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On the high friction limit for the complete Euler system

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Abstract. We show that solutions of the complete Euler system of gas dynamics perturbed by a friction term converge to a solution of the porous medium equation in the high friction/long time limit. The result holds in the largest possible class of generalized solutions–the measure–valued solutions of the Euler system.

1. Introduction

In a recent paper, Lattanzio and Tzavaras [13] consider the singular limit of the isentropic Euler system perturbed by a high friction term. Our goal is to extend this result to the physically more adequate setting of the complete Euler system of gas dynamics, where temperature changes as well as possible singularities resulting in the increase of the total entropy of the system are allowed.

1.1. Euler system of gas dynamics

We consider a scaled *Euler system* of gas dynamics in the form:
Equation of continuity:

$$\varepsilon \partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0. \tag{1.1}$$

Momentum equation:

$$\varepsilon \partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = -\frac{1}{\varepsilon} \mathbf{m}. \tag{1.2}$$

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Energy balance:

$$\varepsilon\partial_t\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}+\varrho e\right)+\mathrm{div}_x\left[\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}+\varrho e+p\right)\frac{\mathbf{m}}{\varrho}\right]=-\frac{1}{\varepsilon}\frac{|\mathbf{m}|^2}{\varrho}. \tag{1.3}$$

The pressure p and the internal energy e are thermodynamic functions interrelated through *Gibbs' equation*

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right), \tag{1.4}$$

where ϑ is the (absolute) temperature and s the (specific) entropy. Here, D refers to differential with respect to the independent thermodynamics variables, e.g. ϱ, ϑ. In the present paper, rather general constitutive relations are supposed, with only minor restrictions specified in Theorem 2.2 below. We consider the fluid mass density $\varrho = \varrho(t,x)$, the momentum $\mathbf{m} = \mathbf{m}(t,x)$, together with the total entropy $S = (\varrho s)(t,x)$ as the basic state variables (unknowns) in the system of equations (1.1)–(1.3). The fluid is confined to a bounded domains $\Omega \subset R^d$ and the problem is formally closed by imposing the impermeability boundary conditions

$$\mathbf{m}\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{1.5}$$

and the initial conditions

$$\varrho(0,\cdot)=\varrho_0,\ \mathbf{m}(0,\cdot)=\mathbf{m}_0,\ S(0,\cdot)=S_0. \tag{1.6}$$

1.2. Large friction limit

Smooth solutions of the Euler system (1.1)–(1.3) conserve entropy, specifically, it follows from Gibbs' relation (1.4)

$$\varepsilon\partial_t S+\mathrm{div}_x\left(S\frac{\mathbf{m}}{\varrho}\right)=0. \tag{1.7}$$

In particular, if $s_0 = s(0,\cdot) = \overline{s}$ is constant, it follows from (1.7) $s(t,\cdot) = \overline{s}$ as long as the solution remains smooth and (1.1)–(1.3) reduces to the so-called *isentropic* Euler system for only two unknowns ϱ and $\mathbf{m}$.

The term $\frac{1}{\varepsilon}\mathbf{m}$ in the momentum equation, together with its counterpart $\frac{1}{\varepsilon}\frac{|\mathbf{m}|^2}{\varrho}$ in the energy balance, represent the effect of "friction" on the gas motion. High friction regularization of the simplified Euler system has been studied by several authors, see Dafermos and Pan [6], Sideris, Thomases, and Wang [15], and the references cited therein.

Lattanzio and Tzavaras [13] identified the high friction limit $\varepsilon \to 0$ in the isentropic Euler system as the porous medium equation:

$$\varrho_\varepsilon \to r,\ \text{where } \partial_t r - \Delta_x p(r, r\overline{s}) = 0,\ \overline{s} - \text{constant}. \tag{1.8}$$

The proof in [13] is formal, conditioned by the existence of weak solutions to the scaled isentropic Euler system with the life span independent of the scaling parameter ε. As a matter of fact, this type of relaxation limits has been studied before in different contexts and with the use of different techniques, see e.g. Marcati and Milani [14]. The reference [13] presents a systematic study of this kind of relaxation limits using the relative energy techniques, again, see also Lattanzio and Tzavaras [12], where the relative energy is also used in the context of measure-valued solutions.

The motion of a gas obeying the physically relevant *complete* Euler system (1.1)–(1.3) is not likely to be isentropic, with possible shocks developed in a finite time violating the entropy equation (1.7). A suitable *admissibility* condition for the general weak solutions is then formulated in terms of the *entropy inequality*

$$\varepsilon \partial_t S + \mathrm{div}_x \left(S \frac{\mathbf{m}}{\varrho} \right) \geq 0,$$

or its *renormalized variant* (see e.g. Chen and Frid [5])

$$\varepsilon \partial_t \left(\varrho \chi \left(\frac{S}{\varrho} \right) \right) + \mathrm{div}_x \left[\chi \left(\frac{S}{\varrho} \right) \mathbf{m} \right] \geq 0 \tag{1.9}$$

for any χ concave, $\chi' \geq 0$, χ bounded above.

Our main goal is to extend the result of Lattanzio and Tzavaras [13] to the complete Euler system. The limit density profile r and the entropy $s = \frac{S}{r}$ (formally) satisfy the following system of equations:

$$\partial_t r - \Delta_x p(r, rs) = 0, \tag{1.10}$$

$$\partial_t s - \frac{1}{r} \nabla_x p(r, rs) \cdot \nabla_x s = 0, \tag{1.11}$$

supplemented with the initial and boundary data

$$r(0, \cdot) = r_0, \ s(0, \cdot) = s_0, \ \nabla_x p(r, rs) \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.12}$$

Accordingly, we suppose that the initial data of the Euler system converge to the initial conditions of (1.10), (1.11) specifically,

$$\varrho_0 = \varrho_{0,\varepsilon} \to r_0, \ S_0 = S_{0,\varepsilon} \to r_0 s_0, \ \text{and } \mathbf{m}_0 = \mathbf{m}_{0,\varepsilon} \to 0. \tag{1.13}$$

Solvability of the problem (1.10), (1.11) is discussed in the forthcoming section.

Although the recent results based on the method of convex integration provide weak solutions (even infinitely many) for a large class of initial data, see e.g. [8], a general existence result that would cover all finite energy initial data is not available so far. For this reason, we consider a larger class of *measure–valued solutions* in the spirit of [3]. The advantage of considering the measure–valued solutions of the Euler system is not only their *existence* that may be shown for all physically admissible data, see e.g. Kröner and Zajączkowski [11] or the more recent adaptation of the same method of construction in [1]. The measure–valued solutions capture a large variety of singular limits including the low diffusion limit of the Navier–Stokes–Fourier system [2] as well as the alternative model proposed by Brenner, see [9, Chapter 10]. The truly measure–valued solutions, unlike the Euler system, can also mimick the behaviour of complete fluid systems in highly turbulent regime, see [7].

The paper is organized as follows. In Sect. 2, we introduce the concept of measure–valued solution and state the main result concerning the high friction limit. In Sect. 3, we recall the *relative energy inequality* introduced in [3] and further developed in [9] and prove the desired convergence.

2. Measure–valued solutions, main result

Besides the general Gibbs' relation (1.4), we suppose that the thermodynamic functions p, e, and s satisfy the *hypothesis of thermodynamic stability*. This can be conveniently formulated in terms of the variables (ϱ, S) as convexity of the internal energy:

$$\begin{aligned} &E_{\rm int} : (\varrho, S) \in R^2 \mapsto \varrho e(\varrho, S) \in [0, \infty] \text{ is a convex l.s.c function,} \\ &E_{\rm int} \in C^2\left(\mathrm{int(dom)}[E_{\rm int}]\right), \ \nabla^2 E_{\rm int} > 0, \end{aligned} \tag{2.1}$$

cf. [9, Chapter 4, Section 4.1.6]. Note that for the standard Boyle-Mariotte equation of state

$$p = (\gamma - 1)\varrho e, \ e = c_v \vartheta, \ c_v = \frac{1}{\gamma - 1}, \ \gamma > 1,$$

we have

$$p(\varrho, S) = (\gamma - 1) E_{\rm int}(\varrho, S) = \begin{cases} \varrho^\gamma \exp\left(\frac{S}{c_v \varrho}\right) & \text{if } \varrho > 0, \\ 0 & \text{if } \varrho = 0, \ S \leq 0, \\ \infty & \text{otherwise,} \end{cases}$$

see [9, Chapter 2, Section 2.2.4].

Similarly, we define the kinetic energy,

$$E_{\rm kin} : (\varrho, \mathbf{m}) \mapsto \begin{cases} \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} & \text{if } \varrho > 0, \\ 0 & \text{if } \varrho = 0, \ \mathbf{m} = 0, \\ \infty & \text{otherwise,} \end{cases}$$

and the total energy

$$\begin{aligned} &E(\varrho, S, \mathbf{m}) = E_{\rm kin}(\varrho, \mathbf{m}) + E_{\rm int}(\varrho, S) \\ &\qquad \text{a convex l.s.c. function on } R^{2+d}, \ \text{ strictly convex on its domain.} \end{aligned}$$

2.1. Measure–valued solutions

Following [3], we define *measure–valued solution* of the Euler system (1.1)–(1.3), (1.9), (1.5)–(1.6) a weakly measurable family of Borel probability measures,

$$\mathcal{V} : (t,x) \in (0,T) \times \Omega \mapsto \mathcal{V}_{t,x} \in \mathfrak{P}[R \times R \times R^d].$$

Moreover, we denote

$$\left\langle \mathcal{V}_{t,x}; F(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}}) \right\rangle = \int_{R^{2+d}} F(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}}) \, \mathrm{d}\mathcal{V}_{t,x},$$

for any Borel measurable function F of the "dummy" variables $(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}}) \in R^{2+d}$.

Definition 2.1. *(Measure–valued solution of the Euler system)*
We say that a parametrized family $(\mathcal{V}_{t,x})_{(t,x)\in(0,T)\times\Omega}$ of probability measures on the Euclidean space R^{2+d} is a *measure–valued solution* to the Euler system (1.1)–(1.3), with the boundary conditions (1.5), and the initial data (1.6), if the following holds:

- **Compatibility:**

$$\mathcal{V}_{t,x} \{\widetilde{\varrho} \geq 0\} = \mathcal{V}_{t,x} \left\{\widetilde{\varrho} = 0, \ \widetilde{S} \geq 0\right\} = 1 \text{ for a.a. } (t,x) \in (0,T) \times \Omega. \tag{2.2}$$

- **Equation of continuity:**

$$\int_0^T \int_\Omega \Big(\varepsilon \left\langle \mathcal{V}_{t,x}; \widetilde{\varrho} \right\rangle \partial_t \varphi(t,x) + \left\langle \mathcal{V}_{t,x}; \widetilde{\mathbf{m}} \right\rangle \cdot \nabla_x \varphi(t,x) \Big) \ \mathrm{d}x \, \mathrm{d}t = -\varepsilon \int_\Omega \varrho_0 \varphi(0,x) \ \mathrm{d}x; \tag{2.3}$$

  for any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$.

- **Momentum equation:** There exists a tensor–valued measure

$$\mathfrak{R} \in L^\infty_{\text{weak}}(0,T; \mathcal{M}^+_{\text{sym}}(\overline{\Omega}; R^{d\times d}))$$

  such that the integral identity

$$\begin{aligned} &\int_0^T \int_\Omega \left(\varepsilon \left\langle \mathcal{V}_{t,x}; \widetilde{\mathbf{m}} \right\rangle \partial_t \boldsymbol{\varphi}(t,x) + \left\langle \mathcal{V}_{t,x}; \mathbb{1}_{\widetilde{\varrho}>0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle : \nabla_x \boldsymbol{\varphi}(t,x) \right) \ \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; p(\widetilde{\varrho}, \widetilde{S}) \right\rangle \mathrm{div}_x \boldsymbol{\varphi}(t,x) \ \mathrm{d}x \, \mathrm{d}t \\ &\quad = \int_0^T \int_\Omega \frac{1}{\varepsilon} \left\langle \mathcal{V}_{t,x}; \widetilde{\mathbf{m}} \right\rangle \cdot \boldsymbol{\varphi}(t,x) \ \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi}(t,x) : \mathrm{d}\mathfrak{R} \, \mathrm{d}t \\ &\quad - \varepsilon \int_\Omega \mathbf{m}_0(x) \cdot \boldsymbol{\varphi}(0,x) \ \mathrm{d}x; \end{aligned} \tag{2.4}$$

  holds for any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; R^d)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

- **Entropy inequality:**

$$\int_0^T \int_\Omega \left(\varepsilon \left\langle \mathcal{V}_{t,x}; \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) \right\rangle \partial_t \varphi(t,x) + \left\langle \mathcal{V}_{t,x}; \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) \frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle \cdot \nabla_x \varphi(t,x) \right) \ \mathrm{d}x \, \mathrm{d}t$$
$$\leq -\varepsilon \int_\Omega \varrho_0(x) \chi \left(\frac{S_0(x)}{\varrho_0(x)} \right) \varphi(0,x) \ \mathrm{d}x \tag{2.5}$$

for any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$, $\varphi \geq 0$, and any $\chi \in C(R)$ concave, $\chi' \geq 0$, $\chi \leq \overline{\chi} \in R$.

- **Energy inequality:** There exists a constant $\Lambda > 0$ such that

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; E(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}}) \right\rangle \ \mathrm{d}x + \Lambda \int_{\overline{\Omega}} \mathrm{d} \ \mathrm{trace}[\mathfrak{R}(\tau, \cdot)] + \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} \right\rangle \ \mathrm{d}x \, \mathrm{d}t$$
$$\leq \int_\Omega E(\varrho_0, S_0, \mathbf{m}_0) \ \mathrm{d}x \tag{2.6}$$

for a.a. $\tau \in (0,T)$.

The measure valued solutions were introduced in [3] to capture the largest class possible of limits of the so–called consistent approximations of the Euler system. Their existence and even possibility to construct a solution semigroup was shown in [1, Proposition 3.8]. Besides, the numerical approximations introduced in [9, Chapter 10] are examples of consistent approximations and the convergence results presented therein may be seen as another way of identifying measure valued solutions.

2.2. Solvability of the limit problem

We point out that the hypothesis of thermodynamic stability enforced through strict convexity of $E_{\mathrm{int}} = E_{\mathrm{int}}(\varrho, S)$ has important consequence on solvability of the system (1.10), (1.11) at least in the case of constant initial entropy $s_0 = \overline{s}$. Indeed the entropy balance (1.11) implies $s(t,x) = \overline{s}$ for any t, x independently of r. Accordingly, the density profile r can be determined as a solution of the problem

$$\partial_t r - \mathrm{div}_x \left[\left(\partial_\varrho p(r, r\overline{s}) + \overline{s} \partial_S p(r, r\overline{s}) \right) \nabla_x r \right], \ r(0,\cdot) = r_0, \ \nabla_x r \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{2.7}$$

It follows form Gibbs' Eq. (1.4) that

$$p(\varrho, S) = \frac{\partial(\varrho e(\varrho, S))}{\partial \varrho} \varrho + \frac{\partial(\varrho e(\varrho, S))}{\partial S} S - \varrho e(\varrho, S).$$

Consequently,

$$\frac{\partial p(\varrho, S)}{\partial \varrho} = \frac{\partial^2(\varrho e(\varrho, S))}{\partial^2 \varrho} \varrho + \frac{\partial^2(\varrho e(\varrho, S))}{\partial \varrho \partial S} S,$$

and, similarly,

$$\frac{\partial p(\varrho, S)}{\partial S} = \frac{\partial^2(\varrho e(\varrho, S))}{\partial \varrho \partial S}\varrho + \frac{\partial^2(\varrho e(\varrho, S))}{\partial^2 S}S.$$

Thus we compute

$$\partial_\varrho p(r, r\overline{s}) + \overline{s}\partial_S p(r, r\overline{s}) = r\left(\frac{\partial^2(re(r, r\overline{s}))}{\partial^2 \varrho} + 2\frac{\partial^2(re(r, r\overline{s}))}{\partial \varrho \partial S}\overline{s} + \frac{\partial^2(re(r, r\overline{s}))}{\partial^2 S}|\overline{s}|^2\right). \tag{2.8}$$

Since the internal energy E_{int} is a strictly convex function of (ϱ, S), we get

$$\frac{\partial^2(re(r, r\overline{s}))}{\partial^2 \varrho} > 0, \quad \frac{\partial^2(re(r, r\overline{s}))}{\partial^2 S} > 0,$$

and

$$\frac{\partial^2(re(r, r\overline{s}))}{\partial^2 \varrho}\frac{\partial^2(re(r, r\overline{s}))}{\partial^2 S} > \left|\frac{\partial^2(re(r, r\overline{s}))}{\partial \varrho \partial S}\right|^2.$$

This yields

$$\left(\partial_\varrho p(r, r\overline{s}) + \overline{s}\partial_S p(r, r\overline{s})\right) > 0 \text{ whenever } r > 0, \tag{2.9}$$

and, consequently, (2.7) is a non–degenerate parabolic equation as soon as $r_0 > 0$.

2.3. Main result, high friction limit

To begin, we introduce the relative energy expressed in the variables $(\varrho, S, \mathbf{m})$,

$$\begin{aligned} &E\left(\varrho, S, \mathbf{m} \middle| r, \widehat{S}, r\mathbf{U}\right) \\ &\quad = \varrho\left|\frac{\mathbf{m}}{\varrho} - \mathbf{U}\right|^2 + \varrho e(\varrho, S) - \frac{\partial(\varrho e(\varrho, S))}{\partial \varrho}(r, \widehat{S})(\varrho - r) - \frac{\partial(\varrho e(\varrho, S))}{\partial S}(r, \widehat{S})(S - \widehat{S}) \\ &\qquad - re(r, \widehat{S}). \end{aligned} \tag{2.10}$$

More precisely, the relative energy for fixed values of the parameters $(r, \widehat{S}, \mathbf{U})$ is a convex l.s.c. function of $(\varrho, S, \mathbf{m})$ defined as

$$\begin{aligned} E\left(\varrho, S, \mathbf{m} \middle| r, \widehat{S}, r\mathbf{U}\right) &= E(\varrho, S, \mathbf{m}) - \partial_{\varrho, S, \mathbf{m}}E(r, \widehat{S}, r\mathbf{U}) \cdot (\varrho - r, S - \widehat{S}, \mathbf{m} - r\mathbf{U}) \\ &\quad - E(r, \widehat{S}, r\mathbf{U}). \end{aligned}$$

In particular, if $(r, \widetilde{S}, r\mathbf{U}) \in \text{int}(\text{dom}(E))$, then

$$E\left(\varrho, S, \mathbf{m} \middle| r, \widehat{S}, r\mathbf{U}\right) = 0 \ \Leftrightarrow \ \varrho = r, \ S = \widehat{S}, \ \mathbf{m} = r\mathbf{U},$$

see [9, Chapter 4, Section 4.1.6] for details.

We are ready to state our main result concerning the high friction limit.

Theorem 2.2. (High friction asymptotic limit)
Let $\Omega \subset R^d$ be a bounded domain of class $C^{2+\nu}$. In addition to Gibbs' relation (1.4) and the hypothesis of thermodynamics stability (2.1), let the thermodynamic functions p, e satisfy

$$p \leq \overline{p}(1 + \varrho e) \text{ for some constant } \overline{p}. \tag{2.11}$$

Finally, suppose that the initial data satisfy

$$\varrho_{0,\varepsilon} > 0, \ S_{0,\varepsilon} \geq \varrho_{0,\varepsilon}\underline{s} \quad \text{for some } \underline{s} \in R,$$
$$\int_\Omega E\left(\varrho_{0,\varepsilon}, S_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon} \,\middle|\, r_0, r_0 s_0, 0\right) \, \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0, \tag{2.12}$$

where

$$r_0, s_0 \in C^2(\overline{\Omega}), \ (r_0, r_0 s_0) \in \mathrm{int}(\mathrm{dom} E_{\mathrm{int}}). \tag{2.13}$$

Let $(\mathcal{V}^\varepsilon_{t,x})_{\varepsilon>0}$ be a family of measure–valued solutions of the Euler system with the initial data $(\varrho_{0,\varepsilon}, S_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon})_{\varepsilon>0}$ in the sense of Definition 2.1. Suppose that the limit system (1.10), (1.11), (1.12) admits a C^2−solution r, s such that

$$(r, rs) \in \mathrm{int}(\mathrm{dom})[E_{\mathrm{int}}] \text{ for any } t \in [0, T]. \tag{2.14}$$

Then

$$\operatorname*{ess\ sup}_{\tau \in (0,T)} \int_\Omega \left\langle \mathcal{V}^\varepsilon_{\tau,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r(\tau, x), rs(\tau, x), 0\right)\right\rangle \, \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0. \tag{2.15}$$

The remaining part of the paper is devoted to the proof of Theorem 2.2.

Remark 2.3. Noticing that the total energy is stricly convex, meaning the relative energy represents a Bregman distance, we can reformulate the convergence statement (2.15) in terms of probability theory:

$$\operatorname*{ess\ sup}_{\tau \in (0,T)} \int_\Omega W_1\left[\mathcal{V}^\varepsilon_{\tau,x}; \delta_{(r(\tau,x), rs(\tau,x), 0)}\right] \, \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0,$$

where W_1 denotes the Monge–Kantorowich (Wasserstein – 1) distance between probability measures, and δ_X stands for the Dirac mass at X.

Remark 2.4. Solvability of the limit problem (1.10), (1.11), (1.12) has been discussed in Sect. 2.2 on condition that the initial entropy s_0 is constant. Note that in this case (2.14) follows from (2.13) by the maximum principle. Another, a rather trivial situation when the limit problem is solvable globally in time, is the choice of the initial data

$$p(r_0, r_0 s_0) = \overline{p} \text{ – a positive constant.}$$

Indeed $r = r_0$, $s = s_0$ then obviously solve the problem. Unlike in Sect. 2.2, the initial entropy distribution may effectively depend on x.

Remark 2.5. The Proof of Theorem 2.2 presented below will actually yield a more exact rate of convergence (2.15):

$$\operatorname{ess}\sup_{\tau\in(0,T)} \int_\Omega \left\langle \mathcal{V}^\varepsilon_{\tau,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r(\tau,x), rs(\tau,x), 0\right)\right\rangle \,\mathrm{d}x \leq \int_\Omega E\left(\varrho_{0,\varepsilon}, S_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon} \,\middle|\, r_0, r_0 s_0, 0\right) \,\mathrm{d}x + c\varepsilon. \tag{2.16}$$

3. High friction limit, proof of the main result

Similarly to Lattanzio and Tzavaras [13], the Proof of Theorem 2.2 leans on stability on the limit solution expressed by means of the relative energy inequality. The relevant version of the latter for the complete Euler system was introduced in [3]. The relative energy inequality in the context of measure-valued solutions for isentropic Euler with friction and also nonlocal terms was studied in [4]. Note that the current studies on complete Euler system do not fall into the general framework of hyperbolic systems studied in [10].

3.1. Entropy minimum principle

We start by exploiting boundedness of the initial entropy, namely

$$S_{0,\varepsilon} \geq \varrho_{0,\varepsilon}\underline{s},$$

yielding the entropy minimum principle in the form

$$\mathcal{V}^\varepsilon_{t,x}\left\{\widetilde{S} \geq \widetilde{\varrho}\underline{s} \,\middle|\, \widetilde{\varrho} > 0\right\} = 1. \tag{3.1}$$

Indeed we may choose χ in the renormalized entropy inequality (2.5) as

$$\chi(Z) < 0 \text{ for } Z \leq \underline{s}, \;\; \chi(Z) = 0 \text{ for } Z \geq \underline{s}.$$

Considering spatially homogeneous test function φ in (2.5) we deduce

$$\int_\Omega \left\langle \mathcal{V}^\varepsilon_{\tau,x}; \widetilde{\varrho}\chi\left(\frac{\widetilde{S}}{\widetilde{\varrho}}\right)\right\rangle \,\mathrm{d}x \geq 0 \text{ for a.a. } \tau \geq 0,$$

which yields the desired conclusion.

Finally, relation (3.1) together with the compatibility condition (2.2) converts (3.1) to an unconditional results

$$\mathcal{V}^\varepsilon_{t,x}\left\{\widetilde{S} \geq \widetilde{\varrho}\underline{s}\right\} = 1. \tag{3.2}$$

3.2. Relative energy inequality

Recalling the definition (2.10) of the relative energy, let us introduce the temperature Θ evaluated by means of the variables r and $\widehat{S}$ through the implicit relation

$$\widehat{S} = rs(r, \Theta). \tag{3.3}$$

Under the condition (3.2), the *relative energy inequality* associated to the Euler system reads, see [3] or [9, Chapter 4, Section 4.1.7, Chapter 6, Section 6.2]:

$$\begin{aligned}
&\int_\Omega \left\langle \mathcal{V}_{\tau,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}\right)(\tau, x)\right\rangle \ \mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\ \mathrm{trace}[\mathfrak{R}](\tau, \cdot) \\
&\quad + \frac{1}{\varepsilon^2}\int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \widetilde{\varrho}\left|\frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}(t,x)\right|^2\right\rangle \ \mathrm{d}x\,\mathrm{d}t \\
&\le \int_\Omega E\left(\varrho_0, S_0, \mathbf{m}_0 \,\middle|\, r, \widehat{S}, r\mathbf{U}\right)(0, x)\ \mathrm{d}x \\
&\quad - \frac{1}{\varepsilon}\int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathbb{1}_{\widetilde{\varrho}>0} \frac{(\widetilde{\varrho}\mathbf{U}(t,x) - \widetilde{\mathbf{m}}) \otimes (\widetilde{\varrho}\mathbf{U}(t,x) - \widetilde{\mathbf{m}})}{\widetilde{\varrho}} : \nabla_x \mathbf{U}(t,x)\right\rangle \ \mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{\varepsilon}\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \big(p(\widetilde{\varrho}, \widetilde{S}) - (\widetilde{\varrho} - r)\partial_\varrho p(r, \widehat{S}) - (\widetilde{S} - \widehat{S})\partial_S p(r, \widehat{S}) \\
&\quad - p(r, \widehat{S})\big)(t, x)\rangle \mathrm{div}_x \mathbf{U}(t,x)\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_{\mathbb{T}^d} \langle \mathcal{V}_{t,x}; \widetilde{\varrho}\mathbf{U}(t,x) - \widetilde{\mathbf{m}}\rangle \left(\partial_t \mathbf{U} + \frac{1}{\varepsilon}\mathbf{U}\cdot\nabla_x \mathbf{U} + \frac{1}{\varepsilon r}\nabla_x p(r, \widehat{S})\right)(t,x)\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; r(t,x) - \widetilde{\varrho}\rangle \frac{1}{r}\partial_\varrho p(r, \widehat{S})\left(\partial_t r + \frac{1}{\varepsilon}\mathrm{div}_x(r\mathbf{U})\right)(t,x)\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; r(t,x) - \widetilde{\varrho}\rangle \frac{1}{r}\partial_S p(r, \widehat{S})\left(\partial_t \widehat{S} + \frac{1}{\varepsilon}\mathrm{div}_x(\widehat{S}\mathbf{U})\right)(t,x)\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \frac{\widetilde{\varrho}}{r}\widehat{S}(t,x) - \widetilde{\varrho}\chi\left(\frac{\widetilde{S}}{\widetilde{\varrho}}\right)\right\rangle \\
&\quad \left(\partial_t \Theta + \frac{1}{\varepsilon}\mathbf{U}\cdot\nabla_x\Theta + \frac{1}{\varepsilon}\partial_S p(r, \widehat{S})\mathrm{div}_x \mathbf{U}\right)(t,x)\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon}\int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left(\widetilde{\varrho}\frac{\widehat{S}}{r}(t,x) - \widetilde{\varrho}\chi\left(\frac{\widetilde{S}}{\widetilde{\varrho}}\right)\right)\left(\frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}(t,x)\right)\right\rangle \cdot \nabla_x \Theta(t,x)\ \mathrm{d}x\,\mathrm{d}t, \\
&\quad - \frac{1}{\varepsilon}\int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U}(t,x) : \mathrm{d}\mathfrak{R}(t,\cdot)\,\mathrm{d}t - \frac{1}{\varepsilon^2}\int_0^\tau \int_\Omega \mathbf{U}(t,x)\cdot\langle \mathcal{V}_{t,x}; \widetilde{\mathbf{m}} - \widetilde{\varrho}\mathbf{U}(t,x)\rangle\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon}\int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \widetilde{\varrho}\chi\left(\frac{\widetilde{S}}{\widetilde{\varrho}}\right) - \widetilde{S}\right\rangle \partial_S p(r, \widehat{S})\mathrm{div}_x \mathbf{U}(t,x)\ \mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.4, 3.5}$$

for a.a. $\tau \in (0, T)$ and any trio of "test functions"

$$\begin{aligned}
&r \in C^1([0,T] \times \overline{\Omega}),\ r > 0,\ \widehat{S} \in C^1([0,T] \times \overline{\Omega}), \\
&\widehat{S} = rs(r, \Theta),\ \mathbf{U} \in C^1([0,T] \times \overline{\Omega}; R^d), \\
&\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0.
\end{aligned} \tag{3.6}$$

4. Convergence, proof of Theorem 2.2

In view of hypothesis (2.12), we consider the following ansatz:

$$\widehat{S} = rs,\ \mathbf{U}^\varepsilon = -\varepsilon \frac{1}{r} \nabla_x p(r, rs),\ \partial_t r + \frac{1}{\varepsilon} \mathrm{div}_x (r \mathbf{U}^\varepsilon) = 0,\ \partial_t s - \frac{1}{\varepsilon} \mathbf{U}^\varepsilon \cdot \nabla_x s = 0. \tag{4.1}$$

As a consequence of (3.3), we get

$$\partial_t \widehat{S} + \frac{1}{\varepsilon} \mathrm{div}_x (\widehat{S} \mathbf{U}^\varepsilon) = 0,$$
$$\partial_t \Theta + \frac{1}{\varepsilon} \mathbf{U}^\varepsilon \cdot \nabla_x \Theta + \frac{1}{\varepsilon} \partial_S p(r, \widehat{S}) \mathrm{div}_x \mathbf{U}^\varepsilon = 0,$$

and

$$\left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \mathbf{U}^\varepsilon(t,x) - \widetilde{\mathbf{m}} \right\rangle \frac{1}{\varepsilon r} \nabla_x p(r, \widehat{S}) - \frac{1}{\varepsilon^2} \mathbf{U}^\varepsilon(t,x) \cdot \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\mathbf{m}} - \widetilde{\varrho} \mathbf{U}^\varepsilon(t,x) \right\rangle = 0.$$

Accordingly, the relative entropy inequality (3.5) simplifies considerably yielding

$$\begin{aligned}
&\int_\Omega \left\langle \mathcal{V}^\varepsilon_{\tau,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right)(\tau, x) \right\rangle \,\mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\,\mathrm{trace}[\mathfrak{R}^\varepsilon](\tau, \cdot) \\
&\quad + \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \left| \frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t,x) \right|^2 \right\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_\Omega E\left(\varrho_{0,\varepsilon}, S_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon} \,\middle|\, r_0, r_0 s_0, r_0 \mathbf{U}^\varepsilon(0,x) \right) \,\mathrm{d}x \\
&\quad - \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \mathbb{1}_{\widetilde{\varrho}>0} \frac{(\widetilde{\varrho} \mathbf{U}^\varepsilon(t,x) - \widetilde{\mathbf{m}}) \otimes (\widetilde{\varrho} \mathbf{U}^\varepsilon(t,x) - \widetilde{\mathbf{m}})}{\widetilde{\varrho}} : \nabla_x \mathbf{U}^\varepsilon(t,x) \right\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \langle \mathcal{V}^\varepsilon_{t,x}; \left(p(\widetilde{\varrho}, \widetilde{S}) - (\widetilde{\varrho} - r) \partial_\varrho p(r, \widehat{S}) - (\widetilde{S} - \widehat{S}) \partial_S p(r, \widehat{S}) - p(r, \widehat{S}) \right)(t,x) \rangle \\
&\quad \mathrm{div}_x \mathbf{U}^\varepsilon(t,x) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \mathbf{U}^\varepsilon(t,x) - \widetilde{\mathbf{m}} \rangle \left(\partial_t \mathbf{U}^\varepsilon + \frac{1}{\varepsilon} \mathbf{U}^\varepsilon \cdot \nabla_x \mathbf{U}^\varepsilon \right)(t,x) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \left(\widetilde{\varrho} \frac{\widehat{S}}{r}(t,x) - \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) \right) \left(\frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t,x) \right) \right\rangle \cdot \nabla_x \Theta(t,x) \,\mathrm{d}x\,\mathrm{d}t, \\
&\quad - \frac{1}{\varepsilon} \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U}^\varepsilon(t,x) : \mathrm{d}\mathfrak{R}^\varepsilon(t, \cdot) \,\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) - \widetilde{S} \right\rangle \partial_S p(r, \widehat{S}) \mathrm{div}_x \mathbf{U}^\varepsilon(t,x) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.2}$$

Next, using the ansatz for $\mathbf{U}^\varepsilon$ from (4.1), we have

$$\begin{aligned}
&- \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \mathbb{1}_{\widetilde{\varrho}>0} \frac{(\widetilde{\varrho} \mathbf{U}^\varepsilon(t,x) - \widetilde{\mathbf{m}}) \otimes (\widetilde{\varrho} \mathbf{U}^\varepsilon(t,x) - \widetilde{\mathbf{m}})}{\widetilde{\varrho}} : \nabla_x \mathbf{U}^\varepsilon(t,x) \right\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \frac{1}{\varepsilon} \int_{\overline{\Omega}} \int_0^\tau \nabla_x \mathbf{U}^\varepsilon(t,x) : \mathrm{d}\mathfrak{R}^\varepsilon(t, \cdot) \,\mathrm{d}t
\end{aligned}$$

$$\lesssim \int_0^\tau \left(\int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right)(\tau, x) \right\rangle \, \mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\, \mathrm{trace}[\mathfrak{R}^\varepsilon](t, \cdot) \right) \mathrm{d}t. \tag{4.3}$$

Similarly, by virtue of hypothesis (2.11),

$$\begin{aligned} &-\frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \left(p(\widetilde{\varrho}, \widetilde{S}) - (\widetilde{\varrho} - r)\partial_\varrho p(r, \widehat{S}) - (\widetilde{S} - \widehat{S})\partial_S p(r, \widehat{S}) - p(r, \widehat{S}) \right)(t, x) \right\rangle \\ &\quad \mathrm{div}_x \mathbf{U}^\varepsilon(t, x) \, \mathrm{d}x \, \mathrm{d}t \\ &\lesssim \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right)(\tau, x) \right\rangle \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.4}$$

The details of the above estimates follow the same lines as e.g. [9, Section 5.4]. Consequently, inequality (4.2) reduces to

$$\begin{aligned} &\int_\Omega \left\langle \mathcal{V}^\varepsilon_{\tau,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right)(\tau, x) \right\rangle \, \mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\, \mathrm{trace}[\mathfrak{R}^\varepsilon](\tau, \cdot) \\ &\quad + \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \left| \frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t, x) \right|^2 \right\rangle \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \int_\Omega E\left(\varrho_{0,\varepsilon}, S_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon} \,\middle|\, r_0, r_0 s_0, r_0 \mathbf{U}^\varepsilon(0, x) \right) \, \mathrm{d}x \\ &\quad + \int_0^\tau \left(\int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right)(\tau, x) \right\rangle \, \mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\, \mathrm{trace}[\mathfrak{R}^\varepsilon](t, \cdot) \right) \mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho}\mathbf{U}^\varepsilon(t, x) - \widetilde{\mathbf{m}} \right\rangle \left(\partial_t \mathbf{U}^\varepsilon + \frac{1}{\varepsilon} \mathbf{U}^\varepsilon \cdot \nabla_x \mathbf{U}^\varepsilon \right)(t, x) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \left(\widetilde{\varrho} \frac{\widehat{S}}{r}(t, x) - \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) \right) \left(\frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t, x) \right) \right\rangle \cdot \nabla_x \Theta(t, x) \, \mathrm{d}x \, \mathrm{d}t, \\ &\quad + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) - \widetilde{S} \right\rangle \partial_S p(r, \widehat{S}) \mathrm{div}_x \mathbf{U}^\varepsilon(t, x). \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{4.5}$$

By virtue of our choice of the initial data (2.12),

$$\int_\Omega E\left(\varrho_{0,\varepsilon}, S_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon} \,\middle|\, r_0, r_0 s_0, r_0 \mathbf{U}^\varepsilon(0, x) \right) \, \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0.$$

Moreover, as E is strictly convex, the energy inequality (2.6) implies boundedness of the first moments

$$\left\langle \mathcal{V}^\varepsilon; \widetilde{\varrho} \right\rangle, \ \left\langle \mathcal{V}^\varepsilon; |\widetilde{\mathbf{m}}| \right\rangle \ \text{in } L^\infty(0, T; L^1(\Omega)). \tag{4.6}$$

Consequently, we deduce from (4.5)

$$\begin{aligned} &\int_\Omega \left\langle \mathcal{V}^\varepsilon_{\tau,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right)(\tau, x) \right\rangle \, \mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\, \mathrm{trace}[\mathfrak{R}^\varepsilon](\tau, \cdot) \\ &\quad + \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \left| \frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t, x) \right|^2 \right\rangle \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^\tau \left(\int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right) \right\rangle \,\mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\,\mathrm{trace}[\mathfrak{R}^\varepsilon](t,\cdot) \right) \mathrm{d}t \\
&\quad + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \left(\widetilde{\varrho} \frac{\widehat{S}}{r}(t,x) - \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) \right) \left(\frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t,x) \right) \right\rangle \cdot \nabla_x \Theta(t,x) \,\mathrm{d}x\,\mathrm{d}t, \\
&\quad + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) - \widetilde{S} \right\rangle \partial_S p(r, \widehat{S}) \mathrm{div}_x \mathbf{U}^\varepsilon(t,x) \,\mathrm{d}x\,\mathrm{d}t + h(\varepsilon),
\end{aligned}
\tag{4.7}
$$

where $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Choosing

$$
\chi(Z) = Z \text{ for } Z \leq M, \text{ where } M \text{ is large enough,}
$$

we get

$$
\begin{aligned}
&\frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \left(\widetilde{\varrho} \frac{\widehat{S}}{r}(t,x) - \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) \right) \left(\frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t,x) \right) \right\rangle \cdot \nabla_x \Theta(t,x) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \frac{1}{2\varepsilon^2} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \left| \frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t,x) \right|^2 \right\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + c \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \left| \chi \left(\frac{\widehat{S}}{r} \right)(t,x) - \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) \right|^2 \right\rangle \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \left| \chi \left(\frac{\widehat{S}}{r} \right)(t,x) - \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) \right|^2 \right\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \lesssim \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right) \right\rangle \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.8}
$$

Similarly

$$
\begin{aligned}
&\frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \chi \left(\frac{\widetilde{S}}{\widetilde{\varrho}} \right) - \widetilde{S} \right\rangle \partial_S p(r, \widehat{S}) \mathrm{div}_x \mathbf{U}^\varepsilon(t,x) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \lesssim \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right) \right\rangle \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Thus (4.7) reduces to

$$
\begin{aligned}
&\int_\Omega \left\langle \mathcal{V}^\varepsilon_{\tau,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right)(\tau,x) \right\rangle \,\mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\,\mathrm{trace}[\mathfrak{R}^\varepsilon](\tau,\cdot) \\
&\quad + \frac{1}{2\varepsilon^2} \int_0^\tau \int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; \widetilde{\varrho} \left| \frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \mathbf{U}^\varepsilon(t,x) \right|^2 \right\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^\tau \left(\int_\Omega \left\langle \mathcal{V}^\varepsilon_{t,x}; E\left(\widetilde{\varrho}, \widetilde{S}, \widetilde{\mathbf{m}} \,\middle|\, r, \widehat{S}, r\mathbf{U}^\varepsilon \right) \right\rangle \,\mathrm{d}x + \Lambda \int_\Omega \mathrm{d}\,\mathrm{trace}[\mathfrak{R}^\varepsilon](t,\cdot) \right) \mathrm{d}t + h(\varepsilon).
\end{aligned}
$$

Consequently, the conclusion of Theorem 2.2 follows by a direct application of Gronwall's lemma and the bounds (4.6).

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REFERENCES

[1] D. Breit, E. Feireisl, and M. Hofmanová. Dissipative solutions and semiflow selection for the complete Euler system. *Comm. Math. Phys.*, **376**(2):1471–1497, 2020.

[2] J. Březina and E. Feireisl. Measure-valued solutions to the complete Euler system revisited. *Z. Angew. Math. Phys.*, 69(3):69-57, 2018.

[3] J. Březina and E. Feireisl. Measure-valued solutions to the complete Euler system. *J. Math. Soc. Japan*, **70**(4):1227–1245, 2018.

[4] J. A. Carrillo, T. Dębiec, P. Gwiazda, A. Świerczewska-Gwiazda. Dissipative measure-valued solutions to the Euler-Poisson equation. *SIAM J. Math. Anal.*, **56**(1): 304–335, 2024

[5] G.-Q. Chen and H. Frid. Uniqueness and asymptotic stability of Riemann solutions for the compressible Euler equations. *Trans. Amer. Math. Soc.*, **353**(3):1103–1117 (electronic), 2001.

[6] C. M. Dafermos and R. Pan. Global BV solutions for the p-system with frictional damping. *SIAM J. Math. Anal.*, **41**(3):1190–1205, 2009.

[7] E. Feireisl and M. Hofmanová. Randomness in compressible fluid flows past an obstacle. *Journal of Statistical Physics*, **186**:32–, 2022.

[8] E. Feireisl, C. Klingenberg, O. Kreml, and S. Markfelder. On oscillatory solutions to the complete Euler system. *J. Differential Equations*, **269**(2):1521–1543, 2020.

[9] E. Feireisl, M. Lukáčová-Medvidová, H. Mizerová, and B. She. *Numerical analysis of compressible fluid flows*. Springer-Verlag, Cham, 2022.

[10] P. Gwiazda, O. Kreml, A. Świerczewska-Gwiazda Dissipative measure-valued solutions for general conservation laws *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **37**(3): 683–707, 2020

[11] D. Kröner and W. M. Zajaczkowski. Measure-valued solutions of the Euler equations for ideal compressible polytropic fluids. *Math. Methods Appl. Sci.*, **19**(3):235–252, 1996.

[12] C. Lattanzio and A. E. Tzavaras. Relative entropy in diffusive relaxation. *SIAM J. Math. Anal.*, **45**(3):1563–1584, 2013.

[13] C. Lattanzio and A. E. Tzavaras. From gas dynamics with large friction to gradient flows describing diffusion theories. *Comm. Partial Differential Equations*, **42**(2):261–290, 2017.

[14] P. Marcati and A. Milani. The one-dimensional Darcy's law as the limit of a compressible Euler flow. *J. Differential Equations*, **84**(1):129–147, 1990.

[15] T. C. Sideris, B. Thomases, and D. Wang. Long time behavior of solutions to the 3D compressible Euler equations with damping. *Comm. Partial Differential Equations*, **28**(3-4):795–816, 2003.

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