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Difference Fourier transforms for nonreduced root systems

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Abstract. In the first part of the paper kernels are constructed which meromorphically extend the Macdonald–Koornwinder polynomials in their degrees. In the second part the kernels associated with rank one root systems are used to define nonsymmetric variants of the spherical Fourier transform on the quantum $\mathrm{SU}(1,1)$ group. Related Plancherel and inversion formulas are derived using double affine Hecke algebra techniques.

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1. Introduction

The Weyl algebra of differential operators with polynomial coefficients on Euclidean n-space is the image of the Heisenberg algebra under the Schrödinger representation. The classical Fourier transform induces an automorphism of the Weyl algebra which interchanges the role of polynomial multiplication operators and constant coefficient differential operators. This automorphism (and its generalizations) is called the *duality isomorphism* in the present paper. Plancherel and inversion formulas for the classical Fourier transform can be easily derived from the above observations by first proving algebraic versions on the cyclic module of the Weyl algebra generated by the Gaussian. See the survey paper [19] for more details and references. We generalize this approach to difference Fourier transforms associated to nonreduced root systems.

Work of Dunkl [13], [14], Heckman [20], Opdam [38], de Jeu [22], Cherednik [2], [3], [6], Macdonald [34], Noumi [35], [36] and others have led to generalizations of the Weyl algebra and the underlying Heisenberg algebra, which are naturally associated to Fourier transforms arising from harmonic analysis on Cartan motion groups, Riemannian symmetric spaces and compact quantum Riemannian spaces. In each case the Weyl algebra is replaced by an algebra consisting of differential (or difference) reflection operators and multiplication operators. This generalized

Weyl algebra may be realized as the faithful image of (an appropriate degeneration of) the double affine Hecke algebra under Cherednik's representation (the analogue of the Schrödinger representation). In each case the generalized Weyl algebras form a powerful tool in the study of the related Fourier transforms. The main goal of the present paper is to extend this picture to include the case of Fourier transforms arising from harmonic analysis on the simplest quantum analogue of a *noncompact* simple Lie group, namely the quantum $\mathrm{SU}(1,1)$ group.

Some remarks are here in order on the theory of locally compact quantum groups. Despite the fact that *compact* quantum groups are well understood, also from the viewpoint of harmonic analysis (see e.g., [36]), this is by far not the case for *noncompact* semisimple quantum groups. Recent developments though essentially settled the theory for the quantum $\mathrm{SU}(1,1)$ group. A quantization of the normalizer of $\mathrm{SU}(1,1)$ in $\mathrm{SL}(2,\mathbb{C})$ was constructed as a locally compact quantum group by Koelink and Kustermans [25] (see Kustermans and Vaes [31] for a detailed account on the general theory of locally compact quantum groups). The harmonic analytic aspects for the quantum $\mathrm{SU}(1,1)$ group were analyzed in [23], [24] and [26]. The harmonic analysis on the quantum $\mathrm{SU}(1,1)$ group in [26] led to an explicit Fourier transform, whose Plancherel and inversion formula were derived by classical function-theoretic methods in [27]. The transform is called the Askey–Wilson function transform since its kernel forms a meromorphic continuation of the well-known Askey–Wilson polynomials (see [1]) in their degrees.

To explore the role of the double affine Hecke algebra for the Askey–Wilson function transform, we first need to construct nonsymmetric variants of the transform which induce the proper analogue of the duality isomorphism on the double affine Hecke algebra. The construction of the associated kernel is developed in this paper for nonreduced root systems of arbitrary rank. Two basic features of the kernel are its explicit series expansion in terms of Macdonald–Koornwinder polynomials, and the fact that it is a meromorphic continuation of the Macdonald–Koornwinder polynomials in their degrees. These results are inspired by Cherednik's paper [5], in which such kernels were introduced for reduced root systems.

Restricting attention to rank one, we use these kernels to define nonsymmetric variants of the Askey–Wilson function transform which induces the analogue of the duality isomorphism on the double affine Hecke algebra. Instead of defining the transforms on compactly supported functions (as was done in the classical approach [27]), we define it now on a space of function consisting of a direct sum of *two* cyclic modules of the double affine Hecke algebra. From harmonic analytic point of view a second cyclic module is needed to take care of the "strange part" of the support of the Plancherel measure, i.e., the part of the support coming from contributions of unitary representations of the quantized universal enveloping algebra $\mathrm{U}_q(\mathrm{su}(1,1))$ which vanish in the classical limit.

Explicit evaluations of the images of the cyclic vectors under the nonsymmetric Askey–Wilson function transform then suffice to prove algebraic inversion and

Plancherel type formulas for the nonsymmetric Askey–Wilson function transform. The image of the cyclic vector of the "classical" module follows from an extension of the polynomial Macdonald–Koornwinder theory. This in particular entails explicit formulas for the Macdonald–Koornwinder type constant term of the product of the inverse of a generalized Gaussian and two Macdonald–Koornwinder polynomials (see [5] for the analogous results in the reduced setup). The computation of the image of the cyclic vector of the "strange" module (see Proposition 8.12) is a key result which combines many of the properties of the kernels with some elementary elliptic function theory. The particularly large number of parameter freedoms, caused by the fact that we are dealing with *nonreduced* root systems, is also used here in an essential way. This may be seen as an extra justification for the special attention to nonreduced root systems in this paper.

Finally we show that the results described in the previous paragraph easily lead to new proofs of the Plancherel and inversion formula for the symmetric Askey–Wilson function transform (see [27] and [43] for the classical function-theoretic approach).

Considering difference analogues of Harish-Chandra transforms only from the viewpoint of double affine Hecke algebras lead to several other self-dual difference Fourier transforms, see e.g., [8] and [9]. This flexibility in choices, combined with the lack in comprehension of non-compact semisimple quantum groups, thus poses essential problems in deciding which difference analogues of the Harish-Chandra transform arise from harmonic analysis on non-compact quantum symmetric spaces. The present, detailed study of the Askey–Wilson function transform from the viewpoint of double affine Hecke algebras hopefully provides some new insights on this matter.

The paper is organized as follows. In Section 2 we introduce the double affine Hecke algebra, the duality isomorphism and the analogue of the Gaussian. In Section 3 we introduce the concept of difference Fourier transforms. We furthermore explain what the main techniques are going to be in the study of such transforms. In Section 4 we show how the polynomial Macdonald–Koornwinder transform (introduced in [41]) fits into this general scheme. In Section 5 we introduce kernels with which explicit integral transforms can be constructed that fit into the concept of difference Fourier transforms. We call the kernels *Cherednik kernels* since they generalize kernels introduced by Cherednik in [5]. In Section 6 we study the main properties of the Cherednik kernels. In particular, we show that the Cherednik kernels meromorphically extend the Macdonald–Koornwinder polynomials in their degrees, and that they satisfy a natural duality property which extends the duality of the Macdonald–Koornwinder polynomials. In Section 7 we study an extension of the Macdonald–Koornwinder transform, acting on the cyclic module generated by the inverse of the Gaussian. It forms the second example of a difference Fourier transform in the sense of Section 3. In Section 8 we restrict attention to rank one. We define the nonsymmetric Askey–Wilson function transform as an integral transform with kernel given by the (rank one) Cherednik kernel. We show that it also qualifies as a difference Fourier transform in the sense of Section 3. We prove inversion and Plancherel Theorems on the algebraic and on the L^2-level. In the Appendix we finally discuss certain bounds for Macdonald–Koornwinder polynomials that are needed for the construction of the Cherednik kernels.

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2. The double affine Hecke algebra

2.1. The affine root system of type $C^\vee C_n$. Denote by ϵ_i $(i = 1, \ldots, n)$ the standard orthonormal basis of the n-dimensional Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. Let

$$\Sigma = \{\pm\epsilon_i \pm \epsilon_j\}_{1\leq i<j\leq n} \cup \{\pm 2\epsilon_i\}_{i=1}^n \subset \mathbb{R}^n$$

be the root system of type C_n. Let $\mathrm{Aff}(\mathbb{R}^n)$ be the space of affine linear transformations $f : \mathbb{R}^n \to \mathbb{R}$. As a vector space, $\mathrm{Aff}(\mathbb{R}^n) \simeq \mathbb{R}^n \oplus \mathbb{R}\delta$ via the formula

$$(v + \lambda\delta)(w) = \langle v, w\rangle + \lambda, \qquad v, w \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}.$$

We extend the form $\langle\cdot,\cdot\rangle$ to a positive semi-definite form on $\mathrm{Aff}(\mathbb{R}^n)$ by requiring the constant function δ to be in the radical of $\langle\cdot,\cdot\rangle$. Then

$$R = \Sigma + \mathbb{Z}\delta \subset \mathrm{Aff}(\mathbb{R}^n)$$

is the *affine root system* of type $\widetilde{C}_n$. Associated with $f \in R$, we have the reflection $r_f \in \mathrm{Gl}_{\mathbb{R}}(\mathrm{Aff}(\mathbb{R}^n))$ defined by

$$r_f(g) = g - 2\frac{\langle f, g\rangle}{\langle f, f\rangle} f, \qquad g \in \mathrm{Aff}(\mathbb{R}^n).$$

The *affine Weyl group* $\mathcal{W}$ is the subgroup of $\mathrm{Gl}(\mathrm{Aff}(\mathbb{R}^n))$ generated by all the reflections r_f $(f \in R)$.

There are two important descriptions of $\mathcal{W}$, namely as a Coxeter group, and as a semi-direct product of a finite reflection group with a lattice. For the presentation of $\mathcal{W}$ as a Coxeter group, we choose the standard basis $\{a_0, a_1, \ldots, a_n\}$ of the affine root system R by

$$a_0 = \delta - 2\epsilon_1, \quad a_i = \epsilon_i - \epsilon_{i+1}, \quad a_n = 2\epsilon_n$$

for $i = 1, \ldots, n-1$, and we set $r_i = r_{a_i}$ for the associated simple reflections. The above choice of basis induces a decomposition of Σ and R in positive roots and negative roots, the positive roots being given by

$$\Sigma_+ = \{\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{2\epsilon_i\}_i, \qquad R_+ = \Sigma_+ \cup \{f \in R \,|\, f(0) > 0\}.$$

Furthermore, it is well known that the affine Weyl group $\mathcal{W}$ is generated by the simple reflections r_i ($i = 0, \ldots, n$). The fundamental relations between the simple reflections r_i are given by the Coxeter relations

$$\begin{aligned} r_i r_{i+1} r_i r_{i+1} &= r_{i+1} r_i r_{i+1} r_i, \qquad i = 0, i = n-1,\\ r_i r_{i+1} r_i &= r_{i+1} r_i r_{i+1}, \qquad i = 1, \ldots, n-2,\\ r_i r_j &= r_j r_i, \qquad |i-j| \geq 2,\\ r_i^2 &= 1, \qquad i = 0, \ldots, n. \end{aligned}$$

Let $W_0 \subset \mathcal{W}$ be the subgroup of $\mathcal{W}$ generated by $r_1, \ldots, r_n$. Then W_0 is the Weyl group of the root system Σ, hence isomorphic to $S_n \ltimes (\pm 1)^n$ with S_n the symmetric group in n letters. Let Λ be the W_0-invariant $\mathbb{Z}$-lattice of $\mathbb{R}^n$ with basis $\{\epsilon_i\}_i$,

$$\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i.$$

The lattice Λ can be naturally identified with the coroot lattice as well as with the weight lattice of the root system Σ. The second description of the affine Weyl group $\mathcal{W}$ is given by

$$\mathcal{W} \simeq W_0 \ltimes \Lambda$$

with the lattice elements $\lambda \in \Lambda$ acting on $\mathrm{Aff}(\mathbb{R}^n)$ by the formula

$$\lambda(f) = f + \langle f, \lambda \rangle \delta, \qquad \lambda \in \Lambda,\ f \in \mathrm{Aff}(\mathbb{R}^n).$$

Finally we note that the set

$$R_{nr} = R \cup \mathcal{W}\frac{a_0}{2} \cup \mathcal{W}\frac{a_n}{2} \subset \mathrm{Aff}(\mathbb{R}^n)$$

is a (nonreduced) affine root system having $\mathcal{W}$ as its associated affine Weyl group. In Macdonald's [33] terminology, R_{nr} is the *affine root system of type* $C^\vee C_n$.

2.2. Difference multiplicity functions. The nonreduced affine root system R_{nr} of type $C^\vee C_n$ has five $\mathcal{W}$-orbits, namely

$$\mathcal{W}a_0, \qquad \mathcal{W}\frac{a_0}{2}, \qquad \mathcal{W}a_i, \qquad \mathcal{W}a_n, \qquad \mathcal{W}\frac{a_n}{2},$$

where i may be arbitrarily chosen from the index set $\{1, \ldots, n-1\}$.

A $\mathcal{W}$-invariant complex-valued function on R_{nr} is called a *multiplicity function.* We denote a multiplicity function by $\mathbf{t} = \{t_f\}_{f \in R_{nr}}$, with $t_f \in \mathbb{C}$ its value at the root $f \in R_{nr}$, and we set $t_f = 1$ when $f \in \mathrm{Aff}(\mathbb{R}^n) \setminus R_{nr}$.

In the theory presented in the next (sub-)sections, a sixth generic parameter $q^{\frac{1}{2}} \in \mathbb{C}^\times$ appears. Its square q is a parameter that arises from the realization of the group algebra $\mathbb{C}[\mathcal{W}]$ as the algebra of difference reflection operators with constant coefficients. The pair $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$, where $\mathbf{t}$ is a multiplicity function and $q^{\frac{1}{2}} \in \mathbb{C}^\times$, is called a *difference multiplicity function*. Alternatively, a difference multiplicity function α can be represented by an ordered six-tuple

$$\alpha = (\mathbf{t}, q^{\frac{1}{2}}) = (t_0, u_0, t_n, u_n, t, q^{\frac{1}{2}}),$$

where

$$t_0 = t_{a_0}, \qquad u_0 = t_{a_0/2}, \qquad t = t_i = t_{a_i}, \qquad t_n = t_{a_n}, \qquad u_n = t_{a_n/2},$$

where i may be any integer from the index set $\{1, \ldots, n-1\}$.

We define two "involutions" σ and τ on difference multiplicity functions, both preserving the parameter $q^{\frac{1}{2}}$. The involution σ acts on $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$ by interchanging the value of $\mathbf{t}$ on the $\mathcal{W}a_0$-orbit with its value on $\mathcal{W}a_n/2$, while τ acts by interchanging the value of $\mathbf{t}$ on the $\mathcal{W}a_0$-orbit with its value on $\mathcal{W}a_0/2$. The new difference multiplicity functions are denoted by α_σ and α_τ, respectively.

Associated with a generic difference multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$, we write $\alpha_\ddagger = (\mathbf{t}^{-1}, q^{-\frac{1}{2}})$ with $\mathbf{t}^{-1} = \{t_f^{-1}\}_{f \in R_{nr}}$ for the difference multiplicity function with inverted parameters.

If an object H depends on a difference multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$, then e.g., $H_{\ddagger\sigma\tau}$ (or $H^{\ddagger\sigma\tau}$) denotes the object H depending on the difference multiplicity function $\alpha_{\ddagger\sigma\tau} = ((\alpha_\ddagger)_\sigma)_\tau$. For example, if

$$H = H_{(t_0, u_0, t_n, u_n, t, q^{\frac{1}{2}})},$$

then

$$H_{\ddagger\sigma\tau} = H_{(u_0^{-1}, u_n^{-1}, t_n^{-1}, t_0^{-1}, t^{-1}, q^{-\frac{1}{2}})}.$$

Throughout the remainder of this section we fix a generic difference multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$.

2.3. Difference reflection operators. Let $\mathcal{A} = \mathbb{C}[x^{\pm 1}]$ be the algebra of Laurent polynomials in n indeterminates $x = (x_1, \ldots, x_n)$, $\mathbb{C}(x)$ the quotient field of $\mathcal{A}$ and $\mathcal{O} = \mathcal{O}\big((\mathbb{C}^\times)^n\big)$ the ring of analytic functions on the complex torus $(\mathbb{C}^\times)^n$. We write $\mathcal{M} = \mathcal{M}\big((\mathbb{C}^\times)^n\big)$ for the field of meromorphic functions on $(\mathbb{C}^\times)^n$. Since $(\mathbb{C}^\times)^n$ is a connected domain of holomorphy, $\mathcal{M}$ is isomorphic to the quotient field of $\mathcal{O}$ (see e.g. [19, Thm. 7.4.6]). We have natural inclusions

$$\mathcal{A} \subset \mathbb{C}(x) \subset \mathcal{M}, \qquad \mathcal{A} \subset \mathcal{O} \subset \mathcal{M}.$$

Let $\mathbb{C}[\mathcal{W}]$ be the group algebra of the affine Weyl group $\mathcal{W}$ over $\mathbb{C}$.

Definition 2.1. The algebra $\mathcal{D}_q$ of q-difference reflection operators with meromorphic coefficients is defined by

$$\mathcal{D}_q = \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{W}]$$

with multiplication

$$(g \otimes v)(h \otimes w) = g(vh) \otimes vw, \qquad g, h \in \mathcal{M}, \;\; w, v \in \mathcal{W},$$

where $\mathcal{W} \simeq W_0 \ltimes \Lambda$ acts as field automorphisms on $\mathcal{M}$ by the formulas

$$\begin{aligned}\big(\lambda h\big)(x) &= h(q^{\lambda_1}x_1, q^{\lambda_2}x_2, \ldots, q^{\lambda_n}x_n),\\ \big(r_i h\big)(x) &= h(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n),\\ \big(r_n h\big)(x) &= h(x_1, \ldots, x_{n-1}, x_n^{-1}),\end{aligned} \tag{2.1}$$

for $h(x) = h(x_1, \ldots, x_n) \in \mathcal{M}$, $\lambda = \sum_j \lambda_j \epsilon_j \in \Lambda$ and for $i = 1, \ldots, n-1$.

Note that the action (2.1) of $\mathcal{W}$ depends on the sixth parameter $q^{\frac{1}{2}}$ of the underlying difference multiplicity function α. When confusion can arise on the underlying difference multiplicity function, we say that $\mathcal{W}$ acts by constant coefficient q-difference reflection operators when the action is given by (2.1) (i.e., when the sixth parameter of the underlying difference multiplicity function is $q^{\frac{1}{2}}$).

The algebra $\mathcal{D}_q$ acts on $\mathcal{M}$ by

$$\big((gw)h\big)(x) = g(x)\big(wh\big)(x), \qquad g \in \mathcal{M},\ w \in \mathcal{W} \tag{2.2}$$

for $h \in \mathcal{M}$, with the action of $\mathcal{W}$ on $\mathcal{M}$ as given by (2.1). Here we use the notation $g(x)w$ or gw for a pure tensor $g \otimes w \in \mathcal{D}_q$. We use the terminology that $\mathcal{D}_q$ (or a subalgebra of $\mathcal{D}_q$) acts as *q-difference reflection operators* on some function space on the complex torus $(\mathbb{C}^\times)^n$ when the action is given by formula (2.2), with the action of $\mathcal{W}$ on functions h defined by (2.1).

We define monomials by

$$x^f = q^{\frac{m}{2}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \in \mathcal{A}$$

for any $f = \lambda + \frac{1}{2}m\delta \in \Lambda + \frac{1}{2}\mathbb{Z}\delta$, where $\lambda = \sum_i \lambda_i \epsilon_i$. Again note here that this definition depends on the sixth parameter $q^{\frac{1}{2}}$ of the underlying difference multiplicity function α, but it will always be clear from the context what the underlying difference multiplicity function is. Associated to any root $f \in R$ and any difference multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$, we now construct explicit difference reflection operators $T_f = T_f^\alpha \in \mathcal{D}_q$ as follows.

Definition 2.2. For $f \in R$, the difference reflection operator $T_f = T_f^\alpha \in \mathcal{D}_q$ is defined by

$$T_f = t_f + t_f^{-1} c_f(x)(r_f - \mathrm{id}),$$

with coefficient $c_f(x) = c_f^\alpha(x) \in \mathbb{C}(x)$ given by

$$c_f(x) = \frac{(1 - t_f t_{f/2} x^{f/2})(1 + t_f t_{f/2}^{-1} x^{f/2})}{(1 - x^f)} \tag{2.3}$$

when $f/2 \in R_{nr}$ and

$$c_f(x) = \frac{(1 - t_f^2 x^f)}{(1 - x^f)} \tag{2.4}$$

when $f/2 \notin R_{nr}$.

Note that the formula (2.4) for c_f in case that $f/2 \notin R_{nr}$ can formally be written as (2.3) in view of the convention that $t_g = 1$ when $g \in \mathrm{Aff}(\mathbb{R}^n) \setminus R_{nr}$.

The following crucial theorem, due to Cherednik in the reduced setup, was proved by Noumi [36].

Theorem 2.3. *The q-difference reflection operators $T_j = T_j^\alpha \in \mathcal{D}_q$ defined by $T_j = T_{a_j}$ $(j = 0, \ldots, n)$, satisfy the fundamental commutation relations of the affine Hecke algebra of type $\widetilde{C}_n$. In other words, they satisfy the braid relations*

$$\begin{aligned} T_iT_{i+1}T_iT_{i+1} &= T_{i+1}T_iT_{i+1}T_i, & i &= 0, i = n-1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, & i &= 1, \ldots, n-2, \\ T_iT_j &= T_jT_i, & |i-j| &\geq 2, \end{aligned} \tag{2.5}$$

and the quadratic relations

$$(T_j - t_j)(T_j + t_j^{-1}) = 0, \qquad j = 0, \ldots, n, \tag{2.6}$$

in the algebra $\mathcal{D}_q$.

Following Noumi [36], one can now define the Y-operators $Y_i = Y_i^\alpha \in \mathcal{D}_q$ for $i = 1, \ldots, n$ by

$$Y_i = T_i \cdots T_{n-1}T_nT_{n-1} \cdots T_1T_0T_1^{-1} \cdots T_{i-1}^{-1}. \tag{2.7}$$

The operator Y_i is naturally associated to the lattice element $\epsilon_i \in \Lambda$ considered as an element of the affine Weyl group $\mathcal{W} \simeq W_0 \ltimes \Lambda$, since

$$\epsilon_i = r_i \cdots r_{n-1}r_nr_{n-1} \cdots r_1r_0r_1 \cdots r_{i-1} \in \mathcal{W}$$

is a reduced expression for $\epsilon_i \in \Lambda \subset \mathcal{W}$. The following result follows from Theorem 2.3 and from the algebraic structure of affine Hecke algebras (see Lusztig [32] and Noumi [36]).

Corollary 2.4. *The operators $Y_i \in \mathcal{D}_q$ $(i = 1, \ldots, n)$ pairwise commute.*

The analogue of Cherednik's double affine Hecke algebra can be defined explicitly in terms of generators and relations, see Sahi [39] (see also [41] and [42] for alternative presentations). We take a shortcut in this paper by defining the double affine Hecke algebra $\mathcal{H}$ directly in its image under Cherednik's faithful realization of $\mathcal{H}$ as q-difference reflection operators and multiplication operators (the analogue of the Schrödinger realization of the Heisenberg algebra). For this we observe that $\mathcal{M}$, and hence $\mathcal{A} \subset \mathcal{M}$, is naturally embedded in $\mathcal{D}_q$ via

$$g \mapsto g \otimes e, \qquad g \in \mathcal{M},$$

where $e \in \mathcal{W}$ is the identity element.

Definition 2.5. The double affine Hecke algebra $\mathcal{H} = \mathcal{H}_\alpha \subset \mathcal{D}_q$ is the unital sub-algebra of $\mathcal{D}_q$ generated by T_j $(j = 0, \ldots, n)$ and $\mathcal{A}$.

It is clear that $\mathcal{H}$ is also generated as algebra by $Y_i^{\pm 1}, x_i^{\pm 1}$ and T_i for $i = 1, \ldots, n$. Observe that $\mathcal{H}$ acts on $\mathcal{A} \subset \mathcal{M}$ and on $\mathcal{O} \subset \mathcal{M}$ by restriction of the natural action of $\mathcal{D}_q$ on $\mathcal{M}$.

2.4. The duality isomorphism. The Heisenberg algebra has a copy of $\mathrm{SL}_2(\mathbb{Z})$ in its automorphism group, generated within the metaplectic representation by the Fourier transform and multiplication by the Gaussian. These two isomorphisms, as well as their realization within the metaplectic representation, generalize to the setup of double affine Hecke algebras.

In this subsection we introduce the isomorphism of the double affine Hecke algebra associated with (the analogue of) the Fourier transform. In the following subsection, we consider the isomorphism associated with multiplication by the generalized Gaussian.

We first recall the so-called ϵ-transform, see Sahi [39].

Theorem 2.6. *There exists a unique, unital algebra isomorphism*

$$\epsilon = \epsilon_\alpha : \mathcal{H} = \mathcal{H}_\alpha \to \mathcal{H}_{\ddagger\sigma}$$

satisfying

$$\epsilon(Y_i) = x_i, \qquad \epsilon(T_i) = T_i^{\ddagger\sigma\, -1}, \qquad \epsilon(x_i) = Y_i^{\ddagger\sigma}$$

for $i = 1, \ldots, n$. *Furthermore,* $\epsilon^{-1} = \epsilon_{\ddagger\sigma}$.

By [41, Lem. 7.3] the assignment

$$x_i \mapsto x_i^{-1}, \qquad T_j \mapsto T_j^{\ddagger\, -1}$$

for $i = 1, \ldots, n$ and $j = 0, \ldots, n$ uniquely extends to a unital algebra isomorphism $\dagger : \mathcal{H} \to \mathcal{H}_\ddagger$. Furthermore, if $I : \mathcal{M} \to \mathcal{M}$ denotes the involution

$$\big(Ig\big)(x) = g(x^{-1}), \qquad g \in \mathcal{M}$$

where $x^{-1} = (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$, then it is easy to check that

$$I \circ X = \dagger(X) \circ I, \qquad X \in \mathcal{H}. \tag{2.8}$$

Composing now ϵ with $\dagger$, we obtain the "duality isomorphism" (see [41, Def. 7.5]) we are looking for.

Corollary 2.7. *The map* $\sigma := \dagger_{\ddagger\sigma} \circ \epsilon : \mathcal{H} \to \mathcal{H}_\sigma$ *is an algebra isomorphism, satisfying*

$$\sigma(x_1^{-1} T_0 Y_1^{-1}) = T_0^\sigma, \qquad \sigma(T_i) = T_i^\sigma, \qquad \sigma(Y_i) = x_i^{-1}$$

for $i = 1, \ldots, n$.

Remark 2.8. Note here that we use the notation σ for the isomorphism σ of the double affine Hecke algebra, as well as for the involution on the (difference) multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$. Which one is used should always be clear from the context (as a generic rule, σ as involution on difference multiplicity functions will always be written as a sub-(or super-)index, in contrast with σ regarded as an isomorphism of the double affine Hecke algebra). Such conventions will also hold for other (anti-)isomorphisms of the double affine Hecke algebra defined at later stages.

In [41, Prop. 8.8] it is proved that the nonsymmetric (polynomial) Macdonald–Koornwinder transform induces the automorphism σ on $\mathcal{H}$, see also Section 4.

2.5. The Gaussian. We assume in this subsection that the parameter $q^{\frac{1}{2}}$ in the generic difference multiplicity function $\alpha = (t_0, u_0, t_n, u_n, t, q^{\frac{1}{2}})$ has modulus unequal to one.

Cherednik introduced an analogue of the Gaussian for double affine Hecke algebras for reduced root systems, see [5] and references therein. In this subsection we generalize Cherednik's construction to the nonreduced setup.

Recall the standard notations for q-shifted factorials (see [16]),

$$\big(z_1, \ldots, z_m; q\big)_k = \prod_{i=1}^{m} \big(z_i; q\big)_k, \qquad \big(z; q\big)_k = \prod_{i=0}^{k-1} (1 - zq^i), \tag{2.9}$$

where $z_1, \ldots, z_m, z \in \mathbb{C}$ and $k \in \mathbb{N} \cup \{\infty\}$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$. Here the modulus of q should be taken < 1 when $k = \infty$.

Definition 2.9. i) For $|q| < 1$ the Gaussian $G(\cdot) = G_\alpha(\cdot) \in \mathcal{M}$ is defined by the infinite product

$$\begin{aligned} G(x) &= \prod_{f \in \mathcal{W}a_0 \cap R_+} (1 + t_f t_{f/2}^{-1} x^{f/2})^{-1} \\ &= \prod_{i=1}^{n} \big(-q^{\frac{1}{2}} t_0 u_0^{-1} x_i, -q^{\frac{1}{2}} t_0 u_0^{-1}/x_i; q\big)_\infty^{-1}. \end{aligned}$$

ii) For $|q| > 1$ the Gaussian $G(\cdot) = G_\alpha(\cdot) \in \mathcal{M}$ is defined by the infinite product

$$\begin{aligned} G(x) &= \prod_{f \in \mathcal{W}a_0 \cap R_-} (1 + t_f t_{f/2}^{-1} x^{f/2}) \\ &= \prod_{i=1}^{n} \big(-q^{-\frac{1}{2}} t_0 u_0^{-1} x_i, -q^{-\frac{1}{2}} t_0 u_0^{-1}/x_i; q^{-1}\big)_\infty. \end{aligned}$$

Remark 2.10. a) We used two different ways of representing the Gaussian in Definition 2.9, the first in terms of the root data, the second in terms of q-shifted factorials. The equivalence of the two expressions can be easily verified using that

$$\mathcal{W}a_0 \cap R_\pm = \Sigma^l + (1 \pm 2\mathbb{N})\delta \tag{2.10}$$

where $\Sigma^l \subset \Sigma$ is the set of roots of length two:

$$\Sigma^l = \{\pm 2\epsilon_i \,|\, i = 1, \ldots, n\}.$$

b) The Gaussians for $|q| < 1$ and $|q| > 1$ are related by the formula

$$G(x) = G_{\ddagger\tau}(x)^{-1}. \tag{2.11}$$

c) If $t_0 = u_0$ and $|q| < 1$, then

$$G(x)^{-1} = \prod_{i=1}^{n} \big(-q^{1/2}x_i, -q^{1/2}x_i^{-1}; q\big)_\infty = \big(q;q\big)_\infty^{-n} \sum_{\lambda\in\Lambda} q^{\langle\lambda,\lambda\rangle/2} x^\lambda \tag{2.12}$$

in view of the Jacobi triple product identity [16, (1.6.1)] for theta functions. Up to the (irrelevant) constant $\big(q;q\big)_\infty^{-n}$, this coincides with the definition of the Gaussian of type $\widetilde{C}_n$ as given by Cherednik, see [5] and references therein.

Proposition 2.11. *The inner automorphism*

$$X \mapsto G\,X\,G^{-1}$$

of $\mathcal{D}_q$ restricts to an algebra isomorphism $\tau = \tau_\alpha : \mathcal{H} \to \mathcal{H}_\tau$. Its action on a set of algebraic generators of $\mathcal{H}$ is given by

$$\tau(x_i) = x_i, \qquad \tau(T_i) = T_i^\tau, \qquad \tau(T_0) = q^{-\frac{1}{2}} x_1 T_0^{\tau\,-1}$$

for $i = 1, \ldots, n$.

Proof. It suffices to show that conjugation by G on the algebraic generators of $\mathcal{H} \subset \mathcal{D}_q$ is as stated in the proposition. Clearly, conjugation by G maps the coordinate functions x_i to themselves. Conjugation with G maps T_i to T_i^τ for $i = 1, \ldots, n$ since G is W_0-invariant and since $T_i = T_i^\tau$ for $i = 1, \ldots, n$ (indeed, τ acts on $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$ by interchanging the parameters $t_0 = t_{a_0}$ and $u_0 = t_{a_0/2}$, which only occur in the generator T_0).

It remains to prove that $\tau(T_0) = q^{-\frac{1}{2}} x_1 T_0^{\tau\,-1}$. Observe that the action of the simple reflection $r_0 \in \mathcal{W} \subset D_q$ on the Gaussian $G \in \mathcal{M}$ is given by

$$(r_0 G)(x) = G(qx_1^{-1}, x_2, \ldots, x_n) = \frac{\big(1 + t_0 u_0^{-1} q^{\frac{1}{2}} x_1^{-1}\big)}{\big(1 + t_0 u_0^{-1} q^{-\frac{1}{2}} x_1\big)} G(x),$$

hence

$$\begin{aligned} t_0^{-1} c_{a_0}(x) \frac{G(x)}{(r_0G)(x)} &= t_0^{-1} \frac{(1 - t_0 u_0 q^{\frac{1}{2}} x_1^{-1})(1 + t_0 u_0^{-1} q^{-\frac{1}{2}} x_1)}{(1 - q x_1^{-2})} \\ &= q^{-\frac{1}{2}} x_1 u_0^{-1} c_{a_0}^\tau(x). \end{aligned} \tag{2.13}$$

Furthermore, observe that

$$\begin{aligned} t_0 - t_0^{-1}c_{a_0}(x) &= \frac{(t_0 - t_0^{-1}) + (u_0 - u_0^{-1})q^{\frac{1}{2}}x_1^{-1}}{1 - qx_1^{-2}} \\ &= q^{-\frac{1}{2}}x_1 \frac{(u_0 - u_0^{-1})qx_1^{-2} + (t_0 - t_0^{-1})q^{\frac{1}{2}}x_1^{-1}}{1 - qx_1^{-2}} \\ &= q^{-\frac{1}{2}}x_1\big(u_0^{-1} - u_0^{-1}c_{a_0}^{\tau}(x)\big). \end{aligned} \tag{2.14}$$

We then compute in $\mathcal{D}_q$, using (2.13) and (2.14),

$$\begin{aligned} G\,T_0\,G^{-1} &= t_0 - t_0^{-1}c_{a_0} + t_0^{-1}c_{a_0}\frac{G}{r_0G}r_0 \\ &= q^{-\frac{1}{2}}x_1\big(u_0^{-1} - u_0^{-1}c_{a_0}^{\tau} + u_0^{-1}c_{a_0}^{\tau}r_0\big) \\ &= q^{-\frac{1}{2}}x_1T_0^{\tau\,-1}, \end{aligned}$$

which completes the proof of the proposition. □

2.6. SL_2-type commutation relations. Let $w \in \mathcal{W}$ and choose a reduced expression $w = r_{i_1}r_{i_2}\cdots r_{i_l}$. We denote

$$T_w = T_{i_1}T_{i_2}\cdots T_{i_l} \in \mathcal{H},$$

which is independent of the choice of reduced expression since the T_j's satisfy the $\widetilde{C}_n$-braid relations. Similarly, $t_w = t_{i_1}t_{i_2}\cdots t_{i_l} \in \mathbb{C}$ is independent of the reduced expression of w, where $t_j = t_{a_j}$ is the value of the multiplicity function $\mathbf{t}$ at the simple root $a_j \in R$.

Let $w_0 \in W_0$ be the longest Weyl group element in W_0. Then for all $X \in \mathcal{H}$, we have the relations

$$\begin{aligned} (\sigma_\sigma \circ \sigma \circ \sigma_\sigma \circ \sigma)(X) = T_{w_0}^{-2}XT_{w_0}^2 &= (\sigma_\sigma \circ \tau_{\sigma\tau} \circ \sigma_{\tau\sigma\tau} \circ \tau_{\tau\sigma} \circ \sigma_\tau \circ \tau)(X), \\ (\sigma_{\tau\sigma} \circ \sigma_\tau \circ \tau)(X) &= (\tau \circ \sigma_\sigma \circ \sigma)(X). \end{aligned} \tag{2.15}$$

Here the composition of isomorphisms on the right-hand side of the first equality in (2.15) is well defined since $\sigma\tau\sigma = \tau\sigma\tau$ when acting on difference multiplicity functions.

The relations (2.15) between the isomorphisms σ and τ of $\mathcal{H}$ should be viewed as the analogues of the characterizing relations

$$\widetilde{\sigma}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (\widetilde{\sigma}\widetilde{\tau})^3, \qquad \widetilde{\sigma}^2\widetilde{\tau} = \widetilde{\tau}\widetilde{\sigma}^2$$

for the generators

$$\widetilde{\sigma} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \widetilde{\tau} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of the modular group $\mathrm{SL}_2(\mathbb{Z})$.

We do not give a detailed proof of the relations (2.15) since they are not needed in the remainder of the paper. We refer to [4, Thm. 2.3] for the analogous result in case of reduced root systems. We only note here that the first equality in (2.15) follows from the simpler formula $(\sigma_\sigma \circ \sigma)(X) = T_{w_0}^{-1} X T_{w_0}$ for $X \in \mathcal{H}$.

3. General remarks on difference Fourier transforms

In this section the concept of Fourier transforms associated with σ is introduced in an informal manner. Rigorous statements will follow in subsequent sections.

Let $V = V_\alpha$ and $W = W_\alpha$ be some function spaces on the complex torus $(\mathbb{C}^\times)^n$ on which $\mathcal{H}$ acts as q-difference reflection operators. We assume that V is stable under the inversion operator $(Ig)(x) := g(x^{-1})$. This in particular implies that $\mathcal{H}_\ddagger$ acts as q^{-1}-difference reflection operators on V. The relation between the two actions on V is given by the formula (2.8).

The starting point of our considerations is the search for explicit linear transformations

$$F = F_\alpha : V \to W_\sigma,$$

which satisfy

$$F \circ X = \sigma(X) \circ F, \qquad \forall\, X \in \mathcal{H}, \tag{3.1}$$

where σ is the duality isomorphism of $\mathcal{H}$, see Corollary 2.7. We call such a linear endomorphism F a *Fourier transform associated with* σ, or sometimes just a *difference Fourier transform*.

We are mainly interested in difference Fourier transforms F which can be realized as integral transforms. The property (3.1) then formally translates to explicit transformation properties of the associated kernels under the action of the double affine Hecke algebra. To be more precise, we first need to recall certain anti-isomorphisms of the double affine Hecke algebra $\mathcal{H}$ which play the role of adjoint maps.

Lemma 3.1. i) *There exists a unique unital antialgebra isomorphism* $\ddagger = \ddagger_\alpha : \mathcal{H} \to \mathcal{H}_\ddagger$ *satisfying*

$$\ddagger(T_j) = T_j^{\ddagger\,-1}, \qquad \ddagger(x_i) = x_i^{-1}$$

for $j = 0\ldots, n$ *and* $i = 1, \ldots, n$. *Furthermore,* $\ddagger^{-1} = \ddagger_\ddagger$.

ii) *There exists a unique unital antialgebra isomorphism* $\iota = \iota_\alpha : \mathcal{H} \to \mathcal{H}$ *satisfying*

$$\iota(T_j) = T_j, \qquad \iota(x_i) = x_i$$

for $j = 0\ldots, n$ *and* $i = 1, \ldots, n$. *Furthermore,* $\iota^{-1} = \iota$.

Proof. i) See [39, Prop. 7.1].

ii) This follows from the fact that $\iota := \ddagger_\ddagger \circ \dagger = \dagger_\ddagger \circ \ddagger : \mathcal{H} \to \mathcal{H}$ is a unital antialgebra isomorphism which fixes the generators T_j and x_i for $j = 0, \ldots, n$ and $i = 1, \ldots, n$. □

Suppose now that we are given a linear map $F : V \to W_\sigma$ of the form

$$\big(Fg\big)(\gamma) = \big(g, \mathfrak{E}_\ddagger(\gamma^{-1}, \cdot)\big), \qquad g \in V \tag{3.2}$$

for some kernel $\mathfrak{E}_\ddagger$, with $\big(\cdot,\cdot\big) = \big(\cdot,\cdot\big)_\alpha$ some bilinear form satisfying

$$\big(Xg, h\big) = \big(g, \ddagger(X)h\big)$$

for $X \in \mathcal{H}$, $g \in V$ and h in some completion of V. At a formal level, the condition that the map $F : V \to W_\sigma$ defines a Fourier transform associated with σ corresponds to the transformation behavior

$$\big(X\mathfrak{E}_\ddagger(\gamma, \cdot)\big)(x) = \big(\psi_\ddagger(X)\mathfrak{E}_\ddagger(\cdot, x)\big)(\gamma), \qquad X \in \mathcal{H}_\ddagger \tag{3.3}$$

of the kernel $\mathfrak{E}_\ddagger$, where $\psi = \psi_\alpha : \mathcal{H} \to \mathcal{H}_\sigma$ is the antiisomorphism

$$\psi = \dagger_{\ddagger\sigma} \circ \sigma_\ddagger \circ \ddagger.$$

Here the double affine Hecke algebra acts by q^{-1}-difference reflection operators on both sides of (3.3). The antiisomorphism ψ is the so-called *duality antiisomorphism*, see [39] and [41]. In particular, it has the following special properties.

Proposition 3.2. *The map $\psi = \dagger_{\ddagger\sigma} \circ \sigma_\ddagger \circ \ddagger$ is the unique unital antiisomorphism $\psi : \mathcal{H} \to \mathcal{H}_\sigma$ satisfying*

$$\psi(x_i) = Y_i^{\sigma\,-1}, \qquad \psi(T_i) = T_i^\sigma, \qquad \psi(Y_i) = x_i^{-1}$$

for $i = 1, \ldots, n$. In particular, $\psi^{-1} = \psi_\sigma$.

Proof. This is an easy verification, see e.g., [39, Sect. 7] for details. □

The transformation behavior (3.3) for a kernel $\mathfrak{E}_\ddagger$ hints at several important (and desirable) properties for $\mathfrak{E}_\ddagger$. For instance the fact that ψ maps the commuting Y-operators to multiplication operators implies that $\mathfrak{E}_\ddagger(\gamma, \cdot)$ is a common eigenfunction of $Y_i^\ddagger \in \mathcal{H}_\ddagger$ with eigenvalue γ_i^{-1} for $i = 1, \ldots, n$. Furthermore, the "involutivity" $\psi_\ddagger^{-1} = \psi_{\ddagger\sigma}$ of the antiisomorphism $\psi_\ddagger$ hints at the symmetric role of the geometric parameter x and the spectral parameter γ in $\mathfrak{E}_\ddagger(\gamma, x)$. In fact, the kernels we encounter indeed turn out to satisfy the *duality property*

$$\mathfrak{E}_\ddagger(\gamma, x) = \mathfrak{E}_{\ddagger\sigma}(x, \gamma) \tag{3.4}$$

after a proper choice of normalization.

The property that a transform F satisfies the transformation behavior (3.1) turns out to be a very strong condition. In fact, for the transforms we encounter in this paper, the corresponding modules V and W are cyclic $\mathcal{H}$-modules, or they consist of a direct sum of two cyclic $\mathcal{H}$-modules. In each case, the cyclic vectors are given explicitly. The transformation behavior (3.1) of F reduces the study of the transform to the explicit evaluation of the image of the cyclic vectors under F. Again by the transformation behavior (3.1) of F, the computation of the image of the cyclic vectors reduce to explicit constant term evaluations (e.g. Macdonald type constant term identities and Macdonald–Mehta type identities).

To analyze inversion formulas for difference Fourier transforms, we make a similar, formal analysis for Fourier transforms $J_\sigma = J_{\alpha_\sigma} : W_\sigma \to V$ associated with the isomorphism σ^{-1}, i.e., linear maps satisfying the opposite transformation behavior

$$J_\sigma \circ X = \sigma^{-1}(X) \circ J_\sigma, \qquad \forall\, X \in \mathcal{H}_\sigma, \tag{3.5}$$

where V and W are as before. When no confusion can arise on the underlying isomorphism, we also use the terminology *difference Fourier transforms* for such transforms J_σ.

We now assume that J_σ is of the form

$$(J_\sigma g)(x) = [g, \mathfrak{E}(\cdot, x)]_\sigma, \tag{3.6}$$

with $[\cdot,\cdot] = [\cdot,\cdot]_\alpha$ a bilinear form satisfying

$$[Xg, h] = [g, \iota(X)h]$$

for $X \in \mathcal{H}$, $g \in W$ and for h in some completion of W. The fact that J_σ is a Fourier transform associated with σ^{-1} then formally relates to the transformation behaviour

$$\big(X\mathfrak{E}(\gamma,\cdot)\big)(x) = \big(\psi(X)\mathfrak{E}(\cdot,x)\big)(\gamma), \qquad X \in \mathcal{H} \tag{3.7}$$

of the kernel $\mathfrak{E}$, since $\psi = \iota_\sigma \circ \sigma$. Here the double affine Hecke algebra acts on both sides of (3.7) by q-difference reflection operators. In view of the (expected) duality (3.4) of the kernels, it is therefore plausible that a given difference Fourier transform $F : V \to W_\sigma$ of the form (3.2) can be inverted by an explicit transform J_σ which has the same kernel as F, but depending now on the difference multiplicity function $\alpha_{\ddagger\sigma}$ instead of α. All these features are shown to be true for the difference Fourier transforms considered in this paper.

It is clear from the above considerations that the first priority should be to construct kernels $\mathfrak{E}$ satisfying the transformation behavior (3.7) under the action of the double affine Hecke algebra. The first example of such a kernel $\mathfrak{E}(\gamma, x)$ can be defined in terms of Macdonald–Koornwinder polynomials, but the spectral parameter γ then runs through the discrete, polynomial spectrum of the operators $Y_i \in \mathcal{H}$. This restrictive kernel can be used to define the so-called polynomial Macdonald–Koornwinder transform (see [41]), which gives a first example of a difference Fourier transform. In Section 4 we explain the concepts introduced in this section for the polynomial Macdonald–Koornwinder transform.

In Section 5 we meromorphically extend this polynomial kernel $\mathfrak{E}$ while preserving the desired transformation behavior (3.7) under the action of the double affine Hecke algebra. This kernel was written down explicitly by Cherednik [5] for reduced root systems. In Section 7 and Section 8 we construct and study related difference Fourier transforms in full detail, following closely the general philosophy as explained in this section.

4. The Macdonald–Koornwinder transform

In this section we recall a known example of a Fourier transform $F : V \to W_\sigma$ associated with σ, the so-called Macdonald–Koornwinder transform (see [41]). In this case F is defined as an integral transform with kernel expressed in terms of nonsymmetric Macdonald–Koornwinder polynomials. In subsection 4.1 we introduce the modules V and W; in Subsection 4.2 we introduce the kernel $\mathfrak{E}$ and in Subsection 4.3 we define the associated bilinear forms $\big(\cdot,\cdot\big)$ and $[\cdot,\cdot]$. In Subsection 4.4 we construct the Macdonald–Koornwinder transform and its inverse. Most results can be found directly or indirectly in [36], [39] or in [41] (see also the lecture notes [42]). I have decided to be quite detailed in this section, since the results play a key role throughout this paper. Furthermore, the theory is presented in such a way that it directly fits into the general scheme of difference Fourier transforms as discussed in Section 3.

Throughout this section we fix a generic multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$. After Subsection 4.2 we impose extra conditions on α, see the beginning of Subsection 4.3.

4.1. The modules. For the Macdonald–Koornwinder transform $F : V \to W_\sigma$, we take V to be the cyclic $\mathcal{H}$-module $\mathcal{A}$, with cyclic vector $1 \in \mathcal{A}$ the Laurent polynomial identically equal to one. The target space $W = W_\alpha = \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$ is the linear space of functions $g : \mathcal{S}_{\ddagger\sigma} \to \mathbb{C}$ with finite support, where $\mathcal{S} = \mathcal{S}_\alpha \subset (\mathbb{C}^\times)^n$ is the spectrum of the commuting elements $Y_1, \ldots, Y_n \in \mathcal{H}_\alpha$ considered as an endomorphism of $\mathcal{A}$ via their action as q-difference reflection operators.

We now introduce the $\mathcal{H}$-module structure on $\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$. In [41, Prop. 8.8] the $\mathcal{H}$-module structure on $\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$ was constructed in an indirect manner, using the Macdonald–Koornwinder transform in an essential way. In order to emphasize the natural order of definitions and results in the study of Fourier transforms associated with σ (see Section 3), we give here a detailed account on a direct construction of the $\mathcal{H}$-module structure on $\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$.

We first need to recall the explicit form of the polynomial spectrum $\mathcal{S} = \mathcal{S}_\alpha$, see e.g., [39] and [41]. It is naturally parametrized by the lattice Λ,

$$\mathcal{S} = \{s_\lambda \,|\, \lambda \in \Lambda\},$$

with $s_\lambda = s_\lambda^\alpha$ given by

$$s_\lambda = (s_{\lambda,1}, s_{\lambda,2}, \ldots, s_{\lambda,n}), \qquad s_{\lambda,i} = (t_n t_0)^{(\rho_l(\lambda),\epsilon_i)} t^{(\rho_m(\lambda),\epsilon_i)} q^{(\lambda,\epsilon_i)},$$

where $\rho_m(\lambda), \rho_l(\lambda) \in \Lambda$ are given by

$$\rho_m(\lambda) = \sum_{\alpha \in \Sigma_m^+} \operatorname{sgn}(\langle \lambda, \alpha \rangle)\alpha^\vee, \qquad \rho_l(\lambda) = \sum_{\alpha \in \Sigma_l^+} \operatorname{sgn}(\langle \lambda, \alpha \rangle)\alpha^\vee.$$

Here $\operatorname{sgn}(m)$ is equal to 1 if $m \in \mathbb{N}$ and equal to -1 if $m \in \mathbb{Z}_{<0}$, Σ_m^+ and Σ_l^+ are the positive roots of Σ of squared length two and four respectively, and $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ is the coroot of α.

Observe that the spectrum $\mathcal{S}$ does not depend on the parameters u_0 and u_n of the difference multiplicity function α, and that the spectral points $s_\lambda = s_\lambda^\alpha$ satisfy

$$s_\lambda^\ddagger = s_\lambda^{-1}, \qquad \forall\, \lambda \in \Lambda.$$

Note also that $s_0 = s_0^\alpha \in \mathcal{S}$ is the spectral point corresponding to the common eigenfunction $1 \in \mathcal{A}$ of the Y-operators $Y_1, \ldots, Y_n$.

Let $\mathcal{F}(\mathcal{S}_{\ddagger\sigma})$ be the space of functions $g : \mathcal{S}_{\ddagger\sigma} \to \mathbb{C}$ (without finiteness conditions). We first introduce an action of $\mathcal{W}$ on $\mathcal{F}(\mathcal{S}_{\ddagger\sigma})$, which we call the dot action. It is defined by

$$(w \cdot g)(s_\lambda^{\ddagger\sigma}) = g(s_{w^{-1}\cdot\lambda}^{\ddagger\sigma}), \qquad w \in \mathcal{W},\ \lambda \in \Lambda$$

for $g \in \mathcal{F}(\mathcal{S}_{\ddagger\sigma})$ with the action of $\mathcal{W}$ on Λ defined by

$$r_j \cdot \lambda = \begin{cases} (-1 - \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n), & j = 0, \\ (\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \lambda_j, \lambda_{j+2}, \ldots, \lambda_n), & 1 \le j \le n-1, \\ (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n), & j = n. \end{cases}$$

Note that $\lambda \cdot \mu = \lambda + \mu$ for $\lambda, \mu \in \Lambda$, where $\lambda \in \Lambda$ is viewed as an affine Weyl group element in $\mathcal{W} \simeq W_0 \ltimes \Lambda$.

Lemma 4.1. *Fix $j \in \{0, \ldots, n\}$ arbitrary.*

a) *For any function $g : (\mathbb{C}^\times)^n \to \mathbb{C}$ we have*

$$\big(r_j g\big)(s_\lambda^{\ddagger\sigma}) = \big(r_j \cdot g|_{\mathcal{S}_{\ddagger\sigma}}\big)(s_\lambda^{\ddagger\sigma})$$

when $\lambda \in \Lambda$ satisfies $r_j \cdot \lambda \neq \lambda$, where $r_j g$ is the action of $r_j \in \mathcal{W}$ on g as constant coefficient q-difference reflection operator (*see* (2.1)).

b) *The rational function $c_{a_j} \in \mathbb{C}(x)$ is regular at the spectral points $s \in \mathcal{S}_{\ddagger\sigma}$. Furthermore,*

$$c_{a_j}(s_\lambda^{\ddagger\sigma}) = 0 \quad \Leftrightarrow \quad r_j \cdot \lambda = \lambda \tag{4.1}$$

for all $\lambda \in \Lambda$.

Proof. a) See the proof of [39, Thm 5.3].

b) It is important to recall here that the difference multiplicity function α is assumed to be generic. The regularity follows then from the explicit expressions for c_{a_j} and $s \in \mathcal{S}_{\ddagger\sigma}$. In a similar manner one checks that $c_{a_0}(s_\lambda^{\ddagger\sigma}) \neq 0$ for all $\lambda \in \Lambda$. Since $r_0 \cdot \lambda \neq \lambda$ for all $\lambda \in \Lambda$, this proves (4.1) for $j = 0$.

For $j \in \{1, \ldots, n-1\}$ we observe that

$$c_{a_j}(s_\lambda^{\ddagger\sigma}) = 0 \quad \Leftrightarrow \quad \big(s_\lambda^{\ddagger\sigma}\big)^{a_j} = (t_n u_n)^{-(\rho_m(\lambda), a_j)} t^{-(\rho_m(\lambda), a_j)} q^{-(\lambda, a_j)} = t^{-2}$$

by the explicit expression for c_{a_j} (see (2.4)). On the other hand, it is easy to check that $\big(s_\lambda^{\ddagger\sigma}\big)^{a_j} = t^{-2}$ if and only if $r_j \cdot \lambda = \lambda$.

For $j = n$ the condition $c_{a_j}(s_\lambda^{\ddagger\sigma}) = 0$ is equivalent to the condition that $\big(s_\lambda^{\ddagger\sigma}\big)^{\epsilon_n}$ is equal to $(t_n u_n)^{-1}$ or to $-t_n^{-1} u_n$, in view of (2.3). By the explicit expression for $s_\lambda^{\ddagger\sigma}$, only the equality $\big(s_\lambda^{\ddagger\sigma}\big)^{\epsilon_n} = (t_n u_n)^{-1}$ can happen for some $\lambda \in \Lambda$. Furthermore, one easily checks that $\big(s_\lambda^{\ddagger\sigma}\big)^{\epsilon_n} = (t_n u_n)^{-1}$ if and only if $r_n \cdot \lambda = \lambda$, which proves (4.1) for $j = n$. □

Using the above lemma we can define an action of $\mathcal{H}$ on $\mathcal{F}(\mathcal{S}_{\ddagger\sigma})$ as follows.

Lemma 4.2. *For $g \in \mathcal{F}(\mathcal{S}_{\ddagger\sigma})$ and $X \in \mathcal{H}$, set*

$$\big(X \cdot g\big)(s) := \big(X\widetilde{g}\big)(s), \qquad s \in \mathcal{S}_{\ddagger\sigma}, \tag{4.2}$$

where $\widetilde{g} : (\mathbb{C}^\times)^n \to \mathbb{C}$ is an arbitrary function satisfying $\widetilde{g}|_{\mathcal{S}_{\ddagger\sigma}} = g$, and with the action of $X \in \mathcal{H}$ on $\widetilde{g}$ in the right-hand side of (4.2) *by q-difference reflection operators. Then* (4.2) *is well defined* (*i.e., independent of the choice of extension $\widetilde{g}$ of g*)*, and it defines a left action of $\mathcal{H}$ on $\mathcal{F}(\mathcal{S}_{\ddagger\sigma})$.*

Proof. It suffices to prove that formula (4.2) is independent of the choice of extension $\widetilde{g}$ of $g \in \mathcal{F}(\mathcal{S}_{\ddagger\sigma})$. This is clear when $X \in \mathcal{H}$ is multiplication by a Laurent polynomial $p \in \mathcal{A}$. Lemma 4.1 shows that

$$\big(T_j\widetilde{g}\big)(s) = t_j g(s) + t_j^{-1} c_{a_j}(s)\big((r_j \cdot g)(s) - g(s)\big), \qquad \forall\, s \in \mathcal{S}_{\ddagger\sigma} \tag{4.3}$$

for $j \in \{0, \ldots, n\}$; hence (4.2) is also independent of the choice of extension $\widetilde{g}$ of g when $X = T_j$ $(j = 0, \ldots, n)$. Since T_j $(j = 0, \ldots, n)$ and the $p \in \mathcal{A}$ generate $\mathcal{H}$ as an algebra, a straightforward inductive argument shows that (4.2) is well defined for all $X \in \mathcal{H}$. □

The standard basis of $\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$ consists of the delta functions $\delta_\mu = \delta_\mu^\alpha \in \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$ for $\mu \in \Lambda$, which are defined by

$$\delta_\mu(s_\lambda^{\ddagger\sigma}) = \begin{cases} 1, & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda \neq \mu. \end{cases}$$

Lemma 4.3. *The subspace $\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma}) \subset \mathcal{F}(\mathcal{S}_{\ddagger\sigma})$ of functions with finite support is a cyclic $\mathcal{H}$-submodule of $\mathcal{F}(\mathcal{S}_{\ddagger\sigma})$, with cyclic vector $\delta_0 \in \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$.*

Proof. By (4.3) it is clear that $\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma}) \subset \mathcal{F}(\mathcal{S}_{\ddagger\sigma})$ is an $\mathcal{H}$-submodule.

Since $\mathcal{W}$ acts transitively on Λ under the dot action, we may define the height $h(\lambda) \in \mathbb{N}$ of $\lambda \in \Lambda$ to be smallest nonnegative integer m such that $w \cdot \lambda = 0$ for some $w \in \mathcal{W}$ of length m. To complete the proof of the lemma we show that $\delta_\lambda \in \mathcal{H} \cdot \delta_0 \subseteq \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$ by induction to the height of $\lambda \in \Lambda$.

It suffices to prove the induction step. Let $m \in \mathbb{Z}_{>0}$ and assume that $\delta_\mu \in \mathcal{H} \cdot \delta_0$ for all $\mu \in \Lambda$ with $h(\mu) < m$. Let $\lambda \in \Lambda$ with $h(\lambda) = m$. Choose a $w \in \mathcal{W}$ satisfying $l(w) = m$ and $w \cdot \lambda = 0$. Let $w = r_{i_1} r_{i_2} \cdots r_{i_m}$ be a reduced expression. We define inductively

$$\lambda_j := r_{i_j} \cdot \lambda_{j+1} \in \Lambda, \qquad j = 1, \ldots, m$$

starting with $\lambda_{m+1} := \lambda$. Observe that $\lambda_1 = 0$, and that the λ_i $(i = 1, \ldots, m)$ are pairwise different. Indeed, if the λ_i are not pairwise different, then $u \cdot \lambda = 0$ for some $u \in \mathcal{W}$ with length strictly smaller than $h(\lambda)$, which is a contradiction. By (4.3) it follows that

$$T_{w^{-1}} \cdot \delta_0 = \big(T_{i_m} T_{i_{m-1}} \cdots T_{i_1}\big) \cdot \delta_0 = c_\lambda \delta_\lambda + \sum_{\substack{\mu \in \Lambda: \\ h(\mu) < h(\lambda)}} c_\mu \delta_\mu$$

for some constants $c_\mu \in \mathbb{C}$, with leading coefficient given explicitly by

$$c_\lambda = t_w^{-1} \prod_{j=1}^{m} c_{i_j}(s^{\ddagger\sigma}_{\lambda_{j+1}}),$$

cf. [41, Lem. 9.2]. Now $c_\lambda \neq 0$ by Lemma 4.1b), since $\lambda_i = r_{i_j} \cdot \lambda_{j+1} \neq \lambda_{j+1}$ for $j = 1, \ldots, m$. Hence $\delta_\lambda \in \mathcal{H} \cdot \delta_0$ by the induction hypothesis. □

4.2. The kernel. The algebra of Laurent polynomials $\mathcal{A}$ decomposes in common eigenspaces of $Y_i \in \mathcal{H}$,

$$\mathcal{A} = \bigoplus_{s \in \mathcal{S}} \mathcal{A}(s)$$

with $\mathcal{A}(s)$ for $s \in \mathcal{S} = \mathcal{S}_\alpha$ the subspace

$$\mathcal{A}(s) = \{p \in \mathcal{A} \,|\, r(Y)p = r(s)p, \ \forall\, r \in \mathcal{A}\}.$$

Here $r(Y) \in \mathcal{H}$ for $r \in \mathcal{A}$ is obtained by replacing the variables $x_1, \ldots, x_n$ by the invertible, commuting operators $Y_1, \ldots, Y_n \in \mathcal{H}_\alpha$. The common eigenspaces $\mathcal{A}(s)$ are one-dimensional. We fix a unique eigenfunction $E(s; \cdot) = E_\alpha(s; \cdot) \in \mathcal{A}(s)$ for $s \in \mathcal{S}$ by requiring the normalization

$$E(s; s_0^{\ddagger\sigma}) = 1. \tag{4.4}$$

In particular, $E(s_0; \cdot) = 1 \in \mathcal{A}$ is the Laurent polynomial identically equal to one. The Laurent polynomials $\{E(s; \cdot) \,|\, s \in \mathcal{S}\}$ are called the Macdonald–Koornwinder polynomials, cf. [39]. We now define a kernel

$$\mathfrak{E}_{\mathcal{A}}(\cdot, \cdot) = \mathfrak{E}_{\mathcal{A},\alpha}(\cdot, \cdot) : \mathcal{S}_\ddagger \times (\mathbb{C}^\times)^n \to \mathbb{C}$$

by

$$\mathfrak{E}_{\mathcal{A},\alpha}(s, x) = E_\alpha(s^{-1}; x), \qquad s \in \mathcal{S}_\ddagger, \ x \in (\mathbb{C}^\times)^n.$$

Proposition 4.4. *For $X \in \mathcal{H}$ and $s \in \mathcal{S}_\ddagger$ we have*

$$\big(X\mathfrak{E}_{\mathcal{A}}(s,\cdot)\big)(x) = \big(\psi(X)\cdot\mathfrak{E}_{\mathcal{A}}(\cdot,x)\big)(s)$$

where $\mathcal{H}$ acts on the left-hand side as q-difference reflection operators.

Proof. This follows from [41, Prop. 7.8] and from the definition of ψ, using formula (4.3). □

The duality between the spectral and geometric parameter of the kernel $\mathfrak{E}_{\mathcal{A}}$ now reads as follows.

Theorem 4.5. *For all $s \in \mathcal{S}_\ddagger$ and $v \in \mathcal{S}_{\ddagger\sigma}$,*

$$\mathfrak{E}_{\mathcal{A}}(s,v) = \mathfrak{E}_{\mathcal{A},\sigma}(v,s). \tag{4.5}$$

Proof. See Sahi [39, Thm. 7.4]. □

4.3. The bilinear forms. In the remainder of this section we choose $0 < q^{\frac{1}{2}} < 1$ and $0 < t \leq 1$ arbitrarily, and add generic parameters $t_0, u_0, t_n, u_n \in \mathbb{C}^\times$ to obtain a generic difference multiplicity function

$$\alpha = (\mathbf{t}, q^{\frac{1}{2}}) = (t_0, u_0, t_n, u_n, t, q^{\frac{1}{2}}).$$

We construct suitable bilinear forms

$$\big(.,.\big)_{\mathcal{A}} = \big(\cdot,\cdot\big)_{\mathcal{A},\alpha} : \mathcal{A}\times\mathcal{A} \to \mathbb{C}$$

and

$$[.,.]_{\mathcal{A}} = [\cdot,\cdot]_{\mathcal{A},\alpha} : \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})\times\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma}) \to \mathbb{C}$$

satisfying the desired transformation properties

$$\big(Xp,r\big)_{\mathcal{A}} = \big(p,\ddagger(X)r\big)_{\mathcal{A}}, \qquad \forall\, p,r \in \mathcal{A},$$

respectively

$$[X\cdot g,h]_{\mathcal{A}} = [g,\iota(X)\cdot h]_{\mathcal{A}}, \qquad \forall\, g,h \in \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$$

under the action of $X \in \mathcal{H}$. We start with the definition of $\big(\cdot,\cdot\big)_{\mathcal{A}}$. Let $\Delta = \Delta_\alpha \in \mathcal{M}$ be the weight function

$$\Delta(x) = \prod_{f\in R_+}\frac{1}{c_f(x)}. \tag{4.6}$$

The fact that $\Delta(x)$ is well defined and meromorphic follows from the fact that $|q| < 1$.

In case that the moduli of the Askey–Wilson parameters

$$\{a,b,c,d\} = \{t_nu_n, -t_nu_n^{-1}, q^{\frac{1}{2}}t_0u_0, -q^{\frac{1}{2}}t_0u_0^{-1}\} \tag{4.7}$$

are strictly less than one, we can define the bilinear form $\big(\cdot,\cdot\big)_{\mathcal{A}}$ by

$$\big(p,r\big)_{\mathcal{A}} = \frac{1}{(2\pi i)^n} \iint_{\mathbb{T}^n} p(x) r(x^{-1}) \Delta(x) \frac{dx}{x}, \qquad p,r \in \mathcal{A},$$

where $\mathbb{T}$ is the counterclockwise oriented unit circle in the complex plane (centered at zero), and $\frac{dx}{x} = \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$ is the standard product measure on $\mathbb{T}^n$. For the bilinear form $\big(\cdot,\cdot\big)_{\mathcal{A}}$ in case that some of the Askey–Wilson parameters have moduli larger than one, one needs to integrate over $\mathcal{T}^n$, where $\mathcal{T} = \mathcal{T}_\alpha \subset \mathbb{C}$ is a suitable deformation of the unit circle $\mathbb{T}$, in order to avoid certain poles of the weight function Δ. For a detailed discussion we refer to the paper [40], where $\mathcal{T}$ is called a **t**-contour. Here we only give the basic properties of such a deformed contour $\mathcal{T}$: it is assumed to be a rectifiable, closed, counterclockwise oriented contour around the origin, which satisfies $\mathcal{T}^{-1} = \mathcal{T}$ (set theoretically), and for which the Askey–Wilson parameters a,b,c and d are contained in the interior of $\mathcal{T}$. The following result follows from [41, Prop. 8.3].

Proposition 4.6. *For all $p,r \in \mathcal{A}$ and $X \in \mathcal{H} = \mathcal{H}_\alpha$,*

$$\big(Xp,r\big)_{\mathcal{A}} = \big(p,\ddagger(X)r\big)_{\mathcal{A}}.$$

Remark 4.7. Proposition 4.6 is also valid for p and r being arbitrary analytic functions on the complex torus $(\mathbb{C}^\times)^n$.

It follows from Proposition 4.6 that

$$\big(E(v;\cdot),E_\ddagger(s^{-1};\cdot)\big)_{\mathcal{A}} = 0, \qquad v,s \in \mathcal{S},\ \ v \neq s \tag{4.8}$$

since $\ddagger(Y_i) = (Y_i^\ddagger)^{-1}$ for $i = 1,\ldots,n$.

Next we proceed by introducing $[\cdot,\cdot]_{\mathcal{A}}$. The Weyl group $W_0 \simeq S_n \ltimes (\pm 1)^n$ acts on Λ by permutations and sign changes of the coordinates. For $\lambda \in \Lambda$ we write $w_\lambda \in S_n \ltimes (\pm 1)^n$ for the element of minimal length such that $w_\lambda^{-1}\lambda \in \Lambda^+$, where

$$\Lambda^+ = \{\lambda = \sum_i \lambda_i \epsilon_i \,|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$$

are the partitions of length $\leq n$. Let u_λ be the S_n-component of w_λ. Let n_λ be the number of parts λ_i of λ which are strictly smaller than zero. The discrete weight function $N = N_\alpha : \mathcal{S}_{\ddagger\sigma} \to \mathbb{C}$ is now defined by

$$N(s_\lambda^{\ddagger\sigma}) = \underset{x=s_\lambda^{\ddagger\sigma}}{\mathbf{Res}} \left(\frac{\Delta(x)}{x_1 \cdots x_n} \right), \qquad \lambda \in \Lambda, \tag{4.9}$$

where the multiple residue $\mathbf{Res}$ is given by

$$\underset{x=s_\lambda^{\ddagger\sigma}}{\mathbf{Res}} \big(\cdot\big) = (-1)^{n_\lambda} \underset{x_{u_\lambda(1)}=s^{\ddagger\sigma}_{\lambda,u_\lambda(1)}}{\mathrm{Res}} \left(\underset{x_{u_\lambda(2)}=s^{\ddagger\sigma}_{\lambda,u_\lambda(2)}}{\mathrm{Res}} \left(\cdots \underset{x_{u_\lambda(n)}=s^{\ddagger\sigma}_{\lambda,u_\lambda(n)}}{\mathrm{Res}} \big(\cdot\big) \cdots \right)\right).$$

The bilinear form $[\cdot,\cdot]_{\mathcal{A}} = [\cdot,\cdot]_{\mathcal{A},\alpha}$ is defined by

$$[g,h]_{\mathcal{A}} = \sum_{s\in\mathcal{S}_{\ddagger\sigma}} g(s)h(s)N(s), \qquad g,h \in \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma}). \tag{4.10}$$

Note that the definition of the bilinear form also makes sense for arbitrary functions $g,h \in \mathcal{F}(\mathcal{S}_{\ddagger\sigma})$, provided that the sum is absolutely convergent. The following proposition follows now from the proof of [41, Prop. 8.9].

Proposition 4.8. *For all $X \in \mathcal{H}$ and all $g,h \in \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$,*

$$[Xg,h]_{\mathcal{A}} = [g,\iota(X)h]_{\mathcal{A}}. \tag{4.11}$$

Remark 4.9. Formula (4.11) also holds true when $g,h \in \mathcal{F}(\mathcal{S}_{\ddagger\sigma})$ as long as absolute convergence of the sums are ensured.

4.4. The difference Fourier transforms. We define the Macdonald–Koorwinder transform $F_{\mathcal{A}} = F_{\mathcal{A},\alpha} : \mathcal{A} \to \mathcal{F}_0(\mathcal{S}_{\ddagger})$ by

$$\big(F_{\mathcal{A}}p\big)(s) = \big(p, \mathfrak{E}_{\mathcal{A},\ddagger}(s^{-1},\cdot)\big)_{\mathcal{A}}, \qquad p \in \mathcal{A}$$

for $s \in \mathcal{S}_{\ddagger}$. Furthermore, we define a linear map $J_{\mathcal{A}} = J_{\mathcal{A},\alpha} : \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma}) \to \mathcal{A}$ by

$$\big(J_{\mathcal{A}}g\big)(x) = [g, \mathfrak{E}_{\mathcal{A},\sigma}(\cdot,x)]_{\mathcal{A}}, \qquad g \in \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma}).$$

The following proposition is now immediate from the previous subsections and Section 3, see also [41, Prop. 8.8 & 8.9].

Proposition 4.10. a) *The Macdonald–Koornwinder transform $F_{\mathcal{A}} : \mathcal{A} \to \mathcal{F}_0(\mathcal{S}_{\ddagger})$ is a Fourier transform associated with σ.*

b) *The map $J_{\mathcal{A}} : \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma}) \to \mathcal{A}$ is a Fourier transform associated with σ_σ^{-1}.*

We mention as an immediate corollary the following result.

Theorem 4.11. a) *$F_{\mathcal{A}} : \mathcal{A} \to \mathcal{F}_0(\mathcal{S}_{\ddagger})$ is a linear bijection with inverse $c_{\mathcal{A}}^{-1}J_{\mathcal{A},\sigma}$, with the non-zero constant $c_{\mathcal{A}}$ given by*

$$c_{\mathcal{A}} = \big(1,1\big)_{\mathcal{A}} N_\sigma(s_0^{-1}).$$

b) *We have*

$$\frac{\big(E(s;\cdot), E_{\ddagger}(s^{-1};\cdot)\big)_{\mathcal{A}}}{\big(1,1\big)_{\mathcal{A}}} = \frac{N_\sigma(s_0^{-1})}{N_\sigma(s^{-1})}$$

for all $s \in \mathcal{S}$.

Proof. See [41, Thm. 8.10]. Since the proof is illustrative for later arguments, we shortly recall the proof.

a) By the orthogonality relations (4.8) of the Macdonald–Koornwinder polynomials, we have

$$F_{\mathcal{A}}(1) = \big(1,1\big)_{\mathcal{A}}\delta_0^\sigma.$$

On the other hand, since $\mathfrak{E}_{\mathcal{A}}(s_0^{-1},x) = E(s_0;x) = 1$,

$$J_{\mathcal{A},\sigma}(\delta_0^\sigma) = N_\sigma(s_0^{-1})\,1 \in \mathcal{A}.$$

The statement now immediately follows from Proposition 4.10, since $\mathcal{A}$ (respectively $\mathcal{F}_0(\mathcal{S}_\ddagger)$) is a cyclic $\mathcal{H}$-module (respectively cyclic $\mathcal{H}_\sigma$-module) with cyclic vector 1 (respectively δ_0^σ).

b) This follows by computing the right-hand side of

$$c_{\mathcal{A}}E(s;\cdot) = J_{\mathcal{A},\sigma}\big(F_{\mathcal{A}}(E(s;\cdot))\big), \qquad s \in \mathcal{S}$$

directly from the definitions of $F_{\mathcal{A}}$ and $J_{\mathcal{A},\sigma}$, using the orthogonality relations (4.8). □

One can reformulate the orthogonality relations in terms of an algebraic Plancherel type theorem by introducing two additional transforms $\widetilde{F}_{\mathcal{A}} = \widetilde{F}_{\mathcal{A},\alpha} : \mathcal{A} \to \mathcal{F}_0(\mathcal{S}_\ddagger)$ and $\widetilde{J}_{\mathcal{A}} = \widetilde{J}_{\mathcal{A},\alpha} : \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma}) \to \mathcal{A}$,

$$\begin{aligned}\big(\widetilde{F}_{\mathcal{A}}p\big)(s) &= \big(\mathfrak{E}_{\mathcal{A}}(s,\cdot),p\big)_{\mathcal{A}},\\ \big(\widetilde{J}_{\mathcal{A}}g\big)(x) &= [I\mathfrak{E}_{\mathcal{A},\ddagger\sigma}(\cdot,x),g]_{\mathcal{A}}\end{aligned}$$

for $p \in \mathcal{A}$, $g \in \mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$ and $s \in \mathcal{S}_\ddagger$, where I is the inversion operator $(Ig)(s) = g(s^{-1})$ mapping $\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})$ onto $\mathcal{F}_0(\mathcal{S}_\sigma)$. By the previous theorem, $\widetilde{F}_{\mathcal{A}} : \mathcal{A} \to \mathcal{F}_0(\mathcal{S}_\ddagger)$ is a linear bijection with inverse $c_{\mathcal{A}}^{-1}\widetilde{J}_{\mathcal{A},\sigma} : \mathcal{F}_0(\mathcal{S}_\ddagger) \to \mathcal{A}$. Furthermore, $\widetilde{J}_{\mathcal{A},\sigma}$ (respectively $\widetilde{F}_{\mathcal{A}}$) is the adjoint of $F_{\mathcal{A}}$ (respectively $J_{\mathcal{A},\sigma}$) in the sense that

$$\begin{aligned}[F_{\mathcal{A}}p,g]_{\mathcal{A},\sigma} &= \big(p,\widetilde{J}_{\mathcal{A},\sigma}g\big)_{\mathcal{A}},\\ \big(J_{\mathcal{A},\sigma}g,p\big)_{\mathcal{A}} &= [g,\widetilde{F}_{\mathcal{A}}p]_{\mathcal{A},\sigma}\end{aligned} \tag{4.12}$$

for all $p \in \mathcal{A}$ and $g \in \mathcal{F}_0(\mathcal{S}_\ddagger)$. This leads to the following Plancherel type theorem.

Corollary 4.12. a) *For all* $p,r \in \mathcal{A}$,

$$[F_{\mathcal{A}}p,\widetilde{F}_{\mathcal{A}}r]_{\mathcal{A},\sigma} = c_{\mathcal{A}}\big(p,r\big)_{\mathcal{A}}.$$

b) *For all* $g,h \in \mathcal{F}_0(\mathcal{S}_\ddagger)$,

$$\big(J_{\mathcal{A},\sigma}g,\widetilde{J}_{\mathcal{A},\sigma}h\big)_{\mathcal{A}} = c_{\mathcal{A}}[g,h]_{\mathcal{A},\sigma}.$$

Proof. a) For $p,r \in \mathcal{A}$ we compute

$$c_{\mathcal{A}}\big(p,r\big)_{\mathcal{A}} = \big(J_{\mathcal{A},\sigma}(F_{\mathcal{A}}p),r\big)_{\mathcal{A}} = [F_{\mathcal{A}}p,\widetilde{F}_{\mathcal{A}}r]_{\mathcal{A}}.$$

The proof of b) is similar. □

We end this section by recalling the explicit form of the weights Δ and $N(s)$ ($s \in \mathcal{S}_{\ddagger\sigma}$) in terms of q-shifted factorials. The expressions take their most natural form by splitting off the nonsymmetric part of the weight functions $\Delta \in \mathcal{M}$ and $N(s)$ ($s \in \mathcal{S}_{\ddagger\sigma}$). Explicitly, we can write for the weight function Δ,

$$\Delta(x) = \mathcal{C}(x)\Delta^+(x), \tag{4.13}$$

with $\mathcal{C} = \mathcal{C}_\alpha \in \mathbb{C}(x)$ and $\Delta^+ = \Delta^+_\alpha \in \mathcal{M}$ given by

$$\mathcal{C}(x) = \prod_{\alpha\in\Sigma_-} c_\alpha(x), \qquad \Delta^+(x) = \prod_{\substack{f\in R:\\ f(0)\geq 0}} \frac{1}{c_f(x)}, \tag{4.14}$$

and Δ^+ is easily seen to be W_0-invariant. In fact, Δ^+ is given explicitly in terms of q-shifted factorials by

$$\begin{aligned}\Delta^+(x) = \prod_{i=1}^n &\frac{\big(x_i^2, x_i^{-2}; q\big)_\infty}{\big(ax_i, ax_i^{-1}, bx_i, bx_i^{-1}, cx_i, cx_i^{-1}, dx_i, dx_i^{-1}; q\big)_\infty} \\ &\times \prod_{1\leq i<j\leq n} \frac{\big(x_ix_j, x_ix_j^{-1}, x_i^{-1}x_j, x_i^{-1}x_j^{-1}; q\big)_\infty}{\big(t^2x_ix_j, t^2x_ix_j^{-1}, t^2x_i^{-1}x_j, t^2x_i^{-1}x_j^{-1}; q\big)_\infty},\end{aligned} \tag{4.15}$$

see e.g., [42, Lem. 3.12].

For the discrete weight $N(s)$ ($s \in \mathcal{S}_{\ddagger\sigma}$), splitting of the nonsymmetric part gives the formula

$$N(s^{\ddagger\sigma}_\lambda) = \mathcal{C}(s^{\ddagger\sigma}_\lambda)N^+(s^{\ddagger\sigma}_{\lambda^+}) \tag{4.16}$$

for $\lambda \in \Lambda$, where $\lambda^+ \in \Lambda^+$ is the unique element in $(W_0 \cdot \lambda) \cap \Lambda^+$, and with $N^+ = N^+_\alpha$ given by

$$N^+(s^{\ddagger\sigma}_\mu) = \mathop{\mathbf{Res}}_{x=s^{\ddagger\sigma}_\mu}\left(\frac{\Delta^+(x)}{x_1\cdots x_n}\right), \qquad \forall\, \mu \in \Lambda^+,$$

see [41, (8.17) & (8.20)]. Then results in [40] (see also [41, Rem. 8.12]) lead to the expression

$$\begin{aligned}\frac{N^+(s^{\ddagger\sigma}_\mu)}{N^+(s^{\ddagger\sigma}_0)} = \prod_{i=1}^n &\left\{\frac{\big(qa^2t^{4(n-i)}; q\big)_{2\mu_i}\big(q^{-1}abcdt^{4(n-i)}\big)^{-\mu_i}}{\big(a^2t^{4(n-i)}; q\big)_{2\mu_i}}\right. \\ &\left.\times\frac{\big(a^2t^{2(n-i)}, abt^{2(n-i)}, act^{2(n-i)}, adt^{2(n-i)}; q\big)_{\mu_i}}{\big(qt^{2(n-i)}, qat^{2(n-i)}/b, qat^{2(n-i)}/c, qat^{2(n-i)}/d; q\big)_{\mu_i}}\right\} \\ \times \prod_{1\leq i<j\leq n} &\left\{\frac{\big(qa^2t^{2(2n-i-j)}, a^2t^{2(2n-i-j+1)}; q\big)_{\mu_i+\mu_j}}{\big(qa^2t^{2(2n-i-j-1)}, a^2t^{2(2n-i-j)}; q\big)_{\mu_i+\mu_j}}\right. \\ &\left.\times\frac{\big(qt^{2(j-i)}, t^{2(j-i+1)}; q\big)_{\mu_i-\mu_j}}{\big(qt^{2(j-i-1)}, t^{2(j-i)}; q\big)_{\mu_i-\mu_j}}\right\}\end{aligned} \tag{4.17}$$

for all $\mu = \sum_i \mu_i \epsilon_i \in \Lambda^+$.

4.5. The symmetric theory. The *symmetrizer* $C_+ = C_+^\alpha \in \mathcal{H}$ is defined by

$$C_+ = \frac{1}{\sum_{w \in W_0} t_w^2} \sum_{w \in W_0} t_w T_w.$$

Observe that $C_+ \in H_0 = H_0^\alpha$, where H_0 is the subalgebra of $\mathcal{H}$ generated by $T_1, \ldots, T_n$ (which is isomorphic to the finite Hecke algebra of type C_n). In fact, the symmetrizer C_+ is the idempotent of H_0 corresponding to the trivial character $T_i \mapsto t_i$ $(i = 1, \ldots, n)$ of H_0.

Observe that H_0 consists of reflection operators only, since the q-difference operators only arise from the affine root a_0. In particular, C_+ only depends on the values t, t_n and u_n of the multiplicity function $\mathbf{t}$.

Let $\mathcal{A}_+ \subset \mathcal{A}$ be the algebra of W_0-invariant Laurent polynomials in $\mathcal{A}$. Then C_+, acting as reflection operator, defines a projection

$$C_+ : \mathcal{A} \to \mathcal{A}_+$$

since $T_i C_+(g) = t_i C_+(g)$ for $i = 1, \ldots, n$ and $g \in \mathcal{A}$. Applying the symmetrizer C_+ leads directly to symmetric variants of the results in the previous subsections, mainly due to the stability of C_+ under the (anti)isomorphisms we have encountered so far:

$$\begin{aligned} \iota(C_+) &= C_+, \qquad \dagger(C_+) = \ddagger(C_+) = C_+^\ddagger, \\ \tau(C_+) &= C_+^\tau, \qquad \sigma(C_+) = \psi(C_+) = C_+^\sigma. \end{aligned} \tag{4.18}$$

In particular, applying the symmetrizer $C_+ \in \mathcal{H}$ to the Macdonald–Koornwinder polynomials leads to the following results.

Proposition 4.13. a) *For all $\lambda \in \Lambda$, we have*

$$C_+ E(s_\lambda; \cdot) = C_+^\ddagger E_\ddagger(s_\lambda^\ddagger; \cdot) \tag{4.19}$$

in $\mathcal{A}_+$. Furthermore, the expression (4.19) *only depends on the W_0-orbit $W_0 \cdot \lambda$ of $\lambda \in \Lambda$.*

b) *Denote $\mathcal{S}^+ = \mathcal{S}_\alpha^+ = \{s_\lambda \,|\, \lambda \in \Lambda^+\}$, and define $E^+(s; \cdot) = E_\alpha^+(s; \cdot)$ for $s \in \mathcal{S}^+$ by*

$$E^+(s; \cdot) = C_+ E(s; \cdot), \qquad \forall\, s \in \mathcal{S}^+.$$

Then $\{E^+(s; \cdot) \,|\, s \in \mathcal{S}^+\}$ is the unique basis of $\mathcal{A}_+$ whose basis elements satisfy the conditions

$$\begin{aligned} p(Y) E^+(s; \cdot) &= p(s) E^+(s; \cdot), \qquad \forall\, p \in \mathcal{A}_+, \\ E^+(s; s_0^\sigma) &= 1. \end{aligned}$$

c) $E^+(s; v) = E_\sigma^+(v; s)$ *for $s \in \mathcal{S}^+$ and $v \in \mathcal{S}_\sigma^+$ (duality).*

Proof. a) This follows from [41, (8.12)] and from the proof of [41, Cor. 8.11].
b) This is well known, see e.g. [36], [39] or [41].
c) See Sahi [39, Thm. 7.4], and references therein. □

We note that the coefficient of x^λ in the monomial expansion of $E^+(s_\lambda;x)$ for $\lambda \in \Lambda^+$ is known explicitly in product form, see [41, Cor. 9.4] and references therein. It is known as the evaluation theorem. In our present notations it reads as follows.

Theorem 4.14. *Let $\lambda = \sum_i \lambda_i\epsilon_i \in \Lambda^+$. The coefficient $c_\lambda = c_\lambda^\alpha \in \mathbb{C}$ of x^λ in the monomial expansion of $E^+(s_\lambda;x)$ is given explicitly by*

$$c_\lambda = \prod_{i=1}^n \frac{\big(q^{-1}abcdt^{4(n-i)};q\big)_{2\lambda_i}(at^{2(n-i)})^{\lambda_i}}{\big(abt^{2(n-i)},act^{2(n-i)},adt^{2(n-i)},q^{-1}abcdt^{2(n-i)};q\big)_{\lambda_i}}$$
$$\times \prod_{1\le i<j\le n} \frac{\big(q^{-1}abcdt^{2(2n-i-j)};q\big)_{\lambda_i+\lambda_j}\big(t^{2(j-i)};q\big)_{\lambda_i-\lambda_j}}{\big(q^{-1}abcdt^{2(2n-i-j+1)};q\big)_{\lambda_i+\lambda_j}\big(t^{2(j-i+1)};q\big)_{\lambda_i-\lambda_j}},$$

where we used the Askey–Wilson parametrization (4.7) *for part of the multiplicity function α.*

The basis elements $E^+(s;\cdot)$ $(s \in \mathcal{S}^+)$ are known as the (normalized) symmetric Macdonald–Koornwinder polynomials. Up to an explicit multiplicative constant, they coincide with Koornwinder's [30] multivariable analogues of the Askey–Wilson polynomials. This can be proved by relating the q-difference reflection operator

$$m_{\epsilon_1}(Y) := Y_1 + \cdots + Y_n + Y_1^{-1} + \cdots + Y_n^{-1} \in \mathcal{H} \tag{4.20}$$

acting on $\mathcal{A}_+$ to Koornwinder's [30] multivariable second-order q-difference operator of Askey–Wilson type, see Noumi [36] (see [42] for a detailed treatment in English). In particular, in the rank one setup $(n=1)$, $E^+(s_m;\cdot)$ for $m \in \Lambda^+ \simeq \mathbb{N}$ is the well known Askey–Wilson polynomial [1] of degree m. In terms of the standard notation [16] for basic hypergeometric series,

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,z\right) = \sum_{k=0}^\infty \frac{\big(a_1,a_2,\ldots,a_r;q\big)_k}{\big(q,b_1,b_2,\ldots,b_s;q\big)_k}\big[(-1)^kq^{\frac{1}{2}k(k-1)}\big]^{1+s-r}z^k,$$

this leads to the explicit series expansion

$$E^+(s_m;x) = {}_4\phi_3\left(\begin{matrix}q^{-m},q^{m-1}abcd,ax,a/x\\ ab,ac,ad\end{matrix};q,q\right) \tag{4.21}$$

for the symmetric, rank one Macdonald–Koornwinder polynomial, see [37] for a detailed account. The Askey–Wilson parameters a,b,c,d play a symmetrical role in both the Askey–Wilson second-order q-difference operator as well as in the polynomial spectrum $\mathcal{S}^+$. Consequently, $E^+(s_m;x)$ is, up to a multiplicative constant, invariant under permuting the Askey–Wilson parameters $\{a, b, c, d\}$. The corresponding identity for the balanced ${}_4\phi_3$ is known as Sear's transformation formula, see [16, (2.10.4)]. This can be generalized to the higher rank setup. We formulate here two cases, the first corresponds to interchanging the role of a and b, the second corresponds to interchanging the role of c and d.

Proposition 4.15. *Let $\{a, b, c, d\}$ be the Askey–Wilson parametrization* (4.7) *of part of the difference multiplicity function $\alpha = (t_0, u_0, t_n, u_n, t, q^{\frac{1}{2}})$.*

a) *Set $\beta = (t_0, u_0, t_n, -u_n^{-1}, t, q^{\frac{1}{2}})$. Then $s_\lambda^\beta = s_\lambda^\alpha$ for all $\lambda \in \Lambda^+$ and*

$$E_\beta^+(s_\lambda; x) = \left(\prod_{i=1}^{n} \frac{\big(act^{2(n-i)}, adt^{2(n-i)}; q\big)_{\lambda_i}}{\big(bct^{2(n-i)}, bdt^{2(n-i)}; q\big)_{\lambda_i}} \left(\frac{b}{a}\right)^{\lambda_i} \right) E_\alpha^+(s_\lambda; x), \qquad \forall\, \lambda \in \Lambda^+.$$

b) *Set $\gamma = (t_0, -u_0^{-1}, t_n, u_n, t, q^{\frac{1}{2}})$. Then $s_\lambda^\gamma = s_\lambda^\alpha$ for all $\lambda \in \Lambda^+$ and*

$$E_\gamma^+(s_\lambda; x) = E_\alpha^+(s_\lambda; x), \qquad \forall\, \lambda \in \Lambda^+.$$

Proof. For $\lambda \in \Lambda^+$ we write $E_\alpha^+(s_\lambda; x) = c_\lambda^\alpha P_\lambda^\alpha(x)$ with $c_\lambda = c_\lambda^\alpha$ given as in Theorem 4.14. The Laurent polynomial P_λ is the unique eigenfunction of $p(Y)$ with eigenvalue $p(s)$ for all $p \in \mathcal{A}_+$ whose coefficient of x^λ in its monomial expansion is equal to one.

Now the operators Y_i $(i = 1, \ldots, n)$ as well as the spectral points $s_\lambda \in \mathcal{S}^+$ $(\lambda \in \Lambda^+)$ are invariant under replacement of u_0 by $-u_0^{-1}$ (respectively u_n by $-u_n^{-1}$). Indeed, for the Y_i this follows from the fact that the q-difference reflection operators $T_j \in \mathcal{H}$ $(j = 0, \ldots, n)$ are invariant under replacement of u_0 by $-u_0^{-1}$ (respectively u_n by $-u_n^{-1}$). For the spectral points $s_\lambda \in \mathcal{S}^+$ this simply follows from the fact that the spectrum is independent of u_0 and of u_n. We conclude that $P_\lambda^\beta(x) = P_\lambda^\gamma(x) = P_\lambda^\alpha(x)$ for all $\lambda \in \Lambda^+$, hence

$$E_\beta^+(s_\lambda; x) = \frac{c_\lambda^\beta}{c_\lambda^\alpha} E_\alpha^+(s_\lambda; x), \qquad E_\gamma^+(s_\lambda; x) = \frac{c_\lambda^\gamma}{c_\lambda^\alpha} E_\alpha^+(s_\lambda; x)$$

for all $\lambda \in \Lambda^+$. The proposition now follows from the explicit expressions for the coefficients c_λ, see Theorem 4.14. □

Let $\mathcal{F}_0^+(\mathcal{S})$ be the space of functions $g \in \mathcal{F}_0(\mathcal{S})$ which are W_0-invariant under the dot action. Observe that $\mathcal{F}_0^+(\mathcal{S})$ may be identified with the space $\mathcal{F}_0(\mathcal{S}^+)$ of functions $g : \mathcal{S}^+ \to \mathbb{C}$ of finite support by the natural restriction map.

The restriction of the Macdonald–Koornwinder transform $F_{\mathcal{A}}$ to $\mathcal{A}_+$ can be written as integral transform with W_0-invariant integrand as follows. By Proposition 4.6, (4.18) and Proposition 4.13 we have

$$\big(F_{\mathcal{A}}p\big)(s_\lambda^{-1}) = \big(p, E^+(s_{\lambda^+}; \cdot)\big)_{\mathcal{A}}, \qquad p \in \mathcal{A}_+,\ \lambda \in \Lambda,$$

where λ^+ is the unique element in $(W_0 \cdot \lambda) \cap \Lambda^+$. The restriction of the bilinear form $\big(\cdot,\cdot\big)_{\mathcal{A}}$ to $\mathcal{A}_+$ can be rewritten using the formula

$$\sum_{w\in W_0}(w\mathcal{C})(x)=\mathcal{C}(s_0^{\ddagger\sigma}), \tag{4.22}$$

see [41, Lem. 8.1 & Lem. 8.2], leading to the identity

$$\big(p,r\big)_{\mathcal{A}}=\frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2^n n!}\big(p,r\big)_{\mathcal{A},+},\qquad p,r\in\mathcal{A}_+, \tag{4.23}$$

with the bilinear form $\big(\cdot,\cdot\big)_{\mathcal{A},+}=\big(\cdot,\cdot\big)_{\mathcal{A},\alpha,+}$ on $\mathcal{A}_+$ given by

$$\big(p,r\big)_{\mathcal{A},+}=\frac{1}{(2\pi i)^n}\iint_{\mathcal{T}^n}p(x)r(x)\Delta^+(x)\frac{dx}{x},\qquad p,r\in\mathcal{A}_+. \tag{4.24}$$

In particular, the restriction of the Macdonald–Koornwinder transform $F_{\mathcal{A}}$ to $\mathcal{A}_+$ becomes

$$\big(F_{\mathcal{A}}p\big)(s_\lambda^{-1})=\frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2^n n!}\big(p,E^+(s_{\lambda^+};\cdot)\big)_{\mathcal{A},+},\qquad p\in\mathcal{A}_+,\ \lambda\in\Lambda^+. \tag{4.25}$$

Similar arguments show that $\widetilde{F}_{\mathcal{A}}|_{\mathcal{A}_+}=F_{\mathcal{A}}|_{\mathcal{A}_+}$. In particular, the Plancherel type formulas for the Macdonald–Koornwinder transform $F_{\mathcal{A}}$ (see Corollary 4.12) reduce to genuine Plancherel formulas when restricting to W_0-invariant functions.

The same arguments can now be applied to handle the restriction of the transform $J_{\mathcal{A}}:\mathcal{F}_0(\mathcal{S}_{\ddagger\sigma})\to\mathcal{A}$ to the subspace $\mathcal{F}_0^+(\mathcal{S}_{\ddagger\sigma})$ of W_0-invariant functions. In this case, one needs Proposition 4.8 and the discretized version

$$\sum_{\mu\in W_0\cdot\lambda}\mathcal{C}(s_\mu^{\ddagger\sigma})=\mathcal{C}(s_0^{\ddagger\sigma}) \tag{4.26}$$

of (4.22), see e.g., [41, Lem. 8.2] for a proof. It leads to the formula

$$J_{\mathcal{A}}g=\widetilde{J}_{\mathcal{A}}g=\mathcal{C}(s_0^{\ddagger\sigma})\sum_{\lambda\in\Lambda^+}g(s_\lambda^{\ddagger\sigma})E_\sigma^+(s_\lambda^\sigma;\cdot)N^+(s_\lambda^{\ddagger\sigma}) \tag{4.27}$$

for $g\in\mathcal{F}_0^+(\mathcal{S}_{\ddagger\sigma})$. By Theorem 4.11a) and formulas (4.25), (4.27) and (4.22), the orthogonality relations of the nonsymmetric Macdonald–Koornwinder polynomials (see (4.8) and Theorem 4.11b)) reduce to the orthogonality relations

$$\frac{\big(E^+(s;\cdot),E(v;\cdot)\big)_{\mathcal{A},+}}{\big(1,1\big)_{\mathcal{A},+}}=\frac{N_\sigma^+(s_0^{-1})}{N_\sigma^+(s^{-1})}\,\delta_{s,v},\qquad\forall\, s,v\in\mathcal{S}^+ \tag{4.28}$$

for the symmetric Macdonald–Koornwinder polynomials, where $\delta_{s,v}$ is the Kronecker delta function on $\mathcal{S}^+$.

Under suitable values of the difference multiplicity function α, the integration contour can be shifted to $\mathbb{T}^n$ on the cost of some extra (partly discrete) contributions to the measure. This in particular leads to a positive orthogonality measure for the symmetric Macdonald–Koornwinder polynomials, see [40] for more details. We also remark that the constant term $\big(1,1\big)_{\mathcal{A},+}$ is exactly the multivariable Askey–Wilson integral, evaluated by Gustafson in [17]. Explicitly, it reads as follows:

$$\begin{aligned}\big(1,1\big)_{\mathcal{A},+} = 2^n n! \prod_{i=1}^{n} \Bigg\{ &\frac{\big(t^2, abcdt^{2(2n-i-1)};q\big)_\infty}{\big(q,t^{2(n-i+1)};q\big)_\infty} \\ &\times \frac{1}{\big(abt^{2(n-i)}, act^{2(n-i)}, adt^{2(n-i)}, bct^{2(n-i)}, bdt^{2(n-i)}, cdt^{2(n-i)};q\big)_\infty}\Bigg\}.\end{aligned} \tag{4.29}$$

5. The construction of the Cherednik kernels

We assume throughout this section that

$$\alpha = (\mathbf{t}, q^{\frac{1}{2}}) = (t_0, u_0, t_n, u_n, t, q^{\frac{1}{2}})$$

is a difference multiplicity function with $0 < q^{\frac{1}{2}} < 1$ and $0 < t \leq 1$, and with generic parameters $t_0, u_0, t_n, u_n \in \mathbb{C}^\times$. As was explained in Section 3, the construction of non-polynomial Fourier transforms associated with σ hinges on the existence of nonzero kernels

$$\begin{aligned}\mathfrak{E} = \mathfrak{E}_\alpha &: (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n \to \mathbb{C}\\ \mathfrak{E}_\ddagger = \mathfrak{E}_{\alpha_\ddagger} &: (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n \to \mathbb{C}\end{aligned}$$

satisfying the transformation behavior

$$\begin{aligned}\big(X\mathfrak{E}(\gamma,\cdot)\big)(x) &= \big(\psi(X)\mathfrak{E}(\cdot,x)\big)(\gamma), \qquad X \in \mathcal{H},\\ \big(X\mathfrak{E}_\ddagger(\gamma,\cdot)\big)(x) &= \big(\psi_\ddagger(X)\mathfrak{E}_\ddagger(\cdot,x)\big)(\gamma), \qquad X \in \mathcal{H}_\ddagger\end{aligned} \tag{5.1}$$

with respect to the duality antiisomorphism ψ. Here e.g., $\big(X\mathfrak{E}(\gamma,\cdot)\big)(x)$ means the action of $X \in \mathcal{H}$ as q-difference reflection operator on the function $x \mapsto \mathfrak{E}(\gamma,x)$, which also makes sense when $\mathfrak{E}$ is assumed to be a meromorphic function on $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$, cf. Subsection 2.3.

If we require $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ to be meromorphic and "as regular as possible", then kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ satisfying (5.1) turn out to be unique up to a multiplicative constant. We call them the *Cherednik kernels* since they generalize kernels introduced by Cherednik in [5] to the setup of nonreduced root systems.

We note that the classical approach to study existence of such a kernel $\mathfrak{E}(\gamma,x) = \mathfrak{E}_\alpha(\gamma,x)$ is by analyzing the solution space of the spectral problem

$$Y_i g = \gamma_i^{-1} g, \qquad i = 1,\ldots,n, \tag{5.2}$$

for the commuting Y-operators $Y_i \in \mathcal{H}$. We follow here a drastically different approach, which is inspired by Cherednik's paper [5]. The philosophy is to *construct* the special solutions $\mathfrak{E}(\gamma,x)$ of the spectral problem (5.2) as an explicit series expansion in terms of the Macdonald–Koornwinder polynomials. The important step in the construction is to associate a so-called *auxiliary kernel* $\mathfrak{F} = \mathfrak{F}_\alpha$ to $\mathfrak{E}$, for which the spectral problem (5.2) translates into a transformation property of the type

$$\big(Y_i\mathfrak{F}(\gamma,\cdot)\big)(x) = \big(Y_i\mathfrak{F}(\cdot,x)\big)(\gamma), \qquad i=1,\ldots,n$$

(with the Y-operators on each side depending on different multiplicity functions). Invoking the polynomial theory, it becomes now plausible that such an auxiliary kernel $\mathfrak{F}$ can be constructed as explicit series expansions involving the Macdonald–Koornwinder polynomials in the variables x as well as in the variables γ. This is indeed the case, as is explained in full detail in this section.

5.1. Auxiliary kernels. Instead of studying kernels $\mathfrak{E}, \mathfrak{E}_\ddagger \in \mathcal{M}\big((\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n\big)$ satisfying the transformation behavior (5.1) directly, it turns out to be convenient to study certain auxiliary kernels $\mathfrak{F} = \mathfrak{F}_\alpha, \mathfrak{F}_\ddagger = \mathfrak{F}_{\alpha_\ddagger} \in \mathcal{M}\big((\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n\big)$ first, which are related to $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ by the formulas

$$\begin{aligned}\mathfrak{F}(\gamma,x) &= G_{\sigma\tau}(\gamma)^{-1}G_\tau(x)^{-1}\mathfrak{E}(\gamma,x),\\ \mathfrak{F}_\ddagger(\gamma,x) &= G_{\ddagger\sigma}(\gamma)G_\ddagger(x)\mathfrak{E}_\ddagger(\gamma^{-1},x),\end{aligned} \tag{5.3}$$

where $G = G_\alpha \in \mathcal{M}$ is the Gaussian. The kernel $\mathfrak{F}_\ddagger$ may be alternatively written as

$$\mathfrak{F}_\ddagger(\gamma,x) = G_{\sigma\tau}(\gamma)^{-1}G_\tau(x)^{-1}\mathfrak{E}_\ddagger(\gamma^{-1},x)$$

in view of (2.11).

Lemma 5.1. *Let $\kappa = \kappa_\alpha : \mathcal{H} \to \mathcal{H}_{\tau\sigma\tau}$ and $\kappa^I = \kappa^I_\alpha : \mathcal{H} \to \mathcal{H}_{\ddagger\tau\sigma\tau}$ be unital anti-isomorphisms defined by*

$$\kappa = \tau_{\tau\sigma\tau}^{-1} \circ \psi_\tau \circ \tau, \qquad \kappa^I = \dagger_{\tau\sigma\tau} \circ \tau_{\tau\sigma} \circ \psi_\tau \circ \tau_\tau^{-1}. \tag{5.4}$$

Under the correspondence (5.3), *the transformation behavior* (5.1) *for the meromorphic kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ are equivalent to the transformation behavior*

$$\big(X\mathfrak{F}(\gamma,\cdot)\big)(x) = \big(\kappa_\tau(X)\mathfrak{F}(\cdot,x)\big)(\gamma), \qquad X \in \mathcal{H}_\tau, \tag{5.5}$$

$$\big(X\mathfrak{F}_\ddagger(\gamma,\cdot)\big)(x) = \big(\kappa^I_{\ddagger\tau}(X)\mathfrak{F}_\ddagger(\cdot,x)\big)(\gamma), \qquad X \in \mathcal{H}_{\ddagger\tau} \tag{5.6}$$

for the auxiliary meromorphic kernels $\mathfrak{F}$ and $\mathfrak{F}_\ddagger$.

Proof. This follows from the fact that conjugation by the Gaussian induces the isomorphism τ on $\mathcal{H}$ (see Proposition 2.11), and that the inversion operator $(Ig)(\gamma) = g(\gamma^{-1})$ induces the isomorphism $\dagger$ on $\mathcal{H}$. As an example, we prove the transformation behavior for $\mathfrak{F}_\ddagger$ under the assumption that $\mathfrak{E}_\ddagger$ satisfies (5.1). We use the formula

$$\mathfrak{F}_\ddagger(\gamma,x) = G_\ddagger(x)\Big(I\big(G_{\ddagger\sigma}(\cdot)\mathfrak{E}_\ddagger(\cdot,x)\big)\Big)(\gamma),$$

which follows from (5.3) since the Gaussian is W_0-invariant. Then we compute for $X \in \mathcal{H}_{\ddagger\tau}$,

$$\begin{aligned}\big(X\mathfrak{F}_\ddagger(\gamma,\cdot)\big)(x) &= G_\ddagger(x)\big(I\big(G_{\ddagger\sigma}(\cdot)\big(\psi_\ddagger\circ\tau_\ddagger^{-1}\big)(X)\mathfrak{E}_\ddagger(\cdot,x)\big)\big)(\gamma)\\ &= G_\ddagger(x)\big(I\big(\big(\tau_{\ddagger\sigma}\circ\psi_\ddagger\circ\tau_\ddagger^{-1}\big)(X)G_{\ddagger\sigma}(\cdot)\mathfrak{E}_\ddagger(\cdot,x)\big)\big)(\gamma)\\ &= \big(\big(\dagger_{\ddagger\sigma\tau}\circ\tau_{\ddagger\sigma}\circ\psi_\ddagger\circ\tau_\ddagger^{-1}\big)(X)\mathfrak{F}_\ddagger(\cdot,x)\big)(\gamma)\\ &= \big(\kappa^I_{\ddagger\tau}(X)\mathfrak{F}_\ddagger(\cdot,x)\big)(\gamma),\end{aligned}$$

which is the desired result. □

In the next lemma we compute the antiisomorphisms κ and κ^I explicitly on suitable algebraic generators of $\mathcal{H}$. In particular, we show that these antiisomorphisms have the desired property that Y-operators are mapped to Y-operators.

We denote

$$U = U_\alpha = T_1\cdots T_{n-1}T_nT_{n-1}\cdots T_1 \in \mathcal{H}, \tag{5.7}$$

so that $Y_1 = UT_0$ in the double affine Hecke algebra $\mathcal{H}$.

Lemma 5.2. a) *For $i = 1,\ldots,n$ we have*

$$\kappa(T_i) = T_i^{\tau\sigma\tau}, \qquad \kappa(Y_i) = Y_i^{\tau\sigma\tau}.$$

Furthermore, $\kappa(T_0) = U_{\tau\sigma\tau}T_0^{\tau\sigma\tau}U_{\tau\sigma\tau}^{-1}$ *and* $\kappa(x_1) = q^{\frac{1}{2}}x_1^{-1}T_0^{\tau\sigma\tau}U_{\tau\sigma\tau}^{-1}$.

b) *For $i = 1,\ldots,n$ we have*

$$\kappa^I(T_i) = T_i^{\ddagger\tau\sigma\tau-1}, \qquad \kappa^I(Y_i) = Y_i^{\ddagger\tau\sigma\tau-1}.$$

Furthermore, $\kappa^I(T_0) = T_0^{\ddagger\tau\sigma\tau-1}$ *and* $\kappa^I(x_1) = q^{\frac{1}{2}}T_0^{\ddagger\tau\sigma\tau-1}x_1U_{\ddagger\tau\sigma\tau}$.

Proof. a) The identities $\kappa(T_i) = T_i^{\tau\sigma\tau}$ for $i = 1,\ldots,n$ are immediately clear from the definitions of τ and ψ. Since $Y_{i+1} = T_i^{-1}Y_iT_i^{-1}$ for $i = 1,\ldots,n-1$, the identities $\kappa(Y_i) = Y_i^{\tau\sigma\tau}$ for $i = 1,\ldots,n$ will follow from $\kappa(Y_1) = Y_1^{\tau\sigma\tau}$. Omitting the parameter dependencies we compute,

$$\begin{aligned}\kappa(Y_1) &= \tau^{-1}\psi\tau(UT_0)\\ &= \tau^{-1}\psi(q^{-\frac{1}{2}}Ux_1T_0^{-1})\\ &= \tau^{-1}\psi(q^{-\frac{1}{2}}Ux_1Y_1^{-1}U)\\ &= \tau^{-1}(q^{-\frac{1}{2}}Ux_1Y_1^{-1}U)\\ &= \tau^{-1}(q^{-\frac{1}{2}}Ux_1T_0^{-1}) = UT_0 = Y_1,\end{aligned}$$

which is the desired result. Furthermore, omitting the parameter dependencies,

$$\kappa(T_0)U = \kappa(UT_0) = \kappa(Y_1) = UT_0,$$

hence $\kappa(T_0) = U_{\tau\sigma\tau}T_0^{\tau\sigma\tau}U_{\tau\sigma\tau}^{-1}$. The identity $\kappa(x_1) = q^{\frac{1}{2}}x_1^{-1}T_0^{\tau\sigma\tau}U_{\tau\sigma\tau}^{-1}$ follows from similar (but easier) computations.

b) The identities $\kappa^I(T_i) = T_i^{\ddagger\tau\sigma\tau^{-1}}$ for $i = 1, \ldots, n$ are clear. For the identities $\kappa^I(Y_i) = Y_i^{\ddagger\tau\sigma\tau^{-1}}$, it then suffices to prove that $\kappa^I(T_0) = T_0^{\ddagger\tau\sigma\tau^{-1}}$. Omitting the parameter dependencies we compute,

$$\begin{aligned}\kappa^I(T_0) &= \dagger\tau\psi\tau^{-1}(T_0)\\ &= \dagger\tau\psi(q^{-\frac{1}{2}}T_0^{-1}x_1)\\ &= \dagger\tau\psi(q^{-\frac{1}{2}}Y_1^{-1}Ux_1)\\ &= \dagger\tau(q^{-\frac{1}{2}}Y_1^{-1}Ux_1)\\ &= \dagger\tau(q^{-\frac{1}{2}}T_0^{-1}x_1)\\ &= \dagger(T_0) = T_0^{-1},\end{aligned}$$

which is the desired result. The proof of the identity $\kappa^I(x_1) = q^{\frac{1}{2}}T_0^{\ddagger\tau\sigma\tau^{-1}}x_1U_{\ddagger\tau\sigma\tau}$ is left to the reader. □

5.2. Auxiliary transforms. Recall that $\mathcal{O} = \mathcal{O}\big((\mathbb{C}^\times)^n\big)$ is the ring of analytic functions on the complex torus $(\mathbb{C}^\times)^n$.

Definition 5.3. a) A nonzero linear mapping $L = L_\alpha : \mathcal{A} \to \mathcal{O}$ is called an auxiliary transform associated to α when

$$L \circ X = \big(\kappa_\tau \circ \ddagger_\tau^{-1}\big)(X) \circ L, \qquad \forall\, X \in \mathcal{H}_{\ddagger\tau}, \tag{5.8}$$

with the usual action of the double affine Hecke algebras on the function spaces $\mathcal{A} \subset \mathcal{O} \subset \mathcal{M}$.

b) A nonzero linear mapping $L_\ddagger = L_{\alpha_\ddagger} : \mathcal{A} \to \mathcal{O}$ is called an auxiliary transform associated to $\alpha_\ddagger$ when

$$L_\ddagger \circ X = \big(\kappa_{\ddagger\tau}^I \circ \ddagger_\tau\big)(X) \circ L_\ddagger, \qquad \forall\, X \in \mathcal{H}_\tau, \tag{5.9}$$

with the usual action of the double affine Hecke algebras on the function spaces $\mathcal{A} \subset \mathcal{O} \subset \mathcal{M}$.

In the following lemma we link auxiliary transforms to *analytic* kernels $\mathfrak{F}$ and $\mathfrak{F}_\ddagger$ satisfying the transformation behavior (5.5) and (5.6), respectively.

Lemma 5.4. *Suppose that $L, L_\ddagger : \mathcal{A} \to \mathcal{O}$ are linear mappings of the form*

$$\big(Lp\big)(\gamma) = \big(\mathfrak{F}(\gamma,\cdot), p\big)_{\mathcal{A},\tau}, \qquad \big(L_\ddagger p\big)(\gamma) = \big(p, \mathfrak{F}_\ddagger(\gamma,\cdot)\big)_{\mathcal{A},\tau}$$

for $p \in \mathcal{A}$, with kernels $\mathfrak{F}, \mathfrak{F}_\ddagger \in \mathcal{O}\big((\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n\big)$ (which are then necessarily unique).

a) *L is an auxiliary transform associated to α if and only if $\mathfrak{F}$ satisfies the transformation behavior* (5.5).

b) *$L_\ddagger$ is an auxiliary transform associated to $\alpha_\ddagger$ if and only if $\mathfrak{F}_\ddagger$ satisfies the transformation behavior* (5.6).

Proof. By Proposition 4.6 and Remark 4.7 we have that $\big(\cdot,\cdot\big)_{\mathcal{A},\tau}$ induces the anti-isomorphism $\ddagger_\tau$ on $\mathcal{H}_\tau$, i.e.,

$$\big(Xp,r\big)_{\mathcal{A},\tau}=\big(p,\ddagger_\tau(X)r\big)_{\mathcal{A},\tau},\qquad X\in\mathcal{H}_\tau$$

when $p,r\in\mathcal{O}$. The lemma is now immediate. □

In this and the next subsection we study auxiliary transforms in detail. We prove that they exist and that they are unique up to a multiplicative constant. Furthermore, we explicitly compute the auxiliary transforms on suitable bases of $\mathcal{A}$. This leads, via the previous lemma, to the explicit construction of *analytic* kernels $\mathfrak{F}$ and $\mathfrak{F}_\ddagger$ satisfying the transformation behaviour (5.5) and (5.6), respectively.

We have the following key lemma.

Lemma 5.5. a) *If an auxiliary transform $L:\mathcal{A}\to\mathcal{O}$ associated to α exists, then it is unique up to a multiplicative constant. Furthermore, $L(1)=c1$ for some constant $c\in\mathbb{C}^\times$, where $1\in\mathcal{A}$ is the Laurent polynomial identically equal to one, and $L(\mathcal{A})\subseteq\mathcal{A}$.*

b) *If an auxiliary transform $L_\ddagger:\mathcal{A}\to\mathcal{O}$ associated to $\alpha_\ddagger$ exists, then it is unique up to a multiplicative constant. Furthermore, $L_\ddagger(1)=c_\ddagger 1$ for some constant $c_\ddagger\in\mathbb{C}^\times$ and $L_\ddagger(\mathcal{A})\subseteq\mathcal{A}$.*

Proof. Let $L,L_\ddagger:\mathcal{A}\to\mathcal{O}$ be nonzero linear maps satisfying the transformation property (5.8) and (5.9), respectively. Since $\mathcal{A}$ is a cyclic module for the action of the double affine Hecke algebra with cyclic vector $1\in\mathcal{A}$, the lemma follows from the formulas $L(1)=c1$ and $L_\ddagger(1)=c_\ddagger 1$ for some constants $c,c_\ddagger\in\mathbb{C}$.

We first show that the functions $L(1),L_\ddagger(1)\in\mathcal{O}$ are W_0-invariant. We focus on $L(1)$ (the proof for $L_\ddagger(1)\in\mathcal{O}$ is the same). Let $i\in\{1,\ldots,n\}$. It suffices to prove that $r_i(L(1))=L(1)$. By the explicit expression of $T_i\in\mathcal{H}\subset\mathcal{D}_q$ as a q-difference reflection operator, we have $T_i^{\ddagger\tau}(1)=t_i^{-1}1$, where (recall) $t_i=t$ for $i=1,\ldots,n-1$. It follows that

$$L(T_i^{\ddagger\tau}1)=t_i^{-1}L(1).$$

On the other hand, by the transformation behavior (5.5) of L under the action of the double affine Hecke algeba, we have

$$\begin{aligned}L(T_i^{\ddagger\tau}1)&=\big(\kappa_\tau\circ\ddagger_\tau^{-1}\big)(T_i^{\ddagger\tau})L(1)\\&=T_i^{\sigma\tau^{-1}}L(1)\\&=t_i^{-1}L(1)+t_i^{-1}c_{a_i}^{\sigma\tau}\big(r_iL(1)-L(1)\big).\end{aligned}$$

Combining the two outcomes we obtain $r_i(L(1))=L(1)$ in $\mathcal{O}$.

The next step is to show that $L(1),L_\ddagger(1)\in\mathcal{O}$ are in fact $\mathcal{W}$-invariant, where $\mathcal{W}$ acts as constant coefficient q-difference reflection operators on $\mathcal{O}$ by (2.1). It suffices to show that $r_0(L(1))=L(1)$ and $r_0(L_\ddagger(1))=L_\ddagger(1)$, where $r_0\in\mathcal{W}$ is the simple reflection associated to the simple root a_0. For $L_\ddagger(1)$ it follows from the identity $\big(\kappa^I_{\ddagger\tau}\circ\ddagger_\tau\big)(T_0^\tau)=T_0^{\sigma\tau}$ that $r_0(L_\ddagger(1))=L_\ddagger(1)$ by repeating the argument of the previous paragraph. For $L(1)$, we observe that

$$L\big(Y_1^{\ddagger\tau}1\big)=L\big(Y_1^{\ddagger\tau}E_{\ddagger\tau}(s_0^{\ddagger\tau};\cdot)\big)=s_{0,1}^{\ddagger\tau}L(1)$$

on the one hand, where

$$s_{0,1}^{\ddagger\tau}=u_0^{-1}t_n^{-1}t^{2(1-n)}$$

is the first coordinate of the spectral point $s_0^{\ddagger\tau}\in\mathcal{S}_{\ddagger\tau}$. On the other hand,

$$\begin{aligned}L(Y_1^{\ddagger\tau}1)&=\big(\kappa_\tau\circ\ddagger_\tau^{-1}\big)(Y_1^{\ddagger\tau})L(1)\\&=Y_1^{\sigma\tau-1}L(1)\\&=T_0^{\sigma\tau-1}U_{\sigma\tau}^{-1}L(1)\\&=t_n^{-1}t^{2(1-n)}T_0^{\sigma\tau-1}L(1),\end{aligned}$$

since we have seen that

$$U_{\sigma\tau}=T_1^{\sigma\tau}T_2^{\sigma\tau}\cdots T_{n-1}^{\sigma\tau}T_n^{\sigma\tau}T_{n-1}^{\sigma\tau}\cdots T_2^{\sigma\tau}T_1^{\sigma\tau}$$

acts as multiplication by $t_nt^{2(n-1)}$ on $L(1)$. Hence $T_0^{\sigma\tau-1}L(1)=u_0^{-1}L(1)$, and a similar argument as before yields $r_0(L(1))=L(1)$.

The completion of the proof of the lemma is now standard: by the change of variables

$$x=e^{2\pi iw}:=(e^{2\pi iw_1},e^{2\pi iw_2},\cdots,e^{2\pi iw_n}),$$

we obtain analytic functions

$$w\mapsto L(1)(e^{2\pi iw}),\qquad w\mapsto L_\ddagger(1)(e^{2\pi iw})$$

on $\mathbb{C}^n/\Gamma_q$, where $\Gamma_q=\mathbb{Z}^n+\mathbb{Z}^nv$ and v is an element in the upper half plane satisfying $q=e^{2\pi iv}$. By Liouville's Theorem, these functions must be constants. □

We fix a convenient normalization for auxiliary transforms as follows.

Definition 5.6. We call an auxiliary transform L (respectively $L_\ddagger$) associated to α (respectively $\alpha_\ddagger$) normalized when $L(1)=G_{\tau\sigma\tau}(s_0^\tau)1$ (respectively $L_\ddagger(1)=G_{\tau\sigma\tau}(s_0^\tau)1$).

Remark 5.7. The normalization constant $G_{\tau\sigma\tau}(s_0^\tau)$ can be written as

$$G_{\tau\sigma\tau}(s_0^\tau)=\prod_{i=1}^n\big(bct^{2(n-i)},dt^{2(i-n)}/a;q\big)_\infty^{-1}$$

in terms of the Askey–Wilson parametrization (4.7) for part of the multiplicity function α.

The fact that the above choice of normalization is the natural one becomes clear in the next subsection. From the previous lemma it follows that a normalized auxiliary transform associated to α (respectively $\alpha_\ddagger$) is unique, provided that it exists.

In the following statement we use the stability

$$s_\lambda^\tau = s_\lambda^{\sigma\tau}, \qquad \forall\, \lambda \in \Lambda, \tag{5.10}$$

which follows from the fact that the spectral points $s_\lambda \in \mathcal{S} = \mathcal{S}_\alpha$ only depend on the values of the multiplicity function $\mathbf{t}$ on the root subsystem $R \subset R_{nr}$.

Corollary 5.8. *Suppose that $L, L_\ddagger : \mathcal{A} \to \mathcal{O}$ are normalized auxiliary transforms associated to α and $\alpha_\ddagger$, respectively. Then for $s \in \mathcal{S}_\tau$ we have*

$$L\big(E_{\ddagger\tau}(s^{-1};\cdot)\big) = d^\tau(s)E_{\sigma\tau}(s;\cdot),$$
$$L_\ddagger\big(E_\tau(s;\cdot)\big) = d^{\ddagger\tau}(s^{-1})E_{\sigma\tau}(s;\cdot)$$

for unique functions $d^\tau \in \mathcal{F}(\mathcal{S}_\tau)$ and $d^{\ddagger\tau} \in \mathcal{F}(\mathcal{S}_{\ddagger\tau})$ satisfying $d^\tau(s_0^\tau) = d^{\ddagger\tau}(s_0^{\ddagger\tau}) = G_{\tau\sigma\tau}(s_0^\tau)$.

Proof. For $i = 1, \ldots, n$ we have

$$\big(\kappa_\tau \circ \ddagger_\tau^{-1}\big)(Y_i^{\ddagger\tau}) = Y_i^{\sigma\tau\,-1},$$

hence for $s \in \mathcal{S}_\tau$,

$$Y_i^{\sigma\tau\,-1} L\big(E_{\ddagger\tau}(s^{-1};\cdot)\big) = L\big(Y_i^{\ddagger\tau}E_{\ddagger\tau}(s^{-1};\cdot)\big) = s_i^{-1}L\big(E_{\ddagger\tau}(s^{-1};\cdot)\big).$$

Furthermore, $L\big(E_{\ddagger\tau}(s^{-1};\cdot)\big) \in \mathcal{A}$, hence by the polynomial Macdonald–Koornwinder theory (see Subsection 4.2) and by (5.10),

$$L\big(E_{\ddagger\tau}(s^{-1};\cdot)\big) = d^\tau(s)E_{\sigma\tau}(s;\cdot)$$

for some constant $d^\tau(s) \in \mathbb{C}$. The normalization $d^\tau(s_0^\tau) = G_{\tau\sigma\tau}(s_0^\tau)$ follows from the normalization of L and the fact that $E_{\ddagger\tau}(s_0^{\ddagger\tau};\cdot) = E_{\sigma\tau}(s_0^{\sigma\tau};\cdot) = 1$ is the Laurent polynomial identically equal to one. The formulas for $L_\ddagger$ are proved similarly, now using the fact that

$$\big(\kappa_{\ddagger\tau}^I \circ \ddagger_\tau\big)(Y_i^\tau) = Y_i^{\sigma\tau}$$

for $i = 1, \ldots, n$. □

Definition 5.9. We call the functions $d^\tau \in \mathcal{F}(\mathcal{S}_\tau)$ and $d^{\ddagger\tau} \in \mathcal{F}(\mathcal{S}_{\ddagger\tau})$ in Corollary 5.8 the generalized eigenvalues associated to the normalized auxiliary transform L and $L_\ddagger$, respectively.

We prove in the next subsection the *existence* of the normalized auxiliary transforms associated to α and $\alpha_\ddagger$ by determining the related generalized eigenvalues d^τ and $d^{\ddagger\tau}$ explicitly.

5.3. The generalized eigenvalues. In this subsection we give full details on determining the generalized eigenvalue $d^\tau \in \mathcal{F}(\mathcal{S}_\tau)$. It leads to the existence of the normalized auxiliary transform associated to α. At the end of the subsection we indicate how a similar procedure leads to the explicit expression for the generalized eigenvalue $d^{\ddagger\tau} \in \mathcal{F}(\mathcal{S}_{\ddagger\tau})$.

Recall that the space $\mathcal{F}(\mathcal{S}_\tau)$ consisting of functions $\phi : \mathcal{S}_\tau \to \mathbb{C}$ is an $\mathcal{H}_{\ddagger\tau\sigma}$-module in view of Lemma 4.2. Since $\mathcal{S}_\tau = \mathcal{S}_{\sigma\tau}$, the space $\mathcal{F}(\mathcal{S}_\tau)$ also has the structure of an $\mathcal{H}_{\ddagger\sigma\tau\sigma}$-module. We consider $\mathcal{F}(\mathcal{S}_\tau)$ as a commutative algebra by pointwise multiplication, and we view $\mathcal{F}(\mathcal{S}_\tau)$ as a subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(\mathcal{S}_\tau))$ by identifying $g \in \mathcal{F}(\mathcal{S}_\tau)$ with the endomorphism of $\mathcal{F}(\mathcal{S}_\tau)$ defined as multiplication by g.

Lemma 5.10. *Let $d^\tau \in \mathcal{F}(\mathcal{S}_\tau)$ and let $L : \mathcal{A} \to \mathcal{A}$ be the linear map defined by*

$$L\big(E_{\ddagger\tau}(s^{-1}; \cdot)\big) = d^\tau(s) E_{\sigma\tau}(s; \cdot), \qquad \forall\, s \in \mathcal{S}_\tau.$$

If

$$X \circ d^\tau = d^\tau \circ \nu_{\ddagger\tau\sigma}(X), \qquad \forall\, X \in \mathcal{H}_{\ddagger\tau\sigma} \tag{5.11}$$

within the endomorphism ring $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(\mathcal{S}_\tau))$, where $\nu_{\ddagger\tau\sigma} : \mathcal{H}_{\ddagger\tau\sigma} \to \mathcal{H}_{\ddagger\sigma\tau\sigma}$ is the unital algebra isomorphism defined by

$$\nu_{\ddagger\tau\sigma} = \dagger_{\sigma\tau\sigma} \circ \psi_{\sigma\tau} \circ \kappa_\tau \circ \ddagger_\tau^{-1} \circ \psi_{\ddagger\tau}^{-1},$$

then L satisfies the transformation behavior (5.8) *under the action of the double affine Hecke algebra $\mathcal{H}_{\ddagger\tau}$.*

Proof. The space $\mathcal{F}(\mathcal{S}_\tau, \mathcal{A})$ consisting of functions $\phi : \mathcal{S}_\tau \to \mathcal{A}$ is an $\mathcal{H}_{\ddagger\tau\sigma}$-module by

$$\big((X \cdot \phi)(\gamma)\big)(x) = \big(X \cdot \phi_x\big)(\gamma), \qquad x \in (\mathbb{C}^\times)^n, \;\; \gamma \in \mathcal{S}_\tau$$

with $\phi_x \in \mathcal{F}(\mathcal{S}_\tau)$ defined by $\phi_x(\gamma) = \big(\phi(\gamma)\big)(x)$. Since $\mathcal{S}_\tau = \mathcal{S}_{\sigma\tau}$, the same formula defines an $\mathcal{H}_{\ddagger\sigma\tau\sigma}$-module structure on $\mathcal{F}(\mathcal{S}_\tau, \mathcal{A})$. Furthermore, pointwise multiplication defines a commutative algebra structure on $\mathcal{F}(\mathcal{S}_\tau, \mathcal{A})$.

The canonical embedding $\mathcal{F}(\mathcal{S}_\tau) \hookrightarrow \mathcal{F}(\mathcal{S}_\tau, \mathcal{A})$ of algebras via the map $\mathbb{C} \to \mathcal{A}$, $\lambda \mapsto \lambda 1$ is compatible with the above actions of the double affine Hecke algebras $\mathcal{H}_{\ddagger\tau\sigma}$ and $\mathcal{H}_{\ddagger\sigma\tau\sigma}$. For $\phi \in \mathcal{F}(\mathcal{S}_\tau, \mathcal{A})$ and $s \in \mathcal{S}_\tau$ we write $(X \cdot \phi)(s)$ for the action of X on ϕ as described above, while we write $X(\phi(s))$ for the action of X on the element $\phi(s) \in \mathcal{A}$.

Fix $d^\tau \in \mathcal{F}(\mathcal{S}_\tau)$ and define a linear map $L : \mathcal{A} \to \mathcal{A}$ by

$$L\big(E_{\ddagger\tau}(s^{-1}; \cdot)\big) = d^\tau(s) E_{\sigma\tau}(s; \cdot), \qquad \forall\, s \in \mathcal{S}_\tau.$$

Assume furthermore that (5.11) holds true for some (yet to be determined) algebra isomorphism $\nu_{\ddagger\tau\sigma} : \mathcal{H}_{\ddagger\tau\sigma} \to \mathcal{H}_{\ddagger\sigma\tau\sigma}$. We define $\phi_1, \phi_2 \in \mathcal{F}(\mathcal{S}_\tau, \mathcal{A})$ by

$$\begin{aligned} \phi_1(s) &= L\big(E_{\ddagger\tau}(s^{-1}; \cdot)\big) = L\big(\mathfrak{E}_{\mathcal{A},\ddagger\tau}(s, \cdot)\big), \\ \phi_2(s) &= E_{\sigma\tau}(s; \cdot) = \mathfrak{E}_{\mathcal{A},\sigma\tau}(s^{-1}, \cdot) \end{aligned}$$

for $s \in \mathcal{S}_\tau$, so that $\phi_1 = d^\tau \phi_2$ in the algebra $\mathcal{F}(\mathcal{S}_\tau, \mathcal{A})$. Then we compute for $s \in \mathcal{S}_\tau$ and $X \in \mathcal{H}_{\ddagger\tau\sigma}$ by Proposition 4.4,

$$(X \cdot \phi_1)(s) = L\big(\psi_{\ddagger\tau}^{-1}(X)\mathfrak{E}_{\mathcal{A},\ddagger\tau}(s,\cdot)\big),$$

where we use the $\mathcal{H}_{\ddagger\tau}$-action on $\mathcal{A}$ in the right-hand side of the equality. On the other hand, under the above assumptions,

$$\begin{aligned}(X \cdot \phi_1)(s) &= \big(X \cdot (d^\tau \phi_2)\big)(s)\\ &= \Big(d^\tau\big(\nu_{\ddagger\tau\sigma}(X) \cdot \phi_2\big)\Big)(s)\\ &= d^\tau(s)\big(\psi_{\sigma\tau}^{-1} \circ \dagger_{\ddagger\sigma\tau\sigma} \circ \nu_{\ddagger\tau\sigma}\big)(X)\big(\phi_2(s)\big)\\ &= \big(\psi_{\sigma\tau}^{-1} \circ \dagger_{\ddagger\sigma\tau\sigma} \circ \nu_{\ddagger\tau\sigma}\big)(X)\big(\phi_1(s)\big).\end{aligned}$$

Equating both outcomes yields

$$L\big(X\mathfrak{E}_{\mathcal{A},\ddagger\tau}(s,\cdot)\big) = \big(\psi_{\sigma\tau}^{-1} \circ \dagger_{\ddagger\sigma\tau\sigma} \circ \nu_{\ddagger\tau\sigma} \circ \psi_{\dagger\tau}\big)(X)\big(\phi_1(s)\big)$$

for $s \in \mathcal{S}_\tau$ and $X \in \mathcal{H}_{\ddagger\tau}$. On the other hand, L satisfies the intertwining property (5.8) when

$$L\big(X\mathfrak{E}_{\mathcal{A},\ddagger\tau}(s,\cdot)\big) = \big(\kappa_\tau \circ \ddagger_\tau^{-1}\big)(X)\big(\phi_1(s)\big)$$

for all $s \in \mathcal{S}_\tau$ and $X \in \mathcal{H}_{\ddagger\tau}$. This will thus be the case when

$$\psi_{\sigma\tau}^{-1} \circ \dagger_{\ddagger\sigma\tau\sigma} \circ \nu_{\ddagger\tau\sigma} \circ \psi_{\dagger\tau} = \kappa_\tau \circ \ddagger_\tau^{-1},$$

i.e., when the isomorphism $\nu_{\ddagger\tau\sigma}$ is given by

$$\nu_{\ddagger\tau\sigma} = \dagger_{\sigma\tau\sigma} \circ \psi_{\sigma\tau} \circ \kappa_\tau \circ \ddagger_\tau^{-1} \circ \psi_{\ddagger\tau}^{-1}.$$

□

We thus search for a function $d^\tau \in \mathcal{F}(\mathcal{S}_\tau)$, normalized by $d^\tau(s_0^\tau) = G_{\tau\sigma\tau}(s_0^\tau)$, which induces the isomorphism $\nu_{\ddagger\tau\sigma} : \mathcal{H}_{\ddagger\tau\sigma} \to \mathcal{H}_{\ddagger\sigma\tau\sigma}$ via the formula (5.11).

Proposition 5.11. i) *We have $\nu_{\ddagger\tau\sigma} = \tau_{\ddagger\tau\sigma}$, with $\tau : \mathcal{H} \to \mathcal{H}_\tau$ the isomorphism as defined in Proposition 2.11.*

ii) *The normalized auxiliary transform $L : \mathcal{A} \to \mathcal{A}$ associated to α exists. Its generalized eigenvalue d^τ is given by*

$$d^\tau(s) = G_{\tau\sigma\tau}(s), \qquad \forall\, s \in \mathcal{S}_\tau.$$

In other words,

$$L\big(E_{\ddagger\tau}(s^{-1};\cdot)\big) = G_{\tau\sigma\tau}(s)E_{\sigma\tau}(s;\cdot), \qquad \forall\, s \in \mathcal{S}_\tau.$$

Proof. i) First observe that $\tau_{\ddagger\tau\sigma}$ defines an algebra isomorphism $\tau_{\ddagger\tau\sigma} : \mathcal{H}_{\ddagger\tau\sigma} \to \mathcal{H}_{\ddagger\sigma\tau\sigma}$ since $\sigma\tau\sigma = \tau\sigma\tau$ when acting upon difference multiplicity functions. Hence we only have to show that the action of $\nu_{\ddagger\tau\sigma}$ and $\tau_{\ddagger\tau\sigma}$ coincides on the algebraic generators $T_i^{\ddagger\tau\sigma}$ $(i=0,\dots,n)$ and x_1 of $\mathcal{H}_{\ddagger\tau\sigma}$. We omit parameter dependencies in the following computations. It is easy to check that

$$\nu(T_i)=T_i=\tau(T_i),\qquad i=1,\dots,n.$$

Furthermore, we have

$$\begin{aligned}\nu(x_1)&=\dagger\psi\kappa\,\ddagger^{-1}\,\psi^{-1}(x_1)=\dagger\psi\kappa\,\ddagger^{-1}\,(Y_1^{-1})\\&=\dagger\psi\kappa(Y_1)=\dagger\psi(Y_1)=\dagger(x_1^{-1})=x_1=\tau(x_1).\end{aligned}$$

Since $Y_1=UT_0$ with $U\in\mathcal{H}$ given by (5.7), we furthermore obtain

$$\begin{aligned}\nu(T_0)&=\dagger\psi\kappa\,\ddagger^{-1}\,\psi^{-1}(T_0)\\&=\dagger\psi\kappa\,\ddagger^{-1}\,\psi^{-1}(U^{-1}Y_1)\\&=\dagger\psi\kappa\,\ddagger^{-1}\,(x_1^{-1}U^{-1})\\&=\dagger\psi\kappa(Ux_1)\\&=\dagger\psi(q^{\frac{1}{2}}x_1^{-1}T_0)\\&=\dagger\psi(q^{\frac{1}{2}}x_1^{-1}U^{-1}Y_1)\\&=\dagger(q^{\frac{1}{2}}x_1^{-1}U^{-1}Y_1)\\&=q^{\frac{1}{2}}x_1T_0^{-1}=\tau(T_0)\end{aligned}$$

where we use in the last equality that the sixth coordinate of the underlying difference multiplicity function is $q^{-\frac{1}{2}}$, so that $\tau_{\ddagger\tau\sigma}(T_0^{\ddagger\tau\sigma})=q^{\frac{1}{2}}x_1T_0^{\ddagger\sigma\tau\sigma-1}$. We conclude that $\nu_{\ddagger\tau\sigma}=\tau_{\ddagger\tau\sigma}$.

ii) Using the realization of the isomorphism τ as conjugation by the Gaussian and using Lemma 4.2, we observe that the function

$$d^{\tau}(s)=G_{\ddagger\tau\sigma}(s)^{-1}=G_{\tau\sigma\tau}(s),\qquad\forall\,s\in\mathcal{S}_{\tau}$$

satisfies (5.11). Furthermore, for this function d^{τ} we have the desired normalization $d^{\tau}(s_0^{\tau})=G_{\tau\sigma\tau}(s_0^{\tau})$. The result now follows from Lemma 5.10. □

We leave it as an exercise to the reader to repeat the arguments of this subsection to determine the normalized auxiliary transform $L_{\ddagger}$ associated to $\alpha_{\ddagger}$. It leads to the transformation behavior

$$\begin{aligned}X\circ d^{\ddagger\tau}&=d^{\ddagger\tau}\circ\big(\psi_{\sigma\tau}\circ\kappa^{I}_{\ddagger\tau}\circ\ddagger_{\tau}\circ\psi_{\tau}^{-1}\big)(X)\\&=d^{\ddagger\tau}\circ\tau_{\tau\sigma\tau}^{-1}(X),\qquad\qquad\forall\,X\in\mathcal{H}_{\tau\sigma}\end{aligned}$$

in $\mathrm{End}_{\mathbb{C}}\big(\mathcal{F}(\mathcal{S}_{\ddagger\tau})\big)$ for the associated generalized eigenvalue $d^{\ddagger\tau}\in\mathcal{F}(\mathcal{S}_{\ddagger\tau})$, and hence to the following result.

Proposition 5.12. *The normalized auxiliary transform $L_{\ddagger}:\mathcal{A}\to\mathcal{A}$ exists. Its generalized eigenvalue $d^{\ddagger\tau}\in\mathcal{F}(\mathcal{S}_{\ddagger\tau})$ is given by*

$$d^{\ddagger\tau}(s)=G_{\tau\sigma\tau}(s)=G_{\tau\sigma\tau}(s^{-1}),\qquad\forall\,s\in\mathcal{S}_{\ddagger\tau}.$$

In other words,

$$L_{\ddagger}(E_\tau(s;\cdot)) = G_{\tau\sigma\tau}(s)E_{\sigma\tau}(s;\cdot), \qquad \forall\, s \in \mathcal{S}_\tau.$$

5.4. The Cherednik kernels. In this subsection we construct meromorphic kernels $\mathfrak{E}, \mathfrak{E}_{\ddagger}$ satisfying the transformation behavior (5.1) under the action of the double affine Hecke algebra. For this it suffices to construct kernels $\mathfrak{F}, \mathfrak{F}_{\ddagger} \in \mathcal{O}\big((\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n\big)$ such that the normalized auxiliary transforms L and $L_{\ddagger}$ associated to α and $\alpha_{\ddagger}$ are given by

$$\big(Lp\big)(\gamma) = \big(\mathfrak{F}(\gamma,\cdot),p\big)_{\mathcal{A},\tau}, \qquad \big(L_{\ddagger}p\big)(\gamma) = \big(p,\mathfrak{F}_{\ddagger}(\gamma,\cdot)\big)_{\mathcal{A},\tau}, \qquad \forall\, p \in \mathcal{A} \qquad (5.12)$$

respectively, cf. Lemma 5.1 and Lemma 5.4. Formal series expansions in Macdonald–Koornwinder polynomials can now immediately be written down for such kernels $\mathfrak{F}$ and $\mathfrak{F}_{\ddagger}$ in view of Proposition 5.11, Proposition 5.12 and the orthogonality relations (4.8) for the Macdonald–Koornwinder polynomials. To ensure that these formal series expansions define analytic kernels, we need to determine bounds for the Macdonald–Koornwinder polynomials $E(s_\lambda;\cdot)$ in their degree $\lambda \in \Lambda$. The following proposition suffices for our purposes.

Proposition 5.13. *For any compact set $K \subset (\mathbb{C}^\times)^n$, there exists a constant $C = C_K > 0$ such that*

$$|E(s_\lambda;x)| \leq C^{N(\lambda)}, \qquad \forall\, x \in K, \quad \forall\, \lambda \in \Lambda,$$

where $N(\lambda) = \sum_{i=1}^n |\lambda_i|$ for $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda$.

The proof of the proposition, which is a bit technical, is based on explicit recurrence relations for the Macdonald–Koornwinder polynomials. These recurrence relations are a direct consequence of Proposition 4.4. For details of the proof, we refer the reader to the appendix.

Remark 5.14. For the symmetric Macdonald–Koornwinder polynomials, we have for any compact set $K \subset (\mathbb{C}^\times)^n$,

$$|E^+(s_\lambda;x)| \leq C^{N(\lambda)}, \qquad \forall\, x \in K, \quad \forall\, \lambda \in \Lambda^+,$$

for some constant $C = C_K > 0$. This follows for instance from Proposition 5.13, Lemma 9.1, the fact that $N(w \cdot \mu) = N(\mu)$ for $w \in W_0$ and $\mu \in \Lambda$ and from the explicit expansion

$$E^+(s_\lambda;x) = \sum_{\mu \in W_0\cdot\lambda} \frac{\mathcal{C}_\sigma(s_\mu^{\ddagger})}{\mathcal{C}_\sigma(s_0^{\ddagger})} E(s_\mu;x), \qquad \lambda \in \Lambda^+,$$

of the symmetric Macdonald–Koornwinder polynomial as a linear combination of the nonsymmetric ones, see [41, Thm. 6.6] or [42, Thm. 3.27].

Corollary 5.15. a) *The series expansion*

$$\mathfrak{F}(\gamma,x):=\sum_{s\in\mathcal{S}_\tau}\frac{G_{\tau\sigma\tau}(s)}{\big(E_\tau(s;\cdot),E_{\ddagger\tau}(s^{-1};\cdot)\big)_{\mathcal{A},\tau}}E_\tau(s;x)E_{\sigma\tau}(s;\gamma) \tag{5.13}$$

converges absolutely and uniformly on compacta of $(\mathbb{C}^\times)^n\times(\mathbb{C}^\times)^n$. *The series expansion* $\mathfrak{F}$ *defined by* (5.13) *is the unique analytic kernel on* $(\mathbb{C}^\times)^n\times(\mathbb{C}^\times)^n$ *such that* $\big(Lp\big)(\gamma)=\big(\mathfrak{F}(\gamma,\cdot),p\big)_{\mathcal{A},\tau}$ *for all* $p\in\mathcal{A}$, *where* L *is the normalized auxiliary transform associated to* α.

b) *The series expansion*

$$\mathfrak{F}_\ddagger(\gamma,x):=\sum_{s\in\mathcal{S}_\tau}\frac{G_{\tau\sigma\tau}(s)}{\big(E_\tau(s;\cdot),E_{\ddagger\tau}(s^{-1};\cdot)\big)_{\mathcal{A},\tau}}E_{\ddagger\tau}(s^{-1};x)E_{\sigma\tau}(s;\gamma) \tag{5.14}$$

converges absolutely and uniformly on compacta of $(\mathbb{C}^\times)^n\times(\mathbb{C}^\times)^n$. *The series expansion* $\mathfrak{F}_\ddagger$ *defined by* (5.14) *is the unique analytic kernel on* $(\mathbb{C}^\times)^n\times(\mathbb{C}^\times)^n$ *such that* $\big(L_\ddagger p\big)(\gamma)=\big(p,\mathfrak{F}_\ddagger(\gamma,\cdot)\big)_{\mathcal{A},\tau}$ *for all* $p\in\mathcal{A}$, *where* $L_\ddagger$ *is the normalized auxiliary transform associated to* $\alpha_\ddagger$.

Proof. We first prove that the series expansions (5.13) and (5.14) converge absolutely and uniformly on compacta of $(\mathbb{C}^\times)^n\times(\mathbb{C}^\times)^n$. For this we need to consider the behavior of the coefficients

$$\frac{G_{\tau\sigma\tau}(s)}{\big(E_\tau(s;\cdot),E_{\ddagger\tau}(s^{-1};\cdot)\big)_{\mathcal{A},\tau}}=\frac{1}{N_{\tau\sigma}(s_0^{\ddagger\tau})\big(1,1\big)_{\mathcal{A},\tau}}G_{\tau\sigma\tau}(s)N_{\tau\sigma}(s^{-1}) \tag{5.15}$$

as a function of $s\in\mathcal{S}_\tau$ in the expansion sums (5.13) and (5.14), see Theorem 4.11 for the second equality in (5.15). By the W_0-invariance of the Gaussian and by Lemma 4.1, we have for $\lambda\in\Lambda$,

$$G_{\tau\sigma\tau}(s_\lambda^\tau)=G_{\tau\sigma\tau}(s_{\lambda^+}^\tau), \tag{5.16}$$

where λ^+ is the unique element in $(W_0\cdot\lambda)\cap\Lambda^+$. Using the explicit expression of the Gaussian G and using that

$$s_\mu^\tau=(t_nu_0t^{2(n-1)}q^{\mu_1},t_nu_0t^{2(n-2)}q^{\mu_2},\ldots,t_nu_0q^{\mu_n}),\qquad \mu=\sum_{i=1}^n\mu_i\epsilon_i\in\Lambda^+$$

it is now easy to show that

$$\frac{G_{\tau\sigma\tau}(s_\mu^\tau)}{G_{\tau\sigma\tau}(s_0^\tau)}=\prod_{i=1}^n\frac{\big(bct^{2(n-i)};q\big)_{\mu_i}}{\big(qat^{2(n-i)}/d;q\big)_{\mu_i}}\left(\frac{-q^{\frac{1}{2}}at^{2(n-i)}}{d}\right)^{\mu_i}q^{\mu_i^2/2} \tag{5.17}$$

for $\mu\in\Lambda^+$ (cf. Remark 5.7), where we used the Askey–Wilson parametrization (4.7) for part of the multiplicity function α. We thus obtain the bounds

$$|G_{\tau\sigma\tau}(s_\lambda^\tau)|\le c_1c_2^{N(\lambda)}q^{\langle\lambda,\lambda\rangle/2},\qquad\forall\,\lambda\in\Lambda \tag{5.18}$$

for certain λ-independent constants $c_1, c_2 > 0$.

We have seen that the factor $N_{\tau\sigma}(s_\lambda^{\ddagger\tau})$ can be evaluated explicitly, see (4.16) and (4.17). From these explicit expressions and from Lemma 9.1 it follows that

$$|N_\tau(s_\lambda^{\ddagger\tau})| \leq d_1 d_2^{N(\lambda)}, \qquad \forall\, \lambda \in \Lambda$$

for certain λ-independent coefficients $d_1, d_2 > 0$. Combined with Proposition 5.13 and the fact that $0 < q < 1$, we see that the "Gaussian term" $q^{\langle\lambda,\lambda\rangle/2}$ in the bounds for $G_{\tau\sigma\tau}$ ensure the absolute and uniform convergence of the sums (5.13) and (5.14) on compact subsets of (γ, x) in $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$. In particular, the series expansions (5.13) and (5.14) define analytic kernels $\mathfrak{F}, \mathfrak{F}_\ddagger \in \mathcal{O}\big((\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n\big)$.

By the orthogonality relations (4.8) for the Macdonald–Koornwinder polynomials and the explicit series expansions (5.13) and (5.14) for $\mathfrak{F}$ and $\mathfrak{F}_\ddagger$, it is immediate that

$$\big(\mathfrak{F}(\gamma,\cdot), E_{\ddagger\tau}(s^{-1};\cdot)\big)_{\mathcal{A},\tau} = G_{\tau\sigma\tau}(s)E_{\sigma\tau}(s;\gamma) = \big(E_\tau(s;\cdot), \mathfrak{F}_\ddagger(\gamma,\cdot)\big)_{\mathcal{A},\tau}, \qquad \forall\, s \in \mathcal{S}_\tau.$$

In view of Proposition 5.11 and Proposition 5.12, we conclude that the normalized auxiliary transforms L and $L_\ddagger$ associated to α and $\alpha_\ddagger$ are given by the integral transforms (5.12), with $\mathfrak{F}$ and $\mathfrak{F}_\ddagger$ the analytic kernels defined by the series expansions (5.13) and (5.14), respectively. Clearly, the linear mappings (5.12) determine the analytic kernels $\mathfrak{F}$ and $\mathfrak{F}_\ddagger$ uniquely (cf. Lemma 5.4). □

Definition 5.16. a) The kernel $\mathfrak{E} = \mathfrak{E}_\alpha \in \mathcal{M}\big((\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n\big)$ defined by

$$\mathfrak{E}(\gamma, x) = G_{\sigma\tau}(\gamma)G_\tau(x)\mathfrak{F}(\gamma, x),$$

with $\mathfrak{F}$ the explicit series expansion (5.13), is called the Cherednik kernel associated to α.

b) The kernel $\mathfrak{E}_\ddagger = \mathfrak{E}_{\alpha_\ddagger} \in \mathcal{M}\big((\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n\big)$ defined by

$$\mathfrak{E}_\ddagger(\gamma, x) = G_{\sigma\tau}(\gamma)G_\tau(x)\mathfrak{F}_\ddagger(\gamma^{-1}, x),$$

with $\mathfrak{F}_\ddagger$ the explicit series expansion (5.14), is called the Cherednik kernel associated to $\alpha_\ddagger$.

We can now state the following main result of this section.

Theorem 5.17. a) *Up to a multiplicative constant, the Cherednik kernel $\mathfrak{E}$ associated to α is the unique non-zero meromorphic kernel such that*

i) *The function $G_{\sigma\tau}(\gamma)^{-1}G_\tau(x)^{-1}\mathfrak{E}(\gamma, x)$ depends analytically on $(\gamma, x) \in (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$.*

ii) *For all $X \in \mathcal{H} = \mathcal{H}_\alpha$ we have $\big(X\mathfrak{E}(\gamma,\cdot)\big)(x) = \big(\psi(X)\mathfrak{E}(\cdot, x)\big)(\gamma)$.*

b) *Up to a multiplicative constant, the Cherednik kernel $\mathfrak{E}_\ddagger$ associated to $\alpha_\ddagger$ is the unique non-zero meromorphic kernel such that*

i) *The function $G_{\sigma\tau}(\gamma)^{-1}G_\tau(x)^{-1}\mathfrak{E}_\ddagger(\gamma, x)$ depends analytically on $(\gamma, x) \in (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$.*

ii) *For all* $X \in \mathcal{H}_\ddagger$ *we have* $\big(X\mathfrak{E}_\ddagger(\gamma,\cdot)\big)(x) = \big(\psi_\ddagger(X)\mathfrak{E}_\ddagger(\cdot,x)\big)(\gamma)$.

Proof. a) It follows immediately from Corollary 5.15, Lemma 5.4 and Lemma 5.1 that $\mathfrak{E}$ satisfies properties i) and ii). Suppose that $\widetilde{\mathfrak{E}}$ is another non-zero meromorphic kernel satisfying i) and ii). Then

$$\widetilde{\mathfrak{F}}(\gamma,x) := G_{\sigma\tau}(\gamma)^{-1}G_\tau(x)^{-1}\widetilde{\mathfrak{E}}(\gamma,x)$$

depends analytically on $(\gamma,x) \in (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$. Furthermore, the linear map $\widetilde{L}: \mathcal{A} \to \mathcal{O}$, defined by $\big(\widetilde{L}p\big)(\gamma) = \big(\widetilde{\mathfrak{F}}(\gamma,\cdot),p\big)_{\mathcal{A},\tau}$ for $p \in \mathcal{A}$, is an auxiliary transform associated to α in view of Lemma 5.1 and Lemma 5.4. Lemma 5.5 then shows that $\widetilde{L} = c\,L$ for some constant $c \in \mathbb{C}^\times$, where L is the normalized auxiliary transform associated to α. By Corollary 5.15 we conclude that $\widetilde{\mathfrak{F}} = c\,\mathfrak{F}$, where $\mathfrak{F}$ is given by the series expansion (5.13), and hence $\widetilde{\mathfrak{E}} = c\,\mathfrak{E}$. The proof of b) is similar to the proof of a). □

6. Properties of the Cherednik kernels

In this section we further analyze the Cherednik kernels. In Subsection 6.1 we prove an evaluation formula, which allows one to normalize the Cherednik kernels in such a manner that they meromorphically extend the Macdonald–Koornwinder polynomials in their degrees (as will be shown in Subsection 6.2). In Subsection 6.1 we furthermore prove the duality of the normalized Cherednik kernels between their geometric and spectral parameters. In Subsection 6.3 we consider symmetric Cherednik kernels. In the case of reduced root systems, many of the results in this section reduce to statements in Cherednik's paper [5].

We keep the same generic conditions on the multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}})$ as in Section 5.

6.1. Evaluation formula and duality. Let $\mathfrak{F}$ and $\mathfrak{F}_\ddagger$ be the auxiliary kernels as defined by the series expansions (5.13) and (5.14), respectively. By the normalization (4.4) of the Macdonald–Koornwinder polynomials and (5.15), we have

$$\mathfrak{F}(s_0^{\ddagger\sigma\tau\sigma}, s_0^{\ddagger\tau\sigma}) = \frac{G_{\tau\sigma\tau}(s_0^\tau)}{\big(1,1\big)_{\mathcal{A},\tau}} \sum_{s\in\mathcal{S}_\tau} \frac{G_{\tau\sigma\tau}(s)N_{\tau\sigma}(s^{-1})}{G_{\tau\sigma\tau}(s_0^\tau)N_{\tau\sigma}(s_0^{\ddagger\tau})} = \mathfrak{F}_\ddagger(s_0^{\ddagger\sigma\tau\sigma}, s_0^{\tau\sigma}). \tag{6.1}$$

By (4.16), (4.26) and by (5.16), the sum in (6.1) can be rewritten as

$$\sum_{s\in\mathcal{S}_\tau} \frac{G_{\tau\sigma\tau}(s)N_{\tau\sigma}(s^{-1})}{G_{\tau\sigma\tau}(s_0^\tau)N_{\tau\sigma}(s_0^{\ddagger\tau})} = \sum_{s\in\mathcal{S}_\tau^+} \frac{G_{\tau\sigma\tau}(s)N_{\tau\sigma}^+(s^{-1})}{G_{\tau\sigma\tau}(s_0^\tau)N_{\tau\sigma}^+(s_0^{\ddagger\tau})}. \tag{6.2}$$

This sum can be evaluated as follows.

Proposition 6.1. *In terms of the Askey–Wilson parametrization* (4.7) *for part of the difference multiplicity function* α*,*

$$\sum_{s\in\mathcal{S}_\tau^+}\frac{G_{\tau\sigma\tau}(s)N_{\tau\sigma}^+(s^{-1})}{G_{\tau\sigma\tau}(s_0^\tau)N_{\tau\sigma}^+(s_0^{\ddagger\tau})}=\prod_{i=1}^n\frac{\left(qabct^{2(2n-i-1)}/d,qt^{2(i-n)}/ad;q\right)_\infty}{\left(qbt^{2(n-i)}/d,qct^{2(n-i)}/d;q\right)_\infty}. \tag{6.3}$$

Proof. We combine (4.17) and (5.17) to obtain the explicit expression

$$\begin{aligned}\sum_{\lambda\in\Lambda^+}\prod_{i=1}^n&\left\{\frac{\left(qabct^{4(n-i)}/d;q\right)_{2\lambda_i}}{\left(abct^{4(n-i)}/d;q\right)_{2\lambda_i}}\left(\frac{-q^{\frac{1}{2}}}{adt^{2(n-i)}}\right)^{\lambda_i}q^{\lambda_i^2/2}\right.\\&\left.\times\frac{\left(abt^{2(n-i)},act^{2(n-i)},abct^{2(n-i)}/d;q\right)_{\lambda_i}}{\left(qt^{2(n-i)},qbt^{2(n-i)}/d,qct^{2(n-i)}/d;q\right)_{\lambda_i}}\right\}\\&\times\prod_{1\le i<j\le n}\left\{\frac{\left(qabct^{2(2n-i-j)}/d,abct^{2(2n-i-j+1)}/d;q\right)_{\lambda_i+\lambda_j}}{\left(qabct^{2(2n-i-j-1)}/d,abct^{2(2n-i-j)}/d;q\right)_{\lambda_i+\lambda_j}}\right.\\&\qquad\left.\times\frac{\left(qt^{2(j-i)},t^{2(j-i+1)};q\right)_{\lambda_i-\lambda_j}}{\left(qt^{2(j-i-1)},t^{2(j-i)};q\right)_{\lambda_i-\lambda_j}}\right\}\end{aligned}\tag{6.4}$$

for the left-hand side of (6.3), where $\lambda_i=\langle\lambda,\epsilon_i\rangle$ for $i=1,\ldots,n$. Now this can be evaluated using the limit case $g_d\to-\infty$ of van Diejen's [12, Thm. 2.2] multiple Roger's ${}_6\phi_5$-sum. After straightforward simplifications (see also [12, (3.7a)]), we obtain the desired evaluation formula. □

It follows from the product formula (6.3) that the special values (6.1) of the auxiliary kernels are non-zero (in view of the generic conditions on the difference multiplicity function α).

Observe that the spectrum $\mathcal{S}$ satisfies the stability conditions

$$s_\lambda^{\tau\sigma}=s_\lambda^\sigma,\qquad s_\lambda^{\sigma\tau\sigma}=s_\lambda^{\tau\sigma\tau}=s_\lambda$$

for all $\lambda\in\Lambda$, since $s\in\mathcal{S}$ only depends on the values of $\mathbf{t}$ on the root subsystem $R\subset R_{nr}$. In particular, the product formula (6.3) leads to an explicit evaluation of the non-zero values $\mathfrak{F}(s_0^\ddagger,s_0^{\ddagger\sigma})$ and $\mathfrak{F}_\ddagger(s_0^\ddagger,s_0^\sigma)$. Since G_τ (respectively $G_{\sigma\tau}$) is regular at s_0^σ (respectively s_0) and $G_\tau(s_0^\sigma)\neq0$ (respectively $G_{\sigma\tau}(s_0)\neq0$), we may define *normalized Cherednik kernels* $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ associated to α and $\alpha_\ddagger$ respectively by requiring

$$\mathfrak{E}(s_0^\ddagger,s_0^{\ddagger\sigma})=1=\mathfrak{E}_\ddagger(s_0,s_0^\sigma). \tag{6.5}$$

This choice of normalization determines the Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ uniquely in view of Theorem 5.17. Furthermore, by Corollary 5.15, Lemma 5.1 and (5.15), the normalized Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ may be explicitly written as

$$\begin{aligned}\mathfrak{E}(\gamma,x)&=G_{\sigma\tau}(\gamma)G_\tau(x)\sum_{s\in\mathcal{S}_\tau}\mu_\tau(s)E_{\sigma\tau}(s;\gamma)E_\tau(s;x),\\ \mathfrak{E}_\ddagger(\gamma,x)&=G_{\sigma\tau}(\gamma)G_\tau(x)\sum_{s\in\mathcal{S}_\tau}\mu_\tau(s)E_{\sigma\tau}(s;\gamma^{-1})E_{\ddagger\tau}(s^{-1};x),\end{aligned}\tag{6.6}$$

with $\mu=\mu_\alpha\in\mathcal{F}(\mathcal{S})$ defined by

$$\mu(s)=C_0\frac{G_{\sigma\tau}(s)N_\sigma(s^{-1})}{G_{\sigma\tau}(s_0)N_\sigma(s_0^\ddagger)},\qquad s\in\mathcal{S}\tag{6.7}$$

and with the (non-zero) normalization constant $C_0=C_0^\alpha\in\mathbb{C}^\times$ chosen in such a way that (6.5) holds true. In other words, C_0 is chosen in such a way that

$$\sum_{s\in\mathcal{S}}\mu(s)=\frac{1}{G_{\tau\sigma\tau}(s_0^\tau)G(s_0^{\tau\sigma})}.\tag{6.8}$$

By Proposition 6.1 and Remark 5.7, the constant C_0 can be evaluated explicitly in terms of the Askey–Wilson parametrization (4.7) of part of the difference multiplicity function α,

$$C_0=\prod_{i=1}^n\frac{\big(adt^{2(n-i)},bdt^{2(n-i)},cdt^{2(n-i)},bct^{2(n-i)},dt^{2(i-n)}/a;q\big)_\infty}{\big(abcdt^{2(2n-i-1)};q\big)_\infty}.\tag{6.9}$$

Combined with (4.16), (4.17) and (5.17), this entails an explicit expression for the weight $\mu\in\mathcal{F}(\mathcal{S})$ in terms of q-shifted factorials, cf. the expression (6.4).

Theorem 6.2 (Duality)**.** *The normalized Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ associated to α and $\alpha_\ddagger$, respectively, satisfy the duality property*

$$\mathfrak{E}(\gamma,x)=\mathfrak{E}_\sigma(x,\gamma),\qquad \mathfrak{E}_\ddagger(\gamma,x)=\mathfrak{E}_{\ddagger\sigma}(x,\gamma).$$

Proof. This follows immediately from the characterization Theorem 5.17 for Cherednik kernels, the fact that $\psi^{-1}=\psi_\sigma$ for the duality anti-isomorphism $\psi:\mathcal{H}\to\mathcal{H}_\sigma$ and the fact that the chosen normalization of the Cherednik kernels is self-dual. □

6.2. Polynomial reduction. Let $v\in\mathcal{S}$ be a spectral point of the Y-operators $Y_i\in\mathcal{H}$ $(i=1,\ldots,n)$, considered as endomorphisms of $\mathcal{A}$. Then we have observed that the Macdonald–Koornwinder polynomial $E(v;\cdot)\in\mathcal{A}$ is the unique Laurent polynomial satisfying

$$p(Y)E(v;\cdot)=p(v)E(v;\cdot),\qquad\forall\,p\in\mathcal{A}\tag{6.10}$$

and satisfying the normalization $E(v;s_0^{\ddagger\sigma})=1$.

On the other hand, the normalized Cherednik function $\mathfrak{E}(\gamma, x)$ associated to α is regular at $\gamma = v^{-1}$, and the meromorphic function $\mathfrak{E}(v^{-1}, \cdot) \in \mathcal{M}$ satisfies

$$p(Y)\mathfrak{E}(v^{-1}; \cdot) = p(v)\mathfrak{E}(v^{-1}, \cdot), \qquad \forall\, p \in \mathcal{A},$$

whence it is tempting to believe that $\mathfrak{E}(v^{-1}, \cdot)$ is a constant multiple of the normalized Macdonald–Koornwinder polynomial $E(v; \cdot)$. In this section, we show that this is indeed the case. The argument essentially amounts to the fact that the meromorphic common eigenfunction $\mathfrak{E}(v^{-1}, \cdot)$ of the Y-operators is "regular enough" to ensure that $\mathfrak{E}(v^{-1}, \cdot) \in \mathcal{A}$ by standard elliptic function theory, and hence it can only be a constant multiple of the Macdonald–Koornwinder polynomial $E(v; \cdot)$.

To make the arguments rigorous and transparent, we study properties of certain explicit function transforms in detail, following similar lines of reasoning as for the auxiliary transforms (see Subsections 5.2 and 5.3).

The starting point for the definitions of these transforms forms the explicit series expansions (6.6) for the normalized Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$. The bound on the associated weight $\mu_\tau \in \mathcal{F}(\mathcal{S}_\tau)$, as derived in the proof of Corollary 5.15, allows us to define the following two transforms in terms of series expansions which absolutely and uniformly converge on compacta of $(\mathbb{C}^\times)^n$.

Definition 6.3. We define linear mappings $H, H_\ddagger : \mathcal{A} \to \mathcal{O}\, G_\tau \subset \mathcal{M}$ by

$$\begin{aligned} \big(Hp\big)(x) &= G_\tau(x) \sum_{s \in \mathcal{S}_\tau} \mu_\tau(s) p(s^{-1}) E_\tau(s; x), \\ \big(H_\ddagger p\big)(x) &= G_\tau(x) \sum_{s \in \mathcal{S}_\tau} \mu_\tau(s) p(s^{-1}) E_{\ddagger\tau}(s^{-1}; x) \end{aligned} \tag{6.11}$$

for all $p \in \mathcal{A}$.

The transforms H and $H_\ddagger$ are linked to $\mathfrak{E}(v^{-1}, \cdot)$ and $\mathfrak{E}_\ddagger(v, \cdot)$ for $v \in \mathcal{S}$ as follows. Observe that the factor $E_{\sigma\tau}(s; v^{-1})$ for $s \in \mathcal{S}_\tau$ occurring in the explicit expansion sum (6.6) for $\mathfrak{E}(v^{-1}, \cdot)$ can be rewritten by the duality (4.5) as

$$E_{\sigma\tau}(s; v^{-1}) = E_{\sigma\tau\sigma}(v; s^{-1}) = E_{\tau\sigma\tau}(v; s^{-1})$$

since $\sigma\tau\sigma = \tau\sigma\tau$ when acting upon difference multiplicity functions, and $s_\lambda^\tau = s_\lambda^{\sigma\tau}$, $s_\lambda = s_\lambda^{\sigma\tau\sigma}$ for all $\lambda \in \Lambda$. This implies that

$$\mathfrak{E}(v^{-1}, x) = G_{\sigma\tau}(v) H\big(E_{\tau\sigma\tau}(v; \cdot)\big)(x), \qquad \forall\, v \in \mathcal{S} \tag{6.12}$$

with $H : \mathcal{A} \to \mathcal{M}$ the linear map defined in (6.11). In exactly the same fashion, we can write

$$\mathfrak{E}_\ddagger(v^{-1}, x) = G_{\sigma\tau}(v) H_\ddagger\big(E_{\tau\sigma\tau}(v^{-1}; \cdot)\big)(x), \qquad \forall\, v \in \mathcal{S}_\ddagger \tag{6.13}$$

with $H_\ddagger : \mathcal{A} \to \mathcal{M}$ the linear map defined in (6.11). The main step now is to prove that the images of the transforms H and $H_\ddagger$ are contained in $\mathcal{A}$. For this, we first compute the intertwining properties of H and $H_{\ddagger}$ under the action of the double affine Hecke algebra. Recall the anti-isomorphisms κ and κ^I defined in Lemma 5.1.

Lemma 6.4. a) *For* $X \in \mathcal{H}_{\sigma\tau\sigma} = \mathcal{H}_{\tau\sigma\tau}$,

$$H \circ X = \big(\dagger_{\ddagger} \circ \kappa^I_{\tau\sigma\tau} \circ \iota_{\tau\sigma\tau}\big)(X) \circ H, \tag{6.14}$$

with the double affine Hecke algebra acting as q-difference reflection operators on both sides.

b) *For* $X \in \mathcal{H}_{\sigma\tau\sigma} = \mathcal{H}_{\tau\sigma\tau}$,

$$H_{\ddagger} \circ X = \big(\kappa_{\ddagger\tau\sigma\tau} \circ \ddagger_{\tau\sigma\tau}\big)(X) \circ H_{\ddagger}, \tag{6.15}$$

with the double affine Hecke algebra acting as q-difference reflection operators on the left-hand side and as q^{-1}-difference reflection operators on the right-hand side.

Proof. a) Up to an (irrelevant) multiplicative constant, Hp can be written as

$$\big(Hp\big)(x) = G_\tau(x)[G_{\tau\sigma\tau}p, \mathfrak{E}_{\mathcal{A},\tau}(\cdot\,,x)]_{\mathcal{A},\tau\sigma}$$

by (6.7), see Subsections 4.2 and 4.3 for the used notations. We thus obtain for $X \in \mathcal{H}_{\tau\sigma\tau}$

$$H \circ X = \big(\tau_\tau \circ \psi_{\tau\sigma} \circ \iota_{\tau\sigma} \circ \tau_{\tau\sigma\tau}\big)(X) \circ H$$

by Proposition 2.11, Proposition 4.8, Remark 4.9, Proposition 4.4 and Lemma 4.2. Now computing the isomorphism $\tau_\tau \circ \psi_{\tau\sigma} \circ \iota_{\tau\sigma} \circ \tau_{\tau\sigma\tau} : \mathcal{H}_{\tau\sigma\tau} \to \mathcal{H}$ on the algebraic generators $T_j^{\tau\sigma\tau}$ $(j = 0, \ldots, n)$ and x_1 of $\mathcal{H}_{\tau\sigma\tau}$ and comparing the outcome with Lemma 5.2 shows that

$$\tau_\tau \circ \psi_{\tau\sigma} \circ \iota_{\tau\sigma} \circ \tau_{\tau\sigma\tau} = \dagger_{\ddagger} \circ \kappa^I_{\tau\sigma\tau} \circ \iota_{\tau\sigma\tau}$$

(both sides map $T_j^{\tau\sigma\tau}$ to T_j for $j = 0, \ldots, n$ and x_1 to $q^{\frac{1}{2}} T_0 x_1^{-1} U^{-1}$).

b) By (2.11), we can write $H_{\ddagger}p$ up to an (irrelevant) multiplicative constant as

$$\big(H_{\ddagger}p\big)(x) = G_{\ddagger}(x)^{-1}[G_{\tau\sigma\tau}p, I\mathfrak{E}_{\mathcal{A},\ddagger\tau}(\cdot\,,x)]_{\mathcal{A},\tau\sigma},$$

where $(Ig)(s) = g(s^{-1})$ for $g \in \mathcal{F}(\mathcal{S}_\tau, \mathcal{A})$ and $s \in \mathcal{S}_{\ddagger\tau}$. A similar computation as for a) now shows that

$$H_{\ddagger} \circ X = \big(\tau_{\dagger}^{-1} \circ \psi_{\ddagger\tau\sigma} \circ \dagger_{\tau\sigma} \circ \iota_{\tau\sigma} \circ \tau_{\tau\sigma\tau}\big)(X) \circ H_{\ddagger}$$

for $X \in \mathcal{H}_{\tau\sigma\tau}$. The lemma follows from the fact that

$$\tau_{\dagger}^{-1} \circ \psi_{\ddagger\tau\sigma} \circ \dagger_{\tau\sigma} \circ \iota_{\tau\sigma} \circ \tau_{\tau\sigma\tau} = \kappa_{\ddagger\tau\sigma\tau} \circ \ddagger_{\tau\sigma\tau}$$

as unital algebra isomorphisms from $\mathcal{H}_{\tau\sigma\tau}$ onto $\mathcal{H}_{\ddagger}$, which again can be checked by computing the images of the algebraic generators $T_i^{\tau\sigma\tau}$ $(i = 1, \ldots, n)$, $Y_1^{\tau\sigma\tau}$ and x_1 of $\mathcal{H}_{\tau\sigma\tau}$ explicitly for both isomorphisms. □

Note the similarities between the maps $H, H_{\ddagger} : \mathcal{A} \to \mathcal{M}$ and the auxiliary transforms $L, L_{\ddagger} : \mathcal{A} \to \mathcal{O}$ defined before: they nearly satisfy the same intertwining properties with respect to the action of the double affine Hecke algebra (up to some elementary (anti-)isomorphisms $\iota, \ddagger, \dagger$). For the auxiliary transforms $L, L_\ddagger : \mathcal{A} \to \mathcal{O}$ we used the intertwining properties to prove that $L1$ and $L_\ddagger 1$ are constant multiples of $1 \in \mathcal{A}$, see Lemma 5.5. A similar argument can now be applied to the transforms $H, H_\ddagger : \mathcal{A} \to \mathcal{M}$.

Lemma 6.5. *The transforms $H, H_\ddagger : \mathcal{A} \to \mathcal{M}$ satisfy*

$$H1 = G_{\sigma\tau}(s_0)^{-1}1 = H_\ddagger 1,$$

where $1 \in \mathcal{A}$ is the Laurent polynomial identically equal to one.

Proof. In view of Lemma 6.4 we can repeat the arguments of the proof of Lemma 5.5 to prove that $H1 \in \mathcal{M}$ is $\mathcal{W}$-invariant (with $\mathcal{W}$ acting as constant coefficient q-difference reflection operators) and that $H_\ddagger 1 \in \mathcal{M}$ is $\mathcal{W}$-invariant (with $\mathcal{W}$ now acting as constant coefficient q^{-1}-difference reflection operators). So in terms of exponential coordinates

$$x = e^{2\pi i w} = \bigl(e^{2\pi i w_1}, e^{2\pi i w_2}, \ldots, e^{2\pi i w_n}\bigr),$$

we obtain meromorphic functions

$$w \mapsto H(1)\bigl(e^{2\pi i w}\bigr), \qquad w \mapsto H_\ddagger(1)\bigl(e^{2\pi i w}\bigr)$$

on the compact torus $\mathbb{C}^n/\Gamma_q$, with $\Gamma_q = \mathbb{Z}^n + \mathbb{Z}^n\, v$ and v an element in the upper half plane satisfying $q = e^{2\pi i v}$. Now both the maps $w \mapsto H(1)\bigl(e^{2\pi i w}\bigr)G_\tau(e^{2\pi i w})^{-1}$ and $w \mapsto H_\ddagger(1)\bigl(e^{2\pi i w}\bigr)G_\tau(e^{2\pi i w})^{-1}$ are analytic in $w \in \mathbb{C}^n$. Since

$$G_\tau(e^{2\pi i w}) = \prod_{j=1}^{n} \frac{1}{\bigl(-q^{\frac{1}{2}}u_0 t_0^{-1} e^{2\pi i w_j}, -q^{\frac{1}{2}}u_0 t_0^{-1} e^{-2\pi i w_j}; q\bigr)_\infty},$$

it now follows that the function $w_j \mapsto H(1)(e^{2\pi i w})$ and $w_j \mapsto H_\ddagger(1)(e^{2\pi i w})$ for fixed, regular $w_k \in \mathbb{C}$ $(k \neq j)$ is elliptic with period lattice $\mathbb{Z} + \mathbb{Z}\, v$, whose possible poles are at most simple and located at

$$\bigl(u + \mathbb{Z} + \mathbb{N}\, v\bigr) \cup \bigl(-u + \mathbb{Z} - \mathbb{N}\, v\bigr), \tag{6.16}$$

where $u \in \mathbb{C}$ is chosen such that $e^{2\pi i u} = -q^{-\frac{1}{2}}u_0^{-1}t_0$. Standard elliptic function theory now implies that the functions $w_j \mapsto H(1)(e^{2\pi i w})$ and $w_j \mapsto H_\ddagger(1)(e^{2\pi i w})$ are constant. Consequently, the meromorphic functions $H1, H_\ddagger 1 \in \mathcal{M}$, regarded as analytic functions on the open, dense set of elements in $(\mathbb{C}^\times)^n$, which do not belong to the zero set of $G_\tau^{-1} \in \mathcal{O}$, are constant. Hence $H1, H_\ddagger 1 \in \mathcal{O}$, and they are constant on $(\mathbb{C}^\times)^n$.

To complete the proof, we note that

$$\begin{aligned} H(1)(s_0^{\ddagger\sigma}) &= G_\tau(s_0^\sigma) \sum_{s \in \mathcal{S}_\tau} \mu_\tau(s) \\ &= G_{\sigma\tau}(s_0)^{-1}\mathfrak{E}(s_0^\ddagger, s_0^{\ddagger\sigma}) = G_{\sigma\tau}(s_0)^{-1} \end{aligned}$$

by the normalization of the Macdonald–Koornwinder polynomials and of the Cherednik kernel. Similarly one can show that $H_{\ddagger}(1)(s_0^{\sigma}) = G_{\sigma\tau}(s_0)^{-1}$. □

The following corollary is immediate from Lemma 6.4 and from (the proof of) Lemma 6.5.

Corollary 6.6. *The linear maps $H, H_{\ddagger} : \mathcal{A} \to \mathcal{O}G_{\tau}$ are, up to a multiplicative constant, uniquely determined by the intertwining properties* (6.14) *and* (6.15) *respectively under the action of the double affine Hecke algebra $\mathcal{H}_{\tau\sigma\tau}$.*

Continuing the same line of arguments as for auxiliary transforms, we can now state the following consequence of Lemma 6.5.

Corollary 6.7. a) *The transform H maps into $\mathcal{A}$. Furthermore,*

$$H\big(E_{\tau\sigma\tau}(v;\cdot)\big) = e(v)E(v;\cdot), \qquad \forall\, v \in \mathcal{S} = \mathcal{S}_{\tau\sigma\tau}$$

for some $e \in \mathcal{F}(\mathcal{S})$.

b) *The transform $H_{\ddagger}$ maps into $\mathcal{A}$. Furthermore,*

$$H_{\ddagger}\big(E_{\tau\sigma\tau}(v^{-1};\cdot)\big) = e_{\ddagger}(v)E_{\ddagger}(v;\cdot), \qquad \forall\, v \in \mathcal{S}_{\ddagger}$$

for some $e_{\ddagger} \in \mathcal{F}(\mathcal{S}_{\ddagger})$.

Proof. By Lemma 6.4 and Lemma 6.5, we clearly have $H(\mathcal{A}) \subseteq \mathcal{A}$ and $H_{\ddagger}(\mathcal{A}) \subseteq \mathcal{A}$. Combining this fact with the formulas

$$\big(\dagger_{\ddagger} \circ \kappa^{I}_{\tau\sigma\tau} \circ \iota_{\tau\sigma\tau}\big)(Y_i^{\tau\sigma\tau}) = Y_i,$$
$$\big(\kappa_{\ddagger\tau\sigma\tau} \circ \ddagger_{\tau\sigma\tau}\big)(Y_i^{\tau\sigma\tau}) = Y_i^{\ddagger-1}$$

for $i = 1, \ldots, n$, and using Lemma 6.4 and the results of Subsection 4.2, we directly obtain the second statement of the corollary (compare with the proof of Corollary 5.8). □

Corollary 6.7 combined with (6.12) and (6.13) already prove that $\mathfrak{E}(s^{-1},\cdot)$ for $s \in \mathcal{S}$ (respectively $\mathfrak{E}_{\ddagger}(v^{-1},\cdot)$ for $v \in \mathcal{S}_{\ddagger}$) is a constant multiple of $E(s;\cdot)$ (respectively $E_{\ddagger}(v;\cdot)$). The next aim is to prove that the constant multiple is in fact always equal to one. For the proof of this result we first need to compute the functions $e \in \mathcal{F}(\mathcal{S})$ and $e_{\ddagger} \in \mathcal{F}(\mathcal{S}_{\ddagger})$, see Corollary 6.7. Again, the method is similar to the computation of the generalized eigenvalues of the auxiliary transforms L and $L_{\ddagger}$, respectively.

Lemma 6.8. *The maps $e \in \mathcal{F}(\mathcal{S})$ and $e_{\ddagger} \in \mathcal{F}(\mathcal{S}_{\ddagger})$ are given explicitly by*

$$e(\cdot) = G_{\sigma\tau}(\cdot)^{-1}|_{\mathcal{S}}, \qquad e_{\ddagger}(\cdot) = G_{\sigma\tau}(\cdot)^{-1}|_{\mathcal{S}_{\ddagger}}.$$

Proof. Let $\widetilde{e} \in \mathcal{F}(\mathcal{S})$ and let $\widetilde{H} : \mathcal{A} \to \mathcal{A}$ be the linear map defined by

$$\widetilde{H}\big(E_{\tau\sigma\tau}(v;\,\cdot)\big) = \widetilde{e}(v)E(v;\,\cdot), \qquad \forall\, v \in \mathcal{S}.$$

Suppose furthermore that $\widetilde{e}$, considered as an endomorphism of $\mathcal{F}(\mathcal{S}) = \mathcal{F}(\mathcal{S}_{\sigma\tau\sigma})$, induces an algebra isomorphism $\widetilde{\nu}_{\ddagger\sigma\tau} : \mathcal{H}_{\ddagger\sigma\tau} \to \mathcal{H}_{\ddagger\sigma}$ by the formula

$$X \circ \widetilde{e} = \widetilde{e} \circ \widetilde{\nu}_{\ddagger\sigma\tau}(X), \qquad \forall\, X \in \mathcal{H}_{\ddagger\sigma\tau}$$

in $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(\mathcal{S}))$, where $\mathcal{H}_{\ddagger\sigma\tau}$ and $\mathcal{H}_{\ddagger\sigma}$ act via the dot action on $\mathcal{F}(\mathcal{S}) = \mathcal{F}(\mathcal{S}_{\sigma\tau\sigma})$.

By repeating the arguments of the proof of Lemma 5.10, we deduce that if the isomorphism $\widetilde{\nu}_{\ddagger\sigma\tau}$ satisfies

$$\psi_\sigma \circ \dagger_{\ddagger\sigma} \circ \widetilde{\nu}_{\ddagger\sigma\tau} \circ \dagger_{\sigma\tau} \circ \psi_{\sigma\tau\sigma} = \dagger_\ddagger \circ \kappa^I_{\tau\sigma\tau} \circ \iota_{\tau\sigma\tau}, \tag{6.17}$$

then the linear map $\widetilde{H}$ has the intertwining property

$$\widetilde{H} \circ X = \big(\dagger_\ddagger \circ \kappa^I_{\tau\sigma\tau} \circ \iota_{\tau\sigma\tau}\big)(X) \circ \widetilde{H} \tag{6.18}$$

for all $X \in \mathcal{H}_{\sigma\tau\sigma} = \mathcal{H}_{\tau\sigma\tau}$. Note now that (6.17) is equivalent to

$$\begin{aligned}\widetilde{\nu}_{\ddagger\sigma\tau} &= \dagger_\sigma \circ \psi \circ \dagger_\ddagger \circ \kappa^I_{\tau\sigma\tau} \circ \iota_{\tau\sigma\tau} \circ \psi_{\sigma\tau} \circ \dagger_{\ddagger\sigma\tau}\\ &= \tau^{-1}_{\dagger\sigma},\end{aligned}$$

where the last equality follows from computing both sides explicitly on suitable algebraic generators of $\mathcal{H}_{\ddagger\sigma\tau}$. By Proposition 2.11, Lemma 4.2 and (2.11), we conclude that if $\widetilde{e}$ equals $G_{\ddagger\sigma}(\cdot)|_{\mathcal{S}} = G_{\sigma\tau}(\cdot)^{-1}|_{\mathcal{S}}$ up to a multiplicative constant, then $\widetilde{e}$ induces the isomorphism $\nu_{\ddagger\sigma\tau}$, and hence the corresponding linear map $\widetilde{H}$ has the intertwining property (6.18). By the uniqueness of such linear maps (see Corollary 6.6), we conclude that these are in fact the only possibilities for $\widetilde{e} \in \mathcal{F}(\mathcal{S})$ for which the associated linear maps $\widetilde{H}$ have the intertwining property (6.18). Applying this result to H and using the normalization $H1 = G_{\sigma\tau}(s_0)^{-1}1$, we conclude that $e(\cdot) = G_{\sigma\tau}(\cdot)^{-1}|_{\mathcal{S}}$.

The proof for $e_\ddagger$ is similar; the arguments lead to the transformation behavior

$$\begin{aligned}X \circ e_\ddagger &= e_\ddagger \circ \big(\dagger_{\ddagger\sigma} \circ \psi_\ddagger \circ \kappa_{\ddagger\tau\sigma\tau} \circ \ddagger_{\tau\sigma\tau} \circ \psi_{\sigma\tau}\big)(X) \circ e_\ddagger\\ &= e_\ddagger \circ \tau_{\sigma\tau}(X), \qquad\qquad \forall\, X \in \mathcal{H}_{\sigma\tau}\end{aligned}$$

in $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(\mathcal{S}_\ddagger))$, where the function $e_\ddagger$ is considered as a multiplication operator in the endomorphism space $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(\mathcal{S}_\ddagger)) = \mathrm{End}_{\mathbb{C}}(\mathcal{F}(\mathcal{S}_{\ddagger\sigma\tau\sigma}))$. It follows from this that $e_\ddagger(\cdot) = G_{\sigma\tau}(\cdot)^{-1}|_{\mathcal{S}_\ddagger}$. □

We are now in the position to prove that the normalized Cherednik kernel meromorphically extends the Macdonald–Koornwinder polynomials in their degrees.

Theorem 6.9. a) *Let $\mathfrak{E}$ be the normalized Cherednik kernel associated to α. For all $v \in \mathcal{S}$,*

$$\mathfrak{E}(v^{-1},\,\cdot) = E(v;\,\cdot),$$

where $E(v;\,\cdot) = E_\alpha(v;\,\cdot)$ *is the normalized Macdonald–Koornwinder polynomial corresponding to the spectral point* $v \in \mathcal{S}$*.*

b) *Let* $\mathfrak{E}_\ddagger$ *be the normalized Cherednik kernel associated to* $\alpha_\ddagger$*. For all* $v \in \mathcal{S}_\ddagger$*,*

$$\mathfrak{E}_\ddagger(v^{-1},\,\cdot) = E_\ddagger(v;\,\cdot),$$

where $E_\ddagger(v;\,\cdot) = E_{\alpha_\ddagger}(v;\,\cdot)$ *is the normalized Macdonald–Koornwinder polynomial corresponding to the spectral point* $v \in \mathcal{S}_\ddagger$*.*

Proof. By formula (6.12), Corollary 6.7 and Lemma 6.8, we have

$$\begin{aligned}\mathfrak{E}(v^{-1},\,\cdot) &= G_{\sigma\tau}(v)H\big(E_{\tau\sigma\tau}(v;\cdot)\big)\\ &= G_{\sigma\tau}(v)e(v)E(v;\cdot)\\ &= E(v;\cdot)\end{aligned}$$

for $v \in \mathcal{S}$. The proof for $\mathfrak{E}_\ddagger$ is similar. □

Several direct consequences can be derived by combining the explicit series expansion (6.6) of the normalized Cherednik kernel with its polynomial reduction (see Theorem 6.9).

Corollary 6.10. a) *The expansion of the inverse Gaussian* $G_\tau^{-1} \in \mathcal{O}$ *in terms of Macdonald–Koornwinder polynomials is given by*

$$\begin{aligned}G_\tau(x)^{-1} &= G_{\sigma\tau}(s_0)\sum_{s\in\mathcal{S}_\tau}\mu_\tau(s)E_\tau(s;x)\\ &= G_{\sigma\tau}(s_0)\sum_{s\in\mathcal{S}_\tau}\mu_\tau(s)E_{\ddagger\tau}(s^{-1};x).\end{aligned}$$

Here the series converges absolutely and uniformly for x *in compacta of* $(\mathbb{C}^\times)^n$*.*

b) *For all* $s \in \mathcal{S} = \mathcal{S}_{\sigma\tau\sigma}$*, we have the identities*

$$\begin{aligned}E_{\sigma\tau\sigma}(s;Y^{\tau-1})\big(G_\tau^{-1}\big) &= \frac{G_{\sigma\tau}(s_0)}{G_{\sigma\tau}(s)}E(s;\cdot)G_\tau^{-1},\\ E_{\sigma\tau\sigma}(s;Y^{\ddagger\tau})\big(G_\tau^{-1}\big) &= \frac{G_{\sigma\tau}(s_0)}{G_{\sigma\tau}(s)}E_\ddagger(s^{-1};\cdot)G_\tau^{-1}\end{aligned}$$

in $\mathcal{O}$*.*

Proof. a) By Theorem 6.9, the normalized Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ satisfy

$$\mathfrak{E}(s_0^{-1},\cdot) = 1 = \mathfrak{E}_\ddagger(s_0,\cdot). \tag{6.19}$$

The identities now follow by substituting the explicit series expansions (6.6) for $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ into (6.19).

b) We prove the first equality, the second is similar. By a), by the polynomial duality $E_{\sigma\tau\sigma}(s;v^{-1}) = E_{\sigma\tau}(v;s^{-1})$ for $v \in \mathcal{S}_\tau = \mathcal{S}_{\sigma\tau}$ and $s \in \mathcal{S} = \mathcal{S}_{\sigma\tau\sigma}$, and by the explicit expansion (6.6) for the Cherednik kernel $\mathfrak{E}$,

$$\begin{aligned} E_{\sigma\tau\sigma}(s;Y^{\tau-1})\big(G_\tau^{-1}\big) &= G_{\sigma\tau}(s_0)\sum_{v\in\mathcal{S}_\tau}\mu_\tau(v)E_{\sigma\tau\sigma}(s;Y^{\tau-1})\big(E_\tau(v;\cdot)\big)\\ &= G_{\sigma\tau}(s_0)\sum_{v\in\mathcal{S}_\tau}\mu_\tau(v)E_{\sigma\tau\sigma}(s;v^{-1})E_\tau(v;\cdot)\\ &= G_{\sigma\tau}(s_0)\sum_{v\in\mathcal{S}_\tau}\mu_\tau(v)E_{\sigma\tau}(v;s^{-1})E_\tau(v;\cdot)\\ &= \frac{G_{\sigma\tau}(s_0)}{G_{\sigma\tau}(s)}\mathfrak{E}(s^{-1},\cdot)G_\tau^{-1}. \end{aligned}$$

The formula then follows from the polynomial reduction $\mathfrak{E}(s^{-1},\cdot)=E(s;\cdot)$ of the Cherednik kernel $\mathfrak{E}$, see Theorem 6.9. □

Remark 6.11. The series expansion of $G_\tau^{-1}\in\mathcal{O}$ in terms of Macdonald–Koornwinder polynomials (see Corollary 6.10a)) may be regarded as a generalization of the Jacobi triple product identity, cf. Remark 2.10c).

6.3. Symmetrization. Recall that the symmetrizer $C_+\in H_0\subset\mathcal{H}$ was used in Subsection 4.5 to derive W_0-invarariant versions of all main results on Macdonald–Koornwinder polynomials. In this subsection we consider the action of the symmetrizer C_+ on the normalized Cherednik kernel $\mathfrak{E}$.

Definition 6.12. Let $\mathfrak{E}$ be the normalized Cherednik kernel associated to α. The meromorphic kernel $\mathfrak{E}^+(\cdot,\cdot)=\mathfrak{E}^+_\alpha(\cdot,\cdot)\in\mathcal{M}\big((\mathbb{C}^\times)^n\times(\mathbb{C}^\times)^n\big)$ defined by

$$\mathfrak{E}^+(\gamma,x)=\big(C_+\mathfrak{E}(\gamma,\cdot)\big)(x)$$

is called the symmetric Cherednik kernel associated to α.

In the following lemma we give some elementary properties of the symmetric Cherednik kernel.

Lemma 6.13. *Let $\mathfrak{E}^+$ be the symmetric Cherednik kernel associated to α.*
a) $\mathfrak{E}^+(\gamma,x)$ *is W_0-invariant in x.*
b) $\mathfrak{E}^+(\gamma,x)=\big(C_+^\sigma\mathfrak{E}(\cdot,x)\big)(\gamma)$.
c) $\mathfrak{E}^+(\gamma,x)$ *is W_0-invariant in γ.*
d) $\mathfrak{E}^+(s_0,s_0^\sigma)=1$ (*normalization*).

Proof. a) This follows from $T_iC_+=t_iC_+$ for $i=1,\ldots,n$ and the explicit form of the reflection operators T_i; compare with the proof of Lemma 5.5.

b) This follows from (4.18) and from the transformation behavior of the Cherednik kernel under the action of the double affine Hecke algebra $\mathcal{H}$, see Theorem 5.17.

c) This is immediately clear from b) and from the proof of a).

d) For any $g \in \mathcal{O}G_{\sigma\tau}$, we have

$$\big(T_i^\sigma g\big)(s_0^\ddagger) = t_i g(s_0^\ddagger), \qquad i = 1, \ldots, n$$

since the rational function $c_{a_i}^\sigma \in \mathbb{C}(x)$ occurring in the definition of T_i^σ vanishes at $s_0^\ddagger$, see Lemma 4.1b). In particular, $(C_+^\sigma g)(s_0^\ddagger) = g(s_0^\ddagger)$ for $g \in \mathcal{O}G_{\sigma\tau}$. It follows that for generic $x \in (\mathbb{C}^\times)^n$ (in particular, for $x = s_0^\sigma$),

$$\mathfrak{E}^+(s_0^\ddagger, x) = \big(C_+^\sigma \mathfrak{E}(\cdot, x)\big)(s_0^\ddagger) = \mathfrak{E}(s_0^\ddagger, x) = 1,$$

where the last equality follows from the polynomial reduction of $\mathfrak{E}$, see Theorem 6.9. In particular, by part c) of the lemma, $\mathfrak{E}^+(s_0, s_0^\sigma) = \mathfrak{E}^+(s_0^\ddagger, s_0^\sigma) = 1$. □

We denote

$$\mu^+(s) = C_0 \frac{G_{\sigma\tau}(s) N_\sigma^+(s^{-1})}{G_{\sigma\tau}(s_0) N_\sigma^+(s_0^\ddagger)}, \qquad s \in \mathcal{S}^+, \tag{6.20}$$

with $C_0 = C_0^\alpha \in \mathbb{C}$ the normalization constant (6.9). Note that the normalization constant is chosen in such a way that

$$\sum_{s \in \mathcal{S}^+} \mu^+(s) = \sum_{s \in \mathcal{S}} \mu(s) = \frac{1}{G_{\tau\sigma\tau}(s_0^\tau) G(s_0^{\tau\sigma})},$$

see (6.2) for the first equality and (6.8) for the second equality. Applying the results of Proposition 4.13 now leads to the following result.

Proposition 6.14. *The symmetric Cherednik kernel $\mathfrak{E}^+$ associated to α is given by the series expansion*

$$\mathfrak{E}^+(\gamma, x) = G_{\sigma\tau}(\gamma) G_\tau(x) \sum_{s \in \mathcal{S}_\tau^+} \mu_\tau^+(s) E_\tau^+(s; x) E_{\sigma\tau}^+(s; \gamma),$$

with the series converging absolutely and uniformly on compacta of $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$.

Proof. Let the symmetrizer C_+, respectively C_+^σ, act on the x-variables, respectively the γ-variables, within the series expansion (6.6) of $\mathfrak{E}(\gamma, x)$, and apply (4.18) and Proposition 4.13. We obtain

$$\mathfrak{E}^+(\gamma, x) = G_{\sigma\tau}(\gamma) G_\tau(x) \sum_{\lambda \in \Lambda^+} \Big(\sum_{\mu \in W_0 \cdot \lambda} \mu_\tau(s_\mu^\tau) \Big) E_\tau^+(s_\lambda^\tau; x) E_{\sigma\tau}^+(s_\lambda^{\sigma\tau}; \gamma).$$

By (4.16), (4.26) and (5.16), and by the explicit expressions (6.7) and (6.20) for μ_τ and μ_τ^+, respectively, we obtain

$$\sum_{\mu \in W_0 \cdot \lambda} \mu_\tau(s_\mu^\tau) = \mu_\tau^+(s_\lambda^\tau), \qquad \forall\, \lambda \in \Lambda^+,$$

from which the explicit series expansion for the symmetric Cherednik function now immediately follows. The convergence properties of the series are clear from e.g., (the proof of) Corollary 5.15 and Remark 5.14. □

Define the spherical subalgebra $\mathcal{H}^+ = \mathcal{H}_\alpha^+$ of the double affine Hecke algebra $\mathcal{H}$ by

$$\mathcal{H}^+ = C_+\mathcal{H}C_+ = \{X \in \mathcal{H} \,|\, C_+X = X = XC_+\}.$$

Observe that $X \in \mathcal{H}$ is an element of $\mathcal{H}^+$ if and only if $T_iX = t_iX = XT_i$ for all $i = 1, \ldots, n$. In particular, any element $X \in \mathcal{H}^+$, considered as endomorphism of $\mathcal{M}$, maps into the field $\mathcal{M}_+$ of W_0-invariant meromorphic functions, and factorizes through the projection $C_+ : \mathcal{M} \to \mathcal{M}_+$. Note furthermore that the duality anti-isomorphism ψ restricts to an anti-isomorphism $\psi : \mathcal{H}^+ \to \mathcal{H}_\sigma^+$ since $\psi(C_+) = C_+^\sigma$.

Symmetrizing the main properties of the normalized Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ leads to the following main result of this subsection.

Theorem 6.15. a) $\mathfrak{E}^+(\gamma, x) = \big(C_+^\ddagger \mathfrak{E}_\ddagger(\gamma, \cdot)\big)(x)$ (*inversion-invariance*).
b) *For* $X \in \mathcal{H}^+$, $\big(X\mathfrak{E}^+(\gamma, \cdot)\big)(x) = \big(\psi(X)\mathfrak{E}^+(\cdot, x)\big)(\gamma)$ (*transformation behavior*).
c) $\mathfrak{E}^+(\gamma, x) = \mathfrak{E}_\sigma^+(x, \gamma)$ (*duality*).
d) $\mathfrak{E}^+(s; x) = E^+(s; x)$ *for* $s \in \mathcal{S}^+$ (*polynomial reduction*).

Proof. a) Similarly as for the normalized Cherednik kernel $\mathfrak{E}$ (see Lemma 6.13), we have $\big(C_+^\ddagger \mathfrak{E}_\ddagger(\gamma, \cdot)\big)(x) = \big(C_+^{\ddagger\sigma} \mathfrak{E}_\ddagger(\cdot, x)\big)(\gamma)$. Using Proposition 4.13, the series expansion (6.6) for $\mathfrak{E}_\ddagger$, (2.11) and (4.18), we can repeat now the arguments of the proof of Proposition 6.14 to show that

$$\begin{aligned}\big(C_+^\ddagger \mathfrak{E}_\ddagger(\gamma, \cdot)\big)(x) &= G_{\sigma\tau}(\gamma)G_\tau(x) \sum_{s \in \mathcal{S}_\tau^+} \mu_\tau^+(s)E_\tau^+(s; x)E_{\sigma\tau}^+(s; \gamma) \\ &= \mathfrak{E}^+(\gamma, x).\end{aligned}$$

b) For $X \in \mathcal{H}^+$ we compute, using the transformation behavior of the normalized Cherednik kernel $\mathfrak{E}$ under the action of $\mathcal{H}$ (see Theorem 5.17),

$$\begin{aligned}\big(X\mathfrak{E}^+(\gamma, \cdot)\big)(x) &= \big(X\mathfrak{E}(\gamma, \cdot)\big)(x) \\ &= \big(\psi(X)\mathfrak{E}(\cdot, x)\big)(\gamma) \\ &= \big(\psi(X)\mathfrak{E}^+(\cdot, x)\big)(\gamma),\end{aligned}$$

where we used Lemma 6.13b) and $\psi(X) \in \mathcal{H}_\sigma^+$ in the last equality.

c) This follows from the duality of the normalized Cherednik kernel $\mathfrak{E}$ (see Theorem 6.2) and Lemma 6.13b),

$$\mathfrak{E}^+(\gamma, x) = \big(C_+\mathfrak{E}(\gamma, \cdot)\big)(x) = \big(C_+\mathfrak{E}_\sigma(\cdot, \gamma)\big)(x) = \mathfrak{E}_\sigma^+(x, \gamma).$$

d) This follows from the polynomial reduction of the normalized Cherednik kernel (see Theorem 6.9) and Lemma 6.13c),

$$\begin{aligned}\mathfrak{E}^+(s, \cdot) &= \mathfrak{E}^+(s^{-1}, \cdot) \\ &= C_+\mathfrak{E}(s^{-1}, \cdot) \\ &= C_+E(s; \cdot) = E^+(s; \cdot)\end{aligned}$$

for all $s \in \mathcal{S}^+$. □

Remark 6.16. Note that $\mathcal{A}_+ \subset \mathcal{H}^+$ (considered as multiplication operators), as well as $\mathcal{A}_+(Y) \subset \mathcal{H}^+$, where $\mathcal{A}_+(Y)$ is the subalgebra of elements $p(Y)$ ($p \in \mathcal{A}_+$) in $\mathcal{H}$ (e.g., by applying the duality anti-isomorphism ψ_σ to the inclusion $\mathcal{A}_+ \subset \mathcal{H}_\sigma^+$). In view of Theorem 6.15b) we thus conclude that

$$p(Y)\mathfrak{E}^+(\gamma,\cdot) = p(\gamma)\mathfrak{E}^+(\gamma,\cdot), \qquad p(Y_\sigma)\mathfrak{E}^+(\cdot,x) = p(x)\mathfrak{E}^+(\cdot,x) \tag{6.21}$$

for all $p \in \mathcal{A}_+$. The algebra of commuting q-difference reflection operators $\mathcal{A}_+(Y)$, considered as endomorphisms of $\mathcal{M}_+$, can be identified with an algebra generated by n algebraically independent, commuting q-difference operators (see van Diejen [10] for an explicit description of these q-difference operators). One of these q-difference operators may be taken to be Koornwinder's [30] multivariable extension of the Askey–Wilson second order q-difference operator, see [36] and Subsection 4.5. The formulas (6.21) thus show that the symmetric Cherednik function $\mathfrak{E}^+(\gamma,\cdot) \in \mathcal{M}_+$ is a common eigenfunction for these q-difference operators.

We note that Corollary 6.10 in symmetrized form reads as follows.

Corollary 6.17. a) *The expansion of the inverse Gaussian $G_\tau^{-1} \in \mathcal{O}$ in terms of symmetric Macdonald–Koornwinder polynomials is given by*

$$G_\tau(x)^{-1} = G_{\sigma\tau}(s_0) \sum_{s \in \mathcal{S}_\tau^+} \mu_\tau^+(s) E_\tau^+(s;x).$$

Here the series converges absolutely and uniformly on compacta of $(\mathbb{C}^\times)^n$.

b) *For all $s \in \mathcal{S}^+ = \mathcal{S}_{\sigma\tau\sigma}^+$, we have the identity*

$$E_{\sigma\tau\sigma}^+(s;Y^\tau)\big(G_\tau^{-1}\big) = \frac{G_{\sigma\tau}(s_0)}{G_{\sigma\tau}(s)} E^+(s;\cdot)G_\tau^{-1}$$

in $\mathcal{O}$.

Proof. The proof is completely analogous to the proof of Corollary 6.10. □

The explicit relations of the symmetric Macdonald–Koornwinder polynomials under permutations of the parameters $\{a,b,c,d\}$ (see Proposition 4.15) can be lifted to the symmetric Cherednik kernel. We give one explicit example.

Proposition 6.18. *With the notation as in Proposition 4.15,*

$$\mathfrak{E}_\beta^+(\gamma,x) = \left(\prod_{i=1}^n \frac{\big(act^{2(n-i)}, qt^{2(i-n)}/bd;q\big)_\infty}{\big(bct^{2(n-i)}, qt^{2(i-n)}/ad;q\big)_\infty}\right) \frac{G_{\beta_{\sigma\tau}}(\gamma)}{G_{\alpha_{\sigma\tau}}(\gamma)} \mathfrak{E}_\alpha^+(\gamma,x).$$

Proof. We use the explicit expansion of $\mathfrak{E}^+$ in terms of symmetric Macdonald–Koornwinder polynomials (see Proposition 6.14) to prove the proposition. We consider what happens to each term in this expansion sum when replacing the difference multiplicity function α by β.

Observe that the spectral points $s_\lambda^\tau \in \mathcal{S}_\tau^+$ are invariant under replacement of α by β. By Proposition 4.15 the Macdonald–Koornwinder polynomials in the expansion sum transform as

$$E^+_{\beta_{\sigma\tau}}(s_\lambda^\tau;\gamma) = E^+_{\alpha_{\sigma\tau}}(s_\lambda^\tau;\gamma),$$
$$E^+_{\beta_\tau}(s_\lambda^\tau;x) = \left(\prod_{i=1}^n \frac{\big(act^{2(n-i)}, qat^{2(n-i)}/d;q\big)_{\lambda_i}}{\big(bct^{2(n-i)}, qbt^{2(n-i)}/d;q\big)_{\lambda_i}}\left(\frac{b}{a}\right)^{\lambda_i}\right) E_{\alpha_\tau}(s_\lambda^\tau;x)$$

for $\lambda \in \Lambda^+$. The weights $\mu_\tau^+(s_\lambda^\tau)$ ($\lambda \in \Lambda^+$) can be expressed in terms of q-shifted factorials in view of the formulas (6.20), (6.9), (4.17) and (5.17); compare with the formula (6.4) in the proof of Proposition 6.1. From this it follows by a straightforward computation that

$$\frac{\mu^+_{\beta_\tau}(s_\lambda^\tau)}{\mu^+_{\alpha_\tau}(s_\lambda^\tau)} = \prod_{i=1}^n \frac{\big(act^{2(n-i)}, qt^{2(i-n)}/bd;q\big)_\infty \big(bct^{2(n-i)}, qbt^{2(n-i)}/d;q\big)_{\lambda_i}}{\big(bct^{2(n-i)}, qt^{2(i-n)}/ad;q\big)_\infty \big(act^{2(n-i)}, qat^{2(n-i)}/d;q\big)_{\lambda_i}}\left(\frac{a}{b}\right)^{\lambda_i}.$$

Since $G_{\beta_\tau}(x) = G_{\alpha_\tau}(x)$, the proposition now follows by a direct computation using the explicit series expansion for $\mathfrak{E}^+$ (see Proposition 6.14). □

Remark 6.19. The behavior of the symmetric Macdonald–Koornwinder polynomials and of the symmetric Cherednik kernel under permutations of the Askey–Wilson parameters $\{a,b,c,d\}$ (see Proposition 4.15 and Proposition 6.18, respectively) can be extended to the nonsymmetric level. For the Macdonald–Koornwinder polynomials, one now uses the evaluation formula for nonsymmetric Macdonald–Koornwinder polynomials as proved in [41, Thm. 9.3]. We do not give the formulas explicitly, since we do not need them in the present paper.

We end this subsection by relating the one variable setup ($n = 1$) to the theory of basic hypergeometric series. Note first that the roots of medium length in R_{nr} are no longer present for rank one. In particular, the parameter t in the difference multiplicity function α disappears. Furthermore, the rank one double affine Hecke algebra involves two difference reflection operators, namely T_0 and $T_n = T_1$. The associated Y-operator is $Y = Y_1 = T_1T_0$, see e.g. [37] for details.

In rank one, the action of $Y + Y^{-1} \in \mathcal{A}_+(Y) \subset \mathcal{H}^+$ on W_0-invariant meromorphic functions on $\mathbb{C}^\times$ (i.e., meromorphic functions f satisfying $f(x^{-1}) = f(x)$), essentially coincides with the action of the standard Askey–Wilson second-order q-difference operator involving four parameters $\{a,b,c,d\}$, which are related to the difference multiplicity function α via formula (4.7), see e.g., [37, Prop. 5.8]. In particular, $\mathfrak{E}^+(\gamma,\cdot) \in \mathcal{M}_+$ is a meromorphic eigenfunction of the Askey–Wilson second order q-difference operator, which admits an explicit series expansion in terms of the Askey–Wilson polynomials, see Proposition 6.14. On the other hand, a basis of eigenfunctions for the Askey–Wilson second-order q-difference operator was given explicitly by Ismail and Rahman [21] in terms of very-well-poised ${}_8\phi_7$ basic hypergeometric series, see also Suslov [44]. Here the very-well-poised ${}_8\phi_7$ basic hypergeometric series, denoted by ${}_8W_7$, is defined by

$$ {}_8W_7\big(a;b,c,d,e,f;q,z\big)=\sum_{k=0}^{\infty}\frac{1-aq^{2k}}{1-a}\frac{\big(a,b,c,d,e,f;q\big)_k z^k}{\big(q,qa/b,qa/c,qa/d,qa/e,qa/f;q\big)_k}, $$

see [16] for details. To relate these solutions with $\mathfrak{E}^+$, we thus need to evaluate the explicit series expansion for $\mathfrak{E}^+$ (see Proposition 6.14) in terms of ${}_8\phi_7$ basic hypergeometric series. This was done in [43]. The result is as follows, see [43, Thm. 4.2] and use (4.21).

Theorem 6.20. *In the one variable setup* ($n=1$)*, the symmetric Cherednik kernel* $\mathfrak{E}^+$ *can be written as*

$$ \begin{aligned} \mathfrak{E}^+(\gamma,x)=&\frac{\big(qax\gamma/\tilde{d},qa\gamma/\tilde{d}x,q/ad,qa/d;q\big)_\infty}{\big(\tilde{a}\tilde{b}\tilde{c}\gamma,q\gamma/\tilde{d},qx/d,q/dx;q\big)_\infty}\\ &\qquad\times {}_8W_7\big(\tilde{a}\tilde{b}\tilde{c}\gamma/q;ax,a/x,\tilde{a}\gamma,\tilde{b}\gamma,\tilde{c}\gamma;q,q/\tilde{d}\gamma\big) \end{aligned} $$

when $|q/\tilde{d}\gamma|<1$*, where we have denoted* $\{a,b,c,d\}$ *and* $\{\tilde{a},\tilde{b},\tilde{c},\tilde{d}\}$ *for the Askey–Wilson parameters* (4.7) *associated to* α *and* α_σ*, respectively.*

Remark 6.21. a) The meromorphic continuation for the expression of $\mathfrak{E}^+$ as a ${}_8W_7$ series can be written explicitly as a sum of two balanced ${}_4\phi_3$'s using Bailey's formula [16, (2.10.10)], see [43, (3.2)].

b) Theorem 6.20 shows that the Cherednik kernel $\mathfrak{E}^+$ in rank one ($n=1$) is the special eigenfunction of the Askey–Wilson second-order q-difference operator named the *Askey–Wilson function* in [27] and [43] (which was defined in these papers in terms of the ${}_8W_7$ series). In [26] the Askey–Wilson function was interpreted as a spherical function on the noncompact quantum group $\mathrm{SU}_q(1,1)$. In particular, the kernel $\mathfrak{E}^+$ in rank one may be regarded as the natural q-analogue of the Jacobi function (see also [28]).

7. An extension of the Macdonald–Koornwinder transform

In this section we study a difference Fourier transform which is closely related to the Macdonald–Koornwinder transform (see Section 4). In fact, one essentially replaces in the Macdonald–Koornwinder transform $F_{\mathcal{A}}$ the kernel $\mathfrak{E}_{\mathcal{A},\ddagger}$ by the normalized Cherednik kernel $\mathfrak{E}_\ddagger$, and the cyclic module $\mathcal{A}$ by the cyclic $\mathcal{H}$-module $\mathcal{A}G^{-1}$. We follow closely the general line of arguments for difference Fourier transforms as explained in Section 3. The results in this section generalize results of Cherednik [5], [8] to the nonreduced setup, as well as results in [43] to the multivariable setup.

Throughout this section we keep the same assumptions on the difference multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}}) = (t_0, u_0, t_n, u_n, t, q^{\frac{1}{2}})$ as in Section 5.

7.1. The transform. Recall the definition of the contour $\mathcal{T} = \mathcal{T}_\alpha$ in Subsection 4.3. We fix in this section such a contour $\mathcal{T}$, satisfying the additional requirement that the parameter $d/q = -q^{-\frac{1}{2}} t_0 u_0^{-1}$ is in the exterior of $\mathcal{T}$. It is convenient to fix the contour $\mathcal{T}$ once and for all, although the main definitions in this section are easily seen to be independent of this choice.

The extra assumption on the fixed contour $\mathcal{T} = \mathcal{T}_\alpha$ implies that the pairings $\big(\cdot,\cdot\big)_{\mathcal{A}} = \big(\cdot,\cdot\big)_{\mathcal{A},\alpha}$ and $\big(\cdot,\cdot\big)_{\mathcal{A},\tau}$ on $\mathcal{A}$ can be written as integrals over the same deformed torus $\mathcal{T}^n$,

$$\begin{aligned} \big(p,r\big)_{\mathcal{A}} &= \frac{1}{(2\pi i)^n} \iint\limits_{\mathcal{T}^n} p(x) r(x^{-1}) \Delta(x) \frac{dx}{x}, \\ \big(p,r\big)_{\mathcal{A},\tau} &= \frac{1}{(2\pi i)^n} \iint\limits_{\mathcal{T}^n} p(x) r(x^{-1}) \Delta_\tau(x) \frac{dx}{x} \end{aligned} \tag{7.1}$$

for $p, r \in \mathcal{A}$. We furthermore use the formulas (7.1) as the definitions of $\big(p,r\big)_{\mathcal{A}}$ and $\big(p,r\big)_{\mathcal{A},\tau}$ for those meromorphic functions p and r which are regular on $\mathcal{T}^n$.

Consider now the subspace $V = V_\alpha = \mathcal{A}G^{-1} \subset \mathcal{O}$. For all $X \in \mathcal{H} = \mathcal{H}_\alpha$ we have

$$X\big(pG^{-1}\big) = \big(\tau(X)p\big)G^{-1}, \qquad p \in \mathcal{A},$$

hence $\mathcal{A}G^{-1}$ is a cyclic $\mathcal{H}$-module, with cyclic vector G^{-1}. Let $\mathfrak{E}_\ddagger$ be the normalized Cherednik kernel associated to α. Since $G_{\sigma\tau}(\gamma)^{-1} G_\tau(x)^{-1} \mathfrak{E}_\ddagger(\gamma, x)$ is analytic at $(\gamma, x) \in (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$ and $G_\tau \in \mathcal{M}$ is regular on $\mathcal{T}^n$, we may define the transform $F = F_\alpha : \mathcal{A}G^{-1} \to \mathcal{O}G_{\sigma\tau}$ by

$$\big(Fg\big)(\gamma) = \big(g, \mathfrak{E}_\ddagger(\gamma^{-1}, \cdot)\big)_{\mathcal{A}}, \qquad g \in \mathcal{A}G^{-1}. \tag{7.2}$$

We collect some elementary properties of F.

Lemma 7.1. a) *For all* $p \in \mathcal{A}$,

$$F\big(pG^{-1}\big)(\gamma) = \big(p, \mathfrak{E}_\ddagger(\gamma^{-1}, \cdot) G_\tau^{-1}\big)_{\mathcal{A},\tau}. \tag{7.3}$$

b) $F : \mathcal{A}G^{-1} \to \mathcal{M}$ *is a Fourier transform associated with* σ.

Proof. a) By a direct computation using the explicit expression of the weight function Δ (see (4.13) and (4.15)) and of the Gaussian G, we have

$$G(x)^{-1} G_\tau(x) \Delta(x) = \Delta_\tau(x). \tag{7.4}$$

The claim is now immediate in view of the special choice of contour $\mathcal{T}$.

b) The advantage of the expression (7.3) is that the kernel

$$\mathfrak{E}_\ddagger(\gamma^{-1}, \cdot) G_\tau^{-1} = \mathfrak{E}_\ddagger(\gamma^{-1}, \cdot) G_\ddagger$$

is analytic on $(\mathbb{C}^\times)^n$ for generic $\gamma\in(\mathbb{C}^\times)^n$, hence the adjoint of $X\in\mathcal{H}_\tau$ acting on p in the pairing (7.3) is $\ddagger_\tau(X)$ (see Remark 4.7). We now compute for $X\in\mathcal{H}$,

$$\begin{aligned}F\big(X(pG^{-1})\big)(\gamma)&=F\big((\tau(X)p)G^{-1}\big)(\gamma)\\&=\big(\tau(X)p,\mathfrak{E}_\ddagger(\gamma^{-1},\cdot)G_\ddagger\big)_{\mathcal{A},\tau}\\&=\big(p,(\ddagger_\tau\circ\tau)(X)\big(\mathfrak{E}_\ddagger(\gamma^{-1},\cdot)G_\ddagger\big)\big)_{\mathcal{A},\tau}\\&=\big(Z\big(F(pG^{-1})\big)\big)(\gamma),\end{aligned}$$

with $Z\in\mathcal{H}_\sigma$ given by

$$Z=(\dagger_{\ddagger\sigma}\circ\psi_\ddagger\circ\tau_\ddagger^{-1}\circ\ddagger_\tau\circ\tau)(X)=\sigma(X).$$

Here the last equality follows by computing the left-hand side and the right-hand side explicitly on a set of algebraic generators of $\mathcal{H}$. □

Consider the subspace $W_\sigma=W_{\alpha_\sigma}=\mathcal{A}G_{\sigma\tau}\subset\mathcal{M}$. For all $X\in\mathcal{H}_\sigma$ we have

$$X(pG_{\sigma\tau})=(\tau_{\sigma\tau}^{-1}(X)p)G_{\sigma\tau},\qquad p\in\mathcal{A},$$

hence $\mathcal{A}G_{\sigma\tau}$ is a cyclic $\mathcal{H}_\sigma$-module with cyclic vector $G_{\sigma\tau}$. The expansion formula (6.6) for the normalized Cherednik kernel $\mathfrak{E}_\ddagger$ leads now to the following result.

Proposition 7.2. *The difference Fourier transform F defines a linear bijection $F:\mathcal{A}G^{-1}\to\mathcal{A}G_{\sigma\tau}$. Explicitly, we have for all $s\in\mathcal{S}_\tau=\mathcal{S}_{\sigma\tau}$,*

$$F\big(E_\tau(s;\cdot)G^{-1}\big)(\gamma)=D_0\,G_{\tau\sigma\tau}(s)E_{\sigma\tau}(s;\gamma)G_{\sigma\tau}(\gamma),\tag{7.5}$$

with D_0 the constant

$$D_0=\mathcal{C}(s_0^{\ddagger\sigma})\prod_{i=1}^n\frac{\big(t^2,bct^{2(n-i)},dt^{2(i-n)}/a,qt^{2(i-n)}/ad;q\big)_\infty}{\big(q,t^{2(n-i+1)},abt^{2(n-i)},act^{2(n-i)};q\big)_\infty},\tag{7.6}$$

where we used the Askey–Wilson parametrization (4.7) *for part of the difference multiplicity function α.*

Proof. We use the series expansion (6.6) for the kernel $\mathfrak{E}_\ddagger$ together with (7.3) and the orthogonality relations (4.8) for the Macdonald–Koornwinder polynomials. Then we obtain for $s\in\mathcal{S}_\tau$,

$$\begin{aligned}F\big(E_\tau(s;\cdot)G^{-1}\big)(\gamma)&=G_{\sigma\tau}(\gamma)\sum_{v\in\mathcal{S}_\tau}\mu_\tau(v)\big(E_\tau(s;\cdot),E_{\ddagger\tau}(v^{-1};\cdot)\big)_{\mathcal{A},\tau}E_{\sigma\tau}(v;\gamma)\\&=\mu_\tau(s)\big(E_\tau(s;\cdot),E_{\ddagger\tau}(s^{-1};\cdot)\big)_{\mathcal{A},\tau}E_{\sigma\tau}(s;\gamma)G_{\sigma\tau}(\gamma).\end{aligned}$$

By the explicit expression (6.7) for μ_τ and by Theorem 4.11b), this simplifies to

$$F\big(E_\tau(s;\cdot)G^{-1}\big)(\gamma)=D_0\,G_{\tau\sigma\tau}(s)E_{\sigma\tau}(s;\gamma)G_{\sigma\tau}(\gamma),$$

with the constant D_0 given by

$$D_0 = \frac{C_0^\tau\big(1,1\big)_{\mathcal{A},\tau}}{G_{\tau\sigma\tau}(s_0^\tau)}.$$

Now D_0 can be evaluated explicitly using (4.23), (4.29), Remark 5.7 and (6.9). Using furthermore the fact that $\mathcal{C}_\tau(s_0^{\ddagger\tau\sigma}) = \mathcal{C}(s_0^{\ddagger\sigma})$, we obtain the explicit expression (7.6) for the constant D_0. □

7.2. The inverse transform. The next step is to invert the difference Fourier transform $F : \mathcal{A}G^{-1} \to \mathcal{A}G_{\sigma\tau}$. We define a transform $J_\sigma = J_{\alpha_\sigma}$ by the formula

$$\big(J_\sigma g\big)(x) := [g, \mathfrak{E}(\cdot, x)]_{\mathcal{A},\sigma} = \sum_{s\in\mathcal{S}_\ddagger} g(s)\mathfrak{E}(s,x)N_\sigma(s) \tag{7.7}$$

for $g \in \mathcal{A}G_{\sigma\tau}$. The defining sum (7.7) converges absolutely and uniformly for x in compacta of $(\mathbb{C}^\times)^n$. This follows from the alternative expression

$$\big(J_\sigma g\big)(x) = [g, \mathfrak{E}_{\mathcal{A}}(\cdot, x)]_{\mathcal{A},\sigma}, \qquad g \in \mathcal{A}G_{\sigma\tau} \tag{7.8}$$

(see Theorem 6.9), combined with the bounds for the Gaussian and for the Macdonald–Koornwinder polynomials $\mathfrak{E}_{\mathcal{A}}(s,x) = E(s^{-1};x)$ ($s \in \mathcal{S}_\ddagger$), see (5.18) and Proposition 5.13 respectively. In particular, J_σ defines a linear map $J_\sigma : \mathcal{A}G_{\sigma\tau} \to \mathcal{O}$.

Lemma 7.3. *$J_\sigma : \mathcal{A}G_{\sigma\tau} \to \mathcal{O}$ is a Fourier transform associated with σ^{-1}.*

Proof. This follows from (7.8), Lemma 4.2, Remark 4.9, Proposition 4.4 and the fact that $\sigma^{-1} = \psi_\sigma \circ \iota_\sigma$ as unital algebra isomorphisms from $\mathcal{H}_\sigma$ to $\mathcal{H}$. □

Proposition 7.4. *The difference Fourier transform J_σ defines a bijection $J_\sigma : \mathcal{A}G_{\sigma\tau} \to \mathcal{A}G^{-1}$. Explicitly, we have for $s \in \mathcal{S}_\tau = \mathcal{S}_{\sigma\tau}$,*

$$J_\sigma\big(E_{\sigma\tau}(s;\cdot)G_{\sigma\tau}\big)(x) = E_0\, G_{\tau\sigma\tau}(s)^{-1}E_\tau(s;x)G(x)^{-1},$$

with the constant $E_0 \in \mathbb{C}$ given by

$$\begin{aligned} E_0 = N_\sigma(s_0^\ddagger)\prod_{i=1}^n \Bigg\{ &\frac{\big(abcdt^{2(2n-i-1)};q\big)_\infty}{\big(adt^{2(n-i)}, bdt^{2(n-i)}, cdt^{2(n-i)};q\big)_\infty} \\ &\times \frac{1}{\big(bct^{2(n-i)}, bct^{2(n-i)}, dt^{2(i-n)}/a, qt^{2(i-n)}/ad;q\big)_\infty}\Bigg\}, \end{aligned} \tag{7.9}$$

where we used the Askey–Wilson parametrization (4.7) *for part of the difference multiplicity function α.*

Proof. In view of (7.8), (6.6), (4.5) and the definition (6.7) for the weight $\mu \in \mathcal{F}(\mathcal{S})$, we easily see that

$$\begin{aligned} J_\sigma\bigl(E_{\sigma\tau}(s;\cdot)G_{\sigma\tau}\bigr)(x) &= \frac{G_{\sigma\tau}(s_0)N_\sigma(s_0^\ddagger)}{C_0}\,G_{\tau\sigma\tau}(s)^{-1}\mathfrak{E}_\tau(s^{-1},x)G(x)^{-1}\\ &= \frac{G_{\sigma\tau}(s_0)N_\sigma(s_0^\ddagger)}{C_0}\,G_{\tau\sigma\tau}(s)^{-1}E_\tau(s;x)G(x)^{-1}, \end{aligned}$$

where we used the polynomial reduction in the second equality (see Theorem 6.9). By Remark 5.7 and (6.9), the coefficient $G_{\sigma\tau}(s_0)N_\sigma(s_0^\ddagger)/C_0$ is easily seen to be equal to the constant E_0 as defined in (7.9). $\square$

Recall the notation $c_{\mathcal{A}} = \bigl(1,1\bigr)_{\mathcal{A}}N_\sigma(s_0^{-1})$ from Theorem 4.11.

Corollary 7.5. *The difference Fourier transform $F : \mathcal{A}G^{-1} \to \mathcal{A}G_{\sigma\tau}$ is a linear bijection, with inverse $c_{\mathcal{A}}^{-1}J_\sigma : \mathcal{A}G_{\sigma\tau} \to \mathcal{A}G^{-1}$.*

Proof. Proposition 7.2 and Proposition 7.4 imply that $F\circ J_\sigma = D_0E_0\,\mathrm{Id}$ on $\mathcal{A}G_{\sigma\tau}$ and $J_\sigma\circ F = D_0E_0\,\mathrm{Id}$ on $\mathcal{A}G^{-1}$. It thus suffices to show that $D_0E_0 = c_{\mathcal{A}}$. Now by (4.29) and by the explicit expressions (7.6), (7.9) for D_0 and E_0, it follows that

$$D_0E_0 = N_\sigma(s_0^\ddagger)\frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2^n n!}\bigl(1,1\bigr)_{\mathcal{A},+} = N_\sigma(s_0^\ddagger)\bigl(1,1\bigr)_{\mathcal{A}} = c_{\mathcal{A}}, \tag{7.10}$$

where the second equality follows from (4.23). $\square$

7.3. Plancherel-type formulas. In this subsection we prove Plancherel-type formulas for the transform F and its inverse $c_{\mathcal{A}}^{-1}J_\sigma$. For this, we introduce two new transforms $\widetilde{F} : \mathcal{A}G^{-1} \to \mathcal{O}G_{\sigma\tau}$ and $\widetilde{J}_\sigma : \mathcal{A}G_{\sigma\tau} \to \mathcal{O}$ by

$$\bigl(\widetilde{F}g\bigr)(\gamma) = \bigl(\mathfrak{E}(\gamma,\cdot),g\bigr)_{\mathcal{A}}, \qquad \bigl(\widetilde{J}_\sigma h\bigr)(x) = [I\mathfrak{E}_\ddagger(\cdot,x),h]_{\mathcal{A},\sigma} \tag{7.11}$$

for $g\in\mathcal{A}G^{-1}$ and $h\in\mathcal{A}G_{\sigma\tau}$, with I the inversion operator $(Ig)(\gamma) = g(\gamma^{-1})$. Repeating the arguments of the previous subsections lead to the following result.

Proposition 7.6. *The transform $\widetilde{F}$ defines a linear bijection $\widetilde{F} : \mathcal{A}G^{-1} \to \mathcal{A}G_{\sigma\tau}$, whose inverse is given by $c_{\mathcal{A}}^{-1}\widetilde{J}_\sigma : \mathcal{A}G_{\sigma\tau} \to \mathcal{A}G^{-1}$. Explicitly, we have*

$$\begin{aligned} \widetilde{F}\bigl(E_{\ddagger\tau}(s^{-1};\cdot)G^{-1}\bigr)(\gamma) &= D_0\,G_{\tau\sigma\tau}(s)E_{\sigma\tau}(s;\gamma)G_{\sigma\tau}(\gamma),\\ \widetilde{J}_\sigma\bigl(E_{\sigma\tau}(s;\cdot)G_{\sigma\tau}\bigr)(x) &= E_0\,G_{\tau\sigma\tau}(s)^{-1}E_{\ddagger\tau}(s^{-1};x)G(x)^{-1} \end{aligned}$$

for $s\in\mathcal{S}_\tau = \mathcal{S}_{\sigma\tau}$, with the constants $D_0\in\mathbb{C}$ and $E_0\in\mathbb{C}$ given by (7.6) *and* (7.9), *respectively.*

The transforms $\widetilde{F}$ and $\widetilde{J}_\sigma$ are defined in such a way that

$$[Fg,h]_{\mathcal{A},\sigma} = \bigl(g,\widetilde{J}_\sigma h\bigr)_{\mathcal{A}}, \qquad \bigl(J_\sigma h,g\bigr)_{\mathcal{A}} = [h,\widetilde{F}g]_{\mathcal{A},\sigma}$$

for all $g \in \mathcal{A}G^{-1}$ and $h \in \mathcal{A}G_{\sigma\tau}$. This can be proved by interchanging integration and summation using Fubini's Theorem, which is justified by the polynomial reduction of the Cherednik kernels (Theorem 6.9) and by the bounds for the Macdonald–Koornwinder polynomials (Proposition 5.13). This leads now immediately to the following Plancherel-type formulas.

Proposition 7.7. a) *Let* $g, h \in \mathcal{A}G^{-1}$. *Then*

$$[Fg, \widetilde{F}h]_{\mathcal{A},\sigma} = c_{\mathcal{A}}\big(g, h\big)_{\mathcal{A}}.$$

b) *Let* $g, h \in \mathcal{A}G_{\sigma\tau}$. *Then*

$$\big(J_\sigma g, \widetilde{J}_\sigma h\big)_{\mathcal{A}} = c_{\mathcal{A}}[g, h]_{\mathcal{A},\sigma}.$$

Combining Proposition 7.7 with Proposition 7.2, Proposition 7.6 and (7.10) leads to the following formulas involving Macdonald–Koornwinder polynomials.

Corollary 7.8. *Let* $s, v \in \mathcal{S}_\tau = \mathcal{S}_{\sigma\tau}$. *Then*

$$D_0 G_{\tau\sigma\tau}(s) G_{\tau\sigma\tau}(v)\, [E_{\sigma\tau}(s;\cdot)G_{\sigma\tau}, E_{\sigma\tau}(v;\cdot)G_{\sigma\tau}]_{\mathcal{A},\sigma} = \\ = E_0\big(E_\tau(s;\cdot)G^{-1}, E_{\ddagger\tau}(v^{-1};\cdot)G^{-1}\big)_{\mathcal{A}},$$

with D_0 *and* E_0 *given by* (7.6) *and* (7.9), *respectively.*

The explicit formulas of Corollary 7.8 are the analogues of the orthogonality relations (4.8) and quadratic norm evaluations (Theorem 4.11b)) for the Macdonald–Koornwinder polynomials.

With the main results for the difference Fourier transform $F : \mathcal{A}G^{-1} \to \mathcal{A}G_{\sigma\tau}$ now established, we can make the connection with the Macdonald–Koornwinder transform $F_{\mathcal{A}}$ and its inverse $c_{\mathcal{A}}^{-1}J_{\mathcal{A},\sigma}$ more explicit. Since the inverse transform J_σ is a discrete transform supported on the polynomial spectrum $\mathcal{S}_\ddagger$, we may as well consider F and J_σ as maps

$$F_{res} : \mathcal{A}G^{-1} \to \big(\mathcal{A}G_{\sigma\tau}\big)|_{\mathcal{S}_\ddagger}, \qquad J_{res,\sigma} : \big(\mathcal{A}G_{\sigma\tau}\big)|_{\mathcal{S}_\ddagger} \to \mathcal{A}G^{-1} \tag{7.12}$$

by restriction of the spectral variable γ to $\mathcal{S}_\ddagger$. Here the space $\big(\mathcal{A}G_{\sigma\tau}\big)|_{\mathcal{S}_\ddagger}$ is again a (cyclic) $\mathcal{H}_\sigma$-submodule of $\mathcal{F}(\mathcal{S}_\ddagger)$ by Lemma 4.2. The transforms F_{res} and $J_{res,\sigma}$ are then Fourier transforms associated to σ and σ^{-1}, respectively, and $c_{\mathcal{A}}^{-1}J_{res,\sigma}$ is the inverse of the transform F_{res}. Moreover, the transforms F_{res} and $J_{res,\sigma}$ can be expressed as

$$\begin{aligned} \big(F_{res}g\big)(s) &= \big(g, \mathfrak{E}_{\mathcal{A},\ddagger}(s^{-1},\cdot)\big)_{\mathcal{A}}, \qquad s \in \mathcal{S}_\ddagger, \\ \big(J_{res,\sigma}h\big)(x) &= [h, \mathfrak{E}_{\mathcal{A}}(\cdot, x)]_{\mathcal{A},\sigma} \end{aligned} \tag{7.13}$$

for $g \in \mathcal{A}G^{-1}$ and $h \in \big(\mathcal{A}G_{\sigma\tau}\big)|_{\mathcal{S}_\ddagger}$, which coincide with the defining formulas for the Macdonald–Koornwinder transform $F_{\mathcal{A}} : \mathcal{A} \to \mathcal{F}_0(\mathcal{S}_\ddagger)$ and the discrete transform $J_{\mathcal{A},\sigma} : \mathcal{F}_0(\mathcal{S}_\ddagger) \to \mathcal{A}$.

In particular, the Macdonald–Koornwinder transform and its extension to the cyclic $\mathcal{H}$-module $\mathcal{A}G^{-1}$ can be treated together in a uniform manner by considering difference Fourier transforms on the $\mathcal{H}$-module $V = V_\alpha$ defined by

$$V = \mathcal{A} \oplus \mathcal{A}G^{-1} \subset \mathcal{M},$$

and on the $\mathcal{H}_\sigma$-submodule $W_\sigma = W_{\alpha_\sigma}$ defined by

$$W_\sigma = \mathcal{F}_0(\mathcal{S}_\ddagger) \oplus \big(\mathcal{A}G_{\sigma\tau}\big)_{|\mathcal{S}_\ddagger} \subset \mathcal{F}(\mathcal{S}_\ddagger)$$

(clearly, the sum is direct in both cases). The transforms $F_{res} : V \to W_\sigma$ and $J_{res,\sigma} : W_\sigma \to V$ are then defined by (7.13), now with $g \in V$ and $h \in W_\sigma$. These extended transforms $F_{res} : V \to W_\sigma$ and $J_{res,\sigma} : W_\sigma \to V$ are Fourier transforms associated to σ and σ^{-1}, respectively, and $c_{\mathcal{A}}^{-1} J_{res,\sigma}$ is the inverse of F_{res}. Furthermore, applying Fubini's Theorem we have the Plancherel-type formulas

$$\begin{aligned} [F_{res}g, \widetilde{F}_{res}h]_{\mathcal{A},\sigma} &= c_{\mathcal{A}}\big(g,h\big)_{\mathcal{A}}, \qquad g,h \in V,\\ \big(J_{res,\sigma}g, \widetilde{J}_{res,\sigma}h\big)_{\mathcal{A}} &= c_{\mathcal{A}}[g,h]_{\mathcal{A},\sigma}, \qquad g,h \in W_\sigma, \end{aligned} \tag{7.14}$$

with $\widetilde{F}_{res} : V \to W_\sigma$ and $\widetilde{J}_{res,\sigma} : W_\sigma \to V$ the transforms

$$\big(\widetilde{F}_{res}g\big)(s) = \big(\mathfrak{E}_{\mathcal{A}}(s,\cdot), g\big)_{\mathcal{A}}, \qquad \big(\widetilde{J}_{res,\sigma}h\big)(x) = [I\mathfrak{E}_{\mathcal{A},\ddagger}(\cdot,x), h]_{\mathcal{A},\sigma}$$

for $s \in \mathcal{S}_\ddagger$, $g \in V$ and $h \in W_\sigma$, where I is the inversion operator $(Ig)(v) = g(v^{-1})$. By the explicit expressions for the images of suitable bases of V and W_σ under F_{res}, $\widetilde{F}_{res}$ and $J_{res,\sigma}$, $\widetilde{J}_{res,\sigma}$, respectively (see Section 4 as well as this section), the Plancherel-type formulas (7.14) lead to the orthogonality relations and quadratic norm evalutions of the Macdonald–Koornwinder polynomials and to the formulas of Corollary 7.8. It also leads to "mixed identities", for which g and h in (7.14) are taken from different summands in V (respectively W_σ). These mixed identities are completely covered by the following integral formulas for Macdonald–Koornwinder polynomials.

Proposition 7.9. *For $v \in \mathcal{S} = \mathcal{S}_{\sigma\tau\sigma}$ and $s \in \mathcal{S}_\tau$,*

$$\big(E(v;\cdot), E_{\ddagger\tau}(s^{-1};\cdot)G^{-1}\big)_{\mathcal{A}} = D_0 G_{\sigma\tau}(v) G_{\tau\sigma\tau}(s) E_{\sigma\tau}(s;v^{-1}).$$

Proof. Let $v \in \mathcal{S}$ and $s \in \mathcal{S}_\tau$. Let $\delta_{v^{-1}} \in \mathcal{F}_0(\mathcal{S}_\ddagger)$ be the function which is one at $v^{-1} \in \mathcal{S}_\ddagger$ and zero otherwise. Then (4.8) and Theorem 4.11b) imply that

$$F_{res}(E(v;\cdot)) = \big(E(v;\cdot), E_\ddagger(v^{-1};\cdot)\big)_{\mathcal{A}} \delta_{v^{-1}} = \frac{c_{\mathcal{A}}}{N_\sigma(v^{-1})}\,\delta_{v^{-1}}.$$

On the other hand,

$$\widetilde{F}_{res}(E_{\ddagger\tau}(s^{-1};\cdot)G^{-1}) = D_0 G_{\tau\sigma\tau}(s) E_{\sigma\tau}(s;\cdot) G_{\sigma\tau}.$$

Combining these two formulas with the first Plancherel-type formula in (7.14) leads to the desired identity. □

7.4. The symmetric theory. The results on the extended Macdonald–Koornwinder transform $F : \mathcal{A}G^{-1} \to \mathcal{A}G_{\sigma\tau}$ and its inverse $J_\sigma : \mathcal{A}G_{\sigma\tau} \to \mathcal{A}G^{-1}$ can be symmetrized in the usual manner by applying the symmetrizer $C_+ \in H_0 \subset \mathcal{H}$. We collect the main formulas in this subsection.

We define the symmetric transforms $F^+ = F^+_\alpha : \mathcal{A}_+G^{-1} \to \mathcal{M}$ and $J^+_\sigma = J^+_{\alpha_\sigma} : \mathcal{A}_+G_{\sigma\tau} \to \mathcal{M}$ by the formulas

$$\begin{aligned} \big(F^+g\big)(\gamma) &= \big(g, \mathfrak{E}^+(\gamma,\cdot)\big)_{\mathcal{A},+}, \\ \big(J^+_\sigma h\big)(x) &= [h, \mathfrak{E}^+(\cdot,x)]_{\mathcal{A},+,\sigma} = [h, \mathfrak{E}^+_\sigma(x,\cdot)]_{\mathcal{A},+,\sigma}, \end{aligned} \tag{7.15}$$

for $g \in \mathcal{A}_+G^{-1}$ and $h \in \mathcal{A}_+G_{\sigma\tau}$, where the integration for the pairing $\big(\cdot,\cdot\big)_{\mathcal{A},+}$ (see (4.24)) is over the deformed torus $\mathcal{T}^n$ with $\mathcal{T}$ as in the previous subsection, and with $[\cdot,\cdot]_{\mathcal{A},+} = [\cdot,\cdot]_{\mathcal{A},+,\alpha}$ defined by

$$[f,g]_{\mathcal{A},+} = \sum_{s \in \mathcal{S}^+_\sigma} f(s)g(s)N^+(s^{-1})$$

for functions f and g such that the sum is absolutely convergent. By standard arguments (cf. Subsection 4.5) we obtain

$$\begin{aligned} \big(Fg\big)(\gamma) = \big(\widetilde{F}g\big)(\gamma) &= \frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2^n n!}\big(F^+g\big)(\gamma), \qquad g \in \mathcal{A}_+G^{-1}, \\ \big(J_\sigma h\big)(x) = \big(\widetilde{J}_\sigma h\big)(x) &= \mathcal{C}_\sigma(s_0^{\ddagger})\big(J^+_\sigma h\big)(x), \qquad h \in \mathcal{A}_+G_{\sigma\tau}. \end{aligned} \tag{7.16}$$

We obtain from these formulas the following result.

Theorem 7.10. *The transform F^+ defines a linear bijection $F^+ : \mathcal{A}_+G^{-1} \to \mathcal{A}_+G_{\sigma\tau}$, whose inverse is given by $(c^+_{\mathcal{A}})^{-1}J^+_\sigma : \mathcal{A}_+G_{\sigma\tau} \to \mathcal{A}_+G^{-1}$, where the constant $c^+_{\mathcal{A}}$ is given by*

$$c^+_{\mathcal{A}} = N^+_\sigma(s_0^{-1})\big(1,1\big)_{\mathcal{A},+}. \tag{7.17}$$

Furthermore, we have the Plancherel formulas

$$\begin{aligned} [F^+g, F^+h]_{\mathcal{A},+,\sigma} &= c^+_{\mathcal{A}}\,\big(g,h\big)_{\mathcal{A},+}, \qquad g,h \in \mathcal{A}_+G^{-1}, \\ \big(J^+_\sigma g, J^+_\sigma h\big)_{\mathcal{A},+} &= c^+_{\mathcal{A}}\,[g,h]_{\mathcal{A},+,\sigma}, \qquad g,h \in \mathcal{A}_+G_{\sigma\tau}. \end{aligned}$$

Proof. First note that for functions $g,h \in \mathcal{F}(\mathcal{S}_\ddagger)$ which are W_0-invariant under the dot-action,

$$[g,h]_{\mathcal{A},\sigma} = \mathcal{C}_\sigma(s_0^{\ddagger})[g,h]_{\mathcal{A},+,\sigma},$$

provided that the sums absolutely converge, cf. (4.27). An analogous statement holds true for $\big(\cdot,\cdot\big)_{\mathcal{A}}$, see (4.23). By the inversion formula and Plancherel formula for F and by (7.16), it then suffices to note that

$$\frac{2^n n!}{\mathcal{C}(s_0^{\ddagger\sigma})\mathcal{C}_\sigma(s_0^{\ddagger})}\,c_{\mathcal{A}} = N^+_\sigma(s_0^{-1})\big(1,1\big)_{\mathcal{A},+}, \tag{7.18}$$

which follows from (4.16) and (4.23). □

We finish the subsection by symmetrizing the explicit formulas in Proposition 7.2, Proposition 7.4, Corollary 7.8 and Proposition 7.9. We define constants D_0^+ and E_0^+ by

$$D_0^+ = 2^n n! \frac{D_0}{\mathcal{C}(s_0^{\ddagger\sigma})}, \qquad E_0^+ = \frac{E_0}{\mathcal{C}_\sigma(s_0^{\ddagger})} \tag{7.19}$$

with D_0 and E_0 given by (7.6) and (7.9), respectively. Note that $D_0^+ E_0^+ = c_{\mathcal{A}}^+$ by (7.10) and (7.18). Using standard symmetrization techniques, one obtains the following proposition.

Proposition 7.11. *Let $s, u \in \mathcal{S}_\tau^+ = \mathcal{S}_{\sigma\tau}^+$ and $v \in \mathcal{S}^+ = \mathcal{S}_{\sigma\tau\sigma}^+$.*

a) *The transform F^+ satisfies*

$$F^+\big(E_\tau^+(s;\cdot)G^{-1}\big)(\gamma) = D_0^+\, G_{\tau\sigma\tau}(s) E_{\sigma\tau}^+(s;\gamma) G_{\sigma\tau}(\gamma).$$

b) *The transform J_σ^+ satisfies*

$$J_\sigma^+\big(E_{\sigma\tau}^+(s;\cdot)G_{\sigma\tau}\big)(x) = E_0^+\, G_{\tau\sigma\tau}(s)^{-1} E_\tau^+(s;x) G(x)^{-1}.$$

c) *The following identities are valid:*

$$D_0^+ G_{\tau\sigma\tau}(s) G_{\tau\sigma\tau}(u)\, [E_{\sigma\tau}^+(s;\cdot)G_{\sigma\tau}, E_{\sigma\tau}^+(u;\cdot)G_{\sigma\tau}]_{\mathcal{A},+,\sigma} = \\ = E_0^+ \big(E_\tau^+(s;\cdot)G^{-1}, E_\tau^+(u;\cdot)G^{-1}\big)_{\mathcal{A},+}.$$

d) *We have the integral evaluations*

$$\big(E^+(v;\cdot), E_\tau^+(s;\cdot)G^{-1}\big)_{\mathcal{A},+} = D_0^+ G_{\sigma\tau}(v) G_{\tau\sigma\tau}(s) E_{\sigma\tau}^+(s;v).$$

8. The (non)symmetric Askey–Wilson function transform

In this section we restrict attention to rank one ($n = 1$). We study nonsymmetric analogues of the (spherical) Fourier transform on the noncompact quantum SU(1, 1) group, following the general philosophy of Section 3.

Recall that the Jacobi function transform is a generalized Fourier transform with kernel given by the Jacobi function. It has an interpretation (for certain discrete parameter values) as the spherical Fourier transform on SU(1, 1), see e.g., [29]. In recent papers of Koelink and the author, see [26], [27] and [28], a Fourier transform was defined and studied which admits an interpretation as a (spherical) Fourier transform on the noncompact quantum group $\mathrm{SU}_q(1,1)$. The transform, named the Askey–Wilson function transform, is an integral transform with kernel given by the so-called Askey–Wilson function.

By Theorem 6.20 and Remark 6.21b), the Askey–Wilson function is precisely the symmetric rank one Cherednik kernel $\mathfrak{E}^+$. This observation naturally leads to nonsymmetric variants of the Askey–Wilson function transform, defined as integral transforms involving the normalized rank one Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_{\ddagger}$. These transforms qualify as difference Fourier transforms in the sense of Section 3. The underlying spaces are given explicitly as direct sums of two cyclic $\mathcal{H}$-modules.

In Subsection 8.1 we define the bilinear forms and the cyclic $\mathcal{H}$-modules. In Subsection 8.2 we define the (non)symmetric Askey–Wilson function transforms. In Subsection 8.3 we analyze the transforms on the first, "classical" cyclic $\mathcal{H}$-module, which reduces to (the rank one case) of the extended Macdonald–Koornwinder transform as discussed in the previous section. In Subsection 8.4 we compute the image of the cyclic vector of the second, "strange" $\mathcal{H}$-module under the transforms. In Subsection 8.5 we prove algebraic Plancherel and inversion formulas for the (non)symmetric Askey–Wilson function transform. In Subsection 8.6 we show how these results can be extended to the L^2-level for the symmetric Askey–Wilson function transform, yielding new proofs for the main results of [27].

In this section we keep the same generic assumptions on the difference multiplicity function $\alpha = (\mathbf{t}, q^{\frac{1}{2}}) = (t_0, u_0, t_1, u_1, q^{\frac{1}{2}})$ as in Section 5. Recall that in the rank one setup ($n = 1$), the roots in R_{nr} of medium length have disappeared, hence the associated parameter t in the multiplicity function $\mathbf{t}$ disappears (see [37] for the detailed treatment of the polynomial theory in the rank one setup). In addition, an extra parameter $e \in \mathbb{C}^\times$ enters in the definition of the nonsymmetric Askey–Wilson function transform, which we only assume to be generic (unless stated explicitly otherwise).

8.1. The bilinear forms. It is well known from the theory of elliptic functions that there exists, up to a multiplicative constant, a unique meromorphic function $\mathcal{P}_e = \mathcal{P}_e^\alpha \in \mathcal{M} = \mathcal{M}(\mathbb{C}^\times)$ satisfying the invariance properties

$$\mathcal{P}_e(x^{-1}) = \mathcal{P}_e(x), \qquad \mathcal{P}_e(qx) = \mathcal{P}_e(x),$$

and with divisor on the elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}}$ (written multiplicatively) given by

$$\mathrm{Div}(\mathcal{P}_e) = (d) + (d^{-1}) - (e) - (e^{-1}).$$

Here and in the remainder of this section we use the Askey–Wilson parametrization

$$\{a, b, c, d\} = \{t_1 u_1, t_1 u_1^{-1}, q^{\frac{1}{2}} t_0 u_0, -q^{\frac{1}{2}} t_0 u_0^{-1}\} \tag{8.1}$$

for the difference multiplicity function α, cf. (4.7). The function $\mathcal{P}_e$ is closely related to the Weierstrass $\mathcal{P}$-function. We fix $\mathcal{P}_e$ here by defining it as a quotient of Jacobi theta-functions. For this we introduce the notation $\theta(y_1, y_2, \ldots, y_n) = \theta(y_1)\theta(y_2)\cdots\theta(y_n)$ for products of the renormalized Jacobi theta function

$$\theta(x) = \big(x, q/x; q\big)_\infty. \tag{8.2}$$

Then we fix $\mathcal{P}_e = \mathcal{P}_e^\alpha \in \mathcal{M}$ uniquely as the quotient

$$\mathcal{P}_e(x) = \frac{\theta(dx, dx^{-1})}{\theta(ex, ex^{-1})}. \tag{8.3}$$

Since $\mathcal{P}_e \in \mathcal{M}$ is $\mathcal{W}$-invariant, the associated multiplication operator in $\mathrm{End}_{\mathbb{C}}(\mathcal{M})$ commutes with the action of the double affine Hecke algebra $\mathcal{H}$ on $\mathcal{M}$.

We now define new weight functions $W_e = W_e^\alpha \in \mathcal{M}$ and $W_e^+ = W_e^{+,\alpha} \in \mathcal{M}$ by

$$W_e(x) = \mathcal{P}_e(x)\Delta(x), \qquad W_e^+(x) = \mathcal{P}_e(x)\Delta^+(x) \tag{8.4}$$

with $\Delta = \Delta_\alpha$ the weight function for the rank one Macdonald–Koornwinder polynomials (see (4.6)) and $\Delta^+ = \Delta_\alpha^+$ the weight function for the symmetric rank one Macdonald–Koornwinder polynomials (see (4.14)). Observe that the weight function W_e^+ is W_0-invariant, i.e., $W_e^+(x^{-1}) = W_e^+(x)$. In terms of q-shifted factorials, the weight function Δ is given by

$$\Delta(x) = \frac{\big(x^2, qx^{-2}; q\big)_\infty}{\big(ax, qax^{-1}, bx, qbx^{-1}, cx, cx^{-1}, dx, dx^{-1}; q\big)_\infty},$$

see (4.13), (4.14) and (4.15), whence $W_e \in \mathcal{M}$ can be expressed as

$$W_e(x) = \frac{\big(x^2, qx^{-2}, qx/d, q/dx; q\big)_\infty}{\big(ax, qax^{-1}, bx, qbx^{-1}, cx, cx^{-1}; q\big)_\infty \theta(ex, ex^{-1})}.$$

Similarly the weight function W_e^+ is given by

$$W_e^+(x) = \frac{\big(x^2, x^{-2}, qx/d, q/dx; q\big)_\infty}{\big(ax, ax^{-1}, bx, bx^{-1}, cx, cx^{-1}; q\big)_\infty \theta(ex, ex^{-1})}.$$

For $\epsilon > 0$ sufficiently small, let $C_\epsilon \subset \mathbb{C}$ be a closed, counterclockwise oriented rectifiable Jordan curve around the origin $0 \in \mathbb{C}$ satisfying $C_\epsilon^{-1} = C_\epsilon$ (set-theoretically), and containing the sequences

$$\{aq^m, bq^m, cq^m, q^{1+m}/d \mid m \in \mathbb{N}\} \cup \{eq^m \mid m \in \mathbb{Z},\ |eq^m| < \epsilon^{-1}\} \tag{8.5}$$

in its interior, respectively the eq^m $(m \in \mathbb{Z})$ with $|eq^m| \geq \epsilon^{-1}$ in its exterior. In the special case that a, b, c and q/d have moduli ≤ 1, we may choose the contour C_ϵ to be the unit circle in the complex plane with two deformations, one to include the poles eq^m with moduli $< \epsilon^{-1}$ and one to exclude the poles $e^{-1}q^m$ with moduli $> \epsilon$.

For such $\epsilon > 0$ sufficiently small, we define now two pairings as follows. For meromorphic functions $g, h \in \mathcal{O}G_\tau$ we define the pairing $\langle g, h\rangle_e^\epsilon = \langle g, h\rangle_e^{\epsilon,\alpha}$ by the formula

$$\langle g, h\rangle_e^\epsilon = \frac{1}{2\pi i}\int_{C_\epsilon} g(x)h(x^{-1})W_e(x)\frac{dx}{x}. \tag{8.6}$$

Similarly, we define the pairing $\langle g, h\rangle_{e,+}^\epsilon = \langle g, h\rangle_{e,+}^{\epsilon,\alpha}$ for meromorphic functions $g, h \in \mathcal{O}G_\tau$ by

$$\langle g, h\rangle_{e,+}^\epsilon = \frac{1}{2\pi i}\int_{C_\epsilon} g(x)h(x)W_e^+(x)\frac{dx}{x}. \tag{8.7}$$

By Cauchy's Theorem, the pairings $\langle g, h\rangle_e^\epsilon$ and $\langle g, h\rangle_{e,+}^\epsilon$ are independent of the choice of contour C_ϵ satisfying the specific defining conditions as stated above.

We now define two subspaces $V_e^{cl} = V_e^{cl,\alpha}$ and $V^{str} = V^{str,\alpha}$ of $\mathcal{O}G_\tau$ by

$$V_e^{cl} = \mathcal{A}\mathcal{P}_e^{-1}G^{-1}, \qquad V^{str} = \mathcal{A}G_\tau. \tag{8.8}$$

Furthermore, we write $V_{e,+}^{cl} = V_{e,+}^{cl,\alpha}$ and $V_+^{str} = V_+^{str,\alpha}$ for the associated subspaces of W_0-invariant functions,

$$V_{e,+}^{cl} = \mathcal{A}_+\mathcal{P}_e^{-1}G^{-1}, \qquad V_+^{str} = \mathcal{A}_+G_\tau. \tag{8.9}$$

The superscripts *cl* and *str* stand for "classical" and "strange" respectively. This terminology is motivated by the fact that V_e^{cl} (respectively V^{str}) covers the contributions of the Plancherel measure of the spherical Fourier transform on the quantum group $\mathrm{SU}_q(1,1)$ which arise from the unitary principal series representations (respectively strange series representations), see e.g., [26] and [43].

Observe that V_e^{cl} and V^{str} are cyclic $\mathcal{H}$-submodules of $\mathcal{M}$ with corresponding cyclic vectors $\mathcal{P}_e^{-1}G^{-1}$ and G_τ, respectively. In fact, for any $p \in \mathcal{A}$ and $X \in \mathcal{H}$ we have

$$\begin{aligned} X\big(p\mathcal{P}_e^{-1}G^{-1}\big) &= \big(\tau(X)p\big)\mathcal{P}_e^{-1}G^{-1}, \\ X\big(pG_\tau\big) &= \big(\tau_\tau^{-1}(X)p\big)G_\tau \end{aligned} \tag{8.10}$$

and the action of $\mathcal{H}_\tau$ preserves the subspace $\mathcal{A}$ of Laurent polynomials. Any function $g \in V_e^{cl}$ vanishes at points $x \in (eq^{\mathbb{Z}})^{\pm 1}$, hence $V_e^{cl} \cap V^{str} = \{0\}$. We define the $\mathcal{H}$-module $M_e = M_e^\alpha$ by

$$M_e = V_e^{cl} \oplus V^{str} \subset \mathcal{O}G_\tau. \tag{8.11}$$

The subspace $M_e^+ = M_e^{+,\alpha} \subset M_e$ consisting of W_0-invariant functions in M_e is given by the direct sum

$$M_e^+ = V_{e,+}^{cl} \oplus V_+^{str}.$$

Observe that the identity $G_{\ddagger\tau} = G^{-1}$ implies that M_e is also $\mathcal{H}_\ddagger$-stable under the natural action of $\mathcal{H}_\ddagger$ on $\mathcal{M}$ as q^{-1}-difference reflection operators. The explicit formulas are given by

$$\begin{aligned} X\big(p\mathcal{P}_e^{-1}G^{-1}\big) &= \big(\tau_{\ddagger\tau}^{-1}(X)p\big)\mathcal{P}_e^{-1}G^{-1}, \\ X\big(pG_\tau\big) &= \big(\tau_\ddagger(X)p\big)G_\tau \end{aligned}$$

for $p \in \mathcal{A}$ and for $X \in \mathcal{H}_\ddagger$.

Lemma 8.1. *For $g, h \in M_e$, the limits*

$$\langle g, h\rangle_e = \lim_{\epsilon \searrow 0} \langle g, h\rangle_e^\epsilon, \qquad \langle g, h\rangle_{e,+} = \lim_{\epsilon \searrow 0} \langle g, h\rangle_{e,+}^\epsilon$$

exist.

Proof. We prove the lemma for the pairing $\langle \cdot, \cdot\rangle_{e,+}$ (the proof for $\langle \cdot, \cdot\rangle_e$ is similar). Let $g, h \in M_e$. Let $\epsilon > 0$ be sufficiently small. We may rewrite the integral over C_ϵ in the definition of $\langle g, h\rangle_{e,+}^\epsilon$ by an integral over an (ϵ-independent) closed, rectifiable

Jordan curve C on the cost of picking up residues at points of the form eq^l and $e^{-1}q^{-l}$ for $m_\epsilon \le l \le l_0$, where $m_\epsilon \in \mathbb{Z}$ is the smallest integer such that $|eq^{m_\epsilon}| < \epsilon^{-1}$, and with $l_0 \in \mathbb{Z}$ some suitably chosen, fixed integer. Hence convergence of the limit will follow from the convergence of the series

$$\sum_{l \le l_0} g(y_l^{\pm 1}) h(y_l^{\pm 1}) \operatorname*{Res}_{x=y_l^{\pm 1}} \left(\frac{W_e^+(x)}{x} \right), \tag{8.12}$$

with $y_l = eq^l$. In case that $g \in V_e^{cl}$ or $h \in V_e^{cl}$ these sums vanish. When $g, h \in V^{str} = \mathcal{A}G_\tau$, then

$$|g(y_l^{\pm 1})| \le c_0 c_1^{|l|} q^{l^2/2}, \qquad \forall\, l \le l_0 \tag{8.13}$$

for some constants $c_0, c_1 > 0$, and similarly for h. Furthermore,

$$\left| \operatorname*{Res}_{x=y_l^{\pm 1}} \left(\frac{W_e^+(x)}{x} \right) \right| \le d_0 d_1^{|l|}, \qquad \forall\, l \le l_0$$

for some constants $d_0, d_1 > 0$ (which e.g., follows from the explicit expression [27, (5.8)] for the residue of the weight function W_e^+ at $y_l^{\pm 1}$). Hence the Gaussian contributions $q^{l^2/2}$ in the asymptotics of g and h force the convergence of (8.12). This completes the proof of the lemma. □

Proposition 8.2. *The bilinear form $\langle \cdot, \cdot \rangle_e$ on M_e induces the anti-isomorphism $\ddagger$ on $\mathcal{H}$. In other words,*

$$\langle Xg, h \rangle_e = \langle g, \ddagger(X) h \rangle_e,$$

for $X \in \mathcal{H}$ and $g, h \in M_e$.

Proof. It suffices to prove the proposition for $X = T_0$ and $X = T_1$. By the explicit form of the operators T_j, we have for $j = 0$ and for $j = 1$,

$$\langle T_j g, h \rangle_e^\epsilon - \langle g, \ddagger(T_j) h \rangle_e^\epsilon = \frac{t_j^{-1}}{2\pi i} \int_{C_\epsilon} \left\{ (r_j g)(x) \widetilde{h}(x) - g(x) (r_j \widetilde{h})(x) \right\} c_{a_j}(x) W_e(x) \frac{dx}{x},$$

with $\widetilde{h}(x) = h(x^{-1})$ and with the r_j's acting as constant coefficient q-difference reflection operators. For $j = 1$, the function $x \mapsto c_{a_1}(x) W_e(x)$ is r_1-invariant, hence the right-hand side vanishes by the inversion-invariance of the contour C_ϵ. Taking the limit $\epsilon \searrow 0$ and using Lemma 8.1 then yields $\langle T_1 g, h \rangle_e = \langle g, \ddagger(T_1) h \rangle_e$.

For $j = 0$, the function $x \mapsto c_{a_0}(x) W_e(x)$ is r_0-invariant, hence

$$\langle T_0 g, h \rangle_e^\epsilon - \langle g, \ddagger(T_0) h \rangle_e^\epsilon = \frac{-t_0^{-1}}{2\pi i} \int_{C_\epsilon - qC_\epsilon} g(x) \widetilde{h}(qx^{-1}) c_{a_0}(x) W_e(x) \frac{dx}{x}.$$

A straightforward computation shows that the poles in $\mathbb{C}^\times$ of the integrand are simple and contained in the zero set of

$$x \mapsto \left(ax, qax^{-1}, bx, qbx^{-1}, cx, qcx^{-1}, qx/d, q^2/dx; q \right)_\infty \theta(ex, ex^{-1}).$$

Now let $m \in \mathbb{Z}$ be the smallest integer satisfying $|eq^m| < \epsilon^{-1}$. By the assumptions on the contour C_ϵ and by Cauchy's Theorem, we pick up poles at eq^m and at $e^{-1}q^{1-m}$ when shifting C_ϵ to qC_ϵ, whence

$$\begin{aligned}\langle T_0 g, h\rangle_e^\epsilon - \langle g, \ddagger(T_0)h\rangle_e^\epsilon &= \\ &= t_0^{-1}\big(g(q^{1-m}e^{-1})\widetilde{h}(eq^m) - g(eq^m)\widetilde{h}(q^{1-m}e^{-1})\big)\operatorname*{Res}_{x=eq^m}\left(\frac{c_{a_0}(x)W_e(x)}{x}\right).\end{aligned}$$

Now take the limit $\epsilon \searrow 0$ and use the bounds derived in the proof of Lemma 8.1. This implies that $\langle T_0 g, h\rangle_e = \langle g, \ddagger(T_0)h\rangle_e$, as desired. □

Remark 8.3. Observe that the conditions on the functions g and h in Lemma 8.1 and Proposition 8.2 may be relaxed. For instance, Lemma 8.1 and Proposition 8.2 hold true when $g \in M_e$ and $h \in \mathcal{O}G_\tau$ with h satisfying

$$|h(y_l^{\pm 1})| \leq c_0 c_1^{|l|}, \qquad \forall\, l \leq l_0 \tag{8.14}$$

for some constants $c_0, c_1 > 0$, where $l_0 \in \mathbb{Z}$ is some arbitrary, fixed integer and $y_l = eq^l$. If $h = h_\gamma \in \mathcal{O}G_\tau$ furthermore depends analytically on an additional parameter $\gamma \in \mathbb{C}^\times$ and (8.14) holds true uniformly for γ in compacta of $\mathbb{C}^\times$, then the proof of Lemma 8.1 in addition implies that $\langle g, h_\gamma\rangle_e$ depends analytically on $\gamma \in \mathbb{C}^\times$, for all $g \in M_e$.

8.2. The transforms. In order to define the (non)symmetric Askey–Wilson function transform we need to establish certain bounds for the normalized Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ associated to α and $\alpha_\ddagger$ respectively, as well as for the normalized symmetric Cherednik kernel $\mathfrak{E}^+$. In fact, we prove bounds for the analytic functions $G_{\sigma\tau}^{-1}\mathfrak{E}(\cdot, y_l^{\pm 1})$, $G_{\sigma\tau}^{-1}\mathfrak{E}_\ddagger(\cdot, y_l^{\pm 1})$ and $G_{\sigma\tau}^{-1}\mathfrak{E}^+(\cdot, y_l)$ in $l \in \mathbb{Z}$, where $y_l = eq^l$.

Lemma 8.4. *For any compact set $K \subset (\mathbb{C}^\times)^n$, there exist constants $C, D > 0$ (depending on K), such that*

$$\begin{aligned}|G_{\sigma\tau}(\gamma)^{-1}\mathfrak{E}(\gamma, y_l^{\pm 1})| &\leq CD^{|l|},\\ |G_{\sigma\tau}(\gamma)^{-1}\mathfrak{E}_\ddagger(\gamma, y_l^{\pm 1})| &\leq CD^{|l|},\\ |G_{\sigma\tau}(\gamma)^{-1}\mathfrak{E}^+(\gamma, y_l)| &\leq CD^{|l|},\end{aligned}$$

for all $l \in \mathbb{Z}$ and all $\gamma \in K$, where $y_l = eq^l$.

Proof. Recurrence relations for the normalized Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_\ddagger$ are essentially the same as Pieri formulas for the polynomial kernel $\mathfrak{E}_{\mathcal{A}}$, since the kernels satisfy the same transformation behaviour under the action of the double affine Hecke algebra (see Proposition 4.4 and Theorem 5.17). Hence the proof of the bounds for the Macdonald–Koornwinder polynomials (see Proposition 5.13 and the appendix) can be easily adjusted to obtain the desired bounds for the analytic functions $G_{\sigma\tau}^{-1}\mathfrak{E}(\cdot, y_l^{\pm 1})$ and $G_{\sigma\tau}^{-1}\mathfrak{E}_\ddagger(\cdot, y_l^{\pm 1})$. We leave the precise details to the reader. The bounds for $\mathfrak{E}^+$ follows easily from the bounds for $\mathfrak{E}$, using e.g., the formula $\mathfrak{E}^+(\gamma, x) = \big(C_+\mathfrak{E}(\gamma,\cdot)\big)(x)$. □

Proposition 8.5. a) *The assignment*

$$\big(\mathcal{F}_e g\big)(\gamma) = \langle g, \mathfrak{E}_\ddagger(\gamma^{-1},\cdot)\rangle_e$$

for $g \in M_e$ defines a linear map $\mathcal{F}_e = \mathcal{F}_e^\alpha : M_e \to \mathcal{O}G_{\sigma\tau}$. Furthermore, $\mathcal{F}_e$ is a Fourier transform associated with σ.

b) *The assignment*

$$\big(\mathcal{J}_e g\big)(\gamma) = \langle g, I\mathfrak{E}(\gamma,\cdot)\rangle_e$$

for $g \in M_e$, with $(Ih)(x) = h(x^{-1})$, defines a linear map $\mathcal{J}_e = \mathcal{J}_e^\alpha : M_e \to \mathcal{O}G_{\sigma\tau}$. Furthermore, $\mathcal{J}_e$ is a Fourier transform associated with σ_σ^{-1}.

Proof. a) Remark 8.3 and Lemma 8.4 imply that the map

$$\gamma \mapsto \langle g, G_{\sigma\tau}(\gamma)^{-1}\mathfrak{E}_\ddagger(\gamma^{-1},\cdot)\rangle_e$$

for fixed $g \in M_e$ defines an analytic function on $\mathbb{C}^\times$. Hence $\mathcal{F}_e$ defines a linear map $\mathcal{F}_e : M_e \to \mathcal{O}G_{\sigma\tau}$. By Proposition 8.2, Remark 8.3 and the general arguments of Section 3 it is clear that $\mathcal{F}_e$ is a Fourier transform associated with σ. The proof of b) is similar. □

Corollary 8.6. *The assignment*

$$\big(\mathcal{F}_e^+ g\big)(\gamma) = \langle g, \mathfrak{E}^+(\gamma,\cdot)\rangle_{e,+}$$

for $g \in M_e^+$ defines a linear map $\mathcal{F}_e^+ = \mathcal{F}_e^{+,\alpha} : M_e^+ \to \mathcal{O}G_{\sigma\tau}$. Furthermore,

$$\mathcal{F}_e g = \mathcal{J}_e g = \frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2}\mathcal{F}_e^+ g$$

for all $g \in M_e^+$.

Proof. By similar arguments as in the proof of Proposition 8.5, we have that $\mathcal{F}_e^+$ defines a linear map $\mathcal{F}_e^+ : M_e^+ \to \mathcal{O}G_{\sigma\tau}$. By (4.18), Lemma 6.13, Proposition 8.2, Remark 8.3 and Theorem 6.15a), we obtain

$$\big(\mathcal{F}_e g\big)(\gamma) = \big(\mathcal{J}_e g\big)(\gamma) = \langle g, \mathfrak{E}^+(\gamma,\cdot)\rangle_e$$

for $g \in M_e^+$. Symmetrizing the integral using (4.22) and the decomposition $W_e(x) = \mathcal{C}(x)W_e^+(x)$ gives

$$\langle g, \mathfrak{E}^+(\gamma,\cdot)\rangle_e = \frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2}\langle g, \mathfrak{E}^+(\gamma,\cdot)\rangle_{e,+}$$

for $g \in M_e^+$, which completes the proof. □

By Theorem 6.20 the transform $\mathcal{F}_e^+$ is, up to a multiplicative constant, precisely the Askey–Wilson function transform as defined and studied in [27]. The only difference is that the transform $\mathcal{F}_e^+$ in [27] is (initially) defined on a certain space of compactly supported functions, while $\mathcal{F}_e^+$ in this subsection is defined on the space M_e^+ (which is a more natural subspace from the viewpoint of double affine Hecke algebras). It turns out though that for restricted parameter values, their continuous extensions to the L^2-level do coincide (see [27], [43] and Subsection 8.6).

In order to distinguish the two transforms $\mathcal{F}_e$ and $\mathcal{F}_e^+$, we use the following terminology throughout the remainder of the paper.

Definition 8.7. a) The transform $\mathcal{F}_e : M_e \to \mathcal{O}G_{\sigma\tau}$ is called the Askey–Wilson function transform.

b) The transform $\mathcal{F}_e^+ : M_e^+ \to \mathcal{O}G_{\sigma\tau}$ is called the symmetric Askey–Wilson function transform.

8.3.The classical part of the transforms. In this subsection we consider the difference Fourier transforms $\mathcal{F}_e$ and $\mathcal{J}_e$ on the cyclic $\mathcal{H}$-submodule $V_e^{cl} \subset M_e$. This is related to the extended (rank one) Macdonald–Koornwinder transforms F and $\widetilde{F}$ of the previous section in the following way.

Lemma 8.8. a) *For* $p \in \mathcal{A}$,

$$\mathcal{F}_e\big(p\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) = F\big(pG^{-1}\big)(\gamma), \qquad \mathcal{J}_e\big(p\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) = \widetilde{F}\big((Ip)G^{-1}\big)(\gamma),$$

where I *is the inversion operator* $(Ig)(x) = g(x^{-1})$.

b) *For* $p \in \mathcal{A}_+$,

$$\mathcal{F}_e^+\big(p\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) = F^+\big(pG^{-1}\big)(\gamma),$$

with F^+ *the extended (rank one) Macdonald–Koornwinder transform defined by* (7.15).

Proof. a) Let $\mathcal{T} = \mathcal{T}_\alpha$ be a deformed circle as defined in Subsection 7.1. For $p \in \mathcal{A}$ we have

$$\begin{aligned}\mathcal{F}_e\big(p\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) &= \frac{1}{2\pi i}\int_{\mathcal{T}_\tau} p(x)\mathfrak{E}_\ddagger(\gamma^{-1}, x^{-1})G(x)^{-1}\Delta(x)\frac{dx}{x}\\ &= \big(pG^{-1}, \mathfrak{E}_\ddagger(\gamma^{-1}, \cdot)\big)_{\mathcal{A}}\\ &= F\big(pG^{-1}\big)(\gamma).\end{aligned}$$

Here the first equality holds by Cauchy's Theorem since $\mathcal{P}_e^{-1}W_e = \Delta$ is regular at the points $x \in (eq^{\mathbb{Z}})^{\pm 1}$. The proof for $\mathcal{J}_e$ is similar.

b) The proof is similar to the proof of a). □

Invoking the main results on the extended Macdonald–Koornwinder transform, we arrive at the following result.

Corollary 8.9. a) *The Askey–Wilson function transform $\mathcal{F}_e$ restricts to a linear bijection $\mathcal{F}_e : V_e^{cl} \to V^{str,\sigma}$. Explicitly, we have*

$$\mathcal{F}_e\big(E_\tau(s;\cdot)\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) = D_0 G_{\tau\sigma\tau}(s)E_{\sigma\tau}(s;\gamma)G_{\sigma\tau}(\gamma) \tag{8.15}$$

for all $s \in \mathcal{S}_\tau = \mathcal{S}_{\sigma\tau}$, with $D_0 = D_0^\alpha$ the constant

$$D_0 = \frac{\big(bc, d/a, q/ad; q\big)_\infty}{\big(q, qab, ac; q\big)_\infty}. \tag{8.16}$$

b) *The difference Fourier transform $\mathcal{J}_e$ restricts to a linear bijection $\mathcal{J}_e : V_e^{cl} \to V^{str,\sigma}$. Explicitly, we have*

$$\mathcal{J}_e\big(I(E_{\ddagger\tau}(s^{-1};\cdot))\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) = D_0 G_{\tau\sigma\tau}(s)E_{\sigma\tau}(s;\gamma)G_{\sigma\tau}(\gamma)$$

for all $s \in \mathcal{S}_\tau$, with $D_0 = D_0^\alpha$ the constant (8.16).

c) *The symmetric Askey–Wilson function transform $\mathcal{F}_e^+$ restricts to a linear bijection $\mathcal{F}_e^+ : V_{e,+}^{cl} \to V_+^{str,\sigma}$. Explicitly, we have*

$$\mathcal{F}_e^+\big(E_\tau^+(s;\cdot)\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) = D_0^+ G_{\tau\sigma\tau}(s)E_{\sigma\tau}^+(s;\gamma)G_{\sigma\tau}(\gamma)$$

for all $s \in \mathcal{S}_\tau^+ = \mathcal{S}_{\sigma\tau}^+$, with $D_0^+ = D_0^{+,\alpha}$ the constant

$$D_0^+ = \frac{2\big(bc, d/a, q/ad; q\big)_\infty}{\big(q, ab, ac; q\big)_\infty}. \tag{8.17}$$

Proof. a) By Lemma 8.8 and Proposition 7.2, formula (8.15) holds with the constant D_0 given by (7.6). Now for rank one ($n = 1$), $\mathcal{C}(s_0^{\ddagger\sigma}) = 1 - ab$, hence (7.6) reduces to (8.16). The proof of b) and c) are similar, now using Proposition 7.6 and Proposition 7.11, respectively. □

8.4. The Fourier transform of the Gaussian. We consider the transforms $\mathcal{F}_e$ and $\mathcal{J}_e$ on the cyclic $\mathcal{H}$-submodule $V^{str} = \mathcal{A}G_\tau$. Since $\mathcal{F}_e$ and $\mathcal{J}_e$ are Fourier transforms associated with σ and σ_σ^{-1} respectively, it suffices to evaluate the image of the cyclic vector G_τ under $\mathcal{F}_e$ and under $\mathcal{J}_e$. This amounts to the same thing, since Corollary 8.6 implies

$$\big(\mathcal{F}_e G_\tau\big)(\gamma) = \big(\mathcal{J}_e G_\tau\big)(\gamma) = \frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2}\big(\mathcal{F}_e^+ G_\tau\big)(\gamma). \tag{8.18}$$

We start with the following one-variable q-analogue of the Macdonald–Mehta integral.

Lemma 8.10. *We have the explicit evaluation*

$$\big(\mathcal{F}_e^+ G_\tau\big)(s_0) = \frac{2}{\big(q, ab, ac, bc; q\big)_\infty}\frac{\theta(abce)}{\theta(ae, be, ce)}.$$

Proof. By the polynomial reduction, we have $\mathfrak{E}^+(s_0,\cdot)=1$, hence

$$\big(\mathcal{F}_e^+G_\tau\big)(s_0)=\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{C_\epsilon}G_\tau(x)W_e^+(x)\frac{dx}{x}. \tag{8.19}$$

Since

$$G_\tau(x)W_e^+(x)=\frac{\big(x^2,1/x^2;q\big)_\infty}{\big(ax,a/x,bx,b/x,cx,c/x;q\big)_\infty\theta(ex,e/x)}, \tag{8.20}$$

the right-hand side of (8.19) can be easily matched with the one variable q-Macdonald–Mehta integral [43, (5.9)], with the parameter u in [43, (5.9)] taken to be e (observe in particular that $\big(\mathcal{F}_e^+G_\tau\big)(s_0)$ does not depend on the parameter d). Its evaluation (see [43, Thm. 5.5]) yields the desired result. □

Remark 8.11. One of the goals in this section is to prove the main results on the symmetric Askey–Wilson function transform (see [27] and [43]) using only affine Hecke algebra techniques and some explicit "constant term" evaluations. It is therefore noteworthy to mention that two proofs of the evaluation of the above Macdonald–Mehta type integral are available which only use some direct basic hypergeometric series manipulations (see [43, Appendix B]).

Proposition 8.12. *The image of the Gaussian G_τ under the symmetric Askey–Wilson function transform $\mathcal{F}_e^+$ is given by*

$$\big(\mathcal{F}_e^+G_\tau\big)(\gamma)=\frac{2\big(q/ad;q\big)_\infty}{\big(q,ab,ac;q\big)_\infty\theta(ae,be,ce,qe/d)}\mathcal{P}^\sigma_{e_\sigma}(\gamma)^{-1}G_\sigma(\gamma)^{-1},$$

with the parameter $e_\sigma\in\mathbb{C}$ defined by

$$e_\sigma=-\frac{q^{\frac12}}{u_0t_1e}.$$

Proof. We consider the W_0-invariant meromorphic function $f\in\mathcal{O}G_\sigma G_{\sigma\tau}$ defined by

$$f(\gamma)=\frac{2}{\mathcal{C}(s_0^{\ddagger\sigma})}\,G_\sigma(\gamma)\big(\mathcal{F}_eG_\tau\big)(\gamma)=G_\sigma(\gamma)\big(\mathcal{F}_e^+G_\tau\big)(\gamma). \tag{8.21}$$

We first show that f is invariant under the action of $r_0\in\mathcal{W}$, i.e., that $f(q\gamma^{-1})=f(\gamma)$. We compute

$$\begin{aligned}(T_0^{\sigma\tau}f)(\gamma)&=\frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2}\,G_\sigma(\gamma)\mathcal{F}_e\big(G_\tau(\tau_\tau^{-1}\circ\sigma^{-1}\circ\tau_\sigma^{-1})(T_0^{\sigma\tau})(1)\big)(\gamma)\\&=\frac{\mathcal{C}(s_0^{\ddagger\sigma})}{2}\,G_\sigma(\gamma)\mathcal{F}_e\big(G_\tau(Y_1^\tau T_1^{\tau\,-1})(1)\big)(\gamma)\\&=u_0f(\gamma),\end{aligned}$$

since $Y_1^\tau T_1^{\tau -1}$ acts on $1 \in \mathcal{A}$ as multiplication by the constant $u_0 t_1 t_1^{-1} = u_0$. By the explicit form of the q-difference reflection operator $T_0^{\sigma\tau}$, we conclude that f is r_0-invariant. Hence $f \in \mathcal{M}$ is $\mathcal{W}$-invariant. So f may be regarded as a meromorphic function on the elliptic curve $T = \mathbb{C}^\times/q^{\mathbb{Z}}$ (written multiplicatively). The only possible poles of f on T are at most simple and located at $-q^{\frac{1}{2}+\mathbb{Z}}u_1u_0^{-1}$ and $-q^{-\frac{1}{2}+\mathbb{Z}}u_1^{-1}u_0$, since $f \in \mathcal{O}G_\sigma G_{\sigma\tau}$ and

$$G_\sigma(\gamma)G_{\sigma\tau}(\gamma) = \frac{1}{\theta\big(-q^{\frac{1}{2}}u_1u_0^{-1}\gamma, -q^{\frac{1}{2}}u_1u_0^{-1}\gamma^{-1}\big)}.$$

By standard elliptic function theory it follows that

$$f(\gamma) = C\mathcal{P}_{\tilde{e}}^\sigma(\gamma)^{-1} = C\,\frac{\theta\big(\tilde{e}\gamma, \tilde{e}\gamma^{-1}\big)}{\theta\big(-q^{\frac{1}{2}}u_1u_0^{-1}\gamma, -q^{\frac{1}{2}}u_1u_0^{-1}\gamma^{-1}\big)} \tag{8.22}$$

for some $\tilde{e} \in \mathbb{C}^\times$ and some $C \in \mathbb{C}$. The choice of $\tilde{e} \in \mathbb{C}^\times$ in (8.22) is not unique. In fact, if (8.22) is valid for $\tilde{e} = u$ for some choice of constant C, then all other possible choices for $\tilde{e}$ are given by $(uq^{\mathbb{Z}})^{\pm 1}$.

We fix now a pair $(\tilde{e}, C)$ such that equation (8.22) is satisfied. The next step is to derive explicit identities for C and $\tilde{e}$ using the evaluation of the Macdonald–Mehta type integral (see Lemma 8.10) and using symmetries in the four Askey–Wilson parameters a, b, c and d. These additional identities in C and $\tilde{e}$ can be solved explicitly and lead to the explicit expressions for C and $\tilde{e}$ as given in the statement of the proposition.

We first evaluate $f(s_0) = f(t_0t_1)$ in two different ways. Since $G_\sigma(t_0t_1) = \big(ad, q/bc; q\big)_\infty^{-1}$, we have by (8.21) and by Lemma 8.10,

$$f(t_0t_1) = \frac{2}{\big(q, ab, ac, ad; q\big)_\infty \theta(bc)}\frac{\theta(abce)}{\theta(ae, be, ce)}.$$

In particular we have $f(t_0t_1) \neq 0$, which implies that $\tilde{e} \notin (t_0t_1q^{\mathbb{Z}})^{\pm 1}$ and $C \neq 0$.

On the other hand, by (8.22),

$$f(t_0t_1) = C\frac{\theta(t_0t_1\tilde{e}, t_0^{-1}t_1^{-1}\tilde{e})}{\theta(ad, bc)}.$$

Combining the two identities, we obtain an expression for the constant C in terms of $\tilde{e}$:

$$C = \frac{2\big(q/ad; q\big)_\infty}{\big(q, ab, ac; q\big)_\infty}\frac{\theta(abce)}{\theta(ae, be, ce, t_0t_1\tilde{e}, t_0^{-1}t_1^{-1}\tilde{e})}. \tag{8.23}$$

In order to find $\tilde{e}$ explicitly, we mimic the previous approach by evaluating $f(-t_0^{-1}t_1)$ in two different ways. In order to do so, we consider a new multiplicity function β by

$$\beta = (t_0, u_0, t_1, -u_1^{-1}, q^{\frac{1}{2}}),$$

i.e., the value u_1 of the multiplicity function $\mathbf{t}$ at the orbit $\mathcal{W}a_1^\vee$ is replaced by $-u_1^{-1}$. In terms of the Askey–Wilson parametrization (8.1), this amounts to interchanging the role of a and b. Note that $\beta_\sigma = (-u_1^{-1}, u_0, t_1, t_0, q^{\frac{1}{2}})$, so that $s_0^{\beta_\sigma} = -u_1^{-1}t_1$. Using the duality and the polynomial reduction of the symmetric Cherednik kernel $\mathfrak{E}^+$ (see Theorem 6.15), as well as the symmetry of $\mathfrak{E}^+$ when interchanging the role of a and b (see Proposition 6.18), we derive

$$\begin{aligned}\mathfrak{E}^+_{\alpha_\sigma}(-u_1^{-1}t_1, \gamma) &= \mathfrak{E}^+_\alpha(\gamma, -u_1^{-1}t_1)\\ &= \frac{\big(bc, q/ad; q\big)_\infty}{\big(ac, q/bd; q\big)_\infty}\frac{G_{\alpha_{\sigma\tau}}(\gamma)}{G_{\beta_{\sigma\tau}}(\gamma)}\mathfrak{E}^+_\beta(\gamma, -u_1^{-1}t_1)\\ &= \frac{\big(bc, q/ad; q\big)_\infty}{\big(ac, q/bd; q\big)_\infty}\frac{G_{\alpha_{\sigma\tau}}(\gamma)}{G_{\beta_{\sigma\tau}}(\gamma)}.\end{aligned}$$

Interchanging the role of t_0 and u_1 (i.e., replacing α by α_σ) then gives the explicit evaluation formula

$$\mathfrak{E}^+(-t_0^{-1}t_1, x) = \frac{\big(qa/d, q/ad; q\big)_\infty}{\big(ac, c/a; q\big)_\infty}\frac{\big(cx, c/x; q\big)_\infty}{\big(qx/d, q/dx; q\big)_\infty}.$$

Hence we obtain

$$\big(\mathcal{F}_e^+ G_\tau\big)(-t_0^{-1}t_1) = \frac{\big(qa/d, q/ad; q\big)_\infty}{\big(ac, c/a; q\big)_\infty}K, \tag{8.24}$$

with K given by

$$K = \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{C_\epsilon} G_\tau(x)\frac{\big(cx, c/x; q\big)_\infty}{\big(qx/d, q/dx; q\big)_\infty}W_e^+(x)\frac{dx}{x}.$$

It follows from the explicit form (8.20) of the integrand $G_\tau W_e^+$ that K is the Macdonald–Mehta type integral $(\mathcal{F}_e^+ G_\tau)(s_0)$ in which the parameter c is replaced by the parameter q/d. In particular, Lemma 8.10 yields an explicit evaluation for K. Formula (8.21) combined with (8.24) then shows that

$$\begin{aligned}f(-t_0^{-1}t_1) &= \frac{1}{\big(d/b, qa/c; q\big)_\infty}\big(\mathcal{F}_e^+ G_\tau\big)(-t_0^{-1}t_1)\\ &= \frac{\big(qa/d, q/ad; q\big)_\infty}{\big(d/b, ac; q\big)_\infty\theta(c/a)}K\\ &= \frac{2\big(q/ad; q\big)_\infty}{\big(q, ab, ac; q\big)_\infty\theta(d/b, c/a)}\frac{\theta(qabe/d)}{\theta(ae, be, qe/d)}.\end{aligned} \tag{8.25}$$

On the other hand, (8.22) and (8.23) show that

$$\begin{aligned} f(-t_0^{-1}t_1) &= C\frac{\theta(-t_0t_1^{-1}\tilde{e}, -t_0^{-1}t_1\tilde{e})}{\theta(d/b, c/a)} \\ &= \frac{2\big(q/ad;q\big)_\infty}{\big(q,ab,ac;q\big)_\infty\theta(d/b,c/a)}\frac{\theta(abce, -t_0t_1^{-1}\tilde{e}, -t_0^{-1}t_1\tilde{e})}{\theta(ae,be,ce,t_0t_1\tilde{e}, t_0^{-1}t_1^{-1}\tilde{e})}. \end{aligned} \tag{8.26}$$

Comparing (8.25) and (8.26) leads to the identity

$$\frac{\theta(-t_0t_1^{-1}\tilde{e}, -t_0^{-1}t_1\tilde{e})}{\theta(t_0t_1\tilde{e}, t_0^{-1}t_1^{-1}\tilde{e})} = \frac{\theta(qabe/d, ce)}{\theta(abce, qe/d)}. \tag{8.27}$$

In other words, if (8.22) holds true for the pair $(\tilde{e}, C)$, then $\tilde{e}$ is necessarily a solution of (8.27).

It is easy to verify that $\tilde{e} := e_\sigma = -q^{\frac{1}{2}}/u_0t_1e$ is a solution of (8.27), as well as $\tilde{e} = e_\sigma^{-1}$. Furthermore, the left-hand side of (8.27) is an elliptic function in $\tilde{e}$ on $T = \mathbb{C}^\times/q^{\mathbb{Z}}$, hence standard elliptic function theory shows that $\big(e_\sigma q^{\mathbb{Z}}\big)^{\pm 1}$ are all possible solutions for $\tilde{e}$ of equation (8.27). Hence we conclude that $\big(e_\sigma q^{\mathbb{Z}}\big)^{\pm 1}$ are all the possible values for $\tilde{e}$ such that (8.22) holds true for some constant C.

For the choice $\tilde{e} = e_\sigma$ in the equation $(\mathcal{F}_e^+G_\tau)(\gamma) = C\mathcal{P}_{\tilde{e}}^\sigma(\gamma)^{-1}G_\sigma(\gamma)^{-1}$, the corresponding constant C is given by

$$C = \frac{2\big(q/ad;q\big)_\infty}{\big(q,ab,ac;q\big)_\infty\theta(ae,be,ce,qe/d)}$$

in view of (8.23). This completes the proof of the proposition. □

8.5. Algebraic inversion and Plancherel formulas. The results of the previous subsections easily lead to an algebraic inversion formula for the Askey–Wilson function transform. For the formulation we introduce normalization constants $K_e = K_e^\alpha$ and $K_e^+ = K_e^{+,\alpha}$ by

$$\begin{aligned} K_e &= \frac{\big(q,qab,ac;q\big)_\infty}{\big(q/ad;q\big)_\infty}\sqrt{\theta(ae,be,ce,qe/d)}, \\ K_e^+ &= \frac{\big(q,ab,ac;q\big)_\infty}{2\big(q/ad;q\big)_\infty}\sqrt{\theta(ae,be,ce,qe/d)}, \end{aligned} \tag{8.28}$$

where we choose an arbitrary branch of the square root. Observe that the constants K_e and K_e^+ are self-dual, i.e., $K_{e_\sigma}^\sigma = K_e$ and $K_e^+ = K_{e_\sigma}^{+,\sigma}$. We define normalized difference Fourier transforms $\widetilde{\mathcal{F}}_e = \widetilde{\mathcal{F}}_e^\alpha$ and $\widetilde{\mathcal{J}}_e = \widetilde{\mathcal{J}}_e^\alpha$ on M_e by

$$\widetilde{\mathcal{F}}_e = K_e\mathcal{F}_e, \qquad \widetilde{\mathcal{J}}_e = K_e\mathcal{J}_e. \tag{8.29}$$

Similarly, we define the normalized symmetric Askey–Wilson function transform $\widetilde{\mathcal{F}}_e^+ = \widetilde{\mathcal{F}}_e^{+,\alpha}$ on M_e^+ by

$$\widetilde{\mathcal{F}}_e^+ = K_e^+\mathcal{F}_e^+. \tag{8.30}$$

The algebraic inversion formulas can now be formulated as follows.

Theorem 8.13. a) *The normalized Askey–Wilson function transform* $\widetilde{\mathcal{F}}_e$ *defines a linear bijection* $\widetilde{\mathcal{F}}_e : M_e \to M^\sigma_{e_\sigma}$*. Its inverse is given by* $\widetilde{\mathcal{J}}^\sigma_{e_\sigma} : M^\sigma_{e_\sigma} \to M_e$*.*

b) *The normalized symmetric Askey–Wilson function transform* $\widetilde{\mathcal{F}}^+_e$ *defines a linear bijection* $\widetilde{\mathcal{F}}^+_e : M^+_e \to M^{+,\sigma}_{e_\sigma}$*. Its inverse is given by* $\widetilde{\mathcal{F}}^{+,\sigma}_{e_\sigma} : M^{+,\sigma}_{e_\sigma} \to M^+_e$*.*

Proof. a) By Corollary 8.9,

$$\mathcal{F}_e\big(\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) = \mathcal{J}_e\big(\mathcal{P}_e^{-1}G^{-1}\big)(\gamma) = \frac{\big(q/ad;q\big)_\infty}{\big(q,qab,ac;q\big)_\infty} G_{\sigma\tau}(\gamma) \tag{8.31}$$

in view of the explicit expression (8.16) for D_0 and Remark 5.7. Furthermore, by (8.18) and Proposition 8.12,

$$\begin{aligned}\big(\mathcal{F}_e G_\tau\big)(\gamma) &= \big(\mathcal{J}_e G_\tau\big)(\gamma) \\ &= \frac{\big(q/ad;q\big)_\infty}{\big(q,qab,ac;q\big)_\infty \theta(ae,be,ce,qe/d)} \mathcal{P}^\sigma_{e_\sigma}(\gamma)^{-1} G_\sigma(\gamma)^{-1}.\end{aligned} \tag{8.32}$$

It follows from (8.31) and (8.32) that $\mathcal{F}_e$ and $\mathcal{J}_e$ map the cyclic vector of V^{cl}_e (respectively V^{str}) to a multiple of the cyclic vector of $V^{str,\sigma}$ (respectively $V^{cl,\sigma}_{e_\sigma}$). Since $\mathcal{F}_e$ and $\mathcal{J}_e$ are Fourier transforms associated with σ and σ_σ^{-1} respectively, $\mathcal{F}_e$ and $\mathcal{J}_e$ thus define linear mappings

$$\mathcal{F}_e, \mathcal{J}_e : M_e \to M^\sigma_{e_\sigma}. \tag{8.33}$$

Observe that the extended parameter map

$$(\alpha, e) \mapsto (\alpha_\sigma, e_\sigma)$$

is an involution. By the definition (8.28) of the constant K_e and by (8.31), (8.32) and (8.33), we obtain

$$\begin{aligned}\big(\mathcal{J}^\sigma_{e_\sigma} \circ \mathcal{F}_e\big)(\mathcal{P}_e^{-1}G^{-1}) = K_e^{-2}\mathcal{P}_e^{-1}G^{-1} = \big(\mathcal{F}^\sigma_{e_\sigma} \circ \mathcal{J}_e\big)(\mathcal{P}_e^{-1}G^{-1}),\\ \big(\mathcal{J}^\sigma_{e_\sigma} \circ \mathcal{F}_e\big)(G_\tau) = K_e^{-2}G_\tau = \big(\mathcal{F}^\sigma_{e_\sigma} \circ \mathcal{J}_e\big)(G_\tau).\end{aligned}$$

Since $\mathcal{F}_e$ and $\mathcal{J}_e$ are Fourier transforms associated with σ and σ_σ^{-1} respectively, and $G_\tau, \mathcal{P}_e^{-1}G^{-1} \in M_e$ generate M_e as $\mathcal{H}$-module, we conclude that

$$\mathcal{J}^\sigma_{e_\sigma} \circ \mathcal{F}_e = K_e^{-2}\mathrm{Id}|_{M_e} = \mathcal{F}^\sigma_{e_\sigma} \circ \mathcal{J}_e.$$

Combined with (8.33), this completes the proof of part a).

b) This follows from a) and from Corollary 8.6. □

We already noticed that $\widetilde{\mathcal{F}}_e$ and $\widetilde{\mathcal{J}}_e$ restrict to linear bijections $\widetilde{\mathcal{F}}_e, \widetilde{\mathcal{J}}_e : V^{cl}_e \to V^{str,\sigma}$, see Corollary 8.9. By (the proof of) the above theorem, the difference Fourier transforms $\widetilde{\mathcal{F}}_e$ and $\widetilde{\mathcal{J}}_e$ also restricts to linear bijections $\widetilde{\mathcal{F}}_e, \widetilde{\mathcal{J}}_e : V^{str} \to$

$V_{e_\sigma}^{cl,\sigma}$. In a similar fashion it follows that the normalized symmetric Askey–Wilson function transform $\widetilde{\mathcal{F}}_e^+$ restricts to a linear bijection $\widetilde{\mathcal{F}}_e^+ : V^{str,+} \to V_{e_\sigma}^{cl,+,\sigma}$. In the following proposition we describe these transforms on a suitable basis of V^{str} (respectively of $V^{str,+}$).

Proposition 8.14. a) *For $s \in \mathcal{S}_\tau$, we have*

$$\widetilde{\mathcal{F}}_e\big(E_\tau(s;\cdot)G_\tau\big)(\gamma) = \frac{1}{D_0^\sigma K_e G_{\tau\sigma}(s)} E_{\ddagger\sigma\tau}(s^{-1};\gamma^{-1})\mathcal{P}_{e_\sigma}^\sigma(\gamma)^{-1}G_\sigma(\gamma)^{-1},$$
$$\widetilde{\mathcal{J}}_e\big(E_\tau(s;\cdot)G_\tau\big)(\gamma) = \frac{1}{D_0^\sigma K_e G_{\tau\sigma}(s)} E_{\sigma\tau}(s;\gamma)\mathcal{P}_{e_\sigma}^\sigma(\gamma)^{-1}G_\sigma(\gamma)^{-1}$$

with D_0 and K_e the explicit constants (8.16) *and* (8.28) *respectively.*

b) *For $s \in \mathcal{S}_\tau^+$, we have*

$$\widetilde{\mathcal{F}}_e^+\big(E_\tau^+(s;\cdot)G_\tau\big)(\gamma) = \frac{1}{D_0^{+,\sigma} K_e^+ G_{\tau\sigma}(s)} E_{\sigma\tau}^+(s;\gamma)\mathcal{P}_{e_\sigma}^\sigma(\gamma)^{-1}G_\sigma(\gamma)^{-1},$$

with D_0^+ and K_e^+ the explicit constants (8.17) *and* (8.28) *respectively.*

Proof. We derive the formulas for $\mathcal{J}_e$, the other cases are derived in a similar fashion. Let $s \in \mathcal{S}_\tau$. Since $\sigma\tau\sigma = \tau\sigma\tau$ when acting on the difference multiplicity function α, we have by Corollary 8.9 and Theorem 8.13,

$$\begin{aligned}\widetilde{\mathcal{J}}_e\big(E_\tau(s;\cdot)G_\tau\big) &= \frac{1}{D_0^\sigma K_e G_{\tau\sigma}(s)}\big(\widetilde{\mathcal{J}}_e \circ \widetilde{\mathcal{F}}_{e_\sigma}^\sigma\big)\big(E_{\sigma\tau}(s;\cdot)\mathcal{P}_{e_\sigma}^{\sigma\ -1}G_\sigma^{-1}\big)\\ &= \frac{1}{D_0^\sigma K_e G_{\tau\sigma}(s)} E_{\sigma\tau}(s;\cdot)\mathcal{P}_{e_\sigma}^{\sigma\ -1}G_\sigma^{-1},\end{aligned}$$

which is the desired result. □

Remark 8.15. Corollary 8.9 and Proposition 8.14 give an explicit description of the image of a suitable basis of M_e (respectively M_e^+) under the (non)symmetric Askey–Wilson function transform. For the symmetric Askey–Wilson function transform, a different proof for these formulas was derived in [43, Thm. 5.2].

We end this subsection with the following algebraic Plancherel type formulas for the (non)symmetric Askey–Wilson function transform.

Theorem 8.16. a) *For all $g, h \in M_e$,*

$$\langle \widetilde{\mathcal{F}}_e g, I\big(\widetilde{\mathcal{J}}_e h\big)\rangle_{e_\sigma}^\sigma = \langle g, Ih\rangle_e,$$

with I the inversion operator $(If)(\gamma) = f(\gamma^{-1})$.

b) *For all $g, h \in M_e^+$,*

$$\langle \widetilde{\mathcal{F}}_e^+ g, \widetilde{\mathcal{F}}_e^+ h\rangle_{e_\sigma,+}^\sigma = \langle g, h\rangle_{e,+}.$$

Proof. a) By a formal computation using the duality of the normalized rank one Cherednik kernels $\mathfrak{E}$ and $\mathfrak{E}_{\ddagger}$ (see Theorem 6.2), we obtain

$$\langle \widetilde{\mathcal{F}}_e g, Ih\rangle_{e_\sigma}^{\sigma} = \langle g, I\big(\widetilde{\mathcal{F}}_{e_\sigma}^{\sigma} h\big)\rangle_e, \qquad \langle \widetilde{\mathcal{J}}_e g, Ih\rangle_{e_\sigma}^{\sigma} = \langle g, I\big(\widetilde{\mathcal{J}}_{e_\sigma}^{\sigma} h\big)\rangle_e \tag{8.34}$$

for $g \in M_e$ and $h \in M_{e_\sigma}^{\sigma}$. For a rigorous proof of formula (8.34) we need to justify that limits and integrations may be interchanged in the formal computation. In case $g \in V_e^{cl}$, this is easily justified by Lemma 8.4 and by Fubini's Theorem, since g vanishes on $(eq^{\mathbb{Z}})^{\pm 1}$. In a similar fashion, (8.34) is proved to be correct when $h \in V_{e_\sigma}^{cl,\sigma}$.

Let now $g \in V^{str}$ and $h \in V^{str,\sigma}$. We write $h = \widetilde{\mathcal{J}}_e f$ with $f \in V_e^{cl}$. Then Theorem 8.13 shows that $f = \widetilde{\mathcal{F}}_{e_\sigma}^{\sigma} h$. We derive that

$$\begin{aligned}
\langle \widetilde{\mathcal{F}}_e g, Ih\rangle_{e_\sigma}^{\sigma} &= \langle \widetilde{\mathcal{F}}_e g, I\big(\widetilde{\mathcal{J}}_e f\big)\rangle_{e_\sigma}^{\sigma} \\
&= \langle \big(\widetilde{\mathcal{J}}_{e_\sigma}^{\sigma} \circ \widetilde{\mathcal{F}}_e\big)(g), If\rangle_e \\
&= \langle g, If\rangle_e \\
&= \langle g, I\big(\widetilde{\mathcal{F}}_{e_\sigma}^{\sigma} h\big)\rangle_e.
\end{aligned}$$

Here the second equality is allowed since $f \in V_e^{cl}$, and the third equality follows from Theorem 8.13. The second identity in (8.34) for $g \in V^{str}$ and $h \in V^{str,\sigma}$ follows by a similar computation.

Formula (8.34) combined with Theorem 8.13 now shows that

$$\langle \mathcal{F}_e g, I\big(\mathcal{J}_e h\big)\rangle_{e_\sigma}^{\sigma} = \langle g, Ih\rangle_e$$

for $g, h \in M_e$.

b) This follows by similar arguments as for the nonsymmetric Askey–Wilson function transform (see part a)). □

8.6. The L^2-theory. The results thus far obtained for the symmetric Askey–Wilson function transform $\mathcal{F}_e^{+}$ are sufficient to derive "analytic" Plancherel and inversion formulas as follows.

We need to restrict the parameter domain first in order to obtain positive measures. As usual, we assume throughout this subsection that $0 < q^{\frac{1}{2}} < 1$. We assume furthermore that the parameter e and the difference multiplicity function α satisfy the conditions

$$\begin{aligned}
&e < 0, && 0 < b, c \leq a < d/q, \\
&bd, cd \geq q, && ab, ac < 1,
\end{aligned} \tag{8.35}$$

where we used the Askey–Wilson parametrization (8.1) for the difference multiplicity function α. These conditions are invariant under the extended involution $(\alpha, e) \mapsto (\alpha_\sigma, e_\sigma)$. For generic parameters satisfying these conditions, we define a measure $m_e(\cdot) = m_e^\alpha(\cdot)$ by

$$\int f(x)dm_e(x) = \frac{1}{2\pi i}\int_{x\in\mathbb{T}} f(x)W_e^+(x)\frac{dx}{x} + \sum_{x\in D_e}\big(f(x)+f(x^{-1})\big)\operatorname*{Res}_{y=x}\left(\frac{W_e^+(y)}{y}\right),$$

where $D_e = D_e^+ \cup D_e^-$ is the discrete set

$$D_e^+ = \{aq^k \,|\, k\in\mathbb{N} :\; aq^k > 1\},$$
$$D_e^- = \{eq^k \,|\, k\in\mathbb{Z} :\; eq^k < -1\}.$$

By continuous extension in the parameters, we obtain a positive measure m_e^α for all parameters α, e satisfying (8.35), see [27] and [43] for details. The support of m_e^α is $\mathbb{T}\cup D_e \cup D_e^{-1}$.

For the remainder of this subsection we fix parameters α and e satisfying the conditions (8.35). Let $L_+^2(m_e)$ be the Hilbert space consisting of L^2-functions f with respect to the measure m_e satisfying $f(x) = f(x^{-1})$ m_e-a.e. Clearly, the space M_e^+ may be considered as subspace of $L_+^2(m_e)$. The following result from [43, Prop. 6.7] tells us exactly when $M_e^+ \subseteq L_+^2(m_e)$ is dense.

Proposition 8.17. *Let $k\in\mathbb{Z}$ be the unique integer such that $1 < |eq^k| \le q^{-1}$. Then $M_e^+ \subseteq L_+^2(m_e)$ is dense if and only if $|e_\sigma^{-1}q^k| \ge 1$.*

Proof. Using the explicit results on the extended, symmetric rank one Macdonald–Koornwinder transform F^+ as derived in Subsection 7.4 (see also [43]), the proof can be reduced to proving the density of some polynomial space in an explicit L^2-space of square integrable functions with respect to a compactly supported measure. The proof then follows from standard density results. For a detailed proof we refer the reader to [43, Prop. 6.7]. □

As a consequence of Proposition 8.17 we arrive at a new proof for the analytic inversion and Plancherel formula for the symmetric Askey–Wilson function transform, see [27, Thm. 1] for the classical approach.

Theorem 8.18. *Let $k\in\mathbb{Z}$ be the unique integer such that $1 < |eq^k| \le q^{-1}$ and assume that $|e_\sigma^{-1}q^k| \ge 1$. The normalized symmetric Askey–Wilson function transform $\widetilde{\mathcal{F}}_e^+$ (see (8.30)) uniquely extends by continuity to a surjective isometric isomorphism*

$$\widetilde{\mathcal{F}}_e^+ : L_+^2(m_e) \to L_+^2(m_{e_\sigma}^\sigma),$$

with inverse given by

$$\widetilde{\mathcal{F}}_{e_\sigma}^{+,\sigma} : L_+^2(m_{e_\sigma}^\sigma) \to L_+^2(m_e).$$

Proof. The conditions on the parameters α, e are invariant under the involution $(\alpha, e)\mapsto(\alpha_\sigma, e_\sigma)$. Hence $L_+^2(m_{e_\sigma}^\sigma)$ is defined, and $M_e^+ \subseteq L_+^2(m_e)$, $M_{e_\sigma}^{+,\sigma} \subseteq$

$L^2_+(m^\sigma_{e_\sigma})$ are both dense by Proposition 8.17. The result now follows immediately from the algebraic inversion and Plancherel formula for $\widetilde{\mathcal{F}}_e^+$, see Theorem 8.13 and Theorem 8.16. $\square$

9. Appendix

In this section we prove the bounds for the Macdonald–Koornwinder polynomials as stated in Proposition 5.13.

Let $\Lambda_i = \{\lambda \in \Lambda \,|\, r_i \cdot \lambda \neq \lambda\}$ and denote

$$\mathcal{S}_i = \{s_\lambda \,|\, \lambda \in \Lambda_i\} \subseteq \mathcal{S}$$

for $i = 0, \ldots, n$. Let $\mathcal{F}(\mathcal{S}_i)$ be the space of functions $g : \mathcal{S}_i \to \mathbb{C}$. Recall the definition of $c_i := c_{a_i} \in \mathbb{C}(x)$ for $i = 0, \ldots, n$, see (2.3) and (2.4). We first observe the following elementary fact.

Lemma 9.1. *The functions $|c_i^{\ddagger\sigma}(\cdot)| \in \mathcal{F}(\mathcal{S})$ and $|c_i^{\ddagger\sigma}(\cdot)|^{-1} \in \mathcal{F}(\mathcal{S}_i)$ are bounded for $i = 0, \ldots, n$.*

Proof. This follows immediately from Lemma 4.1b) and from the explicit expression of the spectral points $s_\lambda \in \mathcal{S}$ (see Subsection 4.1), since the parameters are assumed to be generic. $\square$

If $\lambda = \sum_i \lambda_i \epsilon_i \in \Lambda$, then we call λ_i the ith coordinate of λ. For $\lambda \in \Lambda \setminus \{0\}$ we denote $u_\lambda \in W_0$ for the unique element of minimal length such that the first coordinate of $u_\lambda^{-1} \cdot \lambda$ is strictly negative. There are essentially two cases to consider here; if some of the coordinates of λ are strictly negative, then $u_\lambda = r_{i-1} r_{i-2} \cdots r_1$ with $i \geq 1$ the smallest such that $\lambda_i < 0$ (and $u_\lambda = 1 \in W_0$ the identity element when $i = 1$). If all coordinates of λ are nonnegative and $\lambda_j > 0$, $\lambda_{j+1} = \cdots = \lambda_n = 0$, then $u_\lambda = r_j \cdots r_{n-1} r_n r_{n-1} \cdots r_1$. Finally, for $0 \in \Lambda$ the zero element, we denote $u_0 = r_1 \cdots r_{n-1} r_n r_{n-1} \cdots r_1 \in W_0$. Observe that

$$T_{u_0} = T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_1 = U \in \mathcal{H}$$

(see (5.7)), since $u_0 = r_1 \cdots r_{n-1} r_n r_{n-1} \cdots r_1$ is a reduced expression in W_0. We prove the following refinement of Proposition 5.13.

Lemma 9.2. *Let $K \subset \left(\mathbb{C}^\times\right)^n$ be a compact set. Then there exist positive constants C_1 (independent of K) and C_2 (dependent of K) such that*

$$|E(s_\lambda; x)| \leq C_1^{l(u_\lambda)} C_2^{N(\lambda)}, \qquad \forall x \in K, \ \forall \lambda \in \Lambda.$$

Since $l(u_\lambda) \leq 2n - 1$ for all $\lambda \in \Lambda$ and $E(s_0; \cdot) = 1$, Proposition 5.13 is a direct consequence of Lemma 9.2. We start with a preliminary lemma.

Lemma 9.3. *For all* $\mu \in \Lambda$,

$$u_n c_0^{\ddagger\sigma}(s_\mu) E(s_{r_0\cdot\mu};x) = u_n\big(c_0^{\ddagger\sigma}(s_\mu)-1\big)E(s_\mu;x) + x_1^{-1}\big(U_{\ddagger\sigma}\cdot E(\cdot\,;x)\big)(s_\mu).$$

Proof. By (4.3) and the definition of T_0,

$$\big(T_0^{\ddagger\sigma}E(\cdot\,;x)\big)(s_\mu) = \big(u_n^{-1} - u_n c_0^{\ddagger\sigma}(s_\mu)\big)E(s_\mu;x) + u_n c_0^{\ddagger\sigma}(s_\mu)E(s_{r_0\cdot\mu};x). \tag{9.1}$$

On the other hand,

$$\begin{aligned}\big(T_0^{\ddagger\sigma}\cdot E(\cdot\,;x)\big)(s_\mu) &= \big(T_0^{\ddagger\sigma}\cdot I\mathfrak{E}_{\mathcal{A}}(\cdot,x)\big)(s_\mu)\\ &= \big(T_0^{\sigma-1}\cdot \mathfrak{E}_{\mathcal{A}}(\cdot,x)\big)(s_\mu^{-1})\\ &= \big(\psi_\sigma(T_0^{\sigma-1})E(s_\mu;\cdot)\big)(x)\end{aligned}$$

by Proposition 4.4, where $I:\mathcal{F}(\mathcal{S}_\ddagger)\to\mathcal{F}(\mathcal{S})$ is the "inversion map" $(Ig)(s) = g(s^{-1})$. Now $\psi_\sigma(T_0^{\sigma-1}) = Ux_1 = u_n^{-1} - u_n + x_1^{-1}U^{-1}$, see [39] or [42, Appendix] for the second equality, and $(\dagger_\sigma\circ\psi)(U^{-1}) = U_{\ddagger\sigma}$; hence

$$\big(T_0^{\ddagger\sigma}\cdot E(\cdot\,;x)\big)(s_\mu) = (u_n^{-1}-u_n)E(s_\mu;x) + x_1^{-1}\big(U_{\ddagger\sigma}\cdot E(\cdot\,;x)\big)(s_\mu). \tag{9.2}$$

Subtracting (9.1) from (9.2) gives the result. □

In a similar fashion as in the proof of Lemma 9.3 we obtain from Proposition 4.4 the relations

$$\big(T_wE(s;\cdot)\big)(x) = \big(T_w^{\ddagger\sigma-1}\cdot E(\cdot;x)\big)(s) \tag{9.3}$$

for $w\in W_0$ and $s\in\mathcal{S}$.

Let $\lambda\in\Lambda\setminus\{0\}$ and set $\mu = (r_0u_\lambda^{-1})\cdot\lambda$. Observe that $N(\mu) = N(\lambda)-1$, as follows easily from the definition of the action $\mathcal{W}$ on Λ and from the definition of $u_\lambda\in W_0$. Lemma 9.3 applied to μ now gives the recurrence relation

$$\begin{aligned}u_nc_0^{\ddagger\sigma}(s_{(r_0u_\lambda^{-1})\cdot\lambda})E(s_{u_\lambda^{-1}\cdot\lambda};x) &= x_1^{-1}\big(U_{\ddagger\sigma}\cdot E(\cdot\,;x)\big)(s_{(r_0u_\lambda^{-1})\cdot\lambda})\\ &\quad + u_n\big(c_0^{\ddagger\sigma}(s_{(r_0u_\lambda^{-1})\cdot\lambda})-1\big)E(s_{(r_0u_\lambda^{-1})\cdot\lambda};x).\end{aligned} \tag{9.4}$$

Let $\preceq$ be the Bruhat order on W_0. We apply T_{u_λ}, acting upon $\mathcal{A}$ on the x-variable, to both sides of (9.4). For the left-hand side, we obtain in view of (9.3) an expression of the form

$$K(\lambda)E(s_\lambda;x) + \sum_{\substack{w\in W_0\\ w\prec u_\lambda^{-1}}} e_w(\lambda)E(s_{(u_\lambda w)^{-1}\cdot\lambda};x).$$

The coefficient $K(\lambda)$ can be computed explicitly as follows (compare with the proof of Lemma 4.3). Let $u_\lambda = r_{i_1}r_{i_2}\cdots r_{i_l}$ be a reduced expression for $u_\lambda\in W_0$, so that $l = l(u_\lambda)$, and define

$$\lambda_{j+1} = (r_{i_j}\cdots r_{i_2}r_{i_1})\cdot\lambda,\qquad j=0,\ldots,l,$$

with the convention that $\lambda_1 = \lambda$. Observe that $\lambda_{l+1} = u_\lambda^{-1} \cdot \lambda$, and that $\lambda_j = r_{i_j} \cdot \lambda_{j+1} \neq \lambda_{j+1}$ for $j = 1, \ldots, n$, cf. the proof of Lemma 4.3. Then

$$K(\lambda) = u_n t_{u_\lambda} c_0^{\ddagger\sigma}(s_{(r_0 u_\lambda^{-1})\cdot\lambda}) \prod_{j=1}^{l(u_\lambda)} c_{i_j}^{\ddagger\sigma}(s_{\lambda_{j+1}}) \tag{9.5}$$

and $K(\lambda) \neq 0$ due to Lemma 4.1b). Furthermore, $|K(\lambda)|^{-1}$ is uniformly bounded for $\lambda \in \Lambda \setminus \{0\}$ in view of Lemma 9.1. Similarly, $|e_w(\lambda)|$ is uniformly bounded for $\lambda \in \Lambda \setminus \{0\}$ and $w \in W_0$ with $w \prec u_\lambda^{-1}$. Finally, note that $l(u_\mu) < l(u_\lambda)$ for $\mu = (u_\lambda w)^{-1} \cdot \lambda$ with $\lambda \in \Lambda \setminus \{0\}$, $w \in W_0$ and $w \prec u_\lambda^{-1}$. Hence we conclude that acting by T_{u_λ} on the left-hand side of (9.4) yields an expression of the form

$$K(\lambda)E(s_\lambda; x) + \sum_{\substack{\mu\in W_0\cdot\lambda \\ l(u_\mu)<l(u_\lambda)}} c(\lambda;\mu)E(s_\mu; x) \tag{9.6}$$

with $K(\lambda)$ given by (9.5) and

$$|K(\lambda)|^{-1} \leq L, \qquad |c(\lambda,\mu)| \leq L \tag{9.7}$$

for all $\lambda \in \Lambda \setminus \{0\}$ and $\mu \in W_0 \cdot \lambda$ such that $l(u_\mu) < l(u_\lambda)$, for some constant $L > 0$.

To deal with the action of T_{u_λ} on the right-hand side of (9.4), we need the commutation relations between the T_i and x_j in $\mathcal{H}$ for $i, j = 1, \ldots, n$. They are given by

$$\begin{aligned} &T_i x_j = x_j T_i, && |i-j| > 1,\\ &T_i x_{i-1} = x_{i-1} T_i, && i = 2, \ldots, n,\\ &T_i x_i T_i = x_{i+1}, && i = 1, \ldots, n-1,\\ &(x_n^{-1}T_n^{-1} - u_n)(x_n^{-1}T_n^{-1} + u_n^{-1}) = 0, && \end{aligned}$$

see e.g., [39] or [42, Prop. 6.5]. In view of (9.3), applying T_{u_λ} to the first term in the right-hand side of (9.4) then yields an expression of the form

$$\big(X \cdot E(\cdot; x)\big)(s_{(r_0 u_\lambda^{-1})\cdot\lambda}) + \sum_{\substack{1\leq j\leq n \\ \xi=\pm 1}} x_j^\xi \big(X_{j,\xi} \cdot E(\cdot\,; x)\big)(s_{(r_0 u_\lambda^{-1})\cdot\lambda}) \tag{9.8}$$

with $X, X_{j,\xi} \in H_0^{\ddagger\sigma}$ independent of λ, where (recall) $H_0^{\ddagger\sigma} \subset \mathcal{H}_{\ddagger\sigma}$ is the subalgebra generated by $T_i^{\ddagger\sigma}$ ($i = 1, \ldots, n$). Finally, applying T_{u_λ} to the second term in the right-hand side of (9.4), yields

$$u_n\big(c_0^{\ddagger\sigma}(s_{(r_0 u_\lambda^{-1})\cdot\lambda}) - 1\big)\big(T_{u_\lambda}^{\ddagger\sigma-1} \cdot E(\cdot\,; x)\big)(s_{(r_0 u_\lambda^{-1})\cdot\lambda}). \tag{9.9}$$

Combining (9.6), (9.8) and (9.9), applying T_{u_λ} to (9.4) gives the identity

$$\begin{aligned} K(\lambda)E(s_\lambda; x) = &- \sum_{\substack{\mu\in W_0\cdot\lambda \\ l(u_\mu)<l(u_\lambda)}} c(\lambda;\mu)E(s_\mu; x) + \sum_{\substack{1\leq j\leq n \\ \xi=\pm 1}} x_j^\xi \big(X_{j,\xi} \cdot E(\cdot\,; x)\big)(s_{(r_0 u_\lambda^{-1})\cdot\lambda}) \\ &+ \big(X \cdot E(\cdot; x)\big)(s_{(r_0 u_\lambda^{-1})\cdot\lambda}) + u_n\big(c_0^{\ddagger\sigma}(s_{(r_0 u_\lambda^{-1})\cdot\lambda}) - 1\big)\big(T_{u_\lambda}^{\ddagger\sigma-1} \cdot E(\cdot\,; x)\big)(s_{(r_0 u_\lambda^{-1})\cdot\lambda}). \end{aligned}$$

This formula will now be used inductively. Observe that the terms

$$\big(Z\cdot E(\cdot\,;x)\big)(s_{(r_0u_\lambda^{-1})\cdot\lambda}),\qquad \big(T_{u_\lambda}^{\ddagger\sigma-1}\cdot E(\cdot\,;x)\big)(s_{(r_0u_\lambda^{-1})\cdot\lambda})$$

with $Z=X,X_{j,\xi}$ are linear combinations of $E(s_\mu;x)$ with $\mu\in W_0\cdot((r_0u_\lambda^{-1})\cdot\lambda)$ and with coefficients which are uniformly bounded for all $\lambda\in\Lambda\setminus\{0\}$ and $\mu\in W_0\cdot((r_0u_\lambda^{-1})\cdot\lambda)$, in view of Lemma 9.1. Combined with the uniform bounds (9.7), we conclude that there exists a constant $M\geq 1$ such that

$$\begin{aligned}|E(s_\lambda;x)|\leq M&\sum_{\substack{\mu\in W_0\cdot\lambda\\ l(u_\mu)<l(u_\lambda)}}|E(s_\mu;x)|\\ &+M\Big(1+2n\max_{j,\xi}(|x_j^\xi|)\Big)\sum_{\nu\in W_0\cdot((r_0u_\lambda^{-1})\cdot\lambda)}|E(s_\nu;x)|\end{aligned}\tag{9.10}$$

for all $\lambda\in\Lambda\setminus\{0\}$, where the maximum is taken over $j\in\{1,\ldots,n\}$ and $\xi\in\{\pm1\}$. Now fix a compactum $K\subset\big(\mathbb{C}^\times\big)^n$, and choose a constant $C_1\geq 2|W_0|M$ and a constant $C_2\geq 1$ (depending on K), such that

$$M\big(1+2n\max_{x,j,\xi}(|x_j^\xi|)\big)|W_0|C_1^{2n-1}\leq\frac{C_2}{2},\tag{9.11}$$

where the maximum is taken over $x\in K$, $j\in\{1,\ldots,n\}$ and $\xi\in\{\pm1\}$. Fix $\lambda\in\Lambda\setminus\{0\}$, and assume that

$$|E(s_\mu;x)|\leq C_1^{l(u_\mu)}C_2^{N(\mu)},\qquad\forall\,x\in K$$

when $\mu\in\Lambda$ satisfies either $N(\mu)<N(\lambda)$, or $N(\mu)=N(\lambda)$ and $l(u_\mu)<l(u_\lambda)$. We use now the fact that $N(\nu)=N(w\cdot\nu)$ for all $w\in W_0$ and $\nu\in\Lambda$, and that $N((r_0u_\lambda^{-1})\cdot\lambda)=N(\lambda)-1$ for $\lambda\in\Lambda\setminus\{0\}$, to derive from (9.10) that

$$\begin{aligned}\max_{x\in K}|E(s_\lambda;x)|\leq&MC_2^{N(\lambda)}\sum_{\substack{\mu\in W_0\cdot\lambda\\ l(u_\mu)<l(u_\lambda)}}C_1^{l(u_\mu)}\\ &+M\big(1+2n\max_{x,j,\xi}(|x_j^\xi|)\big)C_2^{N(\lambda)-1}\sum_{\mu\in W_0\cdot((r_0u_\lambda^{-1})\cdot\lambda)}C_1^{l(u_\mu)}\\ \leq&M|W_0|\Big(C_2^{N(\lambda)}C_1^{l(u_\lambda)-1}+\big(1+2n\max_{x,j,\xi}(|x_j^\xi|)\big)C_2^{N(\mu)-1}C_1^{2n-1}\Big)\\ \leq&\frac{1}{2}C_2^{N(\lambda)}C_1^{l(u_\lambda)}+\frac{1}{2}C_2^{N(\mu)}\\ \leq&C_2^{N(\lambda)}C_1^{l(u_\lambda)},\end{aligned}$$

where we used $l(u_\mu)\leq 2n-1$ for $\mu\in\Lambda$ in the second equality, the estimates (9.11) and $C_1\geq 2|W_0|M$ in the third equality, and the fact that $C_1\geq 1$. This proves Lemma 9.2, and hence Proposition 5.13.

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