorem for o-minimal expansions of the real field: the Hausdorff limits of a compact, definable family of sets are definable. While previous proofs of this fact relied on the model-theoretic compactness theorem, our proof explicitly describes the family of all Hausdorff limits in terms of the original family.

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Introduction

Let $\mathcal{R}$ be an o-minimal expansion of the real field; throughout this paper, definable means "definable in $\mathcal{R}$ with parameters from $\mathbb{R}$". We refer the reader to [4] or [5] for the basic properties of o-minimal structures used in this paper; however, since the dimension of definable sets is fundamental to this paper, we quickly give a definition sufficient for the present setting. By Theorem 2.11 in [4, Chapter 3], every definable set $S \subseteq \mathbb{R}^n$ is a finite union of (embedded) topological submanifolds of $\mathbb{R}^n$ $(n \in \mathbb{N})$, each again definable. Therefore, we define

$$\dim S := \max \{\dim M : \ M \subseteq S \text{ is a def. top. submanifold of } \mathbb{R}^n\},$$

where $\dim M$ is the usual topological dimension of M. As verified in [4, Chapter 4], this defines a dimension for definable sets, and for definable manifolds this dimension agrees with the usual manifold dimension.

We now fix an arbitrary bounded, definable set $A \subseteq \mathbb{R}^{m+n}$, and we write $A' = \Pi_m(A)$, where $\Pi_m : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^m$ is the projection onto the first m coordinates, and $A_a = \{x \in \mathbb{R}^n : \ (a,x) \in A\}$ for $a \in \mathbb{R}^m$. In this paper, we describe the Hausdorff limits of the family $(A_a)_{a \in A'}$ as follows.

Theorem. *Assume that A_a is closed for every $a \in A'$. There exist $M \geq m$ and a definable, compact $B \subseteq \mathbb{R}^{M+n}$ such that, with $B' = \Pi_M(B)$,*

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(1) *for every* $a \in A'$ *there is* $b \in B'$ *with* $A_a = B_b$*;*
(2) *for every subsequence* $(b_i)_{i\in\mathbb{N}}$ *of* B' *such that* $\lim b_i = b$*, the limit* $\lim B_{b_i}$ *exists and equals* B_b*;*
(3) $\dim B' = \dim A'$ *and* $\dim\{b \in B' :\ B_b \neq A_a \text{ for all } a \in A'\} < \dim A'$.

L. Bröcker [2] proved the theorem in the case where $\mathcal{R}$ is the real field. D. Marker and C. Steinhorn [7], and later A. Pillay [8] show that types are definable in any $\mathcal{R}$, which implies the theorem. A more precise statement of the theorem along the lines below, as well as a different model-theoretic proof, was recently announced by L. van den Dries [3].

To be more precise, we let $\mathcal{K}_n$ be the space of all compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric. (We consider $\emptyset \in \mathcal{K}_n$ with $d(A,\emptyset) = \infty$ for all nonempty $A \in \mathcal{K}_n$.) Let $F_A : \mathbb{R}^m \longrightarrow \mathcal{K}_n$ be the map defined by $F_A(a) := \mathrm{cl}(A_a)$. We associate a dimension to the image $F_A(A')$ of A' under F_A as follows (see [3] for more details). The quivalence relation $\sim$ defined on A' by $a \sim b$ if and only if $A_a = A_b$ is definable. Thus by Proposition (1.2)(ii) in [4, Ch. 6], there is a definable set $A'' \subseteq A'$ of representatives. Then $F_A(A'') = F_A(A')$, and we put $\dim F_A(A') := \dim A''$. (The reader may easily verify, using [4, Ch. 4], that this dimension is well defined.)

Convention. Given a sequence $(a_i) \in A'$, we say that the sequence (A_{a_i}) **converges to** $C \in \mathcal{K}_n$ if the sequence $(\mathrm{cl}\, A_{a_i})$ converges in $\mathcal{K}_n$ to C, and in this situation we write $C = \lim A_{a_i}$.

Let $F \subseteq \mathcal{K}_n$; we refer to any point in $\mathrm{cl}_{\mathcal{K}_n}(F)$ as a **limit set** of F. Note that $L \in \mathcal{K}_n$ is a limit set of $F_A(A')$ if and only if there is a sequence $(a_i)_{i\in\mathbb{N}}$ of parameters in A' such that $\lim A_{a_i} = L$. We say that $F \subseteq \mathcal{K}_n$ is **definable** if there are k and a definable family $B \subseteq \mathbb{R}^{k+n}$ such that $F = F_B(\Pi_k(B))$.

With these notions, the theorem above can be restated as

Theorem 1. *Let* $F \subseteq \mathcal{K}_n$ *be definable. Then* $\mathrm{cl}_{\mathcal{K}_n}(F)$ *is definable and*

$$\dim(\mathrm{cl}_{\mathcal{K}_n}(F) \setminus F) < \dim(F).$$

Following Van den Dries's lecture at Luminy in June 2001, we came up with a geometric proof of Theorem 1 using the foliation techniques developed in our paper [6]; we now give a summary of the main ideas involved. Theorem 1 suggests that the function F_A is "modelled" by a definable function in the following sense:

Definition. A bounded, definable function $f : A' \longrightarrow \mathbb{R}^k$ **represents** F_A if the map

$$(a, f(a)) \mapsto \mathrm{cl}(A_a) :\ \mathrm{gr}\, f \longrightarrow \mathcal{K}_n$$

has a continuous extension to $\mathrm{cl}(\mathrm{gr}\, f)$. (Throughout the paper, $\mathrm{gr}\, h$ denotes the graph of h, for any function h.) In other words, a definable $f : A' \longrightarrow \mathbb{R}^k$ represents

F_A if and only if for any two sequences (a_i) and (b_j) of parameters in A' such that $\lim(a_i, f(a_i))$ and $\lim(b_j, f(b_j))$ exist, the limits $\lim A_{a_i}$ and $\lim A_{b_j}$ also exist and

$$\lim(a_i, f(a_i)) = \lim(b_j, f(b_j)) \quad \Longrightarrow \quad \lim A_{a_i} = \lim A_{b_j}.$$

Indeed, we prove Theorem 1 by establishing

Theorem 2. *F_A has a definable representation.*

To define a representation of F_A, we would like to uniformly select points in the limit sets of $F_A(A')$; but since the set of all sequences of parameters in A' is not definable, we need another way to characterize these limit sets. Thus, for any set $S \subseteq \mathbb{R}^k$ we define $\mathrm{fr}(S) = \mathrm{cl}(S) \setminus S$, and we put

$$\mathrm{fr}'(A) = \{(a, x) \in A' \times \mathbb{R}^n : \ x \in \mathrm{fr}(A_a)\}.$$

First, by cell decomposition we may assume that A is a C^1-cell. We then view the fibers of A as the leaves of a (trivially obtained) distribution on A. Using the jet space techniques developed in our paper [6], we blow up this distribution on A to obtain a finite collection of new integrable distributions. These distributions depend only on A and not on any particular limit set of $F_A(A')$. We then show (Proposition 8) that every limit set of $F_A(A')$ is, outside of the corresponding limit set of $F_{\mathrm{fr}'(A)}(A')$ (roughly speaking), a union of integral manifolds of these new distributions.

Second, since the fibers of $\mathrm{fr}'(A)$ have strictly lower dimension than the fibers of A, we will be able to assume inductively that $F_{\mathrm{fr}'(A)}$ has a definable representation, say $g : A' \longrightarrow \mathbb{R}^k$. Replacing the parameter space A' by the graph of g (a procedure we call lifting A via g, see Definition 10), we can *definably* subtract the family of all limit sets of $F_{\mathrm{fr}'(A)}(A')$ from the domains of the distributions found above. Inside the remaining smaller domains, any limit set of $F_A(A')$ will actually be a union of leaves of the (correspondingly restricted) distributions.

Third, as any leaf of an integrable distribution is uniquely determined by any one of its points, we can then define a representation f of F_A using a definable choice argument on (the lifting of) A, see Section 4.

What complicates matters is that the above procedure only works as described for those new distributions whose domains are (roughly speaking) open subsets of $\mathrm{cl}(A)$. To deal with this problem, we need to choose a stratification compatible with the domains of the new distributions and then proceed essentially as above by reverse induction on the dimension of the strata. Correspondingly, the notion of representation needs to be relativized to each stratum.

Finally, Theorem 1 follows from Theorem 2 by applying the lifting argument once more (see Corollary 12).

Throughout this paper, we let $\|x\| := \max\{|x_1|, \dots, |x_k|\}$ for $x \in \mathbb{R}^k$. Given $x \in \mathbb{R}^n$ and $\delta > 0$ we put $B(x, \delta) = \{y \in \mathbb{R}^n : \ \|x - y\| < \delta\}$.

1. Limits of η-bounded fibers

Let $A \subseteq \mathbb{R}^{m+n}$ be as in the introduction. In this section, we obtain a local desciption of the limit sets of $F_A(A')$ in the case where the fibers of A are sufficiently smooth. More precisely, we fix $p \geq 0$ and assume in this section that A is a C^p-cell. Thus, for each $a \in A'$, the fiber A_a is a C^p-cell of dimension $d \leq n$, A' is a C^p-cell of dimension $d' \leq m$ and $\dim A = d' + d$.

First, we write A as a finite union of sets of the following nature: given $\eta > 0$, a C^1-manifold $V \subseteq \mathbb{R}^n$ of dimension d is **η-bounded** if for all $x \in V$ there is a matrix $L = (l_{ij}) \in \mathrm{M}_{n-d,d}(\mathbb{R})$ such that $\|L\| := \max_{i,j} |l_{i,j}| \leq \eta$ and $T_xV = \{(u, Lu) : \; u \in \mathbb{R}^d\}$.

Remark. Let $\eta > 0$ and $V \subseteq \mathbb{R}^n$ an η-bounded, embedded manifold of class C^1 and dimension d. Then for any $x \in V$, there is an open box U containing x and an $d\eta$-Lipschitz map $f : \Pi_d(U) \longrightarrow \mathbb{R}^{n-d}$ such that $V \cap U = \mathrm{gr}\, f$ (here the notion "Lipschitz" is used with respect to the norm $\| \cdot \|$).

For the next lemma, we let Σ_n be the set of all permutations on $\{1, \dots, n\}$. For $\sigma \in \Sigma_n$, we also denote by $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the permutation of coordinates $\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

Lemma 3. *Let $E \subseteq \mathbb{R}^n$ be a linear subspace of dimension d. Then there exist $\sigma \in \Sigma_n$ and a matrix $L \in \mathrm{M}_{n-d,d}(\mathbb{R})$ such that $\|L\| \leq 1$ and $\sigma(E) = \{(u, Lu) \in \mathbb{R}^n : \; u \in \mathbb{R}^d\}$.*

Proof (by Stéphane Lamy). Given a basis $\{v_1, \dots, v_d\}$ of E and $\sigma \in \Sigma_n$, we denote by $(v_1, \dots, v_d)_\sigma$ the (signed) volume of the parallelepiped in $\mathbb{R}^d$ spanned by the vectors $\Pi_d(\sigma(v_1))$, $\dots$, $\Pi_d(\sigma(v_d))$, and we choose a $\sigma_0 \in \Sigma$ such that the absolute value of $(v_1, \dots, v_d)_{\sigma_0}$ is maximal. Since the map $(v_1, \dots, v_d) \mapsto (v_1, \dots, v_d)_\sigma$ is d-linear for each σ, we see that σ_0 is independent of the particular basis considered; we claim that the lemma works with $\sigma = \sigma_0$.

To see this, we assume for simplicity of notation that σ_0 is the identity map on $\mathbb{R}^n$. Then $\Pi_d(E) = \mathbb{R}^d$, so there is a matrix $L = (l_{i,j}) \in \mathrm{M}_{n-d,d}(\mathbb{R})$ (with respect to the standard bases for $\mathbb{R}^d$ and $\mathbb{R}^{n-d}$) such that $E = \{(u, Lu) : u \in \mathbb{R}^p\}$. Let $\{e_1, \dots, e_d\}$ be the standard basis of $\mathbb{R}^d$, and consider the vectors $v_k = (e_k, Le_k) \in E$ for $k = 1, \dots, d$; clearly $\{v_1, \dots, v_d\}$ is a basis of E. For $i \in \{1, \dots, n-d\}$ and $j \in \{1, \dots, d\}$ we denote by $\sigma_{i,j} \in \Sigma$ the permutation that exchanges the j-th and the $(p+i)$-th coordinates. Then $l_{i,j} = (v_1, \dots, v_d)_{\sigma_{i,j}}$ for all i, j, and the maximality of $|(v_1, \dots, v_d)_{\sigma_0}|$ gives $|l_{i,j}| \leq |(v_1, \dots, v_d)_{\sigma_0}| = 1$ for all i and j, and hence $\|L\| \leq 1$, as required. $\square$

Corollary 4. *For each $\sigma \in \Sigma_n$ there is a definable open subset A_σ of A such that*

$$A = \bigcup_{\sigma \in \Sigma_n} A_\sigma$$

and for every $a \in \mathbb{R}^m$, the set $\sigma((A_\sigma)_a)$ is a 2-bounded, embedded manifold of dimension d and class C^p.

Proof. Let G_n^d be the Grassmannian of all d-dimensional vector subspaces of $\mathbb{R}^n$, considered as a compact algebraic submanifold of $\mathbb{R}^{n^2}$ (see [1, Section 3.4]; in the next section, we discuss related notions in more detail). The set $\mathcal{E}_2 = \{E \in G_n^d : \exists L \in \mathrm{M}_{n-d,d}(\mathbb{R})$ such that $\|L\| < 2$ and $E = \{(u, Lu) : \ u \in \mathbb{R}^d\}\}$ is an open, semialgebraic subset of G_n^d, and the map $g : A \longrightarrow G_n^d$ defined by $g(a,x) = T_x A_a$ is definable. The corollary now follows from Lemma 3. □

Next, we prove a basic fact about limit sets of $F_A(A')$ in the case where the fibers of A are η-bounded. Let $N \in \mathbb{N}$ (obtained by o-minimality) be such that for any $a \in \mathbb{R}^m$ and any open box $U \subseteq \mathbb{R}^n$, the set $A_a \cap U$ has at most N connected components.

Lemma 5. *Let $\eta > 0$ and assume that for each $a \in A'$, the fiber A_a is η-bounded. Let (a_i) be a sequence in $\mathbb{R}^m$ such that both $\lim A_{a_i}$ and $\lim \mathrm{fr}(A_{a_i})$ exist. Then for every $x \in \lim A_{a_i} \setminus \lim \mathrm{fr}(A_{a_i})$ there are a box $U \subseteq \mathbb{R}^n$ containing x and $d\eta$-Lipschitz functions $f_1, \dots, f_N : \Pi_d(U) \longrightarrow \mathbb{R}^{n-d}$ such that*

$$\lim A_{a_i} \cap U = (\mathrm{gr}\, f_1 \cap U) \cup \cdots \cup (\mathrm{gr}\, f_N \cap U).$$

Proof. For simplicity of notation, we assume throughout the proof that $\eta = 1$; the proof for general η is similar. Let $x \in \lim A_{a_i} \setminus \lim \mathrm{fr}(A_{a_i})$, and choose $\epsilon > 0$ such that $B(x, 3\epsilon) \cap \mathrm{fr}(A_{a_i}) = \emptyset$ for all i (after passing to a subsequence if necessary). We let $U = B(x, \epsilon)$ and $V = \Pi_d(U) \times W$, where

$$W = \left\{ w \in \mathbb{R}^{n-d} : \ |w_k - x_{d+k}| < 3d\epsilon \text{ for } k = 1, \dots, n-d \right\}.$$

We now fix an i. By our assumptions, for any $u \in \Pi_d(U)$

(∗) there is a $\delta > 0$ such that $A_{a_i} \cap (B(u, \delta) \times W)$ is the union of at most N disjoint graphs of d-Lipschitz functions from $B(u, \delta)$ to W.

Let $x \in A_{a_i} \cap U$; we claim that the component C of $A_{a_i} \cap V$ that contains x is the graph of a d-Lipschitz function $g : \Pi_d(U) \longrightarrow W$.

To prove this covering property, we choose δ as in (∗) for $u = \Pi_d(x)$ and let $g : B(u, \delta) \longrightarrow W$ be the corresponding d-Lipschitz function such that $g(u) = (x_{d+1}, \dots, x_n)$. We extend g to all of $\Pi_d(U)$ as follows: for each $v \in \partial \Pi_d(U)$, we let $v' \in [u, v]$ be the point closest to v such that g extends to a d-Lipschitz function g_v along the line segment $[u, v']$ satisfying $\mathrm{gr}(g_v) \subseteq A_{a_i} \cap V$. Then (∗) implies that $v' = v$ for each $v \in \partial \Pi_d(U)$.

Moreover, the extension $g : \Pi_d(U) \longrightarrow W$ defined in this way is continuous (and hence d-Lipschitz): let $v \in \Pi_d(U)$ be such that g is continuous at v' for every $v' \in [u, v)$. Let δ' be obtained for this v in place of u as in $(*)$, and let $h_1, \dots, h_q : B(v, \delta') \longrightarrow W$ be the corresponding distinct d-Lipschitz functions. We assume that $g(v) = h_1(v)$. Shrinking δ' if necessary, we may assume that there is $\mu > 0$ such that for any $s, t \in B(v, \delta')$ and any $1 \le k < l \le q$ we have $|h_k(s) - h_l(t)| > \mu$. Let $v' \in [u, v) \cap B(v, \delta')$ be close enough to v so that $|g(v') - g(v)| < \mu/4$; then $g(v') = h_1(v')$ as well. Since g is continuous at v', it follows that $g(s) = h_1(s)$ for all s sufficiently close to v'. But then the continuity of g along the radial segments $[u, t]$, $t \in \partial\Pi_d(U)$, and our choice of δ' imply that $g = h_1$ in a neighbourhood of v. This proves the claim.

By the claim, for all i there are definable d-Lipschitz functions $f_{1,i}, \dots, f_{N,i} : \Pi_d(V) \longrightarrow \mathbb{R}^{n-d}$ such that every connected component of $A_{a_i} \cap V$ intersecting U is the graph of some $f_{l,i}$ and for all $l, l' \in \{1, \dots, N\}$, either $f_{l,i} = f_{l',i}$ or $\operatorname{gr} f_{l,i} \cap \operatorname{gr} f_{l',i} = \emptyset$, and

$$A_{a_i} \cap U = (\operatorname{gr} f_{1,i} \cap U) \cup \cdots \cup (\operatorname{gr} f_{N,i} \cap U).$$

Passing to a subsequence if necessary, we may therefore assume that each sequence $(f_{l,i})_i$ converges to a d-Lipschitz function $f_l : \Pi_d(V) \longrightarrow \mathbb{R}^{n-d}$. Clearly $\operatorname{gr} f_l \subseteq \lim A_{a_i}$. On the other hand, if $x' \in \lim A_{a_i} \cap U$, then $x' \in \lim(A_{a_i} \cap U)$, so by the above $x' \in \lim(\operatorname{gr} f_{l,i} \cap U)$ for some l, that is, $x' \in \operatorname{gr} f_l$. □

2. Blowing up in jet space

We denote by G_n^d the Grassmannian of all d-dimensional vector subspaces of $\mathbb{R}^n$, considered as a compact algebraic submanifold of $\mathbb{R}^{n^2}$ (see [1, Section 3.4] for details). We shall not notationally distinguish between d-dimensional vector subspaces of $\mathbb{R}^n$ and the corresponding elements of G_n^d.

Let $p \ge 1$ and $V \subseteq \mathbb{R}^{m+n}$. We call V a **fiberwise manifold of class C^p** if V_a is a submanifold of $\mathbb{R}^n$ of class C^p for each $a \in \mathbb{R}^m$. Assume that V is a fiberwise manifold of class C^p and let $q \le p$. A map $g : V \longrightarrow G_n^d$ is called a **fiberwise d-distribution on V of class C^q** if for each $a \in \Pi_m(V)$, the map $g_a : V_a \longrightarrow G_n^d$ given by $g_a(x) = g(a, x)$ is a d-distribution on V_a of class C^q. (If V is a manifold, we can associate a d-distribution $\widetilde{g} : V \longrightarrow G_{m+n}^d$ to g by putting $\widetilde{g}(a, x) = \{0_m\} \times g(a, x)$, where 0_m is the origin of $\mathbb{R}^m$.)

Let $g : V \longrightarrow G_n^d$ be a fiberwise distribution of class C^q. We say that g is **tangent to** V if $g(a, x) \subseteq T_x V_a$ for every $(a, x) \in V$. A manifold $Z \subseteq V$ is an **integral manifold of** g if there is an $a \in \mathbb{R}^m$ such that $Z = \{a\} \times Z_a$ and $T_x Z_a = g(a, x)$ for every $x \in Z_a$. (We do not assume that an integral manifold is embedded or connected, and we consider the empty set to be an integral manifold of any fiberwise distribution). The fiberwise distribution g is **integrable at** $(a, x) \in V$ if there is an integral manifold of g containing (a, x). We simply say that g is

integrable if g is integrable at every $(a,x) \in V$.

Remark 6. As discussed in [6, Section 1], if V and g are definable and of class C^2, then the set $\{(a,x) \in V :\ g \text{ is integrable at } (a,x)\}$ is definable.

Below we put $n_0 := n$ and $n_q := n_{q-1} + n_{q-1}^2$ for $q > 0$, and for $0 \le r \le q$ we let $\pi^q_r : \mathbb{R}^{m+n_q} \longrightarrow \mathbb{R}^{m+n_r}$ be the projection on the first $m + n_r$ coordinates. Clearly the values $n_0, \dots, n_q$ depend on n, but we shall not explicitly indicate this dependence as it will usually be clear from context. Similarly, we generally omit mentioning m (as for instance in the notation "π^q_r"). Finally, we put $J^0 := \mathbb{R}^n$ and $J^q := \mathbb{R}^n \times G^d_{n_0} \times \cdots \times G^d_{n_{q-1}}$ for $1 \le q \le p$.

Definition 7. Assume that g is tangent to V and of class C^q for some $q \le p$. For $r \in \{0,\dots,q\}$, the r-th **blow-up** $\mathrm{b}^r\, g$ of g is obtained as follows: $V^0_g := V$ and $\mathrm{b}^0\, g := g$, and for $r > 0$ we let $V^r_g := \mathrm{gr}(\mathrm{b}^{r-1}\, g)$ and

$$\mathrm{b}^r\, g := \left(\pi^r_{r-1}\big|V^r_g\right)^* \mathrm{b}^{r-1}\, g,$$

the (fiberwise) pull-back of $\mathrm{b}^{r-1}\, g$ via $\pi^r_{r-1}\big|V^r_g$. Note that for each $a \in \mathbb{R}^m$ and $r > 0$ we have $\mathrm{b}^r\, g_a = (\mathrm{b}^r\, g)_a$.

Example. The **fiberwise Gauss map** $g : V \longrightarrow G^d_n$ defined by $g(a,x) = T_x V_a$ is a fiberwise distribution of class C^{p-1} on V that is tangent to V and integrable. In this case, we simply write V^q in place of V^q_g, for $q = 0,\dots,p$.

We now return our attention to the definable set A. We assume throughout this section that A is a C^p-cell; so the set A^q is a C^{p-q}-cell that is also a fiberwise manifold for each $q = 0,\dots,p$, and A^q_a is a C^{p-q}-cell of dimension d. Moreover, $\pi^q_r(\mathrm{fr}'(A^q)) = \mathrm{fr}'(A^r)$ for all $0 \le r \le q$, because each A^q is bounded.

For the next proposition, we write $A = \bigcup A_\sigma$ such that the set $\sigma((A_\sigma)_a)$ is 2-bounded for each $\sigma \in \Sigma_n$, as obtained from Corollary 4, and put

$$B = \bigcup_{\sigma \in \Sigma_n} \mathrm{fr}'\left((A_\sigma)^1\right).$$

Let $C \subseteq \mathbb{R}^{m+n}$ be a definable, fiberwise manifold and $g : C \longrightarrow G^d_n$ a definable, fiberwise distribution on C, both of class C^2. Put $D = \mathrm{gr}(g) \subseteq \mathbb{R}^m \times J^1$. We assume that

(i) $D \subseteq \mathrm{fr}(A^1) \cap \Pi_m^{-1}(\mathrm{fr}\, A')$, and there is a definable $W \subseteq \mathrm{fr}(A^1) \cap \Pi_m^{-1}(\mathrm{fr}\, A')$ such that for any $a \in \mathrm{fr}(A')$, both W_a and $W_a \cup D_a$ are open in $\mathrm{fr}(A^1)_a$;

(ii) if there exists an $(a,x) \in C$ such that $g_a(x) \subseteq T_x C_a$ and g_a is integrable at x, then g is tangent to C and integrable.

(The W in (i) will appear in the proof of Theorem 2 as a result of the necessary relativization described in the introduction.) For any subsequence (a_i) of A' such that $a_i \to a$ and $\lim A^1_{a_i}$ and $\lim B_{a_i}$ exist, we put

$$L_{(a_i)} = (D_a \cap \lim A^1_{a_i}) \setminus \big(\lim B_{a_i} \cup \operatorname{cl}(W_a \cap \lim A^1_{a_i})\big).$$

Proposition 8. *Exactly one of the following holds: either*

(1) $L_{(a_i)} = \emptyset$ *for all subsequences* (a_i) *of* A' *converging to* a *such that* $a_i \to a$ *and* $\lim A^1_{a_i}$ *and* $\lim B_{a_i}$ *exist; or*

(2) g *is tangent to* C *and integrable, and for any subsequence* (a_i) *of* A' *such that* $a_i \to a$ *and* $\lim A^1_{a_i}$ *and* $\lim B_{a_i}$ *exist, the set* $L_{(a_i)}$ *is an embedded integral manifold of* $\mathrm{b}^1 g_a$ *and is open in* $\lim A^1_{a_i}$.

Remark. Let $\sigma \in \Sigma_n$ and write $\sigma(a,x) = (a, x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any $(a,x) \in \mathbb{R}^{m+n}$ and $\sigma S = \{\sigma(a,x) : (a,x) \in S\}$ for any $S \subseteq \mathbb{R}^{m+n}$. Then the conjugate map $g^\sigma = \sigma \circ g \circ \sigma^{-1} : \sigma C \longrightarrow G^d_n$ satisfies

$$g(a,x) \subseteq T_x C_a \quad \text{if and only if} \quad g^\sigma(\sigma(a,x)) \subseteq T_{\sigma x}(\sigma C)_a.$$

Moreover, σ induces a diffeomorphism $\sigma^1 : \mathbb{R}^m \times J^1 \longrightarrow \mathbb{R}^m \times J^1$ defined by $\sigma^1(a,x,x^1) = (\sigma(a,x), \sigma x^1)$ (here we consider x^1 as a subset of $\mathbb{R}^n$ to which σ is applied; note that σ^1 is also just a permutation of coordinates). Then

$$\mathrm{b}^1 g^\sigma = \sigma^1 \circ \mathrm{b}^1 g \circ (\sigma^1)^{-1},$$

and if (a_i) is a subsequence of A' such that $a_i \to a$ and $\lim A^1_{a_i}$ exists, then $\lim \sigma A^1_{a_i}$ also exists and

$$(\sigma D)_a \cap \lim \sigma A^1_{a_i} = \sigma^1(D_a \cap \lim A^1_{a_i}).$$

Proof of Proposition 8. By the remark, after replacing A by A_σ for each $\sigma \in \Sigma_n$, we may assume for the rest of this proof that A is 2-bounded and $B = \operatorname{fr}'(A^1)$.

Let (a_i) be a subsequence of A' such that $a_i \to a$ and both $\lim A^1_{a_i}$ and $\lim \operatorname{fr}(A^1_{a_i})$ exist and write $L = L_{(a_i)}$. Assume that $L \neq \emptyset$, and choose an arbitrary $(x,x^1) \in L$. Since $W_a \cup D_a$ is open in $\operatorname{fr}(A^1)_a$ and $(x,x^1) \notin \operatorname{cl}(W_a \cap \lim A^1_{a_i})$, there is an open box $V \subseteq \mathbb{R}^{n_1}$ such that $(x,x^1) \in V$ and $\operatorname{cl}(V) \cap \lim A^1_{a_i} \subseteq D_a \setminus \big(\lim B_{a_i} \cup \operatorname{cl}(W_a \cap \lim A^1_{a_i})\big)$. Writing $V = V_0 \times V_1$ with $V_0 \subseteq \mathbb{R}^n$, we may also assume that $D_a \cap \big(\operatorname{cl}(V_0) \times \operatorname{fr}(V_1)\big) = \emptyset$, because D_a is the graph of the continuous map g_a and C_a is locally closed.

On the other hand, $V \cap \lim A^1_{a_i} = V \cap \lim(V \cap A^1_{a_i}) = V \cap \lim(A_V)^1_{a_i}$, where $A_V = \{(a,y) \in A : (y, T_y A_a) \in V\}$. We now claim that $x \notin \lim \operatorname{fr}((A_V)_{a_i})$: in fact, the previous paragraph implies that $\operatorname{fr}(A_{a_i}) \cap \operatorname{cl}(A_{V,a_i}) = \emptyset$ for all sufficiently large i, and hence $\operatorname{fr}(A_{V,a_i}) \subseteq \operatorname{fr}(V_0)$ for all sufficiently large i, which proves the claim.

We therefore apply Lemma 5 with A_V in place of A and $\eta = 2$, to obtain a corresponding open neighbourhood $U \subseteq \Pi_n(V)$ of x and $f_1, \dots, f_N$. We let $l \in \{1, \dots, N\}$ be such that $x \in \operatorname{gr}(f_l)$. We claim that f_l is differentiable at $z = \Pi_d(x)$ with $T_x \operatorname{gr}(f_l) = g(a,x)$; since x is arbitrary, this then implies that each $\operatorname{gr}(f_l)$ is an embedded, connected integral manifold of g_a. Assumption (ii) and [6, Lemma 1.6] now imply that g is tangent to C and integrable. Since $(x, x^1) \in L$ was arbitrary, it follows that L is an embedded integral manifold of $\mathrm{b}^1\, g$, as desired.

To prove the claim, let $f_{l,i}$ be the definable functions corresponding to f_l as in the proof of Lemma 5. After a linear change of coordinates if necessary, we may assume that $g_a(x) = \mathbb{R}^d$ (the subspace spanned by the first d coordinates). It now suffices to show that f_l is η-Lipschitz at x for every $\eta > 0$, since then $T_x \operatorname{gr}(f_l) = \mathbb{R}^d$. So let $\eta > 0$; since $\lim(A_V)^1_{a_i} \subseteq D_a = \operatorname{gr}(g_a)$ and $x \in C_a$, and because C_a is locally closed and g_a is continuous, there is a neighborhood $U' \subseteq U$ of x such that $\operatorname{gr}(f_{l,i}) \cap U'$ is $\frac{\eta}{d}$-bounded for all sufficiently large i. Thus by Lemma 5 again, f_l is η-Lipschitz at x, as required. □

3. Lifting

As described in the introduction, we need to make our notion of representation relative: Let $W \subseteq \mathbb{R}^{m+n}$, and let $M \in \mathbb{N}$ and $f : A' \longrightarrow \mathbb{R}^M$. We say that f **represents F_A in W** (or f **is a representation of F_A in W**) if

(i) f is bounded and definable, and
(ii) if (a_i) and (b_i) are subsequences of A' converging to a and $\lim f(a_i)$, $\lim f(b_i)$, $\lim A_{a_i}$ and $\lim A_{b_i}$ exist, then

$$\lim f(a_i) = \lim f(b_i) \quad \Longrightarrow \quad W_a \cap \lim A_{a_i} = W_a \cap \lim A_{b_i}.$$

Note that f represents F_A in $\mathbb{R}^{m+n}$ if and only if f represents F_A in the sense of the introduction. The following observations are elementary; we leave their verification to the reader.

Lemma 9. (1) *Let $A^1, A^2 \subseteq \mathbb{R}^{m+n}$ be definable such that $A = A^1 \cup A^2$, and assume $f_j : A' \longrightarrow \mathbb{R}^{k_j}$ represents F_{A^j}, for $j = 1, 2$. Then $(f_1, f_2) : A' \longrightarrow \mathbb{R}^{k_1+k_2}$ represents F_A.*

(2) *Let $W_j \subseteq \mathbb{R}^{m+n}$ and $f_j : A' \longrightarrow \mathbb{R}^{k_j}$ be a representation of F_A in W_j, for $j = 1, 2$. Then $(f_1, f_2) : A' \longrightarrow \mathbb{R}^{k_1+k_2}$ represents F_A in $W_1 \cup W_2$.*

(3) *Assume that $f : A' \longrightarrow \mathbb{R}^M$ represents F_A in $W \subseteq \mathbb{R}^{m+n}$, and let $k \leq n$. Then for any subsequences (a_i) and (b_i) of A' converging to a such that $\lim f(a_i)$, $\lim f(b_i)$, $\lim A_{a_i}$ and $\lim A_{b_i}$ exist, we have*

$$\lim f(a_i) = \lim f(b_i) \Longrightarrow \Pi_k(W_a \cap \lim A_{a_i}) = \Pi_k(W_a \cap \lim A_{b_i}).$$

Definition 10. Let $f : A' \longrightarrow \mathbb{R}^M$ and $S \subseteq \mathbb{R}^{m+n}$. The **lifting of S via f** is defined as

$$\operatorname{lift}_f S = \left\{(a, b, x) \in \mathbb{R}^{m+M+n} : \ (a, x) \in S \text{ and } (a, b) \in \operatorname{cl}(\operatorname{gr} f)\right\}.$$

Proposition 11. *Let $W \subseteq \mathbb{R}^{m+n}$ and assume that $f : A' \longrightarrow \mathbb{R}^M$ represents F_A in W. Then for any subsequence (a_i) of A' such that $(a_i, f(a_i)) \to (a, b)$ and $\lim A_{a_i}$ exists, we have*

$$W_a \cap \lim A_{a_i} = (\operatorname{lift}_f W)_{(a,b)} \cap \operatorname{cl}(\operatorname{lift}_f A)_{(a,b)}.$$

Proof. Let $V = \operatorname{lift}_f W$ and $B = \operatorname{cl}(\operatorname{lift}_f A)$. Let (a_i) be a subsequence of A' such that $(a_i, f(a_i)) \to (a, b)$ and $\lim A_{a_i}$ exists. Since B is closed, we have $\lim A_{a_i} \subseteq \lim B_{(a_i, f(a_i))} \subseteq B_{(a,b)}$. Hence $W_a \cap \lim A_{a_i} \subseteq V_{(a,b)} \cap B_{(a,b)}$. Conversely, let $x \in V_{(a,b)} \cap B_{(a,b)}$, and let (c_i, x_i) be a subsequence of A such that $(c_i, f(c_i)) \to (a, b)$ and $x_i \to x$. Passing to a subsequence if necessary, we may assume that $\lim A_{c_i}$ exists. Since f represents F_A in W, it follows that $W_a \cap \lim A_{c_i} = W_a \cap \lim A_{a_i}$; in particular, $x \in W_a \cap \lim A_{a_i}$. □

Corollary 12. *Let $f : A' \longrightarrow \mathbb{R}^M$ be a representation of F_A.*

(1) *For any $a \in A'$, we have $\operatorname{cl}(\operatorname{lift}_f A)_{(a,f(a))} = \operatorname{cl}(A_a)$, and for any $\omega_i \in \operatorname{cl}(\operatorname{gr} f)$ such that $\omega_i \to \omega$, we have $\lim \operatorname{cl}(\operatorname{lift}_f A)_{\omega_i} = \operatorname{cl}(\operatorname{lift}_f A)_\omega$.*
(2) *If $D' \subseteq A'$ is such that $f\big|D'$ is continuous, then $\operatorname{cl}'(A) \cap \Pi_m^{-1}(D')$ is closed in $\Pi_m^{-1}(D')$.*

In particular, $\operatorname{cl}_{\mathcal{K}_n}(F_A(A')) = F_B(\Pi_m(B))$ where $B := \operatorname{cl}(\operatorname{lift}_f A)$, which proves Theorem 1 for $F = F_A(A')$.

Proof. We write again $B = \operatorname{cl}(\operatorname{lift}_f A)$. For (1), given $a \in A'$ and taking $a_i = a$ for each i, we obtain from Proposition 11 that $B_{(a,f(a))} = \operatorname{cl}(A_a)$. Let $\omega_i \in \operatorname{cl}(\operatorname{gr} f)$ such that $\omega_i \to \omega$. By Proposition 11 with $W = \mathbb{R}^{m+n}$, there is for each i an $a_i \in A'$ such that

$$\|\omega_i - (a_i, f(a_i))\| < \frac{1}{i} \quad \text{and} \quad d\left(B_{\omega_i}, A_{a_i}\right) < \frac{1}{i}.$$

Then $(a_i, f(a_i)) \to \omega$ and by Proposition 11 again, $\lim B_{\omega_i} = \lim A_{a_i} = B_\omega$.

For (2), let $D' \subseteq A'$ such that $f\big|D'$ is continuous. Then for any subsequence (a_i) of D' such that $a_i \to a \in D'$, we have $f(a_i) \to f(a)$, so $\lim \operatorname{cl}(A_{a_i}) = \lim B_{(a_i, f(a_i))} = B_{(a,f(a))} = \operatorname{cl}(A_a)$ by part (1), which proves part (2). □

4. Defining a representation

Here we consider the special situation that we will obtain in the proof of Theorem 2, after blowing up the fibers of A in jet space and lifting A to a larger parameter space as outlined in the introduction. We assume here that A is a definable, fiberwise manifold of class C^p.

Blowing up the fibers of A in jet space and lifting A will produce (after changing m and n accordingly) a definable set $D \subseteq \operatorname{fr}(A) \cap \Pi_m^{-1}(\operatorname{fr} A')$ and a definable map $g : D \longrightarrow G_n^d$ such that

(i) D is a fiberwise manifold and g a fiberwise distribution on D, both of class C^2, and g is tangent to D and integrable;
(ii) $\mathrm{fr}(D)$ is closed;
(iii) for every subsequence (a_i) of A' such that $a_i \to a \in \Pi_m(D)$ and $\lim A_{a_i}$ exists, the set $D_a \cap \lim A_{a_i}$ is an embedded integral manifold of g_a and an open subset of $\lim A_{a_i}$.

With these assumptions in place, we define a map $\phi : \mathrm{cl}(A) \longrightarrow [0, \infty]$ by

$$\phi(a, x) = d\big((a, x), \mathrm{fr}(D)\big);$$

for each $a \in \mathrm{cl}(A')$, we write $\phi_a : \mathrm{cl}(A)_a \longrightarrow [0, \infty]$ for the map $\phi_a(x) = \phi(a, x)$. (If $\mathrm{fr}(D) = \emptyset$, then $\phi(a, x) = \infty$ for all $(a, x) \in \mathrm{cl}(A)$.) The function ϕ is definable, and hence so is the set

$$C = \{(a, x) \in A : \ \phi_a \text{ attains a local maximum at } x\}.$$

By cell decomposition and definable choice, there are $N \in \mathbb{N}$ and a definable, bounded function $f = (f_1, \dots, f_N) : A' \longrightarrow \mathbb{R}^{Nn}$ such that for every $a \in A'$ and every connected component S_a of C_a, there is a j such that $f_j(a) \in S_a$.

Proposition 13. *The function f represents F_A in D.*

Proof. Let (a_i) be a subsequence of A' such that $\lim a_i = a$, $\lim f_j(a_i) = c_j$ for $j = 1, \dots, N$ and $L = \lim A_{a_i}$ exists. Let Z be a connected component of $L \cap D_a$; since L is an embedded integral manifold of g and is also closed in D_a, it suffices to show that $c_j \in Z$ for some $j \in \{1, \dots, N\}$.

Since Z is closed in D_a, $\phi_a\big|D_a > 0$ and $\phi_a\big|\mathrm{fr}(D_a) = 0$, the function $\phi_a\big|Z$ attains a maximal value λ_0. We let $K_0 = \phi_a^{-1}(\{\lambda_0\}) \cap Z$; note that K_0 is compact. Since $L \cap D_a$ is open in L, there are $0 < \lambda_2 < \lambda_1 < \lambda_0$ and an open neighborhood U of K_0 in $\mathbb{R}^n$ such that
(i) $L \cap \overline{U} = Z \cap \overline{U}$, and
(ii) for all sufficiently large i, we have $\phi_a(x) \le \lambda_2$ for all $x \in A_{a_i} \cap \mathrm{fr}(U)$, and there exists a $y \in A_{a_i} \cap U$ satisfying $\phi_a(y) \ge \lambda_1$.

Thus, for all sufficiently large i, the set C_{a_i} has a connected component S_{a_i} that is contained in U. In particular, $f_{j(i)}(a_i) \in U$ for some $j(i)$ for all sufficiently large i, and it follows that $c_j \in L \cap \overline{U} \subseteq Z$ for some j, as desired. □

5. Proof of Theorem 2

Let $A \subseteq \mathbb{R}^{m+n}$ be bounded and definable, and put $A' = \Pi_m(A)$ and $d = \max\{\dim A_a : \ a \in \mathbb{R}^m\}$. We proceed by induction on d.

We first assume that $d = 0$. Then by o-minimality and Lemma 9, we may assume there is a bounded, continuous, definable function $f : A' \longrightarrow \mathbb{R}^n$ such that $A = \mathrm{gr}(f)$. This f clearly represents F_A.

We now assume that $d > 0$ and that Theorem 2 holds for lower values of d. By Lemma 9, we may assume that A is a definable C^p-cell of dimension $d' + d$, where $d' = \dim(A')$ and $p \geq d' + d$ is fixed; hence each A_a for $a \in A'$ is a definable C^p-cell of dimension d, and we only need worry about the limits $\lim A_{a_i}$ such that $a_i \to a \in \mathrm{fr}(A')$. We now proceed in two main steps to reduce to the special situation of the previous section.

Step 1: blowing up in jet space. Let $\mathcal{G} : A \longrightarrow G^d_n$ be the fiberwise Gauss map of A and write $A^q = A^q_{\mathcal{G}}$ for $q = 0, \dots, p$. Note that $\dim \mathrm{fr}(A^q) < d' + d$, and for any sequence (a_i) in A' such that $a_i \to a \in \mathrm{fr}(A')$ and $\lim A^q_{a_i}$ exists, we have $\lim A^q_{a_i} \subseteq \mathrm{fr}(A^q)_a$.

For each $q = 0, \dots, p$, we let $\mathcal{C}^q$ be a stratification of $\mathrm{fr}(A^q) \cap \Pi^{-1}_m(\mathrm{fr}\, A')$ into definable C^{p+2} cells (see Proposition (1.13) of [4, Ch. 4]) such that, with $\pi = \pi^{q+1}_q$,

(a) if $q < p$, then $\mathcal{C}^q$ is compatible with $\pi(C)$ for each $C \in \mathcal{C}^{q+1}$.

By Remark 6, after refining each $\mathcal{C}^q$ if necessary we may also assume that for every $C \in \mathcal{C}^q$ that is the graph of a definable map $g : \pi(C) \longrightarrow G^d_{n_q}$,

(b) if there is an $(a, z) \in \pi(C)$ such that $g(a, z) \subseteq T_z\, \pi(C)_a$ and g is integrable at (a, z), then g is tangent to $\pi(C)$ and integrable.

By Lemma 9, if F_A has a representation in C, for each $C \in \mathcal{C}^0$, then F_A has a representation in $\mathrm{fr}(A) \cap \Pi^{-1}_m(\mathrm{fr}\, A')$, which in turn implies Theorem 2. We therefore fix an arbitrary $q \in \{0, \dots, p-1\}$ and $C \in \mathcal{C}^q$ such that $\dim(C) \geq q$, and we prove that F_{A^q} has a representation in C. (This in particular implies that F_A has a representation in any $C \in \mathcal{C}^0$, as desired.)

We proceed by reverse induction on $\dim C \leq d' + d - 1$. Thus, if $\dim(C) < d' + d - 1$, we also assume that for any $q' \in \{0, \dots, p-1\}$ and $C' \in \mathcal{C}^{q'}$ such that $\dim C' > \dim C$ and $\dim C' \geq q'$, $F_{A^{q'}}$ has a representation in C'.

Below we write again π in place of π^{q+1}_q. By (a) above, for any sequence (a_i) in A' such that $a_i \to a \in \mathrm{fr}(A')$ and $\lim A^{q+1}_{a_i}$ exists, we have

$$C_a \cap \lim A^q_{a_i} \subseteq \bigcup_{\substack{D \in \mathcal{C}^{q+1} \\ C \subseteq \pi(D)}} \pi\big(D_a \cap \lim A^{q+1}_{a_i}\big).$$

Thus by Lemma 9 it suffices to find, for every $D \in \mathcal{C}^{q+1}$ satisfying $C \subseteq \pi(D)$, a representation of $F_{A^{q+1}}$ in D. If $\dim D > \dim C$ for such a D, then $\dim D \geq q + 1$ as well and we are done by the inductive hypothesis. We therefore put $\mathcal{C}_C = \big\{D \in \mathcal{C}^{q+1} :\ C \subseteq \pi(D) \text{ and } \dim D = \dim C\big\}$ and assume that $D \in \mathcal{C}_C$; so D is the graph of a definable, fiberwise distribution $g : \pi(D) \longrightarrow G^d_{n_q}$ of class C^{p+2}.

As $\mathcal{C}^{q+1}$ is a stratification, the sets $W = \bigcup \big\{E \in \mathcal{C}^{q+1} :\ \dim E > \dim D\big\}$ and $W \cup D$ are open in $\mathrm{fr}(A^{q+1}) \cap \Pi^{-1}_m(\mathrm{fr}(A'))$. Hence by (b) above, Proposition 8 applies with $B = \bigcup_{\sigma \in \Sigma_{n_q}} \mathrm{fr}'\left(((A^q)_\sigma)^1\right)$: either

(i) the set $L_{(a_i)} = (D_a \cap \lim A^{q+1}_{a_i}) \setminus \big(\lim B_{a_i} \cup \mathrm{cl}(W_a \cap \lim A^{q+1}_{a_i})\big)$ is empty for every subsequence (a_i) of A' such that $a_i \to a$ and $\lim A^{q+1}_{a_i}$ and $\lim B_{a_i}$ exist, or

(ii) g is tangent to $\pi(D)$ and integrable (and hence $\mathrm{b}^1 g$ is tangent to D and integrable) and $L_{(a_i)}$ is an embedded integral manifold of $\mathrm{b}^1 g_a$ and an open subset of $\lim A^{q+1}_{a_i}$ for any subsequence (a_i) of A' such that $a_i \to a$ and $\lim A^{q+1}_{a_i}$ and $\lim B_{a_i}$ exist.

Step 2: lifting A^{q+1}. Since $\dim(B_a) < d$ for all $a \in \mathbb{R}^m$ and $A' = \Pi_M(B)$, the inductive hypothesis gives a representation $f_B : A' \longrightarrow \mathbb{R}^{N_B}$ of F_B. Furthermore, by the inductive hypothesis $F_{A^{q+1}}$ has a representation in E for every $E \in \mathcal{C}^{q+1}$ satisfying $\dim E > \dim D$, so by Lemma 9, $F_{A^{q+1}}$ has a representation $f_W : A' \longrightarrow \mathbb{R}^{N_W}$ in W. We define $f' : A' \longrightarrow \mathbb{R}^{N_B+N_W}$ by $f'(a) = (f_B(a), f_W(a))$; then f' represents $F_{A^{q+1}}$ in W and represents F_B. We now lift all our data via f': we write $m' = m + N_B + N_W$ and put

$$\widetilde{A} = \mathrm{lift}_{f'} A^{q+1}, \quad \widetilde{B} = \mathrm{lift}_{f'} B, \quad \widetilde{D} = \mathrm{lift}_{f'} D \quad \text{and} \quad \widetilde{W} = \mathrm{lift}_{f'} W;$$

note that $\widetilde{A}$ and $\widetilde{D}$ are fiberwise manifolds. We shall show that some $f : \mathrm{gr}(f') \longrightarrow \mathbb{R}^N$ represents $F_{\widetilde{A}}$ in $\widetilde{D}$; the function $h : A' \longrightarrow \mathbb{R}^N$ defined by $h(a) = f(a, f'(a))$ then represents $F_{A^{q+1}}$ in D, as desired.

Let (a_i) be any subsequence of A' such that $(a_i, f'(a_i)) \to (a, b)$ and $\lim A_{a_i}$ exists. First, from Proposition 11 with B and $\mathbb{R}^{m+n_{q+1}}$ in place of A and W we obtain

$$\lim B_{a_i} = \mathrm{cl}\left(\widetilde{B}\right)_{(a,b)}. \tag{1}$$

Second, from Proposition 11 with A^{q+1} in place of A we obtain

$$W_a \cap \lim A^{q+1}_{a_i} = \widetilde{W}_{(a,b)} \cap \mathrm{cl}\left(\widetilde{A}\right)_{(a,b)}. \tag{2}$$

Since the right-hand side is equal to $\left(\widetilde{W} \cap \mathrm{cl}\left(\widetilde{A}\right)\right)_{(a,b)}$, we define

$$\widetilde{E} = \widetilde{D} \cap \left(\mathrm{cl}(\widetilde{B}) \cup \mathrm{cl}'\left(\widetilde{W} \cap \mathrm{cl}(\widetilde{A})\right)\right);$$

it follows that, with $L_{(a_i)}$ as in (i) above,

$$\begin{aligned} L_{(a_i)} &= (\widetilde{D}_{(a,b)} \cap \lim \widetilde{A}_{(a_i, f'(a_i))}) \setminus \widetilde{E}_{(a,b)} \\ &= (\widetilde{D} \setminus \widetilde{E})_{(a,b)} \cap \lim \widetilde{A}_{(a_i, f'(a_i))}. \end{aligned}$$

On the other hand, equalities (1) and (2) above also imply that $\widetilde{E}_{(a,b)} \subseteq \lim A^{q+1}_{a_i}$. Since (a_i) is arbitrary, it follows that the function $f'' : \mathrm{gr}(f') \longrightarrow \mathbb{R}^{N_B+N_W}$ defined by $f''(a, f'(a)) = f'(a)$ represents $F_{\widetilde{A}}$ in $\widetilde{E}$. It therefore suffices to find a representation of $F_{\widetilde{A}}$ in $\widetilde{D} \setminus \widetilde{E}$.

Step 3: defining a representation. If $L_{(a_i)} = \emptyset$ for all subsequences (a_i) of A' such that $a_i \to a$ and $\lim A_{a_i}$ exists, then any function $f : \mathrm{gr}(f') \longrightarrow \mathbb{R}^N$ represents $F_{\widetilde{A}}$ in $\widetilde{D} \setminus \widetilde{E}$. We therefore assume from now on that some such $L_{(a_i)}$ is nonempty, and hence by (ii) above that g is tangent to $\pi(D)$ and integrable. We now define $\widetilde{g} : \widetilde{D} \longrightarrow G^d_n$ by $\widetilde{g}(a, b, x) = \mathrm{b}^1 g(a, x)$; then $\widetilde{g}$ is tangent to $\widetilde{D}$ and integrable.

The only remaining problem preventing us from applying Proposition 13 now is that $\mathrm{fr}\,\big(\widetilde{D} \setminus \widetilde{E}\big)$ may not be closed. However, it follows from (1) and (2) above that $\dim \widetilde{E}_{(a,b)} < d$ for any $(a,b) \in \mathrm{cl}(\mathrm{gr}\, f')$. Therefore by the inductive hypothesis, there is a representation $h : \Pi_{m'}(\widetilde{E}) \longrightarrow \mathbb{R}^N$ of $F_{\widetilde{E}}$.

Note that $\Pi_{m'}(\widetilde{E}) \subseteq \Pi_{m'}(\widetilde{D})$, and let $\{D'_1, \dots, D'_k\}$ be a partition of $\Pi_{m'}(\widetilde{D})$ into definable cells compatible with $\Pi_{m'}(\widetilde{E})$ and such that $h\big|D'_j$ is continuous for each j satisfying $D'_j \subseteq \Pi_{m'}(\widetilde{D})$. Then for any j, since $\widetilde{D} = \mathrm{lift}_{f'}\, D$, the set $\widetilde{D}_j = \widetilde{D} \cap \Pi_{m'}^{-1}(D'_j)$ is a cell. Moreover, h represents $F_{\widetilde{E}}$ and $\mathrm{cl}'(\widetilde{E}) = \widetilde{E}$, so Corollary 12(2) implies that the set $\widetilde{E}_j = \widetilde{E} \cap \Pi_{m'}^{-1}(D'_j)$ is a closed subset of $\widetilde{D}_j$; in particular, $\mathrm{fr}\,\big(\widetilde{D}_j \setminus \widetilde{E}_j\big)$ is closed.

Therefore, Proposition 13 applies, with $\widetilde{A}$, $\widetilde{D}_j \setminus \widetilde{E}_j$ and $\widetilde{g}\big|\big(\widetilde{D}_j \setminus \widetilde{E}_j\big)$ in place of A, D and g. Thus, for each j there is a representation $f_j : \mathrm{gr}(f') \longrightarrow \mathbb{R}^{N_j}$ of $F_{\widetilde{A}}$ in $\widetilde{D}_j \setminus \widetilde{E}_j$. It follows that $f = (f_1, \dots, f_k) : \mathrm{gr}(f') \longrightarrow \mathbb{R}^{N_1 + \cdots + N_k}$ represents $F_{\widetilde{A}}$ in $\widetilde{D} \setminus \widetilde{E}$, as desired. This finishes the proof of Theorem 2.

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