J. Evol. Equ. (2022) 22:70

1424-3199/22/030001-24, *published online* August 11, 2022
https://doi.org/10.1007/s00028-022-00831-5

Journal of Evolution Equations

Lack of controllability of the viscous Burgers equation: part I—the L^∞ setting

Boris Andreianov, Shyam Sundar Ghoshal and Konstantinos Koumatos

Abstract. We contribute an answer to a quantitative variant of the question raised in Coron (in: Perspectives in nonlinear partial differential equations. Contemporary mathematics, vol 446, American Mathematical Society, Providence, pp 215–243, 2007) concerning the controllability of the viscous Burgers equation $u_t + (u^2/2)_x = u_{xx}$ for initial and terminal data prescribed for $x \in (0, 1)$. We investigate the (non)-controllability under the additional a priori bound imposed on the (nonlinear) operator that associates the solution to the terminal state. In contrast to typical techniques on the controllability of the viscous Burgers equation invoking the heat equation, we combine scaling and compensated compactness arguments along with observations on the non-controllability of the inviscid Burgers equation to point out wide sets of terminal states non-attainable from zero initial data by solutions of restricted size. We prove in particular that, given $L \geq 1$, for sufficiently large $|C|$ and $T < (1+\Delta)/|C|$ (where $\Delta > 0$ depends on L), the constant terminal state $u(\cdot, T) := C$ is not attainable at time T, starting from zero data, by weak solutions of the viscous Burgers equation satisfying a *bounded amplification restriction* of the form $\|u\|_\infty \leq L|C|$. Our focus on L^∞ solutions is due to the fact that we rely upon the classical theory of Kruzhkov entropy solutions to the inviscid equation. In Part II of this paper, we will extend the non-controllability results to solutions of the viscous Burgers equation in the L^2 setting, upon extending the Kruzhkov theory appropriately.

1. Introduction

We are concerned with the controllability of the viscous Burgers equation

$$u_t + \left(u^2/2\right)_x = u_{xx} \text{ in } D, \qquad \textbf{(BE)}$$

where $(x, t) \in D = \mathbb{R} \times (0, T)$ ("the strip setting") or $(x, t) \in D = (0, 1) \times (0, T)$ ("the box setting"), with a given $T > 0$. Our primary motivation comes from [13], where J.-M. Coron asked the following question (Open Problem 4). Let $T > 0$ and $C \in \mathbb{R}\backslash\{0\}$.

$$\underline{Question}: \textit{Does there exist } u \in L^2((0,1)\times(0,T)) \textit{ satisfying } (\textbf{BE}) \textit{ such that for all } x \in (0,1),\ u(\cdot,0) = 0 \text{ and } u(\cdot,T) = C? \qquad \textbf{(Q)}$$

Mathematics Subject Classification: 93B03, 35L65, 35D30, 47J35
Keywords: Burgers equation, Exact controllability, Scaling, Compensated compactness, Backward characteristics.

With the method explored in this paper and in its sequel [6], we focus on non-controllability issues under additional "bounded amplification" assumptions, which essentially mean that the size of the desired solutions is limited relative to the size of the target datum; this can be witnessed through the "amplification factor" L present in the main statements. The difference with the original question is highlighted in the sequel (see Remark 1.1).

1.1. Overview of the results of the paper, and further investigations

In the present paper, we bring a partial negative answer to (**Q**) (which counter-balances the partial positive answers given in [13,15,21], see Sect. 1.2) in the L^∞ setting instead of the original L^2 setting. We highlight the existence of many triples (M, C, T), with $0 < |C| \leq M$ and $0 < T \leq 1/|C|$, such that the target state $u_T = C$ is not reachable by solutions of (**BE**) satisfying $\|u\|_\infty \leq M$.

We find it convenient to introduce $L = M/|C|$ as an *amplification factor;* we show that for any given $L \geq 1$ there exist pairs (C, T), with roughly speaking $|C|T \leq 1$, such that the system has no solution $u \in L^\infty((0, 1) \times (0, T))$ satisfying the bound $\|u\|_\infty \leq L|C|$. More generally, we point out several families of weakly-$*$ compact sets of states $u_T \in L^\infty$ not attainable at time T, starting from zero data, by solutions of (**BE**), under the a priori *amplification assumption* $\|u\|_\infty \leq L\|u_T\|_\infty$. This happens for small values of T and somewhat large (but smaller than T^{-1}) values of C. The details can be found in Sect. 3 (Corollary 3.16 and more generally, Theorem 3.12). Refinements concerning the non-sharpness of the restriction $CT \leq 1$ and the case of the strip domain are given in Sect. 4.

Note that in the sequel [6], we will extend this negative answer—with the ideas developed in this paper and under the adequate amplification assumption—to question (**Q**) in its original L^2 setting, both for the strip problem and for the box problem. This will require the *ad hoc* amplification assumptions and an additional $L^2 - L^3_{loc}$ regularization assumption on the solutions. The uniqueness theory for unbounded solutions of scalar conservation laws, necessary for the sake of such an extension, will be developed on purpose.

Remark 1.1. Following the arguments developed in the paper, one can see that they apply as well to the classical heat equation replacing the viscous Burgers equation. Indeed, the key scaling observation, the different bounds on solutions, and the underlying non-attainability results of the inviscid case carry on to this linear setting. Here, one can clearly see that the question we answer negatively is different from the mere question of controllability. Indeed, it is classical that the heat equation on an interval is null controllable at any time by boundary controls starting from any initial datum (cf. [19]), which by the linearity means that all constant states are controllable starting from zero initial datum. It is clear that the "cost" of the controls, which can be quantified by amplification factor L as in our assumptions, increases as the desired control time decreases. This is why we should interpret the results obtained in this paper as a quantitative version (with limited "costs") of the original Coron's question

(**Q**). The issue of (non)-controllability of constant states C at arbitrarily small times T for an unbounded cost remains open; the example of the heat equation shows that our method is not suitable for answering negatively this qualitative question.

1.2. The state of the art on controllability of the viscous Burgers equation

Several positive results on exact controllability of constant states for the viscous Burgers equation exist in the literature. One such result is the following:

Theorem 1.2. (See [13,15]) *Let* $T > 0$*. There exists* $N = N(T) > 0$ *such that for every* $|C| \geq N$*, there exists* $u \in \mathrm{L}^2((0,1) \times (0,T))$ *satisfying* (**BE**) *and such that* $u(\cdot, 0) = 0$*,* $u(\cdot, T) = C$ *for* $x \in (0,1)$.

Another related result in the space $\mathrm{L}^\infty((0,1) \times (0,T))$ can be found in Glass and Guerrero [21] where the authors consider boundary controls for the viscous Burgers equation with small dissipation. They prove that any nonzero constant state C can be reached after sufficiently large time. As an immediate consequence of [21], one has the following theorem:

Theorem 1.3. (See [21]) *There exist* $N > 0$ *and* $\beta \geq 1$ *such that for every* $|C| > N$*, there exists* $u \in \mathrm{L}^\infty((0,1) \times (0,T))$ *satisfying* (**BE**) *and such that* $u(\cdot, 0) = 0$*,* $u(\cdot, T) = C$ *for* $x \in (0,1)$ *and all* $T > \beta/|C|$.

In particular, Glass and Guerrero [21] showed that large constant states can be reached in large time by two boundary controls for viscous Burgers equation with small viscosity coefficient. Later Leautaud in [25] extended this result to scalar viscous conservation laws with more general fluxes. Null controllability (Marbach [26]) and small-time local controllability (Fursikov and Imanuvilov [20]) have been achieved with source and one boundary control. It is worth mentioning the result of Guerrero and Imanuvilov [22] where the authors deal with (**BE**) and two boundary controls and show that exact null controllability indeed fails for small time. Also, they prove a negative result to null exact controllability even for large time. In [18] Fernández-Cara and Guerrero have given an estimate of the time of null controllability depending on the L^2-norm of the initial data.

On the other hand, the problem has also been investigated under one control, and we refer the reader to [18,20] and references therein.

Regarding the exact controllability for the inviscid Burgers equation (more generally convex conservation laws or even to some particular hyperbolic systems of conservation laws), one can use tools, such as backward characteristics, in order to construct suitable initial and boundary controls. The theory is nevertheless very delicate due to the occurrence of shocks. For more details, we refer the reader to [1,2,4,7,23,28].

In Theorems 1.2 and 1.3, there is clearly a gap in the range of pairs (C, T) for which the question has not been resolved. Specifically, the range $T < 1/|C|$ is not covered by these results and we stress the fact that at the level of the inviscid Burgers equation, such states cannot be controlled; see [1] and Propositions 3.7 and 4.2. This has been the motivation for the present work in which we provide a partial negative result to question (**Q**) precisely for such pairs (C, T), along with some generalizations directly coming from the techniques we employ.

1.3. Outline of the paper, key ideas and techniques

In this paper, we interpret question (**Q**) as an initial-value problem on $D = \mathbb{R} \times (0, T)$ or as an underdetermined initial-value problem on $D = (0, 1) \times (0, T)$ for solutions understood in the appropriate weak sense (see Sect. 2 for definitions) and under adequate limitations on the L^∞ size of the solution relative to the size of the target state. We give a series of negative answers for couples (C, T) satisfying $T \leq 1/|C|$ (and sometimes $T < (1 + \Delta)/|C|$ with some $\Delta > 0$), for C sufficiently large. More generally, such results concern sufficiently large data and the accordingly small times $(u_T(\cdot), T) \in BV((0, 1)) \times (0, +\infty)$ satisfying properties of the type (**NA**) (see Proposition 3.7); precise statements are given in Sect. 3 (see also [6] for the L^2 versions of the statements). Our method relies on a scaling argument which reduces (**BE**) to the viscous Burgers equation

$$u_t^\epsilon + \left(\frac{(u^\epsilon)^2}{2}\right)_x = \epsilon u_{xx}^\epsilon,$$

while leaving invariant the product $T u_T(\cdot)$ for states $u_T(\cdot)$ attainable at time T.

Note that the scaling argument is restricted to the quadratic nonlinearity. It is appropriate to the Navier–Stokes equation and the corresponding inviscid (Euler) equations (see, e.g., [14] for examples of control problems), which are far beyond the scope of this paper. However, the extension to the L^2 setting ([6]) of the method we develop here for L^∞ solutions (Sects. 3, 4) is motivated in particular by the fact that the L^∞ setting, most natural for scalar problems, is not natural for systems.

The conclusion on non-controllability for the viscous Burgers equation follows, upon a careful use of the scaling (Zoom) of solutions of (**BE**) with $\epsilon = T$ (see Sect. 3), i.e., for small times T, from uniform in ϵ bounds ensuring compactness of sequences of solutions (u_ϵ), and from rather elementary non-controllability results for the inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

The latter is understood in the standard framework of Kruzhkov entropy solutions (Sects. 3, 4) or in the framework of unbounded entropy solutions described for this purpose ([6]). An a priori bound on solutions is required in our argument which we interpret as an amplification assumption limiting the size of the solutions in terms of the size of the target data. The core arguments are given in Sect. 3, in the simplest setting; Sect. 4 presents more technical extensions of the results, achieved with the same strategy of proof (see [6] for further technical extensions).

We would like to emphasize that unlike a large number of works on the controllability of the viscous Burgers equation which invoke the heat equation via the Hopf–Cole transformation, our work is not only motivated by, but also based on the inviscid equation through a vanishing viscosity argument. We remark that for the inviscid equation, the most classical solution space is L^∞ and therefore the focus of this paper is on L^∞ solutions of the viscous Burgers equation. In Part II of this paper ([6]), we will extend the arguments to L^2 solutions by refining the underlying solution concept slightly beyond the classical Kruzhkov setting.

2. Precise setting for question (Q)

In addition to the aforementioned interpretations of the original question (Q) raised in [13] (the quantitative "bounded amplification" assumptions, the choice of L^∞ or L^2 functional framework, the choice $(0, 1)$ or $\mathbb{R}$ for the space domain), we also need to make explicit the underlying notion of solution to the viscous Burgers equation. Let us detail the framework(s) we explore.

We discuss the two following situations:

$$\begin{cases} u_t + \left(u^2/2\right)_x = u_{xx} \text{ in } (0,1)\times(0,T), \\ u(\cdot,0) = 0 \text{ and } u(\cdot,T) = u_T \quad \text{for } x \in (0,1), \end{cases} \qquad (\mathbf{Pb}_{box})$$

and

$$\begin{cases} u_t + \left(u^2/2\right)_x = u_{xx} \text{ in } \mathbb{R}\times(0,T), \\ u(\cdot,0) = u_0 \text{ with } u_0 = 0 \quad \text{for } x \in (0,1), \\ \quad \text{and } u(\cdot,T) = u_T \text{ for } x \in (0,1). \end{cases} \qquad (\mathbf{Pb}_{strip})$$

If one puts aside the assigned terminal conditions for a moment, we recognize in ($\mathbf{Pb}_{strip}$) a standard Cauchy problem. So the question is a particular instance of control by the initial data (which we will instead refer to as *attainability* in the sequel of the paper). Similarly, one possible interpretation of ($\mathbf{Pb}_{box}$) would be in terms of boundary control in the Cauchy–Dirichlet (or even Cauchy–Neumann) setting; however, we prefer to consider ($\mathbf{Pb}_{box}$) as an underdetermined problem with solutions defined locally in $(0, 1) \times [0, T]$ (attention is paid to the initial and terminal times $t = 0, T$ but not to the boundaries $x = 0, 1$). Indeed, prescribing boundary traces of the solution u or of the convection–diffusion flux $u^2/2 - u_x$ at $x = 0, 1$ would restrict the generality of problem ($\mathbf{Pb}_{box}$).

It is obvious that a solution u to the problem ($\mathbf{Pb}_{strip}$) in the strip can be seen as well as a solution to ($\mathbf{Pb}_{box}$): it is enough to consider its restriction $u|_{(0,1)\times(0,T)}$ to the box. Therefore, it is more difficult to attain a given state u_T in the strip setting ($\mathbf{Pb}_{strip}$) than in the box setting ($\mathbf{Pb}_{box}$). Because our focus in this paper is on non-attainability (i.e., on the impossibility to reach the desired states at desired times), we see (Q) in the strip setting ($\mathbf{Pb}_{strip}$) as a simpler question than the same question in its box setting ($\mathbf{Pb}_{box}$).

Next, although question (Q) is originally about L^2 solutions of (BE), our techniques primarily drive us to replace L^2 by L^∞ (see Sect. 3). In order to get closer to the original

L^2 setting, we need to rely upon a theory of unbounded (more precisely, L^2) entropy solutions to the inviscid Burgers equation that we will develop in [6]. At this point, insufficiency of the L^2 uniqueness theory for the Cauchy–Dirichlet problem will push us to consider also the $L^3((0,1)\times(0,T))$ solutions of the viscous Burgers equation.

In our non-attainability results, we will not merely ask that the solutions belong to some L^p spaces, but also that they obey some uniform bounds that we state in terms of "amplification." The amplification constants are denoted by L throughout the paper; their role is to control the size of the solution in terms of the size of the target data u_T.

Finally, the precise meaning of what a "solution" of (**BE**) is in our paper is different from the one found in [13], where solutions are meant in the sense of distributions (usually called "very weak" solutions). Our approach requires that the solution satisfy a local L^2 in time, H^1 in space, energy estimate which means that it should be a weak solution (sometimes called variational solution or finite energy solution) locally in D; moreover, the entropy inequalities of parabolic conservation laws are required. Because for non-degenerate parabolic conservation laws weak formulation implies the entropy formulation (see e.g., [12]), in the sequel we use the term *weak solution of the viscous Burgers equation* (supplemented with initial and terminal data) meaning the following.

Definition 2.1. (*Adopted notion of solution for the viscous Burgers equation*) Let $D = I\times(0,T)$ with $I=(0,1)$ or $I=\mathbb{R}$. Let u_0 and u_T belong to $L^2_{loc}(I)$. A function $u\in L^2(D)$ is called a weak solution of (**BE**) with initial data u_0 and terminal data u_T if $u\in L^2(0,T;H^1_{loc}(I))$ and for all $\xi\in C_c^\infty(I\times[0,T])$, there holds

$$\int_0^T\int_0^1\Big(u\xi_t+\frac{u^2}{2}\xi_x-u_x\xi_x\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}u_0(x)\xi(x,0)\,\mathrm{d}x-\int_{\mathbb{R}}u_T(x)\xi(x,T)\,\mathrm{d}x=0 \tag{1}$$

and, furthermore, for all $\xi\in C_c^\infty(I\times[0,T))$, $\xi\ge 0$, for all $k\in\mathbb{R}$ there holds

$$-\int_0^T\int_0^1\Big(|u-k|\xi_t+|u-k|\frac{u+k}{2}\xi_x-|u-k|_x\xi_x\Big)\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathbb{R}}|u_0-k|\xi(x,0)\,\mathrm{d}x\le 0. \tag{2}$$

In particular, solutions to ($\mathbf{Pb}_{box}$) or ($\mathbf{Pb}_{strip}$) are understood in the sense of Definition 2.1 in the sequel; they will be supplemented with additional bounds in $L^p(D)$ for different choices of p.

Let us stress that the more usual, in the context of such definitions, L^∞ assumption on u is not needed for the above definition to make sense, indeed, under the L^2_{loc} assumptions on u, u_x all terms in (1), (2) are well-defined.

Remark 2.2. The L^2_{loc} regularity of u_x assumed in Definition 2.1 implies in particular that the solution satisfies variants of classical chain rules in space (following from the Sobolev regularity of u in space), like $|u-k|_x=\operatorname{sign}(u-k)u_x$, and chain rules in time (see e.g., [3, Lemma 2.3]); these chain rules are necessary technical ingredients of the entropy formulation (2). In particular, local L^1 estimates on the term $|u_x|^2$ (we refer to the proof of Lemma 3.10), obtained by formally multiplying the equation by u, are justified using chain rules.

In this respect, let us recall that classical solutions to the Burgers equation are related to classical solutions of the heat equation by the Hopf-Cole transformation (see, e.g., [17]) which is a nonlinear change of the unknown; the equivalence relies on chain rules for derivatives. While considering very weak solutions to (**BE**) as suggested in [13], we do not have any kind of chain rule at our disposal; thus not only the classical regularity cannot be derived from the formal link with the heat equation, but also the entropy formulation cannot be guaranteed. For this reason, we cannot rely upon the notion of merely distributional (very weak) solutions to (**BE**).

As a matter of fact, we could go beyond the weak L^2_{loc} setting and even the very weak setting, by considering L^1 data and the appropriate notions of solution developed in the literature.

Remark 2.3. Recall that the L^1 setting is sharp for inviscid conservation laws provided the solutions are interpreted either in abstract semigroup terms (like in [5]) or in the renormalized setting (like in [10,30]) or else, in the setting of the kinetic formulation (see [29] and references therein). Because all these solutions can be seen as pointwise limits of Kruzhkov entropy solutions for truncated data (like the unbounded entropy solutions we construct in the sequel [6] of this paper, the results can be extended to the L^1 setting similarly to what is done in [6] for the L^2 setting. Let us remark that, for example, the notion of kinetic solution can be applied in parallel to the viscous and to the inviscid Burgers equations. In general, these solutions are not even solutions in the sense of distributions (very weak solutions) because $u^2/2$ may fall out of L^1_{loc}, or at least their L^1_{loc} regularity is far from being straightforward ([32]). This line of investigation would provide yet another functional and solution framework for interpretation of question (**Q**).

Making precise the notion of solution strongly impacts the results we can prove concerning question (**Q**); we refer in particular to the final discussion of the paper [6].

3. Sets of terminal data non-attainable by bounded weak solutions

This section is devoted to partial (negative) answers to question (**Q**) in the setting where we assume that the solutions u are bounded and moreover, the ratio of the L^∞ norm of u and the L^∞ norm of the target data u_T is controlled by a constant L given beforehand. Clearly, only $L \geq 1$ makes sense. For general u_T, asking $L = 1$ means roughly speaking that we look for solutions u with the same amplitude as u_T, while letting $L > 1$ allows for a controlled amplification. For this reason, in the sequel we call L the *amplification factor*.

Our argument essentially relies upon the scaling

$$\begin{aligned} &(t, v) \mapsto (\tfrac{t}{\epsilon}, \mathcal{S}_\epsilon v), \\ &\mathcal{S}_\epsilon v := \epsilon v(\cdot, \epsilon \cdot), \quad \text{i.e., } t = \epsilon\tau, \mathcal{S}_\epsilon(v)(x,\tau) = \epsilon v(x, \epsilon\tau), \end{aligned} \tag{Zoom}$$

where v is a function of $(x, t) \in (0, 1) \times (0, T)$; this scaling permits to link (**BE**) to the viscous Burgers equation with viscosity parameter $\epsilon > 0$. The inviscid Burgers equation, under the standard notion of admissibility of solutions, can be seen as the singular limit of the latter as $\epsilon \to 0$. Also note that the inviscid Burgers equation is invariant under the scaling (Zoom).

Remark 3.1. Let us point out that a study analogous to the one we conduct in this section (see [6] for the L^2 extension) can be conducted for the problem

$$u_t + |u|^{p-1} u_x = \left(|u_x|^{p-2} u_x\right)_x$$

for $p \in (1, \infty)$; it possesses a scaling invariance which generalizes (Zoom).

Note that it is well-known that weak (energy) solutions of scalar conservation laws regularized with p-laplacian viscosity $\epsilon(|u_x|^{p-2}u_x)_x$ converge to entropy solutions of the corresponding inviscid problem. Also note that the theory of the Cauchy problem developed in [6] applies to the flux $F(u) = \text{sign}(u)|u|^p/p$ and L^p initial data.

We start by constructing a wide family of non-attainable (from initial data u_0 verifying $u_0 = 0$ in $(0, 1)$) at time $T = 1$ states in the classical setting of Kruzhkov entropy solutions to the *inviscid* Burgers equation. The scaling (Zoom), along with the classical vanishing viscosity characterization of the admissible solutions to the inviscid Burgers equation, will permit to transfer the non-attainability result to our target problem (**BE**). In order to do so, in this paper, we restrict our attention to L^∞ solutions of the latter (we relax this restriction in [6]).

3.1. Non-attainable states for the inviscid Burgers equation in the classical entropy solutions setting

The initial value problem addressed in question (**Q**) is underdetermined (its formulation does not implicitly include boundary data); therefore, we first make precise what we mean by solution of the analogous underdetermined inviscid Burgers problem.

Definition 3.2. A function $u \in L^\infty((0, 1) \times (0, T))$ is a local Kruzhkov entropy solution of the underdetermined problem

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = 0 \ \text{ in } D = (0, 1) \times (0, T), \\ u(x, 0) = 0 \ \text{ on } (0, 1), \end{cases} \qquad (\mathbf{Pb}^0_{box})$$

if for all $k \in \mathbb{R}$, for all $\xi \in C_c^\infty((0, 1) \times [0, T))$, $\xi \geq 0$ there holds

$$-\int_0^T \int_0^1 |u - k|\xi_t + |u - k| \frac{u + k}{2} \xi_x \, dx \, dt - \int_0^1 |k| \xi(x, 0) \, dx \leq 0. \qquad (3)$$

Moreover, $u_T \in L^\infty((0, 1))$ is the terminal state of a local Kruzhkov entropy solution u if for all $\xi \in C_c^\infty((0, 1) \times [0, T])$

$$\int_0^T \int_0^1 u\xi_t + \frac{u^2}{2} \xi_x \, dx \, dt - \int_0^1 u_T(x) \xi(x, T) \, dx = 0.$$

Having in mind a variant of question (**Q**), we are also interested in the Cauchy problem set on the whole real line:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R}\times(0,T), \\ u(x,0) = u_0 & \text{with } u_0(x) = 0 \text{ for } x \in (0,1). \end{cases} \qquad (\mathbf{Pb}^0_{strip})$$

An analogous definition with $u_0 \in \mathrm{L}^\infty(\mathbb{R})$ is classical [24]; we refer to the corresponding solutions as (global) Kruzhkov entropy solutions.

Further, Kruzhkov entropy solutions can be restricted and they can be glued together:

Remark 3.3. Similarly to Definition 3.2, one defines local Kruzhkov entropy solutions on any open domain $D \subset \mathbb{R}\times(0,T)$ by localizing the support of the test functions to $D \cup \left(\overline{D} \cap (\mathbb{R}\times\{0\})\right)$. Nonzero initial data are easily included.

It is obvious that a restriction of a local Kruzhkov entropy solution on an open subdomain $\widetilde{D}$ of $D = (0,1)\times(0,T)$ is a local Kruzhkov entropy solution in $\widetilde{D}$. Further, it is easily checked that gluing continuously local Kruzhkov entropy solutions in domains $\widetilde{D}$, $\widehat{D}$ separated by a Lipschitz curve Γ (by continuity we mean coincidence of strong traces from the right and from the left of Γ), we obtain a Kruzhkov entropy solution in $\widetilde{D} \cup \Gamma \cup \widehat{D}$.

Note that a terminal state exists for every local Kruzhkov entropy solution; further, every such solution can be seen as the solution of the initial-boundary value problem with appropriately chosen boundary data. In the following Remark, we give precise sense to the initial datum and to the Dirichlet boundary data denoted by b_0 (the Dirichlet datum at $x = 0^-$) and b_1 (the Dirichlet datum at $x = 1^-$.) More precisely, we have

Remark 3.4. Local entropy solutions of ($\mathbf{Pb}^0_{box}$) possess the following properties:
(i) $u \in C([0,T]; \mathrm{L}^1((0,1)))$, and in particular, the initial data $u_0 = 0$ and the terminal data $u(\cdot,T) = u_T$ can be understood as traces of u, in the strong L^1 sense, on $(0,1)\times\{0\}$ and on $(0,1)\times\{T\}$, respectively (see [11,27]).
(ii) There exist traces (in the strong L^1 sense) $b_0(\cdot) = u(0^+,\cdot)$ and $b_1(\cdot) = u(1^-,\cdot)$, $b_0, b_1 \in \mathrm{L}^\infty((0,T))$ (see [27,33]), which can also be seen as the boundary data for the Cauchy–Dirichlet problem understood in the BLN sense (see Bardos, LeRoux and Nédélec [9], see also [8]). One can see u as the unique solution in the BLN sense corresponding to the initial data $u_0 = 0$ and boundary data $u(\cdot,0^+)$,$u(\cdot,1^-)$.

Remark 3.5. Recall that the L^1 comparison and contraction property is valid (see e.g., [17,24,31]) for any two Kruzhkov entropy solutions $u, \hat{u}$ corresponding to the $\mathrm{L}^\infty(\mathbb{R})$ initial data $u_0, \hat{u}_0$, respectively:

$$\|(u-\hat{u})^\pm\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1)} \le \|(u_0-\hat{u}_0)^\pm\|_{\mathrm{L}^1}, \qquad (4)$$

where $z^\pm := \max\{0, \pm z\}$. Property (4) makes sense whenever the right-hand side is finite; it implies in particular that smaller initial data ($u_0 \le \hat{u}_0$ on $\mathbb{R}$) give rise to smaller solutions ($u \le \hat{u}$ in $\mathbb{R} \times (0, T)$). Note that the comparison principle under the form

$$u_0 \le \hat{u}_0 \text{ on } (0,1),\ b_0 \le \hat{b}_0,\ b_1 \le \hat{b}_1 \text{ on } (0,T) \implies u \le \hat{u} \text{ in } (0,1) \times (0,T)$$

is known also for Cauchy–Dirichlet problems (see, e.g., [8]), here in addition to the initial data, boundary data b_0, b_1 (respectively, $\hat{b}_0, \hat{b}_1$) for u (resp., for $\hat{u}$) are prescribed.

With the above preliminaries at hand, let us introduce some convenient notation. For $T > 0$, denote by

$$\mathcal{NA}_T^{\mathrm{L}^\infty, box} := \Big\{ u_T \in \mathrm{L}^\infty(0,1) \ \Big|\ \nexists u \ \text{ solution in the sense of Definition 3.2 to problem } (\mathbf{Pb}^0_{box}) \text{ with } u(\cdot, T) = u_T \Big\} \tag{5}$$

the set of states non-attainable at time T by local Kruzhkov entropy solutions of the *inviscid* Burgers equation in $(0,1) \times (0,T)$ with zero initial data. Consider the scaling (Zoom) of the solution of ($\mathbf{Pb}^0_{box}$) where we take $\epsilon = T$ so that the scaled equation is posed in the time interval $\tau \in (0,1)$. It is readily checked that the notion of local Kruzhkov entropy solution is invariant under this scaling. For this reason, we have

$$\mathcal{NA}_T^{\mathrm{L}^\infty, box} = \Big\{ u_T \in \mathrm{L}^\infty(0,1) \ \Big|\ T u_T \in \mathcal{NA}_1^{\mathrm{L}^\infty, box} \Big\} = T^{-1} \mathcal{NA}_1^{\mathrm{L}^\infty, box}, \tag{6}$$

i.e., we can fix $T = 1$ in our study of states non-attainable for the inviscid Burgers equation.

Remark 3.6. It is classical that for ($\mathbf{Pb}^0_{box}$), states that do not belong to $BV_{loc}((0,1))$ are not attainable at any time. In the sequel, and having in mind the constant terminal states of question (**Q**), we will not focus on the BV regularity restrictions but the reader may always suppose that u_T is at least BV_{loc} regular.

Non-attainability for the inviscid equation is naturally studied using the insight from the theory of maximal backward characteristics ([16,17]), see e.g., [4]. In particular, we have the following key observation.

Proposition 3.7. *Let $u_1 \in BV_{loc}((0,1))$ verifying*

$$\begin{array}{l} \exists\, x^* \in [0,1] \text{ such that} \\ \text{either} \quad 0 < \overline{u_1}(x^*) \le x^*, \\ \text{or} \quad -(1 - x^*) \le \underline{u_1}(x^*) < 0, \end{array} \tag{NA}$$

where $\underline{u_1}(\cdot)$ (respectively, $\overline{u_1}(\cdot)$) stands for the left-continuous in x (respectively, right-continuous in x) representative of the BV function $x \mapsto u_1(x)$. Then, there exists no local Kruzhkov entropy solution verifying ($\mathbf{Pb}^0_{box}$) and the terminal datum $u(\cdot, 1) = u_1$. In other words, $u_1 \in \mathcal{NA}_1^{\mathrm{L}^\infty, box}$ and for all $T > 0$, $T^{-1} u_1 \in \mathcal{NA}_T^{\mathrm{L}^\infty, box}$. In particular, for all couples $(C, T) \in (0, +\infty) \times \mathbb{R}_+$ verifying $|C| T \le 1$, there holds $C \in \mathcal{NA}_T^{\mathrm{L}^\infty, box}$.

Proof. We argue by contradiction; let u be a solution of ($\mathbf{Pb}^0_{box}$) corresponding to the terminal data u_1. We can assume without loss of generality that there exists $x^* \in (0, 1]$ such that $u_1(x^*) \leq x^*$, where u_1 is normalized by the right-continuity in the variable x; the case where $u_1(x^*) \geq -(1 - x^*)$ and the normalization is by the left-continuity is fully analogous.

Let $u^* = u_1(x^*)$; we set $x_* = x^* - u^* \in [0, 1)$. We draw from the point $(1, x^*)$ the maximal backward generalized characteristic ([16,17]); it crosses the axis $t = 0$ at the point x_*, see Fig. 1. It follows from the theory of generalized characteristics that $u(x_* + tu^*, t) = u(x^*, 1) = u^*$ for all $t \in [0, 1]$, where we recall that u is normalized to be right-continuous.

Since $u(x, 0) = 0$ for $x \in (0, 1)$, we reach a contradiction whenever $x_* > 0$, which corresponds to the strict inequality $u^* < x^*$. In order to include the special case $x_* = 0$, and also in order to prepare the ground for different extensions of Proposition 3.7 (see Proposition 4.4 and [6]), we construct an auxiliary local entropy solution $\tilde{u}$ of the Burgers equation as follows. We set for $(x, t) \in [0, 1] \times (0, 1]$

$$\tilde{u}(x, t) := \begin{cases} u(x, t), & x \geq x_* + tu^*, \\ (x - x_*)/t, & x_* \leq x \leq x_* + tu^*, \\ 0, & x \leq x_*. \end{cases} \tag{7}$$

In particular, $\tilde{u}$ is continuous across the lines $x = x_*$ and $x = x_* + tu^*$. Because we glued continuously three patches and each of them is a Kruzhkov entropy solution in the corresponding subdomain (a constant, a rarefaction and our solution u, from the left to the right), according to Remark 3.3, we find that $\tilde{u}$ is a local Kruzhkov

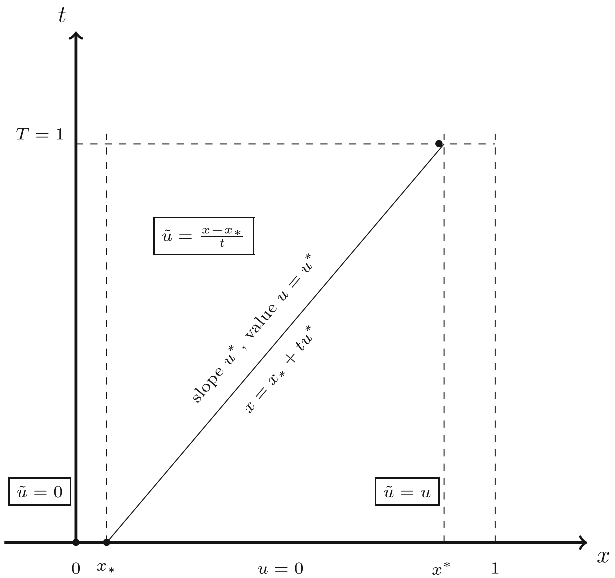

Figure 1. Construction of $\tilde{u}$

entropy solution on $D = (0, 1) \times (0, 1)$. Moreover, $\tilde{u}$ assumes zero initial data and zero boundary data on the left boundary (cf. Remark 3.4).

The finite speed of propagation (recall that $\tilde{u} \in \mathrm{L}^\infty$) ensures that $\tilde{u}$ should be zero in some vicinity of the point $(x_*, 0)$, which is contradictory because for all $t \in (0, 1)$ we have $u(x_* + tu^*, t) = u^* > 0$. This contradiction proves the non-existence of u and the non-attainability of u_1 at time $T = 1$. The remaining claims follow from the fact that $u_1 = CT$ satisfies (**NA**) when $|C|T \leq 1$, and from the scaling observation (6). □

To formulate in an optimal way our non-attainability results for the viscous Burgers equations, we will be interested in compact subsets of $\mathcal{NA}_1^{\mathrm{L}^\infty, box}$ with respect to the weak-* topology of $\mathrm{L}^\infty((0, 1))$. Below are the main examples we consider.

Remark 3.8. The following subsets of $\mathcal{NA}_1^{\mathrm{L}^\infty, box}$ are weakly-* compact in L^∞:

(i) $\mathcal{K}_{\alpha,\beta} := \left\{ u : x \mapsto C \;\middle|\; \alpha \leq |C| \leq \beta \right\}$, for any given α, β with $0 < \alpha \leq \beta \leq 1$;

(ii) $\mathcal{K}^+_{E,m(\cdot)} := \left\{ u \in \mathrm{L}^\infty((0, 1)) \;\middle|\; \forall x \in E \;\; m(x) \leq u(x) \leq x \right\}$, for a given $E \subset (0, 1)$ of nonzero Lebesgue measure and a given measurable $m : E \to (0, 1]$;

(iii) $\mathcal{K}^-_{E,m(\cdot)} := \left\{ u \in \mathrm{L}^\infty((0, 1)) \;\middle|\; \forall x \in E \;\; -(1 - x) \leq u(x) \leq -m(x) \right\}$, for $(E, m(\cdot))$ like in *(ii)*.

In this remark, the fact that $\mathcal{K}_{\alpha,\beta}, \mathcal{K}^\pm_{E,m(\cdot)} \subset \mathcal{NA}_1^{\mathrm{L}^\infty, box}$ follows from Proposition 3.7. Their weak-$*$ precompactness follows from their boundedness; moreover, it is easily seen that they are weakly-$*$ closed. For example, condition $m(x) \leq u(x)$ for a.e. $x \in E$ can be rewritten as

$$\text{for all measurable subsets } F \text{ of } E, \quad \int_F m(x)\,\mathrm{d}x \leq \int_F u(x)\,\mathrm{d}x$$

which is stable with respect to the weak-$*$ convergence in L^∞ because the indicator function $1\!\!1_F$ of F belongs to $\mathrm{L}^1((0, 1))$. Therefore, $\mathcal{K}_{\alpha,\beta}, \mathcal{K}^\pm_{E,m(\cdot)}$ are indeed weakly-$*$ compact in $\mathrm{L}^\infty((0, 1))$.

Remark 3.9. It turns out that the sets $\mathring{\mathcal{K}}_{\alpha,\beta}, \mathring{\mathcal{K}}^\pm_{E,m(\cdot)}$ of $\mathcal{K}_{\alpha,\beta}, \mathcal{K}^\pm_{E,m(\cdot)}$ defined with strict inequalities "$<$" in place of "$\leq$" belong not only to the set $\mathcal{NA}_1^{\mathrm{L}^\infty, box}$—the set of states not attainable by classical (bounded) Kruzhkov entropy solutions—but also to the topological interior of $\mathcal{NA}_1^{\mathrm{L}^\infty, box}$ with respect to the L^1 convergence.

This observation will allow us to extend the non-attainability results to unbounded (L^2) entropy solutions of the inviscid Burgers equation, see [6].

3.2. The viscous Burgers equation inherits non-attainability

The key idea of our work is that in the appropriate regime uncovered via the scaling (Zoom) and under the natural amplification assumptions compatible with this scaling, the viscous Burgers equation inherits the non-attainability of the inviscid one.

We start with the following lemma which is a consequence of Proposition 3.7 and the main technical ingredient of the proof of our main result, Theorem 3.12. The lemma relies on a standard compensated compactness argument.

Lemma 3.10. *Let $\mathcal{K}$ be a subset of $\mathcal{NA}_1^{\mathrm{L}^\infty,box}$ compact in the weak-* topology of* $\mathrm{L}^\infty((0,1))$. *Let $L \geq 1$. Then, there exists $\epsilon_0 = \epsilon_0(\mathcal{K}, L) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and all $u_1 \in \mathcal{K}$, the small viscosity Burgers equation*

$$u_t + \left(u^2/2\right)_x = \epsilon u_{xx} \ \text{ in } D = (0,1) \times (0,1), \qquad (\mathbf{Pb}^{\epsilon}_{box})$$

has no weak solution (in the sense of Definition 2.1, with $\epsilon = 1$ replaced by $\epsilon > 0$ in the diffusion term) with initial data $u(\cdot, 0) = 0$ and terminal data $u(\cdot, 1) = u_1$ within the class of functions verifying $\|u\|_{\mathrm{L}^\infty((0,1)\times(0,1))} \leq L\|u_1\|_{\mathrm{L}^\infty((0,1))}$.

Remark 3.11. Let us stress that in this and the subsequent results on the viscous Burgers equation, we work with the precise notion of weak solution from Definition 2.1, meaning in particular that $u \in \mathrm{L}^2(0, T; H^1_{loc}((0,1)))$, and such solutions verify the entropy formulation proper to parabolic conservation laws (cf. [12] for the L^∞ theory and [3] for extensions that cover, in particular, the L^2_{loc} case). This entropy formulation is an essential tool in our method, due to its link with the Kruzhkov entropy solutions of the inviscid Burgers equations and to its central role in the compensated compactness argument applied below.

While it is obvious that classical solutions of ($\mathbf{Pb}^{\epsilon}_{box}$) are entropy solutions (and classical solutions exist in many situations like the pure Cauchy problem, due to the link between the Burgers equation and the heat equation provided by the Hopf-Cole formula, see e.g., [17]), it is not clear that merely distributional local solutions of the Burgers equation are entropy solutions.

Proof of Lemma 3.10. We argue by contradiction. Assuming that the statement is false, there exists a sequence (which we do not relabel) of values ϵ converging to zero and a sequence $(u_1^\epsilon) \subset \mathcal{K}$ of terminal data such that problem ($\mathbf{Pb}^{\epsilon}_{box}$) has a weak solution u^ϵ with zero initial data and the terminal data u_1^ϵ satisfying the desired L^∞ bound. Due to the assumption of weak-$*$ compactness of $\mathcal{K}$, we can find a subsequence (still not relabelled) such that the corresponding terminal data u_1^ϵ converge weak-$*$ in $\mathrm{L}^\infty((0,1))$ to some $u_1 \in \mathcal{K}$. The associated solutions u^ϵ fulfill $\|u^\epsilon\|_{\mathrm{L}^\infty((0,1)\times(0,1))} \leq const$ because $\mathcal{K}$ is bounded and due to the amplification assumption, therefore up to a further extraction of a subsequence u^ϵ converge weak-$*$ in $\mathrm{L}^\infty((0,1)\times(0,1))$ to some function u.

Using the compensated compactness technique and passing to the limit

- in the local entropy inequalities satisfied by u^ϵ
- in the weak formulation of ($\mathbf{Pb}^{\epsilon}_{box}$) including the terminal and the initial data,

we will show that u is a local Kruzhkov entropy solution of ($\mathbf{Pb}^{0}_{box}$) with the terminal data u_1; this contradicts the non-attainability of $u_1 \in \mathcal{K} \subset \mathcal{NA}_1^{\mathrm{L}^\infty,\mathrm{box}}$.

First, recall that according to Definition 2.1 weak solutions of the viscous Burgers equation satisfy the associated local entropy inequalities. Moreover, the uniform L^∞

bound on u^ϵ implies the uniform $L^1_{loc}((0,1)\times(0,1))$ bound on $\epsilon|u^\epsilon_x|^2$. Indeed, for compact sets of the form $K_\delta = [\delta, 1-\delta]\times[\delta, 1-\delta] \subset (0,1)\times(0,1)$ choose a test function $\xi \in C^\infty_c((0,1)\times(0,1))$ such that $\xi \equiv 1$ on K_δ and $0 \le \xi \le 1$. Recall that arbitrary convex functions can be approximated in the locally uniform sense by linear combinations of Id and Kruzhkov entropies $|\cdot - k|$. Applying this approximation to the convex entropy $\eta : u \mapsto u^2/2$, with the associated entropy flux $q : u \mapsto u^3/3$, we find that the entropy formulation of the type (2) (written for $\epsilon > 0$ in place of $\epsilon = 1$) implies

$$\begin{aligned}\epsilon \iint_K |u^\epsilon_x|^2 \,\mathrm{d}x\,\mathrm{d}t &\le \epsilon \int_0^1\int_0^1 |u^\epsilon_x|^2 \xi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^1\int_0^1 \frac{1}{2}(u^\epsilon)^2\xi_t + \frac{(u^\epsilon)^3}{3}\xi_x + \frac{1}{2}\epsilon(u^\epsilon)^2\xi_{xx}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{8}$$

But (u^ϵ) is bounded in $L^\infty((0,1)\times(0,1))$ so that $(\epsilon|u^\epsilon_x|^2)$ is indeed bounded in $L^1(K_\delta)$; $\delta > 0$ being arbitrary, the claim of $L^1_{loc}((0,1)\times(0,1))$ boundedness is justified.

With these ingredients at hand, standard application of the compensated compactness method (see e.g., [31, Sect. 9.2]) guarantees that (u^ϵ) converges to u a.e. on $(0,1)\times(0,1)$ as $\epsilon \to 0$. Using again the above $L^1_{loc}((0,1)\times(0,1))$ bound on $\epsilon|u^\epsilon_x|^2$ to make the diffusion term vanish in the limit $\epsilon \to 0$, we find that u fulfils the local entropy formulation (3) of ($\mathbf{Pb}^0_{box}$).

As for the terminal data, for $\epsilon > 0$, we write the weak formulation analogous to (1) (it is contained in Definition 2.1); we pass to the limit in the latter, using the a.e. convergence in the integrals over $(0,1)\times(0,1)$ and using the weak-$*$ convergence in the linear in u^ϵ_1 term accounting for the terminal data. The proof is complete. □

We are now in a position to state and prove our central result in the setting of uniformly bounded weak solutions to problem ($\mathbf{Pb}_{box}$).

Theorem 3.12. *Let $\mathcal{K}$ be a subset of $\mathcal{NA}_1^{L^\infty,box}$ compact in the weak-* topology of $L^\infty((0,1))$. Let $L \ge 1$. Then, there exists a constant $\epsilon_0 > 0$ (depending on $\mathcal{K}$ and L only) such that for all couples $(T, u_T) \in (0,\infty)\times L^\infty((0,1))$ satisfying $Tu_T \in \mathcal{K}$, $T \le \epsilon_0$ problem ($\mathbf{Pb}_{box}$) has no weak solution—in the sense of Definition 2.1—satisfying the amplification assumption*

$$\|u\|_{L^\infty((0,1)\times(0,T))} \le L\|u_T\|_{L^\infty((0,1))}. \tag{9}$$

The simplest way to interpret this uniform (over states in $\mathcal{K}$) non-attainability result is to particularize it to singletons $\mathcal{K} = \{u_T\}$. Then, Theorem 3.12 provides information on no-attainability of profiles w *with the shape prescribed by the shape of* u_T (namely, $w = T^{-1}u_T$), the amplitude of w being entangled with the non-attainability times T. In particular, we will do so for constant profiles in Corollary 3.16.

Remark 3.13. Note that given $\mathcal{K}$, the non-attainability times T for $u_T \in T^{-1}\mathcal{K}$ are small. Let us stress that due to this fact, the associated non-attainable data in Theorem 3.12 are somewhat large; indeed, $u_T \in T^{-1}\mathcal{K}$, being understood that the targets in the weakly-$*$ compact subset $\mathcal{K}$ of $\mathcal{NA}_1^{\mathrm{L}^\infty,box}$ satisfy $\mathrm{dist}(\mathcal{K}, 0) > 0$ due to the fact that $0 \notin \mathcal{NA}_1^{\mathrm{L}^\infty,box}$.

Remark 3.14. We underline that in Theorem 3.12, we assume that the solutions are not too large in the L^∞ norm (compared to the L^∞ norm of the desired terminal data), and we assume that the solutions are weak (and not merely very weak) solutions. In the sequel [6] of the paper, we will get closer to the pure L^2 setting suggested in [13]; however, some a priori bound on the size of the desired solutions (measured via the amplification constant L) will always be required.

Proof of Theorem 3.12. It is enough to scale a solution of (**BE**) with terminal data $u(\cdot, T) = u_T$, $u_T \in T^{-1}\mathcal{K}$, by (Zoom) with $\epsilon = T$; we need $T \le \epsilon_0$ in order to apply Lemma 3.10. It is easily checked that $u_\epsilon = \mathcal{S}_\epsilon(u)$ solves ($\mathbf{Pb}_{box}^\epsilon$) (also in the weak sense) on the time interval $(0, 1)$. Also note that the amplification assumption (9) is invariant under this scaling. □

Now, we concentrate on the case of constant solutions addressed in (**Q**); to do so, we apply Theorem 3.12 to the sets $\mathcal{K}_{\alpha,\beta}$ defined in Remark 3.8(i) and we employ the following elementary observation:

Lemma 3.15. *If a constant state C is non-attainable at time T by weak solutions of* ($\mathbf{Pb}_{box}$) *verifying the amplification restriction $\|u\|_{\mathrm{L}^\infty((0,1)\times(0,T))} \le L|C|$, then for all $T' < T$ the state C remains non-attainable, under the restriction $\|u\|_{\mathrm{L}^\infty((0,1)\times(0,T'))} \le L|C|$.*

Proof. Arguing by contradiction, one assumes that $u(\cdot, T') = C$ for some $T' < T$. Gluing continuously u on the time interval $[0, T']$ and the constant function C on the time interval $[T', T]$, we find that the resulting function is a weak solution to ($\mathbf{Pb}_{box}$) with terminal state C; moreover, the amplification restriction at time T is inherited from the one that was assumed at time T'. □

Combining specific choices of $\mathcal{K}$ (Remark 3.8(i)) in Theorem 3.12 with Lemma 3.15, we find the following partial negative answer to (the quantitative version of) the question (**Q**) in the L^∞ setting. We provide two closely related formulations, the first one focusing on individual states and their guaranteed non-attainability times, the second one highlighting the fact that the result naturally applies to non-attainability of families of target states.

Corollary 3.16. *Fix $L \ge 1$ and consider problem* ($\mathbf{Pb}_{box}$) *under the amplification assumption* (9).

(i) *(non-attainability of individual constant states)*
There exists $C_0 = C_0(L)$ such that whenever $|C| \ge C_0$, the state C is non-attainable at all times $T \in [0, 1/|C|]$.

(ii) *(non-attainability of families of constant states at a given time)* *Given* $\alpha \in (0, 1]$ *there exists* $\epsilon_0 = \epsilon_0(\alpha, L)$ *such that for all* $T \le \epsilon_0$ *and all* C *with* $\alpha T^{-1} \le |C| \le T^{-1}$, C *is not attainable at time* T *(as well as at any smaller time).*

Proof. (i) Theorem 3.12 yields, for $\mathcal{K} := \mathcal{K}_{1,1} = \{1\}$, the existence of $\epsilon_0 = \epsilon_0(L)$ such that for $T \le \epsilon_0$, the constant state $C = T^{-1}$ is not attainable at time T; in addition, Lemma 3.15 ensures that this state is not attainable at any smaller time. Setting $C_0 = C_0(L) := 1/\epsilon_0(L)$, we infer claim (i).
(ii) Theorem 3.12 yields, for $\mathcal{K} := \mathcal{K}_{\alpha,1}$, the existence of $\epsilon_0 = \epsilon_0(\alpha, L)$ such that for $T \le \epsilon_0$, the constant states C with $CT \in \mathcal{K}_{\alpha,1}$—i.e., C such that $\alpha \le |C|T \le 1$—are not attainable at time T. Fixing a value $T \le \epsilon_0$ we find non-attainable states C at this time; then by Lemma 3.15 these states are also not attainable at any time smaller than T. This proves claim (ii). □

Remark 3.17. If instead of taking $\mathcal{K} := \mathcal{K}_{1,1} = \{1\}$ we take $\mathcal{K} := \mathcal{K}_{\beta,\beta} = \{\beta\}$ for some $0 < \beta < 1$, it is not difficult to see that we find a smaller threshold C_0 in Corollary 3.16(i). However, in this case the intervals of non-attainability for the target C take the form $[0, \beta/|C|]$ which makes them shorter.

To conclude this paper, let us refine the above result of non-attainability by L^∞ solutions.

4. Some extensions of the non-attainability results by bounded solutions

Within the L^∞ interpretation of (**Q**), in Sect. 4.1 we address the strip setting (**Pb**$_{strip}$); then in Sect. 4.2, we point out the non-optimality of the restriction $T|C| \le 1$ in our non-attainability results.

4.1. Non-attainability by bounded solutions in the strip

We start by extending the non-attainability results to the simpler variant of problem (**BE**), namely for the case $D = \mathbb{R} \times (0, 1)$. We introduce the set $\mathcal{NA}_1^{L^\infty,strip}$ by analogy with $\mathcal{NA}_1^{L^\infty,box}$, replacing in (5) "solutions to (**Pb**$^0_{box}$)" by "solutions to (**Pb**$^0_{strip}$)". It is obvious that states u_T on $(0, 1)$ non attainable at time T by $L^\infty((0, 1) \times (0, T))$ weak solutions of the viscous Burgers equation are also non-attainable by $L^\infty(\mathbb{R} \times (0, T))$ weak solutions, i.e.,

$$\forall T > 0 \;\; \mathcal{NA}_T^{L^\infty,box} \subset \mathcal{NA}_T^{L^\infty,strip}; \tag{10}$$

also note that the scaling property (6) extends to the strip case.

The strip setting (**Pb**$_{strip}$) is a pure initial-value problem, therefore it is simpler than (**Pb**$_{box}$) in many respects. However, note that Lemma 3.15 does not extend to the strip setting. We state the results analogous to Theorem 3.12 and Corollary 3.16 as a reference point for subsequent refinements (see Corollary 4.3 in the next paragraph and further refinements in [6]). Its proof follows the lines of the proofs in Sect. 3.2.

Theorem 4.1. *Let $\mathcal{K}$ be a subset of $\mathcal{NA}_1^{\mathrm{L}^\infty,strip}$ compact in the weak-* topology of $\mathrm{L}^\infty((0,1))$. Let $L \geq 1$. Then, there exists a constant $\epsilon_0 > 0$ (depending on $\mathcal{K}$ and L only) such that for all couples $(T, u_T) \in (0,+\infty) \times \mathrm{L}^\infty((0,1))$ satisfying $T u_T \in \mathcal{K}$, $T \leq \epsilon_0$ there exist no initial data satisfying*

$$\|u_0\|_{\mathrm{L}^\infty(\mathbb{R})} \leq L\|u_T\|_{\mathrm{L}^\infty((0,1))} \tag{11}$$

such that the problem ($\mathbf{Pb}_{strip}$) admits a weak solution in the sense of Definition 2.1.

In particular, for given $\alpha \in (0,1]$ there exists $\epsilon_0 = \epsilon_0(\alpha, L)$ such that for all $T \leq \epsilon_0$ and all C with $\alpha T^{-1} \leq |C| \leq T^{-1}$, the constant state C is not attainable at time T for problem ($\mathbf{Pb}_{strip}$) with initial data fulfilling the amplification restriction $\|u_0\|_{\mathrm{L}^\infty(\mathbb{R})} \leq L|C|$.

The latter conclusion gives a partial negative answer to (the quantitative version of) (**Q**) in the strip setting. Note that it can also be reformulated as follows: given $L \geq 1$ and $\alpha \in (0,1]$, setting $C_0 = C_0(\alpha, L) := 1/\epsilon_0(\alpha, L)$, there holds the following:

$$\begin{aligned}&\text{for all } C \text{ with } |C| \geq C_0, \text{ the constant target state } u_T(\cdot) = C \text{ is not attainable}\\ &\text{by weak solutions of } (\mathbf{Pb}_{strip}) \text{ under the restriction (11) at any } T \in \left[\alpha|C|^{-1}, |C|^{-1}\right].\end{aligned} \tag{12}$$

Note that we cannot extend the non-attainability to times smaller than $\alpha/|C|$ because we don't have the conclusion of Lemma 3.15 in the strip setting.

4.2. Non-attainability for some $T > 1/|C|$

It may seem from the proof of Proposition 3.7 that the non-attainability at $T = 1$ argument is limited to constants $C \leq 1$ (so that the scaling procedure yields the restriction $T \leq 1/|C|$ in the context of Theorem 3.12, Corollary 3.16, Theorem 4.1) because they are based upon Proposition 3.7. Let us point out that this restriction is not sharp. This condition can be weakened due to our introduction of amplification conditions (9), (11) in the context of problems ($\mathbf{Pb}_{box}$), ($\mathbf{Pb}_{strip}$), respectively. Imposing the analogous restrictions in the inviscid setting, in the case $L = 1$ (no amplification), we can extend the result of non-attainability at time $T = 1$ in Proposition 3.7 to constants C with $0 < |C| < 2$, in place of $0 < |C| \leq 1$. More generally, for the case of constant targets, we have the following observation.

Proposition 4.2. *Let $L \geq 1$ be given. For all $C \neq 0$ with $|C| < 1 + \Delta$, $\Delta = \mathrm{L}^{-2}$, there exists no local Kruzhkov entropy solution verifying ($\mathbf{Pb}^0_{box}$), the terminal data $u(\cdot, 1) = C$ and the amplification restriction (9).*

In the case of problem ($\mathbf{Pb}^0_{strip}$), the analogous result (under the amplification restriction (11)) holds with the even larger value of Δ, namely $\Delta = (2L-1)^{-1}$.

Proof. We give the proof in the box setting. The strip setting is similar and we only sketch the argument. We divide the proof for ($\mathbf{Pb}^0_{box}$) into two parts. First, we address the elementary case $L = 1$ and develop the argument based upon the comparison (see Remark 3.5) with an obvious reference solution. Next, we consider $L > 1$ where the construction of an adequate reference solution and an analogous comparison argument yields the desired result. It is enough to consider positive constants C, the case of $C < 0$ being completely analogous (upon exchanging the role of the two boundaries $x = 0$, $x = 1$).

In the case $L = 1$, the function $u_{\text{ref}}(x, t) = C$ for $0 \le x < Ct/2$, $u = 0$ otherwise, is an obvious solution to ($\mathbf{Pb}^0_{box}$) and it attains the terminal data $u_T(\cdot, T) = C$ in $(0, 1)$ if and only if $CT \ge 2$. For any smaller time T, there holds $u_{\text{ref}}(x, T) = 0 < C$ for $x \in (CT/2, 1)$. Now, observe that u_{ref} solves the Cauchy–Dirichlet problem in $(0, 1) \times (0, T)$ with initial data $u_0 = 0$ and boundary data $b_0 = C$ (the Dirichlet datum at $x = 0^+$), $b_1 = C$ (the Dirichlet datum at $x = 1^-$), the boundary data being assumed in the Bardos–LeRoux–Nédélec [9] sense, see Remark 3.4. In the sequel, we rescale this solution to fit our reference setting $T = 1$; this ensures that for $C < 2$, $u_{\text{ref}}(\cdot, 1) = 0 < C$ in $(C/2, 1)$.

Now, fix $0 < C < 2$ and take any local Kruzhkov entropy solution u of (2) attaining the terminal data C at time T; according to Remark 3.4, it corresponds to some boundary data b_0, b_1 which are $[-C, C]$-valued due to the restriction (9) and our assumption $L = 1$. The comparison principle (Remark 3.5) for Cauchy–Dirichlet problems yields $C = u(\cdot, 1) \le u_{\text{ref}}(\cdot, 1)$ which is a contradiction on the interval $(C/2, 1)$. This proves the claim for $L = 1$.

Now, we address the case $L > 1$. Let us indicate the reference solution which achieves the final constant state C precisely at the critical time $T = 1$; it corresponds to the critical value $C = 1 + \mathrm{L}^{-2}$ and takes the following form. Introduce $\delta = 1 - 1/C = 1/(1 + \mathrm{L}^2)$, and define the following curves in $(0, 1) \times (0, 1)$:

$$
\begin{aligned}
\Gamma_1 &:= \{(x, t) \,|\, \delta \le t \le 1,\ x = C(t - \delta)\},\\
\Gamma_2 &:= \{(x, t) \,|\, \delta \le t \le 2\delta,\ x = LC(t - \delta)\},\\
\Gamma_3 &:= \{(x, t) \,|\, 0 \le t \le 2\delta,\ x = LCt/2\},\\
\Gamma_4 &:= \{(x, t) \,|\, 2\delta \le t \le 1,\ x = LC(\delta(t - \delta))^{\frac{1}{2}}\}.
\end{aligned}
$$

Note that the choices $C = 1 + \mathrm{L}^{-2}$, $\delta = 1 - 1/C$ ensure that Γ_1 meets Γ_4 at the point $(x, t) = (1, 1)$.

Then (see Fig. 2), we set $u_{\text{ref}} = C$ above Γ_1, $u_{\text{ref}}(x, t) = x/(t - \delta)$ between Γ_1 and $\Gamma_2 \cup \Gamma_4$, $u_{\text{ref}} = LC$ between Γ_2 and Γ_3, and $u_{\text{ref}} = 0$ below $\Gamma_3 \cup \Gamma_4$. It is easily checked that u_{ref} is a local Kruzhkov entropy solution to ($\mathbf{Pb}^0_{box}$), in particular, the Rankine–Hugoniot and the entropy admissibility conditions on $\Gamma_3 \cup \Gamma_4$ hold true.

It is also easy to verify that for any $C < 1 + \mathrm{L}^{-2}$ the solution constructed in the same way (somewhat abusively, we will keep the notation u_{ref} for this solution) exhibits a crossing of Γ_1 and Γ_4 before $T = 1$, and therefore it attains some state $u_{\text{ref}}(\cdot, 1)$ which takes zero values in a vicinity of $x = 1$. It also assumes the boundary condition $b_1 = LC$ at $x = 1^-$ in the BLN sense.

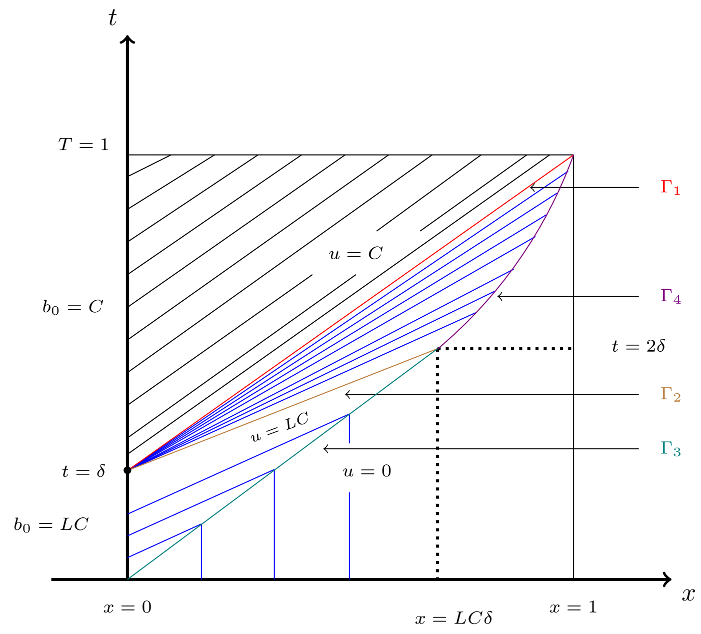

Figure 2. Solution $u_{\rm ref}$ to ($\mathbf{Pb}^0_{strip}$) in the critical case $C = 1+\mathrm{L}^{-2}$, $L > 1$

Now by applying the maximum principle, we conclude that any solution u to ($\mathbf{Pb}^0_{box}$) with $u(\cdot, T) = C$ actually lies below $u_{\rm ref}$. Recalling Remark 3.4(ii), let b_0, b_1 be boundary data that lead to a local Kruzhkov entropy solution to ($\mathbf{Pb}^0_{box}$) with terminal data C at $T = 1$, then $b_0(t) = C$ on $(1 - 1/C, 1)$ (this follows by the backward characteristics construction [16]) and $b_0 \le LC$ on $(0, 1 - 1/C)$, $b_1 \le LC$ on $(0, 1)$ due to assumption (9). Thus, $u_{\rm ref}$ corresponds exactly to the largest possible boundary data; yet in a vicinity of $x = 1$, its terminal state lies strictly below the target state, so also $u(\cdot, 1)$ cannot achieve the target state C. This proves that states $C < 1 + \mathrm{L}^{-2}$ are not attainable for ($\mathbf{Pb}^0_{box}$).

As for problem ($\mathbf{Pb}^0_{strip}$), the initial data leading to the reference solution $u_{\rm ref}$ are given by $u_{0,\rm ref} = 0$ for $x \in (0, 1)$, $u_{0,\rm ref} = C$ for $x \in (-\infty, 1 - C)$, $u_{0,\rm ref} = LC$ for $x \in (1 - C, 0) \cup (1, +\infty)$. With the choice $C = 1 + (2L - 1)^{-1}$, the shock starting from the point $(0, 0)$ encounters the rarefaction starting from the point $(1 - C, 0)$ at $(C - 1, 2(C - 1)/(LC))$, crosses the rarefaction and gets out of the rarefaction precisely at the point $(1, 1)$, quite similarly to what happens in Fig. 2. For any smaller value of C, the shock crosses the rarefaction before $T = 1$ and therefore leads to a reference solution with $u_{\rm ref}(x, 1) = 0$ in some vicinity of $x = 1$.

To conclude using the maximum principle as above, we have to remark first that, if we know that a Kruzhkov entropy solution u assumes the target datum C for $x \in (0, 1)$ and $T = 1$, then the values of u_0 for $x < -C$ do not influence the values of u in the domain $\{(x, t) \,|\, x > C(t - 1)\}$, because the boundary of this domain is a maximal backward generalized characteristic for the solution. For $x > -C$, the initial data $u_{0,\mathrm{ref}}$ taken to generate u_{ref} are the largest ones compatible with the reconstruction of u_0 in $(-C, 1-C)$ by backward characteristics, with the requirement $u(\cdot, 0) = 0$ in $(0, 1)$ and with the amplification constraint (11). □

Using Proposition 4.2 in place of Proposition 3.7, following the same strategies of proof as in Sect. 3.2, we can improve the result of Corollary 3.16(i) by extending the interval of non-attainability times T by the factor $(1+\Delta)$, $\Delta = \mathrm{L}^{-2} \in (0, 1)$:

Corollary 4.3. *Let $L \geq 1$ and restrict attention to weak solutions of (**Pb**$_{box}$) that verify the amplification restriction $\|u\|_{\mathrm{L}^\infty((0,1)\times(0,T))} \leq L|C|$. There exists $C_0 = C_0(L)$ such that whenever $|C| \geq C_0$, the state C is non-attainable at all times $T \in [0, (1+\Delta)/|C|]$ with $\Delta = \mathrm{L}^{-2}$.*

*Similarly, the last conclusion of Theorem 4.1 for problem (**Pb**$_{strip}$) holds for constants C satisfying the weaker restriction $\alpha T^{-1} \leq |C| < (1+\Delta)T^{-1}$, $\Delta = (2L-1)^{-1}$, while the non-attainability of large individual constants C in (12) can be extended to $T \in \left[\alpha|C|^{-1}, (1+\Delta)T^{-1}\right)$.*

For more general target data, we have the following variant of Proposition 3.7. For simplicity, we formulate it for the inviscid Burgers problem in the strip and only for half of the cases covered by assumption (**NA**).

Proposition 4.4. *Let $L, M \geq 1$ and $m > 0$ be given. Consider target states $u_1 \in BV_{loc}((0,1))$, normalized by right-continuity, verifying*

$$\begin{aligned} &\exists\, x^* \in (0, 1] \; such\ that \;\; m \leq u_1(x^*) \leq (1+\Delta)x^* \\ &and\ moreover, \;\; \|u_1\|_\infty \leq M u_1(x^*). \end{aligned}$$

*Assume that $\Delta < m(2LM(LM+1))^{-1}$. Then, there exists no Kruzhkov entropy solution verifying (**Pb**$^0_{strip}$), the terminal data $u(\cdot, 1) = u_1$ in $(0, 1)$ and the bound $\|u\|_\infty \leq L\|u_1\|_\infty$.*

Proof. We write $u^* = u_1(x^*)$ and conduct the construction of the proof of Proposition 3.7, being understood that this time, $x_* = 1 - u^*$ can be negative. In the sequel, we assume that x_* is negative, since otherwise the contradiction is readily given by the argument of Proposition 3.7. We define $\tilde{u}$ by (7), but this time for all $x \in \mathbb{R}$. Define $\bar{t}$ by the relation $x_* + u^*\bar{t} = 1 - LMu^*\bar{t}$; this yields

$$\bar{t} = \frac{1 - x_*}{LMu^* + u^*} = \frac{1}{LM+1},$$

keeping in mind that $1 - x_* = u^*$. Set $\bar{x} = x_* + u^*\bar{t}$; we refer to Fig. 3 for the geometric interpretation of the point $(\bar{x}, \bar{t})$.

In view of the bound $\|\tilde{u}\|_\infty \leq \|u\|_\infty \leq LMu^*$, the classical Kruzhkov propagation estimates [24] imply in particular that

$$\int_{-\infty}^{\bar{x}} |\tilde{u}(x, \bar{t})| \,\mathrm{d}x \leq \int_{-\infty}^{1} |\tilde{u}(x, 0)| \,\mathrm{d}x.$$

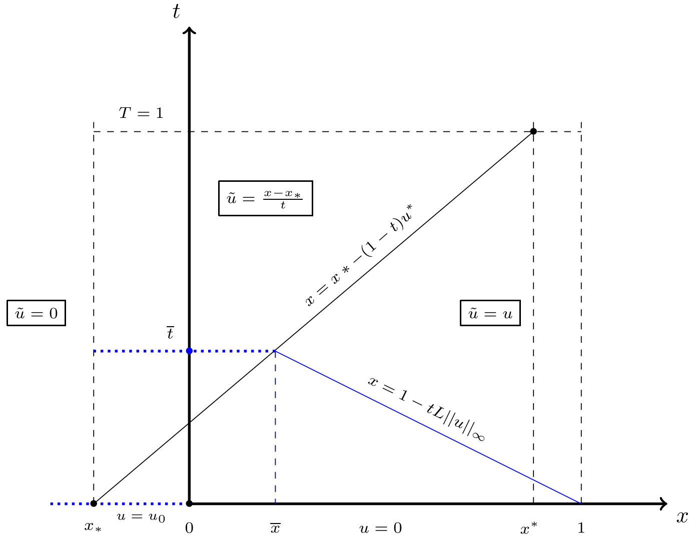

Figure 3. Construction of the point $(\bar{x}, \bar{t})$ in the proof of Proposition 4.4

The expression (7) of $\tilde{u}$ being explicit for $t = \bar{t}$, $x \in (-\infty, \bar{x})$, the calculation of the left-hand side of the above inequality, bearing in mind the bound $|\tilde{u}_0(x)| = |u(x)| \leq L\|u_1\|_\infty$ for $x \in (x_*, 0)$, yields

$$\frac{(u^*)^2}{2(LM+1)} = \frac{(u^*)^2\bar{t}}{2} = \int_{x_*}^{x_*+u^*\bar{t}} \frac{x - x_*}{\bar{t}}\,\mathrm{d}x \leq \int_{x_*}^{0} |\tilde{u}(x,0)| \leq LMu^*|x_*|$$
$$= LMu^*(u^* - 1) \leq LMu^*\Delta.$$

Because $u^* \geq 0$, this leads to a contradiction as soon as $2LM(LM+1)\Delta < m$. □

Note that a qualitatively analogous to Proposition 4.4 result can be formulated for the problem (**Pb**$^0_{box}$); but the explicit bound for Δ in terms of L, M, m is more delicate to compute because the control of the L^1 norm of the solution in terms of the L^1 norm of the boundary data makes the Lipschitz constant of $f : u \mapsto u^2/2$ on $[-\|u\|_\infty, \|u\|_\infty]$ to appear (cf. the stability estimate in [30]). We will not pursue this further.

Using Proposition 4.4 in place of Proposition 3.7, following the same strategy of proof as in Sect. 3.2, one can improve the results of Theorem 4.1 by requiring that

$$T(1+\Delta)u_T \in \mathcal{K}, \quad \Delta < m(2LM(LM+1))^{-1},$$

provided $\mathcal{K}$ consists of states verifying (4.4).

Remark 4.5. While we do not pursue the goal of giving optimal statements in this and related situations, let us stress that the case $|C| = T^{-1}$ (that appeared as critical in the non-attainability statements of Sect. 3.2) is actually situated in the interior of the non-attainable (under amplification restrictions!) set, and not on its boundary (cf.

Remark 3.9); the same is true at least for terminal states $u_T \in \mathcal{NA}_T^{L^\infty,box}$ having the shape (4.4) and the amplitude T^{-1}. This fact will also become important in the sequel of this paper [6].

Acknowledgements

The work in this paper was supported by the French ANR agency (project CoToCoLa No. ANR-11-JS01-006-01) and IFCAM project "Conservation Laws: BV^s, Control, Interfaces". S.S.G. thanks Inspire Faculty-Research grant DST/INSPIRE/04/2016/000237. This paper has been supported by the RUDN University Strategic Academic Leadership Program. The authors thank the anonymous reviewer of the first version of this paper for highlighting the intrinsic limitation of our approach through the example discussed in Remark 1.1. S. S. G would like to thank J-M. Coron for introducing him to the problem. The authors would like to thank GSSI (L'Aquila, Italy) as large part of the work was done during their stay (S. S. G. and K. K.) or visit (B. A.) at GSSI.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

REFERENCES

[1] Adimurthi, S. S. Ghoshal and G. D. Veerappa Gowda, Exact controllability of scalar conservation law with strict convex flux, *Math. Control Relat. Fields*. 04, no. 04, (2014), 401–449.

[2] Adimurthi, S. S. Ghoshal and G. D. Veerappa Gowda, Structure of the entropy solution of a scalar conservation law with strict convex flux, *J. Hyperbolic Differ. Equ.*, 09, no. 04 (2012), 571–611.

[3] K. Ammar and P. Wittbold, Existence of renormalized solutions of degenerate elliptic-parabolic problems, *Proc. Roy. Soc. Edinburgh Sect.* A 133 (2003), no. 3, 477–496.

[4] F. Ancona and A. Marson, On the attainability set for scalar non linear conservation laws with boundary control, *SIAM J. Control Optim.*, 36, no. 1, (1998), 290–312.

[5] F. Ancona and A. Marson, Scalar non-linear conservation laws with integrable boundary data, *Nonlinear Anal.* 35, no. 6, (1999), 687–710.

[6] B. Andreianov, S.S. Ghoshal and K. Koumatos, Lack of controllability of the viscous Burgers equation. Part II: The L^2 setting, with a detour into the well-posedness of unbounded entropy solutions to scalar conservation laws, *submitted*, available as HAL preprint https://hal.archives-ouvertes.fr/hal-03680108

[7] B. Andreianov, C. Donadello, S. S. Ghoshal and U. Razafison, On the attainability set for triangular type system of conservation laws with initial data control, *J. Evol. Equ.*, 15, no. 3, (2015), 503–532.

[8] B. Andreianov and K. Sbihi, Well-posedness of general boundary-value problems for scalar conservation laws, *Transactions AMS* 367, (6), (2015), 3763–3806.

[9] C. Bardos, A. Y. le Roux and J.-C. Nédélec, First order quasilinear equations with boundary conditions, *Comm. Partial Differential Equations* 4 (1979), no. 9, 1017–1034.

[10] Ph. Bénilan, J. Carrillo and P. Wittbold, Renormalized entropy solutions of scalar conservation laws, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 29 (2000), no. 2, 313–327.

[11] C. Cancès and T. Gallouët, On the time continuity of entropy solutions, *J. Evol. Equ.* 11 (2011), no. 1, 43–55.

[12] J. Carrillo, Entropy solutions for nonlinear degenerate problems, *Arch. Ration. Mech. Anal.* 147 (1999), no. 4, 269–361.

[13] J.M. Coron, Some open problems on the control of nonlinear partial differential equations. Perspectives in nonlinear partial differential equations. 215–243, Contemp. Math., 446, Amer. Math. Soc., Providence, RI, 2007.

[14] J.-M. Coron, Control and nonlinearity, Mathematical Surveys and Monographs, 136. American Mathematical Society, Providence, RI, 2007. xiv+426 pp. ISBN: 978-0-8218-3668-2; 0-8218-3668-4

[15] J.-M. Coron and S. Guerrero, Singular optimal control: a linear 1-D parabolic hyperbolic example, *Asymptot. Anal.* 44 (2005), no. 3–4, 237–257.

[16] C. M. Dafermos, Generalized characteristics and the structure of solutions of hyperbolic conservation laws, *Indiana Univ. Math. J.* 26 (1977), no. 6, 1097–1119.

[17] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Fourth edition. Grundl. Math. Wiss., 325. Springer-Verlag, Berlin, 2016

[18] E. Fernandez-Cara and S. Guerrero, Remarks on the null controllability of the Burgers equation, *C. R. Math. Acad. Sci. Paris* 841 (2005), no. 4, 229–232.

[19] H. O. Fattorini and D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.* 43 (1971), 272–292.

[20] A. V. Fursikov and O. Yu. Imanuvilov, Controllability of evolution equations, Lecture Notes Series, vol. 34, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[21] O. Glass and S. Guerrero, On the uniform controllability of the Burgers equation, *SIAM J. Control optim.*, 46, no. 4 (2007), 1211–1238.

[22] S. Guerrero and O. Yu. Immunauvilov, Remarks on global controllability for the Burgers equation with two control forces, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24 (2007), 897–906.

[23] T. Horsin, On the controllability of the Burger equation, *ESIAM, Control optimization and Calculus of variations*, 3 (1998), 83–95.

[24] S. N. Kružkov, First order quasilinear equations with several independent variables, (Russian), *Mat. Sb.*, 81:123 (1970), 228–255. English transl. in *Math. USSR Sb.*, 10 (1970), 217–243.

[25] M. Léautaud, Uniform controllability of scalar conservation laws in the vanishing viscosity limit, *SIAM J. Control Optim.* 50, no. 3, (2012), 1661–1699.

[26] F. Marbach, Small time global null controllability for a viscous Burgers' equation despite the presence of a boundary layer, *J. Math. Pures Appl.* (9) 102, (2014), no. 2, 364–384.

[27] E. Yu. Panov, Existence of strong traces for generalized solutions of multidimensional scalar conservation laws, *J. Hyperbolic Differ. Equ.* 2 (2005), no. 4, 885–908.

[28] V. Perrollaz, Exact controllability of scalar conservation laws with an additional control in the context of entropy solutions, *SIAM J. Control Opt.* 50 (2012), no. 4, 2025–2045.

[29] B. Perthame, *Kinetic formulation of conservation laws*, Math. and Appl. series, vol. 21, Oxford, 2002.

[30] A. Porretta and J. Vovelle, L^1 solutions to first order hyperbolic equations in bounded domains, *Comm. Partial Differential Equations*, 28 (2003), no. 1–2, 381–408.

[31] D. Serre, *Systems of conservation laws. 2. Geometric structures, Oscillations, Initial-boundary value problems*. Cambridge University Press, Cambridge, 1999.

[32] D. Serre and L. Silvestre, Multi-dimensional Burgers equation with unbounded initial data: well-posedness and dispersive estimates, *Arch. Ration. Mech. Anal.*, 234 (2019), 1391–1411.

[33] A. Vasseur, Strong traces for solutions of multidimensional scalar conservation laws, *Arch. Ration. Mech. Anal.* 160 (2001), no. 3, 181–193.

Boris Andreianov
Institut Denis Poisson CNRS UMR7013
Université de Tours, Université d'Orléans
Parc Grandmont 37200 Tours
France
E-mail: boris.andreianov@univ-tours.fr

and

Peoples' Friendship University of Russia
(RUDN University)
6 Miklukho-Maklaya St
Moscow
Russian Federation 117198

Shyam Sundar Ghoshal
Centre For Applicable Mathematics
Tata Institute of Fundamental Research
Sharada Nagar, Chikkabommsandra
Bangalore 560065
India
E-mail: ghoshal@tifrbng.res.in

Konstantinos Koumatos
Department of Mathematics
University of Sussex
Pevensey 2 Building, Falmer
Brighton BN1 9QH
UK
E-mail: K.Koumatos@sussex.ac.uk

Accepted: 17 July 2022