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On the second boundary value problem for a class of fully nonlinear flow III

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Abstract. We study the solvability of the second boundary value problem of the Lagrangian mean curvature equation arising from special Lagrangian geometry. By the parabolic method, we obtain the existence and uniqueness of the smooth uniformly convex solution, which generalizes the Brendle–Warren's theorem about minimal Lagrangian diffeomorphism in Euclidean metric space.

1. Introduction

In this work, we are interested in the long time existence and convergence of convex solutions for special variables, which solves the fully nonlinear equation

$$\frac{\partial u}{\partial t} = F\left(\lambda(D^2u)\right) - f(x), \quad t > 0, \quad x \in \Omega, \tag{1.1}$$

associated with the second boundary value condition

$$Du(\Omega) = \tilde{\Omega}, \quad t > 0, \quad x \in \partial\Omega, \tag{1.2}$$

and the initial condition

$$u = u_0, \quad t = 0, \quad x \in \Omega \tag{1.3}$$

for given F, f and u_0, where Du and D^2u are the gradient and the Hessian matrix of the function u, respectively, Ω and $\tilde{\Omega}$ are two uniformly convex bounded domains with smooth boundary in $\mathbb{R}^n$ and $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of D^2u. One of our main goal to study the flow is to obtain the existence and uniqueness of the smooth uniformly convex solution for the second boundary value problem of the Lagrangian mean curvature equation

$$\begin{cases} F_\tau(\lambda(D^2u)) = \kappa \cdot x + c, \quad x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}, \end{cases} \tag{1.4}$$

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where $\kappa \in \mathbb{R}^n$ is a constant vector, c is a constant to be determined and

$$F_\tau(\lambda) := \begin{cases} \frac{1}{2}\sum_{i=1}^n \ln \lambda_i, & \tau = 0, \\ \frac{\sqrt{a^2+1}}{2b}\sum_{i=1}^n \ln \frac{\lambda_i+a-b}{\lambda_i+a+b}, & 0 < \tau < \frac{\pi}{4}, \\ -\sqrt{2}\sum_{i=1}^n \frac{1}{1+\lambda_i}, & \tau = \frac{\pi}{4}, \\ \frac{\sqrt{a^2+1}}{b}\sum_{i=1}^n \arctan \frac{\lambda_i+a-b}{\lambda_i+a+b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \sum_{i=1}^n \arctan \lambda_i, & \tau = \frac{\pi}{2}, \end{cases} \tag{1.5}$$

where $a = \cot \tau$, $b = \sqrt{|\cot^2 \tau - 1|}$. Regarding the equation, the details can be seen in [32].

Let

$$g_\tau = \sin \tau \delta_0 + \cos \tau g_0, \quad \tau \in \left[0, \frac{\pi}{2}\right]$$

be the linear combined metric of the standard Euclidean metric

$$\delta_0 = \sum_{i=1}^n dx_i \otimes dx_i + \sum_{j=1}^n dy_j \otimes dy_j$$

and the pseudo-Euclidean metric

$$g_0 = \frac{1}{2}\sum_{i=1}^n dx_i \otimes dy_i + \frac{1}{2}\sum_{j=1}^n dy_j \otimes dx_j$$

in $\mathbb{R}^n \times \mathbb{R}^n$.

Under the framework of calibrated geometry in $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$, Warren [1] firstly obtained the special Lagrangian equation as the form

$$F_\tau(\lambda(D^2u)) = c, \tag{1.6}$$

which is a special case of (1.4) when $\kappa \equiv 0$. Then, $(x, Du(x))$ is a minimal Lagrangian graph in $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$.

If $\tau = 0$, (1.6) becomes the famous Monge–Ampère equation

$$\det D^2u = e^{2c}.$$

As for $\tau = \frac{\pi}{2}$, one can show that (1.6) is the classical special Lagrangian equation

$$\sum_{i=1}^n \arctan \lambda_i(D^2u) = c. \tag{1.7}$$

The special Lagrangian Eq. (1.7) was first introduced by Harvey and Lawson in [2] back in 1982. Its solutions u were shown to have the property that the graph $(x, Du(x))$ in $(\mathbb{R}^n \times \mathbb{R}^n, \delta_0)$ is a Lagrangian submanifold which is absolutely volume-minimizing, and the linearization at any solution is elliptic. They proved that a Lagrangian graph $(x, Du(x))$ in $(\mathbb{R}^n \times \mathbb{R}^n, \delta_0)$ is minimal if and only if the Lagrangian angle is a constant, that is, (1.7) holds. Interestingly, several methods for studying the Bernstein-type theorems occured in the literature [3,4]. Jost and Xin [3] used the properties of harmonic maps into convex subsets of Grassmannians. Yuan [4] showed that entire convex solutions of (1.7) must be a quadratic polynomial based on the geometric measure theory.

The Dirichlet problem for the Lagrangian mean curvature equation with various phase constraints had been studied by Collins et al. [5] and Bhattacharya [6]. Bhattacharya and Shankar had obtained the regularity for convex viscosity solutions in [7,8]. We refer the reader to the appendix in [9–11] for interior estimates with critical and supercritical phase. Singular $C^{1,\alpha}$ solutions constructed in [12,13] show that interior regularity is not possible for subcritical phases $|\Theta| < \frac{(n-2)\pi}{2}$, without an additional convexity condition, as in [14–16], and that the Dirichlet problem is not classically solvable for arbitrary smooth boundary data.

Moreover, we now briefly remark on some relevant work about Hessian and gradient estimates of the Lagrangian mean curvature equation. The convex smooth solutions with $C^{1,1}$ phase were obtained in [17]. The C^4 solutions with critical and supercritical phase were considered in [18–21]. Bhattacharya and Wall considered the shrinkers, expanders, translators and rotators of the Lagrangian mean curvature flow in [22].

People have worked on showing the existence of the minimal Lagrangian graphs ($\kappa \equiv 0$), and Du is a diffeomorphism from Ω to $\tilde{\Omega}$. That is,

$$\begin{cases} F_\tau(\lambda(D^2u)) = c, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}. \end{cases} \tag{1.8}$$

Here, Du is a minimal Lagrangian diffeomorphism from Ω to $\tilde{\Omega}$. In the case of $\tau = 0$, in dimension 2, Delanoë [23] obtained a unique smooth solution for the second boundary value problem of the Monge–Ampère equation if both domains are uniformly convex. Later the generalization of Delanoë's theorem to higher dimensions was given by Caffarelli [24] and Urbas [25]. Using the parabolic method, Schnürer and Smoczyk [26] also obtained the existence of solutions to (1.8). As far as $\tau = \frac{\pi}{2}$ is concerned, Brendle and Warren [27] proved the existence and uniqueness of the solution by the elliptic method, and the second author [28] obtained the existence of solution by considering the second boundary value problem for Lagrangian mean curvature flow. Then by the elliptic and parabolic method, the second author with Ou [29], Ye [30] and Chen [31] proved the existence and uniqueness of the solution for $0 < \tau < \frac{\pi}{2}$.

We are now in a position to find out the Lagrangian graph $(x, Du(x))$ prescribed constant mean curvature vector κ in $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$ such that Du is the diffeomorphism between two uniformly convex bounded domains. Thus, it can be described by Eq. (1.4), seeing [32].

By the continuity method, it follows from our early work [32] that we obtain the existence and uniqueness of the smooth uniformly convex solution to (1.4). That is

Theorem 1.1. *For $\tau \in \left(0, \frac{\pi}{2}\right]$, there exists some positive constant ϵ_0 depending only on Ω and $\tilde{\Omega}$, such that if $|\kappa| \le \epsilon_0$, then there exists a uniformly convex solution $u \in C^\infty(\bar{\Omega})$ and a unique constant c solving (1.4), and u is unique up to a constant.*

Theorem 1.1 exhibits an extension of the previous work on $\kappa = 0$ done by Brendle–Warren [27], Huang [28], Huang–Ou [29], Huang–Ye [30] and Chen–Huang–Ye [31].

In the present paper, we pursue a strategy of deriving asymptotic convergence theorem to the solutions of (1.1)–(1.3) for proving Theorem 1.1 based purely on the previous results of Altschuler and Wu [33], Schnürer [34], and Kitagawa [35].

Motivated by the work of Huang–Ou [29] and Huang–Ye [30], we introduce a class of nonlinear functions containing $F_\tau(\lambda)$, $\tau \in (0, \frac{\pi}{2}]$.

For $0 < \alpha_0 < 1$, let $F(\lambda_1, \ldots, \lambda_n)$ be a $C^{2+\alpha_0}$ symmetric function defined on

$$\Gamma_n^+ := \left\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_i > 0,\ i = 1, \ldots, n\right\},$$

and satisfy

$$-\infty < F(0, \ldots, 0) < F(+\infty, \ldots, +\infty) < +\infty, \tag{1.9}$$

$$\frac{\partial F}{\partial \lambda_i} > 0, \quad 1 \le i \le n \quad \text{on} \quad \Gamma_n^+, \tag{1.10}$$

and

$$\left(\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}\right) \le 0 \quad \text{on} \quad \Gamma_n^+. \tag{1.11}$$

For any $(\mu_1, \ldots, \mu_n) \in \Gamma_n^+$, denote

$$\lambda_i = \frac{1}{\mu_i}, \quad 1 \le i \le n,$$

and

$$\tilde{F}(\mu_1, \ldots, \mu_n) := -F(\lambda_1, \ldots, \lambda_n).$$

Assume that

$$\left(\frac{\partial^2 \tilde{F}}{\partial \mu_i \partial \mu_j}\right) \le 0 \quad \text{on} \quad \Gamma_n^+. \tag{1.12}$$

For any $s_1 > 0$, $s_2 > 0$, define

$$\Gamma^+_{]s_1,s_2[} = \left\{(\lambda_1, \ldots, \lambda_n) \in \Gamma_n^+ : 0 \le \min_{1\le i\le n} \lambda_i \le s_1,\ \max_{1\le i\le n} \lambda_i \ge s_2\right\}.$$

We assume that there exist positive constants Λ_1 and Λ_2, depending on s_1 and s_2, such that for any $(\lambda_1, \ldots, \lambda_n) \in \Gamma^+_{]s_1,s_2[}$,

$$\Lambda_1 \le \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \le \Lambda_2, \tag{1.13}$$

and

$$\Lambda_1 \leq \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \leq \Lambda_2. \tag{1.14}$$

Remark 1.2. Since

$$\frac{\partial^2 \tilde{F}}{\partial \mu_i \partial \mu_j} = -\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} \lambda_i^2 \lambda_j^2 - 2\lambda_i^3 \delta_{ij} \frac{\partial F}{\partial \lambda_i},$$

we cannot deduce (1.12) from (1.10) and (1.11).

For $f(x) \in C^{2+\alpha_0}(\bar{\Omega})$, we define

$$\operatorname*{osc}_{\bar{\Omega}}(f) := \max_{x,y \in \bar{\Omega}} |f(x) - f(y)|,$$

and

$$\mathscr{A}_\delta := \left\{ f(x) \in C^{2+\alpha_0}(\bar{\Omega}) : f \text{ is concave, } \operatorname*{osc}_{\bar{\Omega}}(f) \leq \delta \right\}.$$

The constant δ is any positive constant satisfying

$$\delta < \min \left\{ F(+\infty, \ldots, +\infty) - \max_{\bar{\Omega}} F\left(\lambda(D^2 u_0)\right), \min_{\bar{\Omega}} F\left(\lambda(D^2 u_0)\right) - F(0, \ldots, 0) \right\}.$$

Remark 1.3. Let $f(x) = \kappa \cdot x$ and if $|\kappa|$ is sufficiently small, then $f(x) \in \mathscr{A}_\delta$.

Our main results are the following:

Theorem 1.4. *Let F satisfy the structure conditions (1.9)–(1.14) and $f \in \mathscr{A}_\delta$. If*

$$|Df| \leq \frac{\theta \Lambda_1}{2} \cdot \frac{1}{\max_{\bar{\tilde{\Omega}}} |Dh|} \tag{1.15}$$

holds, where θ and h depending only on $\tilde{\Omega}$ appear in Definition 3.1. Then for any given initial function u_0 which is uniformly convex and satisfies $Du_0(\Omega) = \tilde{\Omega}$, the uniformly convex solution of (1.1)–(1.3) exists for all $t \geq 0$ and $u(\cdot, t)$ converges to a function $u^\infty(x, t) = \tilde{u}^\infty(x) + c_\infty \cdot t$ in $C^{1+\zeta}(\bar{\Omega}) \cap C^{4+\alpha}(\bar{D})$ as $t \to \infty$ for any $D \subset\subset \Omega$, $0 < \zeta < 1$ and $0 < \alpha < \alpha_0$. That is,

$$\lim_{t \to +\infty} \|u(\cdot, t) - u^\infty(\cdot, t)\|_{C^{1+\zeta}(\bar{\Omega})} = 0, \quad \lim_{t \to +\infty} \|u(\cdot, t) - u^\infty(\cdot, t)\|_{C^{4+\alpha}(\bar{D})} = 0.$$

And $\tilde{u}^\infty(x) \in C^{1+1}(\bar{\Omega}) \cap C^{4+\alpha_0}(\Omega)$ is a solution of

$$\begin{cases} F\left(\lambda(D^2 u)\right) = f(x) + c_\infty, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}. \end{cases} \tag{1.16}$$

The constant c_∞ depends only on Ω, $\tilde{\Omega}$, u_0, f, δ and F. The solution to (1.16) is unique up to additions of constants.

Especially, if F and f are smooth, then there exist a uniformly convex solution $u_\infty(x) \in C^\infty(\bar{\Omega})$ and a constant c_∞ solving (1.16).

The rest of this article is organized as follows. The next section is to present the structure condition for the operator F_τ and then we can exhibit that Theorem 1.1 is a corollary of Theorem 1.4. To prove the main theorem, we verify the short time existence of the parabolic flow in Sect. 3. Thus, Sect. 4 is devoted to carry out the strictly oblique estimate and the C^2 estimate. Eventually, we give the long time existence and convergence of the parabolic flow in Sect. 5.

Throughout the following, Einstein's convention of summation over repeated indices will be adopted. We denote, for a smooth function u,

$$u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_{ijk} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \ldots.$$

2. Preliminary step of Theorem 1.1

In the following, we are going to describe the analytic structure of the operator F_τ by direct computation.

It is obvious that $F_\tau(\lambda_1, \ldots, \lambda_n)$, $\tau \in \left(0, \frac{\pi}{2}\right]$ is a smooth symmetric function defined on Γ_n^+. For technical reasons, it is necessary to push further the calculation, and we get

$$F_\tau(0, \ldots, 0) = \begin{cases} \frac{n\sqrt{a^2+1}}{2b} \ln \frac{a-b}{a+b}, & 0 < \tau < \frac{\pi}{4}, \\ -\sqrt{2}n, & \tau = \frac{\pi}{4}, \\ \frac{n\sqrt{a^2+1}}{b} \arctan \frac{a-b}{a+b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ 0, & \tau = \frac{\pi}{2}, \end{cases}$$

$$F_\tau(+\infty, \ldots, +\infty) = \begin{cases} 0, & 0 < \tau < \frac{\pi}{4}, \\ 0, & \tau = \frac{\pi}{4}, \\ \frac{n\pi\sqrt{a^2+1}}{4b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \frac{n\pi}{2}, & \tau = \frac{\pi}{2}, \end{cases}$$

$$\frac{\partial F_\tau}{\partial \lambda_i} = \begin{cases} \frac{\sqrt{a^2+1}}{(\lambda_i+a)^2-b^2}, & 0 < \tau < \frac{\pi}{4}, \\ \frac{\sqrt{2}}{(1+\lambda_i)^2}, & \tau = \frac{\pi}{4}, \\ \frac{\sqrt{a^2+1}}{(\lambda_i+a)^2+b^2}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \frac{1}{1+\lambda_i^2}, & \tau = \frac{\pi}{2}, \end{cases}$$

and

$$\frac{\partial^2 F_\tau}{\partial \lambda_i \partial \lambda_j} = \begin{cases} -\frac{2\sqrt{a^2+1}(\lambda_j+a)\delta_{ij}}{\left[(\lambda_i+a)^2-b^2\right]^2}, & 0 < \tau < \frac{\pi}{4}, \\ -\frac{2\sqrt{2}\delta_{ij}}{(1+\lambda_i)^3}, & \tau = \frac{\pi}{4}, \\ -\frac{2\sqrt{a^2+1}(\lambda_j+a)\delta_{ij}}{\left[(\lambda_i+a)^2+b^2\right]^2}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ -\frac{2\lambda_j\delta_{ij}}{\left(1+\lambda_i^2\right)^2}, & \tau = \frac{\pi}{2}, \end{cases}$$

for $i, j = 1, \ldots, n$. Then,

$$-\infty < F_\tau(0, \ldots, 0) < F_\tau(+\infty, \ldots, +\infty) < +\infty, \quad \tau \in \left(0, \frac{\pi}{2}\right], \tag{2.1}$$

$$\frac{\partial F_\tau}{\partial \lambda_i} > 0, \quad 1 \le i \le n \quad \text{on} \quad \Gamma_n^+, \tag{2.2}$$

and

$$\left(\frac{\partial^2 F_\tau}{\partial \lambda_i \partial \lambda_j}\right) \le 0 \quad \text{on} \quad \Gamma_n^+. \tag{2.3}$$

For any $(\lambda_1, \ldots, \lambda_n) \in \Gamma^+_{]s_1,s_2[}$, we get that

$$\sum_{i=1}^n \frac{\partial F_\tau}{\partial \lambda_i} \in \begin{cases} \left[\frac{\sqrt{a^2+1}}{(s_1+a)^2-b^2}, \frac{n\sqrt{a^2+1}}{a^2-b^2}\right], & 0 < \tau < \frac{\pi}{4}, \\ \left[\frac{\sqrt{2}}{(1+s_1)^2}, n\sqrt{2}\right], & \tau = \frac{\pi}{4}, \\ \left[\frac{\sqrt{a^2+1}}{(s_1+a)^2+b^2}, \frac{n\sqrt{a^2+1}}{a^2+b^2}\right], & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \left[\frac{1}{1+s_1^2}, n\right], & \tau = \frac{\pi}{2}, \end{cases} \tag{2.4}$$

and

$$\sum_{i=1}^n \frac{\partial F_\tau}{\partial \lambda_i}\lambda_i^2 \in \begin{cases} \left[\frac{s_2^2\sqrt{a^2+1}}{(s_2+a)^2-b^2}, n\sqrt{a^2+1}\right], & 0 < \tau < \frac{\pi}{4}, \\ \left[\frac{s_2^2\sqrt{2}}{(1+s_2)^2}, n\sqrt{2}\right], & \tau = \frac{\pi}{4}, \\ \left[\frac{s_2^2\sqrt{a^2+1}}{(s_2+a)^2+b^2}, n\sqrt{a^2+1}\right], & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \left[\frac{s_2^2}{1+s_2^2}, n\right], & \tau = \frac{\pi}{2}. \end{cases} \tag{2.5}$$

For any $(\mu_1, \ldots, \mu_n) \in \Gamma_n^+$, denote

$$\lambda_i = \frac{1}{\mu_i}, \quad 1 \le i \le n,$$

and

$$\tilde{F}_\tau(\mu_1, \ldots, \mu_n) := -F_\tau(\lambda_1, \ldots, \lambda_n).$$

Then,

$$\frac{\partial \tilde{F}_\tau}{\partial \mu_i} = \lambda_i^2 \frac{\partial F_\tau}{\partial \lambda_i}, \quad \mu_i^2 \frac{\partial \tilde{F}_\tau}{\partial \mu_i} = \frac{\partial F_\tau}{\partial \lambda_i},$$

and

$$\begin{aligned}\frac{\partial^2 \tilde{F}_\tau}{\partial \mu_i \partial \mu_j} &= -\lambda_i^3 \left(\lambda_i \frac{\partial^2 F_\tau}{\partial \lambda_i^2} + 2 \frac{\partial F_\tau}{\partial \lambda_i} \right) \delta_{ij} \\ &= \begin{cases} -\frac{2\sqrt{a^2+1}(\mu_i+a)}{\left[(1+a\mu_i)^2-(b\mu_i)^2\right]^2}\delta_{ij}, & 0 < \tau < \frac{\pi}{4}, \\ -\frac{2\sqrt{2}\delta_{ij}}{(1+\mu_i)^3}, & \tau = \frac{\pi}{4}, \\ -\frac{2\sqrt{a^2+1}(\mu_i+a)}{\left[(1+a\mu_i)^2+(b\mu_i)^2\right]^2}\delta_{ij}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ -\frac{2\mu_i\delta_{ij}}{\left(1+\mu_i^2\right)^2}, & \tau = \frac{\pi}{2}. \end{cases}\end{aligned}$$

Therefore, we obtain

$$\frac{\partial \tilde{F}_\tau}{\partial \mu_i} > 0, \quad 1 \le i \le n \quad \text{on} \quad \Gamma_n^+,$$

and

$$\left(\frac{\partial^2 \tilde{F}_\tau}{\partial \mu_i \partial \mu_j} \right) \le 0 \quad \text{on} \quad \Gamma_n^+. \tag{2.6}$$

By the discussion above, we have

Proposition 2.1. *For $\tau \in (0, \frac{\pi}{2}]$, the operator $F_\tau(\lambda)$ satisfies the structure conditions* (1.9)–(1.14).

In fact, there are more operators satisfying the structure conditions (1.9)–(1.14). For any constants $\alpha > 1$ and $\varepsilon > 0$, define the operator as follows:

$$S^\alpha(\lambda_1, \ldots, \lambda_n) = -\sum_{i=1}^n \frac{1}{(\varepsilon + \lambda_i)^\alpha}.$$

Therefore, if

$$F\left[D^2 u\right] = S^\alpha\left(\lambda\left(D^2 u\right)\right), \tag{2.7}$$

then $F\left[D^2 u\right]$ satisfies the structure conditions (1.9)–(1.14).

In the next three sections, we are going to prove Theorem 1.4 through the short time existence of the parabolic flow, the strictly oblique estimate and the C^2 estimate based on a Schnürer's convergence result.

3. The short time existence of the parabolic flow

Let $\mathscr{P}_n$ be the set of positive definite symmetric $n \times n$ matrices, and $\lambda_1(A), \ldots, \lambda_n(A)$ be the eigenvalues of A. For $A = (a_{ij}) \in \mathscr{P}_n$, denote

$$F[A] := F\left(\lambda_1(A), \ldots, \lambda_n(A)\right)$$

and

$$(a^{ij}) = (a_{ij})^{-1}, \quad F^{ij} = \frac{\partial F}{\partial a_{ij}}, \quad F^{ij,rs} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{rs}}.$$

Let us recall the relevant Sobolev spaces (cf. Chapter 1 in [36]). For every multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, $\beta_i \geq 0$ for $i = 1, 2, \ldots, n$ with length $|\beta| = \sum_{i=1}^n \beta_i$ and $j \geq 0$, we set

$$D^\beta u := \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}, \quad D^\beta D_t^j u := \frac{\partial^{|\beta|+j} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n} \partial t^j}.$$

We state the definition of the usual functional spaces as follows ($k \geq 0$):

$$\begin{aligned}
&C^k(\Omega) = \{u : \Omega \to \mathbb{R} : \forall \beta,\ |\beta| \leq k,\ D^\beta u \text{ is continuous in } \Omega\},\\
&C^k(\bar\Omega) = \{u \in C^k(\Omega) : \forall \beta, |\beta| \leq k, D^\beta u \text{ can be extended by continuity to } \partial\Omega\},\\
&C^{k,\frac{k}{2}}(\Omega_T) = \{u : \Omega_T \to \mathbb{R} : \forall \beta, j \geq 0, |\beta| + 2j \leq k, D^\beta D_t^j u \text{ is continuous in } \Omega_T\},\\
&C^{k,\frac{k}{2}}(\bar\Omega_T) = \{u \in C^{k,\frac{k}{2}}(\Omega_T) : \forall \beta, j \geq 0, |\beta| + 2j \leq k,\\
&\quad D^\beta D_t^j u \text{ can be extended by continuity to } \partial\Omega_T\}.
\end{aligned}$$

Moreover, $C^k(\bar\Omega)$ and $C^{k,\frac{k}{2}}(\bar\Omega_T)$ are Banach spaces equipped with the norm

$$\|u\|_{C^k(\bar\Omega)} = \sum_{|\beta| \leq k} \sup_{\bar\Omega} |D^\beta u|$$

and

$$\|u\|_{C^{k,\frac{k}{2}}(\bar\Omega_T)} = \sum_{j \geq 0, |\beta|+2j \leq k} \sup_{\bar\Omega_T} |D^\beta D_t^j u|,$$

respectively.

We now present the definition of Hölder spaces. Let $\alpha \in [0, 1]$, define the α-Hölder coefficient of u in Ω as

$$[u]_{\alpha,\Omega} = \sup_{x \neq y, x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

If $[u]_{\alpha,\Omega} < +\infty$, then we call u Hölder continuous with exponent α in Ω. If there are not ambiguity about the domains Ω, we denote $[u]_{\alpha,\Omega}$ by $[u]_\alpha$. Similarly, the $(\alpha, \frac{\alpha}{2})$-Hölder coefficient of u in Ω_T can be defined by

$$[u]_{\alpha,\frac{\alpha}{2},\Omega_T} = \sup_{(x,t) \neq (y,\tau), (x,t), (y,\tau) \in \Omega_T} \frac{|u(x,t) - u(y,\tau)|}{|x - y|^\alpha + |t - \tau|^{\frac{\alpha}{2}}},$$

and u is Hölder continuous with exponent $(\alpha, \frac{\alpha}{2})$ in Ω_T if $[u]_{\alpha,\frac{\alpha}{2},\Omega_T} < +\infty$. Meanwhile, we denote $[u]_{\alpha,\frac{\alpha}{2},\Omega_T}$ by $[u]_{\alpha,\frac{\alpha}{2}}$. We denote $C^{k+\alpha}(\bar{\Omega})$ as the set of functions belonging to $C^k(\bar{\Omega})$ whose k-order partial derivatives are Hölder continuous with exponent α in Ω and $C^{k+\alpha}(\bar{\Omega})$ is a Banach space equipped with the following norm

$$\|u\|_{C^{k+\alpha}(\bar{\Omega})} = \|u\|_{C^k(\bar{\Omega})} + [u]_{k+\alpha},$$

where

$$[u]_{k+\alpha} = \sum_{|\beta|=k} [D^\beta u]_\alpha.$$

Likewise, we denote $C^{k+\alpha,\frac{k+\alpha}{2}}(\bar{\Omega}_T)$ as the set of functions belonging to $C^{k,\frac{k}{2}}(\bar{\Omega}_T)$ whose $(k, \frac{k}{2})$-order partial derivatives are Hölder continuous with exponent $(\alpha, \frac{\alpha}{2})$ in Ω_T and $C^{k+\alpha,\frac{k+\alpha}{2}}(\bar{\Omega}_T)$ is a Banach space equipped with the following norm:

$$\|u\|_{C^{k+\alpha,\frac{k+\alpha}{2}}(\bar{\Omega}_T)} = \|u\|_{C^{k,\frac{k}{2}}(\bar{\Omega}_T)} + [u]_{k+\alpha,\frac{k+\alpha}{2}},$$

where

$$[u]_{k+\alpha,\frac{k+\alpha}{2}} = \sum_{|\beta|+2j=k} [D^\beta D_t^j u]_{\alpha,\frac{\alpha}{2}}.$$

By the methods on the second boundary value problems for equations of Monge–Ampère type [25], the parabolic boundary condition in (1.2) can be reformulated as

$$h(Du) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

where we need

Definition 3.1. A smooth function $h : \mathbb{R}^n \to \mathbb{R}$ is called the defining function of $\tilde{\Omega}$ if

$$\tilde{\Omega} = \{p \in \mathbb{R}^n : h(p) > 0\}, \quad |Dh|_{\partial\tilde{\Omega}} = 1,$$

and there exists $\theta > 0$ such that for any $p = (p_1, \dots, p_n) \in \tilde{\Omega}$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$,

$$\frac{\partial^2 h}{\partial p_i \partial p_j} \xi_i \xi_j \le -\theta |\xi|^2.$$

We can also define $\tilde{h}$ as the defining function of Ω. That is,

$$\Omega = \{\tilde{p} \in \mathbb{R}^n : \tilde{h}(\tilde{p}) > 0\}, \quad |D\tilde{h}|_{\partial\Omega} = 1, \quad D^2\tilde{h} \le -\tilde{\theta} I,$$

where $\tilde{\theta}$ is some positive constant. Thus, the parabolic flow (1.1)–(1.3) is equivalent to the evolution problem

$$\begin{cases} \frac{\partial u}{\partial t} = F\left(\lambda(D^2 u)\right) - f(x), & t > 0, \quad x \in \Omega, \\ h(Du) = 0, & t > 0, \quad x \in \partial\Omega, \\ u = u_0, & t = 0, \quad x \in \Omega. \end{cases} \tag{3.1}$$

To establish the short time existence of classical solutions of (3.1), we use the inverse function theorem in Fréchet spaces and the theory of linear parabolic equations for oblique boundary condition. The method is along the idea of proving the short time existence of convex solutions on the second boundary value problem for Lagrangian mean curvature flow [28]. We include the details for the convenience of the readers.

Lemma 3.2. (Ekeland, see Theorem 2 in [37]) *Let X and Y be Banach spaces with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Suppose*

$$\hbar : X \to Y$$

is continuous and Gâteaux-differentiable, with $\hbar[0] = 0$. Assume that the derivative $D\hbar[x]$ has a right inverse $\mathrm{T}[x]$, uniformly bounded in a neighborhood of 0 in X. That is, for any $y \in Y$,

$$D\hbar[x]\mathrm{T}[x]y = y,$$

and there exist $R > 0$ and $m > 0$ such that

$$\|x\|_1 \leq R \Longrightarrow \|\mathrm{T}[x]\|_2 \leq m.$$

For every $y \in Y$, if

$$\|y\|_2 < \frac{R}{m},$$

then there exists some $x \in X$ such that

$$\|x\|_2 < R,$$

and

$$\hbar[x] = y.$$

As an application of Lemma 3.2, we obtain the following inverse function theorem which will be used to prove the short time existence result for Eq. (3.1).

Lemma 3.3. (See Lemma 2.2 in [30]) *Let X and Y be Banach spaces with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Suppose*

$$J : X \to Y$$

is continuous and Gâteaux-differentiable, with $J(v_0) = w_0$. Assume that the derivative $DJ[v]$ has a right inverse $L[v]$, uniformly bounded in a neighborhood of v_0. That is, for any $y \in Y$,

$$DJ[v]L[v]y = y,$$

and there exist $R > 0$ and $m > 0$ such that

$$\|v - v_0\|_1 \le R \Longrightarrow \|L[v]\|_2 \le m.$$

For every $w \in Y$, if

$$\|w - w_0\|_2 < \frac{R}{m},$$

then there exists some $v \in X$ such that

$$\|v - v_0\|_1 < R,$$

and

$$J(v) = w.$$

We will use the following short time existence and regularity results for linear second-order parabolic equation with strict oblique boundary condition:

Lemma 3.4. (See Theorems 8.8 and 8.9 in [38]) *Assume that $\tilde{f} \in C^{\alpha_0, \frac{\alpha_0}{2}}(\bar{\Omega}_T)$ for some $0 < \alpha_0 < 1$, $T > 0$, and $G(x, p)$, $G_p(x, p)$ are in $C^{1+\alpha_0}(\Xi)$ for any compact subset Ξ of $\partial\Omega \times \mathbb{R}^n$ such that $\inf_{\partial\Omega}\langle G_p, \nu\rangle > 0$ where ν is the inner normal vector of $\partial\Omega$. Let $u_0 \in C^{2+\alpha_0}(\bar{\Omega})$ be strictly convex and satisfy $G(x, Du_0) = 0$. Then, there exists $T' > 0$ $(T' \le T)$ such that we can find a unique solution which is strictly convex in x variable in the class $C^{2+\alpha_0, \frac{2+\alpha_0}{2}}(\bar{\Omega}_{T'})$ to the following equations:*

$$\begin{cases} \frac{\partial u}{\partial t} - a^{ij}(x,t)u_{ij} = \tilde{f}(x,t), & T' > t > 0, \quad x \in \Omega, \\ G(x, Du) = 0, & T' > t > 0, \quad x \in \partial\Omega, \\ u = u_0, & t = 0, \quad x \in \Omega, \end{cases}$$

where $a^{ij}(x,t) \in C^{\alpha_0, \frac{\alpha_0}{2}}(\bar{\Omega}_T)$, $1 \le i, j \le n$ and $[a^{ij}(x,t)] \ge a_0 I$ for some positive constant a_0.

By the property of $C^{2+\alpha_0, \frac{2+\alpha_0}{2}}(\bar{\Omega}_{T'})$ and $u(x,t)|_{t=0} = u_0(x)$, we obtain

$$\lim_{t\to 0} \|u(\cdot, t) - u_0(\cdot)\|_{C^{2+\alpha_0}(\bar{\Omega})} = 0. \tag{3.2}$$

For any $\alpha < \alpha_0$, we have

$$\begin{aligned} &\frac{|(D^2u(x,t) - D^2u_0(x)) - (D^2u(y,\tau) - D^2u_0(y))|}{|x-y|^\alpha + |t-\tau|^{\frac{\alpha}{2}}} \\ &\le \frac{|(D^2u(x,t) - D^2u_0(x)) - (D^2u(y,t) - D^2u_0(y))|}{|x-y|^\alpha} \\ &\quad + |t-\tau|^{\frac{\alpha_0-\alpha}{2}} \frac{|(D^2u(y,t) - D^2u_0(y)) - (D^2u(y,\tau) - D^2u_0(y))|}{|t-\tau|^{\frac{\alpha_0}{2}}}. \end{aligned}$$

Then, we get

$$\|D^2u - D^2u_0\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T'})} \leq \max_{0\leq t\leq T'} \|D^2u(\cdot,t) - D^2u_0(\cdot)\|_{C^\alpha(\bar{\Omega})} + T'^{\frac{\alpha-\alpha_0}{2}} \|D^2u - D^2u_0\|_{C^{\alpha_0,\frac{\alpha_0}{2}}(\bar{\Omega}_{T'})}. \tag{3.3}$$

Combining (3.2) with (3.3), we obtain

$$\lim_{T'\to 0} \|D^2u - D^2u_0\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T'})} = 0, \tag{3.4}$$

which will be used later.

According to the proof in [25], we can verify the oblique boundary condition.

Lemma 3.5. (See Urbas [25]) *Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ be the unit inward normal vector of $\partial\Omega$. If $u \in C^2(\bar{\Omega})$ with $D^2u \geq 0$, then there holds $h_{p_k}(Du)\nu_k \geq 0$.*

Now, we can prove the short time existence of solutions of (3.1), which is equivalent to the problem (1.1)–(1.3).

Proposition 3.6. *According to the conditions in Theorem 1.4, there exist some $T'' > 0$ and $u \in C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{\Omega}_{T''})$ which depend only on Ω, $\tilde{\Omega}$, u_0, f, δ and F, such that u is a solution of (3.1) and is strictly convex in x variable.*

Proof. Denote the Banach spaces

$$X = C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T), \quad Y = C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T) \times C^{1+\alpha,\frac{1+\alpha}{2}}(\partial\Omega \times (0,T]) \times C^{2+\alpha}(\bar{\Omega}),$$

where

$$\|\cdot\|_Y = \|\cdot\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T)} + \|\cdot\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\partial\Omega\times(0,T])} + \|\cdot\|_{C^{2+\alpha}(\bar{\Omega})}.$$

Define a map

$$J: \quad X \to Y$$

by

$$J(u) = \begin{cases} \frac{\partial u}{\partial t} - F[D^2u] + f(x), & (x,t) \in \Omega_T, \\ h(Du), & (x,t) \in \partial\Omega \times (0,T], \\ u, & (x,t) \in \Omega \times \{t=0\}. \end{cases}$$

Thus, the strategy is to use the inverse function theorem to obtain the short time existence result.

The computation of the Gâteaux derivative shows that for any $u, v \in X$,

$$DJ[u](v) := \frac{d}{d\tau} J(u+\tau v)|_{\tau=0} = \begin{cases} \frac{\partial v}{\partial t} - F^{ij}[D^2u]v_{ij}, & (x,t) \in \Omega_T, \\ h_{p_i}(Du)v_i, & (x,t) \in \partial\Omega \times (0,T], \\ v, & (x,t) \in \Omega \times \{t=0\}. \end{cases}$$

Using Lemmas 3.4 and 3.5, there exists $T_1 > 0$ such that we can find

$$\hat{u} \in C^{2+\alpha_0, 1+\frac{\alpha_0}{2}}(\bar{\Omega}_{T_1}) \subset X$$

to be strictly convex in x variable, which satisfies the following equations:

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} - \Delta \hat{u} = F[D^2 u_0] - \Delta u_0 - f, & T_1 > t > 0, \quad x \in \Omega, \\ h(D\hat{u}) = 0, & T_1 > t > 0, \quad x \in \partial\Omega, \\ \hat{u} = u_0, & t = 0, \quad x \in \Omega. \end{cases} \tag{3.5}$$

We see that there exists $R > 0$, such that u is strictly convex in x variable if

$$\|u - \hat{u}\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega}_{T_1})} < R.$$

For each $Z := (\bar{f}, \bar{g}, \bar{w}) \in Y$, using Lemma 3.4 again, we know that there exists a unique $v \in X$ $(T = T_1)$ satisfying $DJ[u](v) = (\bar{f}, \bar{g}, \bar{w})$, that is,

$$\begin{cases} \frac{\partial v}{\partial t} - F^{ij}[D^2 u] v_{ij} = \bar{f}, & T_1 > t > 0, \quad x \in \Omega, \\ h_{p_i}(Du) v_i = \bar{g}, & T_1 > t > 0, \quad x \in \partial\Omega, \\ v = \bar{w}, & t = 0, \quad x \in \Omega. \end{cases}$$

Using Schauder estimates for linear parabolic equation to oblique boundary condition (cf. Theorems 8.8 and 8.9 in [38]), we obtain for some positive constant m,

$$\|v\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega}_{T_1})} \le m \left(\|\bar{f}\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_1})} + \|\bar{g}\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\partial\Omega \times (0, T_1])} + \|\bar{w}\|_{C^{2+\alpha}(\bar{\Omega})} \right).$$

For $T = T_1$, by the definition of the Banach spaces X and Y, we can rewrite the above Schauder estimates as

$$\|v\|_X \le m \|Z\|_Y.$$

If $\|Z\|_Y \le 1$, then we have

$$\|v\|_X \le m.$$

It means that the derivative $DJ[u](v) = Z$ has a right inverse $v = L[u](Z)$ and

$$\|L[u]\| := \sup_{\|Z\|_Y \le 1} \|L[u](Z)\|_X \le m.$$

If we set

$$\hat{f} = \frac{\partial \hat{u}}{\partial t} - F[D^2 \hat{u}] + f, \quad w_0 = (\hat{f}, 0, u_0), \quad w = (0, 0, u_0),$$

then we can show that

$$\begin{aligned} \|\hat{f} - 0\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_1})} &= \|\Delta \hat{u} - \Delta u_0 + F[D^2 u_0] - F[D^2 \hat{u}]\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_1})} \\ &\le \|\Delta \hat{u} - \Delta u_0\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_1})} + \|F[D^2 u_0] - F[D^2 \hat{u}]\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_1})} \\ &\le C \|D^2 \hat{u} - D^2 u_0\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_1})}, \end{aligned}$$

where C is a constant depending only on the known data. Using (3.4), we conclude that there exists $T'' > 0$ ($T'' \leq T_1$) to be small enough such that

$$\|\hat{f} - 0\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T''})} \leq C\|D^2\hat{u} - D^2u_0\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T''})} < \frac{R}{m}.$$

Therefore,

$$\|w - w_0\|_Y = \|0 - \hat{f}\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T''})} < \frac{R}{m}.$$

By Lemma 3.3, we obtain the desired result. □

Remark 3.7. By the strong maximum principle, the strictly convex solution to (3.1) is unique.

4. The strict obliqueness estimate and the C^2 estimate

In this section, the C^2 a priori bound is accomplished by making the second derivative estimates on the boundary for the solutions of fully nonlinear parabolic equations. We also refer to the recent preprint [32] for a proof of separation in elliptic setting with the same criterion as the one used in the present work. This treatment is similar to the problems presented in [25,26,28], but requires some modification to accommodate the more general situation. Specifically, the structure conditions (1.13) and (1.14) are needed in order to derive differential inequalities from barriers which can be used.

For the convenience, we denote $\beta = (\beta^1, \ldots, \beta^n)$ with $\beta^i := h_{p_i}(Du)$, and $\nu = (\nu_1, \ldots, \nu_n)$ as the unit inward normal vector at $x \in \partial\Omega$. The expression of the inner product is

$$\langle \beta, \nu \rangle = \beta^i \nu_i.$$

By Proposition 3.6 and the regularity theory of parabolic equations, we may assume that u is a strictly convex solution of (1.1)–(1.3) in the class $C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T) \cap C^{4+\alpha,2+\frac{\alpha}{2}}(\Omega_T)$ for some $T > 0$.

Lemma 4.1. ($\dot{u}$-estimates) *If the convex solution to* (1.1)–(1.3) *exists and* $f \in \mathscr{A}_\delta$, *then*

$$\min_{\bar{\Omega}} F[D^2u_0] - \max_{\bar{\Omega}} f(x) \leq \dot{u} \leq \max_{\bar{\Omega}} F[D^2u_0] - \min_{\bar{\Omega}} f(x),$$

where $\dot{u} := \frac{\partial u}{\partial t}$.

Proof. From (1.1), a direct computation shows that

$$\frac{\partial(\dot{u})}{\partial t} - F^{ij}\partial_{ij}(\dot{u}) = 0.$$

Using the maximum principle, we see that

$$\min_{\bar{\Omega}_T}(\dot{u}) = \min_{\partial\bar{\Omega}_T}(\dot{u}).$$

Without loss of generality, we assume that $\dot{u} \neq constant$. If there exists $x_0 \in \partial\Omega$, $t_0 > 0$, such that $\dot{u}(x_0, t_0) = \min_{\bar{\Omega}_T}(\dot{u})$. On the one hand, since $\langle \beta, \nu \rangle > 0$, by the Hopf Lemma (cf. [39,40]) for parabolic equations, there must hold in the following:

$$\dot{u}_\beta(x_0, t_0) \neq 0.$$

On the other hand, we differentiate the boundary condition and then obtain

$$\dot{u}_\beta = h_{p_k}(Du)\frac{\partial \dot{u}}{\partial x_k} = \frac{\partial h(Du)}{\partial t} = 0.$$

It is a contradiction. So we deduce that

$$\dot{u} \geq \min_{\bar{\Omega}_T}(\dot{u}) = \min_{\partial\bar{\Omega}_T|_{t=0}}(\dot{u}) = \min_{\bar{\Omega}}\left(F[D^2u_0] - f(x)\right) \geq \min_{\bar{\Omega}} F[D^2u_0] - \max_{\bar{\Omega}} f(x).$$

For the same reason, we have

$$\dot{u} \leq \max_{\bar{\Omega}_T}(\dot{u}) = \max_{\partial\bar{\Omega}_T|_{t=0}}(\dot{u}) = \max_{\bar{\Omega}}\left(F[D^2u_0] - f(x)\right) \leq \max_{\bar{\Omega}} F[D^2u_0] - \min_{\bar{\Omega}} f(x).$$

Putting these facts together, the assertion follows. □

Lemma 4.2. *Let (x, t) be an arbitrary point of Ω_T, and $\lambda_1(x, t), \ldots, \lambda_n(x, t)$ be the eigenvalues of D^2u at (x, t). Suppose that (1.9) and (1.10) hold, if $\operatorname{osc}_{\bar{\Omega}}(f) \leq \delta$ and u is a strictly convex solution to (1.1)–(1.3), then there exists $\mu > 0$ and $\omega > 0$ depending only on $F[D^2u_0]$ and δ such that*

$$\min_{1\leq i\leq n} \lambda_i(x, t) \leq \mu, \quad \max_{1\leq i\leq n} \lambda_i(x, t) \geq \omega.$$

Proof. By condition (1.10) and Lemma 4.1, we obtain

$$\begin{aligned} F\left(\min_{1\leq i\leq n} \lambda_i(x, t), \ldots, \min_{1\leq i\leq n} \lambda_i(x, t)\right) &\leq F[D^2u] = \dot{u} + f(x) \\ &\leq \max_{\bar{\Omega}} F[D^2u_0] + f(x) - \min_{\bar{\Omega}} f(x) \\ &\leq \max_{\bar{\Omega}} F[D^2u_0] + \operatorname{osc}_{\bar{\Omega}}(f) \\ &\leq \max_{\bar{\Omega}} F[D^2u_0] + \delta \\ &< F(+\infty, \ldots, +\infty), \end{aligned}$$

and

$$\begin{aligned}F\left(\max_{1\le i\le n}\lambda_i(x,t),\ldots,\max_{1\le i\le n}\lambda_i(x,t)\right) &\ge F[D^2u]=\dot{u}+f(x)\\ &\ge \min_{\bar\Omega}F[D^2u_0]+f(x)-\max_{\bar\Omega}f(x)\\ &\ge \min_{\bar\Omega}F[D^2u_0]-\operatorname*{osc}_{\bar\Omega}(f)\\ &\ge \min_{\bar\Omega}F[D^2u_0]-\delta\\ &> F(0,\ldots,0).\end{aligned}$$

By the monotonicity of F and condition (1.9), we get the desired result. □

By Lemma 4.2, the points $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ are always in $\Gamma^+_{]\mu,\omega[}$ under the flow. So we can obtain:

Lemma 4.3. *Let (x,t) be an arbitrary point of Ω_T, and $\lambda_1(x,t)$, ..., $\lambda_n(x,t)$ be the eigenvalues of D^2u at (x,t). Suppose that (1.9) and (1.10) hold, if $\operatorname{osc}_{\bar\Omega}(f)\le\delta$ and u is a strictly convex solution to (1.1)–(1.3), then there exists $\Lambda_1>0$ and $\Lambda_2>0$ depending only on $F[D^2u_0]$ and δ such that F satisfies the structure conditions (1.13) and (1.14).*

In the following, we always assume that $\Lambda_1>0$ and $\Lambda_2>0$ are universal constants depending on the known data.

For technical needs below, we introduce the Legendre transformation of u. For any $x\in\mathbb{R}^n$, define

$$\tilde{x}_i:=\frac{\partial u}{\partial x_i}(x),\quad i=1,2,\ldots,n,$$

and

$$\tilde{u}(\tilde{x}_1,\ldots,\tilde{x}_n,t):=\sum_{i=1}^{n}x_i\frac{\partial u}{\partial x_i}(x,t)-u(x,t).$$

In terms of $\tilde{x}_1$, ..., $\tilde{x}_n$ and $\tilde{u}(\tilde{x}_1,\ldots,\tilde{x}_n,t)$, we can easily check that

$$\left(\frac{\partial^2\tilde{u}}{\partial\tilde{x}_i\partial\tilde{x}_j}\right)=\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^{-1}.$$

Let μ_1, ..., μ_n be the eigenvalues of $D^2\tilde{u}$ at $\tilde{x}=Du(x)$. We denote

$$\mu_i=\lambda_i^{-1},\quad i=1,2,\ldots,n.$$

Then,

$$\frac{\partial\tilde{F}}{\partial\mu_i}=\lambda_i^2\frac{\partial F}{\partial\lambda_i},\quad \mu_i^2\frac{\partial\tilde{F}}{\partial\mu_i}=\frac{\partial F}{\partial\lambda_i}.$$

Moreover, it follows from (3.1) that

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} = \tilde{F}(D^2\tilde{u}) + f(D\tilde{u}), & t > 0, \quad \tilde{x} \in \tilde{\Omega}, \\ \tilde{h}(D\tilde{u}) = 0, & t > 0, \quad \tilde{x} \in \partial\tilde{\Omega}, \\ \tilde{u} = \tilde{u}_0, & t = 0, \quad \tilde{x} \in \tilde{\Omega}, \end{cases} \tag{4.1}$$

where $\tilde{h}$ is the defining function of Ω, and $\tilde{u}_0$ is the Legendre transformation of u_0.

Remark 4.4. By Lemma 4.2, if u is a strictly convex solution to (1.1)–(1.3), then the eigenvalues of D^2u and $D^2\tilde{u}$ must be in $\Gamma^+_{]\mu,\omega[}$ and $\Gamma^+_{]\omega^{-1},\mu^{-1}[}$, respectively. Therefore, $\tilde{F}$ also satisfies the structure conditions (1.13) and (1.14).

In order to establish the C^2 estimates, we make use of the method to do the strict obliqueness estimates, a parabolic version of a result of Urbas [25] which was given in [26]. Returning to Lemma 3.5, we get a uniform positive lower bound of the quantity $\inf_{\partial\Omega} h_{p_k}(Du)\nu_k$ which does not depend on t under the structure conditions of F.

Lemma 4.5. *Let F satisfy the structure conditions (1.9)–(1.14) and $f \in \mathscr{A}_\delta$. If u is a strictly convex solution to (1.1)–(1.3) and $|Df|$ satisfies (1.15), then the strict obliqueness estimate*

$$\langle \beta, \nu \rangle \geq \frac{1}{C_1} > 0 \tag{4.2}$$

holds on $\partial\Omega$ for some universal constant C_1, which depends only on F, u_0, Ω, $\tilde{\Omega}$ and δ, and is independent of t.

Remark 4.6. Without loss of generality, in the following, we set $C_1, C_2, \ldots,$ to be constants depending only on the known data.

Proof. The proof follows the similar computations carried out in [32].

Define

$$v = \langle \beta, \nu \rangle + h(Du).$$

Let $(x_0, t_0) \in \partial\Omega \times [0, T]$ such that

$$\langle \beta, \nu \rangle (x_0, t_0) = h_{p_k}(Du(x_0, t_0))\nu_k(x_0, t_0) = \min_{\partial\Omega\times[0,T]} \langle \beta, \nu \rangle.$$

By rotation, we may assume that $t_0 > 0$ and $\nu(x_0, t_0) = (0, 0, \ldots, 1) =: e_n$. By the above assumptions and the boundary condition, we obtain

$$v(x_0, t_0) = \min_{\partial\Omega\times[0,T]} v = \min_{\partial\Omega\times[0,T]} \langle \beta, \nu \rangle = h_{p_n}(Du(x_0, t_0)).$$

By the convexity of Ω and its smoothness, we extend ν smoothly to a tubular neighborhood of $\partial\Omega$ such that in the matrix sense

$$(\nu_{kl}) := (D_k \nu_l) \leq -\frac{1}{C_2} \operatorname{diag}(1, \ldots, 1, 0), \tag{4.3}$$

where C_2 is a positive constant. By Lemma 3.5, we see that $h_{p_n}(Du(x_0,t_0)) \geq 0$.

At (x_0,t_0), we have

$$0 = v_r = h_{p_n p_k} u_{kr} + h_{p_k} \nu_{kr} + h_{p_k} u_{kr}, \quad 1 \leq r \leq n-1. \tag{4.4}$$

At this point, we point out a key estimate

$$v_n(x_0,t_0) \geq -C_3 \tag{4.5}$$

which will be proved later, where C_3 is a constant depending only on Ω, u_0, h, $\tilde{h}$ and δ.

It is not hard to check that (4.5) can be rewritten as

$$h_{p_n p_k} u_{kn} + h_{p_k} \nu_{kn} + h_{p_k} u_{kn} \geq -C_3. \tag{4.6}$$

Multiplying (4.6) with h_{p_n} and (4.4) with h_{p_r}, respectively, and summing up together, we obtain

$$h_{p_k} h_{p_l} u_{kl} \geq -C_3 h_{p_n} - h_{p_k} h_{p_l} \nu_{kl} - h_{p_k} h_{p_n p_l} u_{kl}. \tag{4.7}$$

Using (4.3), and

$$1 \leq r \leq n-1, \quad h_{p_k} u_{kr} = \frac{\partial h(Du)}{\partial x_r} = 0, \quad h_{p_k} u_{kn} = \frac{\partial h(Du)}{\partial x_n} \geq 0, \quad -h_{p_n p_n} \geq 0,$$

we have

$$h_{p_k} h_{p_l} u_{kl} \geq -C_3 h_{p_n} + \frac{1}{C_2}|Dh|^2 - \frac{1}{C_2} h_{p_n}^2 \geq -C_4 h_{p_n} + \frac{1}{C_4} - \frac{1}{C_4} h_{p_n}^2,$$

where we use $|Dh|^2 - h_{p_n}^2 = \sum_{k=1}^{n-1} h_{p_k}^2$ and let $C_4 = \max\{C_2, C_3\}$. For the last term of the above inequality, we distinguish two cases at (x_0,t_0).

Case (i). If

$$-C_4 h_{p_n} + \frac{1}{C_4} - \frac{1}{C_4} h_{p_n}^2 \leq \frac{1}{2C_4},$$

then

$$h_{p_k}(Du)\nu_k = h_{p_n} \geq \sqrt{\frac{1}{2} + \frac{C_4^4}{4}} - \frac{C_4^2}{2}.$$

It shows that there is a uniform positive lower bound for the quantity $\min_{\partial\Omega\times[0,T]} h_{p_k}(Du)\nu_k$.

Case (ii). If

$$-C_4 h_{p_n} + \frac{1}{C_4} - \frac{1}{C_4} h_{p_n}^2 > \frac{1}{2C_4},$$

then we obtain a positive lower bound of $h_{p_k} h_{p_l} u_{kl}$.

Let $\tilde{u}$ be the Legendre transformation of u, then $\tilde{u}$ satisfies

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} = \tilde{F}(D^2\tilde{u}) + f(D\tilde{u}), & T > t > 0, \quad \tilde{x} \in \tilde{\Omega}, \\ \tilde{h}(D\tilde{u}) = 0, & T > t > 0, \quad \tilde{x} \in \partial\tilde{\Omega}, \\ \tilde{u} = \tilde{u}_0, & t = 0, \quad \tilde{x} \in \tilde{\Omega}, \end{cases} \tag{4.8}$$

where $\tilde{h}$ is the defining function of Ω, and $\tilde{u}_0$ is the Legendre transformation of u_0. The unit inward normal vector of $\partial\Omega$ can be expressed by $\nu = D\tilde{h}$. For the same reason, $\tilde{\nu} = Dh$, where $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_n)$ is the unit inward normal vector of $\partial\tilde{\Omega}$.

Let $\tilde{\beta} = (\tilde{\beta}^1, \ldots, \tilde{\beta}^n)$ with $\tilde{\beta}^k := \tilde{h}_{p_k}(D\tilde{u})$. We note that one can also define

$$\tilde{v} = \langle \tilde{\beta}, \tilde{\nu} \rangle + \tilde{h}(D\tilde{u}),$$

in which

$$\langle \tilde{\beta}, \tilde{\nu} \rangle = \langle \beta, \nu \rangle.$$

Denote $\tilde{x}_0 = Du(x_0)$. Then, we obtain

$$\tilde{v}(\tilde{x}_0, t_0) = v(x_0, t_0) = \min_{\partial\tilde{\Omega}\times[0,T]} \tilde{v}.$$

Using the same methods, under the assumption of

$$\tilde{v}_n(\tilde{x}_0, t_0) \geq -C_5, \tag{4.9}$$

we obtain the positive lower bounds of $\tilde{h}_{p_k}\tilde{h}_{p_l}\tilde{u}_{kl}$, or

$$h_{p_k}(Du)\nu_k = \tilde{h}_{p_k}(D\tilde{u})\tilde{\nu}_k = \tilde{h}_{p_n} \geq \sqrt{\frac{1}{2} + \frac{C_5^4}{4}} - \frac{C_5^2}{2}.$$

We notice that

$$\tilde{h}_{p_k}\tilde{h}_{p_l}\tilde{u}_{kl} = \nu_i \nu_j u^{ij}.$$

Then by the positive lower bounds of $h_{p_k}h_{p_l}u_{kl}$ and $\tilde{h}_{p_k}\tilde{h}_{p_l}\tilde{u}_{kl}$, the desired result follows from

$$\langle \beta, \nu \rangle = \sqrt{u^{ij}\nu_i\nu_j h_{p_k}h_{p_l}u_{kl}}, \tag{4.10}$$

which is proved in [25].

It remains to prove the key estimate (4.5) and (4.9).

We prove (4.5) first. By $D^2\tilde{h} \leq -\tilde{\theta} I$ and (1.13), we have

$$L\tilde{h} \leq -\tilde{\theta} \sum_{i=1}^{n} F^{ii}, \tag{4.11}$$

where

$$L := F^{ij}\partial_{ij} - \partial_t.$$

On the other hand,

$$\begin{aligned} Lv = {} & h_{p_k p_l p_m}\nu_k F^{ij} u_{li} u_{mj} + 2h_{p_k p_l} F^{ij} \nu_{kj} u_{li} \\ & + h_{p_k p_l} F^{ij} u_{lj} u_{ki} + h_{p_k p_l}\nu_k L u_l + h_{p_k} L\nu_k + h_{p_k} L u_k. \end{aligned} \tag{4.12}$$

Now, we estimate the first term on the right-hand side of (4.12). By the diagonal basis and (1.14), we have

$$|h_{p_k p_l p_m}\nu_k F^{ij} u_{li} u_{mj}| \le C \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i}\lambda_i^2 \le C_6,$$

where C_6 is a constant depending only on h, Ω, Λ_1, Λ_2, u_0 and δ. Similarly, we also get

$$|h_{p_k p_l} F^{ij} u_{lj} u_{ki}| \le C \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i}\lambda_i^2 \le C_7.$$

For the second term, by Cauchy inequality, we obtain

$$\begin{aligned} |2h_{p_k p_l} F^{ij} \nu_{kj} u_{li}| & \le C \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i}\lambda_i = C \sum_{i=1}^{n} \sqrt{\frac{\partial F}{\partial \lambda_i}}\sqrt{\frac{\partial F}{\partial \lambda_i}}\lambda_i \\ & \le C \left(\sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i}\lambda_i^2\right)^{\frac{1}{2}} \\ & \le C_8. \end{aligned}$$

By (1.1), we have $Lu_l = f_l$. Then, we get

$$|h_{p_k p_l}\nu_k L u_l| \le C_9, \quad |h_{p_k} L u_k| \le C_{10}.$$

It follows from (1.13) that

$$|h_{p_k} L\nu_k| \le C_{11} \sum_{i=1}^{n} F^{ii}.$$

Inserting these into (4.12) and using (1.13), it is immediate to check that there exists a positive constant C_{12} depending only on h, Ω, Λ_1, Λ_2, u_0 and δ, such that

$$Lv \le C_{12} \sum_{i=1}^{n} F^{ii}. \tag{4.13}$$

Denote a neighborhood of x_0 in Ω by

$$\Omega_\rho := \Omega \cap B_\rho(x_0),$$

where ρ is a positive constant such that ν is well-defined in Ω_ρ. To obtain the key estimate, we need to consider the function

$$\Phi(x) := v(x,t) - v(x_0,t_0) + C_0\tilde{h}(x) + A|x - x_0|^2,$$

where C_0 and A are positive constants to be determined. On $\partial\Omega \times [0,T]$, it is clear that $\Phi \geq 0$. Since v is bounded, we can choose A large enough such that on $\big(\Omega \cap \partial B_\rho(x_0)\big) \times [0,T]$

$$\Phi(x) = v(x,t) - v(x_0,t_0) + C_0\tilde{h}(x) + A|x - x_0|^2 \geq v(x,t) - v(x_0,t_0) + A\rho^2 \geq 0.$$

By the strict concavity of $\tilde{h}$, we have

$$\Delta(C_0\tilde{h}(x) + A|x - x_0|^2) \leq C(-C_0\tilde{\theta} + 2A)\sum_{i=1}^n F^{ii}.$$

Then by choosing $C_0 \gg A$, we obtain

$$\Delta(v(x,0) - v(x_0,t_0) + C_0\tilde{h}(x) + A|x - x_0|^2) \leq 0.$$

We apply the maximum principle to get

$$\begin{aligned}
&(v(x,0) - v(x_0,t_0) + C_0\tilde{h}(x) + A|x - x_0|^2)|_{\Omega_\rho}\\
&\geq \min_{(\partial\Omega\cap B_\rho(x_0))\cup(\Omega\cap\partial B_\rho(x_0))} (v(x,0) - v(x_0,t_0) + C_0\tilde{h}(x) + A|x - x_0|^2)\\
&\geq 0.
\end{aligned}$$

Combining (4.11) with (4.13) and letting C_0 be large enough, one yields

$$L\Phi \leq (-C_0\tilde{\theta} + C_{12} + 2A)\sum_{i=1}^n F^{ii} \leq 0.$$

From the above arguments, we verify that Φ satisfies

$$\begin{cases} L\Phi \leq 0, & (x,t) \in \Omega_\rho \times [0,T],\\ \Phi \geq 0, & (x,t) \in (\partial\Omega_\rho \times [0,T]) \cup (\Omega_\rho \times \{t=0\}). \end{cases} \tag{4.14}$$

Using the maximum principle, we deduce that

$$\Phi \geq 0, \quad (x,t) \in \Omega_\rho \times [0,T].$$

Combining it with $\Phi(x_0,t_0) = 0$, we obtain $\langle \nabla\Phi, e_n\rangle|_{(x_0,t_0)} \geq 0$, which gives the desired key estimate (4.5).

Finally, we prove (4.9). The proof of (4.9) is similar to the one of (4.5). Define

$$\tilde{L} = \tilde{F}^{ij}\partial_{ij} + f_{p_i}\partial_i - \partial_t.$$

By (4.8), we see that $\tilde{L}\tilde{u}_l = 0$, and thus

$$\begin{aligned}\tilde{L}\tilde{v} &= \tilde{F}^{ij}\tilde{u}_{mj}\tilde{u}_{li}\tilde{h}_{\tilde{p}_k\tilde{p}_l\tilde{p}_m}\tilde{v}_k + 2\tilde{h}_{\tilde{p}_k\tilde{p}_l}\tilde{F}^{ij}\tilde{u}_{li}\tilde{v}_{kj} + \tilde{F}^{ij}\tilde{h}_{\tilde{p}_k}\tilde{v}_{kij} + \tilde{h}_{\tilde{p}_k\tilde{p}_l}\tilde{F}^{ij}\tilde{u}_{lj}\tilde{u}_{ki}\\ &\quad +\tilde{h}_{\tilde{p}_k} f_{\tilde{p}_i}\tilde{v}_{ki}.\end{aligned}$$

By making use of the following identities

$$\frac{\partial \tilde{F}}{\partial \mu_i} = \lambda_i^2\frac{\partial F}{\partial \lambda_i}, \quad \mu_i^2\frac{\partial \tilde{F}}{\partial \mu_i} = \frac{\partial F}{\partial \lambda_i},$$

we deduce that $\tilde{F}$ satisfies the structure conditions (1.9)–(1.14). Repeating the proof of (4.13), we have

$$\tilde{L}\tilde{v} \le C_{13}\sum_{i=1}^{n}\tilde{F}^{ii}, \tag{4.15}$$

where C_{13} depends only on $\tilde{\Omega}$, Ω, Λ_1, Λ_2, δ and u_0.

Denote a neighborhood of $\tilde{x}_0$ in $\tilde{\Omega}$ by

$$\tilde{\Omega}_r := \tilde{\Omega}\cap B_r(\tilde{x}_0),$$

where r is a positive constant such that $\tilde{v}$ is well-defined in $\tilde{\Omega}_r$. Consider

$$\tilde{\Phi}(y) := \tilde{v}(y,t) - \tilde{v}(\tilde{x}_0,t_0) + \tilde{C}_0 h(y) + \tilde{A}|y-\tilde{x}_0|^2,$$

where $\tilde{C}_0$ and $\tilde{A}$ are positive constants to be determined. It is clear that $\tilde{\Phi} \ge 0$ on $\partial\tilde{\Omega}\times[0,T]$. Since $\tilde{v}$ is bounded, we can choose $\tilde{A}$ large enough such that on $\left(\tilde{\Omega}\cap\partial B_r(\tilde{x}_0)\right)\times[0,T]$

$$\tilde{\Phi}(y) \ge \tilde{v}(y,t) - \tilde{v}(\tilde{x}_0,t_0) + \tilde{A}r^2 \ge 0.$$

By the strict concavity of h, we have

$$\Delta\left(\tilde{C}_0 h(y) + \tilde{A}|y-\tilde{x}_0|^2\right) \le C(-\tilde{C}_0\theta + 2\tilde{A})\sum_{i=1}^{n}\tilde{F}^{ii}.$$

Then by choosing $\tilde{C}_0 \gg \tilde{A}$, we have

$$\Delta\left(\tilde{v}(y,0) - \tilde{v}(\tilde{x}_0,t_0) + \tilde{C}_0 h(y) + \tilde{A}|y-\tilde{x}_0|^2\right) \le 0.$$

It follows from the maximum principle that

$$\begin{aligned}&(\tilde{v}(y,0) - \tilde{v}(\tilde{x}_0,t_0) + \tilde{C}_0 h(y) + \tilde{A}|y-\tilde{x}_0|^2)|_{\tilde{\Omega}_r}\\ &\quad\ge \min_{(\partial\tilde{\Omega}\cap B_r(\tilde{x}_0))\cup(\tilde{\Omega}\cap\partial B_r(\tilde{x}_0))}(\tilde{v}(y,0) - \tilde{v}(\tilde{x}_0,t_0) + \tilde{C}_0 h(y) + \tilde{A}|y-\tilde{x}_0|^2)\\ &\quad\ge 0.\end{aligned}$$

By (1.14) and (4.15), it is not difficult to show that

$$\tilde{L}\tilde{\Phi}(y) \le \left(C_{13} - \frac{\tilde{C}_0\theta}{2} + 2\tilde{A}\right)\sum_{i=1}^{n}\tilde{F}^{ii} + 2\tilde{A} f_{\tilde{p}_i}(y_i - \tilde{x}_{0i}) - \tilde{C}_0\left(\frac{\theta}{2}\sum_{i=1}^{n}\tilde{F}^{ii} - f_{\tilde{p}_i}\partial_i h\right).$$

In order to make

$$\tilde{L}\tilde{\Phi}(y) \le 0,$$

we only need to choose $\tilde{C}_0 \gg \tilde{A}$ and

$$|Df| \le \frac{\theta\Lambda_1}{2}\cdot\frac{1}{\max_{\bar{\tilde{\Omega}}}|Dh|}.$$

Consequently,

$$\begin{cases} \tilde{L}\tilde{\Phi} \le 0, & (y,t)\in\tilde{\Omega}_r\times[0,T],\\ \tilde{\Phi}\ge 0, & (y,t)\in(\partial\tilde{\Omega}_r\times[0,T])\cup(\tilde{\Omega}_r\times\{t=0\}). \end{cases} \tag{4.16}$$

Therefore, we get (4.9) as same as the argument in (4.5). Thus, we complete the proof of the lemma. □

Similar to Proposition 2.6 in [27], by making use of (4.13), we can obtain

Lemma 4.7. *Fix a smooth function $H : \Omega\times\tilde{\Omega}\to R$ and define $\varphi(x,t) = H(x, Du(x,t))$. Then for any $(x,t)\in\Omega_T$,*

$$|L\varphi| \le C\sum_{i=1}^{n}F^{ii}$$

holds for some positive constant C, which depends only on H, Ω, $\tilde{\Omega}$, Λ_1, Λ_2, f and δ.

The following definition provides a basic connection between (4.1) and (3.1) and will be used frequently in the sequel.

Definition 4.8. We say that $\tilde{u}$ in (4.1) is a dual solution to (3.1).

We now proceed to carry out the global C^2 estimate. The strategy is to reduce the C^2 global estimate of u and $\tilde{u}$ to the boundary.

Lemma 4.9. *If u is a strictly convex solution of (3.1) and there hold (1.10), (1.11) and (1.13), then there exists a positive constant C_{14} depending only on n, Ω, $\tilde{\Omega}$, Λ_1, u_0, δ and* diam(Ω)*, such that*

$$\sup_{\Omega_T}|D^2u| \le \max_{\partial\Omega\times[0,T]}|D^2u| + \max_{\bar{\Omega}}|D^2u_0| + C_{14}\sup_{\Omega}|D^2f|. \tag{4.17}$$

Proof. Without loss of generality, we may assume that Ω lies in cube $[0, d]^n$. Let

$$L := F^{ij}\partial_{ij} - \partial_t.$$

For any unit vector ξ, differentiating the equation in (3.1) twice in direction ξ gives

$$Lu_{\xi\xi} + F^{ij,rs}u_{ij\xi}u_{rs\xi} = f_{\xi\xi}.$$

Then by the concavity of F on Γ_n^+, we have

$$Lu_{\xi\xi} = -F^{ij,rs}u_{ij\xi}u_{rs\xi} + f_{\xi\xi} \geq f_{\xi\xi}.$$

Let

$$v = \sup_{\partial\Omega_T} u_{\xi\xi} + \frac{1}{\Lambda_1}\left(ne^d - \sum_{i=1}^n e^{x_i}\right)\sup_{\Omega}|f_{\xi\xi}|.$$

By direct calculation and (1.13), we obtain

$$\begin{aligned} F^{ij}\partial_{ij}v =& -\frac{1}{\Lambda_1}\sup_{\Omega}|f_{\xi\xi}|\left(\sum_{i=1}^n e^{x_i}F^{ii}\right) \\ \leq& -\frac{1}{\Lambda_1}\sup_{\Omega}|f_{\xi\xi}|\left(\sum_{i=1}^n F^{ii}\right) \\ \leq& -\sup_{\Omega}|f_{\xi\xi}|. \end{aligned}$$

Therefore,

$$Lv = F^{ij}\partial_{ij}v - \partial_t v \leq -\sup_{\Omega}|f_{\xi\xi}|,$$

and thus

$$L(v - u_{\xi\xi}) \leq -\left(\sup_{\Omega}|f_{\xi\xi}| + f_{\xi\xi}\right) \leq 0.$$

It is obvious that $v - u_{\xi\xi} \geq 0$ on $\partial\Omega_T$. Then, by the maximum principle, we obtain

$$\begin{aligned} \sup_{\Omega_T} u_{\xi\xi} \leq \sup_{\Omega_T} v \leq& \sup_{\partial\Omega_T} u_{\xi\xi} + \frac{ne^d}{\Lambda_1}\sup_{\Omega}|f_{\xi\xi}| \\ \leq& \max_{\partial\Omega\times[0,T]}|D^2u| + \max_{\bar{\Omega}}|D^2u_0| + C_{14}\sup_{\Omega}|D^2 f|. \end{aligned}$$

This completes the proof of (4.17). □

Next, we estimate the second-order derivative on the boundary. By differentiating the boundary condition $h(Du) = 0$ in any tangential direction ς, we have

$$u_{\beta\varsigma} = h_{p_k}(Du)u_{k\varsigma} = 0. \tag{4.18}$$

The second-order derivative of u on the boundary is controlled by $u_{\beta\varsigma}$, $u_{\beta\beta}$ and $u_{\varsigma\varsigma}$. In the following, we give the arguments as in [25], one can see there for more details.

At $x \in \partial\Omega$, any unit vector ξ can be written in terms of a tangential component $\varsigma(\xi)$ and a component in the direction β by

$$\xi = \varsigma(\xi) + \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle}\beta,$$

where

$$\varsigma(\xi) := \xi - \langle \nu, \xi \rangle \nu - \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle}\beta^T,$$

and

$$\beta^T := \beta - \langle \beta, \nu \rangle \nu.$$

By the strict obliqueness estimate (4.2), we have

$$\begin{aligned} |\varsigma(\xi)|^2 &= 1 - \left(1 - \frac{|\beta^T|^2}{\langle \beta, \nu \rangle^2}\right) \langle \nu, \xi \rangle^2 - 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle} \\ &\le 1 + C_{15}\langle \nu, \xi \rangle^2 - 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle} \\ &\le C_{16}. \end{aligned} \tag{4.19}$$

Denote $\varsigma := \frac{\varsigma(\xi)}{|\varsigma(\xi)|}$, then by (4.18), (4.19) and (4.2), we obtain

$$\begin{aligned} u_{\xi\xi} &= |\varsigma(\xi)|^2 u_{\varsigma\varsigma} + 2|\varsigma(\xi)| \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} u_{\beta\varsigma} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta\beta} \\ &= |\varsigma(\xi)|^2 u_{\varsigma\varsigma} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta\beta} \\ &\le C_{17}(u_{\varsigma\varsigma} + u_{\beta\beta}), \end{aligned} \tag{4.20}$$

where C_{17} depends only on Ω, $\tilde{\Omega}$, Λ_1, Λ_2, δ and the constant C_1 in (4.2). Therefore, we only need to estimate $u_{\beta\beta}$ and $u_{\varsigma\varsigma}$, respectively.

Further, we have

Lemma 4.10. *Let F satisfy the structure conditions (1.9)–(1.14) and $f \in \mathscr{A}_\delta$. If u is a strictly convex solution of (3.1), then there exists a positive constant C_{18} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\max_{\partial\Omega_T} u_{\beta\beta} \le C_{18}. \tag{4.21}$$

Proof. Let $x_0 \in \partial\Omega$, $t_0 \in [0, T]$ satisfy $u_{\beta\beta}(x_0, t_0) = \max_{\partial\Omega_T} u_{\beta\beta}$. Consider the barrier function

$$\Psi := -h(Du) + C_0\tilde{h} + A|x - x_0|^2.$$

For any $x \in \partial\Omega$, $Du(x) \in \partial\tilde{\Omega}$, then $h(Du) = 0$. It is clear that $\tilde{h} = 0$ on $\partial\Omega$. As same as the proof of (4.14), we can find the constants C_0 and A such that

$$\begin{cases} L\Psi \leq 0, & (x,t) \in \Omega_\rho \times [0,T], \\ \Psi \geq 0, & (x,t) \in (\partial\Omega_\rho \times [0,T]) \cup (\Omega_\rho \times \{t=0\}). \end{cases} \tag{4.22}$$

By the maximum principle, we get

$$\Psi \geq 0, \quad (x,t) \in \Omega_\rho \times [0,T].$$

Combining it with $\Psi(x_0,t_0) = 0$, we obtain $\Psi_\beta(x_0,t_0) \geq 0$, which implies

$$\frac{\partial h}{\partial \beta}(Du(x_0,t_0)) \leq C_0.$$

On the other hand, we see that at (x_0,t_0),

$$\frac{\partial h}{\partial \beta} = \langle Dh(Du), \beta \rangle = \frac{\partial h}{\partial p_k} u_{kl} \beta^l = \beta^k u_{kl} \beta^l = u_{\beta\beta}.$$

Let $C_{18} = C_0$. Therefore,

$$u_{\beta\beta} = \frac{\partial h}{\partial \beta} \leq C_{18},$$

whence the result follows. □

Next, we estimate the double tangential derivative.

Lemma 4.11. *Let F satisfy the structure conditions (1.9)–(1.14) and $f \in \mathscr{A}_\delta$. If u is a strictly convex solution of (3.1) and $|Df|$ satisfies (1.15), then there exists a positive constant C_{19} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\max_{\partial\Omega\times[0,T]} \max_{|\varsigma|=1,\langle\varsigma,\nu\rangle=0} u_{\varsigma\varsigma} \leq C_{19}. \tag{4.23}$$

Proof. Without loss of generality, we assume that $x_0 \in \partial\Omega$, $t_0 \in (0,T]$, e_n is the unit inward normal vector of $\partial\Omega$ at x_0, and e_1 is the tangential vector of $\partial\Omega$ at x_0, respectively, such that

$$\max_{\partial\Omega\times[0,T]} \max_{|\varsigma|=1,\langle\varsigma,\nu\rangle=0} u_{\varsigma\varsigma} = u_{11}(x_0,t_0) =: \mathcal{M}.$$

For any $x \in \partial\Omega$, we have by the proof of (4.19),

$$\begin{aligned} u_{\xi\xi} &= |\varsigma(\xi)|^2 u_{\varsigma\varsigma} + \frac{\langle\nu,\xi\rangle^2}{\langle\beta,\nu\rangle^2} u_{\beta\beta} \\ &\leq \left(1 + C_{20}\langle\nu,\xi\rangle^2 - 2\langle\nu,\xi\rangle \frac{\langle\beta^T,\xi\rangle}{\langle\beta,\nu\rangle}\right) M + \frac{\langle\nu,\xi\rangle^2}{\langle\beta,\nu\rangle^2} u_{\beta\beta}. \end{aligned} \tag{4.24}$$

Without loss of generality, we assume that $M \geq 1$. Then by (4.2) and (4.21), we have

$$\frac{u_{\xi\xi}}{M} + 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle} \leq 1 + C_{21} \langle \nu, \xi \rangle^2. \tag{4.25}$$

Let $\xi = e_1$, then

$$\frac{u_{11}}{M} + 2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} \leq 1 + C_{21} \langle \nu, e_1 \rangle^2. \tag{4.26}$$

As in the proof of Proposition 2.14 in [27], let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function satisfying $\eta(s) = s$ for $s \geq \frac{1}{C_1}$ and $\eta(s) \geq \frac{1}{2C_1}$ for all $s \in \mathbb{R}$.

We see that the function

$$w := A|x - x_0|^2 - \frac{u_{11}}{M} - 2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\eta(\langle \beta, \nu \rangle)} + C_{21} \langle \nu, e_1 \rangle^2 + 1 \tag{4.27}$$

satisfies

$$w|_{\partial\Omega \times [0,T]} \geq 0, \quad w(x_0, t_0) = 0.$$

Then, it follows by (4.17) that we can choose the constant A large enough such that

$$w|_{(\partial B_\rho(x_0) \cap \Omega) \times [0,T]} \geq 0.$$

Consider

$$-2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\eta(\langle \beta, \nu \rangle)} + C_{21} \langle \nu, e_1 \rangle^2 + 1$$

as a known function depending on x and Du. Then by Lemma 4.7, we obtain

$$\left| L \left(-2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\eta(\langle \beta, \nu \rangle)} + C_{21} \langle \nu, e_1 \rangle^2 + 1 \right) \right| \leq C_{22} \sum_{i=1}^{n} F^{ii}.$$

Combining the above inequality with the proof of Lemma 4.9, we have

$$Lw \leq C_{23} \sum_{i=1}^{n} F^{ii}.$$

As in the proof of Lemma 4.10, we consider the function

$$\Upsilon := w + C_0 \tilde{h}.$$

A standard barrier argument shows that

$$\Upsilon_\beta(x_0, t_0) \geq 0.$$

Therefore,

$$u_{11\beta}(x_0, t_0) \leq C_{24} M. \tag{4.28}$$

On the other hand, differentiating $h(Du)$ twice in the direction e_1 at (x_0, t_0), we have

$$h_{p_k} u_{k11} + h_{p_k p_l} u_{k1} u_{l1} = 0.$$

The concavity of h yields that

$$h_{p_k} u_{k11} = -h_{p_k p_l} u_{k1} u_{l1} \geq \theta M^2.$$

Combining it with $h_{p_k} u_{k11} = u_{11\beta}$, and using (4.28), we obtain

$$\theta M^2 \leq C_{24} M.$$

Then, we get the upper bound of $M = u_{11}(x_0, t_0)$ and thus the desired result follows. □

By Lemma 4.10, Lemmas 4.11 and (4.20), we obtain the C^2 a priori estimate on the boundary.

Lemma 4.12. *Let F satisfy the structure conditions* (1.9)–(1.14) *and $f \in \mathscr{A}_\delta$. If u is a strictly convex solution of* (3.1) *and $|Df|$ satisfies* (1.15)*, then there exists a positive constant C_{25} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\max_{\partial\Omega_T} |D^2 u| \leq C_{25}. \tag{4.29}$$

In terms of Lemmas 4.9 and 4.12, we readily conclude:

Lemma 4.13. *Let F satisfy the structure conditions* (1.9)–(1.14) *and $f \in \mathscr{A}_\delta$. If u is a strictly convex solution of* (3.1) *and $|Df|$ satisfies* (1.15)*, then there exists a positive constant C_{26} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\max_{\bar{\Omega}_T} |D^2 u| \leq C_{26}. \tag{4.30}$$

In the following, we describe the positive lower bound of $D^2 u$. For (4.1), by considering the Legendre transformation of u, define

$$\tilde{L} := \tilde{F}^{ij} \partial_{ij} + f_{\tilde{p}_i} \partial_i - \partial_t.$$

Then, our goal is to show the upper bound of $D^2 \tilde{u}$ and the argument is very similar to the one used in the proof of Lemma 4.13 by the concavity of f and the condition that $|Df|$ being sufficiently small. For the convenience of readers, we give the details.

At the beginning of the repeating procedure, we have

Lemma 4.14. *Suppose that f is concave on Ω. If $\tilde{u}$ is a strictly convex solution of* (4.1)*, then there holds*

$$\sup_{\tilde{\Omega}_T} |D^2 \tilde{u}| \leq \max_{\partial\tilde{\Omega}_T} |D^2 \tilde{u}|. \tag{4.31}$$

Proof. For any unit vector $\tilde{\xi}$, differentiating the equation in (4.1) twice in direction $\tilde{\xi}$ gives

$$\tilde{L}\tilde{u}_{\tilde{\xi}\tilde{\xi}} + \tilde{F}^{ij,rs}\tilde{u}_{ij\tilde{\xi}}\tilde{u}_{rs\tilde{\xi}} + \frac{\partial^2 f}{\partial \tilde{p}_i \partial \tilde{p}_j}\tilde{u}_{i\tilde{\xi}}\tilde{u}_{j\tilde{\xi}} = 0.$$

Then by the concavity of $\tilde{F}$ on Γ_n^+ and f on Ω, we have

$$\tilde{L}\tilde{u}_{\tilde{\xi}\tilde{\xi}} = -\tilde{F}^{ij,rs}\tilde{u}_{ij\tilde{\xi}}\tilde{u}_{rs\tilde{\xi}} - \frac{\partial^2 f}{\partial \tilde{p}_i \partial \tilde{p}_j}\tilde{u}_{i\tilde{\xi}}\tilde{u}_{j\tilde{\xi}} \geq 0.$$

Then, by the maximum principle, we obtain

$$\sup_{\tilde{\Omega}_T} \tilde{u}_{\tilde{\xi}\tilde{\xi}} \leq \sup_{\partial\tilde{\Omega}_T} \tilde{u}_{\tilde{\xi}\tilde{\xi}}.$$

This completes the proof of (4.31). □

Recall that $\tilde{\beta} = (\tilde{\beta}^1, \dots, \tilde{\beta}^n)$ with $\tilde{\beta}^k := \tilde{h}_{p_k}(D\tilde{u})$ and $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \dots, \tilde{\nu}_n)$ is the unit inward normal vector of $\partial\tilde{\Omega}$. Similar to the discussion of (4.18), (4.19) and (4.20), for any tangential direction $\tilde{\varsigma}$, we have

$$u_{\tilde{\beta}\tilde{\varsigma}} = \tilde{h}_{p_k}(D\tilde{u})\tilde{u}_{k\tilde{\varsigma}} = 0. \tag{4.32}$$

Then, the second-order derivative of $\tilde{u}$ on the boundary is also controlled by $u_{\tilde{\beta}\tilde{\varsigma}}$, $u_{\tilde{\beta}\tilde{\beta}}$ and $u_{\tilde{\varsigma}\tilde{\varsigma}}$.

At $\tilde{x} \in \partial\tilde{\Omega}$, any unit vector $\tilde{\xi}$ can be written in terms of a tangential component $\tilde{\varsigma}(\tilde{\xi})$ and a component in the direction $\tilde{\beta}$ by

$$\tilde{\xi} = \tilde{\varsigma}(\tilde{\xi}) + \frac{\langle \tilde{\nu}, \tilde{\xi} \rangle}{\langle \tilde{\beta}, \tilde{\nu} \rangle}\tilde{\beta},$$

where

$$\tilde{\varsigma}(\tilde{\xi}) := \tilde{\xi} - \langle \tilde{\nu}, \tilde{\xi} \rangle \tilde{\nu} - \frac{\langle \tilde{\nu}, \tilde{\xi} \rangle}{\langle \tilde{\beta}, \tilde{\nu} \rangle}\tilde{\beta}^T,$$

and

$$\tilde{\beta}^T := \tilde{\beta} - \langle \tilde{\beta}, \tilde{\nu} \rangle \tilde{\nu}.$$

We observe that $\langle \tilde{\beta}, \tilde{\nu} \rangle = \langle \beta, \nu \rangle$. Therefore,

$$|\tilde{\varsigma}(\tilde{\xi})| \leq C_{27}, \tag{4.33}$$

and similar to the calculation in (4.24), one should deduce that

$$\tilde{u}_{\tilde{\xi}\tilde{\xi}} \leq C_{28}(\tilde{u}_{\tilde{\varsigma}\tilde{\varsigma}} + \tilde{u}_{\tilde{\beta}\tilde{\beta}}), \tag{4.34}$$

where $\tilde{\varsigma} := \frac{\tilde{\varsigma}(\tilde{\xi})}{|\tilde{\varsigma}(\tilde{\xi})|}$ and C_{28} depends only on Ω, $\tilde{\Omega}$, Λ_1, Λ_2, δ and the constant C_1 in (4.2). Then, we also only need to estimate $\tilde{u}_{\tilde{\beta}\tilde{\beta}}$ and $\tilde{u}_{\tilde{\varsigma}\tilde{\varsigma}}$, respectively.

Indeed, as shown by Lemma 4.10, we state

Lemma 4.15. *Let F satisfy the structure conditions (1.9)–(1.14) and $f \in \mathscr{A}_\delta$. If $\tilde{u}$ is a strictly convex solution of (4.1) and $|Df|$ satisfies (1.15), then there exists a positive constant C_{29} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\max_{\partial\Omega_T} \tilde{u}_{\tilde{\beta}\tilde{\beta}} \leq C_{29}. \tag{4.35}$$

Proof. Let $\tilde{x}_0 \in \partial\tilde{\Omega}$, $t_0 \in [0, T]$ satisfy $\tilde{u}_{\tilde{\beta}\tilde{\beta}}(\tilde{x}_0, t_0) = \max_{\partial\Omega_T} \tilde{u}_{\tilde{\beta}\tilde{\beta}}$. To estimate the upper bound of $\tilde{u}_{\tilde{\beta}\tilde{\beta}}$, we consider the barrier function

$$\tilde{\Psi} := -\tilde{h}(D\tilde{u}) + C_0 h + A|y - \tilde{x}_0|^2.$$

For any $y \in \partial\tilde{\Omega}$, $t \in [0, T]$, $D\tilde{u}(y, t) \in \partial\Omega$, then $\tilde{h}(D\tilde{u}) = 0$. It is clear that $h = 0$ on $\partial\tilde{\Omega}$. Similar to the proof of (4.16), first we have

$$\tilde{L}(C_0 h) = C_0 \left(\tilde{F}^{ij} h_{ij} + f_{\tilde{p}_i} h_i\right) \leq C_0 \left(-\theta \sum_{i=1}^{n} \tilde{F}^{ii} + f_{\tilde{p}_i} h_i\right),$$

and

$$\tilde{L}\left(A|y - \tilde{x}_0|^2\right) = 2A \sum_{i=1}^{n} \tilde{F}^{ii} + 2A f_{\tilde{p}_i} \left(y_i - \tilde{x}_{0i}\right).$$

Similar to the proof of (4.13), we get

$$\tilde{L}\left(-\tilde{h}(D\tilde{u})\right) = \tilde{F}^{ij}\left(-\tilde{h}_{\tilde{p}_k \tilde{p}_l} \partial_{ki}\tilde{u}\, \partial_{lj}\tilde{u}\right) \leq C_{30} \sum_{i=1}^{n} \tilde{F}^{ii}.$$

Therefore, we obtain

$$\begin{aligned} \tilde{L}\tilde{\Psi}(y) \ &\leq \left(C_{30} - \frac{C_0\theta}{2} + 2A\right) \sum_{i=1}^{n} \tilde{F}^{ii} + 2A f_{\tilde{p}_i}(y_i - \tilde{x}_{0i}) \\ &\quad - C_0 \left(\frac{\theta}{2} \sum_{i=1}^{n} \tilde{F}^{ii} - f_{\tilde{p}_i} \partial_i h\right). \end{aligned}$$

As the proof of (4.16) in terms of $|Df|$ satisfying (1.15), we can find the constants C_0 and A such that

$$\begin{cases} \tilde{L}\tilde{\Psi} \leq 0, & (y, t) \in \tilde{\Omega}_r \times [0, T], \\ \tilde{\Psi} \geq 0, & (y, t) \in (\partial\tilde{\Omega}_r \times [0, T]) \cup (\tilde{\Omega}_r \times \{t = 0\}). \end{cases} \tag{4.36}$$

By the maximum principle, we get

$$\tilde{\Psi}(y, t) \geq 0, \quad (y, t) \in \tilde{\Omega}_r \times [0, T].$$

Combining it with $\tilde{\Psi}(\tilde{x}_0, t_0) = 0$, we obtain $\tilde{\Psi}_{\tilde{\beta}}(\tilde{x}_0, t_0) \geq 0$, which implies

$$\frac{\partial \tilde{h}}{\partial \tilde{\beta}}(D\tilde{u}(\tilde{x}_0, t_0)) \leq C_0.$$

On the other hand, we see that at $(\tilde{x}_0, t_0)$,

$$\frac{\partial \tilde{h}}{\partial \tilde{\beta}} = \langle D\tilde{h}(D\tilde{u}), \tilde{\beta}\rangle = \frac{\partial \tilde{h}}{\partial p_k}\tilde{u}_{kl}\tilde{\beta}^l = \tilde{\beta}^k\tilde{u}_{kl}\tilde{\beta}^l = \tilde{u}_{\tilde{\beta}\tilde{\beta}}.$$

Therefore, letting $C_{29} = C_0$, we get

$$\tilde{u}_{\tilde{\beta}\tilde{\beta}} = \frac{\partial \tilde{h}}{\partial \tilde{\beta}} \leq C_{29}.$$

□

Next, we estimate the double tangential derivative of $\tilde{u}$.

Lemma 4.16. *Let F satisfy the structure conditions (1.9)–(1.14) and $f \in \mathscr{A}_\delta$. If $\tilde{u}$ is a strictly convex solution of (4.1) and $|Df|$ satisfies (1.15), then there exists a positive constant C_{31} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\max_{\partial\tilde{\Omega}\times[0,T]} \max_{|\tilde{\varsigma}|=1, \langle\tilde{\varsigma},\tilde{\nu}\rangle=0} \tilde{u}_{\tilde{\varsigma}\tilde{\varsigma}} \leq C_{31}. \tag{4.37}$$

Proof. Assume that $\tilde{x}_0 \in \partial\tilde{\Omega}$, $t_0 \in [0, T]$ and e_n is the unit inward normal vector of $\partial\tilde{\Omega}$ at $\tilde{x}_0$. Let

$$\max_{\partial\tilde{\Omega}\times[0,T]} \max_{|\tilde{\varsigma}|=1, \langle\tilde{\varsigma},\tilde{\nu}\rangle=0} \tilde{u}_{\tilde{\varsigma}\tilde{\varsigma}} = \tilde{u}_{11}(\tilde{x}_0, t_0) =: \tilde{M}.$$

For any $y \in \partial\tilde{\Omega}$, $t \in [0, T]$, we have by the proof of (4.19) and (4.33),

$$\begin{aligned}\tilde{u}_{\tilde{\xi}\tilde{\xi}} &= |\tilde{\varsigma}(\tilde{\xi})|^2\tilde{u}_{\tilde{\varsigma}\tilde{\varsigma}} + \frac{\langle\tilde{\nu}, \tilde{\xi}\rangle^2}{\langle\tilde{\beta}, \tilde{\nu}\rangle^2}\tilde{u}_{\tilde{\beta}\tilde{\beta}} \\ &\leq \left(1 + C_{32}\langle\tilde{\nu}, \tilde{\xi}\rangle^2 - 2\langle\tilde{\nu}, \tilde{\xi}\rangle\frac{\langle\tilde{\beta}^T, \tilde{\xi}\rangle}{\langle\tilde{\beta}, \tilde{\nu}\rangle}\right)\tilde{M} + \frac{\langle\tilde{\nu}, \tilde{\xi}\rangle^2}{\langle\tilde{\beta}, \tilde{\nu}\rangle^2}\tilde{u}_{\tilde{\beta}\tilde{\beta}}.\end{aligned} \tag{4.38}$$

Without loss of generality, we assume that $\tilde{M} \geq 1$. Then, by (4.2) and (4.35), we have

$$\frac{\tilde{u}_{\tilde{\xi}\tilde{\xi}}}{\tilde{M}} + 2\langle\tilde{\nu}, \tilde{\xi}\rangle\frac{\langle\tilde{\beta}^T, \tilde{\xi}\rangle}{\langle\tilde{\beta}, \tilde{\nu}\rangle} \leq 1 + C_{32}\langle\tilde{\nu}, \tilde{\xi}\rangle^2. \tag{4.39}$$

Let $\tilde{\xi} = e_1$, then

$$\frac{\tilde{u}_{11}}{\tilde{M}} + 2\langle\tilde{\nu}, e_1\rangle\frac{\langle\tilde{\beta}^T, e_1\rangle}{\langle\tilde{\beta}, \tilde{\nu}\rangle} \leq 1 + C_{32}\langle\tilde{\nu}, e_1\rangle^2. \tag{4.40}$$

We see that the function

$$\tilde{w} := A|y - \tilde{x}_0|^2 - \frac{\tilde{u}_{11}}{\tilde{M}} - 2\langle\tilde{\nu}, e_1\rangle\frac{\langle\tilde{\beta}^T, e_1\rangle}{\langle\tilde{\beta}, \tilde{\nu}\rangle} + C_{32}\langle\tilde{\nu}, e_1\rangle^2 + 1 \tag{4.41}$$

satisfies

$$\tilde{w}|_{\partial\tilde{\Omega}\times[0,T]} \geq 0, \quad \tilde{w}(\tilde{x}_0, t_0) = 0.$$

Then, by (4.31) we can choose the constant A large enough such that

$$\tilde{w}|_{(\tilde{\Omega}\cap\partial B_r(\tilde{x}_0))\times[0,T]} \geq 0.$$

Consider

$$-2\langle\tilde{\nu}, e_1\rangle\frac{\langle\tilde{\beta}^T, e_1\rangle}{\langle\tilde{\beta}, \tilde{\nu}\rangle} + C_{32}\langle\tilde{\nu}, e_1\rangle^2 + 1$$

as a known function depending on $\tilde{x}$ and $D\tilde{u}$. Then, by the proof of Lemma 4.7, we also obtain

$$\left|\tilde{L}\left(-2\langle\tilde{\nu}, e_1\rangle\frac{\langle\tilde{\beta}^T, e_1\rangle}{\langle\tilde{\beta}, \tilde{\nu}\rangle} + C_{32}\langle\tilde{\nu}, e_1\rangle^2 + 1\right)\right| \leq C_{33}\sum_{i=1}^{n}\tilde{F}^{ii}.$$

By making use of the concavity of $\tilde{F}$ and f, it yields

$$\tilde{L}\tilde{u}_{11} = -\tilde{F}^{ij,rs}\tilde{u}_{ij1}\tilde{u}_{rs1} - \frac{\partial^2 f}{\partial\tilde{p}_i\partial\tilde{p}_j}\tilde{u}_{i1}\tilde{u}_{j1} \geq 0.$$

Combining the above inequality with the proof of Lemma 4.15, by $f \in \mathscr{A}_\delta$ and $|Df|$ satisfying (1.15), we have

$$\tilde{L}\tilde{w} \leq C_{34}\sum_{i=1}^{n}\tilde{F}^{ii}.$$

As in the proof of Lemma 4.15, consider the function

$$\tilde{\Upsilon} := \tilde{w} + C_0 h.$$

A standard barrier argument makes conclusion of

$$\tilde{\Upsilon}_{\tilde{\beta}}(\tilde{x}_0, t_0) \geq 0.$$

Therefore,

$$\tilde{u}_{11\tilde{\beta}}(\tilde{x}_0) \leq C_{35}\tilde{M}. \tag{4.42}$$

On the other hand, differentiating $\tilde{h}(D\tilde{u})$ twice in the direction e_1 at $(\tilde{x}_0, t_0)$, we have

$$\tilde{h}_{p_k}\tilde{u}_{k11} + \tilde{h}_{p_k p_l}\tilde{u}_{k1}\tilde{u}_{l1} = 0.$$

The concavity of $\tilde{h}$ yields that

$$\tilde{h}_{p_k}\tilde{u}_{k11} = -\tilde{h}_{p_k p_l}\tilde{u}_{k1}\tilde{u}_{l1} \geq \tilde{\theta}\tilde{M}^2.$$

Combining it with $\tilde{h}_{p_k}\tilde{u}_{k11} = \tilde{u}_{11\tilde{\beta}}$, and using (4.42), we obtain

$$\tilde{\theta}\tilde{M}^2 \leq C_{35}\tilde{M}.$$

Then, we get the upper bound of $\tilde{M} = \tilde{u}_{11}(\tilde{x}_0, t_0)$ and thus the desired result follows. □

By Lemma 4.15, Lemma 4.16 and (4.34), we obtain the C^2 a priori estimate of $\tilde{u}$ on the boundary.

Lemma 4.17. *Let F satisfy the structure conditions* (1.9)–(1.14) *and $f \in \mathscr{A}_\delta$. If $\tilde{u}$ is a strictly convex solution of* (4.1) *and $|Df|$ satisfies* (1.15)*, then there exists a positive constant C_{36} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\max_{\partial\tilde{\Omega}_T} |D^2\tilde{u}| \leq C_{36}. \tag{4.43}$$

By Lemmas 4.14 and 4.17, we can see that

Lemma 4.18. *Let F satisfy the structure conditions* (1.9)–(1.14) *and $f \in \mathscr{A}_\delta$. If $\tilde{u}$ is a strictly convex solution of* (4.1) *and $|Df|$ satisfies* (1.15)*, then there exists a positive constant C_{37} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\max_{\bar{\tilde{\Omega}}_T} |D^2\tilde{u}| \leq C_{37}. \tag{4.44}$$

By Lemmas 4.13 and 4.18, we conclude that

Lemma 4.19. *Let F satisfy the structure conditions* (1.9)–(1.14) *and $f \in \mathscr{A}_\delta$. If u is a strictly convex solution of* (3.1) *and $|Df|$ satisfies* (1.15)*, then there exists a positive constant C_{38} depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ, such that*

$$\frac{1}{C_{38}} I_n \leq D^2u(x,t) \leq C_{38} I_n, \quad (x,t) \in \bar{\Omega}_T, \tag{4.45}$$

where I_n is the $n \times n$ identity matrix.

5. Longtime existence and convergence

We will need the following proposition, which essentially asserts the convergence of the flow.

Proposition 5.1. (Huang and Ye, see Theorem 1.1 in [41]) *For any $T > 0$, we assume that $u \in C^{4+\alpha, \frac{4+\alpha}{2}}(\bar{\Omega}_T)$ is a unique solution of the nonlinear parabolic Eq.* (3.1)*, which satisfies*

$$\|u_t(\cdot,t)\|_{C(\bar{\Omega})} + \|Du(\cdot,t)\|_{C(\bar{\Omega})} + \|D^2u(\cdot,t)\|_{C(\bar{\Omega})} \leq \tilde{C}_1, \tag{5.1}$$

$$\|D^2u(\cdot,t)\|_{C^\alpha(\bar{D})} \leq \tilde{C}_2, \quad \forall D \subset\subset \Omega, \tag{5.2}$$

and

$$\inf_{x\in\partial\Omega}\left(\sum_{k=1}^{n} h_{p_k}(Du(x,t))\nu_k\right) \geq \frac{1}{\tilde{C}_3}, \tag{5.3}$$

where the positive constants $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ are independent of $t \geq 1$. Then, the solution $u(\cdot,t)$ converges to a function $u^\infty(x,t) = \tilde{u}^\infty(x) + C_\infty \cdot t$ in $C^{1+\zeta}(\bar{\Omega}) \cap C^4(\bar{D})$ as $t \to \infty$ for any $D \subset\subset \Omega$, $\zeta < 1$, that is

$$\lim_{t\to+\infty} \|u(\cdot,t) - u^\infty(\cdot,t)\|_{C^{1+\zeta}(\bar{\Omega})} = 0, \quad \lim_{t\to+\infty} \|u(\cdot,t) - u^\infty(\cdot,t)\|_{C^4(\bar{D})} = 0.$$

And $\tilde{u}^\infty(x) \in C^2(\bar{\Omega})$ is a solution of

$$\begin{cases} F(D^2u) - f(x) = C_\infty, & x \in \Omega, \\ h(Du) = 0, & x \in \partial\Omega. \end{cases} \tag{5.4}$$

The constant C_∞ depends only on Ω f, and F. The solution to (5.4) is unique up to additions of constants.

Now, we can give

Proof of Theorem 1.4. This a standard result by our C^2 estimates and uniformly oblique estimates, but for convenience we include here a proof.

Part 1: The long time existence.

By Lemma 4.19, we know the global $C^{2,1}$ estimates for the solutions of the flow (1.1)–(1.3). Using Theorem 14.22 in Lieberman [38] and Lemma 4.5, we can show that the solutions of the oblique derivative problem (3.1) have global $C^{2+\alpha,1+\frac{\alpha}{2}}$ estimates.

Now, let u_0 be a $C^{2+\alpha_0}$ strictly convex function as in the conditions of Theorem 1.4. We assume that T is the maximal time such that the solution to the flow (3.1) exists. Suppose that $T < +\infty$. Combining Proposition 3.6 with Lemma 4.19 and using Theorem 14.23 in [38], there exists $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)$ which satisfies (3.1) and

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)} < +\infty.$$

Then, we can extend the flow (3.1) beyond the maximal time T. So that we deduce that $T = +\infty$. Then, there exists the solution $u(x,t)$ for all times $t > 0$ to (1.1)–(1.3).

Part 2: The convergence.

By the boundary condition, we have

$$\sup_{\bar{\Omega}_T} |Du| \leq \tilde{C}_4,$$

where $\tilde{C}_4$ is a constant depending on Ω and $\tilde{\Omega}$. Using lemma 4.19, it yields

$$\|u_t(\cdot,t)\|_{C(\bar{\Omega})} + \|Du(\cdot,t)\|_{C(\bar{\Omega})} + \|D^2u(\cdot,t)\|_{C(\bar{\Omega})} \leq \tilde{C}_5,$$

where the constant $\tilde{C}_5$ depending only on u_0, Ω, $\tilde{\Omega}$, Λ_1, Λ_2 and δ. By intermediate Schauder estimates for parabolic equations (cf. Lemma 14.6 and Proposition 4.25 in [38]), for any $D \subset\subset \Omega$, we have

$$[D^2 u]_{\alpha, \frac{\alpha}{2}, D_T} \le C \sup_{\Omega_T} |D^2 u| \le \tilde{C}_6,$$

and

$$\sup_{t\ge 1} \|D^3 u(\cdot, t)\|_{C(\bar{D})} + \sup_{t\ge 1} \|D^4 u(\cdot, t)\|_{C(\bar{D})} + \sup_{x_i \in D, t_i \ge 1} \frac{|D^4 u(x_1, t_1) - D^4 u(x_2, t_2)|}{\max\{|x_1 - x_2|^{\alpha}, |t_1 - t_2|^{\frac{\alpha}{2}}\}} \le \tilde{C}_7,$$

where $\tilde{C}_6$, $\tilde{C}_7$ are constants depending on the known data and $\mathrm{dist}(\partial\Omega, \partial D)$. Using Proposition 5.1 and combining the bootstrap arguments as in [32], we finish the proof of Theorem 1.4. □

Finally, we can present

Proof of Theorem 1.1. By Proposition 2.1 and Remark 1.3, we see that Theorem 1.1 is a direct consequence of Theorem 1.4. □

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Data availability The data that support the findings of this study are available from the authors, upon reasonable request.

Declarations

Conflict of interest We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, and there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

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