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Intersection cohomology of Drinfeld's compactifications

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Abstract. Let X be a smooth complete curve, G be a reductive group and $P \subset G$ a parabolic. Following Drinfeld, one defines a (relative) compactification $\widetilde{\mathrm{Bun}}_P$ of the moduli stack of P-bundles on X. The present paper is concerned with the explicit description of the Intersection Cohomology sheaf of $\widetilde{\mathrm{Bun}}_P$. The description is given in terms of the combinatorics of the Langlands dual Lie algebra $\check{\mathfrak{g}}$.

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Introduction

0.1. This paper merges several points of view on a geometric object introduced by V. Drinfeld. In the incarnation studied here, this is a relative compactification of the moduli stack of principal bundles with respect to a parabolic subgroup (of a given reductive group) over a curve.

Since its discovery several years ago, Drinfeld's compactification has found several remarkable applications in geometric representation theory, some of which are discussed in this introduction. These applications include a construction of the "correct" geometric Eisenstein series functor (cf. [BG]); a geometric study of quantum groups at a root of unity and representations of Lie algebras in positive characteristic; a realization of the combinatorial pattern introduced by Lusztig in [Lu1] in terms of intersection cohomology. In particular, one obtains an interesting and unexpected relation between Eisenstein series and semi-infinite cohomology of quantum groups at a root of unity (cf. [FFKM]).

The main result of this paper is a calculation of the intersection cohomology sheaf of Drinfeld's compactification.

0.2. The space $\widetilde{\mathrm{Bun}}_P$

Let X be a smooth complete curve, G a reductive group and $P \subset G$ a parabolic subgroup. Let us denote by Bun_G the moduli stack of principal G-bundles on X and by Bun_P the moduli stack of P-bundles.

The inclusion of P in G gives rise to a representable morphism $\mathfrak{p} : \mathrm{Bun}_P \to \mathrm{Bun}_G$, and it is a problem that arises most naturally in geometric representation theory to look for a relative compactification of Bun_P along the fibers of this map.

A construction of such a compactification was suggested by Drinfeld. In this way we obtain a new algebraic stack, denoted $\widetilde{\mathrm{Bun}}_P$, endowed with a map $\widetilde{\mathfrak{p}} : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G$ such that $\widetilde{\mathfrak{p}}$ is now *proper*, and Bun_P is contained inside $\widetilde{\mathrm{Bun}}_P$ as an open substack.

The main complication, as well as the source of interest, is the fact that the stack $\widetilde{\mathrm{Bun}}_P$ is singular. Basically, the present paper is devoted to study of singularities of $\widetilde{\mathrm{Bun}}_P$.

0.3. Eisenstein series

The authors of [BG] considered the following problem: For X, G and P as before, let M be the Levi factor of P and Bun_M the corresponding moduli stack. Using Bun_P one can introduce a naive *Eisenstein series* functor, which maps the derived category $\mathrm{D}^b(\mathrm{Bun}_M)$ to $\mathrm{D}^b(\mathrm{Bun}_G)$.

However, since the map $\mathfrak{p}$ is not proper, this functor is "not quite the right one" from the geometric point of view. For example, it does not commute with the Verdier duality and does not preserve purity. It turns out that using the compactified stack $\widetilde{\mathrm{Bun}}_P$, one can introduce another functor Eis_M^G between the same categories, which will have much better properties. The authors of *loc.cit.* called it the geometric Eisenstein series functor.

When our ground field is a finite field $\mathbb{F}_q$, both the naive and the corrected Eisenstein series functor give rise to operators between the corresponding functions spaces, i.e., from $\mathrm{Funct}(\mathrm{Bun}_M(\mathbb{F}_q))$ to $\mathrm{Funct}(\mathrm{Bun}_G(\mathbb{F}_q))$, the former being the usual Eisenstein series operator in the theory of automorphic forms.

The naive operator essentially amounts to taking the trace of the Frobenius element acting on the cohomology of the fibers of the map $\mathfrak{p} : \mathrm{Bun}_P \to \mathrm{Bun}_G$, whereas the operator arising from the corrected functor Eis_M^G corresponds to taking the hyper-cohomology of the *intersection cohomology* sheaf $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ along the fibers of $\widetilde{\mathfrak{p}} : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G$.

One of the results announced (but not proved) in [BG] was Theorem 2.2.12, which compared the naive Eisenstein series operator with the corrected one. Namely, the latter must be the product of the former and an appropriate L-function.

The proof of this theorem essentially amounts to an explicit description of the intersection cohomology sheaf $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$, in terms of the combinatorics of the Langlands dual Lie algebra $\check{\mathfrak{g}}$, or more precisely, in terms of the symmetric algebra $\mathrm{Sym}^\bullet(\check{\mathfrak{u}}(P))$ (here $\check{\mathfrak{u}}(P)$ is the Lie algebra of the unipotent radical of the corresponding parabolic in $\check{G}$), viewed as an $\check{M}$-module.

In the present paper we establish the required explicit description of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$.

0.4. Semi-infinite flag variety and quantum groups

Let us now explain another perspective on the behavior of the above IC sheaf. Consider the semi-infinite flag "variety" $G((t))/B((t))$, where B is the Borel subgroup of G. Since the pioneering work of Feigin and Frenkel [FF], people have been trying to develop the theory of perverse sheaves (constructible with respect to a given stratification) on $G((t))/B((t))$ and, in particular, to compute the IC sheaf on it.

The problem is that $G((t))/B((t))$ is very essentially infinite-dimensional, so that the conventional theory of perverse sheaves, defined for schemes of finite type, was not applicable. Since it was (and still is) not clear whether there exists a direct definition of perverse sheaves on $G((t))/B((t))$, the authors of [FFKM] proposed the following solution.

They introduced certain finite-dimensional varieties, called the Zastava spaces Z^μ in terms of maps of a projective line into the flag variety (the parameter μ is the degree of the map, i.e., it is an element in the coroot lattice of G). They argued that the these spaces provided "models" for $G((t))/B((t))$ from the point of view of singularities of the strata.

Moreover, it was shown in [FFKM] that the stalks of certain perverse sheaves on Z^μ are given by the periodic Kazhdan–Lusztig polynomials of [Lu1], and this agrees with the anticipated answer for $G((t))/B((t))$. Therefore, the Zastava spaces provide a geometric interpretation of Lusztig's combinatorics. This, combined with the earlier work of Feigin and Frenkel, allowed the authors of [FFKM] to come up with a conjecture that ties a certain category of perverse sheaves on $G((t))/B((t))$, thought of as sheaves on Z^μ, with the category of representations of the small quantum group.

The basic characteristic of the Zastava spaces, discovered in [FFKM], is that they are local in nature, which expresses itself in the *factorization property*. Namely, each Z^μ is fibered over the configuration space X^μ, equal to the product of the corresponding symmetric powers of X, i.e., $X^\mu = \prod_{\mathrm{i}} X^{(n_{\mathrm{i}})}$. If $\mu = \mu_1 + \mu_2$ there is an isomorphism $Z^\mu \simeq Z^{\mu_1} \times Z^{\mu_2}$, *after both sides are restricted* to the subspace $(X^{\mu_1} \times X^{\mu_2})_{\mathrm{disj}}$ of $X^{\mu_1} \times X^{\mu_2}$ corresponding to non-intersecting configurations.

0.5. Intersection cohomology of the parabolic Zastava spaces

Now, the key fact for us is that the Zastava spaces Z^μ, appropriately generalized in the case of an arbitrary parabolic P, provide "local models" for the singularities of the stack $\widetilde{\mathrm{Bun}}_P$. More precisely, it turns out that the parabolic Zastava spaces and the stack $\widetilde{\mathrm{Bun}}_P$ are, locally in the smooth topology, isomorphic to one another. Thus, the problem of calculation of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ reduces to an analogous problem for the parabolic Zastava spaces Z^θ (θ now is an element of a certain quotient lattice).

Here we are dealing with the following remarkable phenomenon: the stack $\widetilde{\mathrm{Bun}}_P$ is defined via the global curve X and it classifies P-bundles on our curve which have degenerations at finitely many points. Therefore, one may wonder whether the singularities of $\widetilde{\mathrm{Bun}}_P$ near a point corresponding to such a degenerate P-bundle would depend only on what is happening at the points of degeneration, and this is what our comparison with the Zastava spaces actually proves.

To carry out the calculation of IC_{Z^θ}, we employ an inductive procedure on the parameter θ, using the factorization property mentioned above. (We should note that the present argument is quite different and is in fact simpler than the one used in [FFKM] to treat the case of $P = B$.)

The connection between the stalks of IC_{Z^θ} (and hence, of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$) and the Langlands dual Lie algebra $\check{\mathfrak{g}}$ is explained as follows The fiber of Z^θ over a given point of X^θ is a product of intersections of *semi-infinite orbits* in the *affine Grassmannian* Gr_G corresponding to G. (In fact, the whole space Z^θ can be thought of as a subspace of the corresponding version of the Beilinson–Drinfeld affine Grassmannian.) The required link to $\check{\mathfrak{g}}$ is provided by the Drinfeld–Ginzburg–Lusztig–Mirković–Vilonen theory of spherical perverse sheaves on Gr_G, cf. [MV].

0.6. The naive compactification

To conclude, let us mention that in addition to $\widetilde{\mathrm{Bun}}_P$, the stack Bun_P of parabolic bundles admits another, in a certain sense more naive, relative compactification, which we denote $\overline{\mathrm{Bun}}_P$. This second compactification can be though of as a blow-down of $\widetilde{\mathrm{Bun}}_P$; in particular, we have a proper representable map $\mathfrak{r} : \widetilde{\mathrm{Bun}}_P \to \overline{\mathrm{Bun}}_P$.

By the Decomposition Theorem, $\mathfrak{r}_!(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P})$ contains $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ as a direct summand. In the last section we give an explicit description of both $\mathfrak{r}_!(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P})$ and $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$. The answer for the stalks of $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ is formulated in terms of $\mathrm{Sym}^\bullet(\check{\mathfrak{u}}(P)^f)$ where $f \in \mathrm{Lie}\, \check{M}$ is a principal nilpotent.

Note that the latter vector space is exactly the one appearing in [Lu2]. Therefore, the stack $\overline{\mathrm{Bun}}_P$ provides a geometric object whose singularities reproduce the parabolic version of the periodic Kazhdan–Lusztig polynomials of [Lu2].

0.7. Notation

In this paper we will work over the ground field $\mathbb{F}_q$. However, the reader will readily check that all our results can be automatically carried over to the characteristic 0 situation.

Throughout the paper, G will be a connected reductive group over $\mathbb{F}_q$, assumed split. Moreover, we will assume that its derived group $G' = [G, G]$ is simply connected.

We will fix a Borel subgroup $B \subset G$ and let T be the "abstract" Cartan, i.e., $T = B/U$, where U is the unipotent radical of B. We will denote by Λ the *coweight* lattice of T and by $\check{\Lambda}$–its dual, i.e., the weight lattice; $\langle \cdot, \cdot \rangle$ is the canonical pairing between the two.

The semi-group of dominant coweights (resp., weights) will be denoted by Λ^+_G (resp., by $\check{\Lambda}^+_G$). The set of vertices of the Dynkin diagram of G will be denoted by $\mathcal{I}$; for each $\mathfrak{i} \in \mathcal{I}$ there corresponds a simple coroot $\alpha_\mathfrak{i}$ and a simple root $\check{\alpha}_\mathfrak{i}$. The set of positive coroots will be denoted by Δ and their positive span inside Λ, by Λ^{pos}_G. (Note that, since G is not semi-simple, Λ^+_G is not necessarily contained in Λ^{pos}_G.) By $\check{\rho} \in \check{\Lambda}$ we will denote the half sum of positive roots of G and by w_0 the longest element in the Weyl group of G. For $\lambda_1, \lambda_2 \in \Lambda$ we will write that $\lambda_1 \geq \lambda_2$ if $\lambda_1 - \lambda_2 \in \Lambda^{\mathrm{pos}}_G$, and similarly for $\check{\Lambda}^+_G$.

Let P be a standard proper parabolic of G, i.e., $P \supset B$; let $U(P)$ be its unipotent radical and $M := P/U(P)$–the Levi factor. To M there corresponds a subdiagram $\mathcal{I}_M$ in $\mathcal{I}$. We will denote by $\Lambda^+_M \supset \check{\Lambda}^+_G$, $\Lambda^{\mathrm{pos}}_M \subset \Lambda^{\mathrm{pos}}_G$, $\check{\rho}_M \in \check{\Lambda}$, $w_0^M \in W$, $\underset{M}{\geq}$, etc. the corresponding objects for M.

To a dominant coweight $\check{\lambda} \in \check{\Lambda}$ one attaches the Weyl G-module, denoted $\mathcal{V}^{\check{\lambda}}$, with a fixed highest weight vector $v^{\check{\lambda}} \in \mathcal{V}^{\check{\lambda}}$. For a pair of weights $\check{\lambda}_1, \check{\lambda}_2$, there is a canonical map $\mathcal{V}^{\check{\lambda}_1+\check{\lambda}_2} \to \mathcal{V}^{\check{\lambda}_1} \otimes \mathcal{V}^{\check{\lambda}_2}$ that sends $v^{\check{\lambda}_1+\check{\lambda}_2}$ to $v^{\check{\lambda}_1} \otimes v^{\check{\lambda}_2}$.

Similarly, for $\check{\nu} \in \check{\Lambda}^+_M$, we will denote by $\mathcal{U}^{\check{\nu}}$ the corresponding Weyl module for M. There is a natural functor $\mathcal{V} \mapsto \mathcal{V}^{U(P)}$ from the category of G-modules to that of M-modules and we have a canonical morphism $\mathcal{U}^{\check{\lambda}} \to (\mathcal{V}^{\check{\lambda}})^{U(P)}$, which sends the corresponding highest weight vectors to one another.

Unless specified otherwise, we will work with the perverse t-structure on the category of constructible complexes over various schemes and stacks. If $\mathcal{S}$ is a constructible complex, $h^i(\mathcal{S})$ will denote its i-th perverse cohomology sheaf. The intersection cohomology sheaves are normalized so that they are pure of weight 0. In other words, for a smooth variety Y of dimension n, $\mathrm{IC}_Y \simeq (\overline{\mathbb{Q}_\ell}(\frac{1}{2})[1])^{\otimes n}$, where $\overline{\mathbb{Q}_\ell}(\frac{1}{2})$ corresponds to a chosen once and for all square root of q in $\overline{\mathbb{Q}_\ell}$.

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1. Drinfeld's compactifications

1.1. Let P be a parabolic subgroup of G and let $H \subset P$ be either the unipotent radical $U(P)$ or $[P,P]$. Consider the following functor Schemes $\to$ Categories:

To a scheme S we associate the category of triples $(\mathcal{F}_G, \mathcal{F}_{P/H}, \kappa)$, where $\mathcal{F}_G$ (resp., $\mathcal{F}_{P/H}$) is a principal G-bundle (resp., a principal P/H-bundle) on $X \times S$ and κ is a collection of maps of coherent sheaves

$$\kappa^{\mathcal{V}} : (\mathcal{V}^H)_{\mathcal{F}_{P/H}} \to \mathcal{V}_{\mathcal{F}_G},$$

defined for every G-module $\mathcal{V}$, such that for every geometric point $s \in S$, $\kappa_s^{\mathcal{V}}$ is injective, and such that the Plücker relations hold. This means that for $\mathcal{V}$, being the trivial representation, $\kappa^{\mathcal{V}}$ is the identity map $\mathcal{O}_{X\times S} \to \mathcal{O}_{X \times S}$ and for a morphism $\mathcal{V}_1 \otimes \mathcal{V}_2 \to \mathcal{V}_3$, the diagram

$$\begin{tikzcd}
(\mathcal{V}_1^H)_{\mathcal{F}_{P/H}} \otimes (\mathcal{V}_2^H)_{\mathcal{F}_{P/H}} \arrow[r, "\kappa^{\mathcal{V}_1}\otimes\kappa^{\mathcal{V}_2}"] \arrow[d] & (\mathcal{V}_1 \otimes \mathcal{V}_2)_{\mathcal{F}_G} \arrow[d] \\
(\mathcal{V}_3^H)_{\mathcal{F}_{P/H}} \arrow[r, "\kappa^{\mathcal{V}_3}"] & (\mathcal{V}_3)_{\mathcal{F}_G}
\end{tikzcd}$$

commutes.

Note that the data of κ can be reformulated differently, using Frobenius reciprocity: for a P/H-module $\mathcal{U}$, let $\mathrm{Ind}(\mathcal{U})$ denote the corresponding induced G-module, i.e., $\mathrm{Hom}_{P/H}(\mathcal{U}, \mathcal{V}^H) \simeq \mathrm{Hom}_G(\mathrm{Ind}(\mathcal{U}), \mathcal{V})$ for a G-module $\mathcal{V}$, functorially in $\mathcal{V}$. Then the data of κ is the same as a collection of maps κ defined for every P/H-module $\mathcal{U}$:

$$\kappa^{\mathcal{U}} : \mathcal{U}_{\mathcal{F}_{P/H}} \to (\mathrm{Ind}(\mathcal{U}))_{\mathcal{F}_G},$$

which satisfy the Plücker relations in the sense that this map is again the identity for the trivial representation and for $\mathcal{U}_3 \to \mathcal{U}_1 \otimes \mathcal{U}_2$ the diagram

$$\begin{tikzcd}
(\mathcal{U}_3)_{\mathcal{F}_{P/H}} \arrow[r] \arrow[d] & (\mathrm{Ind}(\mathcal{U}_3))_{\mathcal{F}_G} \arrow[d] \\
(\mathcal{U}_1 \otimes \mathcal{U}_2)_{\mathcal{F}_{P/H}} \arrow[r] & (\mathrm{Ind}(\mathcal{U}_1 \otimes \mathcal{U}_2))_{\mathcal{F}_G}
\end{tikzcd}$$

commutes.

In particular, for $H = [P,P]$, it is sufficient to specify the value of κ on 1-dimensional representations of P/H, since this group is commutative.

For a fixed S, it is clear that triples $(\mathcal{F}_G, \mathcal{F}_{P/H}, \kappa)$ form a groupoid, and for a map $S_1 \to S_2$ there is a natural (pull-back) functor between the corresponding groupoids. In addition, there is a natural forgetful morphism from this functor to the functor represented by the stack Bun_G: out of $(\mathcal{F}_G, \mathcal{F}_{P/H}, \kappa)$ we "remember" only $\mathcal{F}_G$, which is a G-bundle on $X \times S$.

The following facts are proved in [BG]:

Theorem 1.2. *For both choices of H the above functor $S \mapsto (\mathcal{F}_G, \mathcal{F}_{P/H}, \kappa)$ is representable by an algebraic stack, such that its map to* Bun_G *is representable and proper.*

We denote the corresponding stacks by $\widetilde{\mathrm{Bun}}_P$ (when $H = U(P)$) and by $\overline{\mathrm{Bun}}_P$ (when $H = [P,P]$). Their projections to Bun_G will be denoted by $\widetilde{\mathfrak{p}}$ and $\overline{\mathfrak{p}}$, respectively.

Note that we have a natural map from the stack Bun_P to both $\widetilde{\mathrm{Bun}}_P$ and $\overline{\mathrm{Bun}}_P$. Indeed, in both cases, a P-bundle on $X \times S$ is the same as a triple $(\mathcal{F}_G, \mathcal{F}_{P/H}, \kappa)$, for which the maps $\kappa^{\mathcal{V}}$ are injective bundle maps.

Since the condition for a map between vector bundles to be *maximal* (maximal means to be an injective bundle map) is open, the above maps $\mathrm{Bun}_P \to \widetilde{\mathrm{Bun}}_P$ and $\mathrm{Bun}_P \to \widetilde{\mathrm{Bun}}_P$ are open embeddings. The following is also established in *loc. cit*:

Theorem 1.3. Bun_P *is dense in both* $\widetilde{\mathrm{Bun}}_P$ *and* $\overline{\mathrm{Bun}}_P$.

Finally, note that since $U(P)$ is contained in $[P,P]$ we have a natural map $\mathfrak{r} : \widetilde{\mathrm{Bun}}_P \to \overline{\mathrm{Bun}}_P$, which is proper and whose restriction to Bun_P is the identity map.

1.4. By construction, a point of $\widetilde{\mathrm{Bun}}_P$ (with values in a field) defines a P-bundle over an open subset X^0 of the curve X (in fact X^0 is precisely the locus, where the maps $\kappa^{\mathcal{V}}$ are maximal embeddings). We will now describe the partition of these stacks according to the behavior of $(\mathcal{F}_G, \mathcal{F}_{P/H}, \kappa)$ on $X - X^0$. First, we will treat the case of $\overline{\mathrm{Bun}}_P$.

Let M be the Levi factor of P. We choose a splitting $M \hookrightarrow P$; in particular we have a well-defined opposite parabolic P^- such that $P \cap P^- = M$. We will denote by $\mathcal{I}_M$ the corresponding Dynkin subdiagram of $\mathcal{I}$.

Let us denote by $\Lambda_{G,P}$ the lattice of cocharacters of the torus $P/H \simeq M/[M,M]$. We have the natural projection $\Lambda \to \Lambda_{G,P}$, whose kernel is the span of $\alpha_\mathfrak{i}$, $\mathfrak{i} \in \mathcal{I}_M$. We will denote by $\Lambda^{\mathrm{pos}}_{G,P}$ the sub-semigroup of $\Lambda_{G,P}$ spanned by the images of $\alpha_\mathfrak{j}$, $\mathfrak{j} \in \mathcal{I} - \mathcal{I}_M$.

Given $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$, we will denote by $\mathfrak{A}(\theta)$ the elements of the set of decompositions of θ as a sum of non-zero elements of $\Lambda^{\mathrm{pos}}_{G,P}$. In other words, each $\mathfrak{A}(\theta)$ is a way to write $\theta = \sum\limits_m n_m \cdot \theta_m$, where θ_m's belong to $\Lambda^{\mathrm{pos}}_{G,P} - 0$ and are pairwise distinct. The length $|\mathfrak{A}(\theta)|$ of a decomposition $\mathfrak{A}(\theta)$ is by definition the sum $\sum\limits_m n_m$.

For $\mathfrak{A}(\theta)$ we will denote by $X^{\mathfrak{A}(\theta)}$ the corresponding partially symmetrized power of the curve, i.e., $X^{\mathfrak{A}(\theta)} = \prod_m X^{(n_m)}$. We will denote by $\overset{o}{X}{}^{\mathfrak{A}(\theta)}$ the complement of the diagonal divisor in $X^{\mathfrak{A}(\theta)}$.

We will think of a point of $\overset{o}{X}{}^{\mathfrak{A}(\theta)}$ as a collection of pairwise distinct points $x_1, ..., x_n$ of X, and to each x_k there is an assigned element $\theta_k \in \Lambda^{\mathrm{pos}}_{G,P}$.

Proposition 1.5. *There exists a locally closed embedding $j_{\mathfrak{A}(\theta)} : \overset{o}{X}{}^{\mathfrak{A}(\theta)} \times \mathrm{Bun}_P \to \overline{\mathrm{Bun}}_P$. Every field-valued point of $\overline{\mathrm{Bun}}_P$ belongs to the image a unique $j_{\mathfrak{A}(\theta)}$.*

Although the proof is given in [BG], let us indicate the construction of $j_{\mathfrak{A}(\theta)}$.

Proof. Let $\mathcal{F}_P$ be a point of Bun_P and $x^{\mathfrak{A}(\theta)} = \sum_k \theta_k \cdot x_k$ a point of $\overset{o}{X}{}^{\mathfrak{A}(\theta)}$. We define the corresponding point of $\overline{\mathrm{Bun}}_P$ as follows. In the triple $(\mathcal{F}_G, \mathcal{F}_{M/[M,M]}, \kappa)$, $\mathcal{F}_G$ is the G-bundle induced from $\mathcal{F}_P$. Now let $\mathcal{F}'_{M/[M,M]}$ be the $M/[M,M]$-bundle induced from $\mathcal{F}_P$. For each G-dominant weight $\check{\lambda}$ orthogonal to $\mathrm{Span}(\alpha_{\mathfrak{i}})$, $\mathfrak{i} \in \mathcal{I}_M$, let us denote by $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}'_{M/[M,M]}}$ the corresponding associated line bundle. By construction, we have the injective bundle maps

$$\kappa'^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathcal{F}'_{M/[M,M]}} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$$

(here $\mathcal{V}^{\check{\lambda}}$ is the Weyl module corresponding to $\check{\lambda}$), which satisfy Plücker relations.

We set $\mathcal{F}_{M/[M,M]} := \mathcal{F}'_{M/[M,M]}(-\Sigma\, \theta_k \cdot x_k)$. The corresponding line bundles $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}_{M/[M,M]}}$ are then $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}'_{M/[M,M]}}(-\Sigma\, \langle \theta_k, \check{\lambda} \rangle \cdot x_k)$. Thus, by composing we obtain the maps

$$\kappa^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathcal{F}_{M/[M,M]}} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G},$$

which are easily seen to satisfy the Plücker relations, as required. □

Let us denote the image of $j_{\mathfrak{A}(\theta)}$ in $\overline{\mathrm{Bun}}_P$ by ${}_{\mathfrak{A}(\theta)}\overline{\mathrm{Bun}}_P$. It is easy to see that the union of ${}_{\mathfrak{A}(\theta)}\overline{\mathrm{Bun}}_P$ is also a locally closed substack of $\overline{\mathrm{Bun}}_P$, denoted by ${}_{\theta}\overline{\mathrm{Bun}}_P$.

If θ is the projection of $\sum_{i \in \mathcal{I} - \mathcal{I}_M} n_{\mathfrak{i}} \cdot \alpha_{\mathfrak{i}}$, set $X^\theta = \prod_{\mathfrak{i} \in \mathcal{I} - \mathcal{I}_M} X^{(n_{\mathfrak{i}})}$. By definition, X^θ is stratified by the spaces $\overset{o}{X}{}^{\mathfrak{A}(\theta)}$ for all possible $\mathfrak{A}(\theta)$. As in Proposition 1.5, we have a locally closed embedding $X^\theta \times \mathrm{Bun}_P \to \overline{\mathrm{Bun}}_P$, whose image is our ${}_{\theta}\overline{\mathrm{Bun}}_P$.

Let us denote by ${}_{\mathfrak{A}(\theta)}\widetilde{\mathrm{Bun}}_P \subset \widetilde{\mathrm{Bun}}_P$ the preimage of ${}_{\mathfrak{A}(\theta)}\overline{\mathrm{Bun}}_P$ under the map $\mathfrak{r}$. Our next goal is to give a more explicit description of each ${}_{\mathfrak{A}(\theta)}\widetilde{\mathrm{Bun}}_P$.

1.6. Let $x \in X$ be a point. Recall that the affine Grassmannian Gr_G is the ind-scheme representing the functor Schemes $\to$ Sets that attaches to a scheme S the set of pairs $(\mathcal{F}_G, \beta)$, where $\mathcal{F}_G$ is a principal G-bundle on $X \times S$ and β is an isomorphism $\mathcal{F}_G|_{(X-x)\times S} \simeq \mathcal{F}^0_G|_{(X-x)\times S}$, where $\mathcal{F}^0_G$ is *the trivial* G-bundle.

Sometimes, to emphasize the dependence on the point x, we will write $\mathrm{Gr}_{G,x}$. Note that by letting x move along the curve, we obtain a relative version of Gr_G, denoted by $\mathrm{Gr}_{G,X}$.

In the same way one defines the affine Grassmannians for the groups M, P, P^-.

For every G-dominant coweight λ one defines a (finite-dimensional) closed subscheme $\overline{\mathrm{Gr}}^{\lambda}_G \subset \mathrm{Gr}_G$ by the condition that $(\mathcal{F}_G, \beta) \in \overline{\mathrm{Gr}}^{\lambda}_G$ if for every G-module whose weights are $\leq \check{\lambda}$, the meromorphic map $\beta^{\mathcal{V}} : \mathcal{V}_{\mathcal{F}_G} \to \mathcal{V}_{\mathcal{F}^0_G} \simeq \mathcal{V} \otimes \mathcal{O}_{X\times S}$ has a pole of order $\leq \langle \lambda, -w_0(\check{\lambda})\rangle$ along $x \times S$.

Now let θ be an element of $\Lambda^{\mathrm{pos}}_{G,P}$. We define the element $\flat(\theta) \in \Lambda$ as follows: if θ is the projection under $\Lambda \to \Lambda_{G,P}$ of $\widetilde{\theta} \in \mathrm{Span}(\alpha_{\mathfrak{j}})$, $\mathfrak{j} \in \mathcal{I} - \mathcal{I}_M$, then $\flat(\theta) = w^M_0(\widetilde{\theta})$, where w^M_0 is the longest element in the Weyl group of M. Note that by construction, $\flat(\theta)$ is M-dominant; in particular, it makes sense to consider $\overline{\mathrm{Gr}}^{\flat(\theta)}_M$.

Consider the functor that attaches to a scheme S the set of pairs $(\mathcal{F}_M, \beta^{\mathcal{V}})$, where $\beta^{\mathcal{V}}$ is an embedding of coherent sheaves defined for every G-module $\mathcal{V}$:

$$\beta^{\mathcal{V}}_M : \mathcal{V}^{U(P)}_{\mathcal{F}_M} \hookrightarrow \mathcal{V}^{U(P)}_{\mathcal{F}^0_M}$$

such that

1) The Plücker relations hold in the same sense as before.
2) If $\mathcal{V}^{U(P)}$ is 1-dimensional corresponding to the character $\check{\nu}$ of M, then $\beta^{\mathcal{V}}_M$ identifies $\mathcal{V}^{U(P)}_{\mathcal{F}_M} := \mathcal{L}^{\check{\nu}}_{\mathcal{F}_M}$ with $\mathcal{L}^{\check{\nu}}_{\mathcal{F}^0_M}(-\langle\theta,\check{\nu}\rangle\cdot x) \simeq \mathcal{O}_{X\times S}(-\langle\theta,\check{\nu}\rangle\cdot x)$. (In the last formula we have used the fact that $\check{\nu}$ and θ belong to mutually dual lattices.)

The following proposition is proved in [BG], but we will sketch the argument due to its importance:

Proposition 1.7. *The above functor is representable by a finite-dimensional closed subscheme, denoted* $\mathrm{Gr}^{+,\theta}_M$, *of* Gr_M. *We have an inclusion* $\overline{\mathrm{Gr}}^{\flat(\theta)}_M \hookrightarrow \mathrm{Gr}^{+,\theta}_M$, *which induces an isomorphism on the level of reduced schemes.*

Proof. Let $(\mathcal{F}_M, \beta^{\mathcal{V}}_M)$ be an S-point of $\mathrm{Gr}^{+,\theta}_M$. To construct a map of functors, $\mathrm{Gr}^{+,\theta}_M \to \mathrm{Gr}_M$ we must exhibit the maps $\beta^{\mathcal{U}}_M : \mathcal{U}_{\mathcal{F}_M} \to \mathcal{U}_{\mathcal{F}^0_M}$ for all M-modules $\mathcal{U}$ and not just for those of the form $\mathcal{V}^{U(P)}$. However, we can do that because any $\mathcal{U}$ can be tensored with a 1-dimensional representation of M corresponding to a G-dominant weight $\check{\nu}$, so that the new representation will be of the form $\mathcal{V}^{U(P)}$.

By construction, the above map $\mathrm{Gr}^{+,\theta}_M \to \mathrm{Gr}_M$ is a closed embedding. The fact that $\mathrm{Gr}^{+,\theta}_M$ is a scheme (and not an ind-scheme) follows from the fact that we can choose $\mathcal{V}$ such that $\mathcal{V}^{U(P)}$ is faithful as a representation of M.

Let $(\mathcal{F}_M, \beta_M)$ be an S-point of $\overline{\mathrm{Gr}}^{\flat(\theta)}_M$. Then if $\mathcal{V}$ is a G-module (whose weights, we can suppose, are $\leq \check{\lambda}$ for some G-dominant weight $\check{\lambda}$), the maps $\beta^{\mathcal{V}^{U(P)}}_M : \mathcal{V}^{U(P)}_{\mathcal{F}_G} \to \mathcal{V}^{U(P)}_{\mathcal{F}^0_G}$ are regular, since $\langle \flat(\theta), -w^M_0(\check{\lambda})\rangle \leq 0$, by the definition of $\flat(\theta)$.

Hence, $\overline{\mathrm{Gr}}_M^{\flat(\theta)}$ is contained in $\mathrm{Gr}_M^{+,\theta}$. To show that this inclusion is an isomorphism on the level of reduced schemes, one has to check that $\overline{\mathrm{Gr}}_M^\lambda \subset \mathrm{Gr}_M^{+,\theta}$ implies that $\flat(\theta) - \lambda$ is a sum of positive coroots of M, which is obvious. □

1.8. Now let us consider the following relative version of the above situation. Let θ be as above and given an element $\mathfrak{A}(\theta)$ let us consider the space $\mathrm{Gr}_M^{+,\mathfrak{A}(\theta)}$ over $\overset{o}{X}{}^{\mathfrak{A}(\theta)}$, whose fiber over $x^{\mathfrak{A}(\theta)} = \Sigma\, \theta_k \cdot x_k \in \overset{o}{X}{}^{\mathfrak{A}(\theta)}$ equals $\prod\limits_k \mathrm{Gr}_{M,x_k}^{+,\theta_k}$.

In addition, we can generalize this further, by replacing the trivial M-bundle in the definition of Gr_M by an arbitrary background M-bundle $\mathcal{F}_M^b$. By letting $\mathcal{F}_M^b$ vary along the universal family, i.e., Bun_M, we obtain the relative version $\mathcal{H}_M$ of $\mathrm{Gr}_{M,X}$, which is fibered over $X \times \mathrm{Bun}_M$ and is known in the literature as the *Hecke stack*. The relative version of $\mathrm{Gr}_M^{+,\mathfrak{A}(\theta)}$ will be denoted by $\mathcal{H}_M^{+,\mathfrak{A}(\theta)}$ and it is by definition fibered over $\overset{o}{X}{}^{\mathfrak{A}(\theta)} \times \mathrm{Bun}_M$.

Proposition 1.9. *There exists a canonical isomorphism*

$$_{\mathfrak{A}(\theta)}\widetilde{\mathrm{Bun}}_P \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_M^{+,\mathfrak{A}(\theta)},$$

such that the projection $\mathfrak{r}$ *onto* $_{\mathfrak{A}(\theta)}\overline{\mathrm{Bun}}_P$ *on the LHS corresponds to the natural map of the RHS to* $\mathrm{Bun}_P \times \overset{o}{X}{}^{\mathfrak{A}(\theta)}$.

The proof is given in [BG] and is, in fact, an easy consequence of Proposition 1.7 above.

1.10. Finally, we are able to state Theorem 1.12, which is the main result of this paper.

First, let us recall the category of spherical perverse sheaves on Gr_G, which by definition consists of direct sums of perverse sheaves $\mathrm{IC}_{\overline{\mathrm{Gr}}_G^\lambda}$, as λ ranges over the set of G-dominant coweights. It is known that this category possesses a tensor structure, given by the convolution product, and as a tensor category it is equivalent to the category $\mathrm{Rep}(\check{G})$ of finite-dimensional representations of the Langlands dual group $\check{G}$. In particular, we have the functor $\mathrm{Loc} : \mathrm{Rep}(\check{G}) \to \mathrm{Prev}(\mathrm{Gr}_G)$, such that the irreducible representation of $\check{G}$ with h.w. λ goes over under this functor to $\mathrm{IC}_{\overline{\mathrm{Gr}}_G^\lambda}$.

We will use the above definitions for M, rather than for G. Recall that $\Lambda_{G,P}$ can be canonically identified with the lattice of characters of $Z(\check{M})$; for $\theta \in \Lambda_{G,P}$ and an $\check{M}$-representation V, we will denote by V_θ the direct summand of V on which $Z(\check{M})$ acts according to θ. We will call a representation V of $\check{M}$ *positive* if $\forall\, \theta \in \Lambda_{G,P}^{\mathrm{pos}}$, the perverse sheaf $\mathrm{Loc}(V_\theta)$ is supported on $\overline{\mathrm{Gr}}_M^{\flat(\theta)}$.

Recall that the nilpotent radical of the dual parabolic $\check{\mathfrak{u}}(P)$ is naturally a representation of the group $\check{M}$ and let us observe that for $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$, the sub-representation $\check{\mathfrak{u}}(P)_\theta$ is irreducible.

Lemma 1.11. *The representation* $\mathrm{Sym}(\check{\mathfrak{u}}(P))$ *is positive.*

Now, let us fix the notation for the relative versions of the functor Loc. First, we will denote by Loc_X the corresponding functor $\mathrm{Loc} : \mathrm{Rep}(\check{M}) \to \mathrm{Perv}(\mathrm{Gr}_{M,X})$. Secondly, given θ, a decomposition $\mathfrak{A}(\theta)$ and a positive $\check{M}$-representation V, we will denote by $\mathrm{Loc}^{\mathfrak{A}(\theta)}_X(V)$ the perverse sheaf on $\mathrm{Gr}^{+,\mathfrak{A}(\theta)}_M$, whose fiber over $x^{\mathfrak{A}(\theta)} = \Sigma\, \theta_k \cdot x_k \in \overset{o}{X}{}^{\mathfrak{A}(\theta)}$, i.e., over $\prod_k \mathrm{Gr}^{+,\theta_k}_{M,x_k}$, is

$$\boxtimes_k \mathrm{Loc}(V_{\theta_k}) \otimes \left(\overline{\mathbb{Q}_\ell}[1]\left(\frac{1}{2}\right)\right)^{\otimes|\mathfrak{A}(\theta)|}.$$

We will use the symbol $\mathrm{Loc}_{\mathrm{Bun}_M,X}(V)$ (resp., $\mathrm{Loc}^{\mathfrak{A}(\theta)}_{\mathrm{Bun}_M,X}(V)$) to denote the corresponding perverse sheaves on $\mathcal{H}_M$ (resp., $\mathcal{H}^{+,\mathfrak{A}(\theta)}_M$). Furthermore, by considering the tensor product $\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_M$ we can define a perverse sheaf $\mathrm{Loc}_{\mathrm{Bun}_P,X}(V)$ on it, which is $\mathrm{IC}_{\mathrm{Bun}_P}$ "along the base" (i.e., Bun_P) and $\mathrm{Loc}_{\mathrm{Bun}_M,X}(V)$ "along the fiber" (i.e., $\mathcal{H}_M$), and similarly for the perverse sheaf $\mathrm{Loc}^{\mathfrak{A}(\theta)}_{\mathrm{Bun}_P,X}(V)$ on $\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}^{+,\mathfrak{A}(\theta)}_M$.

Theorem 1.12. *The $*$-restriction of* $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ *to* ${}_{\mathfrak{A}(\theta)}\widetilde{\mathrm{Bun}}_P \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}^{\mathfrak{A}(\theta)}_M$ *is isomorphic to*

$$\mathrm{Loc}^{\mathfrak{A}(\theta)}_{\mathrm{Bun}_P,X}(\underset{i\geq 0}{\oplus} \mathrm{Sym}^i(\check{\mathfrak{u}}(P)) \otimes \overline{\mathbb{Q}_\ell}(i)[2i]) \otimes \left(\overline{\mathbb{Q}_\ell}[1]\left(\frac{1}{2}\right)\right)^{\otimes -|\mathfrak{A}(\theta)|},$$

where $\underset{i\geq 0}{\oplus} \mathrm{Sym}^i(\check{\mathfrak{u}}(P)) \otimes \overline{\mathbb{Q}_\ell}(i)[2i]$ *is viewed as a cohomologically graded $\check{M}$-module.*

2. Zastava spaces

Let θ be an element of $\Lambda^{\mathrm{pos}}_{G,P}$. In this section we will introduce the Zastava spaces Z^θ, which will be local models for $\widetilde{\mathrm{Bun}}_P$.

2.1. Let us recall the space X^θ: if $\theta = \underset{i\in\mathcal{I}-\mathcal{I}_M}{\Sigma} n_i \cdot \alpha_i$, $X^\theta = \underset{i\in\mathcal{I}-\mathcal{I}_M}{\Pi} X^{(n_i)}$. One may alternatively view X^θ as the space classifying the data of $(\mathcal{F}_{M/[M,M]}, \beta_{M/[M,M]})$, where $\mathcal{F}_{M/[M,M]}$ is a principal bundle with respect to the group $M/[M,M]$ on X of degree $-\theta$, and $\beta_{M/[M,M]}$ is a system of embeddings defined for every G-dominant weight $\check{\lambda}$ orthogonal to $\mathrm{Span}(\alpha_{\mathfrak{i}})$, $\mathfrak{i} \in \mathcal{I}_M$

$$\beta^{\check{\lambda}}_{M/[M,M]} : \mathcal{L}^{\check{\lambda}}_{\mathcal{F}_{M/[M,M]}} \to \mathcal{L}^{\check{\lambda}}_{\mathcal{F}^0_{M/[M,M]}} \simeq \mathcal{O}_X,$$

such that $\beta^{\check{\lambda}_1} \otimes \beta^{\check{\lambda}_2} = \beta^{\check{\lambda}_1+\check{\lambda}_2}$.

Note that in the product $X \times X^\theta$ there is a natural incidence divisor, denoted by Γ^θ.

Now, let us define the scheme $\mathrm{Mod}^{+,\theta}_M$. By definition, its S-points are pairs $(\mathcal{F}_M, \beta_M)$, where $\mathcal{F}_M$ is an M-bundle on $X \times S$ such that the induced $M/[M,M]$-bundle is of degree $-\theta$, and β_M is a system of embeddings of coherent sheaves defined for every G-module $\mathcal{V}$

$$\beta^{\mathcal{V}}_M : (\mathcal{V}^{U(P)})_{\mathcal{F}_M} \hookrightarrow (\mathcal{V}^{U(P)})_{\mathcal{F}^0_M},$$

such that for a pair of G-modules $\mathcal{V}_1$ and $\mathcal{V}_2$ we have a commutative diagram

$$\begin{array}{ccc}
(\mathcal{V}_1^{U(P)})_{\mathcal{F}_M} \otimes (\mathcal{V}_2^{U(P)})_{\mathcal{F}_M} & \xrightarrow{\beta^{\mathcal{V}_1}_M \otimes \beta^{\mathcal{V}_2}_M} & (\mathcal{V}_1^{U(P)})_{\mathcal{F}^0_M} \otimes (\mathcal{V}_2^{U(P)})_{\mathcal{F}^0_M} \\
\downarrow & & \downarrow \\
(\mathcal{V}_1 \otimes \mathcal{V}_2)^{U(P)}_{\mathcal{F}_M} & \xrightarrow{\beta^{\mathcal{V}_1 \otimes \mathcal{V}_2}_M} & (\mathcal{V}_1 \otimes \mathcal{V}_2)^{U(P)}_{\mathcal{F}^0_M}.
\end{array}$$

It is easy to see, as in Proposition 1.7, that $\mathrm{Mod}^{+,\theta}_M$ is indeed representable by a scheme of finite type.

By construction, we have a natural map $\pi_M : \mathrm{Mod}^{+,\theta}_M \to X^\theta$, which corresponds to taking for $\mathcal{V}$ all possible 1-dimensional M-modules. If $(\mathcal{F}_M, \beta_M)$ is an S-point of $\mathrm{Mod}^{+,\theta}_M$, it follows as in Proposition 1.7 that β_M defines a trivialization of $\mathcal{F}_M$ on $X \times S - \Gamma^\theta_S$, where Γ^θ_S is the preimage of Γ^θ under $X \times S \to X \times \mathrm{Mod}^{+,\theta}_M \to X \times X^\theta$. Moreover, we have:

$$\overset{o}{X}{}^{\mathfrak{A}(\theta)} \underset{X^\theta}{\times} \mathrm{Mod}^{+,\theta}_M \simeq \mathrm{Gr}^{+,\mathfrak{A}(\theta)}_M.$$

In particular, $\mathrm{Mod}^{+,\theta}_M |_{\Delta_X} \simeq \mathrm{Gr}^{+,\theta}_{M,X}$, where $\Delta_X \subset X^\theta$ is the main diagonal.

2.2. Finally, we are ready to define Z^θ. An S-point of Z^θ is a quadruple

$$(\mathcal{F}_G, \mathcal{F}_M, \beta_M, \beta),$$

where $\mathcal{F}_G$ is a G-bundle on $X \times S$, $(\mathcal{F}_M, \beta_M)$ is a point of $\mathrm{Mod}^{+,\theta}_M$ and β is a trivialization of $\mathcal{F}_G$ on $X \times S - \Gamma^\theta_S$, where Γ^θ_S is as above, such that the following two conditions are satisfied:

(1) For every G-module $\mathcal{V}$, the natural map $\mathcal{V} \to \mathcal{V}_{U(P^-)}$ extends to a regular surjective map of vector bundles

$$\mathcal{V}_{\mathcal{F}_G} \xrightarrow{\beta} \mathcal{V}_{\mathcal{F}^0_G} \to (\mathcal{V}_{U(P^-)})_{\mathcal{F}^0_M} \simeq \mathcal{V}_{U(P^-)} \otimes \mathcal{O}_{X \times S}.$$

(2) For every G-module $\mathcal{V}$, the natural map $\mathcal{V}^{U(P)} \to \mathcal{V}$ extends by means of β and β_M to a regular embedding of coherent sheaves

$$(\mathcal{V}^{U(P)})_{\mathcal{F}_M} \to \mathcal{V}_{\mathcal{F}_G}.$$

From the above definition, it follows that Z^θ is representable by an ind-scheme. However, we will see later that Z^θ is in fact a scheme of finite type, cf. Proposition 3.2.

We will denote by π_P the natural map $Z^\theta \to \mathrm{Mod}_M^{+,\theta}$; by π_G we will denote the composition $\pi_M \circ \pi_P : Z^\theta \to X^\theta$.

By definition, Z^θ contains as a subscheme the locus of those $(\mathcal{F}_G, \mathcal{F}_M, \beta_M, \beta)$, for which the maps $(\mathcal{V}^{U(P)})_{\mathcal{F}_M} \to \mathcal{V}_{\mathcal{F}_G}$ are *maximal embeddings*, i.e. are bundle maps. We will denote this subscheme by $Z^\theta_{\max}$.

Observe now that there is a natural closed embedding: $\mathfrak{s}^\theta : \mathrm{Mod}_M^{+,\theta} \to Z^\theta$. Indeed, to $(\mathcal{F}_M, \beta_M) \in \mathrm{Mod}_M^{+,\theta}$ we attach $(\mathcal{F}_G^0, \mathcal{F}_M, \beta_M, \beta^0)$, where β^0 is the tautological trivialization of the trivial bundle.

Remark. Note that for the definition of the Zastava space Z^θ, the curve X need not be complete. Indeed, the only modification is the following. In the definition of $\mathrm{Mod}_M^{+,\theta}$, instead of having pairs $(\mathcal{F}_M, \beta_M)$ we can consider triples $(\mathcal{F}_M, \beta_M, S \to X^\theta)$ where β_M is such that for every 1-dimensional M-module $\mathcal{V}^{\check\lambda}$ (such $\check\lambda$ is automatically orthogonal to $\alpha_\imath$, $\imath \in \mathcal{I}_M$), β_M induces an isomorphism $\mathcal{V}^{\check\lambda}_{\mathcal{F}_M} \simeq \mathcal{O}_{X\times S}(-\langle \theta, \check\lambda\rangle \cdot \Gamma^\theta_S)$.

2.3. Factorization property

The fundamental property of the spaces Z^θ is their local behavior with respect to the base X^θ.

Let $\theta = \theta_1 + \theta_2$ with $\theta_i \in \Lambda^{\mathrm{pos}}_{G,P}$ and let us denote by $(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}}$ the open subset of the direct product $X^{\theta_1} \times X^{\theta_2}$, which corresponds to $x^{\theta_1} \in X^{\theta_1}$, $x^{\theta_2} \in X^{\theta_2}$, such that the supports of x^{θ_1} and x^{θ_2} are disjoint.

We have a natural étale map $(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \to X^\theta$.

Proposition 2.4. *There is a natural isomorphism*

$$(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \underset{X^\theta}{\times} Z^\theta \simeq (X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \underset{X^{\theta_1}\times X^{\theta_2}}{\times} (Z^{\theta_1} \times Z^{\theta_2}).$$

Proof. Let $x^{\theta_1} \times x^{\theta_2}$ be an S-point of $(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}}$. By definition, this means that the divisors $\Gamma^{\theta_1}_S$ and $\Gamma^{\theta_2}_S$ in $X \times S$ do not intersect. Let $(\mathcal{F}_G, \mathcal{F}_M, \beta_M, \beta)$ be an S-point of Z^θ which projects under π_G to the corresponding point of X^θ.

Set $(X \times S)^1 = X \times S - \Gamma^{\theta_1}_S$, $(X \times S)^2 = X \times S - \Gamma^{\theta_2}_S$, $(X \times S)^0 = (X\times S)^1 \cap (X\times S)^2$. By assumption, $(X\times S)^1 \cup (X\times S)^2 = X\times S$. We define a new

G-bundle $\mathcal{F}_G^1$ as follows: over $(X \times S)^1$, $\mathcal{F}_G^1$ is by definition the trivial bundle $\mathcal{F}_G^0$; over $(X \times S)^2$, $\mathcal{F}_G^1$ is identified with $\mathcal{F}_G$; the data of β, being a trivialization of $\mathcal{F}_G$ over $(X \times S)^0$, defines a patching data for $\mathcal{F}_G^1$. By construction, $\mathcal{F}_G^1$ is trivialized off x^{θ_1}; let us denote this trivialization by β^1.

We introduce the second G-bundle $\mathcal{F}_G^2$ in a similar fashion: $\mathcal{F}_2|_{(X\times S)^2} \simeq \mathcal{F}_G|_{(X\times S)^2}$ and $\mathcal{F}_2|_{(X\times S)^1} \simeq \mathcal{F}_G^0|_{(X\times S)^1}$. From the construction, $\mathcal{F}_G^2$ acquires a trivialization $\beta^2 : \mathcal{F}_G^2|_{(X\times S)^2} \simeq \mathcal{F}_G^0|_{(X\times S)^2}$.

In a similar way, from $(\mathcal{F}_M, \beta_M)$ we obtain two pairs $(\mathcal{F}_M^1, \beta_M^1) \in \mathrm{Mod}_M^{+,\theta_1}$ and $(\mathcal{F}_M^2, \beta_M^2) \in \mathrm{Mod}_M^{+,\theta_2}$, which project under π_M to x^{θ_1} and x^{θ_2}, respectively.

Thus, from the S-point $(\mathcal{F}_G, \mathcal{F}_M, \beta_M, \beta)$ we obtain two S-points $(\mathcal{F}_G^1, \mathcal{F}_M^1, \beta_M^1, \beta^1)$ and $(\mathcal{F}_G^2, \mathcal{F}_M^2, \beta_M^2, \beta^2)$ of Z^{θ_1} and Z^{θ_2}, respectively. The map in the opposite direction is constructed in the same way. □

In the course of the proof we have shown that the space $\mathrm{Mod}_M^{+,\theta}$ factorizes as well, i.e. we have a natural isomorphism

$$(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \underset{X^\theta}{\times} \mathrm{Mod}_M^{+,\theta} \simeq (X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \underset{X^{\theta_1}\times X^{\theta_2}}{\times} \mathrm{Mod}_M^{+,\theta_1} \times \mathrm{Mod}_M^{+,\theta_2},$$

compatible with the factorization of Z^θ. In addition, it is clear that the section $\mathfrak{s}^\theta$ is compatible with the factorizations in the natural sense.

2.5. The central fiber

Consider the main diagonal $\Delta_X : X \to X^\theta$. For a fixed point $x \in X$ let us consider the corresponding composition $\Delta_x : \mathrm{pt} \to X \to X^\theta$.

The central fiber $\mathbb{S}^\theta$ of Z^θ is by definition the preimage of the above point under $\pi_G : Z^\theta \to X^\theta$. We will denote by ${}_0\mathbb{S}^\theta$ the intersection $\mathbb{S}^\theta \cap Z^\theta_{\max}$.

For $\theta \in \Lambda_{G,P}$, let Gr_P^θ be the preimage under $\mathrm{Gr}_P \to \mathrm{Gr}_{M/[M,M]}$ of the corresponding point-scheme in $\mathrm{Gr}_{M/[M,M]}$. Both Gr_P^θ and $\mathrm{Gr}_{U(P^-)}$ are locally closed subschemes of Gr_G and let us consider their intersection $\mathrm{Gr}_P^\theta \cap \mathrm{Gr}_{U(P^-)}$.

Proposition 2.6. *There is a natural identification* $\mathrm{Gr}_P^\theta \cap \mathrm{Gr}_{U(P^-)} \simeq {}_0\mathbb{S}^\theta$. *The map* ${}_0\mathbb{S}^\theta \xrightarrow{\pi_P} \mathrm{Gr}_M^{+,\theta} \hookrightarrow \mathrm{Gr}_M$ *corresponds to* $\mathrm{Gr}_P^\theta \cap \mathrm{Gr}_{U(P^-)} \hookrightarrow \mathrm{Gr}_P \to \mathrm{Gr}_M$.

Proof. By construction, an S-point of ${}_0\mathbb{S}^\theta$ is a data of a G-bundle $\mathcal{F}_G$ on $X \times S$, with given reductions $\mathcal{F}_P$ and $\mathcal{F}_{P^-}$ to P and P^-, respectively, such that these reductions are mutually transversal on $(X - x) \times S$, the M-bundle induced from $\mathcal{F}_{P^-}$ is trivialized, and the maps

$$\mathcal{L}^{\check\lambda}_{\mathcal{F}_{M/[M,M]}} \to \mathcal{O}_{X\times S}(-\langle\check\lambda, \theta\rangle \cdot x)$$

are isomorphisms for G-dominant characters $\check\lambda$ of $M/[M,M]$.

Therefore, over $(X-x)\times S$ all the three principal bundles $\mathcal{F}_G$, $\mathcal{F}_P$ and $\mathcal{F}_{P^-}$ are trivialized in a compatible way, and the $M/[M,M]$-bundle induced from $\mathcal{F}_P$ is exactly $\mathcal{F}^0_{M/[M,M]}(-\theta\cdot x)$. Thus, $(\mathcal{F}_P,\mathcal{F}_{P^-})$ indeed defines a point of $\mathrm{Gr}^\theta_P\cap\mathrm{Gr}_{U(P^-)}$.

Conversely, let us be given an S-point of $\mathrm{Gr}^\theta_P\cap\mathrm{Gr}_{U(P^-)}$. I.e., we have a P-bundle $\mathcal{F}_P$ and a $U(P^-)$-bundle $\mathcal{F}_{U(P^-)}$ on $X\times S$, trivialized on $(X-x)\times S$, and an isomorphism of the corresponding induced G-bundles

$$G\underset{P}{\times}\mathcal{F}_P\simeq:\mathcal{F}_G:\simeq G\underset{U(P^-)}{\times}\mathcal{F}_{U(P^-)},$$

compatible with the trivilizations.

Let $\mathcal{F}_M$ be the M-bundle induced from $\mathcal{F}_P$. It is also trivialized over $(X-x)\times S$, and it remains to show that the S-point of Gr_M obtained in this way belongs to $\mathrm{Gr}^{+,\theta}_M$.

By construction, the corresponding $M/[M,M]$-bundle is $\mathcal{F}^0_{M/[M,M]}(-\theta\cdot x)$. Therefore, we only have to show that for an M-module $\mathcal{U}$ of the form $\mathcal{V}^{U(P)}$ for a G-module $\mathcal{V}$, the map $\mathcal{U}_{\mathcal{F}_M}\to\mathcal{U}_{\mathcal{F}^0_M}$ is regular.

For any $\mathcal{V}$ as above we have bundle maps

$$(\mathcal{V}^{U(P)})_{\mathcal{F}_M}\to\mathcal{V}_{\mathcal{F}_G}\to(\mathcal{V}_{U(P^-)})_{\mathcal{F}^0_M},$$

and, therefore, the maps $(\mathcal{V}^{U(P)})_{\mathcal{F}_M}\to(\mathcal{V}_{U(P^-)})_{\mathcal{F}^0_M}$ are regular.

Lemma 2.7. *For any M-module $\mathcal{U}$ isomorphic to $\mathcal{V}^{U(P)}$ for some G-module $\mathcal{V}$, the composition*

$$\mathcal{U}\to(\mathrm{Ind}(\mathcal{U}))^{U(P)}\to(\mathrm{Ind}(\mathcal{U}))_{U(P^-)}$$

is an isomorphism.

(The proof of the lemma is given below.)

Thus, for $\mathcal{V}_1:=\mathrm{Ind}(\mathcal{U})$, we obtain a commutative diagram

$$\begin{array}{ccc}\mathcal{U}_{\mathcal{F}_M} & \longrightarrow & (\mathcal{V}_1^{U(P)})_{\mathcal{F}_M}\\ \big\downarrow & & \big\downarrow\\ \mathcal{U}_{\mathcal{F}^0_M} & \longrightarrow & ((\mathcal{V}_1)_{U(P^-)})_{\mathcal{F}^0_M}.\end{array}$$

in which the upper horizontal and the right vertical arrows are regular, and the lower horizontal arrow is an isomorphism. Hence, the left vertical arrow is also regular, which is what we had to prove. $\square$

As a consequence, we obtain that since Z^θ is a scheme of finite type (which will be proved shortly), the intersection $\mathrm{Gr}^\theta_P\cap\mathrm{Gr}_{U(P^-)}$ is also a scheme of finite type.

Proof (of Lemma 2.7). Note first that when P is the Borel subgroup, our assertion is that for a dominant weight $\check{\lambda}$, the vector space $(\mathcal{V}^{\check{\lambda}})_{U(B^-)}$ is the original 1-dimensional representation of the Cartan group T corresponding to the character $\check{\lambda}$.

To prove the lemma, it will be more convenient to work in the dual set-up. For an M-module $\mathcal{U}$, let $\mathrm{Coind}(\mathcal{U})$ be the corresponding coinduced G-module, i.e., the space of global sections of the G-equivariant vector bundle $\mathfrak{F}_{\mathcal{U}}$ on G/P, whose fiber at $1 \in G/P$ is $\mathcal{U}$.

We need to show that the map $(\mathrm{Coind}(\mathcal{U}))^{U(P^-)} \to \mathcal{U}$ is an isomorphism whenever $\mathcal{U}$ is of the form $\mathcal{V}_{U(P)}$ for a G-module $\mathcal{V}$.

First, the vector space $\mathcal{U}$ can be identified with the space of $U(P^-)$-invariant sections of $\mathfrak{F}_{\mathcal{U}}$ on the open $U(P^-)$-orbit on G/P. In particular, the map $(\mathrm{Coind}(\mathcal{U}))^{U(P^-)} \to \mathcal{U}$ is always injective. Thus, we have to show that for $\mathcal{U} \simeq \mathcal{V}_{U(P)}$, every $U(P^-)$-invariant section of $\mathfrak{F}_{\mathcal{U}}$ on the open $U(P^-)$-orbit extends regularly on the entire G/P.

For that, it is sufficient to show that any such section is regular at the generic point of the complement to the open $U(P^-)$-orbit in G/P. The latter problem reduces to $G = SL(2)$ and P being the Borel subgroup, in which case it is known, cf. above. □

3. Relation of the Zastava spaces with $\widetilde{\mathrm{Bun}}_P$

Our goal now is to show that the space Z^θ models the stack $\widetilde{\mathrm{Bun}}_P$ from the point of view of singularities.

3.1. First, we have to introduce the following relative version of Z^θ.

Let $\mathcal{F}_M^b$ be a fixed M-bundle on X and let $\mathcal{F}_G^b$ be the induced G-bundle under our fixed embedding $M \hookrightarrow G$. The space $Z^\theta_{\mathcal{F}^b_M}$ is defined as follows: it classifies quadruples $(\mathcal{F}_G, \mathcal{F}_M, \beta_M, \beta)$ as in the case of Z^θ with the only difference that the trivial M-bundle $\mathcal{F}_M^0$ is replaced by $\mathcal{F}_M^b$ and the trivial G-bundle $\mathcal{F}_G^0$ is replaced by $\mathcal{F}_G^b$.

Since every M-bundle is locally trivial (cf. [DS]), and due to the factorization property, the spaces Z^θ and $Z^\theta_{\mathcal{F}^b_M}$ are étale-locally isomorphic.

Similarly, if S is a scheme (or a stack) mapping to Bun_M, we can define the space Z^θ_S. When S is smooth, then using [DS] we obtain that Z^θ_S is locally in the smooth topology equivalent to Z^θ. In practice, we will take S to be Bun_M.

Along the same lines, we define the relative version $\mathrm{Mod}^{+,\theta}_{M,S}$ of $\mathrm{Mod}^{+,\theta}_M$. When $S = \mathrm{Bun}_M$, we will denote it by $\mathrm{Mod}^{+,\theta}_{\mathrm{Bun}_M}$.

Recall that the stack Bun_M splits into connected components numbered by the elements of $\Lambda_{G,P}$. By definition, a point $\mathcal{F}_M$ belongs to the connected component Bun^θ_M if the associated $M/[M,M]$-bundle is of degree $-\theta$. We will use the superscript θ to designate the corresponding connected component of the stack $\widetilde{\mathrm{Bun}}_P$ or Bun_{P^-}.

Proposition 3.2. *For every $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$ and $\theta' \in \Lambda_{G,P}$ there is a canonical isomorphism between $Z^\theta_{\mathrm{Bun}^{\theta'}_M}$ and an open sub-stack of $\widetilde{\mathrm{Bun}}^{\theta+\theta'}_P \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\theta'}_{P^-}$.*

From this proposition it follows, in particular, that $Z^\theta_{\mathrm{Bun}_M}$ is a stack locally of finite type, and hence Z^θ is a scheme of finite type (and not just an ind-scheme).

Proof. Let us analyze what it means to have an S-point of the cartesian product

$$\widetilde{\mathrm{Bun}}^{\theta+\theta'}_P \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\theta'}_{P^-}.$$

By definition, we have a G-bundle, a pair of M-bundles $\mathcal{F}_M$ and $\mathcal{F}'_M$ and two systems of maps κ and κ^- for every G-module $\mathcal{V}$:

$$\kappa : (\mathcal{V}^{U(P)})_{\mathcal{F}_M} \to \mathcal{V}_{\mathcal{F}_G}$$
$$\kappa^- : \mathcal{V}_{\mathcal{F}_G} \to (\mathcal{V}_{U(P^-)})_{\mathcal{F}'_M},$$

which satisfy the Plücker relations, with the condition that the κ^-'s are surjective, and the κ's are injective over every geometric point of S.

We define the open substack $(\widetilde{\mathrm{Bun}}^{\theta+\theta'}_P \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\theta'}_{P^-})^0$ by the following condition: for every geometric point $s \in S$, the P- and P^--structures defined on $\mathcal{F}_G|_s$ by means of κ and κ^- *at the generic point of* X are mutually transversal.

Let $(\mathcal{F}_G, \mathcal{F}_M, \mathcal{F}'_M, \beta_M, \beta)$ be an S-point of $Z^\theta_{\mathrm{Bun}^{\theta'}_M}$. It is clear that the maps

$$(\mathcal{V}^{U(P)})_{\mathcal{F}_M} \to \mathcal{V}_{\mathcal{F}_G} \text{ and } \mathcal{V}_{\mathcal{F}_G} \to (\mathcal{V}_{U(P^-)})_{\mathcal{F}'_M},$$

as in the definition of $Z^\theta_{\mathrm{Bun}_M}$ indeed define an S-point of $(\widetilde{\mathrm{Bun}}^{\theta+\theta'}_P \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\theta'}_{P^-})^0$.

The rest of the proof basically repeats that of Proposition 2.6. To define the map in the opposite direction, given an S-point of $(\widetilde{\mathrm{Bun}}^{\theta+\theta'}_P \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\theta'}_{P^-})^0$, we define an S-point of $Z^\theta_{\mathrm{Bun}_M}$ as follows.

First, we set the "background" M-bundle to be $\mathcal{F}'_M$. Let $\mathcal{F}'_G$ denote the induced G-bundle under $M \hookrightarrow G$. Secondly, by construction, there exists an open dense subset $(X \times S)^0 \subset X \times S$, such that $\mathcal{F}_G|_{(X\times S)^0}$ admits reductions simultaneously to P and P^-, which are, moreover, transversal. Hence, over $(X \times S)^0$, we have identifications $\beta : \mathcal{F}_G \simeq \mathcal{F}'_G$ and $\beta_M : \mathcal{F}_M \simeq \mathcal{F}'_M$.

Therefore, it remains to show that β_M is such that the maps $\beta^{\mathcal{U}}_M : \mathcal{U}_{\mathcal{F}_M} \to \mathcal{U}_{\mathcal{F}'_M}$, which are defined on $(X \times S)^0$, extend as regular maps to the entire $X \times S$, provided that $\mathcal{U}$ is of the form $\mathcal{V}^{U(P)}$ for a G-module $\mathcal{V}$. This is done exactly as in Proposition 2.6. $\square$

3.3. Observe that under the isomorphism of the above proposition, the open substack $Z^\theta_{\max \mathrm{Bun}_M}$ coincides with the preimage of $\mathrm{Bun}_P \subset \widetilde{\mathrm{Bun}}_P$. Let us now analyze the behavior of other strata of $\widetilde{\mathrm{Bun}}_P$ under the isomorphism of Proposition 3.2.

Recall that for $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$ we introduced a locally closed substack ${}_\theta\overline{\mathrm{Bun}}_P \subset \overline{\mathrm{Bun}}_P$, as the image of the natural map

$$X^\theta \times \mathrm{Bun}_P \to \overline{\mathrm{Bun}}_P.$$

Let ${}_\theta\widetilde{\mathrm{Bun}}_P \subset \widetilde{\mathrm{Bun}}_P$ denote the preimage of ${}_\theta\overline{\mathrm{Bun}}_P$ under $\mathfrak{r} : \widetilde{\mathrm{Bun}}_P \to \overline{\mathrm{Bun}}_P$. As in Proposition 1.9 one shows that

$${}_\theta\widetilde{\mathrm{Bun}}_P \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Mod}^{+,\theta}_{\mathrm{Bun}_M}.$$

Let $({}_{\theta''}\widetilde{\mathrm{Bun}}_P^{\theta+\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\theta'}_{P^-})^0$ be the preimage of ${}_{\theta''}\widetilde{\mathrm{Bun}}_P^{\theta+\theta'}$ under the natural projection.

Lemma 3.4. *The stack* $({}_{\theta''}\widetilde{\mathrm{Bun}}_P^{\theta+\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\theta'}_{P^-})^0$ *is empty unless* $\theta - \theta'' \in \Lambda^{\mathrm{pos}}_{G,P}$. *For* $\theta'' = \theta$, *the above substack identifies with the image of* $\mathrm{Mod}^{+,\theta}_{\mathrm{Bun}^{\theta'}_M}$ *under* $\mathfrak{s}^\theta : \mathrm{Mod}^{+,\theta}_{\mathrm{Bun}^{\theta'}_M} \hookrightarrow Z^\theta_{\mathrm{Bun}^{\theta'}_M}$.

Proof. Note that an S-point of $\overline{\mathrm{Bun}}_P$ belongs to ${}_{\theta''}\overline{\mathrm{Bun}}_P$ if and only if the following condition holds: for every G-dominant weight $\check{\lambda}$, orthogonal to $\mathrm{Span}(\alpha_\mathfrak{i})$ for $\mathfrak{i} \in \mathcal{I}_M$, the corresponding map

$$\kappa^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathcal{F}_{M/[M,M]}} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$$

is such that there exists a short exact sequence

$$0 \to \mathcal{M}_1 \to \mathrm{coker}(\kappa^{\check{\lambda}}) \to \mathcal{M}_2 \to 0,$$

such that $\mathcal{M}_2$ is a vector bundle on $X \times S$, the support of $\mathcal{M}_1$ is X-finite and over any geometric point $s \in S$, the length of $\mathcal{M}_1|_s$ is exactly $\langle \theta'', \check{\lambda} \rangle$.

Given an S-point of $({}_{\theta''}\widetilde{\mathrm{Bun}}_P^{\theta+\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\theta'}_{P^-})^0$, we can compose the above embedding of sheaves with

$$\mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G} \xrightarrow{\kappa^-} (\mathcal{V}^{\check{\lambda}}_{U(P^-)})_{\mathcal{F}'_M} \to \mathcal{L}^{\check{\lambda}}_{\mathcal{F}'_{M/[M,M]}}$$

and we obtain a map between line bundles, such that over every geometric point $s \in S$ its total amount of zeroes is $\langle \theta'', \check{\lambda} \rangle$. This readily implies the first assertion of the lemma.

To prove the second assertion, observe that

$$({}_{\theta}\widetilde{\mathrm{Bun}}_P^{\theta+\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}^{\theta'})^0 \simeq (\mathrm{Bun}_{P^-}^{\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P^{\theta'})^0 \underset{\mathrm{Bun}_M}{\times} \mathrm{Mod}_{\mathrm{Bun}_M}^{+,\theta}.$$

However, the condition on the degree forces that

$$(\mathrm{Bun}_{P^-}^{\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P^{\theta'})^0 \simeq \mathrm{Bun}_M^{\theta'}.$$

Hence,

$$({}_{\theta}\widetilde{\mathrm{Bun}}_P^{\theta+\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}^{\theta'})^0 \simeq \mathrm{Mod}_{\mathrm{Bun}_M^{\theta'}}^{+,\theta}$$

and the fact that its embedding into $Z^{\theta}_{\mathrm{Bun}_M}$ coincides with $\mathfrak{s}^{\theta}$ follows from the construction. □

3.5. For an element θ' with $\theta-\theta' \in \Lambda^{\mathrm{pos}}_{G,P}$, let us denote by ${}_{\theta'}Z^{\theta}$ the corresponding locally closed subvariety of Z^{θ}, which is the trace of ${}_{\theta'}\widetilde{\mathrm{Bun}}_P$ under the isomorphism of Proposition 3.2. In particular, ${}_0Z^{\theta} = Z^{\theta}_{\max}$.

As in Proposition 1.9 above, we obtain

$${}_{\theta'}Z^{\theta} \simeq Z^{\theta-\theta'}_{\max} \underset{\mathrm{Bun}_M}{\times} \mathrm{Mod}_M^{+,\theta'},$$

where the map $Z^{\theta-\theta'}_{\max} \to \mathrm{Bun}_M$ used in the definition of the fiber product is $(\mathcal{F}_M, \beta_M) \in Z^{\theta-\theta'} \mapsto \mathcal{F}_M \in \mathrm{Bun}_M$.

Let us denote by ${}_{\theta'}\mathbb{S}^{\theta}$ the intersection of ${}_{\theta'}Z^{\theta}$ with the central fiber $\mathbb{S}^{\theta}$. We obtain the following description of ${}_{\theta'}\mathbb{S}^{\theta}$:

Let Conv_M denote the convolution diagram of the affine Grassmannian of the group M. By definition, Conv_M classifies quadruples $(\mathcal{F}_M, \mathcal{F}'_M, \widetilde{\beta}_M, \beta'_M)$, where $\widetilde{\beta}_M$ is an isomorphism $\mathcal{F}_M|_{X-x} \simeq \mathcal{F}'_M|_{X-x}$ and β'_M is an isomorphism $\mathcal{F}'_M|_{X-x} \simeq \mathcal{F}^0_M|_{X-x}$. We have a natural projection $pr' : \mathrm{Conv}_M \to \mathrm{Gr}_M$, which sends $(\mathcal{F}_M, \mathcal{F}'_M, \widetilde{\beta}_M, \beta'_M) \mapsto (\mathcal{F}'_M, \beta'_M)$ and the projection $pr : \mathrm{Conv}_M \to \mathrm{Gr}_M$, which sends $(\mathcal{F}_M, \mathcal{F}'_M, \widetilde{\beta}_M, \beta'_M) \mapsto (\mathcal{F}_M, \beta'_M \circ \widetilde{\beta}_M)$.

Projection pr' makes Conv_M a fibration over Gr_M with the typical fiber isomorphic to Gr_M and we will denote by $\mathrm{Conv}_M^{+,\theta}$ the closed subscheme of Conv, which is a fibration over Gr_M with the typical fiber $\mathrm{Gr}_M^{+,\theta}$.

Using Proposition 2.6 we obtain

$${}_{\theta'}\mathbb{S}^{\theta} \simeq (\mathrm{Gr}_P^{\theta-\theta'} \cap \mathrm{Gr}_{U(P^-)}) \underset{\mathrm{Gr}_M}{\times} \mathrm{Conv}_M^{+,\theta'},$$

where $\mathrm{Gr}_M \leftarrow \mathrm{Conv}_M^{+,\theta'}$ is the map pr'.

3.6. Smoothness issues

Above we have constructed the map

$$Z^\theta_{\mathrm{Bun}_M^{\theta'}} \simeq (\widetilde{\mathrm{Bun}}_P^{\theta+\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}^{\theta'})^0 \to \widetilde{\mathrm{Bun}}_P^{\theta+\theta'}.$$

We do not *a priori* know whether this map is smooth, since $\mathrm{Bun}_{P^-} \to \mathrm{Bun}_G$ is not smooth. We will now construct an open substack in $Z^\theta_{\mathrm{Bun}_M}$, which will map smoothly onto $\widetilde{\mathrm{Bun}}_P^{\theta+\theta'}$.

Let $\mathfrak{u}(P)$ be the Lie algebra of $U(P)$ viewed as an M-module. We define the open substack $\mathrm{Bun}_M^r \subset \mathrm{Bun}_M$ to consist of those M-bundles $\mathcal{F}_M$, for which $H^1(X, \mathcal{U}) = 0$, for all M-modules $\mathcal{U}$, which appear as sub-quotients of $\mathfrak{u}(P)$. Let $\mathrm{Bun}_{P^-}^r$ be the preimage of Bun_M^r under the natural projection $\mathfrak{q} : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M$.

Lemma 3.7. *The restriction of the natural map* $\mathrm{Bun}_{P^-} \to \mathrm{Bun}_G$ *to* $\mathrm{Bun}_{P^-}^r$ *is smooth.*

Proof. Since both Bun_{P^-} and Bun_G are smooth, it is enough to check the surjectivity on the level of tangent spaces. Thus, let $\mathcal{F}_{P^-}$ be a P^--bundle and let $\mathcal{F}_G$ be the induced G-bundle. We must show that

$$H^1(X, \mathfrak{p}^-_{\mathcal{F}_{P^-}}) \to H^1(X, \mathfrak{g}_{\mathcal{F}_G})$$

is surjective if $\mathfrak{q}(\mathcal{F}_{P^-}) \in \mathrm{Bun}_M^r$.

In general, the cokernel of this map is $H^1(X, (\mathfrak{g}/\mathfrak{p}^-)_{\mathcal{F}_{P^-}})$. However, the irreducible subquotients of $\mathfrak{g}/\mathfrak{p}^-$ as a P^--module are all M-modules, which appear in the Jordan–Hölder series of $\mathfrak{u}(P)$. Hence, the assertion of the proposition follows from the definition of Bun_M^r. $\square$

Let $Z^\theta_{\mathrm{Bun}_M^r}$ denote the corresponding open substack of $Z^\theta_{\mathrm{Bun}_M}$. From Proposition 3.2 we obtain an isomorphism $Z^\theta_{\mathrm{Bun}_M^r} \simeq \underset{\theta'}{\cup} (\widetilde{\mathrm{Bun}}_P^{\theta+\theta'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}^{\theta',r})^0$. From the above lemma, we see that the resulting map $Z^\theta_{\mathrm{Bun}_M^r} \to \widetilde{\mathrm{Bun}}_P$ is smooth. In particular, since the stack Bun_P is smooth, we obtain the following corollary:

Corollary 3.8. *The open subscheme* $Z^\theta_{\max}$ *of* Z^θ *is smooth.*

It is well-known (cf. [DS]) that every open substack of Bun_G of finite type belongs to the image of some $\mathrm{Bun}_{P^-}^{\theta',r}$, when $-\theta'$ is large enough. Similarly, it is easy to see that every open substack of finite type of $\widetilde{\mathrm{Bun}}_P$ belongs to the image of $Z^\theta_{\mathrm{Bun}_M^r}$ for θ large enough. Hence, in order to understand the singularities of $\widetilde{\mathrm{Bun}}_P$, it is sufficient to analyze the singularities of Z^θ.

4. Computation of IC_{Z^θ}: Statements

4.1. For $\theta \in \Lambda^{\mathrm{pos}}_{G,P} \simeq \mathrm{Span}^+(\alpha_\imath,\, \imath \in \mathcal{I} - \mathcal{I}_M)$, let $\mathfrak{P}(\theta)$ denote an element of the set of partitions of θ as a sum $\theta = \sum_k \theta_k$, where each θ_k is a projection under $\Lambda \to \Lambda_{G,P}$ of a coroot of G belonging to $\mathrm{Span}^+(\alpha_\imath,\, \imath \in \mathcal{I} - \mathcal{I}_M)$.

We emphasize the difference between $\mathfrak{P}(\theta)$ and $\mathfrak{A}(\theta)$: in the latter case we decompose θ as a sum of arbitrary non-zero elements of $\Lambda^{\mathrm{pos}}_{G,P}$.

For a fixed $\mathfrak{P}(\theta)$, let $X^{\mathfrak{P}(\theta)}$ denote the corresponding partially symmetrized power of the curve. In other words, if $\theta = \sum_k n_m \cdot \theta_m$, where θ_m's are pairwise distinct, $X^{\mathfrak{P}(\theta)} = \prod_m X^{(n_m)}$.

Now we need to introduce a version of the Beilinson-Drinfeld affine Grassmannian $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}$. First, consider the ind-scheme $\mathrm{Gr}_M^{\mathfrak{P}(\theta),\infty}$, which classifies triples $(x^{\mathfrak{P}(\theta)}, \mathcal{F}_M, \beta_M)$, where $x^{\mathfrak{P}(\theta)} \in X^{\mathfrak{P}(\theta)}$, $\mathcal{F}_M$ is an M-bundle on X and β_M is the trivialization of $\mathcal{F}_M$ away from the support of $x^{\mathfrak{P}(\theta)}$. (We leave it to the reader to formulate the above definition in terms of S-points, in the spirit of what we have done before.)

Consider the open subset $\overset{o}{X}{}^{\mathfrak{P}(\theta)}$ of $X^{\mathfrak{P}(\theta)}$ equal to the complement of all the diagonals. Inside $\mathrm{Gr}_M^{\mathfrak{P}(\theta),\infty}|_{\overset{o}{X}{}^{\mathfrak{P}(\theta)}}$ we define the closed subset $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}|_{\overset{o}{X}{}^{\mathfrak{P}(\theta)}}$ as follows: For $x^{\mathfrak{P}(\theta)} = \Sigma\, \theta_k \cdot x_k$ with all the x_k's distinct, the fiber of $\mathrm{Gr}_M^{\mathfrak{P}(\theta),\infty}$ over it is just the product of the affine Grassmannians $\prod_k \mathrm{Gr}_{M,x_k}$ and the fiber of $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}$ is set to be $\prod_k \overline{\mathrm{Gr}}{}^{\flat(\theta_k)}_{M,x_k}$, where $\flat(\theta_k)$ is as in Proposition 1.7. The entire $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}$ is defined as a closure of $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}|_{\overset{o}{X}{}^{\mathfrak{P}(\theta)}}$ inside $\mathrm{Gr}_M^{\mathfrak{P}(\theta),\infty}$.

By construction, if $(x^{\mathfrak{P}(\theta)}, \mathcal{F}_M, \beta_M)$ belongs to $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}$, then among the rest, the trivialization β_M has the following property: for every G-module $\mathcal{V}$ the map

$$\beta_M^{\mathcal{V}^{U(P)}} : (\mathcal{V}^{U(P)})_{\mathcal{F}_M} \to (\mathcal{V}^{U(P)})_{\mathcal{F}_M^0} \simeq \mathcal{V}^{U(P)} \otimes \mathcal{O}_X,$$

which is defined *a priori* on $X - x^{\mathfrak{P}(\theta)}$ extends to a regular map on X. Therefore, we obtain a map $i_{\mathfrak{P}(\theta)} : \mathrm{Gr}_M^{\mathfrak{P}(\theta)} \to \mathrm{Mod}_M^{+,\theta}$, which covers the natural map $X^{\mathfrak{P}(\theta)} \to X^\theta$. It is easy to see that the above map $i_{\mathfrak{P}(\theta)}$ is finite.

4.2. Let us denote by $\mathrm{IC}^{\mathfrak{P}(\theta)}$ the intersection cohomology sheaf on $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}$. We need to understand more explicitly the behavior of $\mathrm{IC}^{\mathfrak{P}(\theta)}$ over the diagonals in $X^{\mathfrak{P}(\theta)}$.

Thus, let $\Delta_X \subset X^{\mathfrak{P}(\theta)}$ be the main diagonal. By construction, $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}|_{\Delta_X}$ is a subscheme of the relative affine Grassmannian $\mathrm{Gr}_{M,X}$. Recall that Loc_X denotes the localization functor from $\mathrm{Rep}(\check{M})$ to the category of perverse sheaves on $\mathrm{Gr}_{M,X}$.

Lemma 4.3. *If* $\mathfrak{P}(\theta)$ *corresponds to* $\theta = \sum_m n_m \cdot \theta_m$, *then the* $*$*-restriction of* $\mathrm{IC}^{\mathfrak{P}(\theta)}$ *to* $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}|_{\Delta_X} \subset \mathrm{Gr}_{M,X}$ *can be canonically identified with*

$$\mathrm{Loc}_X(\underset{m}{\otimes} \mathrm{Sym}^{n_m}(\check{\mathfrak{u}}(P)_{\theta_m})) \otimes \left(\overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]\right)^{\otimes|\mathfrak{P}(\theta)|-1},$$

where $|\mathfrak{P}(\theta)| = \sum_m n_m$.

Proof. Consider the corresponding non-symmetrized power of the curve $\prod_m X^{n_m}$. Over it we can consider the scheme $\prod_m (\overline{\mathrm{Gr}}_{M,X}^{\flat(\theta_m)})^{\times n_m}$ and we have a natural proper map

$$\mathrm{sym} : \prod_m (\overline{\mathrm{Gr}}_{M,X}^{\flat(\theta_m)})^{\times n_m} \to \mathrm{Gr}_M^{\mathfrak{P}(\theta)},$$

which covers the usual symmetrization map $\prod_m X^{n_m} \to X^{\mathfrak{P}(\theta)}$. Let us denote temporarily by $\mathcal{S}$ the direct image $\mathrm{sym}_!(\mathrm{IC}_{\prod_m (\overline{\mathrm{Gr}}_{M,X}^{\flat(\theta_m)})^{\times n_m}})$.

The fact that the map which defines convolution of perverse sheaves on the usual affine Grassmannian Gr_M is semi-small implies that the above map sym is small. Hence, $\mathcal{S}$ is the Goresky–MacPherson extension of its restriction to the open subscheme $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}|_{\overset{o}{X}{}^{\mathfrak{P}(\theta)}}$. In particular, it carries a canonical action of the product of symmetric groups $\prod_m S^{n_m}$, because this is obviously so over $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}|_{\overset{o}{X}{}^{\mathfrak{P}(\theta)}}$, and $\mathrm{IC}^{\mathfrak{P}(\theta)}$ coincides with the invariants $(\mathcal{S})^{\prod_m S^{n_m}}$.

By construction, the $*$-restriction of $\mathcal{S}$ to $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}|_{\Delta_X}$ can be identified with

$$\mathrm{Loc}_X(\underset{m}{\otimes} (\check{\mathfrak{u}}(P)_{\theta_m})^{\otimes n_m}) \otimes \left(\overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]\right)^{\otimes \sum_m n_m - 1}.$$

Therefore, it remains to see that the $\prod_m S^{n_m}$-action on $\mathcal{S}|_{\Delta_X}$ corresponds to the natural action of the group on $\underset{m}{\otimes} (\check{\mathfrak{u}}(P)_{\theta_m})^{\otimes n_m}$. We prove the latter fact as follows:

Since taking the global cohomology is a fiber functor for the category of spherical perverse sheaves on Gr_M, it suffices to analyze the $\prod_m S^{n_m}$-action on the direct image of $\mathcal{S}$ under $\mathrm{Gr}_M^{\mathfrak{P}(\theta)} \to X^{\mathfrak{P}(\theta)}$, in which case the assertion becomes obvious. □

4.4. The main theorem

Our main technical result is the following theorem:

Theorem 4.5. *The* $!$*-restriction of* IC_{Z^θ} *under* $\mathfrak{s}^\theta : \mathrm{Mod}_M^{+,\theta} \to Z^\theta$ *can be identified with*

$$\underset{\mathfrak{P}(\theta)}{\oplus} i_{\mathfrak{P}(\theta)*}(\mathrm{IC}^{\mathfrak{P}(\theta)}) \otimes \left(\overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]\right)^{-|\mathfrak{P}(\theta)|}.$$

Remark. Let us explain to what extent the isomorphism stated in this theorem is canonical. (In fact, it is not!) The LHS carries the cohomological filtration (*filtration canonique*), which corresponds to the filtration on the RHS according to $|\mathfrak{P}(\theta)|$. Unfortunately, our proof does not even give a canonical identification for the associated graded quotients: each $i_{\mathfrak{P}(\theta)*}(\mathrm{IC}^{\mathfrak{P}(\theta)})$ appears up to tensoring with a 1-dimensional vector space.

To prove this theorem, we will proceed by induction on $|\theta| := \sum_{\mathfrak{i}\in\mathcal{I}-\mathcal{I}_M} n_{\mathfrak{i}}$ if $\theta = \sum_{\mathfrak{i}\in\mathcal{I}-\mathcal{I}_M} n_{\mathfrak{i}} \cdot \alpha_{\mathfrak{i}}$. But first, we will derive from it various facts about $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$, since we will use them to perform the induction step.

4.6. Observe that since IC_{Z^θ} is Verdier self-dual, from Theorem 4.5 we obtain the description of $\mathfrak{s}^{\theta*}(\mathrm{IC}_{Z^\theta})$ as well. By translating this description to $\widetilde{\mathrm{Bun}}_P$ using Proposition 3.2, we obtain the following corollary:

Corollary 4.7. *The $*$-restriction of* $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ *to* ${}_\theta\widetilde{\mathrm{Bun}}_P \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Mod}^{+,\theta}_{\mathrm{Bun}_M}$ *can be identified with*

$$\underset{\mathfrak{P}(\theta)}{\oplus} (\mathrm{id} \times i_{\mathfrak{P}(\theta)})_*(\mathrm{IC}_{\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_M^{\mathfrak{P}(\theta)}}) \otimes \left(\overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]\right)^{|\mathfrak{P}(\theta)|},$$

where $\mathcal{H}_M^{\mathfrak{P}(\theta)}$ *is the corresponding relative version of* $\mathrm{Gr}_M^{\mathfrak{P}(\theta)}$ *over* Bun_M.

From this corollary one easily deduces Theorem 1.12:

Proof (of Theorem 1.12). To simplify the notation, we will take the element $\mathfrak{A}(\theta) = \mathfrak{A}^0(\theta)$ corresponding to the decomposition which consists of one element: $\theta = \theta$. We need to calculate the $*$-restriction of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ to

$${}_{\mathfrak{A}^0(\theta)}\widetilde{\mathrm{Bun}}_P \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_M^{\mathfrak{A}^0(\theta)} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}^{+,\theta}_{M,X}.$$

By definition, our embedding $j_{\mathfrak{A}^0(\theta)}$ factors through $\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Mod}^{+,\theta}_{\mathrm{Bun}_M} \simeq {}_\theta\widetilde{\mathrm{Bun}}_P$. Note that $\mathcal{H}^{+,\theta}_{M,X} \subset \mathrm{Mod}^{+,\theta}_{\mathrm{Bun}_M}$ is exactly the preimage of the main diagonal $\Delta_X \subset X^\theta$. Therefore, the sought-for complex is, according to Corollary 4.7, the direct sum over $\mathfrak{P}(\theta)$ of

$$(\mathrm{id} \times i_{\mathfrak{P}(\theta)})_*(\mathrm{IC}_{\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_M^{\mathfrak{P}(\theta)}})|_{\Delta_X} \otimes \left(\overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]\right)^{|\mathfrak{P}(\theta)|}.$$

Using Lemma 4.3, we obtain that $(\mathrm{id} \times i_{\mathfrak{P}(\theta)})_*(\mathrm{IC}_{\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_M^{\mathfrak{P}(\theta)}})|_{\Delta_X}$ corresponding to $\mathfrak{P}(\theta)$ with $\theta = \sum_m n_m \cdot \theta_m$ equals

$$\mathrm{Loc}_{\mathrm{Bun}_P,X}(\underset{m}{\otimes} \mathrm{Sym}^{n_m}(\check{\mathfrak{u}}(P)_{\theta_m})) \otimes (\overline{\mathbb{Q}_\ell}(1)[2])^{\otimes|\mathfrak{P}(\theta)|}) \otimes \left(\overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]\right)^{\otimes -1}.$$

However,

$$\underset{\mathfrak{P}(\theta)}{\oplus}\big(\underset{m}{\otimes}\operatorname{Sym}^{n_m}(\check{\mathfrak{u}}(P)_{\theta_m})\otimes(\overline{\mathbb{Q}_\ell}(1)[2])^{\otimes|\mathfrak{P}(\theta)|}\big)\simeq\underset{i\geq 0}{\oplus}\operatorname{Sym}^i(\check{\mathfrak{u}}(P))_\theta\otimes(\overline{\mathbb{Q}_\ell}(1)[2])^{\otimes i},$$

which is what we had to show. □

4.8. The following result is an interesting byproduct of Corollary 4.7. In order to save notation, we will formulate it for $\mathfrak{A}(\theta)=\mathfrak{A}^0(\theta)$, although the generalization to an arbitrary $\mathfrak{A}(\theta)$ is straightforward.

Consider the *hyperbolic* restriction of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ to the stratum ${}_{\mathfrak{A}^0(\theta)}\widetilde{\mathrm{Bun}}_P$. By definition, this is the !-restriction of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ to ${}_\theta\widetilde{\mathrm{Bun}}_P$ followed by the further $*$-restriction from ${}_\theta\widetilde{\mathrm{Bun}}_P$ to ${}_{\mathfrak{A}^0(\theta)}\widetilde{\mathrm{Bun}}_P$.

Corollary 4.9. *The hyperbolic restriction (in the above sense) of* $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ *to* ${}_{\mathfrak{A}^0(\theta)}\widetilde{\mathrm{Bun}}_P\simeq\mathrm{Bun}_P\underset{\mathrm{Bun}_M}{\times}\mathcal{H}^{+,\theta}_{M,X}$ *is isomorphic to*

$$\mathrm{Loc}_{\mathrm{Bun}_P,X}(\operatorname{Sym}(\check{\mathfrak{u}}(P))_\theta)\otimes\left(\overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]\right)^{\otimes-1}.$$

Let us draw the reader's attention to the fact that Corollary 4.9 implies that the hyperbolic restriction of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ to ${}_{\mathfrak{A}^0(\theta)}\widetilde{\mathrm{Bun}}_P$ is a perverse sheaf, up to a cohomological shift.

The proof of this corollary repeats the above proof of Theorem 1.12, using the fact that

$$\underset{\mathfrak{P}(\theta)=\sum\limits_m n_m\theta_m}{\oplus}\big(\underset{m}{\otimes}\operatorname{Sym}^{n_m}(\check{\mathfrak{u}}(P)_{\theta_m})\big)\simeq\operatorname{Sym}(\check{\mathfrak{u}}(P))_\theta.$$

Remark. We remark again that, due to the non-canonicity of the direct sum decomposition stated in Theorem 4.5, the isomorphism of Corollary 4.9 is non-canonical either. We only can claim that the LHS carries a canonical filtration, which on the RHS coincides with the filtration by the degree. However, in the course of the proof of Theorem 4.5, we will show that the above hyperbolic restriction can be *canonically* identified with $\mathrm{Loc}_X(U(\check{\mathfrak{u}}(P))_\theta)\otimes(\overline{\mathbb{Q}_\ell}(\frac{1}{2})[1])^{\otimes-1}$. It seems natural to guess (although our proof does not imply it) that our filtration on the LHS corresponds under this isomorphism to the canonical filtration on $U(\check{\mathfrak{u}}(P))$.

5. Computation of IC_{Z^θ}: Proofs

5.1. The goal of this section is to prove Theorem 4.5. Our strategy will be as follows: from the induction hypothesis we will obtain an almost complete description of how $\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta})$ looks like away from the main diagonal. Then we will explicitly compute the "contribution" at the main diagonal and hence prove the theorem.

The crucial idea of the proof is the following assertion:

Proposition 5.2. *There is a canonical isomorphism* $\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta}) \simeq \pi_{P!}(\mathrm{IC}_{Z^\theta})$.

We will deduce this proposition from the following well-known lemma:

Let $\pi : Y' \to Y$ be a map of schemes and $\mathfrak{s} : Y \to Y'$ a section. Assume now that the group $\mathbb{G}_m$ acts on Y' in such a way that it "contracts" Y' onto Y. This means that the action map $\mathbb{G}_m \times Y' \to Y'$ extends to a regular map $\mathbb{A}^1 \times Y' \to Y'$, such that the composition

$$0 \times Y' \to \mathbb{A}^1 \times Y' \to Y'$$

coincides with $\mathfrak{s} \circ \pi : Y' \to Y \to Y'$. Let now $\mathcal{S}$ be a $\mathbb{G}_m$-equivariant complex on Y'.

Lemma 5.3. *Under the above circumstances,* $\pi_!(\mathcal{S}) \simeq \mathfrak{s}^!(\mathcal{S})$.

We apply this lemma to $Y' = Z^\theta$ and $Y = \mathrm{Mod}_M^{+,\theta}$. To construct a $\mathbb{G}_m$-action we proceed as follows. Let $\mathbb{G}_m \to Z(M)$ be a 1-parameter subgroup, which acts as contraction on $U(P^-)$. In this way $\mathbb{G}_m$ acts on the trivial M-bundle $\mathcal{F}_M^0$ on X and hence on Z^θ. It remains to verify that $\mathbb{G}_m$ indeed contracts Z^θ onto $\mathrm{Mod}_M^{+,\theta}$. We do that as follows:

Let $\mathrm{Gr}_G^{\theta,\infty}$ be the Beilinson–Drinfeld affine Grassmannian over X^θ. In other words, a point of $\mathrm{Gr}_G^{\theta,\infty}$ is a triple $(x^\theta, \mathcal{F}_G, \beta)$, where $x^\theta \in X^\theta$, $\mathcal{F}_G$ is a G-bundle on X and β is a trivialization of $\mathcal{F}_G$ off the support of x^θ. We have the unit section $X^\theta \to \mathrm{Gr}_G^{\theta,\infty}$ which sends x^θ to $(x^\theta, \mathcal{F}_G^0, \beta^0)$, where β^0 is the tautological trivialization of the trivial bundle.

In the same way we can consider a Beilinson–Drinfeld version of the affine Grassmannian for the group $U(P^-)$ (denote it $\mathrm{Gr}_{U(P^-)}^{\theta,\infty}$), and we have a locally closed embedding: $\mathrm{Gr}_{U(P^-)}^{\theta,\infty} \hookrightarrow \mathrm{Gr}_G^{\theta,\infty}$.

By construction, our Z^θ is a closed subscheme inside $\mathrm{Mod}_M^{+,\theta} \underset{X^\theta}{\times} \mathrm{Gr}_{U(P^-)}^{\theta,\infty}$. The image of $\mathfrak{s}^\theta$ is the product of $\mathrm{Mod}_M^{+,\theta}$ and the unit section of $\mathrm{Gr}_{U(P^-)}^{\theta,\infty}$.

The above $\mathbb{G}_m$-action on Z^θ comes from a natural action of this group on $\mathrm{Gr}_{U(P^-)}^{\theta,\infty}$, while the action on $\mathrm{Mod}_M^{+,\theta}$ is trivial. Therefore, to prove our assertion we have to show that $\mathbb{G}_m$ contracts $\mathrm{Gr}_{U(P^-)}^{\theta,\infty}$ to the unit section. However, this easily follows from the fact that our $\mathbb{G}_m \to Z(M)$ contracts $U(P^-)$ to $1 \in U(P^-)$.

5.4. Having established Proposition 5.2, we obtain the following corollary:

Corollary 5.5. *When we pass from* $\mathbb{F}_q$ *to* $\overline{\mathbb{F}_q}$, *the complex* $\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta}) \simeq \pi_{P!}(\mathrm{IC}_{Z^\theta})$ *splits as a direct sum of (cohomologically shifted) irreducible perverse sheaves.*

Proof. According to [BBD], $\pi_{P!}(\mathrm{IC}_{Z^\theta})$ has weights ≤ 0, since IC_{Z^θ} is pure of weight 0. At the same time, [BBD] implies that $\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta})$ has weights ≥ 0. Hence, we obtain that $\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta}) \simeq \pi_{P!}(\mathrm{IC}_{Z^\theta})$ is pure of weight 0.

Hence, the assertion of the corollary follows from the Decomposition Theorem. □

From now until Section 5.13, we will disregard the $\mathbb{F}_q$-structure on $\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta})$ and will prove the isomorphism stated in Theorem 4.5 over $\overline{\mathbb{F}_q}$. In Section 5.13 we will show that the direct sum decomposition holds over $\mathbb{F}_q$ as well.

Now let us use the induction hypothesis, i.e., our knowledge about $\mathfrak{s}^{\theta'!}(\mathrm{IC}_{Z^{\theta'}})$ for all θ' with $\theta-\theta' \in \Lambda^{\mathrm{pos}}_{G,P}$. It is easy to see that the *factorization property* of Proposition 2.4 implies that locally over $X^\theta-\Delta_X$ we do obtain an isomorphism

$$\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta}) \simeq \underset{\mathfrak{P}(\theta),\,|\mathfrak{P}(\theta)|\neq 1}{\oplus} i_{\mathfrak{P}(\theta)*}(\mathrm{IC}^{\mathfrak{P}(\theta)})[-|\mathfrak{P}(\theta)|].$$

However, globally we can *a priori* have a non-trivial monodromy: suppose that $\mathfrak{P}(\theta)$ corresponds to $\theta = 2\cdot\theta'$, where θ' is the image of a coroot in $\mathrm{Span}(\alpha_j,\, \mathrm{j}\in \mathcal{I}-\mathcal{I}_M)$. Then the preimage of $X^\theta-\Delta_X$ in $X^{\mathfrak{P}(\theta)}$ is $X^{(2)}-\Delta_X$, and we can have an order 2 monodromy. Therefore, so far we can only claim that

$$\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta})|_{X^\theta-\Delta_X} \simeq \underset{\mathfrak{P}(\theta),\,|\mathfrak{P}(\theta)|\neq 1}{\oplus} i_{\mathfrak{P}(\theta)*}(\mathrm{IC}^{\mathfrak{P}(\theta)}\otimes \pi_M^*(\mathcal{E}_{\mathfrak{P}(\theta)}))[-|\mathfrak{P}(\theta)|]|_{X^\theta-\Delta_X},$$

where $\mathcal{E}_{\mathfrak{P}(\theta)}$ is an order 2 local system on the preimage of $X^\theta-\Delta_X$ in $X^{\mathfrak{P}(\theta)}$, which can be non-trivial only for $\mathfrak{P}(\theta)$ of the form specified above. Of course, later we will have to show that all the $\mathcal{E}_{\mathfrak{P}(\theta)}$'s are necessarily trivial.

By combining this with Corollary 5.5 we obtain:

$$\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta}) \simeq \underset{\mathfrak{P}(\theta),\,|\mathfrak{P}(\theta)|\neq 1}{\oplus} i_{\mathfrak{P}(\theta)*}(\mathcal{K}^{\mathfrak{P}(\theta)})[-|\mathfrak{P}(\theta)|] \oplus \mathcal{K}^\theta, \tag{1}$$

where $\mathcal{K}^\theta$ is a complex supported on $\mathrm{Mod}^{+,\theta}_M|_{\Delta_X} \simeq \mathrm{Gr}^{+,\theta}_{M,X}$ and $\mathcal{K}^{\mathfrak{P}(\theta)}$ is the Goresky–MacPherson extension of $\mathrm{IC}^{\mathfrak{P}(\theta)}|_{X^\theta-\Delta_X}\otimes\pi_M^*(\mathcal{E}_{\mathfrak{P}(\theta)})$ to the whole $\mathrm{Gr}^{\mathfrak{P}(\theta)}_M$.

To prove the theorem we have to understand the complex $\mathcal{K}^\theta$. This will be done by analyzing the $*$-restriction of the LHS of (1) to $\mathrm{Mod}^{+,\theta}_M|_{\Delta_X}$.

5.6. Recall that for $x\in X$ we denoted by Δ_x the embedding $\mathrm{pt}\hookrightarrow X \overset{\Delta_X}{\hookrightarrow} X^\theta$. To simplify the notation, instead of $\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta})|_{\Delta_X}$ we will compute $\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta})|_{\Delta_x}$. We will prove the following assertion:

Proposition 5.7. $\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta})|_{\Delta_x}$*, being a complex of sheaves on* $\mathrm{Mod}^{+,\theta}_M|_{\Delta_x} \simeq \mathrm{Gr}^{+,\theta}_M$*, is concentrated in perverse cohomological degrees* ≤ 0*. Its* 0*-th perverse cohomology can be identified with* $\mathrm{Loc}(U(\check{\mathfrak{u}}(P))_\theta)$*.*

Remark. Note that the *a posteriori* proven Corollary 4.9 implies that the above complex has perverse cohomology only in dimension 0.

Recall our definition of the central fiber $\mathbb{S}^\theta$. Using Proposition 5.2 and base change, we obtain that

$$\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta})|_{\Delta_x} \simeq \pi_{P!}(\mathrm{IC}_{Z^\theta}|_{\mathbb{S}^\theta}).$$

The following is a refinement of Proposition 5.7:

Proposition 5.8. *Let θ' be as above.*

(1) $\pi_{P!}(\mathrm{IC}_{Z^\theta}|_{{}_{\theta'}\mathbb{S}^\theta})$ *lives in strictly negative cohomological dimensions if* $\theta' \neq 0$.
(2) *The complex* $\pi_{P!}(\mathrm{IC}_{Z^\theta}|_{{}_0\mathbb{S}^\theta})$ *lives in cohomological dimensions* ≤ 0.
(3) $h^0(\pi_{P!}(\mathrm{IC}_{Z^\theta}|_{{}_0\mathbb{S}^\theta})) \simeq \mathrm{Loc}(U(\check{\mathfrak{u}}(P))_\theta)$.

The proof of Proposition 5.8 will use the following facts about the geometry of the affine Grassmannian, whose proofs will be given in Section 6.

Let us denote by $\mathfrak{t}^\theta$ the natural map $\mathrm{Gr}_P^\theta \to \mathrm{Gr}_M^\theta$.

Theorem 5.9. *We have:*

(1) $\mathfrak{t}^\theta_!(\overline{\mathbb{Q}_\ell}_{\mathrm{Gr}_P^\theta \cap \mathrm{Gr}_{U(P^-)}})$ *as a complex of sheaves on* $\mathrm{Gr}_M^{+,\theta}$ *lives in the perverse cohomological dimensions* $\leq \langle \theta, 2(\check{\rho}_G - \check{\rho}_M)\rangle$.
(2) $h^{\langle \theta, 2(\check{\rho}_G - \check{\rho}_M)\rangle}(\mathfrak{t}^\theta_!(\overline{\mathbb{Q}_\ell}_{\mathrm{Gr}_P^\theta \cap \mathrm{Gr}_{U(P^-)}})) \simeq \mathrm{Loc}(U(\check{\mathfrak{u}}(P))_\theta)$.

5.10. Proof of Proposition 5.8

First, from Proposition 3.2 we can compute the dimension of Z^θ and we obtain $\langle \theta, 2(\check{\rho}_G - \check{\rho}_M)\rangle$. Since ${}_0Z^\theta$ is contained in $Z^\theta_{\max}$ (and $Z^\theta_{\max}$ is smooth, according to Corollary 3.8),

$$\mathrm{IC}_{Z^\theta}|_{{}_0\mathbb{S}^\theta} \simeq \overline{\mathbb{Q}_\ell}_{{}_0\mathbb{S}^\theta}[\langle \theta, 2(\check{\rho}_G - \check{\rho}_M)\rangle].$$

Therefore, points 2 and 3 of the proposition follow immediately from Theorem 5.9 combined with Proposition 2.6.

To prove point 1 of the proposition, let us first take $\theta' \neq \theta$. However, ${}_{\theta'}\mathbb{S}^\theta$ is contained in ${}_{\theta'}Z^\theta$, we will be able to use the induction hypothesis to calculate $\mathrm{IC}_{Z^\theta}|_{{}_{\theta'}\mathbb{S}^\theta}$:

Recall the identification

$${}_{\theta'}\mathbb{S}^\theta \simeq (\mathrm{Gr}_P^{\theta-\theta'} \cap \mathrm{Gr}_{U(P^-)}) \underset{\mathrm{Gr}_M}{\times} \mathrm{Conv}_M^{+,\theta'}$$

of Section 3.5.

Let us recall also the following construction. Projection pr' realizes the convolution diagram Conv_M as a fibration over Gr_M with the typical fiber isomorphic to Gr_M itself. Hence, starting with a spherical perverse sheaf $\mathcal{S}$ on Gr_M and an arbitrary complex $\mathcal{S}'$ on Gr_M, we can define their twisted external product $\mathcal{S}\tilde{\boxtimes}\mathcal{S}' \in \mathrm{D}(\mathrm{Conv}_M)$, which is "$\mathcal{S}'$ along the base", and "$\mathcal{S}$ along the fiber". The convolution of $\mathcal{S}$ and $\mathcal{S}'$ is by definition the complex on Gr_M equal to $pr_!(\mathcal{S}\tilde{\boxtimes}\mathcal{S}')$. It is a basic fact (cf. [Ga]) that if $\mathcal{S}'$ is a perverse sheaf, then its convolution with any $\mathcal{S}$ as above is perverse as well.

Similarly, if $\mathcal{S}$ is a spherical perverse sheaf on Gr_M and $\mathcal{S}''$ is an arbitrary complex on $\mathrm{Gr}_P^{\theta-\theta'} \cap \mathrm{Gr}_{U(P^-)}$ we can construct the complex $\mathcal{S}\tilde{\boxtimes}\mathcal{S}''$ on

$$\mathrm{Conv}_M^{+,\theta'} \underset{\mathrm{Gr}_M}{\times} (\mathrm{Gr}_P^{\theta-\theta'} \cap \mathrm{Gr}_{U(P^-)}).$$

By combining Theorem 1.12 with Lemma 4.3 and Proposition 3.2 we obtain that $\mathrm{IC}_{Z^\theta}|_{_{\theta'}\mathbb{S}^\theta}$ is the direct sum

$$\bigoplus_{\mathfrak{P}(\theta')} (\mathrm{Loc}(\bigotimes_m \mathrm{Sym}^{n'_m}(\check{\mathfrak{u}}(P)_{\theta'_m}))[2\cdot|\mathfrak{P}(\theta')|])\widetilde{\boxtimes}\overline{\mathbb{Q}_\ell}[\langle\theta-\theta', 2(\check{\rho}-\check{\rho}_M)\rangle],$$

where $\theta' = \Sigma n'_m \cdot \theta'_m$.

Now, projection $\pi_P : {_{\theta'}\mathbb{S}^\theta} \to \mathrm{Gr}_M^{+,\theta}$ in the above description of ${_{\theta'}\mathbb{S}^\theta}$ corresponds to $\mathrm{Conv}_M^{+,\theta'} \underset{\mathrm{Gr}_M}{\times} (\mathrm{Gr}_P^{\theta-\theta'} \cap \mathrm{Gr}_{U(P^-)}) \to \mathrm{Conv}_M^{+,\theta'} \xrightarrow{pr} \mathrm{Gr}_M$. Therefore, $\pi_{P!}(\mathrm{IC}_{Z^\theta}|_{_{\theta'}\mathbb{S}^\theta})$ is the sum over $\mathfrak{P}(\theta')$ of convolutions of

$$\mathfrak{t}_!^{\theta-\theta'}(\overline{\mathbb{Q}_\ell}_{\mathrm{Gr}_P^{\theta-\theta'}\cap\mathrm{Gr}_{U(P^-)}})[2\cdot|\mathfrak{P}(\theta')| + \langle\theta-\theta', 2(\check{\rho}-\check{\rho}_M)\rangle]$$

with the spherical perverse sheaf $\mathrm{Loc}(\bigotimes_m \mathrm{Sym}^{n'_m}(\check{\mathfrak{u}}(P)_{\theta'_m}))$. The important thing is that $|\mathfrak{P}(\theta')| > 0$: using Theorem 5.9(2), we obtain that $\pi_{P!}(\mathrm{IC}_{Z^\theta}|_{_{\theta'}\mathbb{S}^\theta})$ lies in strictly negative cohomological degrees.

Since the convolution of a complex lying in negative perverse cohomological dimensions on Gr_M with a spherical perverse sheaf is again a complex lying in negative cohomological dimensions, point 1 of the proposition follows for $\theta' \neq \theta$.

Finally, let us consider $\theta' = \theta$. In this case, $\pi_P : {_\theta\mathbb{S}^\theta} \to \mathrm{Gr}_M^{+,\theta}$ is an isomorphism and it suffices to observe, that by the very definition of intersection cohomology, $\mathrm{IC}_{_\theta\mathbb{S}^\theta}$ lives in strictly negative cohomological degrees.

Thus, Proposition 5.8 is proved modulo Theorem 5.9, which will be dealt with later.

5.11. Let us go back to the isomorphism of Equation (1). At this point we are ready to prove that the local systems $\mathcal{E}_{\mathfrak{P}(\theta)}$ are all trivial. For that purpose, we can assume that $\mathfrak{P}(\theta)$ corresponds to $\theta = 2\cdot\theta'$, as above. Consider

$$\overline{\mathrm{Gr}}_{M,X}^{\flat(\theta')} \times \overline{\mathrm{Gr}}_{M,X}^{\flat(\theta')} \to \mathrm{Gr}_M^{\mathfrak{P}(\theta)} \xrightarrow{i_{\mathfrak{P}(\theta)}} \mathrm{Mod}_M^{+,\theta}.$$

By induction hypothesis (in the incarnation of Corollary 4.9) and Proposition 2.4 we have that over $X \times X - \Delta_X$,

$$(\mathfrak{s}^{\theta!}(\mathrm{IC}_{Z^\theta}))|_{\overline{\mathrm{Gr}}_{M,X}^{\flat(\theta')}\times\overline{\mathrm{Gr}}_{M,X}^{\flat(\theta')}} \simeq \mathrm{Loc}_X(\mathrm{Sym}(\check{\mathfrak{u}}(P))_{\theta'}) \boxtimes \mathrm{Loc}_X(\mathrm{Sym}(\check{\mathfrak{u}}(P))_{\theta'})[-2]. \quad (2)$$

The group $\mathbb{Z}_2$ acts in a natural way on $\overline{\mathrm{Gr}}_{M,X}^{\flat(\theta')} \times \overline{\mathrm{Gr}}_{M,X}^{\flat(\theta')}$ and we have to show that the $\mathbb{Z}_2$-equivariant structure on the LHS of (2) corresponds to the tautological $\mathbb{Z}_2$-equivariant structure on the RHS.

Let us apply a relative version of Proposition 5.8 for θ', in which instead of a fixed $x \in X$ we have a pair of distinct points on X. We obtain an isomorphism of complexes over $X \times X - \Delta_X$

$$\begin{aligned}&(\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta}))|_{\overline{\mathrm{Gr}}^{\flat(\theta')}_{M,X} \times \overline{\mathrm{Gr}}^{\flat(\theta')}_{M,X}}[2] \simeq (\pi_{P!}(\mathrm{IC}_{Z^\theta}))|_{\overline{\mathrm{Gr}}^{\flat(\theta')}_{M,X} \times \overline{\mathrm{Gr}}^{\flat(\theta')}_{M,X}}[2] \simeq \\ &h^{top}((\mathfrak{t}^{\theta'} \boxtimes \mathfrak{t}^{\theta'})_!(\overline{\mathbb{Q}_\ell}_{\mathrm{Gr}^{\theta'}_{P,X} \cap \mathrm{Gr}_{U(P^-),X} \times \mathrm{Gr}^{\theta'}_{P,X} \cap \mathrm{Gr}_{U(P^-),X}})) \simeq \\ &\mathrm{Loc}_X(U(\check{\mathfrak{u}}(P))_{\theta'}) \boxtimes \mathrm{Loc}_X(U(\check{\mathfrak{u}}(P))_{\theta'}),\end{aligned}$$

where $top = 2 \cdot (1 + \langle \theta', 2(\check{\rho}_G - \check{\rho}_M)\rangle)$ and $\mathrm{Gr}^{\theta}_{P,X}$ and $\mathrm{Gr}_{U(P^-),X}$ are the corresponding relative (over X) versions of Gr^{θ}_P and $\mathrm{Gr}_{U(P^-)}$, respectively.

The last isomorphism, by construction, intertwines the natural $\mathbb{Z}_2$-structure on $(\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta}))|_{\overline{\mathrm{Gr}}^{\flat(\theta')}_{M,X} \times \overline{\mathrm{Gr}}^{\flat(\theta')}_{M,X}}$ and the tautological $\mathbb{Z}_2$-structure on the external product $\mathrm{Loc}(U(\check{\mathfrak{u}}(P))_{\theta'}) \boxtimes \mathrm{Loc}(U(\check{\mathfrak{u}}(P))_{\theta'})$. By comparing with (2) we obtain the required assertion.

5.12. To prove the theorem over $\overline{\mathbb{F}_q}$ it remains to analyze the term $\mathcal{K}^\theta$. Note that there is not more than one $\mathfrak{P}(\theta)$ with $|\mathfrak{P}(\theta)| = 1$. We will denote it by $\mathfrak{P}^0(\theta)$. By definition, $\mathrm{Gr}_M^{\mathfrak{P}^0(\theta)} \simeq \overline{\mathrm{Gr}}^{\flat(\theta)}_{M,X}$.

We have to show that

$$\mathcal{K}^\theta \simeq i_{\mathfrak{P}^0(\theta)*}(\mathrm{Loc}_X(\check{\mathfrak{u}}(P)_\theta))[-1].$$

By the definition of IC, since $\mathcal{K}^\theta|_{\Delta_X}$ is a direct summand of $\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta})$, it can have perverse cohomology only in degrees ≥ 1. Let us now restrict both sides of (1) to $\mathrm{Gr}^{+,\theta}_{M,X} \simeq \mathrm{Mod}^{+,\theta}_M|_{\Delta_X}$ and apply the cohomological truncation $\tau^{\geq 1}$. Using Lemma 4.3 on the one hand, and the relative (over X) version of Proposition 5.7 on the other hand, we obtain

$$\mathrm{Loc}_X((U(\check{\mathfrak{u}}(P)))_\theta)[-1] \simeq \bigoplus_{\mathfrak{P}(\theta),\, |\mathfrak{P}(\theta)| \neq 1} \mathrm{Loc}_X(\bigotimes_k \mathrm{Sym}^{n_k}(\check{\mathfrak{u}}(P)_{\theta_k}))[-1] \oplus \mathcal{K}^\theta|_{\Delta_X}.$$

Hence, $\mathcal{K}^\theta[1]$ is a perverse sheaf. Moreover, since $U(\check{\mathfrak{u}}(P))$ and $\mathrm{Sym}(\check{\mathfrak{u}}(P))$ are (non-canonically) isomorphic as $\check{M}$-modules, the comparison of multiplicities forces $\mathcal{K}^\theta[1]|_{\Delta_X} \simeq \mathrm{Loc}_X(\check{\mathfrak{u}}(P)_\theta)$.

5.13. Now let us restore the $\mathbb{F}_q$-structure on $\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta})$. To complete the proof of the theorem, by induction, it suffices to show that the arrow

$$\mathcal{K}^\theta|_{\Delta_X} \simeq \mathrm{Loc}_X(\check{\mathfrak{u}}(P)_\theta) \otimes \left(\overline{\mathbb{Q}_\ell}[1]\left(\frac{1}{2}\right)\right)^{\otimes -1} \to \mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta}),$$

which is known to split over $\overline{\mathbb{F}_q}$, splits over $\mathbb{F}_q$ as well. For that, it is enough to show that the complex $\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta})|_{\Delta_X}$ is semisimple.

We know already that $\mathfrak{s}^{\theta !}(\mathrm{IC}_{Z^\theta})|_{\Delta_X} \otimes (\overline{\mathbb{Q}_\ell}[1](\frac{1}{2}))$ has perverse cohomology only in dimension 0, which is equal, as in Proposition 5.8, to

$$h^0\left(\mathfrak{t}^\theta_!(\overline{\mathbb{Q}_\ell}_{\mathrm{Gr}^\theta_{P,X}\cap \mathrm{Gr}_{U(P^-),X}}) \otimes \left(\overline{\mathbb{Q}_\ell}[1]\left(\frac{1}{2}\right)\right)^{\langle \theta, 2(\check\rho_G - \check\rho_M)\rangle}\right).$$

The needed result follows from the fact that the isomorphism of Theorem 5.9(3) is compatible with the $\mathbb{F}_q$-structure in the sense that

$$h^0\left(\mathfrak{t}^\theta_!(\overline{\mathbb{Q}_\ell}_{\mathrm{Gr}^\theta_{P,X}\cap \mathrm{Gr}_{U(P^-),X}}) \otimes \left(\overline{\mathbb{Q}_\ell}[1]\left(\frac{1}{2}\right)\right)^{\langle \theta, 2(\check\rho_G - \check\rho_M)\rangle}\right) \simeq \mathrm{Loc}_X((U(\check{\mathfrak{u}}(P)))_\theta).$$

6. Intersections of semi-infinite orbits in the affine Grassmannian

6.1. The restriction functors

Let $\mathcal{O}_x$ (resp., $\mathcal{K}_x$) denote the completed local ring (resp., local field) at x. We can form the group-schemes $G(\mathcal{O}_x)$, $P(\mathcal{O}_x)$, $U(P)(\mathcal{O}_x)$ and the corresponding group-ind-schemes $G(\mathcal{K}_x)$, $P(\mathcal{K}_x)$, $U(P)(\mathcal{K}_x)$. Note, however, that the latter is not only a group-ind-scheme, but also an ind-group-scheme, i.e., an inductive limit of group-schemes.

Let $\nu \in \Lambda$ be M-dominant and let θ be its image under $\Lambda \to \Lambda_{G,P}$. Let us denote by Gr^ν_P the preimage $(\mathfrak{t}^\theta)^{-1}(\mathrm{Gr}^\nu_M) \subset \mathrm{Gr}^\theta_P$. The schemes Gr^ν_P are nothing but orbits of the group $U(P)(\mathcal{K}_x)\cdot M(\mathcal{O}_x)$ on Gr_G. We will denote by $\mathfrak{t}^\nu$ the restriction of the projection $\mathfrak{t}^\theta : \mathrm{Gr}^\theta_P \to \mathrm{Gr}_M$ to Gr^ν_P.

The goal of this section is to prove Theorem 5.9. The starting point is the following result, which describes the intersections of Gr^θ_P with Gr^λ_G inside the affine Grassmannian Gr_G (cf. [BD], [BG] and [MV]).

For a G-dominant (resp., M-dominant) coweight λ, let V^λ (resp., V^λ_M) denote the corresponding irreducible representation of $\check G$ (resp., $\check M$).

Theorem 6.2. *Let λ be a dominant integral coweight of G.*

(1) *The intersection $\mathrm{Gr}^\nu_P \cap \mathrm{Gr}^\lambda_G$ has dimension $\leq \langle \nu+\lambda, \check\rho_G\rangle$.*

(2) *The irreducible components of $\mathrm{Gr}^\nu_P \cap \mathrm{Gr}^\lambda_G$ of dimension $\langle \nu+\lambda, \check\rho_G\rangle$ form a basis for $\mathrm{Hom}_{\check M}(V^\nu_M, \mathrm{Res}^{\check G}_{\check M}(V^\lambda))$.*

6.3. The case $P = B$

We will first consider the situation when $P = B$. Note that in our notation $\mathrm{Bun}^0_{B^-}$ is the same as $\mathrm{Bun}_{U(B^-)}$.

In this case, $\Lambda_{G,P} = \Lambda$ and for two elements $\nu, \mu \in \Lambda$ let us consider the intersection $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}$.

First, it is easy to see that the action of $t^\mu \in T(\mathcal{K}_x)$ on Gr_G identifies $\mathrm{Gr}_B^\nu \cap \mathrm{Gr}_{B^-}^0$ with $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}$ for any $\mu \in \Lambda$. To prove the theorem, it suffices to show that for a given $\nu \in \Lambda^{pos}$ and some $\mu \in \Lambda$, the intersection $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}$ is of dimension $\leq \langle \nu, \check{\rho}_G \rangle$ and

$$H_c^{2\langle \nu, \check{\rho} \rangle}(\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}) \simeq U(\check{\mathfrak{u}})_\nu.$$

Proposition 6.4. *For a fixed ν and μ deep enough in the dominant chamber, the intersection $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}$ is contained inside $\mathrm{Gr}_G^{-w_0(\mu)}$.*

Proof. Let us identify $\mathrm{Gr}_{B^-}^0$ with the quotient $U(B^-)(\mathcal{K}_x)/U(B^-)(\mathcal{O}_x)$.

Since we know already that $\mathrm{Gr}_B^\nu \cap \mathrm{Gr}_{B^-}^0$ is a scheme of finite type, the preimage of $\mathrm{Gr}_B^\nu \cap \mathrm{Gr}_{B^-}^0$ under the projection $U(B^-)(\mathcal{K}_x) \to U(B^-)(\mathcal{K}_x)/U(B^-)(\mathcal{O}_x)$ is contained inside the subgroup

$$\mathrm{Ad}_{t^\mu}(U(B^-)(\mathcal{O}_x)) \subset U(B^-)(\mathcal{K}_x),$$

for μ deep enough in the dominant chamber.

Let us now consider $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}$, which via the action of t^μ can be identified with $\mathrm{Gr}_B^\nu \cap \mathrm{Gr}_{B^-}^0$.

We can view $\mathrm{Gr}_{B^-}^{-\mu}$ as a quotient $U(B^-)(\mathcal{K}_x)/\mathrm{Ad}_{t^{-\mu}}(U(B^-)(\mathcal{O}_x))$, via the action of $U(B^-)(\mathcal{K}_x)$ on $t^{-\mu}$, viewed as an element of $\mathrm{Gr}_T \subset \mathrm{Gr}_{B^-}^{-\mu} \subset \mathrm{Gr}_G$. We obtain that the preimage of $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}$ in $U(B^-)(\mathcal{K}_x)$ is contained in $U(B^-)(\mathcal{O}_x)$. Hence,

$$\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu} \subset U(B^-)(\mathcal{O}_x) \cdot t^{-\mu} \subset G(\mathcal{O}_x) \cdot t^{-\mu} = \mathrm{Gr}_G^{-w_0(\mu)}.$$

□

The above proposition implies the dimension estimate $\dim(\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}) \leq \langle \nu, \check{\rho}_G \rangle$ right away.

Indeed, we may assume that μ is such that $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu} \subset \mathrm{Gr}_G^{-w_0(\mu)}$. However, Theorem 6.2(1) implies that $\dim(\mathrm{Gr}_G^{-w_0(\mu)} \cap \mathrm{Gr}_B^{\nu-\mu}) \leq \langle \nu, \check{\rho}_G \rangle$.

To prove the other statements of the theorem, observe that for μ as above, the irreducible components of $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}$ of dimension $\langle \nu, \check{\rho}_G \rangle$ are naturally a subset among the irreducible components of $\mathrm{Gr}_G^{-w_0(\mu)} \cap \mathrm{Gr}_B^{\nu-\mu}$ of the same dimension.

Let us show that the generic point of every irreducible component K of the intersection $\mathrm{Gr}_G^{-w_0(\mu)} \cap \mathrm{Gr}_B^{\nu-\mu}$ of dimension $\langle \nu, \check{\rho}_G \rangle$ is contained in $\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}$.

Suppose the contrary. Then there exists $\mu' \in \Lambda$, such that the generic point of K is contained in $\mathrm{Gr}_G^{-w_0(\mu)} \cap \mathrm{Gr}_{B^-}^{-\mu'}$. However, it is easy to see that $\mathrm{Gr}_G^{-w_0(\mu)} \cap \mathrm{Gr}_{B^-}^{-\mu'} \neq \emptyset$ implies $\mu - \mu' \in \Lambda^{\mathrm{pos}}$.

However, as we have shown above, $\dim(\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu'}) \leq \langle \nu+\mu'-\mu, \check{\rho}_G\rangle$, which is smaller than the dimension of K.

Thus, we obtain that

$$H_c^{2\langle\nu,\check{\rho}\rangle}(\mathrm{Gr}_B^{\nu-\mu} \cap \mathrm{Gr}_{B^-}^{-\mu}) \simeq H_c^{2\langle\nu,\check{\rho}\rangle}(\mathrm{Gr}_G^{-w_0(\mu)} \cap \mathrm{Gr}_B^{\nu-\mu}).$$

However, according to Theorem 6.2(2), the RHS of the above equation can be canonically identified with the $\nu-\mu$-weight space in the irreducible $\check{G}$-representation with highest weight $-w_0(\mu)$. The latter, when μ is large compared to ν, is isomorphic to $U(\check{\mathfrak{u}})_\nu$ via the action on the lowest weight vector.

6.5. The general case

We fix θ and $\nu \in \Lambda$ such that $\mathrm{Gr}_M^\nu \subset \mathrm{Gr}_M^{+,\theta}$. Since Gr_M^ν is simply-connected, it suffices to show that each intersection $\mathrm{Gr}_P^\nu \cap \mathrm{Gr}_{U(P^-)}$ is of dimension $\leq \langle\nu, \check{\rho}_G\rangle$ and that the number of its irreducible components of dimension exactly $\langle\nu, \check{\rho}_G\rangle$ equals the dimension of $\mathrm{Hom}_{\check{M}}(V_M^\nu, U(\check{\mathfrak{u}}(P)))$.

For an M-dominant weight μ let us consider the corresponding $\mathrm{Gr}_{P^-}^\mu \subset \mathrm{Gr}_G$. Note that for $\mu = 0$ this subscheme coincides with $\mathrm{Gr}_{U(P^-)}$.

Let $\Lambda'_{G,P} \subset \Lambda_G$ denote the lattice of cocharacters of the center $Z(M)$ of M. If $\mu' \in \Lambda'_{G,P}$, the action of the corresponding $t^{\mu'} \in Z(M)(\mathcal{K}_x)$ identifies $\mathrm{Gr}_P^\nu \cap \mathrm{Gr}_{P^-}^\mu$ with $\mathrm{Gr}_P^{\nu-\mu'} \cap \mathrm{Gr}_{P^-}^{\mu-\mu'}$.

Proposition 6.6. *Let $\mu' \in \Lambda'_{G,P}$ be G-dominant and deep enough on the corresponding wall of the Weyl chamber. Then the intersection $\mathrm{Gr}_P^{\nu-\mu'} \cap \mathrm{Gr}_{P^-}^{\mu-\mu'}$ is contained in $\mathrm{Gr}_G^{w_0(w_0^M(\mu-\mu'))}$.*

Proof. The initial observation is that each $\mathrm{Gr}_P^\nu \cap \mathrm{Gr}_{P^-}^\mu$ is a scheme of finite type. We know this fact for $\mu = 0$, since the above intersection is a locally closed subscheme in the Zastava space Z^θ.

In general, this assertion can be proven either by introducing the corresponding analog of the Zastava space over a global curve, or by a straightforward local argument.

Let us view Gr_M^μ as a sub-scheme of $\mathrm{Gr}_{P^-}^\mu$, such that $\mathrm{Gr}_{P^-}^\mu = U(P^-)(\mathcal{K}_x)\cdot\mathrm{Gr}_M^\mu$. As in the case $P = B$, we obtain that that the preimage of $\mathrm{Gr}_P^\nu \cap \mathrm{Gr}_{P^-}^\mu$ under

$$U(P^-)(\mathcal{K}_x) \times \mathrm{Gr}_M^\mu \to \mathrm{Gr}_{P^-}^\mu$$

is contained in a subscheme of the form $\mathrm{Ad}_{t^{\mu'}}(U(P^-)(\mathcal{O}_x)) \times \mathrm{Gr}_M^{\mu-\mu'}$. Hence, the action of $t^{-\mu'}$ maps $\mathrm{Gr}_P^\nu \cap \mathrm{Gr}_{P^-}^\mu$ inside

$$U(P^-)(\mathcal{O}_x) \cdot \mathrm{Gr}_M^{\mu-\mu'} \subset \mathrm{Gr}_G^{w_0(w_0^M(\mu-\mu'))}.$$

□

The rest of the proof is similar to the case of $P = B$:

From the above proposition we obtain that the intersection $\mathrm{Gr}^\nu_P \cap \mathrm{Gr}^\mu_{P^-}$ is of dimension $\leq \langle \nu - w_0^M(\mu), \check{\rho}_G \rangle$. In particular, $\mathrm{Gr}^\nu_P \cap \mathrm{Gr}_{U(P^-)}$ is of dimension $\leq \langle \nu, \check{\rho}_G \rangle$.

Moreover, as in the previous case, we obtain that there is a bijection between the set of irreducible components of $\mathrm{Gr}^\nu_P \cap \mathrm{Gr}_{U(P^-)}$ of dimension $\langle \nu, \check{\rho}_G \rangle$ and the set of irreducible components of the same dimension of $\mathrm{Gr}^{\nu-\mu'}_P \cap \mathrm{Gr}^{-w_0(\mu')}_G$, where μ' is large enough. However, Theorem 6.2(2) implies that the latter set parameterizes a basis of

$$\mathrm{Hom}_{\check{M}}(V_M^{\nu-\mu'}, V_G^{-w_0(\mu')}),$$

which, since μ' is large compared to ν, can be identified with $\mathrm{Hom}_{\check{M}}(V^\nu_M, U(\check{\mathfrak{u}}(P)))$.

7. Intersection cohomology of $\overline{\mathrm{Bun}}_P$

In this section we are concerned with describing explicitly the intersection cohomology sheaf on $\overline{\mathrm{Bun}}_P$. First, we introduce an analogue, $Z^{nv,\theta}$, of the Zastava spaces for $\overline{\mathrm{Bun}}_P$ (here the superscript nv stands for "naive").

By definition, $Z^{nv,\theta}$ is a scheme classifying the data of $(x^\theta, \mathcal{F}_G, \beta)$, where $x^\theta \in X^\theta$, $\mathcal{F}_G$ is a G-bundle and β is a trivialization of $\mathcal{F}_G$ off the support of x^θ, such that for every G-dominant weight $\check{\lambda}$ orthogonal to $\mathrm{Span}(\alpha_\mathfrak{i})$, $\mathfrak{i} \in \mathcal{I}_M$ the induced meromorphic maps

$$\mathcal{L}^{\check{\lambda}}_{\mathcal{F}^0_{M/[M,M]}} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G} \text{ and } \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G} \to \mathcal{L}^{\check{\lambda}}_{\mathcal{F}^0_{M/[M,M]}}$$

induce a regular map $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}^0_{M/[M,M](-x^\theta)}} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ and a regular and *surjective* map $\mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G} \to \mathcal{L}^{\check{\lambda}}_{\mathcal{F}^0_{M/[M,M]}}$.

There is a natural proper map $Z^\theta \to Z^{nv,\theta}$, which corresponds to "forgetting" the data of $(\mathcal{F}_M, \beta_M)$. In addition, $Z^{nv,\theta}$ contains an open subscheme $Z^{nv,\theta}_{\max}$ corresponding to the locus, where the maps $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}^0_{M/[M,M](-x^\theta)}} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ are maximal embeddings, over which we have an isomorphism $Z^\theta_{\max} \to Z^{nv,\theta}_{\max}$.

As in the case of Z^θ, one easily establishes the *factorization property* for $Z^{nv,\theta}$:

$$(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \underset{X^\theta}{\times} Z^{nv,\theta} \simeq (X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \underset{X^{\theta_1} \times X^{\theta_2}}{\times} (Z^{nv,\theta_1} \times Z^{nv,\theta_2}). \quad (3)$$

Finally, the spaces $Z^{nv,\theta}$ model the singularities of $\overline{\mathrm{Bun}}_P$ in the same sense as Z^θ models the singularities of $\widetilde{\mathrm{Bun}}_P$. In other words, an analogue of Proposition 3.2 holds, whose formulation we leave to the reader.

7.1. It turns out, that although the stack $\overline{\mathrm{Bun}}_P$ is "simpler" than $\widetilde{\mathrm{Bun}}_P$, the description of its intersection cohomology sheaf is more involved, and in particular it relies on the description of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$.

To formulate the main theorem, we introduce the following notation. Recall that the equivalence Loc between the category of spherical perverse sheaves on Gr_M and the category of $\check{M}$-modules admits a quasi-inverse given by the global cohomology functor: $\mathcal{S} \mapsto H^\bullet(\mathrm{Gr}_M, \mathcal{S})$.

Under this equivalence, the multiplication by the first Chern class $c_1(\det)$ of the determinant line bundle on Gr_M corresponds to the action of a principal nilpotent element $e \in \mathrm{Lie}(\check{M})$. Moreover, the cohomological grading of $H^\bullet$ corresponds to the action of a semisimple $h \in \mathrm{Lie}(\check{M})$ contained in a uniquely defined principal $\mathfrak{sl}_2$-triple (e, f, h) in $\mathrm{Lie}(\check{M})$. For a $\check{M}$-module V the $\mathbb{Z}$-grading arising from the action of h is given by the following rule: the weight subspace V_η has degree $\langle \eta, 2\check{\rho}_M \rangle$. This $\mathbb{Z}$-grading on V will be called *the principal grading*.

For V as above, we will denote by V^f the subspace annihilated by f. We will consider it as a graded vector space, via the principal grading.

We define the functor $\overline{\mathrm{Loc}}\,(V)$ from the category of $\mathbb{Z}$-graded vector spaces to complexes over $\mathrm{Spec}(\mathbb{F}_q)$ by setting

$$\overline{\mathrm{Loc}}\,(V) = \underset{n}{\oplus} V_n \otimes [-n](-\frac{n}{2}).$$

In particular, we will apply the functor $\overline{\mathrm{Loc}}$ to $\check{M}$-modules V (or their direct summands, such as V^f), endowed with the principal grading.

Theorem 7.2. *The restriction of* $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ *to*

$$_\theta\overline{\mathrm{Bun}}_P \simeq X^\theta \times \mathrm{Bun}_P$$

can be identified with the direct sum over $\mathfrak{P}(\theta) = \sum_m n_m \cdot \theta_m$ *of the direct images under* $X^{\mathfrak{P}(\theta)} \times \mathrm{Bun}_P \to X^\theta \times \mathrm{Bun}_P$ *of*

$$\mathrm{IC}_{\mathrm{Bun}_P} \boxtimes \left(\underset{m}{\boxtimes} \left(\overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)^f_{\theta_m}) \right)^{(n_m)} \right) \otimes \left(\overline{\mathbb{Q}_\ell} \left(\frac{1}{2} \right) [1] \right)^{\otimes 2 \cdot |\mathfrak{P}(\theta)|},$$

where each $\overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)_{\theta_m})$ *is viewed as a constant local system on* X *and the superscript* (n_m) *designates the* n_m*-th symmetric power.*

As a corollary, we obtain the following description of the restriction of $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ to the strata $_{\mathfrak{A}(\theta)}\overline{\mathrm{Bun}}_P$:

Theorem 7.3. *Let* $\mathfrak{A}(\theta)$ *be a partition* $\theta = \sum_m n_m \cdot \theta_m$. *The* $*$*-restriction of* $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ *to* $_{\mathfrak{A}(\theta)}\overline{\mathrm{Bun}}_P \simeq \overset{o}{X}{}^{\mathfrak{A}(\theta)} \times \mathrm{Bun}_P$ *is isomorphic to* $\mathrm{IC}_{_{\mathfrak{A}(\theta)}\overline{\mathrm{Bun}}_P}$ *tensored by the complex*

$$\bigotimes_m \left(\underset{i \geq 0}{\oplus} \overline{\mathrm{Loc}} \left(\mathrm{Sym}^i (\check{\mathfrak{u}}(P)^f)_{\theta_m} \right) (i)[2i] \right)^{\otimes n_m} \otimes \left(\overline{\mathbb{Q}_\ell} \left(\frac{1}{2} \right) [1] \right)^{\otimes - |\mathfrak{A}(\theta)|}.$$

Remark. Suppose $G = SL_n$. Then the parabolic subgroups of G are numbered by the ordered partitions $n = n_1 + \ldots + n_k,\ n_i > 0$. Suppose a parabolic subgroup P corresponds to a non-decreasing partition $n = n_1 + \ldots + n_k,\ 0 < n_1 \leq n_2 \leq \ldots \leq n_k$. In this case Theorem 7.3 was proved by A. Kuznetsov in the summer of 1997 (unpublished). His proof made use of *Laumon's compactification* Bun_P^L [La]. Namely, Bun_P^L is always smooth (see *loc. cit.*) and equipped with a natural dominant representable projective morphism $\varpi :\ \mathrm{Bun}_P^L \to \overline{\mathrm{Bun}}_P$. In case P corresponds to a non-decreasing partition, A. Kuznetsov proved that ϖ is small, and computed the cohomology of its fibers.

Let us mention that in case $G = SP(4)$, and P corresponding to the Dynkin subdiagram formed by the *long* simple root, $\overline{\mathrm{Bun}}_P$ *does not admit* a small resolution, as can be seen from the calculation of IC stalks in codimension 5 (the existence of such resolution would imply that a fiber has cohomology $\overline{\mathbb{Q}_\ell} \oplus \overline{\mathbb{Q}_\ell}[-4]$).

7.4. We will deduce Theorem 7.2 from Corollary 4.7. Let $\mathcal{Q}$ denote the direct image of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ under $\mathfrak{r} : \widetilde{\mathrm{Bun}}_P \to \overline{\mathrm{Bun}}_P$. On the one hand, from Corollary 4.7 and Lusztig's computation [Lu] of global cohomology of perverse sheaves on affine Grassmannians, we obtain:

Corollary 7.5. *The $*$-restriction of $\mathcal{Q}$ to ${}_\theta\overline{\mathrm{Bun}}_P \simeq X^\theta \times \mathrm{Bun}_P$ is isomorphic to the direct sum over $\mathfrak{P}(\theta) = \sum n_m \cdot \theta_m$ of the direct images under $X^{\mathfrak{P}(\theta)} \times \mathrm{Bun}_P \to X^\theta \times \mathrm{Bun}_P$ of*

$$\mathrm{IC}_{\mathrm{Bun}_P} \boxtimes \left(\underset{m}{\boxtimes} \left(\overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)_{\theta_m})\right)^{(n_m)}\right) \otimes \left(\overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]\right)^{\otimes 2\cdot|\mathfrak{P}(\theta)|}.$$

On the other hand, by the Decomposition Theorem, $\mathcal{Q}$ is a pure complex, which contains $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ as a direct summand. Therefore, Theorem 7.2 amounts to identifying the corresponding direct summand in the formula for $\mathcal{Q}|_{{}_\theta\overline{\mathrm{Bun}}_P}$ of the above corollary. In particular, we obtain that $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}|_{{}_\theta\overline{\mathrm{Bun}}_P}$ is a pure complex.

Consider the main diagonal $X \times \mathrm{Bun}_P \to X^\theta \times \mathrm{Bun}_P$, which corresponds to the partition $\mathfrak{A}(\theta) = \mathfrak{A}^0(\theta)$. Let $\mathcal{S}_\theta$ be the direct summand of $\mathrm{IC}_{\overline{\mathrm{Bun}}_P}|_{{}_\theta\overline{\mathrm{Bun}}_P}$, supported on $X \times \mathrm{Bun}_P$. By induction and the *factorization property*, it suffices to show that

$$\mathcal{S}_\theta \simeq \mathrm{IC}_{X\times\mathrm{Bun}_P} \otimes \overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)^f_\theta) \otimes \overline{\mathbb{Q}_\ell}(\frac{1}{2})[1].$$

7.6. We proceed as follows:

Let $\overline{{}_{\mathfrak{P}(\theta)}\overline{\mathrm{Bun}}_P}$ denote the closure of the image of $X^{\mathfrak{P}(\theta)} \times \mathrm{Bun}_P \to X^\theta \times \mathrm{Bun}_P \simeq {}_\theta\overline{\mathrm{Bun}}_P$ in $\overline{\mathrm{Bun}}_P$. By Corollary 7.5 and the Decomposition Theorem, we have

$$\mathcal{Q} = \bigoplus_{\theta\in\Lambda^{\mathrm{pos}}_{G,P},\, \mathfrak{P}(\theta)} \mathcal{Q}_{\mathfrak{P}(\theta)},$$

where each $\mathcal{Q}_{\mathfrak{P}(\theta)}$ is a complex on $\overline{{}_{\mathfrak{A}(\theta)}\overline{\mathrm{Bun}}_P}$. In particular, we obtain that

$$\mathcal{Q}|_{{}_\theta\overline{\mathrm{Bun}}_P} \simeq \bigoplus_{\theta\in\Lambda^{\mathrm{pos}}_{G,P},\,\mathfrak{P}(\theta)} \mathcal{Q}_{\mathfrak{P}(\theta)}|_{{}_\theta\overline{\mathrm{Bun}}_P}.$$

Lemma 7.7. *For $0 < \theta' < \theta$, none of the $\mathcal{Q}_{\mathfrak{P}(\theta')}|_{{}_\theta\overline{\mathrm{Bun}}_P}$ has a direct summand supported on the main diagonal $X \times \mathrm{Bun}_P \subset {}_\theta\overline{\mathrm{Bun}}_P$.*

Proof. First, from Corollary 4.7, it is easy to see that each $\mathcal{Q}_{\mathfrak{P}(\theta')}$ has the following form: it is the intersection cohomology sheaf of $\overline{{}_{\mathfrak{P}(\theta')}\overline{\mathrm{Bun}}_P}$ tensored with a complex over $\mathrm{Spec}(\mathbb{F}_q)$.

There is a finite map $X^{\mathfrak{P}(\theta')} \times \overline{\mathrm{Bun}}_P \to \overline{\mathrm{Bun}}_P$, defined as in Proposition 1.5, which normalizes $\overline{{}_{\mathfrak{P}(\theta')}\overline{\mathrm{Bun}}_P}$. Hence, it suffices to analyze the $*$-restriction to ${}_\theta\overline{\mathrm{Bun}}_P$ of the direct image of $\mathrm{IC}_{X^{\mathfrak{P}(\theta')}\times\overline{\mathrm{Bun}}_P}$ under this map.

However, the preimage of ${}_\theta\overline{\mathrm{Bun}}_P$ in $X^{\mathfrak{P}(\theta')} \times \overline{\mathrm{Bun}}_P \to \overline{\mathrm{Bun}}_P$ is $X^{\mathfrak{P}(\theta')} \times {}_{\theta-\theta'}\overline{\mathrm{Bun}}_P$ and we can assume that the $*$-restriction of $\mathrm{IC}_{X^{\mathfrak{P}(\theta')}\times\overline{\mathrm{Bun}}_P}$ to this substack is known by induction.

In particular, all its direct summands are supported on substacks of the form

$$X^{\mathfrak{P}(\theta')} \times X^{\mathfrak{P}(\theta-\theta')} \times \mathrm{Bun}_P\,.$$

Since, $\theta' \neq 0$ and $\theta - \theta' \neq 0$, none of these sub-stacks maps onto the main diagonal in ${}_\theta\overline{\mathrm{Bun}}_P$. □

Thus, from Corollary 7.5, we obtain that

$$\mathrm{IC}_{X\times\mathrm{Bun}_P} \otimes \left(\overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)_\theta)\right) \otimes \overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1] \simeq \mathcal{S}_\theta \oplus \mathcal{Q}_{\mathfrak{P}^0(\theta)}.$$

Hence, it suffices to see that if we decompose $\check{\mathfrak{u}}(P)_\theta$ as

$$\check{\mathfrak{u}}(P)_\theta = \check{\mathfrak{u}}(P)^f_\theta \oplus \mathrm{Im}(e : \check{\mathfrak{u}}(P) \to \check{\mathfrak{u}}(P))_\theta,$$

then the induced map

$$\mathcal{S}_\theta \to \mathrm{IC}_{X\times\mathrm{Bun}_P} \otimes \left(\overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)_\theta)\right) \otimes \overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1] \to \mathrm{IC}_{X\times\mathrm{Bun}_P} \otimes$$
$$\left(\overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)^f_\theta)\right) \otimes \overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]$$

is an isomorphism.

The latter is established as follows:

Consider the line bundle L on $\widetilde{\mathrm{Bun}}_P$ equal to the ratio of the pull-backs of the determinant line bundles under the maps $\widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G$ and $\widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_M$, respectively. Its restriction to the fibers of $\mathfrak{r}_P : \widetilde{\mathrm{Bun}}_P \to \overline{\mathrm{Bun}}_P$ over ${}_{\mathfrak{A}^0(\theta)}\overline{\mathrm{Bun}}_P$ is equal to a positive power of the determinant line bundle det on Gr_M. Hence L is a relatively ample line bundle for $\widetilde{\mathrm{Bun}}_P$ over $\overline{\mathrm{Bun}}_P$. The relative hard Lefschetz theorem [BBD] asserts that the multiplication by $c_1(L)^i$ induces an isomorphism from $\mathcal{Q}^{-i}$ to $\mathcal{Q}^i(i)$ where $\mathcal{Q}^i$ denotes the direct summand of $\mathcal{Q}$ in perverse cohomological degree i.

Let us restrict the action of $c_1(L)$ to the direct summand of $\mathcal{Q}|_{\theta\overline{\mathrm{Bun}}_P}$ supported on the main diagonal $X \times \mathrm{Bun}_P$. Under the identification of this summand with

$$\mathrm{IC}_{X\times\mathrm{Bun}_P} \otimes \left(\overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)_\theta)\right) \otimes \overline{\mathbb{Q}_\ell}\left(\frac{1}{2}\right)[1]$$

this action coincides, up to a scalar, with the action of e, by the very definition. Let us disregard Tate twists and view the above direct summand as a semisimple graded perverse sheaf.

We have the following general lemma:

Lemma 7.8. *Let $A^\bullet$ be a graded semisimple object of an abelian category, equipped with an endomorphism $e:\ A^\bullet \to A^{\bullet+2}$ such that $e^i:\ A^{-i} \to A^i$ is an isomorphism. Suppose $A[1] = B \oplus C$ where B is concentrated in negative degrees, and $e^k:\ C^{-k} \to C^k$ is an isomorphism for any $k \geq 0$. Then*

(a) *There is a unique endomorphism $f:\ A^\bullet \to A^{\bullet-2}$ satisfying the relations of $\mathfrak{sl}_2$ together with e, h where $h|_{A^i} = i$;*

(b) *$C = \mathrm{Im}(e)$, and the projection $B \to A[1]/C = A[1]/\mathrm{Im}(e) \widetilde{\leftarrow} \mathrm{Ker}(f)$ identifies B with $\mathrm{Ker}(f)$.*

The proof of Theorem 7.2 is concluded by applying this lemma to

$$A^\bullet = \mathrm{IC}_{X\times\mathrm{Bun}_P} \otimes \left(\overline{\mathrm{Loc}}\,(\check{\mathfrak{u}}(P)_\theta)\right),\ B = \mathcal{S}_\theta \text{ and } C = \mathcal{Q}_{\mathfrak{P}^0(\theta)}.$$

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