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Solvability of the Cauchy problem for fractional semilinear parabolic equations in critical and doubly critical cases

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Abstract. Let $0 < \theta \leq 2$, $N \geq 1$ and $T > 0$. We are concerned with the Cauchy problem for the fractional semilinear parabolic equation

$$\begin{cases} \partial_t u + (-\Delta)^{\theta/2} u = f(u) & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \mathbb{R}^N. \end{cases}$$

Here, $f \in C[0, \infty)$ denotes a rather general growing nonlinearity and u_0 may be unbounded. We study local in time solvability in the so-called critical and doubly critical cases. In particular, when $f(u) = u^{1+\theta/N} \left[\log(u + e)\right]^a$, we obtain a sharp integrability condition on u_0 which explicitly determines local in time existence/nonexistence of a nonnegative solution.

1. Introduction

Let $0 < \theta \leq 2$, $N \geq 1$ and $T > 0$. We study existence and nonexistence of a local in time solution of the Cauchy problem for the fractional semilinear parabolic equation

$$\begin{cases} \partial_t u + (-\Delta)^{\theta/2} u = f(u) & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u_0 \geq 0$ and $f \in C[0, \infty)$ is assumed to be nonnegative and nondecreasing.

First, we consider classical semilinear parabolic equations, *i.e.,* $\theta = 2$. When $u_0 \in L^\infty(\mathbb{R}^N)$, (1.1) always has a local in time solution for an arbitrary locally Lipschitz continuous function f (cf. [4,15]). On the other hand, if u_0 is unbounded, then solvability depends on the integrability properties of u_0 and the growth rate of f. Weissler [20] studied solvability of (1.1) with possibly unbounded and sign-changing initial function $u_0 \in L^r(\mathbb{R}^N)$.

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Proposition 1.1. ([20]) *Let $\theta = 2$, $N \geq 1$ and $f(u) = |u|^{p-1}u$, $p > 1$.*

(i) Assume that one of the following holds:
  (1) (subcritical case) $r \geq 1$ and $r > \frac{N}{2}(p-1)$,
  (2) (critical case) $r > 1$ and $r = \frac{N}{2}(p-1)$.
  Then for each $u_0 \in L^r(\mathbb{R}^N)$, (1.1) has a local in time classical solution $u \in C^{2,1}(\mathbb{R}^N \times (0,T)) \cap C([0,T), L^r(\mathbb{R}^N))$.

(ii) (supercritical case) For each $1 \leq r < \frac{N}{2}(p-1)$, there is $u_0 \in L^r(\mathbb{R}^N)$ such that (1.1) has no local in time nonnegative classical solution.

In Proposition 1.1 the classical solution u satisfies $u(t) \to u_0$ in $L^r(\mathbb{R}^N)$ as $t \to 0$.

This proposition was generalized to a wide class of nonlinearities and to various functional spaces. For example, in [8,9] an optimal growth rate of f was obtained such that (1.1) with $\theta = 2$ has a solution for all $u_0 \in L^r(\mathbb{R}^N)$. In this paper we mainly study an optimal integrability condition on u_0 for an existence of a solution of (1.1) when f is given.

We need some notation to mention further studies. For $1 \leq r \leq \infty$, define a uniformly local L^r space $L^r_{\mathrm{ul}}(\mathbb{R}^N)$ by

$$L^r_{\mathrm{ul}}(\mathbb{R}^N) := \left\{ u \in L^r_{\mathrm{loc}}(\mathbb{R}^N);\ \|u\|_{L^r_{\mathrm{ul}}(\mathbb{R}^N)} < \infty \right\}.$$

Here,

$$\|u\|_{L^r_{\mathrm{ul}}(\mathbb{R}^N)} := \begin{cases} \sup_{y\in\mathbb{R}^N} \left(\int_{B(y,1)} |u(x)|^r dx \right)^{1/r} & \text{if } 1 \leq r < \infty, \\ \operatorname{esssup}_{y\in\mathbb{R}^N} \|u\|_{L^\infty(B(y,1))} & \text{if } r = \infty, \end{cases}$$

and $B(y,\rho) := \left\{ x \in \mathbb{R}^N;\ |x-y| < \rho \right\}$. We easily see that $L^\infty_{\mathrm{ul}}(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ and $L^{r_1}_{\mathrm{ul}}(\mathbb{R}^N) \subset L^{r_2}_{\mathrm{ul}}(\mathbb{R}^N)$ if $1 \leq r_2 \leq r_1 < \infty$. For $1 \leq r < \infty$, let $\mathcal{L}^r_{\mathrm{ul}}(\mathbb{R}^N)$ denote the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ in the space $L^r_{\mathrm{ul}}(\mathbb{R}^N)$, *i.e.*,

$$\mathcal{L}^r_{\mathrm{ul}}(\mathbb{R}^N) := \overline{BUC(\mathbb{R}^N)}^{\|\cdot\|_{L^r_{\mathrm{ul}}(\mathbb{R}^N)}}.$$

In [12, Proposition 2.2] we obtain basic properties of $\mathcal{L}^r_{\mathrm{ul}}(\mathbb{R}^N)$. See also [4, Lemma 2.2]. It follows that $\mathcal{L}^r_{\mathrm{ul}}(\mathbb{R}^N) \subsetneq L^r_{\mathrm{ul}}(\mathbb{R}^N)$ for $1 \leq r < \infty$.

In this paper we assume the following:

Assumption A. The function $f \in C^1(0,\infty) \cap C[0,\infty)$ satisfies

$$f(u) > 0,\ f'(u) > 0 \text{ and } F(u) < \infty \ \text{ for } u > 0, \quad \text{where } F(u) := \int_u^\infty \frac{d\tau}{f(\tau)}$$

and

$$\text{the limit } q := \lim_{u\to\infty} f'(u)F(u) \text{ exists.}$$

It was proved in [4,13] that $q \geq 1$ if the limit q exists. Let p denote a growth rate of f defined by $p := \lim_{u\to\infty} \frac{uf'(u)}{f(u)}$. When $q > 1$, by L'Hospital's rule,

$$p = \lim_{u\to\infty} \frac{uf'(u)F(u)}{f(u)F(u)} = q \cdot \lim_{u\to\infty} \frac{(u)'}{(f(u)F(u))'} = \frac{q}{q-1}, \text{ and hence } \frac{1}{p} + \frac{1}{q} = 1.$$

The exponent q can be considered as a conjugate of the growth rate of f. If $q > 1$, then f has an algebraic growth. However, the case $q = 1$ is special. Super-power nonlinearities are included in this case. The following functions satisfy Assumption A with $q = 1$:

$$f(u) = \exp(u^r)\ (r > 0),\ \ f(u) = \exp(\underbrace{\cdots \exp(u) \cdots}_{n \text{ times}}) \text{ and}$$
$$f(u) = \exp(|\log u|^{r-1} \log u)\ (r > 1).$$

Let $G = G(x,t)$ be the fundamental solution of

$$\partial_t u + (-\Delta)^{\theta/2} u = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.2}$$

where $0 < \theta \leq 2$. We recall various properties of G in Section 2 and set

$$[S(t)w](x) := \int_{\mathbb{R}^N} G(x-y,t)w(y)dy \quad \text{for } w \in L^1_{\mathrm{ul}}(\mathbb{R}^N).$$

Fujishima-Ioku [4] studied (1.1) with nonnegative initial function u_0 under Assumption A:

Proposition 1.2. ([4]) *Let $\theta = 2$, $N \geq 1$ and $u_0 \geq 0$. Suppose that f satisfies Assumption A and $f'(u)F(u) \leq q$ for large $u > 0$.*

(i) Assume that one of the following holds:
  (1) (subcritical case) $F(u_0)^{-r} \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ for some $r > \frac{N}{2}$ and $r \geq q-1$,
  (2) (critical case) $F(u_0)^{-r} \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ with $r = \frac{N}{2}$ and $\frac{N}{2} > q-1$.
  Then (1.1) has a local in time nonnegative classical solution $u \in C^{2,1}(\mathbb{R}^N \times (0,T))$ in the following sense:

$$\begin{aligned} &\lim_{t\to 0} \|u(t) - S(t)u_0\|_{L_{ul}^{\frac{r}{q-1}}(\mathbb{R}^N)} = 0 \quad \text{if } q > 1, \\ &\lim_{t\to 0} \|u(t) - S(t)u_0\|_{L^\infty(\mathbb{R}^N)} = 0 \quad \text{if } q = 1. \end{aligned} \tag{1.3}$$

(ii) (supercritical case) Assume that $f \in C^2([0,\infty))$ is convex and $\frac{N}{2} > q-1$. For any $r \in [q-1, \frac{N}{2})$ if $q > 1$ or $r \in (0, \frac{N}{2})$ if $q = 1$, there is a nonnegative initial function u_0 such that $F(u_0)^{-r} \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ and (1.1) has no local in time nonnegative classical solution satisfying (1.3).

Note that in Proposition 1.2 the classical solution u satisfies (1.3). This is a slightly different initial condition from Proposition 1.1.

When $f(u) = u^p$, the condition $F(u_0)^{-r} \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ implies that $u_0 \in L^{r(p-1)}_{\mathrm{ul}}(\mathbb{R}^N)$. Then Proposition 1.2 is a generalization of Proposition 1.1.

We define a local in time solution.

Definition 1.3. (Local in time solution) Let $u_0 \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ be a nonnegative initial function. By a solution of (1.1) on $(0, T)$ we mean that $u(t) \in L^\infty_{\mathrm{loc}}((0, T), L^\infty(\mathbb{R}^N)) \cap L^\infty((0, T), L^1_{\mathrm{ul}}(\mathbb{R}^N))$ and u satisfies

$$\infty > u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds \quad \text{for a.e. } x \in \mathbb{R}^N, \ \text{for } t \in (0, T). \tag{1.4}$$

We call u a supersolution for (1.1) if u is measurable and satisfies (1.4) with $=$ replaced by $\geq$.

In the previous paper of the first author [6] fractional semilinear parabolic equations were studied.

Proposition 1.4. ([6]) *Let* $0 < \theta \leq 2$, $N \geq 1$ *and* $u_0 \geq 0$. *Assume that Assumption A holds.*

(i) (subcritical case) Suppose that $F(u_0)^{-r} \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ *for some* $r > \frac{N}{\theta}$, *where*
- $r > q-1$, *or* $r = q-1$ *and* $f'(u)F(u) \leq q$ *for large* $u > 0$ *if* $q > 1$,
- $f(u)$ *is convex and* $f'(u)F(u) \leq 1$ *for large* $u > 0$ *if* $q = 1$.

Then (1.1) has a local in time nonnegative solution in the sense of Definition 1.3.

(ii) (supercritical case) Assume that f *is convex. For any* $0 < r < \frac{N}{\theta}$, *there is a nonnegative initial function* u_0 *such that* $F(u_0)^{-r} \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ *and (1.1) has no local in time nonnegative solution in the sense of Definition 1.3.*

It should be noted that the definition of a solution in Proposition 1.4 is weaker than that in Proposition 1.2, and hence Proposition 1.4 is not a direct generalization even for $\theta = 2$. See [10,11] for existence and nonexistence results in the Lebesgue space $L^r(\mathbb{R}^N)$.

In the proof of Proposition 1.2 a change of variables $v(x, t) = \mathcal{F}(u(x, t))$ plays a crucial role in constructing a supersolution. However, it does not work for $0 < \theta < 2$, because of the nonlocal term $(-\Delta)^{\theta/2}$. In the proof of Proposition 1.4 a supersolution was constructed without a change of variables. The critical case $r = \frac{N}{\theta}$ was not easy to analyze and it was not covered by Proposition 1.4. In order to study the critical case we study in detail a relationship between the integrability properties of u_0 and the growth rate of f.

We need two functions J and K satisfying the following:

$$J \in C^1[0,\infty),\ \ J(u) > 0 \text{ and } J'(u) > 0 \ \text{ for } u > 0 \quad \text{and } J'(u) \text{ is nondecreasing for large } u > 0. \tag{J}$$

$$K \in C[0,\infty),\ \ K(u) > 0 \text{ for } u > 0 \text{ and } K(u) \text{ is concave and increasing for large } u > 0. \tag{K}$$

The first main result in the present paper is the following:

Theorem 1.5. (Local in time existence) *Let* $0 < \theta \le 2$, $N \ge 1$, $u_0 \ge 0$ *and* $c > 0$. *Suppose that* $f \in C[0,\infty)$ *is nonnegative and nondecreasing, and that* J *satisfies (J). If there exists a function* K *satisfying (K) such that one of the following holds:*

(i) $J(u_0) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ *and*

$$\lim_{v\to\infty} \hat{J}(v) \int_v^\infty \frac{\hat{f}(\tau) J'(\tau) d\tau}{J(\tau)^{1+\frac{\theta}{N}}} = 0, \tag{1.5}$$

(ii) $J(u_0) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ *and*

$$\limsup_{v\to\infty} \hat{J}(v) \int_v^\infty \frac{\hat{f}(\tau) J'(\tau) d\tau}{J(\tau)^{1+\frac{\theta}{N}}} < \infty, \tag{1.6}$$

where

$$\hat{J}(v) := \max_{c\le w\le v} \frac{J'(w) K(J(w))}{J(w)} \ \ \textit{and} \ \ \hat{f}(\tau) := \max_{c\le w\le \tau} \frac{f(w)}{K(J(w))},$$

then there exists $T > 0$ *such that (1.1) has a local in time nonnegative solution* u *in the sense of Definition 1.3 on the interval* $(0, T)$. *Moreover, there exists* $C > 0$ *such that*

$$\|J(u(t))\|_{L^1_{ul}(\mathbb{R}^N)} \le C \quad \textit{for} \quad 0 < t < T. \tag{1.7}$$

When $\theta = 2$, $J(u) = u$ and $K(u) = u$, Theorem 1.5 corresponds to [8, Theorem 4.4].

By Theorem 1.5 we can derive an existence result in the critical case.

Theorem 1.6. (Critical case) *Let* $0 < \theta \le 2$, $N \ge 1$ *and* $u_0 \ge 0$. *Suppose that* f *satisfies Assumption A. If* $F(u_0)^{-N/\theta} \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ *and* $\frac{N}{\theta} > q - 1$, *then (1.1) has a local in time nonnegative solution* u *in the sense of Definition 1.3 on the interval* $(0, T)$. *Moreover, there exists* $C > 0$ *such that*

$$\left\| F(u(t))^{-N/\theta} \right\|_{L^1_{ul}(\mathbb{R}^N)} \le C \quad \textit{for} \quad 0 < t < T. \tag{1.8}$$

Theorem 1.6 corresponds to [4, Theorem 1.1 (ii)] when $\theta = 2$. In [4, Theorem 1.1] they impose $f'(u)F(u) \le q$, which is not assumed in Theorem 1.6. However, it should be noted that the definition of a solution does not include (1.3), which is different from that of [4].

In [6] the critical case is considered only when $f(u) = u^p$ and $f(u) = e^u$. Theorem 1.6 can be applied to a general nonlinear term f.

Let us again go back to the classical case, *i.e.*, $\theta = 2$ and $f(u) = |u|^{p-1}u$. Proposition 1.1 does not cover the case where $r = 1$ and $r = \frac{N}{2}(p-1)$, *i.e.*, $p = 1 + \frac{2}{N}$. This case is called *a doubly critical case* in [2], and is known to be quite delicate. In this case there is a nonnegative initial function $u_0 \in L^1(\mathbb{R}^N)$ such that (1.1) does not have a nonnegative solution. Nonexistence results can be found in [2,3,8,9,21]. On the other hand, the following optimal integrability condition was recently obtained in [14]: (1.1) has a local in time solution if a possibly sign-changing initial function u_0 satisfies $u_0 \in Z_{N/2}$, where

$$Z_r := \left\{ \phi(x) \in L^1(\mathbb{R}^N);\ \int_{\mathbb{R}^N} |\phi| \left(\log(|\phi| + e)\right)^r dx < \infty \right\}.$$

One can check that (1.1) does not have a nonnegative solution with a nonnegative initial function u_0 given in [1], which satisfies $u_0 \in Z_r$ for each $r \in [0, \frac{N}{2})$, and hence $Z_{N/2}$ is optimal. In the case of time-fractional semilinear parabolic equations critical and doubly critical cases are studied in [5].

Now, we also consider the fractional case $0 < \theta < 2$. In the doubly critical case we focus on the nonlinearity

$$f_a(u) := u^{1+\theta/N} \left[\log(u+e)\right]^a \quad \text{and} \quad F_a(u) := \int_u^\infty \frac{d\tau}{f_a(\tau)}, \tag{1.9}$$

where $a \ge -\left(1 + \frac{\theta}{N}\right)\kappa$ and κ is the largest positive root of

$$\log \kappa + 2 = \kappa, \quad i.e., \quad \kappa \simeq 3.146.$$

Since $a \ge -\left(1 + \frac{\theta}{N}\right)\kappa$, we can check that $f_a'(u) \ge 0$ for $u \ge 0$.

Using Theorem 1.5, we give an optimal integrability condition on u_0 for the nonlinearity f_a and obtain a complete classification for an existence and nonexistence result.

Theorem 1.7. *(Doubly critical case) Let* $0 < \theta \le 2$, $N \ge 1$, $u_0 \ge 0$ *and* $a \ge -\left(1 + \frac{\theta}{N}\right)\kappa$. *For* $b \ge 0$, *we set*

$$J_{a,b}(u) := F_a(u)^{-N/\theta} \left[\log\left(F_a(u)^{-N/\theta} + e\right)\right]^b,$$

where $F_a(u)$ *is defined by (1.9). Then the following hold:*

(i) $\underline{a > -1}$

(a) *(Existence) If* $J_{a,b}(u_0) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ *for some* $b > \frac{N}{\theta}$ *or* $J_{a,b}(u_0) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ *with* $b = \frac{N}{\theta}$, *then (1.1) with* $f(u) = f_a(u)$ *has a local in time nonnegative solution in the sense of Definition 1.3.*

(b) *(Nonexistence) For each* $b \in [0, \frac{N}{\theta})$, *there exists a nonnegative function* $u_0 \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ *satisfying* $J_{a,b}(u_0) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ *such that, for every* $T > 0$, *(1.1) with* $f(u) = f_a(u)$ *admits no local in time nonnegative solution in the sense of Definition 1.3 on* $(0, T)$.

(ii) $\underline{a = -1}$

(a) *(Existence) If* $J_{a,b}(u_0) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ *for some* $b > \frac{N}{\theta}$, *then (1.1) with* $f(u) = f_a(u)$ *has a local in time nonnegative solution in the sense of Definition 1.3.*

(b) *(Nonexistence) For each* $b \in [0, \frac{N}{\theta}]$, *there exists a nonnegative function* $u_0 \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ *satisfying* $J_{a,b}(u_0) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ *such that, for every* $T > 0$, *(1.1) with* $f(u) = f_a(u)$ *admits no local in time nonnegative solution in the sense of Definition 1.3 on* $(0, T)$.

(iii) $\underline{-\left(1 + \frac{\theta}{N}\right)\kappa \le a < -1}$

If $u_0 \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$, *then (1.1) with* $f(u) = f_a(u)$ *has a local in time nonnegative solution in the sense of Definition 1.3.*

Theorem 1.7 corresponds to [14, Theorem 1.3] when $\theta = 2$ and $a = 0$.

As mentioned above, the doubly critical case is quite delicate. In Theorem 1.7 (i) and (ii) the borderline value of b is $\frac{N}{\theta}$. However, there exists a nonlinear term such that the borderline value of b is less than or equal to $\frac{N}{2\theta}$. Thus, $b = \frac{N}{\theta}$ is not necessarily a critical exponent in the doubly critical case. See Section 5 for details.

Let us explain a sketch of the proofs. The main points of the proofs are a supersolution for the existence part and the contradiction argument for the nonexistence part.

The proof of Theorem 1.5 proceeds in a similar manner to [19]. We construct a function with u_0, $S(t)$ and J. In order to show that this is indeed a supersolution, we estimate an integral term corresponding to (1.4) and the other term. Theorem 1.6 and the existence part of Theorem 1.7 are shown by applying Theorem 1.5.

In the proof of Theorem 1.7 (i) (b) we improve the method of [7] to obtain an upper bound of the integral value of constructed u_0 over $B(0, \rho)$. Since the singularity of u_0 is strong, this integral value increases faster than the upper bound as $\rho \to 0$. This is a contradiction. Theorem 1.7 (ii) (b) is based on the nonexistence result of [8]. Specifically, the divergence of the L^1-norm of the integral term in (1.4) as $t \to 0$ causes a contradiction.

This paper is organized as follows. In Section 2 we show some properties of $S(t)$ and $\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^N)$, and recall properties of the fundamental solution G. In Section 3 we use these properties and prove Theorems 1.5, 1.6 and 1.7 (i) (a), (ii) (a) and (iii). In Section 4 we prove Theorem 1.7 (i) (b) and (ii) (b) by contradiction. In Section 5 we discuss local in time solvability with a condition obtained from Theorem 1.5.

2. Preliminaries

After this section, let $0 < \theta \le 2$ and $N \ge 1$ unless otherwise noted.

Proposition 2.1. (Monotone iterative method) *Let $u_0 \ge 0$ and $0 < T < \infty$. Suppose that $f \in C[0,\infty)$ is nonnegative and nondecreasing. If there exists a nonnegative function $\bar{u} \in L^\infty_{\mathrm{loc}}((0,T), L^\infty(\mathbb{R}^N)) \cap L^\infty((0,T), L^1_{\mathrm{ul}}(\mathbb{R}^N))$ such that*

$$\bar{u}(t) \ge [\mathcal{F}(\bar{u})](t) := S(t)u_0 + \int_0^t S(t-s) f(\bar{u}(s))ds \quad \text{for a.e. } x \in \mathbb{R}^N, \text{ for } t \in (0,T),$$

then (1.1) has a solution u in the sense of Definition 1.3 on the interval $(0,T)$ and $0 \le u(t) \le \bar{u}(t)$.

We show the proof for readers' convenience. See [16, Theorem 2.1] for details.

Proof. Put $u_1 := S(t)u_0$ and $u_n := \mathcal{F}(u_{n-1})$ for $n = 2, 3, \ldots$. Let $t \in (0,T)$. By induction we have

$$0 \le u_1(t) \le u_2(t) \le \cdots \le u_n(t) \le \cdots \le \bar{u}(t) < \infty \quad \text{for a.e. } x \in \mathbb{R}^N.$$

This indicates that the limit $\lim_{n\to\infty} u_n(x,t)$ which is denoted by $u(x,t)$ exists for a.e. $x \in \mathbb{R}^N$, for $t \in (0,T)$. Then it follows from the monotone convergence theorem that

$$\lim_{n\to\infty} \mathcal{F}(u_{n-1}) = \mathcal{F}(u),$$

and hence $u = \mathcal{F}(u)$. Moreover, we obtain $0 \le u(t) \le \bar{u}(t)$. □

We recall useful properties of the fundamental solution G of (1.2). It is represented by

$$G(x,t) = \begin{cases} (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{if } \theta = 2, \\ \int_0^\infty g_{t,\frac{\theta}{2}}(s)(4\pi s)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4s}\right) ds & \text{if } 0 < \theta < 2, \end{cases}$$

where $g_{t,\frac{\theta}{2}}(s)$ is a nonnegative function on $[0,\infty)$ defined by

$$g_{t,\frac{\theta}{2}}(s) := \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \exp\left(zs - tz^{\frac{\theta}{2}}\right) dz, \quad \sigma > 0,\ t > 0.$$

The fundamental solution G is a positive smooth function in $\mathbb{R}^N \times (0,\infty)$. See [6,7,17]. Moreover, G has the following properties:

$$G(x,t) = t^{-\frac{N}{\theta}} G(t^{-\frac{1}{\theta}}x, 1), \tag{2.1}$$

$$C^{-1}(1+|x|)^{-N-\theta} \le G(x,1) \le C(1+|x|)^{-N-\theta} \quad \text{if } 0 < \theta < 2, \tag{2.2}$$

$$G(\cdot,1) \text{ is radially symmetric and } G(x,1) \le G(y,1) \text{ if } |x| \ge |y|, \tag{2.3}$$

$$G(x,t)=\int_{\mathbb{R}^N} G(x-y,t-s)G(y,s)dy, \tag{2.4}$$

$$\int_{\mathbb{R}^N} G(x,t)dx=1 \tag{2.5}$$

for $x, y \in \mathbb{R}^N$ and $0 < s < t$.

Proposition 2.2. (cf. [6, Propositions 2.4 and 2.5]) *The following (i) and (ii) hold:*

(i) Let $1 \le p \le q \le \infty$. Then there exists $C > 0$ such that

$$\|S(t)w\|_{L^q_{\mathrm{ul}}(\mathbb{R}^N)} \le C\left(t^{-\frac{N}{\theta}\left(\frac{1}{p}-\frac{1}{q}\right)}+1\right)\|w\|_{L^p_{\mathrm{ul}}(\mathbb{R}^N)}$$

for $t > 0$ and $w \in L^p_{\mathrm{ul}}(\mathbb{R}^N)$. In particular, there exists $C > 0$ such that

$$\|S(t)w\|_{L^q_{\mathrm{ul}}(\mathbb{R}^N)} \le Ct^{-\frac{N}{\theta}\left(\frac{1}{p}-\frac{1}{q}\right)}\|w\|_{L^p_{\mathrm{ul}}(\mathbb{R}^N)}$$

for small $t > 0$ and $w \in L^p_{\mathrm{ul}}(\mathbb{R}^N)$.

(ii) Let $1 \le p < q \le \infty$, $C_* > 0$ and $w \in \mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^N)$. Then there exists $t_0 = t_0\left(C_*, \|w\|_{L^p_{\mathrm{ul}}(\mathbb{R}^N)}, \frac{N}{\theta}\left(\frac{1}{p}-\frac{1}{q}\right)\right)$ such that

$$\|S(t)w\|_{L^q_{ul}(\mathbb{R}^N)} \le C_* t^{-\frac{N}{\theta}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for} \quad 0 < t < t_0.$$

Note that the assertion (i) follows from [12, Corollary 3.1] with minor modifications and that the assertion (ii) is proved on the basis of the proof in [2, Lemma 8]. In the assertion (ii) the constant C_* can be taken arbitrarily small.

Proposition 2.3. *Let $M \ge 0$. Then the following (i) and (ii) hold:*

(i) Suppose that $J \in C[M,\infty)$ is a nonnegative convex function. If $v \ge M$ in $\mathbb{R}^N$, $v \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ and $J(v) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$, then

$$J([S(t)v](x)) \le [S(t)J(v)](x) \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

(ii) Suppose that $K \in C[M,\infty)$ is a nonnegative increasing concave function. If $w \ge M$ in $\mathbb{R}^N$ and $w \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$, then

$$K([S(t)w](x)) \ge [S(t)K(w)](x) \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

Proof. We prove (i). Let $(x,t) \in \mathbb{R}^N \times (0,\infty)$. By (2.5) we have $\int_{\mathbb{R}^N} G(x-y,t)dy = 1$. Then it follows from Jensen's inequality that

$$\begin{aligned} J([S(t)v](x)) &= J\left(\int_{\mathbb{R}^N} G(x-y,t)v(y)dy\right) \\ &\le \int_{\mathbb{R}^N} G(x-y,t)J(v(y))dy = [S(t)J(v)](x). \end{aligned}$$

We prove (ii). It suffices to prove the case where $M = 0$. The assertion with $M = 0$ is assumed to hold. We consider the general case where $M \geq 0$. Put $\hat{K}(u) := K(u + M)$ and $\hat{w} := w - M$. We see that $\hat{K} \in C[0, \infty)$, $0 \leq \hat{w} \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ and $\hat{K}(\hat{w}) = K(w)$ hold. Then it follows that

$$\hat{K}([S(t)\hat{w}](x)) \geq [S(t)\hat{K}(\hat{w})](x),$$

which yields

$$K([S(t)(w - M)](x) + M) \geq [S(t)K(w)](x).$$

We deduce from $[S(t)M](x) = M \int_{\mathbb{R}^N} G(x - y, t)dy = M$ in $\mathbb{R}^N \times (0, \infty)$ that

$$K([S(t)w](x)) \geq [S(t)K(w)](x) \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

In order to prove (ii) with $M = 0$, we start with the case where $K \in C[0, n]$ and $0 \leq w \leq n$ in $\mathbb{R}^N$ for each $n \in \{1, 2, \ldots\}$. Since K is increasing and concave, $K^{-1} \in C[K(0), K(n)]$ is convex. Note that the inequality in (i) holds when $J \in C[M, L]$ and $M \leq v \leq L$ in $\mathbb{R}^N$, where $M < L$. Then we obtain from the assertion (i) that for $K(0) \leq v \leq K(n)$ in $\mathbb{R}^N$,

$$K^{-1}([S(t)v](x)) \leq [S(t)K^{-1}(v)](x) \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

Let $w := K^{-1}(v)$. By the monotonicity of K we have $0 \leq w \leq n$ in $\mathbb{R}^N$ and

$$[S(t)K(w)](x) \leq K([S(t)w](x)) \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

Thus the desired inequality can be derived in the case where $K \in C[0, n]$ and $0 \leq w \leq n$ in $\mathbb{R}^N$. Then we can consider the case where $K \in C[0, \infty)$ and $w \geq 0$ in $\mathbb{R}^N$. For $n \in \{1, 2, \ldots\}$, define $w_n := \min\{w, n\}$. We see that

$$[S(t)K(w_n)](x) \leq K([S(t)w_n](x)) \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

Taking $n \to \infty$, we also obtain the desired inequality in this case, which follows from the monotone convergence theorem and $K \in C[0, \infty)$. □

Lemma 2.4. *Let* $M > 0$. *If* $w \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$, *then* $\max\{w, M\} \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$.

Note that Lemma 2.4 follows from [18, Lemma 2.5 (ii)] with $p = 1$.

3. Existence result

After this section, for any set X and the mappings $a = a(x)$ and $b = b(x)$ from X to $[0, \infty)$, we say

$$a(x) \lesssim b(x) \quad \text{for all } x \in X$$

if there exists a positive constant C such that $a(x) \leq Cb(x)$ for all $x \in X$.

Proof of Theorem 1.5. Let $M > c$ be large such that $J(u)$ is convex for $u \geq M$. Put $u_1(x) := \max\{u_0(x), M\}$. In the case (i) $J(u_1) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ holds. In the case (ii) by Lemma 2.4 $J(u_1) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ holds. Let $\sigma > 0$ be a constant. We define

$$\bar{u}(t) := J^{-1}\left((1+\sigma)S(t)J(u_1)\right). \tag{3.1}$$

Since $J(u_1) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$, the latter estimate in Proposition 2.2 (i) implies that for small $t > 0$,

$$\|\bar{u}(t)\|_{L^\infty(\mathbb{R}^N)} \leq J^{-1}\left((1+\sigma)\cdot Ct^{-\frac{N}{\theta}}\,\|J(u_1)\|_{L^1_{\mathrm{ul}}(\mathbb{R}^N)}\right) =: v(t) < \infty. \tag{3.2}$$

Moreover, for $T > 0$, $J(\bar{u}(t)) \in L^\infty_{\mathrm{loc}}((0,T), L^\infty(\mathbb{R}^N)) \cap L^\infty((0,T), L^1_{\mathrm{ul}}(\mathbb{R}^N))$ follows from the former estimate in Proposition 2.2 (i). Since $J(u)$ is convex for $u \geq M$ and $\bar{u}(t) \geq M$, we obtain $J'(M)(\bar{u}(t)-M)+J(M) \leq J(\bar{u}(t))$, which yields

$$\bar{u}(t) \in L^\infty_{\mathrm{loc}}((0,T), L^\infty(\mathbb{R}^N)) \cap L^\infty((0,T), L^1_{\mathrm{ul}}(\mathbb{R}^N)) \quad \text{for } T > 0.$$

It follows from Proposition 2.3 (i) and the monotonicity of J that

$$[S(t)u_1](x) \leq J^{-1}\left([S(t)J(u_1)](x)\right) \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

This along with the mean value theorem yields

$$\begin{aligned}\bar{u}(t) - S(t)u_0 &\geq \bar{u}(t) - S(t)u_1\\ &\geq \bar{u}(t) - J^{-1}\left(S(t)J(u_1)\right)\\ &= J^{-1}\left((1+\sigma)S(t)J(u_1)\right) - J^{-1}\left(S(t)J(u_1)\right)\\ &= (J^{-1})'\left((1+\rho\sigma)S(t)J(u_1)\right)\sigma S(t)J(u_1)\end{aligned} \tag{3.3}$$

for some $\rho = \rho(x,t) \in [0,1]$. Since $J(u)$ is convex for $u \geq M$, $J^{-1}(u)$ is concave for $u \geq J(M)$. We have

$$\begin{aligned}(J^{-1})'\left((1+\rho\sigma)S(t)J(u_1)\right)\sigma S(t)J(u_1) &\geq (J^{-1})'\left((1+\sigma)S(t)J(u_1)\right)\sigma S(t)J(u_1)\\ &= \frac{\sigma S(t)J(u_1)}{J'\left(J^{-1}\left((1+\sigma)S(t)J(u_1)\right)\right)}\\ &= \frac{\sigma}{1+\sigma}\frac{J(\bar{u}(t))}{J'(\bar{u}(t))}.\end{aligned} \tag{3.4}$$

By (3.3) and (3.4) we have

$$\bar{u}(t) - S(t)u_0 \geq \frac{\sigma}{1+\sigma}\frac{J(\bar{u}(t))}{J'(\bar{u}(t))}. \tag{3.5}$$

On the other hand, let $t > 0$ be small. We see that

$$\int_0^t S(t-s)f(\bar{u}(s))ds \le \int_0^t \left\| \frac{f(\bar{u}(s))}{K(J(\bar{u}(s)))} \right\|_{L^\infty(\mathbb{R}^N)} S(t-s)K(J(\bar{u}(s)))ds. \tag{3.6}$$

Let $s \in (0, t)$. It follows from Proposition 2.3 (ii) that

$$\begin{aligned} S(t-s)K(J(\bar{u}(s))) &= S(t-s)\{K((1+\sigma)S(s)J(u_1))\} \\ &\le K(S(t-s)\{(1+\sigma)S(s)J(u_1)\}) \\ &\le K((1+\sigma)S(t)J(u_1)) \\ &= K(J(\bar{u}(t))). \end{aligned} \tag{3.7}$$

Using (3.6) and (3.7), we have

$$\begin{aligned} \int_0^t S(t-s)f(\bar{u}(s))ds &\le K(J(\bar{u}(t))) \int_0^t \left\| \frac{f(\bar{u}(s))}{K(J(\bar{u}(s)))} \right\|_{L^\infty(\mathbb{R}^N)} ds \\ &\le \frac{J(\bar{u}(t))}{J'(\bar{u}(t))} \hat{J}(\bar{u}(t)) \int_0^t \hat{f}\left(\|\bar{u}(s)\|_{L^\infty(\mathbb{R}^N)}\right) ds. \end{aligned} \tag{3.8}$$

We prove the case (i). It follows from (3.2) that

$$\begin{aligned} &\hat{J}(\bar{u}(t)) \int_0^t \hat{f}\left(\|\bar{u}(s)\|_{L^\infty(\mathbb{R}^N)}\right) ds \le \hat{J}(v(t)) \int_0^t \hat{f}(v(s))ds \\ &= \frac{\theta}{N}\{C(1+\sigma)\|J(u_1)\|_{L^1_{\mathrm{ul}}(\mathbb{R}^N)}\}^{\frac{\theta}{N}} \hat{J}(v(t)) \int_{v(t)}^\infty \frac{\hat{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+\frac{\theta}{N}}}, \end{aligned} \tag{3.9}$$

where we used a change of variables $\tau := v(s) = J^{-1}\left((1+\sigma)\cdot Cs^{-\frac{N}{\theta}}\|J(u_1)\|_{L^1_{\mathrm{ul}}(\mathbb{R}^N)}\right)$. Since $t > 0$ is small, it follows from (1.5) that

$$\frac{\theta}{N}\{C(1+\sigma)\|J(u_1)\|_{L^1_{\mathrm{ul}}(\mathbb{R}^N)}\}^{\frac{\theta}{N}} \hat{J}(v(t)) \int_{v(t)}^\infty \frac{\hat{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+\frac{\theta}{N}}} \le \frac{\sigma}{1+\sigma}. \tag{3.10}$$

Due to (3.8), (3.9) and (3.10), we obtain

$$\int_0^t S(t-s)f(\bar{u}(s))ds \le \frac{\sigma}{1+\sigma} \frac{J(\bar{u}(t))}{J'(\bar{u}(t))}. \tag{3.11}$$

By (3.5) and (3.11) we have

$$\bar{u}(t) - S(t)u_0 \ge \int_0^t S(t-s)f(\bar{u}(s))ds. \tag{3.12}$$

Thus $\bar{u}(t)$ is a supersolution. By Proposition 2.1 there exists $T > 0$ such that (1.1) has a local in time nonnegative solution u in the sense of Definition 1.3 on the interval $(0, T)$ and $0 \le u(t) \le \bar{u}(t)$. Since $\sigma > 0$ is a constant, (1.7) follows from (3.1).

We prove the case (ii). In the case (ii) the inequality (3.8) also holds. Since $J(u_1) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$, it follows from Proposition 2.2 (ii) that (3.2) holds with C and $\|J(u_1)\|_{L^1_{\mathrm{ul}}(\mathbb{R}^N)}$

replaced by C_* and 1, respectively. By the same calculation we have (3.9) with the same replacement. Due to (1.6) and Proposition 2.2 (ii), we can take a small $C_* > 0$ such that for small $t > 0$,

$$\frac{\theta}{N}\{C_*(1+\sigma)\}^{\frac{\theta}{N}}\hat{J}(v(t))\int_{v(t)}^{\infty}\frac{\hat{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+\frac{\theta}{N}}} \le \frac{\sigma}{1+\sigma}. \tag{3.13}$$

Then the inequality (3.11) follows from (3.8), (3.9) with the same replacement and (3.13). By (3.5) and (3.11) we have (3.12). Thus $\bar{u}(t)$ is a supersolution. The rest of the proof is the same as (i). We complete the proof. □

Lemma 3.1. *Suppose that f satisfies Assumption A. Then the following (i) and (ii) hold:*

(i) If $a > q$, then $f(u)F(u)^a \to 0$ as $u \to \infty$.

(ii) If $b > q-1$, then $uF(u)^b \to 0$ and $\int_u^\infty F(\tau)^b d\tau \to 0$ as $u \to \infty$.

Proof. We prove (i). Let $\varepsilon > 0$ be chosen such that $a > q + \varepsilon$. It follows from $q = \lim_{u\to\infty} f'(u)F(u)$ that

$$\frac{d}{du}\left(f(u)F(u)^{a-\varepsilon}\right) = F(u)^{a-\varepsilon-1}(f'(u)F(u) - a + \varepsilon) < 0 \quad \text{for large } u > 0.$$

Then $f(u)F(u)^{a-\varepsilon}$ is decreasing for large $u > 0$. This along with $F(u)^\varepsilon \to 0$ as $u \to \infty$ yields $f(u)F(u)^a \to 0$ as $u \to \infty$.

We prove (ii). By L'Hospital's rule and the assertion (i) we have

$$\lim_{u\to\infty} uF(u)^b = \lim_{u\to\infty}\frac{\frac{d}{du}u}{\frac{d}{du}(F(u)^{-b})} = \lim_{u\to\infty}\frac{1}{b}f(u)F(u)^{b+1} = 0.$$

Moreover, it follows from $b > q-1$ that we choose $\delta > 0$ such that $\frac{b}{1+\delta} > q-1$. Then we also obtain $\lim_{u\to\infty} uF(u)^{\frac{b}{1+\delta}} = 0$, and hence $F(u)^b \lesssim u^{-1-\delta}$ for large $u > 0$. This leads to

$$\int_u^\infty F(\tau)^b d\tau \lesssim \int_u^\infty \tau^{-1-\delta}d\tau = \frac{u^{-\delta}}{\delta} \to 0 \quad \text{as } u \to \infty.$$

We complete the proof. □

Proof of Theorem 1.6. Put $r := \frac{N}{\theta}$. Since $q < r+1$, we can choose $\alpha \in \left(\frac{q-1}{r}, \min\{1, \frac{q}{r}\}\right)$. Set $J(u) := F(u)^{-r}$ and $K(u) := u^\alpha$. By direct calculation we have

$$J'(u) = \frac{r}{f(u)F(u)^{r+1}} > 0 \text{ and } J''(u) = \frac{r(r+1-f'(u)F(u))}{f(u)^2F(u)^{r+2}} \quad \text{for } u > 0.$$

Due to $r+1 > q = \lim_{u\to\infty} f'(u)F(u)$, we obtain $r+1-f'(u)F(u) > 0$ for large $u > 0$. Then $J''(u) > 0$ for large $u > 0$ and J satisfies (J).

We show that $\frac{J'(w)}{J(w)^{1-\alpha}}\left(=\frac{J'(w)K(J(w))}{J(w)}\right)$ and $\frac{f(w)}{J(w)^{\alpha}}\left(=\frac{f(w)}{K(J(w))}\right)$ are nondecreasing for large $w>0$. It follows that

$$\frac{d}{dw}\left(\frac{J'(w)}{J(w)^{1-\alpha}}\right)=(1-\alpha)J(w)^{\alpha-1}J''(w)\left(\frac{1}{1-\alpha}-\frac{J'(w)^2}{J(w)J''(w)}\right).$$

Here we deduce

$$\frac{J'(w)^2}{J(w)J''(w)}=\frac{r}{r+1-f'(w)F(w)}\to\frac{r}{r+1-q}\quad\text{as } w\to\infty.$$

Since $\alpha>\frac{q-1}{r}$, $\frac{1}{1-\alpha}-\frac{r}{r+1-q}>0$ holds and hence $\frac{d}{dw}\left(\frac{J'(w)}{J(w)^{1-\alpha}}\right)>0$ for large $w>0$. In addition, since $\alpha<\frac{q}{r}$,

$$\frac{d}{dw}\left(\frac{f(w)}{J(w)^{\alpha}}\right)=(f'(w)F(w)-r\alpha)F(w)^{r\alpha-1}>0\quad\text{for large } w>0.$$

Thus we can take a sufficiently large $c>0$ such that $\frac{J'(w)}{J(w)^{1-\alpha}}$ and $\frac{f(w)}{J(w)^{\alpha}}$ are nondecreasing for $w\geq c$. If $v>c$, then

$$\hat{J}(v)\int_v^\infty\frac{\hat{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+\frac{\theta}{N}}}=\frac{J'(v)}{J(v)^{1-\alpha}}\int_v^\infty\frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{1+\alpha+\frac{\theta}{N}}}=\frac{r^2}{f(v)F(v)^{r\alpha+1}}\int_v^\infty F(\tau)^{r\alpha}d\tau.$$

Note that we use $J(\tau)^{1+\alpha+\frac{\theta}{N}}=J(\tau)^{1+\alpha+\frac{1}{r}}$. Since $r\alpha+1>q$, it follows from Lemma 3.1 that $f(v)F(v)^{r\alpha+1}$ and $\int_v^\infty F(\tau)^{r\alpha}d\tau$ converge to 0 as $v\to\infty$. Then L'Hospital's rule is applicable and we obtain

$$\lim_{v\to\infty}\frac{\int_v^\infty F(\tau)^{r\alpha}d\tau}{f(v)F(v)^{r\alpha+1}}=\lim_{v\to\infty}\frac{\frac{d}{dv}\left(\int_v^\infty F(\tau)^{r\alpha}d\tau\right)}{\frac{d}{dv}\left(f(v)F(v)^{r\alpha+1}\right)}=\frac{1}{r\alpha+1-q}.$$

By Theorem 1.5 (ii) there exists $T>0$ such that (1.1) has a local in time nonnegative solution u in the sense of Definition 1.3 on the interval $(0,T)$. Moreover, (1.8) holds. □

Lemma 3.2. *Let $0<\theta\leq 2$, $N\geq 1$ and $a\in\mathbb{R}$. Let f_a and F_a be defined by (1.9). Then the following hold:*

$$\lim_{u\to\infty}\frac{\log\left(F_a(u)^{-\frac{N}{\theta}}+e\right)}{\log(u+e)}=1\ \textit{and}$$

$$\lim_{u\to\infty}\left(1+\frac{N}{\theta}-f_a'(u)F_a(u)\right)\log(u+e)=\left(\frac{N}{\theta}\right)^2a.$$

In particular, $\lim_{u\to\infty}\left(1+\frac{N}{\theta}-f_a'(u)F_a(u)\right)\log\left(F_a(u)^{-\frac{N}{\theta}}+e\right)=\left(\frac{N}{\theta}\right)^2a.$

Proof. By direct calculation we have

$$\frac{\frac{d}{du}\log\left(F_a(u)^{-\frac{N}{\theta}}+e\right)}{\frac{d}{du}\log(u+e)}=\frac{N}{\theta}\frac{u+e}{f_a(u)F_a(u)}\cdot\frac{1}{1+eF_a(u)^{\frac{N}{\theta}}}$$

and

$$\frac{\frac{d}{du}(u+e)}{\frac{d}{du}(f_a(u)F_a(u))}=\frac{1}{f_a'(u)F_a(u)-1}.$$

Since $f_a'(u)F_a(u)\to 1+\frac{N}{\theta}$ as $u\to\infty$, it follows that $\lim_{u\to\infty}\frac{d}{du}(f_a(u)F_a(u))=\frac{N}{\theta}>0$, and hence $f_a(u)F_a(u)\to\infty$ as $u\to\infty$. Then we obtain from L'Hospital's rule that

$$\begin{aligned}\lim_{u\to\infty}\frac{\log\left(F_a(u)^{-\frac{N}{\theta}}+e\right)}{\log(u+e)}&=\frac{N}{\theta}\lim_{u\to\infty}\frac{u+e}{f_a(u)F_a(u)}\cdot\lim_{u\to\infty}\frac{1}{1+eF_a(u)^{\frac{N}{\theta}}}\\&=\frac{N}{\theta}\lim_{u\to\infty}\frac{1}{f_a'(u)F_a(u)-1}=1.\end{aligned}$$

Next we mention the latter limit. With the help of integration by parts we have

$$F_a(u)=\frac{N}{\theta}u^{-\frac{\theta}{N}}\left[\log(u+e)\right]^{-a}-\frac{N}{\theta}aI(u),$$

where

$$I(u):=\int_u^\infty\tau^{-\frac{\theta}{N}}(\tau+e)^{-1}\left[\log(\tau+e)\right]^{-a-1}d\tau.$$

Then it follows that

$$\begin{aligned}f_a'(u)F_a(u)&=1+\frac{N}{\theta}+\frac{N}{\theta}a\frac{u}{u+e}\left[\log(u+e)\right]^{-1}\\&\quad-\left(1+\frac{N}{\theta}\right)au^{\frac{\theta}{N}}\left[\log(u+e)\right]^{a}I(u)-\frac{N}{\theta}a^2\frac{u^{1+\frac{\theta}{N}}}{u+e}\left[\log(u+e)\right]^{a-1}I(u),\end{aligned}$$

which yields

$$\begin{aligned}&\left(1+\frac{N}{\theta}-f_a'(u)F_a(u)\right)\log(u+e)\\&=\left(1+\frac{N}{\theta}\right)au^{\frac{\theta}{N}}\left[\log(u+e)\right]^{a+1}I(u)+\frac{N}{\theta}a^2\frac{u^{1+\frac{\theta}{N}}}{u+e}\left[\log(u+e)\right]^{a}I(u)-\frac{N}{\theta}a\frac{u}{u+e}.\end{aligned}$$

We observe from L'Hospital's rule that

$$\lim_{u\to\infty}u^{\frac{\theta}{N}}\left[\log(u+e)\right]^{a+1}I(u)=\lim_{u\to\infty}\frac{I'(u)}{\frac{d}{du}\left\{u^{-\frac{\theta}{N}}\left[\log(u+e)\right]^{-a-1}\right\}}=\frac{N}{\theta},$$

which implies that $\lim_{u\to\infty}\frac{u^{1+\frac{\theta}{N}}}{u+e}\left[\log(u+e)\right]^{a}I(u)=0$. Then by calculation we can derive the desired limit. □

Proof of the existence part in Theorem 1.7. We prove (i) (a) and (ii) (a). Put $J(u) := J_{a,b}(u)$. In order to apply Theorem 1.5 with $K(u) = u$ and a sufficiently large $c > 0$, it suffices to show the following properties:

(A) $J''(v) > 0$ for large $v > 0$,

(B) $\hat{J}(v) = J'(v)$ and $\hat{f}(\tau) = \frac{f_a(\tau)}{J(\tau)}$,

(C) $J'(v) \to \infty$ and $\int_v^\infty \frac{f_a(\tau)J'(\tau)d\tau}{J(\tau)^{2+\frac{\theta}{N}}} \to 0$ as $v \to \infty$,

(D) $\lim_{v\to\infty} \frac{f_a(v)J'(v)^3}{J(v)^{2+\frac{\theta}{N}}J''(v)} = 0$ for $b > \frac{N}{\theta}$ (when $a \geq -1$ and $J(u_0) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$),

(E) $\lim_{v\to\infty} \frac{f_a(v)J'(v)^3}{J(v)^{2+\frac{\theta}{N}}J''(v)} < \infty$ with $b = \frac{N}{\theta}$ (when $a > -1$ and $J(u_0) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$).

Indeed, (C) and (D) imply that we can apply L'Hospital's rule and obtain

$$\lim_{v\to\infty} J'(v) \int_v^\infty \frac{f_a(\tau)J'(\tau)d\tau}{J(\tau)^{2+\frac{\theta}{N}}} = \lim_{v\to\infty} \frac{\frac{d}{dv}\left(\int_v^\infty \frac{f_a(\tau)J'(\tau)d\tau}{J(\tau)^{2+\frac{\theta}{N}}}\right)}{\frac{d}{dv}\left(\frac{1}{J'(v)}\right)}$$

$$= \lim_{v\to\infty} \frac{f_a(v)J'(v)^3}{J(v)^{2+\frac{\theta}{N}}J''(v)} = 0.$$

By this together with (A) and (B) we see that all the assumptions of Theorem 1.5 (i) are satisfied. In the same way, if (E) holds instead of (D), then Theorem 1.5 (ii) can be applied.

Note that we omit the details of the former of (B) and the former of (C), since they can be seen from (A).

We start with the latter of (B). By direct calculation we have

$$\frac{d}{dv}\left(\frac{f_a(v)}{J(v)}\right) = \frac{f_a'(v)J(v) - f_a(v)J'(v)}{J(v)^2}.$$

Put $h(v) := F_a(v)^{-\frac{N}{\theta}}$. It follows that $J(v) = h(v)\left[\log\left(h(v)+e\right)\right]^b$ and

$$J'(v) = C_1(v)\frac{h(v)^{1+\frac{\theta}{N}}}{f_a(v)}\left[\log\left(h(v)+e\right)\right]^b, \tag{3.14}$$

where

$$C_1(v) := \frac{N}{\theta}\left\{1 + \frac{bh(v)}{(h(v)+e)\log\left(h(v)+e\right)}\right\} \to \frac{N}{\theta} \quad \text{as } v \to \infty. \tag{3.15}$$

Since $f_a'(v) \geq \left(1+\frac{\theta}{N}\right)v^{\frac{\theta}{N}}\left[\log(v+e)\right]^a$ and (3.14) hold, we have

$$\begin{aligned} &f_a'(v)J(v) - f_a(v)J'(v) \\ &\geq \left(1+\frac{\theta}{N}\right)v^{\frac{\theta}{N}}\left[\log(v+e)\right]^a \cdot h(v)\left[\log\left(h(v)+e\right)\right]^b - C_1(v)h(v)^{1+\frac{\theta}{N}}\left[\log\left(h(v)+e\right)\right]^b \\ &= h(v)^{1+\frac{\theta}{N}}\left[\log\left(h(v)+e\right)\right]^b\left\{\left(1+\frac{\theta}{N}\right)v^{\frac{\theta}{N}}\left[\log(v+e)\right]^a \cdot h(v)^{-\frac{\theta}{N}} - C_1(v)\right\}. \end{aligned} \tag{3.16}$$

We see from L'Hospital's rule that

$$\lim_{v\to\infty} v^{\frac{\theta}{N}} \left[\log(v+e)\right]^a \cdot h(v)^{-\frac{\theta}{N}} = \lim_{v\to\infty} \frac{F_a'(v)}{\frac{d}{dv}\left\{v^{-\frac{\theta}{N}}\left[\log(v+e)\right]^{-a}\right\}} = \frac{N}{\theta}. \quad (3.17)$$

By (3.15) and (3.17) we derive

$$\left(1+\frac{\theta}{N}\right) v^{\frac{\theta}{N}} \left[\log(v+e)\right]^a \cdot h(v)^{-\frac{\theta}{N}} - C_1(v) \to 1 \quad \text{as } v \to \infty. \quad (3.18)$$

It follows from (3.16) and (3.18) that $\lim_{v\to\infty} f_a'(v)J(v) - f_a(v)J'(v) = \infty$. Thus we have $\frac{d}{dv}\left(\frac{f_a(v)}{J(v)}\right) > 0$ for large $v > 0$, which leads to $\hat{f}(\tau) = \frac{f_a(\tau)}{J(\tau)}$.

We mention the latter of (C). It follows from (3.14) that

$$\frac{f_a(\tau)J'(\tau)}{J(\tau)^{2+\frac{\theta}{N}}} = \frac{C_1(v)h(v)^{1+\frac{\theta}{N}}\left[\log\left(h(v)+e\right)\right]^b}{h(v)^{2+\frac{\theta}{N}}\left[\log\left(h(v)+e\right)\right]^{\left(2+\frac{\theta}{N}\right)b}} = \frac{C_1(v)}{h(v)\left[\log\left(h(v)+e\right)\right]^{\left(1+\frac{\theta}{N}\right)b}}.$$

By (3.15), (3.17) and Lemma 3.2 we obtain

$$C_1(v) \lesssim 1,\ h(v) \gtrsim v\left[\log(v+e)\right]^{\frac{N}{\theta}a} \gtrsim (v+e)\left[\log(v+e)\right]^{\frac{N}{\theta}a} \text{ and}$$
$$\log\left(h(v)+e\right) \gtrsim \log(v+e) \quad \text{for large } v > 0, \text{ respectively.}$$

Then we derive

$$\frac{f_a(\tau)J'(\tau)}{J(\tau)^{2+\frac{\theta}{N}}} \lesssim \frac{1}{(\tau+e)\left[\log(\tau+e)\right]^{\frac{N}{\theta}a+\left(1+\frac{\theta}{N}\right)b}} \quad \text{for large } \tau > 0. \quad (3.19)$$

It follows in both cases (i) (a) and (ii) (a) that $\frac{N}{\theta}a+\left(1+\frac{\theta}{N}\right)b > -\frac{N}{\theta}+\left(1+\frac{\theta}{N}\right)\frac{N}{\theta} = 1$, which yields

$$\int_v^\infty \frac{d\tau}{(\tau+e)\left[\log(\tau+e)\right]^{\frac{N}{\theta}a+\left(1+\frac{\theta}{N}\right)b}} \le \frac{(v+e)^{1-\frac{N}{\theta}a-\left(1+\frac{\theta}{N}\right)b}}{\frac{N}{\theta}a+\left(1+\frac{\theta}{N}\right)b-1} \to 0 \quad \text{as } v \to \infty.$$

Thus (3.19) leads to $\int_v^\infty \frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{2+\frac{\theta}{N}}} \to 0$ as $v \to \infty$.

We mention (A). Differentiating (3.14) with (3.15) gives

$$J''(v) = C_2(v)\frac{h(v)^{1+\frac{2\theta}{N}}}{f_a(v)^2}\left[\log\left(h(v)+e\right)\right]^{b-1}. \quad (3.20)$$

Here $C_2(v)$ is defined by

$$C_2(v) := C_3(v)\left(1+\frac{N}{\theta}-f_a'(v)F_a(v)\right)\log\left(h(v)+e\right)+C_4(v),$$

where

$$C_3(v) := \frac{N}{\theta}\left\{1 + \frac{bh(v)}{(h(v)+e)\log(h(v)+e)}\right\} \to \frac{N}{\theta}$$

and

$$C_4(v) := \left(\frac{N}{\theta}\right)^2 \frac{bh(v)}{h(v)+e}\left\{2 - \frac{h(v)}{h(v)+e} + \frac{(b-1)h(v)}{\log(h(v)+e)(h(v)+e)}\right\} \to \left(\frac{N}{\theta}\right)^2 b$$

as $v \to \infty$. We see from Lemma 3.2 that

$$C_2(v) \to \left(\frac{N}{\theta}\right)^3 a + \left(\frac{N}{\theta}\right)^2 b \quad \text{as } v \to \infty.$$

In both cases (i) (a) and (ii) (a), $\left(\frac{N}{\theta}\right)^3 a + \left(\frac{N}{\theta}\right)^2 b > 0$ holds. Thus we obtain $J''(v) > 0$ for large $v > 0$.

It follows from (3.14) and (3.20) that

$$\frac{f_a(v)J'(v)^3}{J(v)^{2+\frac{\theta}{N}}J''(v)} = \frac{C_1(v)^3}{C_2(v)\left[\log(h(v)+e)\right]^{\frac{\theta}{N}b-1}}.$$

Since $C_1(v) \to \frac{N}{\theta}$ and $C_2(v) \to \left(\frac{N}{\theta}\right)^3 a + \left(\frac{N}{\theta}\right)^2 b > 0$ as $v \to \infty$, the properties (D) and (E) hold.

It remains to prove (iii). Put $J(u) := u$ and $K(u) := u$. Then $\hat{J}(v) = 1$ and $\hat{f}(\tau) = \frac{f_a(\tau)}{\tau}$ hold. Indeed, we see that $\frac{d}{dw}\left(\frac{f_a(w)}{K(J(w))}\right) = \frac{d}{dw}\left\{w^{\frac{\theta}{N}}\left[\log(w+e)\right]^a\right\} > 0$ for large $w > 0$. Since $a < -1$, we obtain

$$\begin{aligned}\hat{J}(v)\int_v^\infty \frac{\hat{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+\frac{\theta}{N}}} &= \int_v^\infty \frac{\left[\log(u+e)\right]^a d\tau}{\tau} \\ &\lesssim \int_v^\infty \frac{\left[\log(\tau+e)\right]^a d\tau}{\tau+e} = -\frac{\left[\log(v+e)\right]^{a+1}}{a+1} \to 0 \quad \text{as } v \to \infty.\end{aligned}$$

By Theorem 1.5 (i) there exists $T > 0$ such that (1.1) has a local in time nonnegative solution u in the sense of Definition 1.3 on the interval $(0, T)$. □

4. Nonexistence result

Proof of Theorem 1.7 (i) (b). Let $\varepsilon > 0$ be chosen such that $\varepsilon < \min\left\{\frac{N}{\theta}(a+1), \frac{N}{\theta} - b\right\}$. We define

$$u_0(x) := \max\left\{M,\ \chi_{B(0,e^{-1})}(x)\cdot|x|^{-N}\left(\log\frac{1}{|x|}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon}\right\},$$

where $M > 0$ is a constant such that $f_a(u)$ is convex for $u \geq M$, and $\chi_{B(0,e^{-1})}$ is a characteristic function. Let $\rho > 0$. We deduce from $\varepsilon < \frac{N}{\theta}(a+1)$ that

$$\int_{B(0,\rho)} |x|^{-N}\left(\log\frac{1}{|x|}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} dx$$
$$= \int_0^\rho \frac{1}{r}\left(\log\frac{1}{r}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} dr = \left(\frac{N}{\theta}(a+1)-\varepsilon\right)^{-1}\cdot\left(\log\frac{1}{\rho}\right)^{-\frac{N}{\theta}(a+1)+\varepsilon}, \tag{4.1}$$

and hence $|x|^{-N}\left(\log\frac{1}{|x|}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} \in L^1(B(0,\rho))$. Then $u_0 \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ holds. Moreover, it follows that for large $u > 0$,

$$F_a(u) = \int_u^\infty \frac{d\tau}{\tau^{1+\frac{\theta}{N}}[\log(\tau+e)]^a} \geq \frac{N}{2\theta}\int_u^\infty \frac{\frac{\theta}{N}\log(\tau+e)+\frac{a\tau}{\tau+e}}{\tau^{1+\frac{\theta}{N}}[\log(\tau+e)]^{a+1}}d\tau$$
$$= \frac{N}{2\theta}u^{-\frac{\theta}{N}}[\log(u+e)]^{-a},$$

and hence $F_a(u)^{-\frac{N}{\theta}} \leq (\frac{2\theta}{N})^{\frac{N}{\theta}}u[\log(u+e)]^{\frac{N}{\theta}a}$. This estimate implies that

$$J_{a,b}(u) \lesssim u(\log u)^{\frac{N}{\theta}a+b} \quad \text{for large } u > 0,$$

which yields

$$J_{a,b}(u_0) \lesssim |x|^{-N}\left(\log\frac{1}{|x|}\right)^{b-\frac{N}{\theta}-1+\varepsilon} \qquad \text{for small } |x| > 0.$$

Since $\varepsilon < \frac{N}{\theta} - b$, we obtain in the same way as (4.1) that $|x|^{-N}\left(\log\frac{1}{|x|}\right)^{b-\frac{N}{\theta}-1+\varepsilon} \in L^1(B(0,1))$. Then $J_{a,b}(u_0) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ also holds.

The proof is by contradiction. Assume that there exists $T > 0$ such that (1.1) with $f(u) = f_a(u)$ has a local in time nonnegative solution u in the sense of Definition 1.3 on the interval $(0, T)$. Let $0 < \tau < t < T$. It follows from the Fubini theorem that

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^\tau S(t-s)f_a(u(s))ds + \int_\tau^t S(t-s)f_a(u(s))ds \\ &= S(t-\tau)S(\tau)u_0 + S(t-\tau)\int_0^\tau S(\tau-s)f_a(u(s))ds + \int_\tau^t S(t-s)f_a(u(s))ds \\ &= S(t-\tau)u(\tau) + \int_\tau^t S(t-s)f_a(u(s))ds. \end{aligned}$$

By convexity of $f_a(u)$ for $u \geq M$ together with $u(t) \geq S(t)u_0 \geq M$ we deduce in a similar way to [7, Eq. (3.22)] that if $\rho > 0$ is sufficiently small, then

$$w(t) \geq c_* M_\tau 3^{-\frac{N}{\theta}}G(0,1)t^{-\frac{N}{\theta}} + 2^{-\frac{N}{\theta}}t^{-\frac{N}{\theta}}\int_{\rho^\theta}^t s^{\frac{N}{\theta}}f_a(w(s))ds \tag{4.2}$$

for almost all $0 < \tau < \rho^\theta$ and $\rho^\theta < t < \frac{1}{3}\left\{T - (2\rho)^\theta\right\}$, where $c_* > 0$ is a constant depending only on N,

$$w(t) := \int_{\mathbb{R}^N} u\left(x, t + (2\rho)^\theta\right) G(x,t)dx \quad \text{and} \quad M_\tau := \int_{B(0,\rho)} u(y,\tau)dy.$$

Note that we use (2.3) to obtain [7, Eq. (3.22)].

We prepare another estimate of w. Let $0 < \rho < 1$ be small and let $\rho^\theta < s < \rho^{\frac{\theta}{2}}$. It follows that

$$\begin{aligned} w(s) &\geq \int_{\mathbb{R}^N} \left[S\left(s + (2\rho)^\theta\right) u_0\right](x) G(x,s)dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(x - y, s + (2\rho)^\theta\right) u_0(y)dy\, G(x,s)dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(x - y, s + (2\rho)^\theta\right) G(x,s)dx\, u_0(y)dy \\ &= \int_{\mathbb{R}^N} G\left(y, 2s + (2\rho)^\theta\right) u_0(y)dy. \end{aligned} \tag{4.3}$$

Here we use $G(y, t+s) = \int_{\mathbb{R}^N} G(y-x,t)G(x,s)dx = \int_{\mathbb{R}^N} G(x-y,t)G(x,s)dx$ for $t > 0$, which follows from (2.3) and (2.4). Hereafter, we set $s_* := 2s + (2\rho)^\theta$. Since $s < \rho^{\frac{\theta}{2}}$ and $\rho > 0$ is small, $s_*^{\frac{1}{\theta}} < e^{-1}$ holds. When $\theta = 2$, we obtain from putting $y = \sqrt{s_*}z$ that

$$\begin{aligned} &\int_{\mathbb{R}^N} G(y, s_*)\, u_0(y)dy \\ &\geq (4\pi s_*)^{-\frac{N}{2}} \int_{\left\{\frac{\sqrt{s_*}}{2} \leq |y| \leq \sqrt{s_*}\right\}} e^{-\frac{|y|^2}{4s_*}} |y|^{-N} \left(\log\frac{1}{|y|}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} dy \\ &\geq (4\pi s_*)^{-\frac{N}{2}} \int_{\left\{\frac{1}{2} \leq |z| \leq 1\right\}} e^{-\frac{|z|^2}{4}} |z|^{-N} \left(\log\frac{1}{\sqrt{s_*}|z|}\right)^{-\frac{N}{2}(a+1)-1+\varepsilon} dz \\ &\gtrsim s_*{}^{-\frac{N}{2}} \left(\log\frac{4}{s_*}\right)^{-\frac{N}{2}(a+1)-1+\varepsilon}. \end{aligned} \tag{4.4}$$

On the other hand, when $0 < \theta < 2$, it follows that

$$G(y, s_*) = s_*^{-\frac{N}{\theta}} G\left(s_*^{-\frac{1}{\theta}} y, 1\right) \gtrsim s_*^{-\frac{N}{\theta}} \left(1 + s_*^{-\frac{1}{\theta}}|y|\right)^{-N-\theta},$$

which yields

$$
\begin{aligned}
&\int_{\mathbb{R}^N} G\,(y, s_*)\, u_0(y) dy \\
&\quad \gtrsim s_*^{-\frac{N}{\theta}} \int_{\left\{\frac{s_*^{\frac{1}{\theta}}}{2} \le |y| \le s_*^{\frac{1}{\theta}}\right\}} \left(1 + s_*^{-\frac{1}{\theta}} |y|\right)^{-N-\theta} |y|^{-N} \left(\log \frac{1}{|y|}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} dy \\
&\quad = s_*^{-\frac{N}{\theta}} \int_{\left\{\frac{1}{2} \le |z| \le 1\right\}} (1 + |z|)^{-N-\theta} |z|^{-N} \left(\log \frac{1}{s_*^{\frac{1}{\theta}} |z|}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} dz \\
&\quad \gtrsim s_*^{-\frac{N}{\theta}} \left(\log \frac{2^\theta}{s_*}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} .
\end{aligned}
\tag{4.5}
$$

By (4.3), (4.4) and (4.5) we have $w(s) \gtrsim s_*^{-\frac{N}{\theta}} \left(\log \frac{2^\theta}{s_*}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon}$, which yields

$$
s^{\frac{N}{\theta}} w(s) \gtrsim \left\{\frac{s}{2s + (2\rho)^\theta}\right\}^{\frac{N}{\theta}} \left(\log \frac{1}{2^{1-\theta} s + \rho^\theta}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} . \tag{4.6}
$$

Since the right hand side of (4.6) is nondecreasing with respect to s, it follows from $s > \rho^\theta$ that

$$
s^{\frac{N}{\theta}} w(s) \gtrsim \frac{1}{\left(2 + 2^\theta\right)^{\frac{N}{\theta}}} \left(\log \frac{1}{\left(1 + 2^{1-\theta}\right) \rho^\theta}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} ,
$$

and hence there exists $C_1 > 0$ such that, for $\rho^\theta < s < \rho^{\frac{\theta}{2}}$,

$$
w(s) \ge C_1 s^{-\frac{N}{\theta}} \left(\log \frac{1}{\left(1 + 2^{1-\theta}\right) \rho^\theta}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} . \tag{4.7}
$$

Since $s < \rho^{\frac{\theta}{2}}$ and $\rho > 0$ is small,

$$
\begin{aligned}
&C_1 s^{-\frac{N}{2\theta}} \left(\log \frac{1}{\left(1 + 2^{1-\theta}\right) \rho^\theta}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} \\
&> C_1 \rho^{-\frac{N}{4}} \left(\log \frac{1}{\left(1 + 2^{1-\theta}\right) \rho^\theta}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon} > 1.
\end{aligned}
$$

This along with (4.7) and $s < \rho^{\frac{\theta}{2}}$ yields

$$
\log(w(s) + e) > \log w(s) > \frac{N}{2\theta} \log \frac{1}{s} > \frac{N}{2\theta} \log \frac{s}{\rho^\theta}. \tag{4.8}
$$

Therefore, another estimate of w is derived.

We are now ready to estimate M_τ. Combining (4.2) with (4.8), we obtain

$$w(t) \geq c_* M_\tau 3^{-\frac{N}{\theta}} G(0,1) t^{-\frac{N}{\theta}} + 2^{-\frac{N}{\theta}} \left(\frac{N}{2\theta}\right)^a t^{-\frac{N}{\theta}} \int_{\rho^\theta}^t s^{\frac{N}{\theta}} \left(\log \frac{s}{\rho^\theta}\right)^a w(s)^{1+\frac{\theta}{N}} ds \tag{4.9}$$

for almost all $0 < \tau < \rho^\theta$ and $\rho^\theta < t < \rho^{\frac{\theta}{2}}$. Put $p := 1 + \frac{\theta}{N}$,

$$c_1 := c_* 3^{-\frac{N}{\theta}} G(0,1) \quad \text{and} \quad c_{k+1} := 2^{-\frac{N}{\theta}} \left(\frac{N}{2\theta}\right)^a c_k{}^p \frac{p-1}{\left(p^k - 1\right)(a+1)}, \ k = 1, 2, \ldots.$$

By (4.9) and induction we deduce in a similar way to [7, p.122] that

$$w(t) \geq g_k(t) := c_k M_\tau{}^{p^{k-1}} t^{-\frac{N}{\theta}} \left(\log \frac{t}{\rho^\theta}\right)^{\frac{\left(p^{k-1}-1\right)(a+1)}{p-1}}, \ k = 1, 2, \ldots$$

and that

$$\begin{aligned} \infty > w(t) \geq g_{k+1}(t) &\geq \left[\beta^p M_\tau \left(\log \frac{t}{\rho^\theta}\right)^{\frac{a+1}{p-1}}\right]^{p^k} t^{-\frac{N}{\theta}} \left(\log \frac{t}{\rho^\theta}\right)^{-\frac{a+1}{p-1}} \\ &\geq \left[\beta^p M_\tau \left(\frac{\theta}{4} \log \frac{1}{\rho}\right)^{\frac{a+1}{p-1}}\right]^{p^k} t^{-\frac{N}{\theta}} \left(\log \frac{t}{\rho^\theta}\right)^{-\frac{a+1}{p-1}} \end{aligned}$$

for almost all $\rho^{\frac{3}{4}\theta} < t < \rho^{\frac{\theta}{2}}$, where $\beta > 0$ is a constant such that $c_k \geq \beta^{p^k}$, $k = 1, 2, \ldots$. Then we derive

$$\beta^p M_\tau \left(\frac{\theta}{4} \log \frac{1}{\rho}\right)^{\frac{a+1}{p-1}} \leq 1. \tag{4.10}$$

It follows that

$$M_\tau = \int_{B(0,\rho)} u(y,\tau) dy \geq \int_{B(0,\rho)} [S(\tau) u_0](y) dy \geq \int_{B(0,\rho)} \left[S(\tau)\left(u_0 \chi_{B(0,\rho)}\right)\right](y) dy. \tag{4.11}$$

By (4.1) we have $u_0 \chi_{B(0,\rho)} \in L^1(\mathbb{R}^N)$, which yields $\left\| S(\tau)\left(u_0 \chi_{B(0,\rho)}\right) - u_0 \chi_{B(0,\rho)} \right\|_{L^1(\mathbb{R}^N)} \to 0$ as $\tau \to 0$. Then there exists a subsequence $\left\{S(\tilde{\tau})\left(u_0 \chi_{B(0,\rho)}\right)\right\}_{\tilde{\tau}}$ such that $S(\tilde{\tau})\left(u_0 \chi_{B(0,\rho)}\right) \to u_0 \chi_{B(0,\rho)}$ as $\tilde{\tau} \to 0$ a.e. in $\mathbb{R}^N$. Thus it follows from

Fatou's lemma, (4.10) and (4.11) that

$$\begin{aligned}1 &\gtrsim \left(\log\frac{1}{\rho}\right)^{\frac{a+1}{p-1}}\int_{B(0,\rho)}\left[S(\tau)\left(u_0\chi_{B(0,\rho)}\right)\right](y)dy\\ &\geq \left(\log\frac{1}{\rho}\right)^{\frac{a+1}{p-1}}\liminf_{\tilde{\tau}\to 0}\int_{B(0,\rho)}\left[S(\tilde{\tau})\left(u_0\chi_{B(0,\rho)}\right)\right](y)dy\\ &\geq \left(\log\frac{1}{\rho}\right)^{\frac{a+1}{p-1}}\int_{B(0,\rho)}u_0\chi_{B(0,\rho)}dy\\ &= \left(\log\frac{1}{\rho}\right)^{\frac{N}{\theta}(a+1)}\int_{B(0,\rho)}u_0(y)dy. \end{aligned}\tag{4.12}$$

On the other hand, by (4.1) we have

$$\int_{B(0,\rho)}u_0(y)dy \geq \int_{B(0,\rho)}|y|^{-N}\left(\log\frac{1}{|y|}\right)^{-\frac{N}{\theta}(a+1)-1+\varepsilon}dx \gtrsim \left(\log\frac{1}{\rho}\right)^{-\frac{N}{\theta}(a+1)+\varepsilon}. \tag{4.13}$$

Using (4.12) and (4.13), we obtain

$$1 \gtrsim \left(\log\frac{1}{\rho}\right)^{\varepsilon} \to \infty \quad \text{as } \rho \to 0.$$

This is a contradiction. We complete the proof. □

Proposition 4.1. (cf. [8, Theorem 4.1 and Lemma 4.2]) *Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Suppose that $f:[0,\infty)\to[0,\infty)$ is nondecreasing. If*

$$\int_1^{\infty}\frac{\tilde{f}(\tau)d\tau}{\tau^{1+\frac{2}{N}}} = \infty, \quad \textit{where} \quad \tilde{f}(\tau) := \sup_{1\leq w\leq\tau}\frac{f(w)}{w}, \tag{4.14}$$

then there is a nonnegative function $\phi \in L^1(\Omega)$ such that, for each small $t > 0$,

$$\left\|\int_0^t S_\Omega(t-s)f(S_\Omega(s)\phi)ds\right\|_{L^1(\Omega)} = \infty. \tag{4.15}$$

Here $S_\Omega(t)\phi$ is the solution of the heat equation in Ω with Dirichlet boundary conditions

$$\partial_t u - \Delta u = 0, \quad u|_{\partial\Omega} = 0 \quad \textit{and} \quad u(x,0) = \phi.$$

Proposition 4.2. *Suppose that $f:[0,\infty)\to[0,\infty)$ is nondecreasing. Let $0<\theta<2$ and let $\tilde{f}(\tau)$ be defined by (4.14). If*

$$\int_1^{\infty}\frac{\tilde{f}(\tau)d\tau}{\tau^{1+\frac{\theta}{N}}} = \infty, \tag{4.16}$$

then there exists a nonnegative function u_0 satisfying $u_0 \in L^1(\mathbb{R}^N)$ such that, for each small $t > 0$,

$$\left\| \int_0^t S(t-s)f(S(s)u_0)ds \right\|_{L^1(B(0,1))} = \infty. \tag{4.17}$$

Proposition 4.2 is based on [8, Theorem 4.1 and Lemma 4.2] as well as Proposition 4.1. Since (4.17) is derived in a similar way to [8, Theorem 4.1], we omit the details. In the proof of this theorem estimates using the semigroup $S(t)$ with $\theta = 2$ are applied ([8, Lemma 2.1 and Corollary 2.2]). Here we provide corresponding lemma and corollary in the case where $0 < \theta < 2$.

Lemma 4.3. *Let $0 < \theta < 2$. Then there exists a constant $c_1 > 0$ such that for $r > 0$,*

$$S(t)\chi_{B(0,r)} \geq c_1 \left(\frac{r}{r + t^{\frac{1}{\theta}}} \right)^N \chi_{B(0,r+t^{\frac{1}{\theta}})} \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{4.18}$$

Moreover, there exists a constant $c_2 > 0$ such that for $r > 0$,

$$S(t)\chi_{B(0,r)} \geq c_2 \frac{1}{(1 + r^{-\theta}t)^{\frac{N}{\theta}}} \chi_{B(0,r+t^{\frac{1}{\theta}})} \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{4.19}$$

Corollary 4.4. *There exists a constant $c > 0$ such that for $r > 0$ and $t > 0$,*

$$\int_{\mathbb{R}^N} S(t)\chi_{B(0,r)} \geq cr^N.$$

Note that Corollary 4.4 follows from integrating (4.18) over $\mathbb{R}^N$.

Proof of Lemma 4.3. It follows from (2.1) and (2.2) that

$$\begin{aligned} G(x-y,t) &= t^{-\frac{N}{\theta}} G(t^{-\frac{1}{\theta}}(x-y), 1) \\ &\geq C^{-1} t^{-\frac{N}{\theta}} (1 + t^{-\frac{1}{\theta}}|x-y|)^{-N-\theta} \\ &= C^{-1} t (t^{\frac{1}{\theta}} + |x-y|)^{-N-\theta} \quad \text{for } x \in \mathbb{R}^N,\ y \in \mathbb{R}^N \text{ and } t \in (0, \infty). \end{aligned} \tag{4.20}$$

We consider the case where $r \geq t^{\frac{1}{\theta}}$. Let $x \in B(0, r + t^{\frac{1}{\theta}})$ be fixed. Then there exists $a \in B(0,r)$ satisfying $B(a, t^{\frac{1}{\theta}}) \subset B(0,r)$ such that $|x-y| < 3t^{\frac{1}{\theta}}$ for $y \in B(a, t^{\frac{1}{\theta}})$. By (4.20) we have

$$G(x-y,t) \gtrsim t^{-\frac{N}{\theta}} \quad \text{for } y \in B(a, t^{\frac{1}{\theta}}),$$

which implies that

$$[S(t)\chi_{B(0,r)}](x) = \int_{B(0,r)} G(x-y,t)dy \geq \int_{B(a,t^{\frac{1}{\theta}})} G(x-y,t)dy$$

$$\gtrsim \int_{B(a,t^{\frac{1}{\theta}})} t^{-\frac{N}{\theta}}\,dy \gtrsim 1.$$

We consider the case where $r \le t^{\frac{1}{\theta}}$. Let $x \in B(0, r + t^{\frac{1}{\theta}})$ and $y \in B(0,r)$. Since $|x-y| < 2r + t^{\frac{1}{\theta}}$, it follows from (4.20) that

$$G(x-y,t) \ge C^{-1} t (t^{\frac{1}{\theta}} + 2r + t^{\frac{1}{\theta}})^{-N-\theta} \gtrsim t(r+t^{\frac{1}{\theta}})^{-N-\theta}.$$

This along with $r \le t^{\frac{1}{\theta}}$ yields

$$\begin{aligned}[S(t)\chi_{B(0,r)}](x) &\gtrsim \int_{B(0,r)} t(r+t^{\frac{1}{\theta}})^{-N-\theta}\,dy \gtrsim r^N t(r+t^{\frac{1}{\theta}})^{-N-\theta}\\ &= \frac{r^N}{(r+t^{\frac{1}{\theta}})^N}\frac{t}{(r+t^{\frac{1}{\theta}})^\theta} \ge \frac{r^N}{(r+t^{\frac{1}{\theta}})^N}\frac{t}{(t^{\frac{1}{\theta}}+t^{\frac{1}{\theta}})^\theta} \gtrsim \left(\frac{r}{r+t^{\frac{1}{\theta}}}\right)^N.\end{aligned}$$

Therefore, in both cases (4.18) holds.

It remains to prove (4.19). Since $\left(\frac{r}{r+t^{\frac{1}{\theta}}}\right)^N = \frac{1}{(1+r^{-1}t^{\frac{1}{\theta}})^{\theta\cdot\frac{N}{\theta}}}$, it suffices to show

$$(1+r^{-1}t^{\frac{1}{\theta}})^\theta \lesssim 1 + r^{-\theta} t.$$

Let $p \ge 1$. It follows that $1 + v^p \le (1+v)^p$ for $v \ge 0$. When $0 < \theta \le 1$, putting $v = r^{-\theta} t$ and $p = \frac{1}{\theta}$, we have $(1+r^{-1}t^{\frac{1}{\theta}})^\theta \le 1 + r^{-\theta}t$. When $1 \le \theta < 2$, it follows that $\left(\frac{1+w}{2}\right)^\theta \le \frac{1+w^\theta}{2}$ for $w \ge 0$. Putting $w = r^{-1}t^{\frac{1}{\theta}}$, we obtain $(1+r^{-1}t^{\frac{1}{\theta}})^\theta \le 2^{\theta-1}(1+r^{-\theta}t)$. Therefore, in both cases (4.19) holds. □

Lemma 4.5. *Let $a = -1$ and $b \in [0, \frac{N}{\theta}]$. If $w \ge 0$ in $\mathbb{R}^N$ and $w \in L^1(\mathbb{R}^N)$, then $J_{a,b}(w) \in L^1(\mathbb{R}^N)$.*

Proof. It suffices to prove $J_{a,b}(v) \lesssim v$ for $v \ge 0$. By (3.14) we have

$$J'_{a,b}(v) = C_1(v)\frac{F_a(v)^{-\frac{N}{\theta}-1}}{f_a(v)}\left[\log\left(F_a(v)^{-\frac{N}{\theta}} + e\right)\right]^b.$$

Since $a = -1$, we obtain

$$F_a(v) \ge \log(v+e)\int_v^\infty \frac{d\tau}{\tau^{1+\frac{\theta}{N}}} = \frac{N}{\theta} v^{-\frac{\theta}{N}} \log(v+e),$$

which yields $F_a(0)^{-\frac{N}{\theta}} = \lim_{v\to 0} F_a(v)^{-\frac{N}{\theta}} = 0$ and

$$\begin{aligned}J'_{a,b}(v) &\lesssim C_1(v)\left[\log(v+e)\right]^{-\frac{N}{\theta}}\left[\log\left(F_a(v)^{-\frac{N}{\theta}}+e\right)\right]^b\\ &= C_1(v)\left[\frac{\log\left(F_a(v)^{-\frac{N}{\theta}}+e\right)}{\log(v+e)}\right]^b\left[\log(v+e)\right]^{b-\frac{N}{\theta}}.\end{aligned}$$

It follows from (3.15), Lemma 3.2 and $b \le \frac{N}{\theta}$ that $J'_{a,b}(v) \lesssim 1$ for large $v > 0$. Moreover, we observe from (3.15) and $\lim_{v\to 0} F_a(v)^{-\frac{N}{\theta}} = 0$ that $\lim_{v\to 0} C_1(v) = \frac{N}{\theta}$. Then $J'_{a,b}(v) \lesssim 1$ holds for small $v > 0$. Thus we have $J'_{a,b}(v) \lesssim 1$ for $v > 0$, which leads to $J_{a,b}(v) \lesssim v$ for $v \ge 0$. Note that we use $J_{a,b}(0) = \lim_{v\to 0} J_{a,b}(v) = 0$, which follows from $F_a(0)^{-\frac{N}{\theta}} = 0$. □

Proof of Theorem 1.7 (ii) (b). We prove the case where $\theta = 2$. Since there exists $\sigma > 1$ such that $\frac{f_a(w)}{w} = w^{\frac{2}{N}}\left[\log(w+e)\right]^{-1}$ is nondecreasing for $w \ge \sigma$, and $\frac{f_a(w)}{w} \to \infty$ as $w \to \infty$, $\tilde{f}(\tau) = \frac{f_a(\tau)}{\tau} = \tau^{\frac{2}{N}}\left[\log(\tau+e)\right]^{-1}$ for $\tau \ge \sigma$. Then we obtain

$$\int_1^\infty \frac{\tilde{f}(\tau)d\tau}{\tau^{1+\frac{2}{N}}} \ge \int_\sigma^\infty \frac{d\tau}{\tau\log(\tau+e)} \ge \int_\sigma^\infty \frac{d\tau}{(\tau+e)\log(\tau+e)} = \left[\log\log(\tau+e)\right]_\sigma^\infty = \infty.$$

Let $\Omega = B(0,1)$. By Proposition 4.1 there is a nonnegative function $\phi \in L^1(\Omega)$ such that (4.15) holds for each small $t > 0$. We define $u_0(x) := \phi(x)$ if $x \in \Omega$, and $u_0(x) := 0$ if $x \in \mathbb{R}^N\setminus\Omega$. Then $u_0 \in L^1(\mathbb{R}^N)$ holds, and $J_{a,b}(u_0) \in L^1(\mathbb{R}^N)$ holds by Lemma 4.5. It follows that $L^1(\mathbb{R}^N) \subset \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$, since $C_0^\infty(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$ is dense and $C_0^\infty(\mathbb{R}^N) \subset BUC(\mathbb{R}^N)$. Thus we have $u_0 \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ and $J_{a,b}(u_0) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$.

The proof is by contradiction. Assume that there exists $T > 0$ such that (1.1) with $f(u) = f_a(u)$ has a local in time nonnegative solution u in the sense of Definition 1.3 on $(0,T)$. In particular, $u(t) \in L^\infty((0,T), L^1_{\mathrm{ul}}(\mathbb{R}^N))$ follows. By (1.4) we have $u(t) \ge S(t)u_0$ and

$$u(t) \ge \int_0^t S(t-s)f(u(s))ds \ge \int_0^t S(t-s)f(S(s)u_0)ds. \tag{4.21}$$

We see that $S(t)u_0 \ge S_\Omega(t)u_0|_\Omega = S_\Omega(t)\phi$ in $\Omega \times (0,\infty)$, and hence

$$u(t) \ge \int_0^t S_\Omega(t-s)f(S_\Omega(s)\phi)ds \quad \text{in } \Omega \times (0,T).$$

This along with (4.15) yields $\|u(t)\|_{L^1(\Omega)} = \infty$ for small $t > 0$. Then we obtain $u(t) \notin L^\infty((0,T), L^1_{\mathrm{ul}}(\mathbb{R}^N))$, which is a contradiction.

We prove the case where $0 < \theta < 2$. In the same way as in the case where $\theta = 2$ we obtain (4.16). By Proposition 4.2 there exists a nonnegative function u_0 satisfying $u_0 \in L^1(\mathbb{R}^N)$ such that, for each small $t > 0$, (4.17) holds. For the same reason as in the case where $\theta = 2$, $u_0 \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ and $J_{a,b}(u_0) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ hold.

The proof is also by contradiction. Assume that (1.1) with $f(u) = f_a(u)$ has a local in time nonnegative solution u in the sense of Definition 1.3. Due to (4.17) and (4.21), we have $\|u(t)\|_{L^1(B(0,1))} = \infty$ for small $t > 0$. Then we obtain $u(t) \notin L^\infty((0,T), L^1_{\mathrm{ul}}(\mathbb{R}^N))$, which is a contradiction. □

5. Discussion

In this paper we consider a local in time nonnegative solution of the equation (1.1) in critical and doubly critical cases. In our theorems we derive the solvability for a wider class of nonlinear terms. In the doubly critical case we obtain a complete classification for the solvability, using the integrability condition on u_0 when $f(u)$ is given by (1.9).

The existence result is based on Theorem 1.5. In this section we focus on (1.5). Let J satisfy (J). When $K(u) = u$ and $\frac{f(w)}{J(w)}$ is nondecreasing for large $w > 0$, by taking $c > 0$ sufficiently large it follows that

$$\hat{J}(v) \int_v^\infty \frac{\hat{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+\frac{\theta}{N}}} = J'(v) \int_v^\infty \frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{2+\frac{\theta}{N}}}.$$

As mentioned in the proof of the existence part in Theorem 1.7, we obtain from L'Hospital's rule that

$$\lim_{v\to\infty} J'(v) \int_v^\infty \frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{2+\frac{\theta}{N}}} = \lim_{v\to\infty} \frac{\frac{d}{dv}\left(\int_v^\infty \frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{2+\frac{\theta}{N}}}\right)}{\frac{d}{dv}\left(\frac{1}{J'(v)}\right)}$$
$$= \lim_{v\to\infty} \frac{f(v)J'(v)^3}{J(v)^{2+\frac{\theta}{N}} J''(v)} =: \alpha.$$

Let $\theta = 2$. Then the following hold:

(i) When $J(v) = v^r$ $(r > 1)$, $\alpha < \infty$ is equivalent to $\lim_{v\to\infty} \frac{f(v)}{v^{1+\frac{2}{N}r}} < \infty$. Thus this case corresponds to [8, Theorem 3.4].

(ii) Let f satisfy Assumption A. When $J(v) = F(v)^{-r}$ and $q < r + 1$, we have $\alpha = 0$ if $r > \frac{N}{2}$, $0 < \alpha < \infty$ if $r = \frac{N}{2}$, and $\alpha = \infty$ if $0 < r < \frac{N}{2}$. Thus this case corresponds to [4, Theorems 1.1 and 1.2].

The finiteness of α leads to the existence of a local in time solution of (1.1).

Example 5.1. Let $0 < \theta \le 2$ and

$$f(u) = (u+1)^{1+\frac{\theta}{N}} \exp\left(\sqrt{\log(u+1)}\right) \left(\frac{\theta}{N} + \frac{1}{2\sqrt{\log(u+1)}}\right)^{-1}.$$

Then it follows that $F(u) = (u+1)^{-\frac{\theta}{N}} \exp\left(-\sqrt{\log(u+1)}\right)$ and that f satisfies Assumption A with $q = 1 + \frac{N}{\theta}$. Let

$$J(u) = J_b(u) := F(u)^{-\frac{N}{\theta}} \left[\log\left(F(u)^{-\frac{N}{\theta}} + e\right)\right]^b.$$

Then we have

$$\alpha = 0 \text{ if } b > \frac{N}{2\theta}, \quad 0 < \alpha < \infty \text{ if } b = \frac{N}{2\theta}, \quad \text{and } \alpha = \infty \text{ if } 0 < b < \frac{N}{2\theta}.$$

Thus Theorem 1.5 implies that a local in time nonnegative solution of (1.1) exists if $J_b(u_0) \in L^1_{\mathrm{ul}}(\mathbb{R}^N)$ for some $b > \frac{N}{2\theta}$, or $J_b(u_0) \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R}^N)$ with $b = \frac{N}{2\theta}$.

Example 5.1 indicates that the borderline value of b is less than or equal to $\frac{N}{2\theta}$, and hence the borderline value of b is different from that in Theorem 1.7 (i) and (ii). Since Theorem 1.5 can be applied to a wide class of f and J, further studies are needed in the doubly critical case.

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