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On self-similar singularity formation for the binormal flow

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Abstract. The aim of this article is to establish a concise proof for a stability result of self-similar solutions of the binormal flow, in some more restrictive cases than in Banica and Vega (Ann Sci Éc Norm Supér 48:1421–1453, 2015). This equation, also known as the Local Induction Approximation, is a standard model for vortex filament dynamics, and its self-similar solution describes the formation of a corner singularity on the filament. Our approach strongly uses the link that Hasimoto pointed out in 1972 between the solution of the binormal flow and the one of the 1-D cubic Schrödinger equation, as well as the existence results associated to the latter.

1. Introduction

In this paper, we propose a new proof of the stability of self-similar solutions of the binormal flow

$$\chi_t = \chi_x \wedge \chi_{xx}. \tag{1}$$

In terms of physics, $\chi(t, x)$ belongs to $\mathbb{R}^3$, t represents the time, and x is the arc-length variable. This equation was proposed in 1906 by DaRios [14] and re-discovered in 1965 by Arms and Harma [1], for modeling a vortex filament dynamic under Euler equations.

In a few words, its formal derivation goes as follows. If we consider the velocity of an incompressible fluid u and its vorticity ω, the Biot-Savart law tells us that:

$$u(t, x) = \int_{\mathbb{R}^3} \frac{(x - y) \wedge \omega(t, y)}{4\pi |x - y|^3} \mathrm{d}y.$$

Then, if we suppose that $\omega(t)$ belongs to a 1D curve (i.e., $\omega = \Gamma \chi_x \delta_\chi$) with χ_x of norm 1, we can write:

$$u(t, x) = \int_{-\infty}^{\infty} \frac{(x - \chi(t, s)) \wedge \omega(t, \chi(t, s))}{4\pi |x - \chi(t, s)|^3} \mathrm{d}s.$$

Keywords: Binormal flow, Singularity, Stability, Vortex filament.

Conducting a Taylor expansion around zero on the space variable and restricting the domain of integration to $[-L, L]$ approximates the previous integral by:

$$
\begin{aligned}
u(t,0) &\approx \frac{\Gamma}{4\pi}\int_{-L}^{L}\frac{((x_1,x_2,0)-s\chi_s(t,0)-\frac{s^2}{2}\chi_{ss}(t,0))\wedge(\chi_s(t,0)+s\chi_{ss}(t,0))}{|(x_1,x_2,-s)|^3}\mathrm{d}s\\
&= \frac{\Gamma}{4\pi}\frac{(-x_2,x_1,0)}{\epsilon^2}\int_{-\frac{L}{\epsilon}}^{\frac{L}{\epsilon}}\frac{\mathrm{d}s}{(1+s^2)^{\frac{3}{2}}}+\frac{\Gamma}{4\pi}(x_1,x_2,0)\wedge\chi_{ss}(t,0)\int_{-L}^{L}\frac{s}{|\epsilon^2+s^2|^{\frac{3}{2}}}\mathrm{d}s\\
&\quad -\frac{\Gamma}{8\pi}\chi_s(t,0)\wedge\chi_{ss}(t,0)\int_{-\frac{L}{\epsilon}}^{\frac{L}{\epsilon}}\frac{s^2}{|1+s^2|^{\frac{3}{2}}}\mathrm{d}s.
\end{aligned}
$$

The first term corresponds to a fluid rotating around a still vertical axis, the second term vanishes by a parity argument, and the third term gives us (1), after a time-renormalization. This model is sometimes called the Local Induction Approximation (LIA) or vortex filament equation (VFE), and is the subject of further discutions in [6,13] and more recently by Jerrard and Seis [10] with stronger assumptions but rigorous arguments.

In 1972, Hasimoto linked the solutions $\chi(t,x)$ of (1) to solutions of a 1-D cubic Schrödinger equation by using the Frenet and parallel frames in [9]. This transformation is in the same spirit as the Madelung transform.

Conversly, for a given real potential a and a given solution ψ of

$$
i\psi_t+\psi_{xx}+\frac{1}{2}(|\psi|^2-a(t))\psi=0, \tag{2}
$$

the Hasimoto transformation is reversible by using Frenet frames for non-vanishing curvatures vortices. However, the calculations are much faster and work for any curvatures by constructing first parallel frames $(T,e_1,e_2)(t,x)$ that satisfy:

$$
T_x=\Re(\overline{\psi}N),\quad N_x=-\psi T,\quad T_t=\Im(\overline{\psi_x}N),\quad N_t=-i\psi_xT-\frac{i}{2}(|\psi|^2-a(t))N, \tag{3}
$$

with $N=e_1+ie_2$, and any orthonormal basis as initial data. It follows that the vector T satisfies the 1-D Schrödinger map with values in $\mathbb{S}^2$:

$$
T_t=T\wedge T_{xx},
$$

and can be integrated into a solution χ of the binormal flow (1) starting at a point P at (t_0,x_0) with the formula:

$$
\chi(t,x)=P+\int_{t_0}^{t}(T\wedge T_x)(\tau,x_0)\mathrm{d}\tau+\int_{x_0}^{x}T(t,s)\mathrm{d}s,\quad \forall(t,x).
$$

In this paper, we study the stability of the self-similar solutions $\{\chi_\alpha\}_{\alpha>0}$ of (1) determined for $t>0$ by a curvature of $\frac{\alpha}{\sqrt{t}}$ and a torsion of $\frac{x}{2t}$. The behavior of $\chi_\alpha(t,s)$ for $t>0$ was exhibited by physicists in [11,12], and a numeric study on it was done in [7]. In [8], it has been proven that they are solutions of (2), smooth as long as $t > 0$ and have a trace at $t = 0$ forming a one corner polygonal line of angle θ such that

$$\sin\frac{\theta}{2} = \mathrm{e}^{-\pi\frac{\alpha^2}{2}}. \tag{4}$$

This class of solutions correspond to solutions of 1-D cubic NLS solutions

$$\psi_\alpha(t,x) = \alpha\frac{\mathrm{e}^{i\frac{x^2}{4t}}}{\sqrt{t}},$$

taking $a(t) = \frac{\alpha^2}{t}$ in (2).

Theorem 1. (The initial value problem for the binormal flow) *Let χ_0 a smooth arc-length parametrized curve of $\mathbb{R}^3$, except at one point located at arc-length $x = 0$ where it forms a corner of angle θ. Let c be the curvature of χ_0, τ its torsion and α given by* (4)*.*

If α defined from θ by (4) *is small enough, and if*

$$\begin{aligned}
&c \in W^{3,1}(\mathbb{R}) \cap H^2(\mathbb{R}), \quad \frac{c}{x} \in W^{2,1}(\mathbb{R}) \cap H^2(\mathbb{R}),\\
&x^2c \in W^{3,1}(\mathbb{R}) \cap H^2(\mathbb{R}), \quad (1+x^2)c \in L^2(\mathbb{R}),\\
&x^{-2}c \in L^2(\mathbb{R}), \quad \tau \in H^2(\mathbb{R}) \quad \textit{and} \quad \tau^2 \in H^1(\mathbb{R}),
\end{aligned}$$

then there exists $t_0 > 0$ and

$$\chi(t,x) \in \mathcal{C}([-t_0,t_0], Lip) \cap \mathcal{C}([-t_0,t_0]\backslash\{0\}, \mathcal{C}^4), \tag{5}$$

a solution of the binormal flow (1) *on $(0, t_0]$, having χ_0 as a limit at time $t = 0$, and there exists $C > 0$ such that:*

$$\sup_x |\chi(t,x) - \chi_0(x)| \le C\sqrt{t}. \tag{6}$$

Moreover, the tangent vector $T = \partial_x\chi$ has a limit at time zero with the same time-decay rate:

$$\forall t > 0 \quad \forall x \in \mathbb{R} \quad \exists C(x) \quad |T(t,x) - \partial_x\chi_0(t)| \le C(x)t^{\frac{1}{4}}. \tag{7}$$

This type of result has already been proven by Banica and Vega in Theorem 1.2 of [5], under weaker assumptions on the curvature and torsion of χ_0. As a counterpart, the corresponding scattering results for (2) (existence of wave operator and asymptotic completeness) obtained in [3] are with weaker decay. As a consequence, the proof requires to obtain asymptotic space states for $T(t,x)$ and $N(t,x)$ when $x \to \pm\infty$, and a much more technical iterative argument to obtain the limit for T and N at time $t = 0$.

Here, we will use a stronger convergence rate (obtained in the existence result of the wave operator in [2]) to give a concise proof of Theorem 1.

We note that even under more restrictive hypothesis than in [5], we do not have an asymptotic completeness result with better decay, that would allow us to give also a concise proof of Theorem 1.3 of the second stability result in [5].

Let us streamline here the constructive proof of Theorem 1. Denoting T_0 the tangent vector to χ_0, we define the complex-valued functions $g \in \mathbb{C}$ and $N_0 \in \mathbb{S}^2 + i\mathbb{S}^2$ defined by the parallel frame system:

$$\begin{cases} T_{0x}(x) = \Re(g(x)N_0(x)) \\ N_{0x}(x) = -\overline{g}(x)T_0(x) \end{cases}, \tag{8}$$

with initial data (A_α^+, B_α^+) for $x > 0$ and (A_α^-, B_α^-) for $x < 0$, where $A_\alpha^\pm$ and $B_\alpha^\pm$ stand for the complex vectors appearing in the asymptotics of the normals vectors of the same self-similar solution χ_α (see Theorem 1 of [8]).

Let us note that, using Frenet frame, there exists $\gamma \in [0, 2\pi]$ such that:

$$g(x) = c(x)e^{i(\int_0^x \tau(s)ds + \gamma)}, \tag{9}$$

as explained in Remark 2.1 of [4].

Now set:

$$u_+ = \mathcal{F}^{-1}\sqrt{i}\left(g(2\cdot)e^{i\alpha^2 \log|\cdot|}\right). \tag{10}$$

The hypothesis of Theorem 1 on c and τ allows u_+ to belong to some particular Sobolev spaces in order to use the existence of a wave operator for (2) proved in Theorem 1.4 of [2]. More precisely, u_+ is in $\dot{H}^{-2} \cap H^2 \cap W^{2,1}$ and α is small, so there exists $t_0 > 0$ and a unique solution of (2) on $(0, t_0]$ of the form:

$$\psi(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}\left(\alpha + \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)\right), \tag{11}$$

with u being a perturbation that writes:

$$u(t, x) = e^{it\partial_x^2}u_+(x) + r(t, x). \tag{12}$$

The proof of this result uses scattering methods after performing a pseudo-conformal transformation, and allows us to have the following control on the time decay of the remainder term r, for $k = 1$ and $k = 2$:

$$\|r(t)\|_{L_x^2} = \mathcal{O}(t^{-\frac{1}{2}}) \quad , \quad \|\nabla^k r(t)\|_{L_x^2} = \mathcal{O}(t^{-1}). \tag{13}$$

The next step in our proof is to use the parallel frame (3) with the function ψ given by (11) to construct a solution χ of (1) on $(0, t_0]$.

Then, we consider the vectors T and N given by (3), as well as $\tilde{N}$ a modulated version of N defined later. We prove in Sect. 2.2 that T and $\tilde{N}$ admit a trace at time $t = 0$, thanks to bounds on the perturbation u given in Corollary 1, consequence of bound (13).

Then, in Sect. 2.3 we find the ODE system verified by $T_{|t=0}$ and $\tilde{N}_{|t=0}$ for $x \neq 0$ that turns out to be the same as the one of T_0 and N_0, due to (10). Sections 2.2 and 2.3 are the part of the proof that simplify consistently the proof in [5].

Finally, in Sect. 2.4, we use self-similar paths to determine $T_{|t=0}$ and $\tilde{N}_{|t=0}$ at $x = 0^+$ and $x = 0^-$ for the ODE system, that coincides with the corner singularity directions of χ_0 and complete the Cauchy Problem. These last results allows us to conclude in Sect. 3 that we recovered χ_0 at time $t = 0$.

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

2. Construction of perturbed self-similar solution of the binormal flow

As announced in the introduction, we first define the complex-valued function g with the system verified by χ_0's tangent and normal vectors T_0 and N_0:

$$\begin{cases} T_{0x}(x) = \Re(g(x)N_0(x)) \\ N_{0x}(x) = -\overline{g}(x)T_0(x) \end{cases}, \tag{14}$$

with initial data (A_α^+, B_α^+) for $x > 0$ and (A_α^-, B_α^-) for $x < 0$, and consider

$$u_+ = \mathcal{F}^{-1}\sqrt{i}\left(g(2\cdot)e^{i\alpha^2 \log|\cdot|}\right). \tag{15}$$

We now deduce regularity on u_+ from the the hypothesis of Theorem 1 on c and τ, which is the purpose of the following lemma.

Lemma 1. *Consider the curvature c and the torsion τ of a parametrized curve. Define u_+ by formula (15) and recall expression (9) of g.*
If

$$c \in W^{3,1}(\mathbb{R}) \cap H^2(\mathbb{R}), \quad \frac{c}{x} \in W^{2,1}(\mathbb{R}) \cap H^2(\mathbb{R}),$$
$$x^2 c \in W^{3,1}(\mathbb{R}) \cap H^2(\mathbb{R}), \quad (1 + x^2)c \in L^2(\mathbb{R}),$$

and

$$x^{-2}c \in L^2(\mathbb{R}), \quad \tau \in H^2(\mathbb{R}) \quad \textit{and} \quad \tau^2 \in H^1(\mathbb{R}),$$

then

$$u_+ \in W^{1,2}(\mathbb{R}) \cap H^2(\mathbb{R}) \cap \dot{H}^{-2}(\mathbb{R}) \quad \textit{and} \quad (1 + x^2)u_+ \in L^\infty(\mathbb{R}),$$
$$(1 + x^2)xu_+ \in L^\infty(\mathbb{R}).$$

This lemma will allow us to apply a wave operator existence theorem right after, but also to use the weighted L^∞ bound on u_+ in the proof of Corollary 1.

Proof. The idea of the proof is to write the inverse Fourier transform formula and perform integration by parts on it, to gain decay. We have by definition:

$$u_+(x) = \int_{\mathbb{R}} \mathrm{e}^{-ixy} \sqrt{i} c(2y) \mathrm{e}^{i(\int_0^{2y} \tau(s)\mathrm{d}s+\gamma)} \mathrm{e}^{i\alpha^2 \log|y|} \mathrm{d}y,$$

so integrating by parts to times leads to:

$$\begin{aligned} u_+(x) &= -\int_{\mathbb{R}} \frac{\mathrm{e}^{-ixy}}{-ix} \sqrt{i}(2c'(2y) + ic(2y)\tau(2y) + i\alpha^2 \frac{c(2y)}{y}) \mathrm{e}^{i(\int_0^{2y} \tau(s)\mathrm{d}s+\gamma)} \mathrm{e}^{i\alpha^2 \log|y|} \mathrm{d}y \\ &= \int_{\mathbb{R}} \frac{\mathrm{e}^{-ixy}}{x^2} \sqrt{i}(4c''(2y) + i2c'(2y)\tau(2y) + i2c(2y)\tau'(2y)) \mathrm{e}^{i(\int_0^{2y} \tau(s)\mathrm{d}s+\gamma)} \mathrm{e}^{i\alpha^2 \log|y|} \mathrm{d}y \\ &\quad + \int_{\mathbb{R}} \frac{\mathrm{e}^{-ixy}}{x^2} \sqrt{i} i\alpha^2 \frac{2yc'(2y) + c(2y)}{y^2} \mathrm{e}^{i(\int_0^{2y} \tau(s)\mathrm{d}s+\gamma)} \mathrm{e}^{i\alpha^2 \log|y|} \mathrm{d}y \\ &\quad + \int_{\mathbb{R}} \frac{\mathrm{e}^{-ixy}}{x^2} \sqrt{i} i\tau(2y)(2c'(2y) + ic(2y)\tau(2y) + i\alpha^2 \frac{c(2y)}{y}) \mathrm{e}^{i(\int_0^{2y} \tau(s)\mathrm{d}s+\gamma)} \mathrm{e}^{i\alpha^2 \log|y|} \mathrm{d}y \\ &\quad + \int_{\mathbb{R}} \frac{\mathrm{e}^{-ixy}}{x^2} \sqrt{i} \frac{i\alpha^2}{y}(2c'(2y) + ic(2y)\tau(2y) + i\alpha^2 \frac{c(2y)}{y}) \mathrm{e}^{i(\int_0^{2y} \tau(s)\mathrm{d}s+\gamma)} \mathrm{e}^{i\alpha^2 \log|y|} \mathrm{d}y. \end{aligned}$$

Because all of the terms in those integrals are by hypothesis either L^1, or a product of two L^2 functions, it all converges and we deduce that $u_+ \in L^1$ and $(1+x^2)u_+ \in L^\infty$.

Then, it is straightforward to check that $(1 + x^2)xu_+ \in L^\infty$ with an additional integration by parts. To obtain $\nabla u_+ \in L^1$, we write:

$$\nabla u_+(x) = -i \int_{\mathbb{R}} \mathrm{e}^{-ixy} y \sqrt{i} c(2y) \mathrm{e}^{i(\int_0^{2y} \tau(s)\mathrm{d}s+\gamma)} \mathrm{e}^{i\alpha^2 \log|y|} \mathrm{d}y,$$

and perform as well two integration by parts. We similarly show that $\nabla^2 u_+ \in L^1$. Finally, for the L^2 hypothesis, we use Parseval identity to claim that $(1 + x^2)c \in L^2$ and $x^{-2}c \in L^2$ imply that $u_+ \in H^2 \cap \dot{H}^{-2}$. □

Thanks to this lemma, we have that u_+ is in $W^{1,2} \cap H^2 \cap \dot{H}^{-2}$ under the hypothesis of Theorem 1. Therefore, we can apply Theorem 1.2 of [2], to obtain a unique solution of (2) on $(0, t_0]$ that writes:

$$\psi(t, x) = \frac{\mathrm{e}^{i\frac{x^2}{4t}}}{\sqrt{t}} \left(\alpha + \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)\right), \tag{16}$$

where:

$$u(t, x) = \mathrm{e}^{it\partial_x^2} u_+(x) + r(t, x), \tag{17}$$

with r satisfying (13).

Then, equations (3) of Hasimoto's construction allow us to construct χ, a solution of (1) on $(0, t_0]$ by its tangent and normal vectors T and N. However, in order to identify the trace of $\chi(t)$ at time $t = 0$, we need a better understanding of the perturbation u.

2.1. Preliminary bound

In order to obtain a bound on u that is sharp enough, we shall use the decay given by (13).

Corollary 1. (L^∞ bound on the perturbation u) *Let u defined by* (17)*. Under the hypothesis of Theorem* 1*, we have the following bound on u and its derivative as t goes to zero:*

$$\left|u\left(\frac{1}{t},\frac{x}{t}\right)\right| \leq t^{\frac{1}{2}}, \quad \text{with} \quad \left|r\left(\frac{1}{t},\frac{x}{t}\right)\right| \leq t^{\frac{3}{4}},$$

and

$$\left|\partial_x u\left(\frac{1}{t},\frac{x}{t}\right)\right| \leq \frac{x}{\sqrt{t}} + t^{\frac{1}{2}}, \quad \text{with} \quad \left|\partial_x r\left(\frac{1}{t},\frac{x}{t}\right)\right| \leq t^{\frac{1}{2}}.$$

Moreover, we have:

$$\left|\frac{ix}{2t}e^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right) - \left[e^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right)\right]_x\right| \leq t^{\frac{1}{2}}. \tag{18}$$

The last estimate comes from a cancelation, and gives us more decay that expected.

Proof. First, we give a bound of the remainder term r and its derivative using the decay (13) given in Theorem 1.2 of [2] (wave operator existence). For this, we apply the Gagliardo Niremberg interpolation inequality:

$$\left|r\left(\frac{1}{t},\frac{x}{t}\right)\right| \leq t^{\frac{1}{4}}\left\|r\left(\frac{1}{t},\cdot\right)\right\|_{L^2}^{\frac{1}{2}} t^{-\frac{1}{4}}\left\|\partial_x r\left(\frac{1}{t},\cdot\right)\right\|_{L^2}^{\frac{1}{2}} \leq t^{\frac{3}{4}},$$

and similarly:

$$\left|\partial_x r\left(\frac{1}{t},\frac{x}{t}\right)\right| \leq t^{\frac{1}{2}}.$$

Next, we simply write:

$$\left|e^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right)\right| = \left|\int \sqrt{t}e^{i\frac{t}{4}(\frac{x}{t}-y)^2}u_+(y)\mathrm{d}y\right| \leq \sqrt{t}\|u_+\|_{L^1},$$

and for the other term we use the fact that $xu_+(x) \in L^1$, obtained in Lemma 1:

$$\left|\partial_x e^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right)\right| = \left|\partial_x \int \sqrt{t}e^{i\frac{t}{4}(\frac{x}{t}-y)^2}u_+(y)\mathrm{d}y\right| = \left|\frac{ix}{2t}e^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right)\right| + \left|\int \sqrt{t}e^{i\frac{t}{4}(\frac{x}{t}-y)^2}\frac{iy}{2}u_+(y)\mathrm{d}y\right|,$$

that ensures:

$$\left|\partial_x u\left(\frac{1}{t},\frac{x}{t}\right)\right| \le \frac{x}{\sqrt{t}} + \sqrt{t}.$$

Finally, (18) comes directly from the previous expression, as we write:

$$\left|\frac{ix}{2t}\mathrm{e}^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right) - \left[\mathrm{e}^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right)\right]_x\right| = \left|\int \sqrt{t}\mathrm{e}^{i\frac{t}{4}(\frac{x}{t}-y)^2}\frac{iy}{2}u_+(y)\mathrm{d}y\right|.$$

□

We are now ready to tackle our proof.

2.2. Limit at time $t = 0$

As announced, the next step is to prove the existence of a limit for vectors T and N, up to a phase.

Lemma 2. (Limit of vector T) *The tangent vector T of χ has a limit at time zero with a convergence rate given by:*

$$\forall t_0 \ge t_2 \ge t_1 > 0 \quad \forall x \in \mathbb{R}^* \quad |T(t_2,x) - T(t_1,x)| \le x t_2^{\frac{1}{4}} + t_2^{\frac{3}{4}} + \frac{\sqrt{t_2}}{x}.$$

This lemma gives us the convergence rate (7) announced in Theorem 1.

Proof. Now let $t_2 \ge t_1 > 0$,

$$\begin{aligned}
|T(t_2,x) - T(t_1,x)| &= \left|\int_{t_1}^{t_2} T_t(t,x)\mathrm{d}t\right| = \left|\Im\int_{t_1}^{t_2}\overline{\psi_x}N(t,x)\mathrm{d}t\right| \\
&= \left|\Im\int_{t_1}^{t_2}\frac{\mathrm{e}^{-i\frac{x^2}{4t}}}{\sqrt{t}}\left(\frac{-ix}{2t}u\left(\frac{1}{t},\frac{x}{t}\right) - i\frac{x\overline{\alpha}}{2t} + \left[u\left(\frac{1}{t},\frac{x}{t}\right)\right]_x\right)N(t,x)\mathrm{d}t\right| \\
&\le x t_2^{\frac{1}{4}} + t_2 + \left|\Im\int_{t_1}^{t_2}\mathrm{e}^{-i\frac{x^2}{4t}}\frac{ix\alpha}{2t\sqrt{t}}N(t,x)\mathrm{d}t\right| \\
&\quad + \left|\Im\int_{t_1}^{t_2}\frac{\mathrm{e}^{-i\frac{x^2}{4t}}}{\sqrt{t}}\left(\frac{-ix}{2t}\mathrm{e}^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right) + \left[\mathrm{e}^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right)\right]_x\right)N(t,x)\mathrm{d}t\right|,
\end{aligned}$$

where the terms with the remainder r have provided enough decay. Then, if we use (18), we have that:

$$\left|\Im\int_{t_1}^{t_2}\frac{\mathrm{e}^{-i\frac{x^2}{4t}}}{\sqrt{t}}\left(\frac{-ix}{2t}\mathrm{e}^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right) + \left[\mathrm{e}^{i\frac{1}{t}\partial_x^2}u_+\left(\frac{x}{t}\right)\right]_x\right)N(t,x)\mathrm{d}t\right| \le t_2.$$

For the other term, we integrate by parts:

$$\left|\Im\int_{t_1}^{t_2} \mathrm{e}^{-i\frac{x^2}{4t}}\frac{ix\alpha}{2t\sqrt{t}}N(t,x)\mathrm{d}t\right| \le \left|\Im\left[\mathrm{e}^{-i\frac{x^2}{4t}}\frac{2\sqrt{t}\alpha}{x}N(t,x)\right]_{t_1}^{t_2}\right| + \left|\Im\int_{t_1}^{t_2}\mathrm{e}^{-i\frac{x^2}{4t}}\frac{\alpha}{x\sqrt{t}}N(t,x)\mathrm{d}t\right|$$
$$+\left|\Im\int_{t_1}^{t_2}\mathrm{e}^{-i\frac{x^2}{4t}}\frac{2\sqrt{t}\alpha}{x}N_t(t,x)\mathrm{d}t\right|$$
$$\le \frac{2\alpha\sqrt{t_2}}{x} + \left|\Im\int_{t_1}^{t_2}\mathrm{e}^{-i\frac{x^2}{4t}}\frac{2\sqrt{t}\alpha}{x}N_t(t,x)\mathrm{d}t\right|.$$

We must now expand the term in N_t:

$$\left|\Im\int_{t_1}^{t_2}\mathrm{e}^{-i\frac{x^2}{4t}}\frac{2\sqrt{t}\alpha}{x}N_t(t,x)\mathrm{d}t\right|$$
$$\le \left|\Im\int_{t_1}^{t_2}\mathrm{e}^{-i\frac{x^2}{4t}}\frac{2\sqrt{t}\alpha}{x}i\frac{\mathrm{e}^{i\frac{x^2}{4t}}}{\sqrt{t}}\left(\frac{ix}{2t}\overline{u}\left(\frac{1}{t},\frac{x}{t}\right)+i\frac{x\alpha}{2t}+\left[\overline{u}\left(\frac{1}{t},\frac{x}{t}\right)\right]_x\right)T(t,x)\mathrm{d}t\right|$$
$$+\left|\Im\int_{t_1}^{t_2}\mathrm{e}^{-i\frac{x^2}{4t}}\frac{2\sqrt{t}\alpha}{x}\frac{i}{2}\left(\frac{|u\left(\frac{1}{t},\frac{x}{t}\right)|^2}{t}+\frac{2\Re(u\left(\frac{1}{t},\frac{x}{t}\right)\alpha)}{t}\right)N(t,x)\mathrm{d}t\right|$$
$$\le t_2^{\frac{3}{4}}+\frac{t_2}{x},$$

using both (18) and the fact that T is real, so we have:

$$\Im\int_{t_1}^{t_2}\frac{\alpha^2}{t}T(t,x)\mathrm{d}t = 0.$$

To sum up, we showed that:

$$\forall t_0 \ge t_2 \ge t_1 > 0 \quad \forall x\in\mathbb{R}^* \quad |T(t_2,x)-T(t_1,x)| \le xt_2^{\frac{1}{4}}+t_2^{\frac{3}{4}}+\frac{\sqrt{t_2}}{x},$$

and the lemma is proven.

Note that, for self-similar paths, we also obtained that $T(t,x\sqrt{t})$ has a limit as t, $\frac{1}{x}$ and $x\sqrt{t}$ simultaneously go to zero. □

In order for N to converge, we must add a phase.

Lemma 3. (Limit of vector N) *Let us write*

$$\tilde{N}(t,x) = \mathrm{e}^{i\alpha^2\ln\frac{|x|}{\sqrt{t}}}N(t,x) = \mathrm{e}^{i\phi}N,$$

where N is the normal vector of χ. Then $\tilde{N}$ has a limit at time zero with a convergence rate given by:

$$\forall t_0 \ge t_2 \ge t_1 > 0 \quad \forall x\in\mathbb{R}^* \quad |\tilde{N}(t_2,x)-\tilde{N}(t_1,x)| \le xt_2^{\frac{1}{4}}+t_2^{\frac{1}{2}}+\frac{\sqrt{t_2}}{x}+\frac{t_2}{x^2}.$$

Note that the factor $|x|$ in ϕ could be replaced by anything independent of t, but is chosen for assuring properties at time $t = 0$ as we will see in Lemma 5.

Proof. To follow the proof, the reader must only keep in mind that $|u\left(\frac{1}{t}, \frac{x}{t}\right)|$ behaves at worse like $t^{\frac{1}{2}}$ and $|\partial_x u\left(\frac{1}{t}, \frac{x}{t}\right)|$ at worse like $\sqrt{t} + tx\sqrt{t}$.
Recalling that:

$$\tilde{N}_t = \mathrm{e}^{i\phi} N_t - i\frac{\alpha^2}{2t} N(t,x)\mathrm{e}^{i\phi},$$

given $0 < t_1 \le t_2 \le t_0$, we have:

$$\begin{aligned}
&\tilde{N}(t_2, x) - \tilde{N}(t_1, x) \\
&= \int_{t_1}^{t_2} \tilde{N}_t(t,x)\mathrm{d}t = \int_{t_1}^{t_2} -i\psi_x T\mathrm{e}^{i\phi} + \frac{i}{2}(|\psi|^2 - \frac{\alpha^2}{t})N\mathrm{e}^{i\phi}\mathrm{d}t - i\frac{\alpha^2}{2t}N(t,x)\mathrm{e}^{i\phi} \\
&= -\int_{t_1}^{t_2} i\frac{\mathrm{e}^{i\frac{x^2}{4t}}}{\sqrt{t}}\left(\frac{ix}{2t}\overline{u}\left(\frac{1}{t},\frac{x}{t}\right) + i\frac{x\alpha}{2t} + \left[\overline{u}\left(\frac{1}{t},\frac{x}{t}\right)\right]_x\right)T(t,x)\mathrm{e}^{i\phi}\mathrm{d}t \\
&\quad + \frac{i}{2}\int_{t_1}^{t_2}\left(\frac{\cancel{\alpha^2}}{\cancel{t}} + \frac{|u\left(\frac{1}{t},\frac{x}{t}\right)|^2}{t} + \frac{2\Re(u\left(\frac{1}{t},\frac{x}{t}\right)\alpha)}{t} - \frac{\cancel{\alpha^2}}{\cancel{t}}\right)N(t,x)\mathrm{e}^{i\phi}\mathrm{d}t. \\
&\quad - \int_{t_1}^{t_2} i\frac{\alpha^2}{2t}N(t,x)\mathrm{e}^{i\phi}\mathrm{d}t.
\end{aligned}$$

As before, we use (18) so terms with u in the first integral partially cancel with each other. Using bounds of Corollary 1, we are now left with only a difference to study:

$$|\tilde{N}(t_2,x) - \tilde{N}(t_1,x)| \le xt_2^{\frac{1}{4}} + t_2^{\frac{1}{2}} + \left|-\int_{t_1}^{t_2} i\frac{\mathrm{e}^{i\frac{x^2}{4t}}}{\sqrt{t}}\frac{ix}{2t}\alpha T(t,x)\mathrm{e}^{i\phi}\mathrm{d}t - \int_{t_1}^{t_2} i\frac{\alpha^2}{2t}N(t,x)\mathrm{e}^{i\phi}\mathrm{d}t\right|.$$

For that, we integrate by parts the first term:

$$\begin{aligned}
&\int_{t_1}^{t_2}\frac{\mathrm{e}^{i\frac{x^2}{4t}}x}{2t\sqrt{t}}\alpha T(t,x)\mathrm{e}^{i\phi}\mathrm{d}t \\
&\quad = \left[\mathrm{e}^{i\frac{x^2}{4t}}\frac{2\sqrt{t}}{ix}\alpha T(t,x)\mathrm{e}^{i\phi}\right]_{t_1}^{t_2} \\
&\qquad - \int_{t_1}^{t_2}\mathrm{e}^{i\frac{x^2}{4t}}\frac{1}{ix\sqrt{t}}\alpha T(t,x)\mathrm{e}^{i\phi}\mathrm{d}t - \int_{t_1}^{t_2}\mathrm{e}^{i\frac{x^2}{4t}}\frac{2\sqrt{t}}{ix}\alpha T_t(t,x)\mathrm{e}^{i\phi}\mathrm{d}t \\
&\qquad + \int_{t_1}^{t_2}\mathrm{e}^{i\frac{x^2}{4t}}\frac{\alpha^2}{x\sqrt{t}}\alpha T(t,x)\mathrm{e}^{i\phi}\mathrm{d}t,
\end{aligned}$$

and get:

$$\begin{aligned}
|\tilde{N}(t_2,x) - \tilde{N}(t_1,x)| &\le xt_2^{\frac{1}{4}} + t_2^{\frac{1}{2}} + \frac{\sqrt{t_2}}{x} \\
&\quad + \left|\int_{t_1}^{t_2}\mathrm{e}^{i\frac{x^2}{4t}}\frac{2\sqrt{t}}{ix}\alpha T_t(t,x)\mathrm{e}^{i\phi}\mathrm{d}t - \int_{t_1}^{t_2} i\frac{\alpha^2}{2t}N(t,x)\mathrm{e}^{i\phi}\mathrm{d}t\right|.
\end{aligned}$$

We then use the fact that $T_t = \Im(\overline{\psi_x} N) = \frac{1}{2i}(\overline{\psi_x} N - \psi_x \overline{N})$ to write:

$$\begin{aligned}
&\int_{t_1}^{t_2} \mathrm{e}^{i\frac{x^2}{4t}} \frac{2\sqrt{t}}{ix} \alpha T_t(t,x) \mathrm{e}^{i\phi} \mathrm{d}t \\
&\quad = \frac{1}{2i} \int_{t_1}^{t_2} \frac{2}{ix} \alpha \left(\frac{-ix}{2t} u\left(\frac{1}{t}, \frac{x}{t}\right) - i\frac{x\alpha}{2t} + \left[u\left(\frac{1}{t}, \frac{x}{t}\right)\right]_x \right) N(t,x) \mathrm{e}^{i\phi} \mathrm{d}t \\
&\qquad - \frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{x^2}{4t}} \frac{2}{ix} \alpha \mathrm{e}^{i\frac{x^2}{4t}} \left(\frac{ix}{2t} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right) + i\frac{x\alpha}{2t} + \left[\overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)\right]_x \right) \overline{N}(t,x) \mathrm{e}^{i\phi} \mathrm{d}t.
\end{aligned}$$

Again, thanks to Corollary 1, only the terms without u are worth studying. Moreover, the first term cancels with the term coming from the phase ϕ. Therefore we have:

$$|\tilde{N}(t_2,x) - \tilde{N}(t_1,x)| \leq x t_2^{\frac{1}{4}} + t_2^{\frac{1}{2}} + \frac{\sqrt{t_2}}{x} + \left| \frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{2x^2}{4t}} \frac{\alpha^2}{t} \overline{N}(t,x) \mathrm{e}^{i\phi} \mathrm{d}t \right|.$$

The other one has a phase, so we perform a second integration by parts on it:

$$\begin{aligned}
&\frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{2x^2}{4t}} \frac{\alpha^2}{t} \overline{N}(t,x) \mathrm{e}^{i\phi} \mathrm{d}t \\
&\quad = \frac{1}{2i} \left[\mathrm{e}^{i\frac{2x^2}{4t}} \frac{2\alpha^2 t}{ix^2} \overline{N}(t,x) \mathrm{e}^{i\phi} \right]_{t_1}^{t_2} + \frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{2x^2}{4t}} \frac{2\alpha^2}{ix^2} \overline{N}(t,x) \mathrm{e}^{i\phi} \mathrm{d}t \\
&\qquad - \frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{2x^2}{4t}} \frac{2\alpha^2 t}{ix^2} \overline{N}_t(t,x) \mathrm{e}^{i\phi} \mathrm{d}t + \frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{2x^2}{4t}} \frac{\alpha^2\alpha^2}{x^2} \overline{N}(t,x) \mathrm{e}^{i\phi} \mathrm{d}t.
\end{aligned}$$

We finally expand the N_t term and observe that it has the desired behavior:

$$\begin{aligned}
&- \frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{2x^2}{4t}} \frac{2\alpha^2 t}{ix^2} \overline{N}_t(t,x) \mathrm{e}^{i\phi} \mathrm{d}t \\
&\quad = + \frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{x^2}{4t}} \frac{2\alpha^2 t}{ix^2} \frac{i}{\sqrt{t}} \left(\frac{-ix}{2t} u\left(\frac{1}{t}, \frac{x}{t}\right) - i\frac{x\overline{\alpha}}{2t} - \left[u\left(\frac{1}{t}, \frac{x}{t}\right)\right]_x \right) \overline{T}(t,x) \mathrm{e}^{i\phi} \mathrm{d}t \\
&\qquad - \frac{1}{2i} \int_{t_1}^{t_2} \mathrm{e}^{i\frac{2x^2}{4t}} \frac{2\alpha^2 t}{ix^2} \left(\frac{|u\left(\frac{1}{t}, \frac{x}{t}\right)|^2}{t} + \frac{2\Re\left(\overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)\overline{\alpha}\right)}{t} \right) N(t,x) \mathrm{e}^{i\phi} \mathrm{d}t.
\end{aligned}$$

To sum up, we proved that:

$$\forall t_0 \geq t_2 \geq t_1 > 0 \quad \forall x \in \mathbb{R}^* \quad |\tilde{N}(t_2,x) - \tilde{N}(t_1,x)| \leq x t_2^{\frac{1}{4}} + t_2^{\frac{1}{2}} + \frac{\sqrt{t_2}}{x} + \frac{t_2}{x^2}.$$

As for T, we also obtained that, for self-similar paths, $\tilde{N}(t, x\sqrt{t})$ has a limit as t, $\frac{1}{x}$ and $x\sqrt{t}$ simultaneously go to zero. □

2.3. More information about the tangents vectors at time $t = 0$

The aim of this section is to quantify the evolution of $T_{|t=0}$ and $\tilde{N}_{|t=0}$ with respect to the space variable. More precisely, we will show that:

$$\begin{cases} T_x(0,x) = \Re \frac{1}{\sqrt{i}} \widehat{u_+}\left(\frac{x}{2}\right) \mathrm{e}^{-i\alpha^2 \log|x|} \tilde{N}(0,x), \\ \tilde{N}_x(0,x) = -\frac{1}{\sqrt{i}} \overline{\widehat{u_+}\left(\frac{x}{2}\right) \mathrm{e}^{-i\alpha^2 \log|x|}} T(0,x), \end{cases} \quad \forall x \neq 0.$$

Those two claims can be proved separately and that is what we are going to do.

Lemma 4. (Properties of $T_{|t=0}$) *Let $x \in \mathbb{R}^*$, then we have:*

$$T_x(0,x) = \lim_{t\to 0} T_x(t,x) = \Re \frac{1}{\sqrt{i}} \widehat{u_+}\left(\frac{x}{2}\right) \mathrm{e}^{-i\alpha^2 \log|x|} \tilde{N}(0,x).$$

Proof. Let $(x_1, x_2) \in {\mathbb{R}_+^*}^2$. We are going to write the variation of T at $t > 0$ between x_1 and x_2, with the idea to make t go to zero:

$$\begin{aligned} T(t,x_2) - T(t,x_1) &= \int_{x_1}^{x_2} T_x(t,s)\mathrm{d}s = \int_{x_1}^{x_2} \Re(\overline{\psi} N)(t,s)\mathrm{d}s \\ &= \Re \int_{x_1}^{x_2} \frac{\mathrm{e}^{-i\frac{s^2}{4t}}}{\sqrt{t}} (u\left(\frac{1}{t}, \frac{s}{t}\right) + \alpha) N(t,s)\mathrm{d}s \\ &= \Re \left[\mathrm{e}^{-i\frac{s^2}{4t}} \frac{2\sqrt{t}}{is} \alpha N(t,s) \right]_{x_1}^{x_2} + \Re \int_{x_1}^{x_2} \mathrm{e}^{-i\frac{s^2}{4t}} \frac{2\sqrt{t}}{is^2} \alpha N(t,s)\mathrm{d}s \\ &\quad + \Re \int_{x_1}^{x_2} \mathrm{e}^{-i\frac{s^2}{4t}} \frac{2}{is^2} \mathrm{e}^{i\frac{s^2}{4t}} \alpha^2 T(t,s)\mathrm{d}s \\ &\quad + \Re \int_{x_1}^{x_2} \mathrm{e}^{-i\frac{s^2}{4t}} \frac{2}{is^2} \mathrm{e}^{i\frac{s^2}{4t}} \overline{u}\left(\frac{1}{t}, \frac{s}{t}\right) T(t,s)\mathrm{d}s \\ &\quad + \Re \int_{x_1}^{x_2} \frac{\mathrm{e}^{-i\frac{s^2}{4t}}}{\sqrt{t}} u\left(\frac{1}{t}, \frac{s}{t}\right) N(t,s)\mathrm{d}s. \end{aligned}$$

The last term will provide us the differential equation that we are looking for. The term in α^2 vanishes since it is an imaginary term inside the $\Re$ operator. All the other terms go to zero with t thanks to Corollary 1.

Now, recall that $u\left(\frac{1}{t}, \frac{x}{t}\right) = \mathrm{e}^{i\frac{1}{t}\partial_x^2} u_+\left(\frac{x}{t}\right) + r\left(\frac{1}{t}, \frac{x}{t}\right)$. If we write:

$$\mathrm{e}^{i\frac{1}{t}\partial_x^2} u_+\left(\frac{x}{t}\right) = \frac{\mathrm{e}^{i\frac{x^2}{4t}}}{\sqrt{\frac{i}{t}}} \int \mathrm{e}^{-i\frac{xy}{2}} \mathrm{e}^{i\frac{y^2}{4}t} u_+(y)\mathrm{d}y,$$

we have:

$$\frac{\mathrm{e}^{-i\frac{x^2}{4t}}}{\sqrt{t}} u\left(\frac{1}{t}, \frac{x}{t}\right) = \frac{1}{\sqrt{i}} \int \mathrm{e}^{-i\frac{xy}{2}} \mathrm{e}^{i\frac{y^2}{4}t} u_+(y)\mathrm{d}y + \frac{\mathrm{e}^{-i\frac{x^2}{4t}}}{\sqrt{t}} r\left(\frac{1}{t}, \frac{x}{t}\right) \underset{t\to 0}{\longrightarrow} \frac{1}{\sqrt{i}} \widehat{u_+}\left(\frac{x}{2}\right),$$

since $\|r\left(\frac{1}{t},\frac{x}{t}\right)\|_{L^\infty}\leq t^{\frac{3}{4}}$. Note that in [5], r decays like $t^{\frac{1}{4}}$ so the present argument is not enough.

Then, let us consider $(t_n)_{n\in\mathbb{Z}}$ such that $\forall n\in\mathbb{N},\quad \mathrm{e}^{i\alpha^2\log\sqrt{t_n}}=1$ and $t_n \underset{n\to\infty}{\longrightarrow} 0$,

$$N(t_n,x)=\mathrm{e}^{-i\phi(t_n,x)}\tilde{N}(t_n,x)=\mathrm{e}^{-i\alpha^2\log\frac{|x|}{\sqrt{t_n}}}\tilde{N}(t_n,x)\underset{n\to\infty}{\longrightarrow}\mathrm{e}^{-i\alpha^2\log|x|}\tilde{N}(0,x),$$

so by multiplying the limits:

$$\Re\frac{\mathrm{e}^{-i\frac{x^2}{4t_n}}}{\sqrt{t_n}}u\left(\frac{1}{t_n},\frac{x}{t_n}\right)N(t_n,x)\underset{n\to\infty}{\longrightarrow}\Re\frac{1}{\sqrt{i}}\widehat{u_+}\left(\frac{x}{2}\right)\mathrm{e}^{-i\alpha^2\log|x|}\tilde{N}(0,x),$$

and by dominated convergence:

$$\Re\int_{x_1}^{x_2}\mathrm{e}^{-i\frac{s^2}{4t_n}}\frac{u\left(\frac{1}{t_n},\frac{x}{t_n}\right)}{\sqrt{t_n}}N(t_n,s)\mathrm{d}s\underset{n\to\infty}{\longrightarrow}\Re\int_{x_1}^{x_2}\frac{1}{\sqrt{i}}\widehat{u_+}\left(\frac{x}{2}\right)\mathrm{e}^{-i\alpha^2\log|x|}\tilde{N}(0,x).$$

To sum up, we proved that:

$$T(t_n,x_2)-T(t_n,x_1)\underset{n\to\infty}{\longrightarrow}\Re\int_{x_1}^{x_2}\frac{1}{\sqrt{i}}\widehat{u_+}\left(\frac{x}{2}\right)\mathrm{e}^{-i\alpha^2\log|x|}\tilde{N}(0,x)\mathrm{d}x,$$

and the conclusion of the lemma is obtained by taking $x_1=x$, $x_2=x+h$, dividing by h, using Lemma 2 and choosing n large with respect to h. □

Lemma 5. (Properties of $\tilde{N}_{|t=0}$) *For $x\neq 0$, we have:*

$$\tilde{N}_x(0,x)=\lim_{t\to 0}\tilde{N}_x(t,x)=\frac{1}{\sqrt{i}}\overline{\widehat{u_+}\left(\frac{s}{2}\right)\mathrm{e}^{-i\alpha^2\log|x|}}T(0,s).$$

Proof. Let $(x_1,x_2)\in{\mathbb{R}_+^*}^2$, we write:

$$\tilde{N}(t,x_2)-\tilde{N}(t,x_1)=\int_{x_1}^{x_2}\tilde{N}_x(t,s)\mathrm{d}s=\int_{x_1}^{x_2}(-\psi T+i\frac{\alpha^2}{s}N)\mathrm{e}^{i\phi}\mathrm{d}s.$$

The term produced by the phase will help removing an otherwise non-vanishing term, so we start by looking at the integral of N_x:

$$\begin{aligned}\int_{x_1}^{x_2}\psi(t,s)T(t,s)\mathrm{e}^{i\phi}\mathrm{d}s&=\int_{x_1}^{x_2}\frac{\mathrm{e}^{i\frac{s^2}{4t}}}{\sqrt{t}}\alpha T(t,s)\mathrm{e}^{i\phi}\mathrm{d}s+\int_{x_1}^{x_2}\frac{\mathrm{e}^{i\frac{s^2}{4t}}}{\sqrt{t}}\overline{u}\left(\frac{1}{t},\frac{s}{t}\right)\mathrm{e}^{i\phi}T(t,s)\mathrm{d}s\\&=\left[\mathrm{e}^{i\frac{s^2}{4t}}\frac{2\sqrt{t}}{is}\alpha T(t,s)\mathrm{e}^{i\phi}\right]_{x_1}^{x_2}+\int_{x_1}^{x_2}\mathrm{e}^{i\frac{s^2}{4t}}\frac{2\sqrt{t}}{is^2}\alpha T(t,s)\mathrm{e}^{i\phi}\mathrm{d}s\\&\quad-\int_{x_1}^{x_2}\mathrm{e}^{i\frac{s^2}{4t}}\frac{2\sqrt{t}}{is}\alpha T(t,s)i\frac{\alpha^2}{s}\mathrm{e}^{i\phi}\mathrm{d}s-\int_{x_1}^{x_2}\mathrm{e}^{i\frac{s^2}{4t}}\frac{2\sqrt{t}}{is}\alpha T_s(t,s)\mathrm{e}^{i\phi}\mathrm{d}s\\&\quad+\int_{x_1}^{x_2}\frac{\mathrm{e}^{i\frac{s^2}{4t}}}{\sqrt{t}}\overline{u}\left(\frac{1}{t},\frac{x}{t}\right)T(t,s)\mathrm{e}^{i\phi}\mathrm{d}s.\end{aligned}$$

As with T, we will treat the term with u at the end, first we have to make sure that the T_s term goes to zero with t, using that $T_s = \Re(\overline{\psi} N)$:

$$\int_{x_1}^{x_2} e^{i\frac{s^2}{4t}} \frac{2\sqrt{t}}{is} \alpha T_s(t,s) e^{i\phi} ds = \int_{x_1}^{x_2} e^{i\frac{s^2}{4t}} \frac{2\sqrt{t}}{is} \alpha \frac{e^{i\frac{s^2}{4t}}}{\sqrt{t}} \overline{u}\left(\frac{1}{t}, \frac{s}{t}\right) \overline{N}(t,s) e^{i\phi} ds$$
$$+ \int_{x_1}^{x_2} e^{2i\frac{s^2}{4t}} \frac{1}{is} \alpha^2 \overline{N}(t,s) e^{i\phi} ds + \int_{x_1}^{x_2} \frac{1}{is} \alpha^2 N(t,s) e^{i\phi} ds$$
$$+ \int_{x_1}^{x_2} e^{i\frac{s^2}{4t}} \frac{1}{is} \alpha e^{-i\frac{s^2}{4t}} u\left(\frac{1}{t}, \frac{s}{t}\right) N(t,s) e^{i\phi} ds.$$

The first term is treated with Cauchy Schwarz, as well as the fourth. The third one is canceled by the phase. For the second term, an IBP shows that it goes to zero with t, using that $N_s = -\psi T$:

$$\int_{x_1}^{x_2} e^{2i\frac{s^2}{4t}} \frac{1}{is} \alpha^2 \overline{N}(t,s) e^{i\phi} ds = \left[e^{2i\frac{s^2}{4t}} \frac{t}{-s^2} \alpha^2 \overline{N}(t,s) e^{i\phi} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} e^{2i\frac{s^2}{4t}} \frac{2t}{-s^3} \alpha^2 \overline{N}(t,s) e^{i\phi} ds$$
$$+ \int_{x_1}^{x_2} e^{2i\frac{s^2}{4t}} \frac{t}{-s^2} \alpha^2 \overline{\psi}(t,s) \overline{T}(t,s) e^{i\phi} ds - \int_{x_1}^{x_2} e^{2i\frac{s^2}{4t}} \frac{t}{-s^2} \alpha^2 \overline{N}(t,s) \frac{i\alpha^2}{s} e^{i\phi} ds.$$

We shall now obtain the differential equation verified by $\tilde{N}$. Again, using $(t_n)_{n\in\mathbb{Z}}$ such that $e^{i\alpha^2 \log\sqrt{t_n}} = 1$ and $t_n \underset{n\to\infty}{\longrightarrow} 0$,

$$e^{i\phi(t_n,x)} T(t_n, x) \underset{n\to\infty}{\longrightarrow} e^{i\alpha^2 \log|x|} T(0,x),$$

and by multiplying the limits under the integral we write:

$$\int_{x_1}^{x_2} \frac{e^{i\frac{s^2}{4t_n}}}{\sqrt{t_n}} \overline{u}\left(\frac{1}{t_n}, \frac{s}{t_n}\right) e^{i\phi} T(t,s) ds \underset{n\to\infty}{\longrightarrow} \int_{x_1}^{x_2} \frac{1}{\sqrt{i}} \overline{\widehat{u_+}}\left(\frac{s}{2}\right) e^{i\alpha^2 \log|x|} T(0,s).$$

Hence:

$$\tilde{N}(t_n, x_2) - \tilde{N}(t_n, x_1) \underset{n\to\infty}{\longrightarrow} -\int_{x_1}^{x_2} \frac{1}{\sqrt{i}} \overline{\widehat{u_+}\left(\frac{s}{2}\right) e^{-i\alpha^2 \log|s|} T(0,s)} ds,$$

and the conclusion of the lemma is obtained by taking $x_1 = x$, $x_2 = x + h$, dividing by h, using Lemma 3 and chosing n large with respect to h. □

2.4. Description of the angles via self-similar paths

For the description of the angles, we will follow the same proof as for Proposition 5.1 of [4]. For the sake of completeness, we recall here the proof. As recalled in the introduction, we denote by $A_\alpha^\pm \in \mathbb{S}^2$ the directions of the corner generated at time $t = 0$ by the canonical self-similar solution $\chi_\alpha(t,x)$ of the binormal flow of curvature $\frac{\alpha}{\sqrt{t}}$:

$$A_\alpha^\pm := \partial_x \chi_\alpha(0, 0^\pm).$$

The frame of the profile $\chi(1)$ satisfies the system:

$$\begin{cases} \partial_x T_\alpha(1,x) = \Re(\alpha e^{-i\frac{x^2}{4}} N_\alpha(1,x)), \\ \partial_x N_\alpha(1,x) = \quad -\alpha e^{i\frac{x^2}{4}} T_\alpha(1,x), \end{cases} \tag{19}$$

and for $x \to \pm\infty$, there exists $B_\alpha^\pm \perp A_\alpha^\pm$, with $\Re(B_\alpha^\pm), \Im(A_\alpha^\pm) \in \mathbb{S}^2$ such that:

$$T_\alpha(1,x) = A_\alpha^\pm + \mathcal{O}\left(\frac{1}{x}\right) \quad \text{and} \quad e^{i\alpha^2 \log|x|} N_\alpha(1,x) = B_\alpha^\pm + \mathcal{O}\left(\frac{1}{x}\right).$$

Lemma 6. (Self-similar paths) *Let t_n be a sequence of positive times converging to zero. Up to a subsequence, there exists for all $x \in \mathbb{R}$ a limit given by:*

$$(T_*(x), N_*(x)) = \lim_{t\to 0}(T(t_n, x\sqrt{t_n}), N(t_n, x\sqrt{t_n})),$$

such that $(T_*, N_*(x))$ satisfies system (19) in the strong sense.

Then, there exists a unique rotation Θ, such that, for $x \to \pm\infty$:

$$T_*(x) = \Theta(A_\alpha^\pm) + \mathcal{O}\left(\frac{1}{|x|}\right), \quad N_*(x) = \Theta(B_\alpha^\pm) + \mathcal{O}\left(\frac{1}{|x|}\right).$$

Proof. Let $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ a sequence of positive times converging to 0. As explained in [4], $u \in L^4((1,\infty), L^\infty)$ so we can chose $(t_n)_{n\in\mathbb{N}}$ such that $\|u(1/t_n)\|_{L^\infty}$ goes to zero.

We now naturally define the following sequences:

$$\forall n \in \mathbb{N} \quad (T_n, N_n) = (T(t_n, x\sqrt{t_n}), N(t_n, x\sqrt{t_n})).$$

Since $\|T\|_{L^\infty} \leq 1$ and $\|N\|_{L^\infty} \leq 2$ it is obvious that those sequences are bounded. Let us prove their equicontinuity.

For all $n \in \mathbb{N}$, $\quad T_n$ is derivable and using that $T_x = \Re(\overline{\psi} N)$ and $N_x = -\psi T$,

$$T_n'(x) = \sqrt{t_n}\Re(\overline{\psi}N)(t_n, x\sqrt{t_n}) = \Re\left[\alpha e^{-i\frac{x^2}{4}} N(t_n, x\sqrt{t_n})\right] + o(1)N_n(x).$$

Similarly, for all $x \in \mathbb{R}$,

$$N_n'(x) = \sqrt{t_n}(-\psi N)(t_n, x\sqrt{t_n}) = -\alpha e^{i\frac{x^2}{4}} T(t_n, x\sqrt{t_n}) + o(1)T_n(x).$$

Sequences (T_n', N_n') are uniformly bounded, so (T_n, N_n) are equicontinuous.
By d'Arzela-Ascoli theorem on $\mathcal{T} = \{T_n, n \in \mathbb{N}\}$ and $\mathcal{N} = \{N_n, n \in \mathbb{N}\}$, there exists a subsequence of (T_n, N_n), converging toward $(T_*(x), N_*(x))$. For convenience, we will not write the extractice.

As the coefficients involved in the ODE are analytic, we conclude that $(T_*, N_*(x))$ satisfies system (19) in the strong sense, as $(T_\alpha(x), N_\alpha(x))$.

Therefore, there exists an unique rotation Θ such that

$$\begin{cases} T_*(x) = \Theta(T_\alpha(x)), \\ \Re(N_*(x)) = \Theta(\Re(N_\alpha(x))), \\ \Im(N_*(x)) = \Theta(\Im(N_\alpha(x))). \end{cases}$$

So we conclude that for $x \to \pm\infty$:

$$T_*(x) = \Theta(A_\alpha^\pm) + \mathcal{O}\left(\frac{1}{|x|}\right), \quad N_*(x) = \Theta(B_\alpha^\pm) + \mathcal{O}\left(\frac{1}{|x|}\right).$$

□

Lemma 7. (Description of the singularity) *We have*

$$T(0, 0^\pm) = \Theta(A_\alpha^\pm) \quad \textit{and} \quad e^{i\alpha^2 \log|x|}\tilde{N}(0, 0^\pm) = \Theta(B_\alpha^\pm),$$

where Θ has been introduced in Lemma 6.

The proof of this lemma uses all we did in the previous section concerning the limit of vectors $\tilde{N}$ and T.

Proof. Let $\varepsilon > 0$. The main idea of this proof is to write

$$\begin{aligned} |T(0, 0^+) - \Theta(A_\alpha^+)| \le |T(0, 0^+) - T(0, x\sqrt{t_n})| + |T(0, x\sqrt{t_n}) - T(t_n, x\sqrt{t_n})| \\ + |T(t_n, x\sqrt{t_n}) - T_*(x)| + |T_*(x) - \Theta(A_\alpha^+)|. \end{aligned}$$

First, we chose x big enough, such that $|T_*(x) - \Theta(A_\alpha^+)| \le \frac{\varepsilon}{4}$, thanks to Lemma 6. Then we chose n big enough, such that $|T(t_n, x\sqrt{t_n}) - T_*(x)| \le \frac{\varepsilon}{4}$ thanks to convergence, such that $|T(0, x\sqrt{t_n}) - T(t_n, x\sqrt{t_n})| \le \frac{\varepsilon}{4}$ thanks to Lemma 2 and finally such that $|T(0, 0^+) - T(0, x\sqrt{t_n})| \le \frac{\varepsilon}{4}$, using Lemma 4:

$$|T(0, 0^+) - T(0, x\sqrt{t_n})| \le \|T_x\|_\infty x\sqrt{t_n} \le C(u_+)x\sqrt{t_n}.$$

So we have $|T(0, 0^+) - \Theta(A_\alpha^+)| \le \varepsilon$, i.e.,

$$T(0, 0^+) = \Theta(A_\alpha^+).$$

Similarly, for $x < 0$ we prove that $T(0, 0^-) = \Theta(A_\alpha^-)$.

For $\tilde{N}$ we follow the same path, taking care to handle the phases. For $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ converging to zero, such that

$$\exp(i\alpha^2 \log\sqrt{t_n}) = 1,$$

we have:

$$\begin{aligned} &|\Theta(B_\alpha^+) - \tilde{N}(0, k+)| \\ &\quad \le |\Theta(B_\alpha^+) - \mathrm{e}^{i\alpha^2 \log|x|}N_*(x)| + |\mathrm{e}^{i\alpha^2 \log|x|}N_*(x) - \mathrm{e}^{i\alpha^2 \log|x|}N(t_n, x\sqrt{t_n})| \\ &\qquad + |\mathrm{e}^{i\alpha^2 \log|x|}N(t_n, x\sqrt{t_n}) - \mathrm{e}^{i\alpha^2 \ln\frac{|x\sqrt{t_n}|}{\sqrt{t_n}}}N(t_n, x\sqrt{t_n})| + |\tilde{N}(t_n, x\sqrt{t_n}) - \tilde{N}(0, x\sqrt{t_n})| \\ &\qquad + |\tilde{N}(0, x\sqrt{t_n}) - \tilde{N}(0, k+)|. \end{aligned}$$

The first term is small for x big enough thanks to Lemma 6. The second is small for n big enough thanks to Lemma 6. The third term is zero, the fourth term is small when t_n is small enough using Lemma 3. Finally, the last term is controlled by $C(u)x\sqrt{t_n}$ due to Lemma 5, and we have the desired result. □

3. Recovering the initial curve χ_0

In this section, we prove that the curve χ is equal to χ_0 at time zero, combining the results of the two previous parts and the choice of u_+ in the introduction.

The system that verify N and T at time zero is the following:

$$\begin{cases} T_x(0,x) = \Re \frac{1}{\sqrt{i}} \widehat{u_+}\left(\frac{x}{2}\right) \mathrm{e}^{-i\alpha^2 \log|x|} \tilde{N}(0,x), \\ \tilde{N}_x(0,x) = -\frac{1}{\sqrt{i}} \overline{\widehat{u_+}\left(\frac{x}{2}\right) \mathrm{e}^{-i\alpha^2 \log|x|}} T(0,x), \end{cases} \quad \forall x \neq 0,$$

with initial value given by

$$T(0,0^{\pm}) = \Theta(A_\alpha^{\pm}) \quad \text{and} \quad \mathrm{e}^{i\alpha^2 \log|x|} \tilde{N}(0,0^{\pm}) = \Theta(B_\alpha^{\pm}).$$

Recalling the definition of u_+ given by (10), $T(0)$ and $\tilde{N}(0)$ satisfy the same Cauchy system (8) as T_0 and N_0, hence $\chi(0) = \chi_0$.

Finally, we are left to prove the convergence rate (6) of $\chi(t,x)$ as t goes to zero. Since $\chi_t(t,x) = c(t,x)$ and $c(t,x) = |\psi(t,x)| \leq \frac{C}{\sqrt{t}}$, we have:

$$|\chi(t_2,x) - \chi(t_1,x)| \leq \int_{t_1}^{t_2} \frac{C}{\sqrt{t}} \mathrm{d}t \leq C\sqrt{t_2},$$

and Theorem 1 is proven.

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