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Weak solutions and supersolutions in L^1 for reaction-diffusion systems

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Abstract. We prove here that limits of nonnegative solutions to reaction-diffusion systems whose nonlinearities are bounded in L^1 always converge to supersolutions of the system. The motivation comes from the question of global existence in time of solutions for the wide class of systems preserving positivity and for which the total mass of the solution is uniformly bounded. We prove that, for a large subclass of these systems, weak solutions exist globally.

1. Introduction

This paper is motivated by the general question of global existence in time of solutions to reaction-diffusion systems of the form:

$$\left.\begin{aligned} u_t - d_1 \Delta u = f(u, v) \ \text{ on } \ Q, \\ v_t - d_2 \Delta v = g(u, v) \ \text{ on } \ Q, \\ u(0, \cdot) = u_0(\cdot) \geq 0,\ v(0, \cdot) = v_0(\cdot) \geq 0, \\ u, v \ \text{ satisfy some good boundary conditions on } \ \partial\Omega, \end{aligned}\right\} \tag{1}$$

where $Q = (0, +\infty) \times \Omega$, Ω is a regular bounded open subset of $\mathbb{R}^N$, $d_1, d_2 > 0$ and f, g are regular functions whose nonlinear structure is such that two main properties occur:

- the nonegativity of solutions of (1) is preserved in time,
- the total mass of the solutions is uniformly bounded in time.

The functions f, g may also depend on time and space variable ($f = f(t, x, u, v)$).

With good boundary conditions on $\partial\Omega$ like for instance $u = v = 0$ or $\partial_n u = \partial_n v = 0$ (where ∂_n is the normal derivative at the boundary), nonnegativity will be preserved in time, like for systems of ordinary differential equations, as soon as

$$\forall u, v \geq 0,\ \ f(0, v) \geq 0,\ \ g(u, 0) \geq 0. \tag{2}$$

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The second property will occur for instance when

$$f + g \leq 0. \tag{3}$$

Indeed, by just integrating the sum of the two equations, we obtain

$$\int_\Omega u(t) + v(t) \leq \int_\Omega u_0 + v_0.$$

Together with nonnegativity, this yields an L^1-bound of the solution uniformly in time. A general question is to understand how these two properties help to provide global existence in time of solutions.

Note that, if we had uniform L^∞-bounds rather than L^1-bounds, we would deduce global existence in time of "classical" solutions, by standard results for reaction-diffusion systems. By "classical" solution, we mean "bounded" solution, so that, by well-known regularity results, a "classical" solution also has classical derivatives at least *a.e.* and the equations are understood pointwise.

The point here is that bounds are a priori only in L^1 and one cannot apply the L^∞-approach even if the initial data are regular.

This situation frequently comes out in applications where positivity of the unknowns u, v is implicit from their definition (they are densities, concentrations, normalized temperatures,...) and where the total mass is preserved or, at least, controlled in time. This explains why these systems have been studied in several places in the literature. Let us refer here to [11, 17] for a survey and references.

To help understand the situation, let us mention two particular examples of the nonlinearities we are considering:

$$f(u, v) = u^3v^2 - u^2v^3, \quad g(u, v) = -u^3v^2 + \gamma u^2v^3, \quad \text{where } 0 \leq \gamma \leq 1. \tag{4}$$

$$f(u, v) = c_1(x, t)u^\alpha v^\beta, \quad g(u, v) = c_2(x, t)u^\alpha v^\beta, \tag{5}$$

where $\alpha, \beta > 1$, and c_1, c_2 are regular functions such that

$$a.e.\ (t, x) \in Q, c_1(x, t) + c_2(x, t) \leq 0. \tag{6}$$

Obviously, for bounded initial data, we will have local existence of classical solutions. With some extra assumptions, like for instance $\gamma = 0$ in (4) or $c_2 \leq 0$ in (5), global existence of classical solutions may be proved. It is not straightforward: several approaches may be found in [13, 7, 8, 3, 14].

But for our purpose here, the main fact to remember is that, although one has an uniform L^1-bound in time, "classical" solutions may not globally exist when the diffusion coefficients d_1, d_2 are not equal (global existence obviously holds if they are equal). As surprisingly proved in [16, 17], it may indeed happen that, under assumptions (2), (3), solutions blow up in finite time in L^∞! In particular, classical bounded solutions do not exist globally in time.

We emphasize the fact that, in the examples of blow up provided in [16, 17], not only (3) holds, but even

$$f + \lambda_0 g \leq 0 \text{ for some } \lambda_0 \in [0, 1). \tag{7}$$

Note that (3) together with (7) imply that

$$f + \lambda\, g \leq 0 \quad \text{for} \quad \lambda \in [\lambda_0, 1].$$

As we will see in more details later, under this more restrictive assumption, not only u, v are bounded in $L^1(\Omega)$, but the nonlinear terms $f(u, v)$, $g(u, v)$ are also bounded in $L^1(Q_T)$ for all $0 < T < \infty$ and $Q_T = (0, T) \times \Omega$.

A first main purpose of this paper is to prove that, under the latter stronger assumption, *global existence on* $[0, \infty)$ *of weak solutions holds* for the above considered systems (although these solutions may blow up in L^∞ at some time). By "weak" solution, we essentially mean solution in the sense of distributions or, equivalently here, solution in the sense of the variation of constant formula with the corresponding semigroups (see Appendix). In particular, classical derivatives may not exist. Such weak L^1-solutions had already been considered in [15, 10, 3] to handle initial data in L^1. However, an extra condition of "triangular" structure of the nonlinearities was required (which would, for instance, imply $\gamma = 0$ in example (4)).

Concerning the above examples, our result here means that weak solutions exist globally for the nonlinearities (4) if $\gamma \in [0, 1)$ and for the nonlinearities (5) if, moreover, $c_1 + \lambda_0 c_2 \leq 0$ for some $\lambda_0 \geq 0$, $\lambda_0 \neq 1$. But, according to [16, 17], weak solutions in example (5) may blow up in finite time in $L^p(\Omega)$ for p large. We do not know specifically what happens for example (4), but we know that similar polynomial nonlinearities do lead to blow up in finite time [16, 17].

One of the main steps in the proof turns out to be interesting by itself for reaction-diffusion systems. One knows that maximum principle is valid for equations, but generally not for systems. It turns out that systems do nevertheless share some order properties with equations, no matter their structure: this is also a purpose of this paper to point it out.

To explain this point, let us first consider an equation and a sequence of *nonnegative* regular solutions of

$$\partial u_n/\partial t - \Delta u^n = F_n(u_n) \text{ on } Q_T$$

where $F_n : [0, +\infty) \to \mathbb{R}$ converges uniformly on bounded sets to the continuous function $F : [0, +\infty) \to \mathbb{R}$. Assume that

$$F_n(u_n) \text{ is bounded in } L^1(Q_T) \text{ independently of } n.$$

Assume also that u_n satisfies, for instance, the boundary condition $u_n(t,\cdot) = 0$ on $\partial\Omega$ and that $u_n(0,.)$ is bounded in L^1.

Then, up to a subsequence, u_n converges in $L^1(Q_T)$ to a *supersolution* u of the equation, namely

$$\partial u/\partial t - \Delta u \geq F(u) \text{ on } Q_T, \tag{8}$$

in the sense of distributions.

The proof of this fact goes essentially as follows. Thanks to the L^1 bound on the nonlinear term $F_n(u_n)$ and to the parabolic boundary conditions, u_n is relatively compact in $L^1(Q_T)$. Up to a subsequence, one can assume that u_n converges in $L^1(Q_T)$ and almost everywhere to a function u. Then, $F_n(u_n)$ converges pointwise to the integrable function $F(u)$. Unfortunately, this is not enough to pass to the limit in the equation.

Then, let us introduce a truncation procedure: for $k \geq 1$ and $r \geq 0$, set $\tau_k(r) = \min\{r, k\}$. By a simple computation, we obtain for all k, n:

$$\partial\tau_k(u_n)/\partial t \;-\; \Delta\tau_k(u^n) \;\geq\; \tau_k'(u_n)F_n(u_n) \text{ on } Q_T.$$

But, $\tau_k'(u_n) = 0$ where $u_n > k$. For k fixed, since $F_n(u_n)$ is bounded independently of n on the set where $u_n \leq k$, then, $\tau_k'(u_n)F_n(u_n)$ converges, not only pointwise, but also in $L^1(Q_T)$ to $\tau_k'(u)F(u)$, so that

$$\partial\tau_k(u)/\partial t - \Delta\tau_k(u) \geq \tau_k'(u)F(u) \text{ on } Q_T.$$

We now let k go to ∞ to obtain (8).

Obviously, this approach does not extend as such to a sequence u_n, v_n of solutions of a 2×2 system since, multiplying the first equation by $T_k'(u_n)$ does not take care of unbounded values of v_n. However, we are able to prove that the same result holds and this is another main goal of this paper: when the nonlinearities remain bounded in $L^1(Q_T)$, *the limit is a supersolution of the system.*

2. The main results

Let Ω be a bounded open subset of $\mathbb{R}^N$ with regular boundary. We denote $Q := (0,+\infty)\times\Omega$, and for $T \in (0,+\infty)$, $Q_T := (0,T)\times\Omega$.

Let $f, g : Q \times [0,+\infty)^2 \to \mathbb{R}$ satisfy the usual local Lipschitz conditions:

$$\left.\begin{aligned} f, g \text{ are measurable, } \forall T > 0,\ f(\cdot,\cdot,0,0),\ g(\cdot,\cdot,0,0) \in L^1(Q_T),\\ \exists K : [0,+\infty) \to [0,+\infty) \text{ nondecreasing such that}\\ a.e.\ (t,x) \in [0,+\infty)\times\Omega,\ \forall M > 0, \forall r, s, \hat{r}, \hat{s} \in (0,M),\\ |f(t,x,r,s) - f(t,x,\hat{r},\hat{s})| + |g(t,x,r,s) - g(t,x,\hat{r},\hat{s})| \leq \ldots\\ \ldots K(r)(|r-\hat{r}| + |s-\hat{s}|).\end{aligned}\right\} \tag{9}$$

The condition (2) will take the form

$$a.e.\ (t,x) \in Q,\ \forall u, v \geq 0,\ f(t,x,0,v) \geq 0,\ g(t,x,u,0) \geq 0. \tag{10}$$

We will also assume that conditions (3)+(7) are satisfied in a weaker sense:

$$\left.\begin{array}{r}\exists\,\lambda_0 \in [0,1),\ \text{such that}\ \ \forall\,\lambda \in [\lambda_0, 1],\\ a.e.(t,x) \in Q, \forall\, r, s \geq 0,\\ f(t,x,r,s) + \lambda\, g(t,x,r,s) \leq \sigma\,(r+s) + h(t,x),\\ \text{where}\ \sigma \geq 0,\ \text{and}\ h \in L^1(Q_T), \forall\, T > 0,\ h \geq 0.\end{array}\right\} \tag{11}$$

THEOREM 2.1. *Let f, g be given as in* (9) *and let $d_1, d_2 > 0$. Assume that f, g satisfy the positivity property* (10) *and the structure condition* (11). *Then, for all $u_0, v_0 \in L^1(\Omega)$, $u_0, v_0 \geq 0$, there exists a global nonnegative solution (u, v) on $[0,+\infty)$ of*

$$\left.\begin{array}{r}u, v \in C([0,+\infty); L^1(\Omega)) \cap L^1_{loc}([0,+\infty); W_0^{1,1}(\Omega)),\\ u(0,\cdot) = u_0,\quad v(0,\cdot) = v_0,\\ \forall\, T > 0,\ f(\cdot,\cdot,u(\cdot,\cdot),v(\cdot,\cdot)),\ g(\cdot,\cdot,u(\cdot,\cdot),v(\cdot,\cdot)) \in L^1(Q_T),\\ u_t - d_1\Delta u = f(t,x,u,v)\ \ \text{in}\ \ \mathcal{D}'(Q),\\ v_t - d_2\Delta v = g(t,x,u,v)\ \ \text{in}\ \ \mathcal{D}'(Q).\end{array}\right\} \tag{12}$$

Here and hereafter, equations are understood in the sense of distributions $\mathcal{D}'(Q)$, that is, for all test-function φ in the space $C_0^\infty(Q)$ of infinitely differentiable functions with compact support in Q, we have:

$$-\int_Q u\,(\varphi_t + d_1\Delta\varphi) = \int_Q \varphi\, f,$$

and similarly for v. It is well-known that (12) is equivalent to the variation of constant formula, that is to say (see Appendix)

$$u(t) = S_{d_1}(t)u_0 + \int_0^t S_{d_1}(t-s) f(s,\cdot,u(s,\cdot),v(s,\cdot))\,ds,$$

where $S_{d_1}(\cdot)$ is the semigroup generated in $L^1(\Omega)$ by the Laplacian operator with homogeneous boundary conditions (and the similar formula for v).

REMARK. The boundary condition $u = v = 0$ on $\partial\Omega$ is understood here in the sense that $a.e.t, u(t), v(t) \in W_0^{1,1}(\Omega)$. As usual, for all $1 \leq p < +\infty$, $W_0^{1,p}(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ equipped with the norm

$$\|w\|_{W_0^{1,p}} := \{\|w\|^p_{L^p(\Omega)} + \|\nabla w\|^p_{L^p(\Omega)^N}\}^{1/p}.$$

As it will be clear from the proof, a similar result could be stated for Neumann boundary conditions or for more general boundary conditions. One must however be careful when choosing two different boundary conditions for u and v (see [4, 12]).

As announced in the introduction, the second main result of this paper deals with limits of approximate solutions of systems when the nonlinearities are bounded in L^1.

THEOREM 2.2. *Let* (u_n, v_n) *be a sequence of (regular)* nonegative *solutions of*

$$\left.\begin{aligned} u_n, v_n \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); W_0^{1,2}(\Omega)), \\ u_{nt}, v_{nt}, \Delta u_n, \Delta v_n \in L^2(Q_T), \\ F_n(\cdot,\cdot,u_n(\cdot),v_n(\cdot)),\ G_n(\cdot,\cdot,u_n(\cdot),v_n(\cdot)) \in L^2(Q_T), \\ \partial u_n/\partial t - d_1 \Delta u_n = F_n(t,x,u_n,v_n) \text{ on } Q_T, \\ \partial v_n/\partial t - d_2 \Delta v_n = G_n(t,x,u_n,v_n) \text{ on } Q_T, \end{aligned}\right\} \tag{13}$$

where $F_n, G_n : Q \times [0,+\infty)^2 \to \mathbb{R}$ *converge in the following sense to* $F, G : Q \times [0,+\infty)^2 \to \mathbb{R}$ *satisfying* (9)*: for all* $M > 0$, ϵ_M^n *tends to zero in* $L^1(Q_T)$ *as* $n \to +\infty$ *where*

$$\epsilon_M^n = \sup_{0 \le r,s \le M} \{|F_n(\cdot,\cdot,r,s) - F(\cdot,\cdot,r,s)| + |G_n(\cdot,\cdot,r,s) - G(\cdot,\cdot,r,s)|\}. \tag{14}$$

Assume that $F_n(\cdot,\cdot,u_n(\cdot),v_n(\cdot))$, $G_n(\cdot,\cdot,u_n(\cdot),v_n(\cdot))$ *are bounded in* $L^1(Q_T)$ *independently of n. Assume also that* $u_n(0), v_n(0)$ *are bounded in* L^1.

Then, up to a subsequence, u_n, v_n *converge to* u, v *in* $L^1(Q_T)$ *satisfying*

$$\left.\begin{aligned} u, v \in L^\infty((0,T); L^1(\Omega)) \cap L^1((0,T); W_0^{1,1}(\Omega)), \\ F(\cdot,\cdot,u(\cdot,\cdot),v(\cdot,\cdot)),\ G(\cdot,\cdot,u(\cdot,\cdot),v(\cdot,\cdot)) \in L^1(Q_T), \\ \partial u/\partial t - d_1 \Delta u \ge F(t,x,u,v) \ in\ \mathcal{D}'(Q_T), \\ \partial v/\partial t - d_2 \Delta v \ge G(t,x,u,v) \ on\ \mathcal{D}'(Q_T). \end{aligned}\right\} \tag{15}$$

Moreover, if $u_n(0), v_n(0)$ *converge to* u_0, v_0 *in* $L^1(\Omega)$*, then, for all nonnegative* $\varphi \in C_0^\infty(\Omega)$*, we have*

$$\liminf_{t\to 0} \int_\Omega u(t)\varphi \ge \int_\Omega u_0\varphi, \quad \liminf_{t\to 0} \int_\Omega v(t)\varphi \ge \int_\Omega v_0\varphi. \tag{16}$$

REMARK. Although it is not essential, we assume here that the solutions are "regular", in the sense that they have derivatives $u_{nt}, \Delta u_n$ in L^2. This allows to make direct computations. Without L^2-regularity, we could also do it by using one more approximation process. Theorem 2 (which we prove first) will be sufficient for the approximate solutions considered in Theorem 1 which are regular by construction.

The above inequations are understood in the sense of distributions in Q: this means that for all nonnegative test-functions φ of $C_0^\infty(Q)$,

$$-\int_Q u\,(\varphi_t + d_1 \Delta\varphi) \ge \int_Q \varphi\, F,$$

and the same for v.

Obviously, estimate (16) says that the initial data of u, v are above the limit of the initial data of u_n, v_n. More precisely, since $u(t)$ is bounded in $L^1(\Omega)$, one may find a nonnegative Radon measure u_{0+} and a sequence t_k tending to zero as k tends to $+\infty$ such that $u(t_k)$ converges to u_{0+} in the sense of measures. Then, for all such limit u_{0+}, by (16), we have $u_{0+} \geq u_0$, and the same for v.

3. The proofs

We will use the following more or less classical compactness lemma for the heat operator (a proof may be found e.g. in [2]; comments are also given in Appendix)

LEMMA 3.1. *Let $d > 0, w_0 \in L^1(\Omega), H \in L^1(Q_T)$. Then there exists a unique solution of*

$$\left.\begin{array}{l} w \in C([0,T]; L^1(\Omega)) \cap L^1((0,T); W_0^{1,1}(\Omega)), \\ \partial w/\partial t - d\Delta w = H \text{ in } \mathcal{D}'(Q_T), w(0,\cdot) = w_0. \end{array}\right\} \tag{17}$$

Moreover, for all $s, q \geq 1$ with $2s^{-1} + Nq^{-1} > N+1$, there exists $C = C(q, s, \Omega, d)$ such that

$$\|w\|_{L^\infty(0,T;L^1(\Omega))} + \|w\|_{L^s(0,T;W_0^{1,q}(\Omega))} \leq C\,[\|H\|_{L^1(Q_T)} + \|w_0\|_{L^1(\Omega)}]. \tag{18}$$

Finally, the mapping $(H, w_0) \to w$ is compact from $L^1(Q_T) \times L^1(\Omega)$ into $L^1(Q_T)$.

Proof of Theorem 2.2. By Lemma 3.1, if (u_n, v_n) is the sequence considered in Theorem 2.2, up to a subsequence, we may assume that u_n, v_n converge in $L^1(Q_T)$ and *a.e.* to $u, v \in L^\infty(0,T; L^1(\Omega)) \cap L^1(0,T; W_0^{1,1}(\Omega))$. According to the type of convergence of F_n, G_n to F, G, $F_n(t,x,u_n,v_n)$ converge a.e. to $F = F(t,x,u,v)$ and the same for G_n, G, but this pointwise convergence is not sufficient by itself to pass to the limit in the equations and this is where the more difficult step starts.

We introduce truncation functions. For technical reasons, we need them to be a little more regular than in the introduction. For all $k > 0$, we define a C^2- function T_k, such that

$$\left.\begin{array}{rr} \forall r \in [0,k], T_k(r) = r \;\; ; & \forall r \geq k,\; T_k(r) \leq k+1, \\ \forall r \geq 0,\; 0 \leq T_k'(r) \leq 1 \;\; ; & \forall r \geq k+1, T_k'(r) = 0, \\ & 0 \leq -T_k''(r) \leq C(k). \end{array}\right\}$$

For instance, we may choose T_k as $T_k(r) = r$ on $[0,k]$ and

$$\forall r \in [k, k+1], T_k(r) = (r-k)^4/2 - (r-k)^3 + r; \forall r > k+1, T_k(r) = k + 1/2.$$

Now, we fix $\eta > 0$ and we introduce $z_n := T_k(u_n + \eta\, v_n)$. Using the equations satisfied by u_n, v_n, we obtain

$$\partial z_n/\partial t - d_1 \Delta z_n = T_k'(u_n + \eta\, v_n)(F_n + \eta G_n) + \eta(d_2 - d_1)S_n^1 + d_1 S_n^2, \tag{19}$$

where

$$S_n^1 = T_k'(u_n + \eta\, v_n)\Delta v_n,\ \ S_n^2 = -T_k''(u_n + \eta\, v_n)|\nabla(u_n + \eta\, v_n)|^2 \geq 0. \tag{20}$$

The main point is to pass to the limit as n tends to ∞ in the equation (19), η and k being fixed. Let us look successively at the four terms involved.

Note first that the last term is nonnegative so that we may just forget it.

Since z_n tends to $z = T_k(u + \eta\, v)$ in $L^1(Q_T)$, $\partial z_n/\partial t - d_1 \Delta z_n$ converges in the sense of distributions to $\partial z/\partial t - d_1 \Delta z$.

Next, by the type of convergence of F_n, G_n to F, G (see (14)) and by the continuity property of F, G, $T_k'(u_n + \eta\, v_n)(F_n + \eta\, G_n)$ converges pointwise and in $L^1(Q_T)$ to $T_k'(u+\eta\, v)(F+\eta\, G)$: indeed, on one hand, $T_k'(u_n+\eta\, v_n) = 0$ on the set $[u_n+\eta\, v_n \geq k+1]$. On the other hand, on the set $[u_n + \eta\, v_n \leq k + 1]$, we have:

$$|F_n(t, x, u_n, v_n) - F(t, x, u_n, v_n)| \leq \epsilon^n_{(k+1)(1+n^{-1})}(t, x),$$

and the right hand side tends to zero in $L^1(Q_T)$ and a.e. as n tends to $+\infty$. Moreover,

$$F(t, x, u_n(t, x), v_n(t, x)) \to F(t, x, u(t, x), v(t, x)),\ a.e.(t, x),$$

and remains bounded on the set $[u_n + \eta\, v_n \leq k + 1]$ by

$$\zeta(t, x) := \sup_{r\leq k+1, s\leq (k+1)\eta^{-1}} |F(t, x, r, s)|,$$

which is in $L^1(Q_T)$ by the conditions (9) on F (and similarly for G_n, G).

Note that, by Fatou's Lemma and the L^1-bounds on F_n, G_n, F, G are in $L^1(Q_T)$.

Now we are left with the main step: estimating S_n^1. For this, we need the following lemma.

LEMMA 3.2. *There exists C depending only on the bounds on $\|F_n\|_{L^1(Q_T)}$, $\|G_n\|_{L^1(Q_T)}$, $\|u_0\|_{L^1(\Omega)}$, $\|v_0\|_{L^1(\Omega)}$ such that*

$$\forall\, k \geq 1, \int_{[u_n\leq k]} |\nabla u_n|^2, \int_{[v_n\leq k]} |\nabla v_n|^2 \leq Ck. \tag{21}$$

Let us postpone the proof of this lemma and continue the proof of Theorem 2.2. We will denote by $C(k)$ all positive constants depending only on k. Let $\varphi \in C_0^\infty([0,T]\times\Omega)$. Then, for all $t \in [0,T]$

$$\int_{Q_t} \varphi S_n^1 = -\int_{Q_t} \nabla v_n[T_k'(u_n+\eta\, v_n)\nabla\varphi + \varphi\, T_k''(u_n+\eta\, v_n)\nabla(u_n+\eta\, v_n)],$$

so that, using the properties of T_k

$$\begin{aligned}|\textstyle\int_{Q_t}\varphi S_n^1| \le \{\int_{[u_n+\eta\, v_n\le k+1]}|\nabla v_n|^2\}^{1/2}[\{\int_{Q_T}|\nabla\varphi|^2\}^{1/2}\\ +C(k)\|\varphi\|_{L^\infty(Q_T)}\{\textstyle\int_{[u_n+\eta\, v_n\le k+1]}|\nabla u_n+\eta\, v_n|^2\}^{1/2}].\end{aligned}$$

Note that $[u_n+\eta\, v_n \le k+1]$ is included in $[u_n \le k+1]$ and in $[v_n \le (k+1)\eta^{-1}]$. We bound the last term from above as follows:

$$\begin{aligned}\{\textstyle\int_{[u_n+\eta\, v_n\le k+1]}|\nabla u_n+\eta\, v_n|^2\}^{1/2} \le \{\int_{[u_n\le k+1]}|\nabla u_n|^2\}^{1/2}\\ +\eta\{\textstyle\int_{[v_n\le(k+1)\eta^{-1}]}|\nabla v_n|^2\}^{1/2}.\end{aligned}$$

Setting $D(\varphi) := \{\int_{Q_T}|\nabla\varphi|^2\}^{1/2} + \|\varphi\|_{L^\infty(Q_T)}$ and using Lemma 3.2, we deduce

$$\left|\int_{Q_t}\varphi S_n^1\right| \le D(\varphi)C(k)\eta^{-1/2}[1+\eta^{1/2}]. \tag{22}$$

We can now let n tend to $+\infty$ in (19). We will denote by $\langle Z,\varphi\rangle$ the result of a distribution Z of $\mathcal{D}'(Q_T)$ applied to the C_0^∞-test-function φ. We obtain for any *nonnegative* test-function φ

$$\langle z_t - d_1\Delta z - T_k'(u+\eta\, v)(F+\eta\, G),\varphi\rangle \;\ge\; -C(k)\eta^{1/2}D(\varphi).$$

Now, we let η tend to zero in the above inequality. Since, $z = T_k(u+\eta\, v)$ tends to $T_k(u)$ in $L^1(Q_T)$ and since $T_k'(u+\eta\, v)$ remains uniformly bounded by 1 and tends a.e. to $T_k'(u)$, we can pass to the limit in the sense of distributions to find

$$\partial T_k(u)/\partial t - d_1\Delta T_k(u) \ge T_k'(u)\,F \;\text{ in }\; \mathcal{D}'(Q_T).$$

Finally, we let k tend to $+\infty$: by monotonicity, $T_k(u)$ tends to u in $L^1(Q_T)$ and $T_k'(u)$ tends a.e. to 1; since $F \in L^1(Q_T)$, we can pass to the limit and obtain

$$\partial u/\partial t - d_1\Delta u \ge F \;\text{ in }\; \mathcal{D}'(Q_T).$$

Let us now look at the **initial data:** we assume that $u_n(0)$, $v_n(0)$ tend in $L^1(\Omega)$ to u_0, v_0. We go back to equation (19) and multiply by $\varphi \in C_0^\infty(\Omega)$, nonnegative, to obtain

$$\int_\Omega [z_n(t)-z_n(0)]\varphi \ge \int_{Q_t} d_1 z_n\Delta\varphi + \varphi[\eta(d_2-d_1)S_n^1 + T_k'(u_n+\eta\, v_n)(F_n+\eta\, G_n)].$$

We recall (22) which gives a bound from below for S_n^1. Letting n tend to ∞ leads to

$$\int_\Omega [z(t) - T_k(u_0 + \eta\, v_0)]\varphi \geq \int_{Q_t} d_1 z \Delta\varphi + \varphi T_k'(u + \eta\, v)(F + \eta\, G) - C(k, \varphi)\eta^{1/2}.$$

We let η tend to zero, then k tend to ∞: as before, using that $F \in L^1(Q_T)$, we may pass to the limit to obtain

$$\int_\Omega [u(t) - u_0]\varphi \geq \int_{Q_t} d_1 u \Delta\varphi + \varphi F.$$

Letting now t tend to zero gives

$$\liminf_{t\to 0} \int_\Omega u(t)\varphi \geq \int_\Omega u_0\varphi.$$

This is the statement of (16).

Proof of Lemma 3.2. We choose T_k as above. Multiplying the equation in u_n by $T_k(u_n)$ gives

$$\frac{\partial}{\partial t}\int_\Omega j_k(u_n) + d_1 \int_\Omega T_k'(u_n)|\nabla u_n|^2 = \int_\Omega T_k(u_n)\, F_n, \tag{23}$$

where $j_k(r) = \int_0^r T_k(s)\, ds$. Note that $j_k(r) \leq (k+1)\, r$. After integrating (23) in time, we obtain

$$d_1 \int_{[u_n \leq k]} |\nabla u_n|^2 \leq (k+1)\left\{\int_{Q_T} |F_n| + \int_\Omega u_n(0)\right\},$$

whence the estimate (21) for u_n. The proof is the same for v_n.

Proof of Theorem 2.1. The first step is to truncate the data in order to solve an approximate problem. We set $u_{0n} := \inf\{u_0, n\}$, $v_{0n} := \inf\{v_0, n\}$. We truncate the nonlinearities f, g in such a way that they be bounded and that they keep satisfy the same conditions (9, 10, 11). For this, we introduce a C_0^∞ function $\psi_1 : [0, +\infty)^2 \to [0, 1]$ satisfying

$$\forall\, 0 \leq r, s \leq 1,\ \psi_1(r, s) = 1;\ \forall\, r, s \geq 2,\ \psi_1(r, s) = 0.$$

Next, we set $\psi_n(r, s) = \psi_1(r/n, s/n)$. With this choice, for all n, $0 \leq \psi_n \leq 1$, and ψ_n tends pointwise to 1 as n tends to ∞.

In order to take care of the fact that $f(\cdot, \cdot, 0, 0)$ is only in $L^1(Q_T)$, we also truncate it and, for technical reasons, we introduce

$$\mathcal{F}_n(t, x) := \tau_{K(n)2n}(f(t, x, 0, 0)),\ \ \mathcal{G}_n(t, x) := \tau_{K(n)2n}(g(t, x, 0, 0)),$$

where $\tau_k(r) = r$ if $|r| \leq k$, $\tau_k(r) = k$ if $r > k$, $\tau_k(r) = -k$ if $r < -k$.

Finally, we define

$$\left.\begin{aligned} f_n(t,x,r,s) &:= \psi_n(r,s)[f(t,x,r,s) - f(t,x,0,0)] + \mathcal{F}_n(t,x), \\ g_n(t,x,r,s) &:= \psi_n(r,s)[g(t,x,r,s) - g(t,x,0,0)] + \mathcal{G}_n(t,x). \end{aligned}\right\} \tag{24}$$

One easily verifies that f_n, g_n converge to f, g in the sense (14) and that f_n, g_n satisfies the same conditions (9,10,11) as f, g. Note that f_n, g_n are even globally Lipschitz continuous with respect to r, s with a constant depending on n. Note also that f_n, g_n satisfy (11) with the same λ_0, σ, h as for f, g. For the nonegativity condition (10), we remark that, if $f(t,x,0,0) > K(n)\,2n$, then

$$f_n(t,x,0,s) \geq -K(n)\,2n + \mathcal{F}_n(t,x) = 0,$$

and otherwise $\mathcal{F}_n(t,x) \geq f(t,x,0,0)$ so that

$$f_n(t,x,0,s) \geq \psi_n(0,s) f(t,x,0,s) + [1 - \psi_n(0,s)] f(t,x,0,0) \geq 0,$$

whence (10) and similarly for g_n.

Note finally that f_n, g_n are uniformy bounded since

$$\forall\, r, s,\ |f_n(t,x,r,s)| + |g_n(t,x,r,s)| \leq 3\,K(2n)\,2n. \tag{25}$$

By a classical fixed point theorem (see e.g. Appendix), there exists a unique "classical" and nonnegative solution u_n, v_n of

$$\left.\begin{aligned} u_n, v_n \in C([0,+\infty); L^2(\Omega)) \cap L^2_{loc}([0,+\infty); W_0^{1,2}(\Omega)), \\ u_n(0,\cdot) = u_{0n}, \quad v_n(0,\cdot) = v_{0n}, \\ \forall\, T \in (0,+\infty),\ u_{nt}, v_{nt}, \Delta u_n, \Delta v_n \in L^2(Q_T), \\ f_n(\cdot,\cdot,u_n(\cdot,\cdot), v_n(\cdot,\cdot)),\ g_n(\cdot,\cdot,u_n(\cdot,\cdot), v_n(\cdot,\cdot)) \in L^\infty(Q_T), \\ \partial u_n/\partial t - d_1 \Delta u_n = f(t,x,u_n,v_n) \ \text{ in } Q, \\ \partial v_n/\partial t - d_2 \Delta v_n = g(t,x,u_n,v_n) \ \text{ on } Q. \end{aligned}\right\} \tag{26}$$

Adding the two equations in $u_n, \lambda v_n$, we obtain for all $\lambda \in [\lambda_0, 1]$,

$$\int_\Omega u_n(t) + \lambda v_n(t) - \int_{Q_t} f_n + \lambda g_n \leq \int_\Omega u_{0n} + \lambda v_{0n}. \tag{27}$$

(Here, we used, $\int_\Omega \Delta u_n \leq 0$ and the same for v_n which is true because u_n, v_n are nonnegative on Ω and equal to zero on $\partial\Omega$, see e.g. [6]). Using now (11), we deduce in particular that

$$\int_\Omega u_n(t) + \lambda v_n(t) - \int_{Q_t} \sigma\,(u_n(s) + v_n(s)) + h(s) \leq \int_\Omega u_{0n} + \lambda v_{0n}. \tag{28}$$

From this linear Gronwall type inequality, we deduce that

$$\sup_{t \in (0,T)} \{\|u_n(t)\|_{L^1(\Omega)} + \|v_n(t)\|_{L^1(\Omega}\} \leq C(T, \|h\|_{L^1}, \sigma, \|u_{0n}\|_{L^1}, \|v_{0n}\|_{L^1}). \tag{29}$$

Now, the hypothesis (11) implies that

$$\|f_n + \lambda g_n\|_{L^1(Q_T)} \leq -\int_{Q_T} f_n + \lambda g_n + \|\sigma\,(u_n + v_n) + h\|_{L^1(Q_T)}.$$

Together with (27) and (29), this implies that $\|f_n + \lambda g_n\|_{L^1(Q_T)}$ is bounded for all $\lambda \in [\lambda_0, 1]$. Since, $\lambda_0 \neq 1$, this implies that f_n and g_n are separately bounded in $L^1(Q_T)$.

As a consequence, we are in the conditions of application of Theorem 2.2. It follows that, up to a subsequence, u_n, v_n converge a.e. on Q and in $L^1(Q_T)$ for all $T > 0$ to a supersolution (u, v) of our problem in the sense of (15) with F, G replaced by f, g.

To go from a supersolution to a solution, we argue as follows. Let φ be a C_0^∞, *nonnegative* test-function. We already know that

$$-\int_{Q_T} u\,(\varphi_t + d_1\Delta\varphi) \geq \int_{Q_T} \varphi\, f,\ -\int_{Q_T} v\,(\varphi_t + d_2\Delta\varphi) \geq \int_{Q_T} \varphi\, g.$$

To obtain the reverse inequality for each, it is sufficient to prove that

$$-\int_{Q_T} (u + v)\varphi_t + (d_1 u + d_2 v)\Delta\varphi \leq \int_{Q_T} \varphi(f + g), \tag{30}$$

starting from

$$-\int_{Q_T} (u_n + v_n)\varphi_t + (d_1 u_n + d_2 v_n)\Delta\varphi = \int_{Q_T} \varphi(f_n + g_n). \tag{31}$$

By L^1-convergence of u_n, v_n, the left hand side of (31) does converge to the left hand side of (30). Since $f_n + g_n$ converges a.e. to $f + g$, and since, by (11), we have the pointwise estimate

$$\sigma(u_n + v_n) + h - (f_n + g_n) \geq 0,$$

by Fatou' Lemma, we deduce that

$$\liminf_{n\to\infty} \int_{Q_T} [\sigma(u_n + v_n) + h - (f_n + g_n)]\varphi \geq \int_{Q_T} [\sigma(u + v) + h - (f + g)]\varphi.$$

This gives the expected reverse inequality (30).

We now have to verify that u, v have the right initial data u_0, v_0. We already have one inequality by Theorem 2.2. (see 16). The bound in $L^\infty(0, T; L^1(\Omega))$ implies that $\{u(t), v(t), t \in (0, T)\}$ are compact for the weak convergence of measures on Ω (i.e. against continuous test-functions φ with compact support in Ω). If u_{0^+}, v_{0^+} is a weak-limit for a subsequence $u(t_k), v(t_k)$ where $\lim_{k\to+\infty} t_k = 0$, we already now (see Theorem 2.2) that

$$u_{0^+} \geq u_0, \quad v_{0^+} \geq v_0.$$

We will prove that

$$\limsup_{t\to 0} \int_\Omega u(t) + v(t) \leq \int_\Omega u_0 + v_0. \tag{32}$$

It will then follow that $u_{0^+} = u_0$, $v_{0^+} = v_0$: this uniqueness of the possible weak limits and the fact that there is no loss of mass imply that $u(t), v(t)$ converge as $t \to 0$, for the narrow convergence of measures, to u_0, v_0, namely (see Appendix)

$$\forall \varphi \in C_b(\Omega),\ \lim_{t\to 0} \int_\Omega \varphi u(t) = \int_\Omega \varphi u_0,$$

and the same for v. But, by the uniqueness Lemma 5.1 of the Appendix, we may then deduce that $u, v \in C([0, T]; L^1(\Omega))$ for all $T > 0$, which finishes the proof of Theorem 2.1.

To prove (32), we start from (28) with $\lambda = 1$ and we pass to the limit in n:

$$a.e.t,\ \int_\Omega u(t) + v(t) \leq \int_{Q_t} \sigma\,(u(s) + v(s)) + h(s) + \int_\Omega u_0 + v_0.$$

Then, (32) follows directly from this inequality.

4. Some comments

The question of global existence for systems (1) when only (2) and (3) hold remains open (it is the case of example (4) when $\gamma = 1$ and of example (5)). A main new difficulty is that no more L^1 bound on the nonlinear terms f, g is available. It is likely that some kind of weak solutions exist globally in time, but to prove it would first require to introduce a quite weaker notion of solution. It could be possible that some notion of "renormalized" solution may work where the nonlinearities are truncated in the definition.

The analysis made here is not particular to 2×2 systems. Theorem 2 may be generalized to $N \times N$ systems where all the N nonlinearities are bounded in $L^1(Q_T)$. The idea is to replace in the proof $T_k(u + \eta\, v)$ by $T_k(u_1 + \eta[u_2 + \ldots u_N])$ if we denote by $(u_1, ..., u_N)$ the unknown of the system. Theorem 1 also extends: we then have to assume that N linearly independent relations of the form $\sum_{i=1}^N \lambda_i f_i \leq$ with $\lambda_i \geq 0$ hold. This will provide the L^1 bound on all the f_i.

One could also replace the Laplacian operators $d_1\Delta, d_2\Delta$ by more general elliptic operators.

Elliptic versions of the same results may be proved for systems of the form

$$\left.\begin{array}{r} u + A_1\, u = f(u, v) + \mathcal{F} \ \text{ on } \ \Omega, \\ v + A_2\, v = g(u, v) + \mathcal{G} \ \text{ on } \ \Omega, \\ u, v \ \text{ satisfy some good boundary conditions on } \ \partial\Omega, \end{array}\right\}$$

where $\mathcal{F}, \mathcal{G}$ are given nonegative functions on Ω, A_1, A_2 are good elliptic operators and f, g satisfy (2,3,7). As a nontrivial case, we might think for instance to the simple choice $A_1 = -\Delta, A_2 = -\Delta - du_{x_1x_1}$ with d not too small. Here the difficulty is the proof of existence of solutions and the question is quite similar to proving global existence for the parabolic system (1). Then by the same technique, we can prove existence of weak solutions under similar hypotheses on the nonlinearities. One may also state an elliptic version of Theorem 2.2. The limit case (only (3)) is open as well in this elliptic situation.

5. Appendix

About Lemma 3.1 *let us first comment on the proof of Lemma* 3.1. *A starting point may be the L^2-theory: for $u_0 \in L^2(\Omega)$, $H \in L^2(Q_T)$, there exists a unique solution of*

$$\left.\begin{aligned} u \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); W_0^{1,2}(\Omega)) \\ u_t,\ \Delta u \in L^2(Q_T), \\ \partial u/\partial t - d\Delta u = H \text{ in } L^2(Q_T),\quad u(0) = u_0. \end{aligned}\right\} \tag{33}$$

Moreover,

$$u(t) = S_d(t)\,u_0 + \int_0^t S_d(t-s)\,H(s)\,ds, \tag{34}$$

where $S_d(\cdot)$ is the semigroup in $L^2(\Omega)$ whose infinitesimal generator is the Laplacian operator $-\Delta$, with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$.

This may be found in several places in the literature, as well as the contraction property

$$\forall\, p \in [1, +\infty],\ \ \|S_d(t)u_0\|_{L^p(\Omega)} \le \|u_0\|_{L^p(\Omega)}, \tag{35}$$

(see e.g. [9, 1, 5]).

Thanks to the contraction property in $L^1(\Omega)$, the solution of (33) may be extended to $u_0 \in L^1(\Omega)$, $H \in L^1(Q_T)$ with $u \in C([0,T]; L^1(\Omega))$ at least. It is also given by the formula (34) where, now, $S_d(\cdot)$ is the realization of the heat semigroup in $L^1(\Omega)$ (see e.g. [6]) -we also denote it by $S_d(\cdot)$-. To obtain that $u \in L^1(0,T; W_0^{1,1}(\Omega))$, we need the estimates (18). They may be obtained by duality from the L^∞-estimates for the heat operator (see e.g. [9], Th. III.7.1), namely

$$\|u\|_{L^\infty(Q_T)} \le C \sum_{i=1}^N \|h_i\|_{L^{s'}(O,T;L^{q'}(\Omega))},$$

for the solution of (33) with $H = \sum_{i=1}^N \partial h_i/\partial x_i$.

For the uniqueness part, one has to be careful when working in an L^1-setting. Remember, for instance, that there is not uniqueness for the problem

$$u \in W_0^{1,1}(\mathcal{O}),\quad \Delta u = 0 \text{ in } D'(\mathcal{O}),$$

without any regularity on the open subset $\mathcal{O}$ of $\mathbb{R}^N$ (a counterexample is given by $u(x) = 1 - |x|^{2-N}$ for $N \geq 3$ and $\mathcal{O} = \{x \in \mathbb{R}^N \setminus \{0\};\ |x| < 1\}$).

But we do have uniqueness if $\mathcal{O} = \Omega$ is regular -which we assumed throughout this paper- (see e.g. [6]). Again, this uniqueness result relies, by duality, on regularizing properties of the Laplacian in good domains.

We also have uniqueness for the parabolic problem. We will state it in a general way that we actually need in this paper. We denote by $C_b(\Omega)$ the continuous and bounded functions from Ω into $\mathbb{R}$.

LEMMA 5.1. *Let* $w \in L^\infty(0, T; L^1(\Omega)) \cap L^1_{loc}((0, T], W_0^{1,1}(\Omega))$ *be a solution of*

$$\left.\begin{array}{r} \partial w/\partial t - d\Delta w = H \ \text{ in } \ D'(Q_T), \\ \forall \varphi \in C_b(\Omega),\ \varphi \geq 0,\ \lim_{t\to 0} \int_\Omega \varphi\, w(t) = \int_\Omega \varphi\, w_0. \end{array}\right\} \tag{36}$$

Then,

$$w(t) = S_d(t) w_0 + \int_0^t S_d(t-s)\, H(s)\, ds.$$

In particular, $w \in C([0, T]; L^1(\Omega)) \cap L^1(0, T; W_0^{1,1}(\Omega))$.

We refer e.g. to [2] for details of the proof. As explained above, it is based on regularity property of the dual problem.

Note that the initial data are understood here in the sense of the "narrow" convergence for measures, that is to say that the test-functions φ in (36) have to be taken in $C_b(\Omega)$. The uniqueness *would not be true* if they were to be taken only among continuous functions with compact support in Ω.

About existence of regular solutions for systems: we consider the functions f_n, g_n defined in (24). Since they are defined only for $r, s \in [0, +\infty)^2$, we extend them in the variable r, s to $\mathbb{R}^2$ by $\Pi \circ f_n$, $\Pi \circ g_n$ where Π is the projection in $\mathbb{R}^2$ onto $[0, +\infty)^2$. We still denote by f_n, g_n this extension. The main point is that f_n, g_n are globally Lipschitz continuous on $\mathbb{R}^2$.

To prove existence of a classical solution to the system (26), for all $T \in (0, +\infty)$, we consider the mapping $\mathcal{S}$ from $X = C\big([0, T]; L^1(\Omega)\big)^2$ into itself which to $(\hat{U}, \hat{V})$ associates (U, V) defined by

$$U(t) = S_{d_1}(t) u_{0n} + \int_0^t S_{d_1}(t-s) f_n(s, \cdot, \hat{U}(s), \hat{V}(s))\, ds,$$
$$V(t) = S_{d_2}(t) v_{0n} + \int_0^t S_{d_2}(t-s) g_n(s, \cdot, \hat{U}(s), \hat{V}(s))\, ds.$$

Since, f_n, g_n are globally Lipschitz in r, s, one easily proves that there exists p such that $\mathcal{S}^p$ is a strict contraction from X into itself, whence the existence of a unique weak solution to the system. By the above uniqueness results and the L^2-theory, since all the data u_{0n}, v_{0n}, f_n, g_n are uniformly bounded and therefore in L^2, the solution has the regularity announced in (26).

For the positivity, classically we multiply the equation in u_n by $-u_n^- =: inf\{u_n, 0\}$ to obtain

$$\partial/\partial t \int_\Omega \left(u_n^-\right)^2 \leq -2 \int_{[u_n<0]} u_n f_n(t, x, u_n, v_n).$$

But, by construction, on $[u_n < 0]$, $f_n(t, x, u_n, v_n) \geq 0$, since it is equal either to f_n $(t, x, 0, v_n)$ if $v_n \geq 0$ or to $f_n(t, x, 0, 0)$ if $v_n \leq 0$. We deduce that $u_n^- = 0$ and similarly $v_n^- = 0$.

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