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An L^1-theory for a nonlinear temporal periodic problem involving $p(x)$-growth structure with a strong dependence on gradients

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Abstract. We investigate the existence of a time-periodic solution to a nonlinear evolution equation involving $p(x)$-growth conditions with irregular data. We tackle our problem in a suitable functional setting by considering the so-called variable exponent Lebesgue and Sobolev spaces. By assuming that the data belongs only to L^1, we prove the existence of a renormalized time-periodic solution to the studied model.

1. Introduction

The study of nonlinear partial differential equations has undergone a great revolution in several fields of applied sciences. Relevant examples can be found in the modeling of many biological, physical, ecological, and chemical phenomena. Among these are temperature distribution, heat diffusion, population growth, heat control, and cellular neural networks. As well known, modeling such phenomena requires the use of some mathematical tools. In particular, standard Lebesgue and Sobolev spaces L^p and $W^{1,p}$ with constant exponent p can be used to model many materials with sufficient certainty. But when we talk about nonhomogeneous materials (we refer for example to "smart fluids"), this approach failed and proved its limitations in applications. The class of Lebesgue and Sobolev spaces $L^{p(x)}$ and $W^{1,p(x)}$ with variable exponent $p(x)$ are the most adequate spaces to describe these kinds of materials. The application fields of these spaces are various and so rich. Worth mentioning are the study of electrorheological fluids [29,42,44], robotics and thermorheological fluids [30,36,43], epidemiology modeling [10,11,45] and image processing [5,20].

Motivated by the application fields of PDEs with variable exponent, we aim in this work to investigate the existence of a time-periodic solution to a nonlinear parabolic equation with $p(x)$-growth structure whose model is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + g(t,x,u,\nabla u) = f(t,x) & \text{in } Q_T \\ u(0,\cdot) = u(T,\cdot) & \text{in } \Omega \\ u = 0 & \text{on } \Sigma_T. \end{cases} \tag{1}$$

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where Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $T > 0$ is the period, $Q_T := (0, T) \times \Omega$, $\Sigma_T := (0, T) \times \partial\Omega$, f is a measurable function periodic in time with period T and belonging to $L^1(Q_T)$, $g : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, periodic in time with period T and enjoying some growth assumptions to be specified later and $p(\cdot)$ is a continuous function on $\overline{\Omega}$ such that $\inf_{x\in\overline{\Omega}} p(x) > 1$.

The interest in taking into account the study of periodic parabolic problems comes from several facts. Indeed, periodic behavior may exhibit space dispersion of many physical, ecological, chemical, or biological processes. It is often used to describe real phenomena which behave periodically over time, see for example [6,7,27]. To proceed with our presentation, we propose to recollect some previous works in the literature which rely strongly on our problems. We begin with some former papers involving specific cases of (1) with initial conditions u_0. In Bendahmane et al. [9] studied problem (1) with an initial data u_0 belonging to L^1 but without nonlinearity ($g \equiv 0$). The authors proved the existence of a unique renormalized solution by combining semi-group theory with the approximation method. They established also the equivalence between the renormalized and entropy notions of solutions to this kind of boundary value problem. Further, in the same equation was treated by Zhang and Zhou Bendahmane et al. The authors proposed another theoretical approach to show the existence and uniqueness of a renormalized and entropy solution. Their method was based on the combination of a time semi-discretization method with the approximation approach. In Li and Go [34] generalized the both mentioned works [9,50] by adding a lower-order term which depends on the solutions and their gradients. By assuming that the nonlinearity meets some growth structure with a sign condition, the authors showed the existence of a renormalized solution. Likewise to the above-mentioned papers, the existence was done via the approximations method. In [28], a study on semilinear parabolic equations with $p(x)$-growth conditions has been considered by Rădulescu et al. They tackled an initial-boundary value problem involving $p(x)$-Laplacian operator with a nonlinear source term $f(u)$. The authors examined the existence of a weak solution by assuming that the variable exponent $p(x)$ satisfies some conditions. They also investigated uniqueness when $f(u)$ is a locally Lipschitz continuous function with respect to u. Quasilinear parabolic problems with variable exponent were also examined in recent years. We mainly refer the readers to see [1,16,26,47,49] and the references therein.

In parallel, manifold works have been dedicated to investigating the existence, uniqueness and asymptotic behaviors of periodic solutions to parabolic problems. We start by recalling some papers dealing on periodic PDEs but with constant exponent. The book by Hess [31] introduced a qualitative analysis of periodic solutions to partial differential equations with regular data. The authors used the sub-and super-solutions method to prove the existence of classical periodic solutions when the data belongs to suitable Hölder space. The book by Lions [35] offers the reader a comprehensive introduction to periodic parabolic problems with constant exponent $p(x) \equiv p$. The author displayed the existence, uniqueness and regularities properties of weak periodic solutions to p-Laplacian parabolic equations via monotone operator theory. In Deuel and Hess [22] studied Eq. (1) when the nonlinearity g satisfy a particular growth assumption and f belongs to $L^{\frac{p}{p-1}}(Q_T)$ for a constant $p > 1$. Under the fact that a pair of bounded sub- and super-solutions are known, the authors showed existence of a periodic bounded solution in the weak sense. Besides the aforementioned papers, there is an abundance of literature regarding the case $p(x) = 2$ see [17–19]. Various theoretical approaches have been considered to investigate the existence of periodic solutions to different kinds of nonlinear parabolic equations. However, there are rare finding results concerning periodic parabolic equations with $p(x)$-growth conditions. Only a few papers have been recently dedicated to investigating this topic. In the paper, Fragnelli [27] studied a system of $(p(x), q(x))$-Laplacian parabolic equations involving having time-periodic conditions with nonlocal terms. Under the assumptions that $\inf_{x\in\overline{\Omega}} p(x) > 2$ and $\inf_{x\in\overline{\Omega}} q(x) > 2$, the author disused the existence of a positive weak periodic solution in L^2 framework. Their method was based on the implementation of Leray-Schauder's topological degree. The paper [2] by Akagi and Matsuura dealt with the existence and uniqueness of a periodic solution to (1) when $g = 0$. The authors assumed that f belongs to $L^2(Q_T)$ and $\inf_{x\in\overline{\Omega}} p(x) > \max\{1, 2N/N+2\}$ to tackle the problem in L^2-setting. They employed the subdifferential approach to derive the existence and uniqueness of periodic L^2-solution. Recently, Charkaoui et al. [4] investigated a periodic $M \times M$ system involving Leray-Lions-type operators with a variable exponent. By combining the sub- and super-solution method with Leray-Schauder topological degree, they proved the existence of weak periodic solutions when the data are regular enough. We point out that until now the investigation of periodic problems with $p(x)$-growth structure and irregular data are more limited in the literature. Let us mention that, we have recently intended this gap in our work [15] by considering (1) with $g \equiv 0$ and f belongs only to $L^1(Q_T)$. We have discussed two existence and uniqueness results of periodic solutions to the considered problem. In the first one, we tackled the problem when f belongs to a suitable Bochner space. Based on monotone operator theory, we showed the existence and uniqueness of the weak periodic solution. For the second result, we studied the existence and uniqueness of periodic solution when f belongs only to $L^1(Q_T)$. This fact forced us to adapt a novel notion of solution which we have called a *renormalized periodic* solution. Our approach was based on the approximation method and involves some technical estimates.

Motivated by the above discussion, we are going to study the solvability of problem (1) in a more general framework by taking weak regularity on the data f associated with natural assumptions on the nonlinearity g. We will assume that the variable exponent satisfies $\inf_{x\in\overline{\Omega}} p(x) > 1$ and the source term f belongs only to $L^1(Q_T)$. As well known, we require to suppose that g meets a sign condition hypothesis. Further, we shall assume that g has nonstandard growth conditions which involve the variable exponent $p(x)$. All these assumptions guarantee that problem (1) contributes to enriching literature on periodic parabolic equations not only about problems having $p(x)$-growth conditions but also those involving constant growth. Note that the consideration of this hypothesis kinds complicates the existence proof which leads to managing several major difficulties. More precisely, the presence of the L^1 term causes difficulty to deal with problem (1) via the standard weak notion of periodic solution. To manage this difficulty, we shall follow our recent work [15] to adapt the notion of *renormalized periodic* solutions to problem (1). We would like to mention that this notion was first suggested by DiPerna and Lions [25] to investigate the Boltzmann's equation. And it was extended to different types of PDEs such as linear and nonlinear, more precisely those involving irregular terms see [3,12,13,32,38,39,48]. Another difficulty appears from the $p(x)$-growth structure of the nonlinearity g which leads problem (1) to behave completely differently from the considered problems in [9,15,50]. To manage the last one, we will inspire by the classical Porretta's work [40] to develop a new adequate approach. We also point out that one of the major difficult parts arises from the time-periodic condition appearing in (1). In fact, we need to build a sequence (u_n) of approximate solutions to (1) which is periodic in time. To overcome this we shall ensure the existence of such sequence via the result of Charkaoui et al. [4]. Thus, we should carefully estimate (u_n) to derive good compactness results and therefore prove that the limit of this sequence is time-periodic in a suitable sense.

We have structured the rest of our paper as follows. We start Sect. 2 with a brief recall of the variable exponent Lebesgue-Sobolev spaces as well as some basic properties and useful relationships. Section 3 is devoted to enunciating our main results. We will state the needed assumptions to deal with (1), we will put forward the notion of a *renormalized periodic* solution to (1) and we state our existence theorem. In Sect. 4, we detail the proof of our main results. We will divide this part into four subsections. In the first one, we introduce a well-posed approximate problem to (1). The second subsection will be reserved to establish prior estimates on the approximate solution. In the third subsection, we demonstrate the strong convergence of the truncation in a suitable Banach space. And finally, the fourth subsection deal with the passing to the limit on the approximate problem.

2. Mathematical backgrounds

For the reader's convenience, we will briefly recall some definitions, properties and valuable relationships of Lebesgue and Sobolev spaces with variable exponent. For a more detailed presentation, we refer the readers to see the books by Antontsev and Shmarev [8], Diening et al. [24], Rădulescu and Repovš [41].

2.1. Lebesgue and Sobolev spaces with variable exponent

For a given $p \in \mathcal{C}\left(\overline{\Omega}\right)$, we define the following real values

$$p^- = \inf_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^+ = \sup_{x\in\overline{\Omega}} p(x).$$

Further, we introduce the sets

$$\mathcal{E}_1 := \left\{p \in \mathcal{C}\left(\overline{\Omega}\right) : p^- > 1\right\}, \qquad \mathcal{M}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable}\}.$$

The Lebesgue space with variable exponent $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega) = \left\{ u \in \mathcal{M}(\Omega) \ : \ \rho_{p(x)}(u) < \infty \right\},$$

where $\rho_{p(\cdot)}$ designates the following convex modular

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

We consider on $L^{p(x)}(\Omega)$ the so-called Luxemburg norm

$$\|u\|_{p(x)} = \inf \left\{ \mu > 0, \quad \rho_{p(x)}\left(\frac{u}{\mu}\right) \leq 1 \right\}.$$

Among the interesting properties, we find that when $p(\cdot)$ belongs to $\mathcal{E}_1$, the space $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a separable, reflexive Banach space. In addition, for any $p(\cdot) \in \mathcal{E}_1$, we set $p'(x) = \frac{p(x)}{p(x)-1}$ as the conjugate exponent of $p(\cdot)$. Further, we designate by $L^{p'(x)}(\Omega)$ the dual space of $L^{p(x)}(\Omega)$. The succeeding proposition summarizes some useful inequalities known by $p(x)$-Hölder inequalities.

Proposition 1. *Let $p(\cdot) \in \mathcal{E}_1$. For any couple $(u, v) \in L^{p(x)}(\Omega) \times L^{p'(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \leq 2 \|u\|_{p(x)} \|v\|_{p'(x)}.$$

Moreover, if we have $\frac{1}{p(x)} + \frac{1}{p'(x)} + \frac{1}{p''(x)} = 1$, then

$$\left| \int_\Omega uvw \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} + \frac{1}{(p^-)''} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \|w\|_{p''(x)}$$
$$\leq 3 \|u\|_{p(x)} \|v\|_{p'(x)} \|w\|_{p''(x)},$$

for all $(u, v, w) \in L^{p(x)}(\Omega) \times L^{p'(x)}(\Omega) \times L^{p''(x)}(\Omega)$.

We extend the variable exponent $p : \overline{\Omega} \to (1, +\infty)$ to $\overline{Q_T} = [0, T] \times \overline{\Omega}$ by setting $p(t, x) := p(x)$ for all $(t, x) \in \overline{Q_T}$. Extending the variable exponent comes with a great advantage. It allows the obtainment of an interpolation result that will serve to acquire important a-priori-estimates (see [9] for more details). On that account, the variable exponent Lebesgue space $L^{p(x)}(Q_T)$ is presented as

$$L^{p(x)}(Q_T) = \left\{ u \in \mathcal{M}(Q_T) \ : \ \int_{Q_T} |u(t, x)|^{p(x)} dx \, dt < \infty \right\}.$$

Equipped with the following norm

$$\|u\|_{p(x)} = \inf \left\{ \mu > 0, \quad \int_{Q_T} \left| \frac{u(t, x)}{\mu} \right|^{p(x)} dx \, dt \leq 1 \right\},$$

the space $\left(L^{p(x)}(Q_T), \|\cdot\|_{p(x)}\right)$ is a separable, reflexive Banach. The Sobolev space with variable exponent is defined follows as

$$W^{1,\,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega), \quad |\nabla u| \in \left(L^{p(x)}(\Omega)\right)^N \right\}.$$

The associated standard norm is given by

$$\|u\|_{1,\,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

A popular equivalent norm is given as

$$\|u\|_{1,p(x)} = \inf \left\{ \mu > 0, \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

Hereinafter referred, $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e., there exists a constant C such that

$$|p(x_1) - p(x_2)| \leq \frac{C}{-log|x_1 - x_2|}, \ \forall x_1, x_2 \in \Omega, \ \text{ with } |x_1 - x_2| < \frac{1}{2}. \tag{2}$$

The last assumption was considered by Zhikov in [51] to deal with the Lavrentiev phenomenon. Furthermore, hypothesis (2) ensures that the space of smooth functions $\mathcal{C}_c^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$. For added convenience, we define $W_0^{1,p(x)}(\Omega) := \overline{\mathcal{C}_c^\infty(\Omega)}^{W^{1,p(x)}(\Omega)}$ and we denote $\left(W_0^{1,p(x)}(\Omega)\right)^*$ its dual space. We will designate by $\langle \cdot, \cdot \rangle$ the duality pairing between $\left(W_0^{1,p(x)}(\Omega)\right)^*$ and $W_0^{1,p(x)}(\Omega)$. The spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach's. Further, for any $u \in W_0^{1,p(x)}(\Omega)$, we have the following $p(x)$-Poincaré inequality

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)}, \tag{3}$$

where C is a constant depending only on $p(\cdot)$ and Ω. Subsequently, the following is a validated norm on $W_0^{1,p(x)}(\Omega)$

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}.$$

The following assertions describe several useful properties and relationships of Lebesgue and Sobolev spaces with variable exponents.

Proposition 2. *1. For any $u \in L^{p(x)}(\Omega)$, we have the following relationships*

$$\min\left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+} \right\} \leq \rho_{p(x)}(u) \leq \max\left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+} \right\}. \tag{4}$$

$$\min\left\{ \rho_{p(x)}^{\frac{1}{p^-}}(u), \rho_{p(x)}^{\frac{1}{p^+}}(u) \right\} \leq \|u\|_{p(x)} \leq \max\left\{ \rho_{p(x)}^{\frac{1}{p^-}}(u), \rho_{p(x)}^{\frac{1}{p^+}}(u) \right\}. \tag{5}$$

2. Let (u_n) be a sequence in $L^{p(x)}(\Omega)$, then the following statements are equivalent:

(i) $\lim_{n\to+\infty} \|u_n - u\|_{p(x)} = 0.$

(ii) $\lim_{n\to+\infty} \rho_{p(x)}(u_n - u) = 0.$

(iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to+\infty} \rho_{p(x)}(u_n) = \rho_{p(x)}(u).$

Proposition 3. *1. Let $p_1(\cdot), p_2(\cdot) \in \mathcal{E}_1$ such that $p_1(x) \le p_2(x)$ almost everywhere in Ω. Then, we have the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.*

2. Let $p(\cdot), q(\cdot) \in \mathcal{E}_1$ such that $1 \le q(x) < p^*(x)$, for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N \\ +\infty, & p(x) \ge N. \end{cases}$$

2.2. Functional framework

In this segment, we present the functional framework which will be considered in the solvability of problem (1). For any $0 < T < +\infty$, we define the time space

$$L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)) = \left\{ u \in L^{p(x)}(Q_T) : \int_0^T \|\nabla u\|_{p(x)}^{p^-} dt < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p^-}\left(0,T;W_0^{1,p(x)}(\Omega)\right)} = \left(\int_0^T \|\nabla u\|_{p(x)}^{p^-} dt \right)^{\frac{1}{p^-}}.$$

We recall the customary space $\mathcal{U}(Q_T)$ often considered in the studies of parabolic problems with variable exponent

$$\mathcal{U}(Q_T) = \left\{ u \in L^{p^-}\left(0, T; W_0^{1,p(x)}(\Omega)\right) : |\nabla u| \in L^{p(x)}(Q_T)^N \right\},$$

and of which, the associated norm reads

$$\|u\|_{\mathcal{U}(Q_T)} = \|\nabla u\|_{L^{p(x)}(Q_T)}.$$

Due to $p(x)$-Poincaré inequality (3) and the continuity of the embedding $L^{p(x)}(Q_T) \hookrightarrow L^{p^-}(0, T; L^{p(x)}(\Omega))$ the norm $\|\cdot\|_{\mathcal{U}(Q_T)}$ is equivalent to the following standard norm

$$\|u\|_{\mathcal{U}(Q_T)} = \|u\|_{L^{p^-}\left(0,T;W_0^{1,p(x)}(\Omega)\right)} + \|\nabla u\|_{L^{p(x)}(Q_T)}.$$

The space $\mathcal{U}(Q_T)$ is a separable reflexive Banach and we designates by $\mathcal{U}(Q_T)^*$ its dual space. The following Lemma relatively resumes the interesting properties of the space to our work.

Lemma 1. [9] *Let $\mathcal{U}(Q_T)$ be the space defined as above. Then,* *(i) The following is a continuous dense embedding*

$$L^{p^+}(0,T;W_0^{1,p(x)}(\Omega)) \hookrightarrow \mathcal{U}(Q_T) \hookrightarrow L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)). \tag{6}$$

In particular, and since $\mathcal{C}_c^\infty(Q_T)$ *is dense in* $L^{p^+}(0,T;W_0^{1,p(x)}(\Omega))$, *then it is also dense in* $\mathcal{U}(Q_T)$. *Similarly, the corresponding dual spaces satisfy*

$$L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega))^*) \hookrightarrow \mathcal{U}(Q_T)^* \hookrightarrow L^{(p^+)'}(0,T;(W_0^{1,p(x)}(\Omega))^*). \tag{7}$$

(ii) The elements of $\mathcal{U}(Q_T)^*$ *are represented in this fashion: For all* $\zeta \in \mathcal{U}(Q_T)^*$, *there exists* $\xi = (\xi_1, \ldots, \xi_N) \in (L^{p'(x)}(Q_T))^N$ *such that:* $\zeta = \mathrm{div}(\xi)$ *and*

$$< \zeta, \varphi >_{\mathcal{U}(Q_T)^*, \mathcal{U}(Q_T)} = \int_{Q_T} \xi \nabla\varphi dxdt,$$

for any $\varphi \in \mathcal{U}(Q_T)$. *Furthermore, we have*

$$\|\zeta\|_{\mathcal{U}(Q_T)^*} = max\{\|\xi_i\|_{L^{p(x)}(Q_T)}, i = 1, \ldots, N\}.$$

(iii) For any $u \in \mathcal{U}(Q_T)$, *we have*

$$\min\left\{\|u\|^{p^-}_{\mathcal{U}(Q_T)}, \|u\|^{p^+}_{\mathcal{U}(Q_T)}\right\} \le \int_{Q_T} |\nabla u|^{p(x)}\, dxdt \le \max\left\{\|u\|^{p^-}_{\mathcal{U}(Q_T)}, \|u\|^{p^+}_{\mathcal{U}(Q_T)}\right\}. \tag{8}$$

Now, we are ready to introduce the following functional space.

$$\mathcal{W}(Q_T) := \left\{u \in \mathcal{U}(Q_T); \quad \frac{\partial u}{\partial t} \in \mathcal{U}(Q_T)^* + L^1(Q_T)\right\}.$$

The interest into taking account the introduction of space $\mathcal{W}(Q_T)$ can be viewed in the following lemma which gives some interesting embedding results.

Lemma 2. *Let* $\mathcal{W}(Q_T)$ *be the space defined above. Then, we have*

$$\mathcal{W}(Q_T) \hookrightarrow \mathcal{C}\left([0,T];L^1(\Omega)\right). \tag{9}$$

$$\mathcal{W}(Q_T) \cap L^\infty(Q_T) \hookrightarrow \mathcal{C}\left([0,T];L^2(\Omega)\right). \tag{10}$$

We can show the result of Lemma 2 by following the same lines as the proof of the case $p(\cdot) = p$ constant, see Theorem 1.1 from [40].

2.3. Some truncation functions

In this paragraph, we state the used truncation functions in our paper. Furthermore, we exhibit some fundamental results of functional analysis.

(i) For any integer $k > 0$, we define $T_k : \mathbb{R} \to \mathbb{R}$ the truncation function at height k as

$$T_k(r) = \begin{cases} r & \text{if } |r| \le k \\ k\, sign(r) & \text{if } |r| > k, \end{cases} \tag{11}$$

where,

$$sign(r) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ -1 & \text{if } r < 0. \end{cases}$$

(ii) For given integers $a, k > 0$, we introduce the truncation function $T_{k,a}(s) = T_a\,(s - T_k(s))$ which reads as

$$T_{k,a}(s) = \begin{cases} s - k\,\text{sign}(s) & \text{if } k \le |s| < k + a \\ a\,\text{sign}(s) & \text{if } |s| \ge k + a \\ 0 & \text{if } |s| \le k. \end{cases} \tag{12}$$

We will designate by $\theta_k\,(\cdot)$ and $\theta_{k,a}\,(\cdot)$, respectively, the primitive functions of $T_k\,(\cdot)$ and $T_{k,a}\,(\cdot)$, which given by

$$\theta_k(r) = \int_0^r T_k(s)ds, \quad \theta_{k,a}(r) = \int_0^r T_{k,a}(s)ds.$$

(iii) Let us consider $s(\cdot)$ a positive $\mathcal{C}^\infty(\mathbb{R})$ function such that

$$s(z) = \begin{cases} 1 & \text{if } |z| \le 1 \\ 0 & \text{if } |z| \ge 2 \end{cases} \quad \text{and} \quad 0 \le s(z) \le 1 \text{ for all } z \in \mathbb{R}.$$

For any integer $i \ge 2$, we define the truncation function $S_i(r) = \int_0^r s_i(z)dz$ where

$$s_i(z) = \begin{cases} 1 & \text{if } |z| \le i - 1 \\ s(z - (i-1)sign(z)) & \text{if } |z| \ge i - 1. \end{cases}$$

We can easy to verify that for $i \ge 2$, the truncation function $S_i(\cdot)$ fulfills the following properties

$$\begin{cases} S_i(r) = S_i(T_{i+1}(r)), \quad \|S_i'\|_{L^\infty(\mathbb{R})} \le \|s\|_{L^\infty(\mathbb{R})}, \quad \|S_i''\|_{L^\infty(\mathbb{R})} \le 1 \\ \\ \text{supp } S_i' \subset [-(i+1), (i+1)], \quad \text{supp } S_i'' \subset [-(i+1), -i] \cup [i, (i+1)]. \end{cases} \tag{13}$$

In the ensuing lemma, we recall the famous Lebesgue generalized convergence theorem which will be frequently used in several limit processes.

Lemma 3. ([23]) *Let (f_n) be a sequence of measurable functions and f a measurable function such that $f_n \to f$ a.e. in Q_T. Let $(g_n) \subset L^1(Q_T)$ such that for all $n \in \mathbb{N}$, we have $|f_n| \le g_n$ a.e. in Q_T and $g_n \to g$ in $L^1(Q_T)$. Then*

$$\int_{Q_T} f_n dxdt \to \int_{Q_T} f dxdt.$$

Due to the presence of nonlinear terms in (1), we will need the following technical lemma to get prior estimates on the gradient of the solution.

Lemma 4. ([14,37]) *Let* $\phi_\lambda(s) = se^{\lambda s^2}$, $s \in \mathbb{R}$, $\lambda \geq 0$; *and let* $\Phi_\lambda(s) = \int_0^s \phi_\lambda(\xi)\,d\xi$. *Then*

$$\phi_\lambda(0) = 0, \quad \Phi_\lambda(s) \geq 0, \quad \phi'_\lambda(s) > 0.$$

When $\lambda \geq \frac{b^2}{4a^2}$ *is fixed, the following relationships hold true*

$$a\phi'_\lambda(s) - b|\phi_\lambda(s)| \geq \frac{a}{2}, \quad \forall s \in \mathbb{R}. \tag{14}$$

For any $(\mathfrak{a}, \mathfrak{b}) \in \mathbb{R}^N \times \mathbb{R}^N$, we recall the following well-known inequality

$$\left(|\mathfrak{a}|^{p(x)-2}\mathfrak{a} - |\mathfrak{b}|^{p(x)-2}\mathfrak{b}\right) \cdot (\mathfrak{a} - \mathfrak{b}) \geq \begin{cases} 2^{2-p^+}|\mathfrak{a} - \mathfrak{b}|^{p(x)}, & \text{if } p(x) \geq 2 \\ \left(p^- - 1\right) \frac{|\mathfrak{a}-\mathfrak{b}|^2}{(|\mathfrak{a}|+|\mathfrak{b}|)^{2-p(x)}}, & \text{if } 1 < p(x) < 2, \end{cases} \tag{15}$$

In the whole of this paper, we denote by χ_ω the characteristic function of a measurable set ω and we use for simplicity the notation $\{|\psi| > k\}$ to designate the measurable subset $\{(t, x) \in Q_T,\ |\psi(t, x)| > k\}$. Furthermore, we denote by C every generic and positive constant. The value of this constant can change in different situations. It may depend on the given data but always remains independent of the estimated sequence index.

3. Main results

Throughout this paper, we assume that $p(\cdot)$ belongs to $\mathcal{E}_1$ and satisfies the log-Hölder continuity condition (2). In addition, we present in the following items our hypothesis on the source data f and on the nonlinearity term. We assume that

($\mathcal{A}_1$): f is a measurable function belonging to $L^1(Q_T)$.

($\mathcal{A}_2$): $g : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carath éodory function, namely

$$(s, r) \mapsto g(t, x, s, r) \text{ is continuous for a.e } (t, x) \in Q_T, \tag{16}$$

$$(t, x) \mapsto g(t, x, s, r) \text{ is measurable for all } (s, r) \in \mathbb{R} \times \mathbb{R}^N. \tag{17}$$

($\mathcal{A}_3$): there exists a nonnegative measurable function H belonging to $L^1(Q_T)$ such that

$$|g(t, x, s, r)| \leq \mathfrak{c}(|s|)\left(H(t, x) + |r|^{p(x)}\right), \tag{18}$$

for all (s, r) in $\mathbb{R} \times \mathbb{R}^N$ and for a.e (t, x) in Q_T, with $\mathfrak{c} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

($\mathcal{A}_4$): g enjoys the following sign condition

$$g(t,x,s,r)s \geq 0 \quad \text{for all } (s,r) \in \mathbb{R} \times \mathbb{R}^N \text{ and for a.e } (t,x) \in Q_T. \tag{19}$$

To solve our problem, we need to define an adapted notion of solution to (1) which involves the above assumptions. For this reason, we begin initially by giving the notion of a very weak gradient.

Proposition 4. *Let $u : Q_T \to \mathbb{R}$ be a measurable function such that for every $k > 0$, we have $T_k(u) \in \mathcal{U}(Q_T)$. Then, there exists a unique measurable function $v : Q_T \to \mathbb{R}^N$ called the very weak gradient of u and denote $v = \nabla u$, which satisfies for all $k > 0$*

$$v = \nabla T_k(u) \ \text{ a.e on the set } \{|u| < k\}.$$

Moreover, when u belongs to $L^1\left(0, T; W_0^{1,1}(\Omega)\right)$ the very weak gradient v coincides with the gradient of u.

To prove Proposition 4, we use the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega)$ and we follow immediately the same reasoning from Lemma 2.1 in [12]. In the following definition, we present the notion of *renormalized periodic* solution which we will consider to solve problem (1).

Definition 1. We call *renormalized periodic* solution to problem (1) all measurable function $u : Q_T \to \mathbb{R}$ which satisfies for every $k > 0$

$$T_k(u) \in \mathcal{U}(Q_T), \tag{20}$$

$$g(t,x,u,\nabla u) \in L^1(Q_T), \tag{21}$$

$$\lim_{k \to +\infty} \int_{\{k \leq |u| \leq k+1\}} |\nabla u|^{p(x)} dxdt = 0, \tag{22}$$

and, for any function $S \in W^{2,\infty}(\mathbb{R})$ which is $\mathcal{C}^1$-piecewise such that S' has a compact support, we have

$$\begin{aligned} &\frac{\partial S(u)}{\partial t} - \operatorname{div}\left(S'(u)|\nabla u|^{p(x)-2}\nabla u\right) \\ &+S''(u)|\nabla u|^{p(x)} + g(t,x,u,\nabla u)S'(u) = fS'(u) \ \text{ in } \mathcal{D}'(Q_T). \end{aligned} \tag{23}$$

Moreover, the periodicity condition is fulfilled in the following sense

$$S(u)(0) = S(u)(T) \text{ a.e in } \Omega. \tag{24}$$

Remark 1.

1. Observe that if u is a *renormalized periodic* solution to (1). Then, $S(u) \in \mathcal{U}(Q_T) \cap L^\infty(Q_T)$ and $\frac{\partial S(u)}{\partial t} \in \mathcal{U}(Q_T)^* + L^1(Q_T)$. Employing this fact with (10), we derive that $S(u) \in \mathcal{C}([0,T]; L^2(\Omega))$. This proves that *renormalized periodic* condition (24) makes sense.

2. Using the density property of $\mathcal{C}_c^\infty(Q_T)$ in $\mathcal{U}(Q_T)$, we can approach any function $\varphi \in \mathcal{U}(Q_T) \cap L^\infty(Q_T)$ by a sequence (φ_n) of regular functions belonging in $\mathcal{C}_c^\infty(Q_T)$ such that (φ_n) converges to φ strongly in $\mathcal{U}(Q_T)$ and weak-* in $L^\infty(Q_T)$. This fact allows us to take test functions of (23) not only in $\mathcal{C}_c^\infty(Q_T)$ but also in $\mathcal{U}(Q_T) \cap L^\infty(Q_T)$. And therefore we can reformulate Eq. (23) as follows

$$\left\langle \frac{\partial S(u)}{\partial t}, \varphi \right\rangle + \int_{Q_T} S'(u)\, |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dxdt + \int_{Q_T} S''(u)\, |\nabla u|^{p(x)} \varphi \, dxdt$$
$$+ \int_{Q_T} g(t,x,u,\nabla u) S'(u)\, \varphi \, dxdt = \int_{Q_T} f S'(u)\, \varphi \, dxdt,$$

for all test function $\varphi \in \mathcal{U}(Q_T) \cap L^\infty(Q_T)$. Where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathcal{U}(Q_T)^* + L^1(Q_T)$ and $\mathcal{U}(Q_T) \cap L^\infty(Q_T)$.

The succeeding theorem summarized the main results of our work.

Theorem 1. *Under assumptions* $(\mathcal{A}_1)$*–*$(\mathcal{A}_4)$*, problem* (1) *has a renormalized periodic solution* u *which satisfies the conditions of Definition* 1.

4. Proof of the main results

We are concerned by the proof of Theorem 1. To do this, we will follow the approximation method. This approach sits on the construction of an approximate problem of (1) by truncating the nonlinearity g and the source term f to become regular enough. Afterward, we shall establish some prior estimates on the approximate periodic solution. After that, we will pass the limit in all the terms of the approximate scheme by employing the strong convergences of truncations.

4.1. Approximation scheme

Let $n \in \mathbb{N}^*$, we start by approximating the nonlinearity g and the source term f as follows

$$g_n(t,x,s,r) := \frac{g(t,x,s,r)}{1 + \frac{1}{n}\,|g(t,x,s,r)|}, \quad f_n(t,x) = T_n\left(f(t,x)\right),$$

for all (s,r) in $\mathbb{R} \times \mathbb{R}^N$ and for almost (t,x) in Q_T. One has no difficulty verifying that (g_n) and (f_n) are bounded for every fixed $n \in \mathbb{N}^*$. Moreover, the truncated function (g_n) meets the same hypothesis of g such as $(\mathcal{A}_2)$-$(\mathcal{A}_4)$. Furthermore, we can easy to check that

$$(f_n) \to f \text{ in } L^1(Q_T) \quad \text{and} \quad \|f_n\|_{L^1(Q_T)} \le \|f\|_{L^1(Q_T)}. \tag{25}$$

Now, we are ready to approximate problem (1) as follows

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}\left(|\nabla u_n|^{p(x)-2}\nabla u_n\right) + g_n(t,x,u_n,\nabla u_n) = f_n(t,x) & \text{in } Q_T \\ u_n(0,\cdot) \;=\; u_n(T,\cdot) & \text{in } \Omega \\ u_n = 0 & \text{on } \Sigma_T. \end{cases} \tag{26}$$

Since $p(\cdot)$ belongs to $\mathcal{E}_1$, we have $L^{p^-}(Q_T) \hookrightarrow L^1(Q_T)$ which allows us to deduce that $L^\infty(Q_T) \hookrightarrow L^{(p^-)'}(Q_T)$. Then, by using the fact that (f_n) and (g_n) are bounded, we can employ the results of Theorem 1 from [4] to ensure the existence of $u_n \in \mathcal{U}(Q_T) \cap \mathcal{C}([0,T]; L^2(\Omega))$ a weak periodic solution to (26) which satisfies the following conditions

$$\begin{aligned} &\frac{\partial u_n}{\partial t} \in \mathcal{U}(Q_T)^*, \quad u_n(0,\cdot) = u_n(T,\cdot) \text{ in } L^2(\Omega) \\ &\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\varphi\, dxdt \\ &\qquad + \int_{Q_T} g_n(t,x,u_n,\nabla u_n)\varphi\, dxdt = \int_{Q_T} f_n\varphi\, dxdt, \end{aligned} \tag{27}$$

for all test function $\varphi \in \mathcal{U}(Q_T)$.

4.2. A priori estimates

In this paragraph, we establish some a priori estimates on the approximate solution (u_n).

Lemma 5. *Let (u_n) be the sequence defined as above. Then, we have*

(i)

$$\min\left\{ \|T_k(u_n)\|^{p^-}_{\mathcal{U}(Q_T)}, \|T_k(u_n)\|^{p^+}_{\mathcal{U}(Q_T)} \right\} \le k\|f\|_{L^1(Q_T)}. \tag{28}$$

(ii)

$$(g_n(t,x,u_n,\nabla u_n)) \text{ is bounded in } L^1(Q_T). \tag{29}$$

(iii)

$$\lim_{k\to+\infty} \operatorname{meas}\{|u_n| > k\} = 0. \tag{30}$$

Proof. (i) We take $\varphi = T_k(u_n) \in \mathcal{U}(Q_T)$ as an admissible choice of test function in (27), we obtain

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n) \right\rangle dt + \int_{Q_T} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla T_k(u_n) dxdt \\ &\qquad + \int_{Q_T} g_n(t,x,u_n,\nabla u_n) T_k(u_n) dxdt = \int_{Q_T} f_n T_k(u_n) dxdt. \end{aligned}$$

From the sign condition $(\mathcal{A}_4)$ and (25), one gets

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n) \right\rangle dt + \int_{Q_T} |\nabla T_k(u_n)|^{p(x)} dxdt$$
$$+k \int_{Q_T \cap \{|u_n|>k\}} |g_n(t,x,u_n,\nabla u_n)|\, dxdt \leq k\|f\|_{L^1(Q_T)}. \tag{31}$$

By using the periodicity property of u_n, one gets

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n) \right\rangle dt = \int_\Omega \theta_k(u_n)(T,x)dx - \int_\Omega \theta_k(u_n)(0,x)dx = 0.$$

Therefore, inequality (31) becomes

$$\int_{Q_T} |\nabla T_k(u_n)|^{p(x)} dxdt + k \int_{Q_T \cap \{|u_n|>k\}} |g_n(t,x,u_n,\nabla u_n)|\, dxdt \leq k\|f\|_{L^1(Q_T)}. \tag{32}$$

With the help of relationship (8), we derive that (28) holds true.

(ii) Firstly, let us split the desired integral on the sets where $|u_n| > 1$ and where $|u_n| \leq 1$, we have

$$\int_{Q_T} |g_n(t,x,u_n,\nabla u_n)|\, dxdt = \int_{Q_T \cap \{|u_n|\leq 1\}} |g_n(t,x,u_n,\nabla u_n)|\, dxdt$$
$$+ \int_{Q_T \cap \{|u_n|>1\}} |g_n(t,x,u_n,\nabla u_n)|\, dxdt. \tag{33}$$

We deal with the first integral by using growth assumption $(\boldsymbol{\mathcal{A}}_3)$ and (32), one has

$$\int_{Q_T \cap \{|u_n|\leq 1\}} |g_n(t,x,u_n,\nabla u_n)|\, dxdt$$
$$\leq \mathfrak{c}(1) \left(\int_{Q_T} H(t,x)dxdt + \int_{Q_T} |\nabla T_1(u_n)|^{p(x)} dxdt \right)$$
$$\leq \mathfrak{c}(1) \left(\|H\|_{L^1(Q_T)} + \|f\|_{L^1(Q_T)} \right). \tag{34}$$

On the other hand, inequality (32) implies that

$$\int_{Q_T \cap \{|u_n|>1\}} |g_n(t,x,u_n,\nabla u_n)|\, dxdt \leq \|f\|_{L^1(Q_T)}. \tag{35}$$

According to (33), (34) and (35), we derive that $(g_n(t,x,u_n,\nabla u_n))$ is bounded in $L^1(Q_T)$.

(iii) Let us remark that for $0 < \varepsilon \leq k$, one has

$$\{|u_n| \geq \varepsilon\} = \{|T_k(u_n)| \geq \varepsilon\}.$$

Then, we can write

$$\|T_k(u_n)\|^{p^-}_{L^{p^-}(Q_T)} = \int_{[\{u_n|\geq \varepsilon\}} |T_k(u_n)|^{p^-}\,dxdt + \int_{\{|u_n|<\varepsilon\}} |T_k(u_n)|^{p^-}\,dxdt.$$

Consequently, we get

$$\text{meas}\{|u_n| > \varepsilon\} \leq \left(\frac{\|T_k(u_n)\|_{L^{p^-}(Q_T)}}{\varepsilon}\right)^{p^-}.$$

Using the continuous embedding $\mathcal{U}(Q_T) \hookrightarrow L^{p^-}(Q_T)$, one obtains

$$\text{meas}\{|u_n| > \varepsilon\} \leq C\left(\frac{\|T_k(u_n)\|_{\mathcal{U}(Q_T)}}{\varepsilon}\right)^{p^-}. \tag{36}$$

Therefore, we have two cases to discuss: If $\|T_k(u_n)\|_{\mathcal{U}(Q_T)} \geq 1$, we deal with (36) via estimate (28), one gets

$$\text{meas}\{|u_n| > \varepsilon\} \leq C\left(\frac{\|T_k(u_n)\|^{p^-}_{\mathcal{U}(Q_T)}}{\varepsilon^{p^-}}\right) \leq C\left(\frac{k\|f\|_{L^1(Q_T)}}{\varepsilon^{p^-}}\right). \tag{37}$$

If $\|T_k(u_n)\|_{\mathcal{U}(Q_T)} \leq 1$, relation (36) reduced to

$$\text{meas}\{|u_n| > \varepsilon\} \leq C\left(\frac{1}{\varepsilon}\right)^{p^-}. \tag{38}$$

Therefore, setting $\varepsilon = k$ and taking the limit as k goes to $+\infty$ in (37) and (38), we get for each case that

$$\lim_{k\to+\infty} \text{meas}\{|u_n| > k\} = 0.$$

□

Lemma 6. *Let (u_n) be the weak periodic solution to (26). Then, we have*

(i)

$$\lim_{k\to+\infty} \sup_{n>0}\left(\int_{Q_T\cap\{|u_n|>k\}} |g_n(t,x,u_n,\nabla u_n)|\,dxdt\right) = 0. \tag{39}$$

(ii) there exists $u : Q_T \to \mathbb{R}$ a measurable function, such that (up to subsequence)

$$u_n \to u, \ \text{a.e. in } Q_T. \tag{40}$$

Proof. (i) Let $0 < h < k$, by choosing $\varphi = T_k(u_n) \in \mathcal{U}(Q_T)$ as a test function in (27) and following the same reasoning of (i) from Lemma 5, we arrive at

$$\begin{aligned} k\int_{Q_T\cap\{|u_n|>k\}} |g_n(t,x,u_n,\nabla u_n)|\,dxdt &\leq \int_{Q_T} |f_n T_k(u_n)|\,dxdt \\ &\leq \int_{\{|u_n|>h\}} |f_n T_k(u_n)|\,dxdt + \int_{\{|u_n|\leq h\}} |f_n T_k(u_n)|\,dxdt \\ &\leq k\int_{Q_T} |f|\,\chi_{\{|u_n|>h\}}dxdt + h\int_{Q_T} |f|\,dxdt. \end{aligned}$$

Which implies that

$$\int_{Q_T\cap\{|u_n|>k\}} |g_n(t,x,u_n,\nabla u_n)|\,dxdt \le \int_{Q_T} |f|\,\chi_{\{|u_n|>h\}}dxdt + \frac{h}{k}\|f\|_{L^1(Q_T)}. \tag{41}$$

Using the equi-integrability of f in $L^1(Q_T)$, we deduce that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any subset $E \subset Q_T$, one has

$$\text{if} \quad \text{meas}\,(E) < \delta, \quad \text{then} \quad \int_E |f|\,dxdt.$$

According to the limit result (30), we derive that for each $\varepsilon > 0$ there exists $h_\varepsilon > 0$ such that for all $h \ge h_\varepsilon$, we have

$$\sup_{n>0}\left(\int_{Q_T} |f|\,\chi_{\{|u_n|>h\}}dxdt\right) \le \varepsilon. \tag{42}$$

Therefore, by taking $h = h_\varepsilon$ and letting first k to $+\infty$ in (41), after using (42) and letting h to $+\infty$, we conclude that

$$\lim_{k\to+\infty}\sup_{n>0}\left(\int_{Q_T\cap\{|u_n|>k\}} |g_n(t,x,u_n,\nabla u_n)|\,dxdt\right) = 0.$$

(ii) As a first step, we introduce the truncation function $\sigma_k \in \mathcal{C}^2(\mathbb{R})$ defined as follows

$$\sigma_k(r) = \begin{cases} r & \text{if} \ |r| \le \frac{k}{2} \\ k\ \text{sign}(r) & \text{if} \ |r| > k. \end{cases}$$

An interesting feature of $\sigma_k(\cdot)$ is that $\sigma_k'(\cdot)$ and $\sigma_k''(\cdot)$ has a compact support in $[-k,k]$. Our interest goes to proving firstly that the sequence $(\sigma_k(u_n))$ is relatively compact in a certain Lebesgue space. To do this, let $\varphi = \sigma_k'(u_n)\zeta$ as a test function in the weak formulation (27) with $\zeta \in \mathcal{C}_c^\infty(Q_T)$. By a direct computation, we get in the distributional sense

$$\begin{aligned} \frac{\partial\sigma_k(u_n)}{\partial t} - \text{div}\left(\sigma_k'(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n\right) + \sigma_k''(u_n)|\nabla u_n|^{p(x)} \\ +g(t,x,u_n,\nabla u_n)\sigma_k'(u_n) = f_n\sigma_k'(u_n) \text{ in } \mathcal{D}'(Q_T). \end{aligned} \tag{43}$$

By taking into account the above stated properties of $\sigma_k(\cdot)$, it comes that

$$\nabla\sigma_k(u_n) = \sigma_k'(u_n)\nabla T_k(u_n) \ \text{ a.e in } Q_T, \tag{44}$$

$$\sigma_k'(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n = \sigma_k'(u_n)|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) \ \text{ a.e in } Q_T, \tag{45}$$

$$\sigma_k''(u_n)|\nabla u_n|^{p(x)} = \sigma_k''(u_n)|\nabla T_k(u_n)|^{p(x)} \ \text{ a.e in } Q_T. \tag{46}$$

According to estimate (28), we derive that $(T_k(u_n))$ is bounded in $\mathcal{U}(Q_T)$ and by combining the embedding (6) with (44), one gets

$$(\sigma_k(u_n)) \ \text{ is bounded in } \ L^{p^-}\left(0,T;W_0^{1,p(x)}(\Omega)\right). \tag{47}$$

With the help of estimate (28), one has no difficulty showing that

$$\left(\sigma_k'(u_n)|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\right) \text{ is bounded in } \left(L^{p'(x)}(Q_T)\right)^N. \tag{48}$$

By comparing (45) and (48), it follows that

$$\operatorname{div}\left(\sigma_k'(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n\right) \text{ is bounded in } \mathcal{U}(Q_T)^*. \tag{49}$$

We use again the result of (28) with (46), we arrive at

$$\left(\sigma_k''(u_n)|\nabla u_n|^{p(x)}\right) \text{ is bounded in } L^1(Q_T). \tag{50}$$

In view of Eq. (43) and by employing (29), (49) and (50), we deduce that

$$\left(\frac{\partial \sigma_k(u_n)}{\partial t}\right) \text{ is bounded in } \mathcal{U}(Q_T)^* + L^1(Q_T). \tag{51}$$

Now, we need the use of some known embedding relationships. Let s be fixed such that $s > \frac{N}{2} + 1$, we recall that

- $s > \frac{N}{2}$, we have $H_0^s(\Omega) \hookrightarrow L^\infty(\Omega)$, and then $L^1(\Omega) \hookrightarrow H^{-s}(\Omega)$
- $s - 1 > \frac{N}{2}$, one has $H_0^s(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$, consequently, $W^{-1,p'(x)}(\Omega) \hookrightarrow H^{-s}(\Omega)$.

It is clear that

$$W_0^{1,p(x)}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^{p(x)}(\Omega) \hookrightarrow H^{-s}(\Omega). \tag{52}$$

And by virtue of (7) and (51), we easily check that

$$\left(\frac{\partial \sigma_k(u_n)}{\partial t}\right) \text{ is bounded in } L^1\left(0, T; H^{-s}(\Omega)\right). \tag{53}$$

Thanks to (53), one may apply the compactness argument of Corollary 4, page 85 from [46] to deduce that $(\sigma_k(u_n))$ is relatively compact in $L^{p^-}(Q_T)$. This fact allows us to conclude that for every $k > 0$ and up to a subsequence, $(\sigma_k(u_n))$ is a Cauchy sequence in measure. Hence, to establish that (u_n) converges almost everywhere in Q_T, we shall show that (u_n) is a Cauchy sequence in measure, namely:

$$\forall \delta > 0, \forall \epsilon > 0, \exists \mathfrak{n}_0 \text{ such that } \forall n, m \geq \mathfrak{n}_0,\ \operatorname{meas}\{|u_n - u_m| > \delta\} \leq \epsilon.$$

Let $\delta > 0$, for each $n, m \in \mathbb{N}$, we observe that

$$\{|u_n - u_m| > \delta\} \subset \left\{|u_n| > \frac{k}{2}\right\} \cup \left\{|u_m| > \frac{k}{2}\right\} \cup \{|\sigma_k(u_n) - \sigma_k(u_m)| > \delta\}.$$

We then have

$$\begin{aligned} \operatorname{meas}\{|u_n - u_m| > \delta\} \leq & \operatorname{meas}\left\{|u_n| > \frac{k}{2}\right\} + \operatorname{meas}\left\{|u_m| > \frac{k}{2}\right\} \\ & + \operatorname{meas}\{|\sigma_k(u_n) - \sigma_k(u_m)| > \delta\}. \end{aligned}$$

With the help of (30), one may choose $k^\star$ sufficiently large enough such that for $\epsilon > 0$, one has

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \epsilon + \text{meas}\{|\sigma_{k^\star}(u_n) - \sigma_{k^\star}(u_m)| > \delta\}.$$

On the other hand, by using the fact that $(\sigma_{k^\star}(u_n))$ is a Cauchy sequence in measure, we deduce the existence of a measurable function $u : Q_T \to \mathbb{R}$ and a subsequence of (u_n) which denote again by (u_n) for simplicity such that

$$u_n \to u \text{ a.e in } Q_T.$$

□

Remark 2. 1. It is worth mentioning that $T_k(\cdot)$ is continuous and bounded by k. Then, by combining (40) with Lebesgue's dominated convergence Theorem, one gets $(T_k(u_n)) \to T_k(u)$ strongly in $L^{p^-}(Q_T)$ for each $k > 0$. Therefore, by following the same lines used in the proof of (30), we establish that

$$\lim_{k\to+\infty} \text{meas}\{|u| > k\} = 0. \tag{54}$$

2. According to estimate (28) one has $(T_k(u_n))$ is bounded in $\mathcal{U}(Q_T)$, which is equivalent to say that $(\nabla T_k(u_n))$ is bounded in $\left(L^{p(x)}(Q_T)\right)^N$. On top of that, the almost everywhere convergence (40) suggests that for every $k > 0$, we have

$$(\nabla T_k(u_n)) \rightharpoonup \nabla T_k(u) \text{ weakly in } \left(L^{p(x)}(Q_T)\right)^N. \tag{55}$$

Lemma 7. *Let (u_n) be the sequence defined as above. Then*

$$\lim_{k\to\infty} \int_{\{k\leq|u|\leq k+1\}} |\nabla u|^{p(x)}\, dxdt = 0. \tag{56}$$

Proof. Testing the weak formulation (27) with $\varphi = T_{k,a}(u_n) \in \mathcal{U}(Q_T) \cap L^\infty(Q_T)$ yields, for all $a,\ k > 0$, the equation

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_{k,a}(u_n)\right\rangle dt + \int_{Q_T} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla T_{k,a}(u_n) dxdt$$
$$+ \int_{Q_T} g_n(t,x,u_n,\nabla u_n) T_{k,a}(u_n) dxdt = \int_{Q_T} f_n T_{k,a}(u_n) dxdt. \tag{57}$$

The periodicity property of u_n allows us to obtain

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_{k,a}(u_n)\right\rangle dt = \int_\Omega \theta_{k,a}(u_n)(T,x)dx - \int_\Omega \theta_{k,a}(u_n)(0,x)dx = 0. \tag{58}$$

On the other hand, by (12) and through the sign condition $(\mathcal{A}_4)$, we have

$$\int_{Q_T} g_n(t,x,u_n,\nabla u_n) T_{k,a}(u_n) dxdt \geq 0. \tag{59}$$

According to (12), (58) and (59), Eq. (57) becomes

$$\int_{Q_T} \left|\nabla T_{k,a}(u_n)\right|^{p(x)} dxdt \le \int_{Q_T} f_n T_{k,a}(u_n)dxdt \le a\|f\|_{L^1(Q_T)}. \quad (60)$$

Following the same reasoning of (55), one gets

$$\left(\nabla T_{k,a}\left(u_n\right)\right) \rightharpoonup \nabla T_{k,a}(u) \text{ weakly in } \left(L^{p(x)}(Q_T)\right)^N \text{ as } n \to \infty. \quad (61)$$

In view of expression (12), we check easily that $|T_{k,a}(u_n)| \le a$. This fact implies that $\left(T_{k,a}\left(u_n\right)\right)$ converges to $T_{k,a}(u)$ weak-* in $L^\infty\left(Q_T\right)$ as $n \to \infty$. Furthermore, by the strong convergence (25), one may pass to the limit in (60) as $n \to \infty$. We have

$$\limsup_{n\to\infty} \int_{Q_T} \left|\nabla T_{k,a}(u_n)\right|^{p(x)} dxdt \le \int_{Q_T} f T_{k,a}(u)dxdt. \quad (62)$$

In accordance with (12), we recall that $\left(T_{k,a}\left(u\right)\right)$ converges to 0 pointwise as $k \to \infty$ and $\left|T_{k,a}\left(u\right)\right| \le a$. Further, by employing Lebesgue's dominated convergence theorem, it results that

$$\int_{Q_T} f T_{k,a}(u)dxdt \to 0 \text{ as } k \to \infty. \quad (63)$$

We use (63) when passing to the limit as $k \to \infty$ in (62). The result is

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_{Q_T} \left|\nabla T_{k,a}\left(u_n\right)\right|^{p(x)} dxdt \le 0. \quad (64)$$

As well known, $\|\cdot\|_{p(x)}$ is weakly lower semi-continuous, combining this fact with the results of (5), (61) yields

$$\begin{aligned} 0 &\le \min\left\{\left(\int_{Q_T} \left|\nabla T_{k,a}(u)\right|^{p(x)} dxdt\right)^{\frac{1}{p^{\pm}}}\right\} \\ &\le \|\nabla T_{k,a}(u)\|_{L^{p(x)}(Q_T)} \\ &\le \liminf_{n\to\infty} \|\nabla T_{k,a}\left(u_n\right)\|_{L^{p(x)}(Q_T)} \\ &\le \limsup_{n\to\infty} \max\left\{\left(\int_{Q_T} \left|\nabla T_{k,a}\left(u_n\right)\right|^{p(x)} dxdt\right)^{\frac{1}{p^{\pm}}}\right\}. \end{aligned} \quad (65)$$

Taking advantage of (62) and (65) leads to

$$0 \le \min\left\{\left(\int_{Q_T} \left|\nabla T_{k,a}(u)\right|^{p(x)} dxdt\right)^{\frac{1}{p^{\pm}}}\right\} \le \max\left\{\left(\int_{Q_T} f T_{k,a}(u)dxdt\right)^{\frac{1}{p^{\pm}}}\right\}. \quad (66)$$

With the help of (64), we can arrive to the following convergence by letting $k \to \infty$ in (66)

$$\lim_{k\to\infty} \int_{\{k\le|u|\le k+a\}} |\nabla u|^{p(x)} dxdt = 0.$$

By taking $a = 1$ in the last equality, we finish the proof. □

4.3. Strong convergence of truncations

This subsection provides the strong convergence of truncations $(T_k(u_n))$ in $\mathcal{U}(Q_T)$. We have the following result.

Lemma 8. *Let* (u_n) *be the weak periodic solution to* (26)*. Then, for any* $k > 0$*, we have*

$$(T_k\,(u_n)) \to T_k(u) \text{ strongly in } \mathcal{U}(Q_T) \text{ as } n \to \infty. \tag{67}$$

Proof. Let us observe that showing (67) is equivalent to establish that

$$(\nabla T_k\,(u_n)) \to \nabla T_k(u) \text{ strongly in } \left(L^{p(x)}(Q_T)\right)^N. \tag{68}$$

To this aim, we propose to use Landes's time-regularization method developed in [33] for initial parabolic problems with constant exponent and which was recently generalized in [15] for a periodic problem with variable exponent. Let us start by considering $\left(u_0^\mu\right)$ a sequence of functions such that

$$\begin{aligned}
&u_0^\mu \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega), \quad \left\|u_0^\mu\right\|_{L^\infty(\Omega)} \le k\\
&u_0^\mu \to T_k\,(u)\,(T) \text{ a.e. in } \Omega \text{ as } \mu \to \infty\\
&\lim_{\mu\to\infty} \frac{1}{\mu}\left\|u_0^\mu\right\|_{W_0^{1,p(x)}(\Omega)} = 0.
\end{aligned}$$

For any $\mu > 0$, we define $(T_k(u))_\mu$ the time regularized function of $T_k(u) \in \mathcal{U}(Q_T)$ as the unique solution to the following problem

$$\begin{cases}
(T_k(u))_\mu \in \mathcal{U}(Q_T) \cap L^\infty(Q_T)\\
\dfrac{\partial T_k(u)_\mu}{\partial t} = \mu\left(T_k(u) - T_k(u)_\mu\right) & \text{in } \mathcal{D}'(Q_T)\\
T_k(u)_\mu(0) = u_0^\mu & \text{in } \Omega.
\end{cases} \tag{69}$$

By taking advantage on the solvability of (69), we infer that $(T_k(u))_\mu$ has the following form

$$(T_k(u))_\mu\,(t,x) := \mu \int_{-\infty}^{t} e^{\mu(s-t)} T_k(u(s,x))ds + u_0^\mu e^{-\mu t},$$

with $T_k(u)$ is belonging in $\mathcal{U}(Q_T)$ and extending by 0 when $s < 0$. According to [33], one has no difficulty verifying that $(T_k(u))_\mu$ meets the following properties

$$\begin{aligned}
&\frac{\partial T_k(u)_\mu}{\partial t} \in \mathcal{U}(Q_T) \cap L^\infty\,(Q_T)\,, \quad \left\|(T_k(u))_\mu\right\|_{L^\infty(Q_T)} \le k,\\
&(T_k(u))_\mu \to T_k(u) \text{ a.e. in } Q_T \text{ and weak-* in } L^\infty\,(Q_T) \text{ as } \mu \to \infty,\\
&(T_k(u))_\mu \to T_k(u) \text{ strongly in } \mathcal{U}(Q_T) \text{ as } \mu \to \infty.
\end{aligned} \tag{70}$$

Let us set $\psi_{n,\mu} = T_k(u_n) - (T_k(u))_\mu$ and take $\varphi = S_i'(u_n)\phi_\lambda(\psi_{n,\mu})$ as an admissible choice of test function in (27), where $\phi_\lambda(\cdot)$ is the function defined in (14) with parameter λ to be fixed later. We therefore have

$$\mathcal{I}^1_{n,\mu,i} + \mathcal{I}^2_{n,\mu,i} + \mathcal{I}^3_{n,\mu,i} + \mathcal{I}^4_{n,\mu,i} = \mathcal{I}^5_{n,\mu,i}. \tag{71}$$

where

$$\mathcal{I}^1_{n,\mu,i} = \int_0^T \left\langle \frac{\partial u_n}{\partial t}, S_i'(u_n)\phi_\lambda(\psi_{n,\mu}) \right\rangle dt, \tag{72}$$

$$\mathcal{I}^2_{n,\mu,i} = \int_{Q_T} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \psi_{n,\mu} S_i'(u_n)\phi_\lambda'(\psi_{n,\mu})\, dx\, dt, \tag{73}$$

$$\mathcal{I}^3_{n,\mu,i} = \int_{Q_T} |\nabla u_n|^{p(x)} S_i''(u_n)\phi_\lambda(\psi_{n,\mu})\, dx\, dt, \tag{74}$$

$$\mathcal{I}^4_{n,\mu,i} = \int_{Q_T} g(t,x,u_n,\nabla u_n) S_i'(u_n)\phi_\lambda(\psi_{n,\mu})\, dx\, dt, \tag{75}$$

$$\mathcal{I}^5_{n,\mu,i} = \int_{Q_T} f_n S_i'(u_n)\phi_\lambda(\psi_{n,\mu})\, dx\, dt. \tag{76}$$

Now, we are concerned with the passage to the limit into infinity in (71). To do so, we propose to deal with the five integrals (72)-(76) separately, where the limit order is first n, then μ and finally i. For the reader's convenience, we will use the notation $\omega(n,\mu,i)$ to designate all quantities (which can be possibly different from line to line) such that

$$\lim_{i\to\infty} \lim_{\mu\to\infty} \lim_{n\to\infty} \omega(n,\mu,i) = 0.$$

Following the same arguments of [13,21] for the constant exponent case, see also [34] for the variable exponent case, we can extend that for any $i \geq k$, we have

$$\mathcal{I}^1_{n,\mu,i} \geq \omega(n,\mu). \tag{77}$$

In view to the expression of $\psi_{n,\mu}$ and by taking into account (40), (55), it results that

$$\begin{cases} \left\| \psi_{n,\mu} \right\|_{L^\infty(Q_T)} \leq 2k \text{ for any } n > 0 \\ \psi_{n,\mu} \rightharpoonup \psi_\mu \text{ weakly in } \mathcal{U}(Q_T) \text{ as } n \to \infty \\ \psi_{n,\mu} \rightharpoonup \psi_\mu \text{ a.e. in } Q_T \text{ and weak-* in } L^\infty(Q_T) \text{ as } n \to \infty. \end{cases} \tag{78}$$

where $\psi_\mu = T_k(u) - (T_k(u))_\mu$. Furthermore, by combining (70) and (78) with the boundness properties of the functions $S_i(\cdot)$ and $\phi_\lambda(\cdot)$, one may deduce that when $\mu, n \to \infty$, we have

$$\begin{cases} S_i'(u_n)\phi_\lambda(\psi_{n,\mu}) \to 0, \text{ a.e. in } Q_T, \text{ weak-* in } L^\infty(Q_T) \\ S_i'(u_n)\phi_\lambda'(\psi_{n,\mu}) \to S_i'(u), \text{ a.e. in } Q_T \text{ and weak-* in } L^\infty(Q_T) \\ S_i''(u_n)\phi_\lambda(\psi_{n,\mu}) \to 0, \text{ a.e. in } Q_T, \text{ weak-* in } L^\infty(Q_T). \end{cases} \tag{79}$$

We focus our interest on the integral $\mathcal{I}^3_{n,\mu,i}$. According to (13), we have $S''_i \subset [-(i+1), -i] \cup [i, i+1]$. Therefore, for any $i \in \mathbb{N}$ and $\mu > 0$, one has

$$|\mathcal{I}^3_{n,\mu,i}| \leq \left\| S''_i\,(u_n) \right\|_{L^\infty(\mathbb{R})} \left\| \phi_\lambda(\psi_{n,\mu}) \right\|_{L^\infty(Q_T)} \int_{\{i \leq |u_n| \leq i+1\}} |\nabla u_n|^{p(x)}\, dx\, dt.$$

According to (13) and (78) the last inequality becomes

$$|\mathcal{I}^3_{n,\mu,i}| \leq C \int_{\{i \leq |u_n| \leq i+1\}} |\nabla u_n|^{p(x)}\, dx\, dt,$$

where C is a nonnegative constant independent of the index n, μ and i. Further, the result of (64) yield

$$|\mathcal{I}^3_{n,\mu,i}| \leq \omega(n, \mu, i). \tag{80}$$

Next, we combine convergences results of (25) with (79), we arrive at

$$\mathcal{I}^5_{n,\mu,i} = \omega(n, \mu, i). \tag{81}$$

Let us back to deal with $\mathcal{I}^2_{n,\mu,i}$, by splitting this integral on the sets where $|u_n| < k$ and where $|u_n| \geq k$, it follows that

$$\begin{aligned}
\mathcal{I}^2_{n,\mu,i} &= \int_{Q_T} |\nabla u_n|^{p(x)-2}\, \nabla u_n \cdot \nabla \psi_{n,\mu} S'_i(u_n) \phi'_\lambda \left(\psi_{n,\mu}\right) dx\, dt \\
&= \int_{\{|u_n|<k\}} |\nabla T_k(u_n)|^{p(x)-2}\, \nabla T_k(u_n) \cdot \nabla (T_k\,(u_n) \\
&\quad - (T_k(u))_\mu) S'_i(u_n) \phi'_\lambda \left(\psi_{n,\mu}\right) dx\, dt \\
&\qquad - \int_{\{|u_n|\geq k\}} |\nabla u_n|^{p(x)-2}\, \nabla u_n \cdot \nabla \,(T_k(u))_\mu\, S'_i(u_n) \phi'_\lambda \left(\psi_{n,\mu}\right) dx\, dt \\
&:= \mathcal{J}_{n,\mu,i} - \mathcal{L}_{n,\mu,i}.
\end{aligned} \tag{82}$$

We begin initially by studying the integral $\mathcal{L}_{n,\mu,i}$. Using the fact that supp $S'_i \subset [-(i+1), (i+1)]$ and splitting $\mathcal{L}_{n,\mu,i}$ on the sets where $|u| \geq k$ and where $|u| < k$, one obtains

$$\begin{aligned}
\mathcal{L}_{n,\mu,i} = &\int_{Q_T} |\nabla T_{i+1}\,(u_n)|^{p(x)-2}\, \nabla T_{i+1}\,(u_n) \\
&\cdot \nabla \,(T_k(u))_\mu\, S'_i\,(u_n)\, \phi'_\lambda \left(\psi_{n,\mu}\right) \chi_{\{|u_n|\geq k\}} \chi_{\{|u|\geq k\}} dx\, dt \\
+ &\int_{Q_T} |\nabla T_{i+1}\,(u_n)|^{p(x)-2}\, \nabla T_{i+1}\,(u_n) \\
&\cdot \nabla \,(T_k(u))_\mu\, S'_i\,(u_n)\, \phi'_\lambda \left(\psi_{n,\mu}\right) \chi_{\{|u_n|\geq k\}} \chi_{\{|u|<k\}} dx dt.
\end{aligned}$$

In accordance with (13) and (78), it comes that $S'_i\,(u_n)\, \phi'_\lambda \left(\psi_{n,\mu}\right)$ is uniformly bounded with respect to the index n and μ. This fact implies the existence of C a nonnegative constant independent of n and μ such that

$$\begin{aligned}\mathcal{L}_{n,\mu,i} &\leq C\int_{Q_T}\left|\nabla T_{i+1}(u_n)\right|^{p(x)-1}\left|\nabla (T_k(u))_\mu\,\chi_{\{|u|\geq k\}}\right|dxdt\\ &+C\int_{Q_T}\left|\nabla T_{i+1}(u_n)\right|^{p(x)-1}\left|\nabla (T_k(u))_\mu\,\chi_{\{|u_n|\geq k\}}\chi_{\{|u|<k\}}\right|dxdt\\ &:=\mathcal{L}^1_{n,\mu,i}+\mathcal{L}^2_{n,\mu,i}.\end{aligned}$$

It follows from $p(x)$-Hölder's inequality that

$$\begin{aligned}|\mathcal{L}^1_{n,\mu,i}| &\leq C\left\||\nabla T_{i+1}(u_n)|^{p(x)-1}\right\|_{L^{p'(x)}(Q_T)}\left\|\nabla (T_k(u))_\mu\,\chi_{\{|u|\geq k\}}\right\|_{L^{p(x)}(Q_T)}\\ &\leq C\max\left\{\left(\int_{Q_T}|\nabla T_{i+1}(u_n)|^{p(x)}\,dxdt\right)^{\frac{1}{(p')^{\pm}}}\right\}\\ &\quad\times\max\left\{\left(\int_{Q_T}\left|\nabla (T_k(u))_\mu\,\chi_{\{|u|\geq k\}}\right|^{p(x)}\,dxdt\right)^{\frac{1}{p^{\pm}}}\right\}\\ &\leq C\max\left\{\left(\int_{Q_T}\left|\nabla (T_k(u))_\mu\,\chi_{\{|u|\geq k\}}\right|^{p(x)}\,dxdt\right)^{\frac{1}{p^{\pm}}}\right\}.\end{aligned}\tag{83}$$

We recall that $\nabla T_k(u)\chi_{\{|u|\geq k\}}=0$. Then, it result from (70) that $|\nabla (T_k(u))_\mu\,\chi_{\{|u|\geq k\}}|^{p(x)}\to 0$ almost everywhere in Q_T as $\mu\to\infty$. Therefore, by a direct application of Lemma 3, one gets $\mathcal{L}^1_{n,\mu,i}=\omega(n,\mu)$. By following similar arguments, we can show that $\mathcal{L}^2_{n,\mu,i}=\omega(n,\mu)$. In accordance with the above results, one obtains

$$\mathcal{L}_{n,\mu,i}=\omega(n,\mu).\tag{84}$$

Back to the integral $\mathcal{J}_{n,\mu,i}$, we remark that for any $i>k$, it follows that $S'_i(u_n)=1$ on the set $\{|u_n|<k\}$. Which yields

$$\begin{aligned}\mathcal{J}_{n,\mu,i} &=\int_{Q_T}|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\\ &\qquad\cdot\nabla\left(T_k(u_n)-(T_k(u))_\mu\right)\phi'_\lambda\left(\psi_{n,\mu}\right)dxdt\\ &=\int_{Q_T}\left(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right)\\ &\qquad\cdot\nabla\left(T_k(u_n)-T_k(u)\right)\phi'_\lambda\left(\psi_{n,\mu}\right)dxdt\\ &+\int_{Q_T}\left(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right)\\ &\qquad\cdot\nabla\left(T_k(u)-(T_k(u))_\mu\right)\phi'_\lambda\left(\psi_{n,\mu}\right)dxdt\\ &+\int_{Q_T}|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\\ &\qquad\cdot\nabla\left(T_k(u_n)-(T_k(u))_\mu\right)\phi'_\lambda\left(\psi_{n,\mu}\right)dxdt\\ &:=\mathcal{J}^1_n+\mathcal{J}^2_{n,\mu}+\mathcal{J}^3_{n,\mu}.\end{aligned}\tag{85}$$

By the mean of estimate (28), we can achieve that $\left(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\right)_n$ is bounded in $L^{p'(x)}(Q_T)$. Accordingly, we deduce the existence of $\zeta_k : Q_T \to \mathbb{R}^N$ a measurable function such that (up to a subsequence)

$$\left(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\right) \rightharpoonup \zeta_k \text{ weakly in } \left(L^{p'(x)}(Q_T)\right)^N.$$

Therefore,

$$\lim_{n\to\infty} \mathcal{J}^2_{n,\mu} = \int_{Q_T} \left(\zeta_k - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right)\cdot \nabla\left(T_k(u) - (T_k(u))_\mu\right)\phi'_\lambda\left(\psi_\mu\right)dxdt. \tag{86}$$

In view of (70), passing to the limit in (86) as $\mu\to\infty$ yields

$$\mathcal{J}^2_{n,\mu} = \omega(n,\mu). \tag{87}$$

Following the same procedures, we can prove that

$$\mathcal{J}^3_{n,\mu} = \omega(n,\mu). \tag{88}$$

Thus, the results of (85), (87) and (88) leading to obtain

$$\mathcal{J}_{n,\mu,i} = \mathcal{J}^1_n + \omega(n,\mu). \tag{89}$$

In accordance with (82), (84) and (89), we arrive at

$$\mathcal{I}^2_{n,\mu,i} = \mathcal{J}^1_n + \omega(n,\mu). \tag{90}$$

Now, we study the fourth integral $\mathcal{I}^4_{n,\mu,i}$. We use again the splitting into two integrals. One obtains

$$\begin{aligned}\mathcal{I}^4_{n,\mu,i} &= \int_{\{|u_n|<k\}} g(t,x,u_n,\nabla u_n)S'_i(u_n)\,\phi_\lambda\left(\psi_{n,\mu}\right)dx\,dt\\ &\quad+\int_{\{|u_n|\geq k\}} g(t,x,u_n,\nabla u_n)S'_i(u_n)\,\phi_\lambda\left(\psi_{n,\mu}\right)dx\,dt\\ &:= \mathcal{K}^1_{n,\mu,i} + \mathcal{K}^2_{n,\mu,i}.\end{aligned} \tag{91}$$

By a simple computation, we have

$$\phi_\lambda\left(\psi_{n,\mu}\right) = \begin{cases}\left(-k-(T_k(u))_\mu\right)e^{\lambda\left(-k-(T_k(u))_\mu\right)^2} & \text{if } u_n\leq -k\\ \\ \left(k-(T_k(u))_\mu\right)e^{\lambda\left(k-(T_k(u))_\mu\right)^2} & \text{if } u_n \geq k.\end{cases} \tag{92}$$

Which proves that $\phi_\lambda\left(\psi_{n,\mu}\right)$ and u_n have the same sign one the set where $|u_n|\geq k$. Further, by combining the positivity property of $S'_i(\cdot)$ with sign condition (19), we derive that

$$\mathcal{K}^2_{n,\mu,i} \geq 0. \tag{93}$$

We observe that for any $i > k$, we have $S_i'(u_n) = 1$ on the set $\{|u_n| < k\}$. On the other hand, combining this fact with growth assumption (18) yields to

$$\begin{aligned}\left|\mathcal{K}^1_{n,\mu,i}\right| &\leq \int_{Q_T} |g(t,x,T_k(u_n),\nabla T_k(u_n))| \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| dx\,dt \\ &\leq \mathfrak{c}(k) \int_{Q_T} H(t,x) \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| dx\,dt + \mathfrak{c}(k) \\ &\quad \int_{Q_T} |\nabla T_k(u_n)|^{p(x)} \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| dx\,dt \\ &\leq \mathcal{N}^1_{n,\mu} + \mathcal{N}^2_{n,\mu} + \mathcal{N}^3_{n,\mu,} + \mathcal{N}^4_{n,\mu},\end{aligned} \tag{94}$$

where,

$$\begin{aligned}\mathcal{N}^1_{n,\mu} &= \mathfrak{c}(k) \int_{Q_T} H(t,x) \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| dx\,dt, \\ \mathcal{N}^2_{n,\mu} &= \mathfrak{c}(k) \int_{Q_T} \left(|\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u)\right) \\ &\qquad \times \nabla\left(T_k(u_n) - T_k(u)\right) \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| dxdt, \\ \mathcal{N}^3_{n,\mu} &= \mathfrak{c}(k) \int_{Q_T} |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \cdot \nabla\left(T_k(u_n) - T_k(u)\right) \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| dxdt, \\ \mathcal{N}^4_{n,\mu} &= \mathfrak{c}(k) \int_{Q_T} |\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) \cdot \nabla T_k(u) \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| dxdt.\end{aligned}$$

Thanks to (70) and (78), one may apply Lebesgue's dominated convergence Theorem to get

$$\mathcal{N}^1_{n,\mu} = \omega(n,\mu). \tag{95}$$

From (78), one can have as $n \to \infty$

$$|\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| \to |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \left|\phi_\lambda\left(\psi_\mu\right)\right| \text{ a.e in } Q_T.$$

Further, by a simple application of Vitali's Lemma, one obtains as $n \to \infty$

$$\begin{aligned}&|\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| \\ &\to |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \left|\phi_\lambda\left(\psi_\mu\right)\right| \text{ strongly in } \left(L^{p'(x)}(Q_T)\right)^N.\end{aligned} \tag{96}$$

We combine (55) and (96), it results that

$$\mathcal{N}^3_{n,\mu} = \omega(n,\mu). \tag{97}$$

To investigate $\mathcal{N}^4_{n,\mu}$, we use $p(x)$-Hölder's inequality with the help of (5) and (28). We then have

$$\begin{aligned}\left|\mathcal{N}^4_{n,\mu}\right| &\leq \mathfrak{c}(k)\int_{Q_T} |\nabla T_k(u_n)|^{p(x)-1} |\nabla T_k(u)| \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| dxdt \\ &\leq 2\mathfrak{c}(k)\left\| |\nabla T_k(u_n)|^{p(x)-1}\right\|_{L^{p'(x)}(Q_T)} \left\| |\nabla T_k(u)| \left|\phi_\lambda\left(\psi_{n,\mu}\right)\right| \right\|_{L^{p(x)}(Q_T)} \\ &\leq 2\mathfrak{c}(k)\max\left\{\left(\int_{Q_T} |\nabla T_k(u_n)|^{p(x)} dxdt\right)^{\frac{1}{(p')^{\pm}}}\right\} \\ &\quad\times \max\left\{\left(\int_{Q_T} \left|\nabla T_k(u)\phi_\lambda\left(\psi_{n,\mu}\right)\right|^{p(x)} dxdt\right)^{\frac{1}{p^{\pm}}}\right\} \\ &\leq C\max\left\{\left(\int_{Q_T} \left|\nabla T_k(u)\phi_\lambda\left(\psi_{n,\mu}\right)\right|^{p(x)} dxdt\right)^{\frac{1}{p^{\pm}}}\right\}. \end{aligned} \tag{98}$$

Apply Lebesgue's dominated convergence Theorem in (98) yields

$$\mathcal{N}^4_{n,\mu} = \omega(n,\mu). \tag{99}$$

By taking into account (94), (95), (97) and (99), we achieve that for $i > k$, we have

$$\mathcal{K}^1_{n,\mu,i} \geq -\mathcal{N}^2_{n,\mu} + \omega(n,\mu). \tag{100}$$

According to (91) and (93) inequality (100) implies that for any $i > k$, one has

$$\mathcal{I}^4_{n,\mu,i} \geq -\mathcal{N}^2_{n,\mu} + \omega(n,\mu). \tag{101}$$

Therefore, based on the obtained results in (77), (80), (81), (90) and (101), we notice that

$$\mathcal{J}^1_n - \mathcal{N}^2_{n,\mu} \leq \omega(n,\mu,i), \qquad \text{for } i > k.$$

Which is equivalent to say that for $i > k$, we have

$$\begin{aligned}&\int_{Q_T} \left(\phi'_\lambda\left(\psi_{n,\mu}\right) - \mathfrak{c}(k)\left|\phi_\lambda\left(\psi_{n,\mu}\right)\right|\right) \\ &\qquad\left(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right) \\ &\qquad\times\nabla\left(T_k(u_n) - T_k(u)\right) dxdt \leq \omega(n,\mu,i). \end{aligned} \tag{102}$$

Choosing $\lambda \geq \frac{(\mathfrak{c}(k))^2}{4}$ in Lemma 14, we get $\left(\phi'_\lambda\left(\psi_{n,\mu}\right) - \mathfrak{c}(k)\left|\phi_\lambda\left(\psi_{n,\mu}\right)\right|\right) \geq \frac{1}{2}$. Furthermore, by using (15) and successively passing to the limit in (102) as $n \to \infty$, $\mu \to \infty$ and then $i \to \infty$, one obtains

$$\begin{aligned}&\lim_{n\to\infty}\int_{Q_T}\left(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right)\cdot\nabla\left(T_k(u_n)\right. \\ &\left.-T_k(u)\right) dxdt \leq 0. \end{aligned} \tag{103}$$

Following the same used approach in [15], we arrive at

$$\lim_{n\to\infty}\int_{Q_T}|\nabla T_k(u_n)-\nabla T_k(u)|^{p(x)}\,dxdt=0.$$

Hence, we have

$$(T_k\,(u_n))\to T_k(u)\text{ strongly in }\mathcal{U}(Q_T),$$

which completes our proof. □

4.4. Passing to the limit

In the subsequent section, we aim to establish that u the limit of the sequence (u_n) is a *renormalized periodic* solution to (1) satisfying all the conditions of Definition 1. To this aim, we start by proving the following convergence results.

Lemma 9. *Let* (u_n) *be the sequence defined as above. We then have*

(i)

$$(\nabla u_n)\to\nabla u\ a.e\ in\ Q_T. \tag{104}$$

(ii)

$$(g_n(t,x,u_n,\nabla u_n))\to g(t,x,u,\nabla u)\ strongly\ in\ L^1(Q_T). \tag{105}$$

Proof. (i) To establish the almost everywhere convergence (104), we shall show that (∇u_n) converges to ∇u in measure, namely

$$\forall\delta>0,\forall\epsilon>0,\exists\mathfrak{n}_0\text{ such that }\forall n\geq\mathfrak{n}_0,\ \text{meas}\{|\nabla u_n-\nabla u|>\delta\}\leq\epsilon.$$

Let then $\delta>0$, we remark that for any $n\in\mathbb{N}$, we have the following inclusion

$$\{|\nabla u_n-\nabla u|>\delta\}\subset\{|u_n|>k\}\cup\{|u|>k\}\cup\{|\nabla T_k(u_n)-\nabla T_k(u)|>\delta\}.$$

Which leads to obtain

$$\begin{aligned}\text{meas}\,\{|\nabla u_n-\nabla u|>\delta\}&\leq\text{meas}\,\{|u_n|>k\}+\text{meas}\{|u|>k\}\\&\quad+\text{meas}\{|\nabla T_k(u_n)-\nabla T_k(u)|>\delta\}.\end{aligned}$$

By taking into account (30), (54) and (67), we conclude that (∇u_n) converges to ∇u in measure. Which implies that holds (104).

(ii) According to almost everywhere convergence (40), (104) and (16), one gets

$$(g_n(t,x,u_n,\nabla u_n))\to g(t,x,u,\nabla u)\text{ a.e in }Q_T.$$

Then proving (105) is equivalent to showing that $(g_n(t,x,u_n,\nabla u_n))$ is equi-integrable in $L^1(Q_T)$, namely

$$\forall\varepsilon>0,\exists\delta>0,\forall E\subset Q_T,\ \text{if }|E|<\delta\text{ then }\int_E|g_n(t,x,u_n,\nabla u_n)|dxdt\leq\varepsilon.$$

Let then E be a measurable subset of Q_T, $\varepsilon > 0$ and $k > 0$. We write

$$\begin{aligned}\int_E |g_n(t,x,u_n,\nabla u_n)|dxdt = &\int_{E\cap\{u_n>k\}} |g_n(t,x,u_n,\nabla u_n)|dxdt \\ &+ \int_{E\cap\{u_n\leq k\}} |g_n(t,x,u_n,\nabla u_n)|dxdt \\ :=&\mathfrak{I}_n^1(k) + \mathfrak{I}_n^2(k). \end{aligned} \tag{106}$$

For the first integral, we have

$$\mathfrak{I}_n^1(k) \leq \int_{Q_T\cap\{u_n>k\}} |g_n(t,x,u_n,\nabla u_n)|dxdt.$$

By way of Limit (39), there exists a $k^\star > 0$ such that for all $k \geq k^\star$,

$$\mathfrak{I}_n^1(k) \leq \frac{\varepsilon}{3}. \tag{107}$$

The second integral $\mathfrak{I}_n^2(k)$ is dealt through the growth assumption $(\mathcal{A}_3)$. We have for all $k \geq k^\star$

$$\mathfrak{I}_n^2(k) \leq \mathfrak{c}\,(k) \int_E H(t,x)dxdt + \mathfrak{c}\,(k) \int_E |\nabla T_k\,(u_n)\,|^{p(x)}dxdt. \tag{108}$$

The fact that $H \in L^1(Q_T)$ implies that it is equi-integrable in $L^1(Q_T)$. Hence, there exists $\delta_1 > 0$ such that if $|E| \leq \delta_1$, we have

$$\mathfrak{c}\,(k) \int_E H(t,x)dxdt \leq \frac{\varepsilon}{3}. \tag{109}$$

At the same time, convergence results from 67 and 68 imply that $\left(|\nabla T_k(u_n)|^{p(x)}\right)$ is equi-integrable in $L^1(Q_T)$. Thus, there exists a $\delta_2 > 0$ such that if $|E| \leq \delta_2$, we have

$$\mathfrak{c}\,(k) \int_E |\nabla T_k\,(u_n)\,|^{p(x)}dxdt \leq \frac{\varepsilon}{3}. \tag{110}$$

By choosing $\delta^\star = \min\{\delta_1, \delta_2\}$, we deduce from (108), (109) and (110) that if $|E| \leq \delta^\star$, then

$$\mathfrak{I}_n^2(k) \leq \frac{2\varepsilon}{3}. \tag{111}$$

Both integrals are bounded hence $(g_n(t,x,u_n,\nabla u_n))$ is indeed equi-integrable in $L^1(Q_T)$. Which finishes our proof.

□

To proceed with our aim, we begin initially by considering $S \in W^{2,\infty}(\Omega)$ a function $\mathcal{C}^1$-piecewise such that supp $S' \subset [-M, M]$ for some $M > 0$. We take $\varphi = S'\,(u_n)\,\zeta$

in the weak formulation (27) with $\zeta \in \mathcal{C}_c^\infty(Q_T)$, one gets

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, S'(u_n)\,\zeta \right\rangle dt + \int_{Q_T} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla (S'(u_n)\,\zeta) dx dt$$
$$+ \int_{Q_T} g_n(t, x, u_n, \nabla u_n) S'(u_n)\,\zeta\, dx dt$$
$$= \int_{Q_T} f_n S'(u_n)\,\zeta\, dx dt.$$

By simple computations, we arrive at

$$\int_0^T \left\langle \frac{\partial S(u_n)}{\partial t}, \zeta \right\rangle dt$$
$$+ \int_{Q_T} S'(u_n)\,|\nabla u_n|^{p(x)-2} \nabla u_n \nabla \zeta\, dx dt + \int_{Q_T} S''(u_n)\,|\nabla u_n|^{p(x)} \zeta\, dx dt$$
$$+ \int_{Q_T} g_n(t, x, u_n, \nabla u_n) S'(u_n)\,\zeta\, dx dt = \int_{Q_T} f_n S'(u_n)\,\zeta\, dx dt.$$

Which implies that

$$\frac{\partial S(u_n)}{\partial t} - \operatorname{div}\left(S'(u_n)\,|\nabla u_n|^{p(x)-2}\,\nabla u_n\right) + S''(u_n)\,|\nabla u_n|^{p(x)}$$
$$+ g_n(t, x, u_n, \nabla u_n) S'(u_n) = f_n S'(u_n) \text{ in } \mathcal{D}'(Q_T). \tag{112}$$

We now focus on passing to the limit in each term of (112) as $n \to \infty$ (in the distributional sense). In view of the almost everywhere convergence (40) and by taking into account the properties of $S(\cdot)$, it results that

$$S(u_n) \to S(u), \text{ a.e. in } Q_T \text{ and weak-* in } L^\infty(Q_T), \tag{113}$$
$$S'(u_n) \to S'(u), \text{ a.e. in } Q_T \text{ and weak-* in } L^\infty(Q_T), \tag{114}$$
$$S''(u_n) \to S''(u), \text{ a.e. in } Q_T \text{ and weak-* in } L^\infty(Q_T). \tag{115}$$

By employing (113), we deduce that

$$\frac{\partial S(u_n)}{\partial t} \to \frac{\partial S(u)}{\partial t} \text{ in } \mathcal{D}'(Q_T).$$

Using the fact that supp $S' \subset [-M, M]$, one obtains

$$S'(u_n)\,|\nabla u_n|^{p(x)-2}\,\nabla u_n = S'(u_n)\,|\nabla T_M(u_n)|^{p(x)-2}\,\nabla T_M(u_n), \text{ a.e. in } Q_T.$$

Thus, strong convergence of truncation (67) allows us to get

$$|\nabla T_M(u_n)|^{p(x)-2}\,\nabla T_M(u_n) \to |\nabla T_M(u)|^{p(x)-2}\,\nabla T_M(u),$$
$$\text{strongly in } \left(L^{p'(x)}(Q_T)\right)^N.$$

This result leads to obtain

$$-\operatorname{div}\left(S'(u_n)\,|\nabla T_M(u_n)|^{p(x)-2}\,\nabla T_M(u_n)\right) \to -\operatorname{div}\left(S'(u)\,|\nabla T_M(u)|^{p(x)-2}\,\nabla T_M(u)\right).$$

strongly in $\mathcal{U}(Q_T)^*$, as $n \to \infty$. As a result, we have

$$-\operatorname{div}\left(S'(u_n)\,|\nabla u_n|^{p(x)-2}\,\nabla u_n\right) \to -\operatorname{div}\left(S'(u)\,|\nabla u|^{p(x)-2}\,\nabla u\right) \text{ strongly in } \mathcal{U}(Q_T)^*.$$

Again, since supp $S'' \subset [-M, M]$, we have

$$S''(u_n)\,|\nabla u_n|^{p(x)} = S''(u_n)\,|\nabla T_M(u_n)|^{p(x)} \text{ a.e. in } Q_T.$$

We will use Lemma (3) to achieve the three followings results. With the help (67) and (115), one gets

$$S''(u_n)\,|\nabla T_M(u_n)|^{p(x)} \to S''(u)\,|\nabla T_M(u)|^{p(x)} \text{ strongly in } L^1(Q_T).$$

The strong convergence (105) and (114) yields that

$$g_n(t,x,u_n,\nabla u_n)S'(u_n) \to g(t,x,u,\nabla u)S'(u) \text{ strongly in } L^1(Q_T).$$

The results of (25) and (114) give us

$$f_n S'(u_n) \to f S'(u) \text{ strongly in } L^1(Q_T).$$

By recapping the above obtained convergences, we pass to the limit in all the terms of (112). We therefore have

$$\begin{aligned} \frac{\partial S(u)}{\partial t} &- \operatorname{div}\left(S'(u)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + S''(u)\,|\nabla u|^{p(x)} \\ &+ g(t,x,u,\nabla u)S'(u) = f S'(u) \text{ in } \mathcal{D}'(Q_T). \end{aligned} \tag{116}$$

To finish our proof, we need to check periodicity condition (24). It is clear that

$$\nabla S(u_n) = S'(u_n)\nabla T_M(u_n) \quad \text{and} \quad S'(u)\nabla T_M(u) = \nabla S(u). \tag{117}$$

We employ once again Lemma (3) alongside (68) to deduce that

$$S'(u_n)\,\nabla T_M(u_n) \to S'(u)\nabla T_M(u) \text{ strongly in } \left(L^{p(x)}(Q_T)\right)^N. \tag{118}$$

It results from (117) and (118) that

$$S(u_n) \to S(u) \text{ strongly in } \mathcal{U}(Q_T). \tag{119}$$

Furthermore, the aforementioned limit processes for (112) lead to conclude that

$$\frac{\partial S(u_n)}{\partial t} \to \frac{\partial S(u)}{\partial t} \text{ strongly in } \mathcal{U}(Q_T)^* + L^1(Q_T). \tag{120}$$

According to (119), (120) and (9), we derive that

$$S(u_n) \to S(u) \text{ strongly in } \mathcal{C}\left([0,T]; L^1(\Omega)\right). \tag{121}$$

Using the fact that u_n is periodic with respect to time, we arrive at

$$S(u)(T) = S(u)(0) \text{ a.e in } \Omega.$$

This completes the proof of Theorem 1.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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