J. Evol. Equ. (2022) 22:67

1424-3199/22/030001-25, *published online* July 29, 2022
https://doi.org/10.1007/s00028-022-00821-7

Journal of Evolution Equations

Existence, uniqueness, and exponential stability for the Kirchhoff equation in whole hyperbolic space

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Abstract. In this paper, we prove the existence, uniqueness, and exponential stability for a damped nonlinear wave equation of Kirchhoff type which is defined in whole hyperbolic space $\mathbb{B}^N$. Our strategy consists of changing the problem into a singular problem defined in the unitary ball of $\mathbb{R}^N$ endowed with the Euclidean metric. One difficulty is to prove the existence of solution and the Faedo–Galerkin method was our main tool. It is well known that when we deal with the Kirchhoff model defined in whole space $\mathbb{R}^N$, the exponential stability is not expected. In this work, we prove that, in the hyperbolic space, the problem is exponentially stable. The main tool to reach the result is to combine the classical Nakao's techniques with the use of Hardy inequality.

1. Introduction

In this paper we study the following problem

$$u_{tt} - M\left(\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2\, dV\right)\Delta_{\mathbb{B}^N} u + \delta u_t = 0 \ \text{ in } \mathbb{B}^N \times (0,\infty), \tag{1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \ x \in \mathbb{B}^N, \tag{2}$$

where $\nabla_{\mathbb{B}^N}$ and $\Delta_{\mathbb{B}^N}$ are, respectively, the gradient and the Laplace–Beltrami operator in the disc model of the hyperbolic space $\mathbb{B}^N$; $M : [0,\infty) \to \mathbb{R}$ is a known function; u_0 and u_1 are the initial data and δ is a positive constant.

The space $\mathbb{B}^N$ is the unit disc $\{x \in \mathbb{R}^N : |x| < 1\}$ of $\mathbb{R}^N$ endowed with the Riemannian metric g given by $g_{ij} = p^2\delta_{ij}$, where $p(x) = \frac{2}{1-|x|^2}$ and $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$. The hyperbolic gradient $\nabla_{\mathbb{B}^N}$ and the hyperbolic Laplacian $\Delta_{\mathbb{B}^N}$ are given by

$$\nabla_{\mathbb{B}^N} u = \frac{\nabla u}{p} \quad \text{and} \quad \Delta_{\mathbb{B}^N} u = p^{-N} div(p^{N-2}\nabla u) = p^{-2}\Delta + \frac{(N-2)}{p} x\cdot\nabla, \tag{3}$$

where $\cdot$ is the standard scalar product in $\mathbb{R}^N$; and ∇ and Δ are the usual gradient and Laplacian of $\mathbb{R}^N$. Details can be found in the references [15,16,31,32].

There are large literature concerned with existence, uniqueness, and stability of Kirchhoff model. We can cite the works of Bae and Nakao [1], Cavalcanti, Domingos

Keywords: Stability, Kirchhoff equation, Hyperbolic space, Unbounded coefficients.

Cavalcanti and Soriano [10], Cavalcanti *et al.* [11,12], Ghisi [17,18], Louredo and Miranda [21], Miranda and Jutuca [22], Miranda, Louredo and Medeiros [23], Perla Menzala [19], Ono [27–29], Nishihara [26], Yamada [33], and references therein.

We would like to emphasize the work of Miranda and Jutuca [22] where the authors proved the existence, uniqueness and decay for the problem with boundary damping. Precisely, they studied the problem

$$u_{tt} - M\left(t, \int_\Omega |\nabla u|^2\, dx\right)\Delta u = 0 \ \text{ in } \Omega \times (0,\infty), \tag{4}$$

$$u = 0 \ \text{ on } \Gamma_0 \times (0,\infty), \tag{5}$$

$$\frac{\partial u}{\partial \nu} + \delta u_t = 0 \ \text{ on } \Gamma_1 \times (0,\infty), \tag{6}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \ \ x \in \Omega, \tag{7}$$

where the domain Ω is an open and bounded subset of $\mathbb{R}^N$ and its boundary is given by $\Gamma = \Gamma_0 \cup \Gamma_1$. To prove the existence of solution, the authors used fixed point theorem combined with the use of Faedo–Galerkin method. They proved the exponential decay for the strong energy associated to the problem.

In [10], Cavalcanti, Domingos Cavalcanti and Soriano extended the results of Miranda and Jutuca [22] to the nonlinear case. Precisely, they studied the problem

$$u_{tt} - M\left(t, \int_\Omega |\nabla u|^2\, dx\right)\Delta u = 0 \ \text{ in } \Omega \times (0,\infty), \tag{8}$$

$$u = 0 \ \text{ on } \Gamma_0 \times (0,\infty), \tag{9}$$

$$\frac{\partial u}{\partial \nu} + g(u_t) = 0 \ \text{ on } \Gamma_1 \times (0,\infty), \tag{10}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \ \ x \in \Omega, \tag{11}$$

where the domain Ω is an open, bounded star-shaped subset of $\mathbb{R}^N$ and its boundary is given by $\Gamma = \Gamma_0 \cup \Gamma_1$. The authors proved the existence and uniqueness of regular solutions without the classical assumption involving the smallness on the initial data.

When the domain is whole space $\mathbb{R}^N$, there are some additional difficulties and the exponential stabilization is not expected. In fact, it is well known that an ingredient to prove the exponential stability (without restriction on the initial data) is to use the Poincaré inequality, which does not hold in whole $\mathbb{R}^N$. In this direction, we can cite the work of Ono [30], where the author proved that the energy decays with polynomial rate.

Recently Dias Silva, Pitot, and Vicente [14] studied the Kirchhoff equation defined on whole $\mathbb{R}^N$ space. Inspired on work of Bjorland and Schonbek [2], they defined suitable Hilbert spaces V, H and an operator $A = -\Delta$ by the triple $\{V, H, a(u,v)\}$, where $a(u,v)$ is a bilinear, continuous and coercive form defined in V. This strategy allows the authors to prove that the energy decays exponentially.

On the other hand, studies involving the wave equation defined in whole hyperbolic space can be found in [34], where Wang, Ning, and Yang considered the following problem

$$u_{tt} - \Delta_g u + a(x)u_t = 0 \ \text{in}\ \mathbb{H}^N \times (0, \infty), \tag{12}$$

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ \ x \in \mathbb{H}^N, \tag{13}$$

where Δ_g is the Laplace–Beltrami operator in $\mathbb{H}^N$. Using multiplier methods and compactness-uniqueness arguments, they proved the exponential stabilization. An important tool to prove the stability of (12)–(13) is the following Poincaré inequality

$$\int_{\mathbb{H}^N} u^2\, dx_g \le C \int_{\mathbb{H}^N} |\nabla_g u|_g^2\, dx_g, \tag{14}$$

for $u \in H^1(\mathbb{H}^N)$, where ∇_g is the gradient operator associated to the Riemannian metric g. As described before, in whole $\mathbb{R}^N$ the inequality above does not hold. Therefore, it is not possible to prove the exponential stability without restriction on the initial data.

Another way to prove the exponential stability of wave equation defined in $\mathbb{B}^N$ was proved by Carrião, Miyagaki, and Vicente [9]. Indeed, the authors considered the semilinear problem with localized damping

$$u_{tt} - \Delta_{\mathbb{B}^N} u + f(u) + a(x)u_t = 0 \ \text{in}\ \mathbb{B}^N \times (0, \infty), \tag{15}$$

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x) \ \text{for}\ x \in \mathbb{B}^N, \tag{16}$$

where a, f, u_0 and u_1 are known functions and $\Delta_{\mathbb{B}^N}$ is the Laplace–Beltrami operator in the disc model of the Hyperbolic $\mathbb{B}^N$. Making the appropriate change $v := p^{\frac{N-2}{2}} u$, we have that u satisfies (15)–(16) if, and only if, v satisfies the following singular problem

$$p^2 v_{tt} - \Delta v + \beta_0 p^2 v + p^{\frac{N+2}{2}} f(p^{-\frac{N}{2}+1} v) + a(x)p^2 v_t = 0 \ \text{in}\ B_1 \times (0, \infty), \tag{17}$$

$$v(x, 0) = v_0(x),\ v_t(x, 0) = v_1(x) \ \text{for}\ x \in B_1, \tag{18}$$

where $\beta_0 = \frac{N(N-2)}{4}$ and B_1 is the unit disc $\{x \in \mathbb{R}^N : \ |x| < 1\}$ of $\mathbb{R}^N$ endowed with the Euclidean metric. Therefore, the authors worked with (17)–(18) which is a singular problem in B_1. To overcome the difficulty of deal with the singularities, they used the Hardy inequality, in a version due to the Brezis and Marcus [4] and Brezis, Marcus, and Shafrir [5]. To best of our knowledge, the technique used by Carrião, Miyagaki, and Vicente [9] is new in the context of hyperbolic equations. Before that, only elliptic equations were studied with this tool. See Carrião *et al.* [6–8].

As described before, the main tool used in [9] is Hardy's inequality. Precisely, using this inequality, it is possible to prove that

$$\int_{B_1} p^2 w^2\, dx \le C \int_{B_1} |\nabla w|^2\, dx, \tag{19}$$

for all $w \in H_0^1(B_1)$, which is a kind of Poincaré's inequality with a weight p. This inequality is shown in Lemma 1 and it will be used many times in the paper.

We also would like to cite the work of Bortot *et al.* [3], where they studied the Klein Gordon equation, subject to a nonlinear and localized damping, in a complete and non-compact Riemannian manifold without boundary. Precisely, they studied the problem

$$u_{tt} - \Delta u + f(u) + a(x)g(u_t) = 0 \text{ in } \mathcal{M} \times (0, \infty), \tag{20}$$

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x) \text{ for } x \in \mathcal{M}, \tag{21}$$

where $\mathcal{M}$ (endowed with a Riemannian metric) is a complete and non-compact N dimensional Riemannian manifold without boundary and Δ denotes the Laplace–Beltrami operator. The function a, responsible by the damping localization, acts in $\mathcal{M} \setminus \overline{\Omega}$, where Ω is an arbitrary open and bounded set in $\mathcal{M}$.

In the present paper, we use the strategy of [9] to change the original problem into a singular problem. Therefore, defining $v := p^{\frac{N-2}{2}} u$ and observing (3), we have that u satisfies (1)–(2) if, and only if, v satisfies the following singular problem

$$p^2 v_{tt} - M(\lambda(t)) \left(\Delta v - \beta_0 p^2 v\right) + \delta p^2 v_t = 0 \text{ in } B_1 \times (0, \infty), \tag{22}$$

$$v = 0 \text{ on } \partial B_1 \times (0, \infty), \tag{23}$$

$$v(x, 0) = v_0(x),\ v_t(x, 0) = v_1(x) \text{ for } x \in B_1, \tag{24}$$

where $\beta_0 = \frac{N(N-2)}{4}$ and B_1 is the unit disc $\{x \in \mathbb{R}^N : |x| < 1\}$ of $\mathbb{R}^N$ endowed with the Euclidean metric and

$$\lambda(t) = \int_{B_1} \left(|\nabla v|^2 - (N-2)\langle \nabla v, xpv\rangle + \left(\frac{N-2}{2}\right)^2 p^2 v^2 |x|^2 \right) dx. \tag{25}$$

We work, equivalently, with (22)–(24) which is a singular problem in B_1. Therefore, since $p(s) \to \infty$, as $|s| \to 1$, the difficulty of deal with the wave equation in whole space $\mathbb{B}^N$ is replaced by the difficulty of deal with a singular problem in B_1.

The main goal of the present paper is bring the technique used by Carrião, Miyagaki, and Vicente [9] and Carrião *et al.* [6–8] to the context of Kirchhoff equation. This strategy allows us to prove that the energy associated to the problem decay exponentially.

The main technical difficulty of the present paper is to control the singularities. It is well known that when Kirchhoff equation is in place, it is not possible to use semigroups techniques to prove the existence of solution. In the $\mathbb{R}^N$ case many authors have been used fixed point methods and Faedo–Galerkin method. But, due to the presence of singularities, some usual calculus does not hold here and it is necessary some additional arguments. Indeed, when the Kirchhoff model is considered in a domain Ω of $\mathbb{R}^N$ it is usual to use Faedo–Galerkin method with special bases (given by eigenvector of the Laplace operator). This allows to estimate the norm of the sequence of approximate solution in the space $L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$. But, the singularities does not allow us to take the same way. This difficult can be seen in the proof of the existence of solution. To overcome this problem, it is necessary to make four estimates. It will be clarified in Sect. 4.

Our paper is organized as follows. In Sect. 2 we present the notation, assumptions, and preliminaries. We also enunciate the theorem which gives the exponential stability. Moreover, we enunciate a result which gives the existence and uniqueness of solution. The stability is proved in Sect. 3. In Sect. 4 we prove the existence and uniqueness of solution.

2. Preliminaries and main result

As described in the introduction, in this section we establish some notations and the main result. We also enunciate the tool which is the main novelty in the context of class of Kirchhoff equations, a class of Hardy inequality. This inequality is used many times into the paper. The reader can see that Lemma 1 is called many times through the paper.

Thus, we start defining some usual spaces. Let $L^2(B_1)$ be endowed with the norm and inner product

$$\|u\|_2 = \left(\int_{B_1} u^2\,dx\right)^{\frac{1}{2}} \quad \text{and} \quad (u,v) = \int_{B_1} uv\,dx.$$

In the space $H_0^1(B_1)$ we consider the norm and inner product defined by

$$\|u\|_{H_0^1(B_1)} = \|\nabla u\|_2 \quad \text{and} \quad (u,v) = \int_{B_1} \nabla u \cdot \nabla v\,dx.$$

Now, we enunciate the Lemma 1, which gives us a Hardy inequality class and, after this, the classical Nakao's Lemma.

Lemma 1. *There exists a positive constant C_H such that the following inequality holds*

$$\int_{B_1} p^2 w^2\,dx \le C_H^2 \int_{B_1} |\nabla w|^2\,dx, \tag{26}$$

for all $w \in H_0^1(B_1)$.

Proof. See Carrião, Miyagaki, and Vicente [9]. See also Carrião *et al.* [6–8].

It is well known that Nakao's lemma is an important tool to prove the stability to problems involving the Kirchhoff equation. Below, we enunciate the lemma whose proof can be found in Nakao [24,25]. □

Lemma 2. (Nakao's Lemma) *Let $\phi : [0,\infty) \to [0,\infty)$ be a bounded function satisfying*

$$\sup_{t\le s\le t+1} \phi(s) \le C_0(\phi(t) - \phi(t+1)), \tag{27}$$

for $t \geq 0$, where C_0 is a positive constant. Then, it holds

$$\phi(t) \leq \theta_1 e^{-\theta_2 t} \tag{28}$$

for all $t \geq 0$, where θ_1 and θ_2 are positive real number which depends on known constants.

Now, it is important to observe that, from the inequality

$$(N-2)\langle \nabla v, xpv \rangle \leq |\nabla v|^2 + \left(\frac{N-2}{2}\right)^2 p^2 v^2 |x|^2$$

we infer that

$$\lambda(t) \geq 0, \quad \text{for all } t \geq 0.$$

Thus, we have the control in the sign of $\lambda(t)$, but we do not control the sign of each term of $\lambda(t)$, and it is one difficulty that needs being overcome.

Throughout this paper, we denote the specific constants by $C_1, C_2, \ldots$ and the generic ones only by C.

Below, we enunciate two assumptions. The first involves the function M and the second one is called into the literature of assumption of small initial data, and it is well used in the context of Kirchhoff models.

Assumption 1. $M : [0, \infty) \to (0, \infty)$ is an increasing and continuously differentiable function. There exist positive constants m_0, C_1 and a real number $q \geq 1$ such that

$$0 < m_0 \leq M(s), \quad \text{for all } s \in [0, \infty) \tag{29}$$

and

$$|M(s)| \leq C_1 |s|^q, \quad \text{for all } s \in [1, \infty). \tag{30}$$

Thus, it holds

$$|M(s)| \leq C_1(1 + |s|^q), \quad \text{for all } s \in [0, \infty). \tag{31}$$

We define the energy associated to the problem (22)–(24) by

$$E(t) = \int_{B_1} p^2 v_t^2 \, dV + \overline{M}(\lambda(t)), \tag{32}$$

where

$$\overline{M}(s) = \int_0^s M(\xi) \, d\xi. \tag{33}$$

We also define the following auxiliary functional

$$\Psi(t) = E(t) + \frac{\delta}{2} \int_{B_1} p^2 v_t v \, dx + \frac{\delta^2}{4} \int_{B_1} p^2 v^2 \, dx. \tag{34}$$

Assumption 2. We suppose that the initial data $(v_0, v_1) \in H_0^1(B_1) \cap H^2(B_1) \times H_0^1(B_1)$ satisfy

$$\max\left\{C_2, C_3\Psi^q(0)\right\} < \delta, \tag{35}$$

where

$$C_2 = \frac{4C_1C_6}{\sqrt{m_0}}, \quad C_3 = \frac{\sqrt{2}C_2}{2m_0^q}, \tag{36}$$

here

$$C_6 = \frac{(N-2)(N+5)C_H}{2} + 2(N-2).$$

To prove the stability of the problem, we need working with $\Psi(t)$ instead of $E(t)$. First, we observe that

$$\left|\frac{\delta}{2}\int_{B_1} p^2 v_t v\, dx\right| \leq \frac{1}{2}\int_{B_1} p^2 v_t^2\, dx + \frac{\delta^2}{8}\int_{B_1} p^2 v^2\, dx. \tag{37}$$

Thus,

$$\frac{\delta}{2}\int_{B_1} p^2 v_t v\, dx \geq -\frac{1}{2}\int_{B_1} p^2 v_t^2\, dx - \frac{\delta^2}{8}\int_{B_1} p^2 v^2\, dx. \tag{38}$$

We also have

$$\overline{M}\left(\lambda(t)\right) \geq m_0\lambda(t) \geq 0, \tag{39}$$

for all $t \geq 0$.

Thus, we infer

$$\Psi(t) \geq \frac{1}{2}\int_{B_1} p^2 v_t^2\, dx + \overline{M}\left(\lambda(t)\right) + \frac{\delta^2}{8}\int_{B_1} p^2 v^2\, dx. \tag{40}$$

Therefore,

$$\Psi(t) \geq \frac{1}{2}\left[\int_{B_1} p^2 v_t^2\, dx + \overline{M}\left(\lambda(t)\right)\right] = \frac{1}{2}E(t). \tag{41}$$

From (41), we see that to show that the energy associated to (22)–(24) decay exponentially, it is enough to prove that there exist positive constants α_1 and α_2 such that

$$\Psi(t) \leq \alpha_1 e^{-\alpha_2 t}, \tag{42}$$

for all $t \geq 0$. Therefore, we can enunciate the following result which gives us the exponential decay for the energy associated to (22)–(24).

Theorem 1. *Assume that Assumptions 1 and 2 hold. Let v be a solution of* (22)–(24) *in the class*

$$\begin{gathered} v \in L^\infty(0,T; H_0^1(B_1) \cap H^2(B_1)), \quad v_t \in L^\infty(0,T; H_0^1(B_1)), \\ p v_{tt} \in L^\infty(0,T; L^2(B_1)). \end{gathered} \tag{43}$$

Then there exist positive constants α_1 and α_2 such that

$$\Psi(t) \le \alpha_1 e^{-\alpha_2 t}, \quad \text{for all } t \ge 0. \tag{44}$$

Next task is to establish a result which ensures the existence and uniqueness of solution to (22)–(24). For this purpose, we need an additional assumption. First, we define

$$\begin{aligned} \widetilde{\Psi}(0) = \Psi(0) + \Bigg[\frac{M_0}{2} \left(\int_{B_1} |\Delta v_0|^2 \, dx \right)^{\frac{1}{2}} + \beta_0 M_0 \left(\int_{B_1} p^2 |v_0|^2 \, dx \right)^{\frac{1}{2}} \\ + \delta \left(\int_{B_1} p^2 |v_1|^2 \, dx \right)^{\frac{1}{2}} \Bigg]^2 + \int_{B_1} |\nabla v_1|^2 \, dx + \beta_0 \int_{B_1} \frac{p^2 |v_1|^2}{M(\lambda(0))} \, dx, \end{aligned}$$

where

$$M_0 = \max_{0 \le s \le \left(\frac{\delta^2 m_0}{8 C_1^2 C_6^2} \right)^{\frac{1}{2q}}} |M(s)|.$$

Thus, we consider the following assumption

Assumption 3. We suppose that the initial data $(v_0, v_1) \in H_0^1(B_1) \cap H^2(B_1) \times H_0^1(B_1)$ satisfy

$$C_4 \widetilde{\Psi}^{\frac{1}{2}}(0) + \delta C_5 \widetilde{\Psi}(0) < \delta, \tag{45}$$

where

$$C_4 = \frac{M_1 \sqrt{L_2} [8 + 8 C_H (N-2) + C_H^2 (N-2)^2]}{m_0} \quad \text{and} \quad C_5 = \frac{C_4^2}{m_0},$$

here

$$M_1 = \max_{0 \le s \le \left(\frac{\delta^2 m_0}{8 C_1^2 C_6^2} \right)^{\frac{1}{2q}}} |M'(s)|$$

and

$$L_2 = 2 \left(\frac{1}{m_0} + \frac{3(N-2)^2}{4} \right) \frac{\Psi(0)}{\min\left\{ \frac{1}{2}, \frac{\delta^2}{8} \right\}}. \tag{46}$$

We observe that it is possible to find δ, M, v_0, and v_1 such that the Assumptions 2 and 3 hold. Indeed, given M, we can calculate C_2 and C_3. After this, we choose $\delta > C_2$. Now, as $\Psi(0)$ and $\widetilde{\Psi}(0)$ depend of v_0 and v_1, it is possible to take a couple of initial data (v_0, v_1) sufficiently small such that $C_3\Psi^q(0)$ and $C_4\widetilde{\Psi}^{\frac{1}{2}}(0) + \delta C_5\widetilde{\Psi}(0)$ become so small that (35) holds.

Theorem 2. (Existence and uniqueness of solution) *If Assumptions 1, 2, and 3 are in place, then there exists a unique solution of* (22)–(24) *in the class* (43).

3. Exponential decay

In this section, we prove the exponential decay for the energy associated to the problem (22)–(24). We start with a lemma.

Lemma 3. *Let v the solution of* (22)–(24) *in the class* (43)*. It holds*

$$\frac{1}{2}\frac{d}{dt}\Psi(t) + \frac{\delta}{4}\int_{B_1} p^2 v_t^2\,dx + \frac{\beta_0 m_0 \delta}{4}\int_{B_1} p^2 v^2\,dx \leq 0, \tag{47}$$

for all $t \geq 0$.

Proof. Multiplying (22) by v_t and integrating over B_1, we have

$$\frac{1}{2}\frac{d}{dt}\int_{B_1} p^2 v_t^2\,dx + \frac{1}{2}M(\lambda(t))\frac{d}{dt}\left[\int_{B_1}\left(|\nabla v|^2 + \beta_0 p^2 v^2\right)dx\right]$$
$$+\,\delta\int_{B_1} p^2 v_t^2\,dx = 0. \tag{48}$$

□

We observe that

$$\begin{aligned}
&\frac{1}{2}M(\lambda(t))\frac{d}{dt}\left[\int_{B_1}\left(|\nabla v|^2 + \beta_0 p^2 v^2\right)dx\right] = \frac{1}{2}M(\lambda(t))\frac{d}{dt}\lambda(t)\\
&\quad + \frac{1}{2}M(\lambda(t))\frac{d}{dt}\left\{\int_{B_1}\left[\left(\beta_0 - \beta_1|x|^2\right)p^2v^2 + (N-2)\langle\nabla v, xpv\rangle\right]dx\right\}\\
&= \frac{1}{2}\frac{d}{dt}\overline{M}(\lambda(t))\\
&\quad + \frac{1}{2}M(\lambda(t))\frac{d}{dt}\left\{\int_{B_1}\left[\left(\beta_0 - \beta_1|x|^2\right)p^2v^2 + (N-2)\langle\nabla v, xpv\rangle\right]dx\right\}\\
&= \frac{1}{2}\frac{d}{dt}\overline{M}(\lambda(t)) + M(\lambda(t))\left\{\int_{B_1}\left[\left(\beta_0 - \beta_1|x|^2\right)p^2 v v_t\right.\right.\\
&\quad \left.\left. + (N-2)(\langle\nabla v, xpv_t\rangle + \langle\nabla v_t, xpv\rangle)\right]dx\right\}.
\end{aligned} \tag{49}$$

Combining (48) with (49), we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\int_{B_1} p^2|v_t|^2\,dx + \overline{M}(\lambda(t))\right] + \delta\int_{B_1} p^2|v_t|^2\,dx$$
$$= -M(\lambda(t))\Big\{\int_{B_1}\Big[\left(\beta_0-\beta_1|x|^2\right)p^2 v v_t$$
$$+(N-2)(\langle\nabla v, xpv_t\rangle + \langle\nabla v_t, xpv\rangle)\Big]\,dx\Big\}. \tag{50}$$

Now, we are going to estimate the right hand side of (50). Using (31), Hölder's inequality, and Lemma 1, we have

$$M(\lambda(t))\int_{B_1}\left(\beta_0-\beta_1|x|^2\right)p^2 v v_t\,dx$$
$$\le C_1\left(1+|\lambda(t)|^q\right)(\beta_0+\beta_1)\left(\int_{B_1} p^2 v^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_1} p^2|v_t|^2\,dx\right)^{\frac{1}{2}}. \tag{51}$$

From (51) and using Lemma 1, we infer

$$M(\lambda(t))\int_{B_1}\left(\beta_0-\beta_1|x|^2\right)p^2 v v_t\,dx$$
$$\le C_1\left(1+|\lambda(t)|^q\right)(\beta_0+\beta_1)\,C_H\left(\int_{B_1}|\nabla v|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_1} p^2|v_t|^2\,dx\right)^{\frac{1}{2}}. \tag{52}$$

Now, from (31) and Hölder's inequality, we also obtain

$$M(\lambda(t))\int_{B_1}\langle\nabla v, xpv_t\rangle\,dx$$
$$\le C_1\left(1+|\lambda(t)|^q\right)\left(\int_{B_1}|\nabla v|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_1} p^2|v_t|^2\,dx\right)^{\frac{1}{2}}. \tag{53}$$

Using (31), Lemma 1, Gauss' theorem, and Hölder's inequality, we have

$$M(\lambda(t))\int_{B_1}\langle\nabla v_t, xpv\rangle\,dx = -M(\lambda(t))\sum_{i=1}^{n}\int_{B_1} v_t\frac{\partial(x_i pv)}{\partial x_i}\,dx$$
$$= -M(\lambda(t))\int_{B_1} v_t\left(p\langle x,\nabla v\rangle + p^2 v|x|^2 + pv\right)dx$$
$$\le C_1\left(1+|\lambda(t)|^q\right)\int_{B_1}\left(p|v_t||\nabla v| + p^2|v_t||v| + \frac{p^2}{p}|v||v_t|\right)dx$$
$$\le C_1\left(1+|\lambda(t)|^q\right)(1+3C_H)\left(\int_{B_1}|\nabla v|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_1} p^2|v_t|^2\,dx\right)^{\frac{1}{2}}. \tag{54}$$

From (50), (52)–(54), we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\int_{B_1} p^2|v_t|^2\,dx + \overline{M}(\lambda(t))\right] + \delta\int_{B_1} p^2|v_t|^2\,dx$$
$$\leq C_1\left(1+|\lambda(t)|^q\right)C_6\left(\int_{B_1}|\nabla v|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_1} p^2|v_t|^2\,dx\right)^{\frac{1}{2}}, \tag{55}$$

where

$$C_6 = \frac{(N-2)(N+5)C_H}{2} + 2(N-2).$$

On the other hand, multiplying (22) by v, we have

$$\int_{B_1} p^2 v_{tt} v\,dx + M(\lambda(t))\int_{B_1}|\nabla v|^2\,dx + \beta_0 M(\lambda(t))\int_{B_1} p^2 v^2\,dx$$
$$+\delta\int_{B_1} p^2 v v_t\,dx = 0. \tag{56}$$

We observe that

$$\int_{B_1} p^2 v_{tt} v\,dx = \frac{d}{dt}\int_{B_1} p^2 v_t v\,dx - \int_{B_1} p^2|v_t|^2\,dx. \tag{57}$$

From (56) and (57), we infer

$$\frac{d}{dt}\left[\int_{B_1} p^2 v_t v\,dx + \frac{\delta}{2}\int_{B_1} p^2 v^2\,dx\right] - \int_{B_1} p^2|v_t|^2\,dx + M(\lambda(t))\int_{B_1}|\nabla v|^2\,dx$$
$$+\beta_0 M(\lambda(t))\int_{B_1} p^2 v^2\,dx = 0. \tag{58}$$

Multiplying (58) by $\frac{\delta}{4}$, adding the resultant equation with (55), and observing the definition of Ψ (see (34)), we have

$$\frac{1}{2}\frac{d}{dt}\Psi(t) + \frac{3\delta}{4}\int_{B_1} p^2|v_t|^2\,dx + \frac{\delta}{4}M(\lambda(t))\int_{B_1}|\nabla v|^2\,dx + \frac{\delta\beta_0}{4}M(\lambda(t))\int_{B_1} p^2 v^2\,dx$$
$$\leq C_1 C_6\left(1+|\lambda(t)|^q\right)\left(\int_{B_1}|\nabla v|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_1} p^2|v_t|^2\,dx\right)^{\frac{1}{2}}. \tag{59}$$

From the elementary inequality $2ab \leq a^2 + b^2$, we have

$$C_1 C_6\left(\int_{B_1}|\nabla v|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_1} p^2|v_t|^2\,dx\right)^{\frac{1}{2}}$$
$$\leq \frac{C_1^2 C_6^2}{\delta}\int_{B_1}|\nabla v|^2\,dx + \frac{\delta}{4}\int_{B_1} p^2|v_t|^2\,dx \tag{60}$$

and

$$C_1 C_6|\lambda(t)|^q\left(\int_{B_1}|\nabla v|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_1} p^2|v_t|^2\,dx\right)^{\frac{1}{2}}$$

$$\leq \frac{m_0\delta}{8}\int_{B_1}|\nabla v|^2\,dx + \frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0\delta}\int_{B_1}p^2|v_t|^2\,dx. \tag{61}$$

From (59)–(61), we conclude that

$$\frac{1}{2}\frac{d}{dt}\Psi(t) + \frac{1}{\delta}\left(\frac{\delta^2}{2} - \frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0}\right)\int_{B_1}p^2|v_t|^2\,dx$$
$$+\left(\frac{\delta m_0}{8} - \frac{C_1^2C_6^2}{\delta}\right)\int_{B_1}|\nabla v|^2\,dx + \frac{\delta m_0\beta_0}{4}\int_{B_1}p^2v^2\,dx \leq 0.$$

Since $\delta > C_2$, we infer

$$\frac{\delta m_0}{8} - \frac{C_1^2C_6^2}{\delta} > \frac{m_0\delta}{16}.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\Psi(t) + \frac{1}{\delta}\left(\frac{\delta^2}{2} - \frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0}\right)\int_{B_1}p^2|v_t|^2\,dx$$
$$+\frac{\delta m_0}{16}\int_{B_1}|\nabla v|^2\,dx + \frac{\delta m_0\beta_0}{4}\int_{B_1}p^2v^2\,dx \leq 0. \tag{62}$$

As $\lambda(t) \geq 0$, for all $t \in [0, T]$, and $0 < m_0 \leq M(s)$, for all $s \geq 0$, we have

$$\frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0} \leq \frac{2C_1^2C_6^2[\overline{M}(\lambda(t))]^{2q}}{m_0^{2q+1}}, \tag{63}$$

for all $t \in [0, T]$. We observe that

$$-\frac{\delta}{2}\int_{B_1}p^2v_tv\,dx \leq \frac{1}{2}\int_{B_1}p^2|v_t|^2\,dx + \frac{\delta^2}{8}\int_{B_1}p^2v^2\,dx. \tag{64}$$

Thus,

$$\overline{M}(\lambda(t)) \leq \overline{M}(\lambda(t)) + \frac{\delta}{2}\int_{B_1}p^2v_tv\,dx + \frac{1}{2}\int_{B_1}p^2|v_t|^2\,dx + \frac{\delta^2}{8}\int_{B_1}p^2v^2\,dx$$
$$\leq \Psi(t). \tag{65}$$

Therefore, from (63)–(65), we have

$$\frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0} \leq \frac{2C_1^2C_6^2\Psi^{2q}(t)}{m_0^{2q+1}}, \tag{66}$$

for all $t \in [0, T]$. Using (66) and the Assumption 2, we infer

$$\frac{2C_1^2C_6^2|\lambda(0)|^{2q}}{m_0} \leq \frac{2C_1^2C_6^2\Psi^{2q}(0)}{m_0^{2q+1}} < \frac{\delta^2}{4}. \tag{67}$$

Next task is to prove that

$$\frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0} < \frac{\delta^2}{4}, \tag{68}$$

for all $t \in [0, T]$. We suppose that (68) does not hold. From (67) and of the continuity of the function $t \mapsto \frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0}$ there exists $t^* \in (0, T]$ such that

$$\frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0} < \frac{\delta^2}{4}, \tag{69}$$

for all $t \in [0, t^*)$, and

$$\frac{2C_1^2C_6^2|\lambda(t^*)|^{2q}}{m_0} = \frac{\delta^2}{4}. \tag{70}$$

Integrating (62) from 0 to t^*, we have

$$\frac{1}{2}\left[\Psi_m(t^*) - \Psi(0)\right] + \frac{1}{\delta}\int_0^{t^*}\left(\frac{\delta^2}{2} - \frac{2C_1^2C_6^2|\lambda(t)|^{2q}}{m_0}\right)\int_{B_1} p^2|v_t|^2\,dx\,dt \le 0. \tag{71}$$

Combining (69) with (71), we obtain

$$\Psi(t^*) \le \Psi(0). \tag{72}$$

The estimate (63), (65), (72), and the Assumption 2 give us that

$$\frac{2C_1^2C_6^2|\lambda(t^*)|^{2q}}{m_0} \le \frac{2C_1^2C_6^2\Psi^{2q}(0)}{m_0^{2q+1}} < \frac{\delta^2}{4}, \tag{73}$$

which is a contradiction with (70). Thus, (68) holds.

Therefore, (62) and (68) allow us to conclude that (47) holds. □

Proof of Theorem. 1. Observing Nakao's Lemma, it is enough to prove that here exists a positive constant C such that

$$\Psi(t) \le C\,(\Psi(t) - \Psi(t+1)) \tag{74}$$

for all $t \ge 0$.

Thus, let $t \ge 0$ be a fixed real number. To simplify the notation, we define $F^2(t) = \Psi(t) - \Psi(t+1)$. Integrating (47) from t to $t+1$, we have

$$\int_t^{t+1}\int_{B_1} p^2 v_t^2\,dx\,dt \le \frac{2}{\delta}F^2(t), \tag{75}$$

for all $t \ge 0$. Using the mean value theorem for integrals, there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\int_t^{t+\frac{1}{4}}\int_{B_1} p^2 v_t^2\,dx\,dt = \frac{1}{4}\int_{B_1} p^2(x)v_t^2(x,t_1)\,dx \tag{76}$$

and

$$\int_{t+\frac{3}{4}}^{t+1}\int_{B_1} p^2 v_t^2\,dx\,dt = \frac{1}{4}\int_{B_1} p^2(x)v_t^2(x,t_2)\,dx. \tag{77}$$

Thus, (75)–(77) give us

$$\int_{B_1} p^2(x)v_t^2(x,t_1)\,dx + \int_{B_1} p^2(x)v_t^2(x,t_2)\,dx \le \frac{8}{\delta}F^2(t). \tag{78}$$

On the other hand, multiplying (22) by v and integrating over $B_1\times(t_1,t_2)$, we have

$$\begin{aligned}&\int_{t_1}^{t_2} M(\lambda(t))\int_{B_1}\left[|\nabla v|^2+\beta_0 p^2 v^2\right]dx\,dt = \int_{B_1} p^2(x)v_t(x,t_1)v(x,t_1)\,dx\\ &-\int_{B_1} p^2(x)v_t(x,t_2)v(x,t_2)\,dx + \int_{t_1}^{t_2}\int_{B_1} p^2 v_t^2\,dx\,dt - \delta\int_{t_1}^{t_2}\int_{B_1} p^2 v_t v\,dx\,dt.\end{aligned} \tag{79}$$

Using the assumption that M is an increasing function, we obtain

$$\begin{aligned}\overline{M}(\lambda(t)) &= \int_0^{\lambda(t)} M(\xi)\,d\xi \le M(\lambda(t))\lambda(t)\\ &= M(\lambda(t))\left[\int_{B_1}\left(|\nabla v|^2-(N-2)\langle\nabla v, xpv\rangle+\beta_1 p^2v^2|x|^2\right)dx\right]\\ &\le CM(\lambda(t))\int_{B_1}\left(|\nabla v|^2+p^2v^2|x|^2\right)dx\\ &\le CM(\lambda(t))\int_{B_1}\left(|\nabla v|^2+\frac{N(N-2)}{4}p^2v^2\right)dx.\end{aligned} \tag{80}$$

From (79) and (80), we have

$$\begin{aligned}\overline{M}(\lambda(t)) \le C\Bigg(&\int_{B_1} p^2(x)|v_t(x,t_1)||v(x,t_1)|\,dx + \int_{B_1} p^2(x)|v_t(x,t_2)||v(x,t_2)|\,dx\\ &+\int_{t_1}^{t_2}\int_{B_1} p^2v_t^2\,dx\,dt + \delta\int_{t_1}^{t_2}\int_{B_1} p^2|v_t||v|\,dx\,dt\Bigg).\end{aligned} \tag{81}$$

Now, we are going to estimate the right-hand side of (81). Observing (40) and that Ψ is decreasing (see Lemma 3), we obtain

$$\int_{B_1} p^2(x)v^2(x,t_i)\,dx \le \frac{8}{\delta^2}\Psi(t_i) \le \frac{8}{\delta^2}\Psi(t), \tag{82}$$

for $i=1,2$.

Using (78), (82), and the elementary inequality $2ab\le \varepsilon a^2+\frac{b^2}{\varepsilon}$, for each $\varepsilon>0$, we have

$$\int_{B_1} p^2(x)v_t(x,t_i)v(x,t_i)\,dx \le \frac{4}{\varepsilon\delta}F^2(t)+\frac{4\varepsilon}{\delta^2}\Psi(t), \tag{83}$$

for $i = 1, 2$. Using one more time the inequality $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ and (75), we infer

$$\delta \int_{t_1}^{t_2} \int_{B_1} p^2 v_t v \, dx \, dt \leq \frac{\delta}{2\varepsilon} \int_{t_1}^{t_2} \int_{B_1} p^2 v_t^2 \, dx \, dt + \frac{\varepsilon\delta}{2} \sup_{t \leq \xi \leq t+1} \int_{B_1} p^2 v^2 \, dx$$
$$\leq \frac{1}{\varepsilon} F^2(t) + \frac{\varepsilon\delta}{2} \Psi(t). \tag{84}$$

Since Ψ is decreasing, we can use Lemma 3 to conclude that

$$\int_{t_1}^{t_2} \int_{B_1} p^2 v^2 \, dx \, dt \leq \frac{2}{\delta\beta_0 m_0} F^2(t). \tag{85}$$

Integrating (34) from t_1 to t_2, and observing (75), (80), (84), and (85), we have

$$\int_{t_1}^{t_2} \Psi(t) \, dt \leq C(\varepsilon) F^2(t) + \varepsilon \left(\frac{8}{\delta^2} + \frac{3\delta}{4} \right) \Psi(t). \tag{86}$$

Using the mean value theorem for integrals, there exists $\tau^* \in [t_1, t_2]$ such that

$$\frac{1}{2} \Psi(\tau^*) \leq (t_2 - t_1) \Psi(\tau^*) = \int_{t_1}^{t_2} \Psi(t) \, dt. \tag{87}$$

Combining (86) with (87), we infer

$$\Psi(\tau^*) \leq C(\varepsilon) F^2(t) + 2\varepsilon \left(\frac{8}{\delta^2} + \frac{3\delta}{4} \right) \Psi(t). \tag{88}$$

Taking the same way of (50) and (58), we infer

$$\frac{1}{2} \frac{d}{dt} \Psi(t) + \frac{3\delta}{4} \int_{B_1} p^2 v_t^2 \, dx + \frac{\delta}{4} M(\lambda(t)) \int_{B_1} |\nabla v|^2 \, dx$$
$$+ \frac{\delta\beta_0}{4} M(\lambda(t)) \int_{B_1} p^2 v^2 \, dx + M(\lambda(t)) \left\{ \int_{B_1} \left[\left(\frac{N(N-2)}{4} - \beta_1 |x|^2 \right) p^2 v v_t \right. \right.$$
$$\left. \left. + (N-2)(\langle \nabla v, x p v_t \rangle + \langle \nabla v_t, x p v \rangle) \right] dx \right\} = 0. \tag{89}$$

Integrating (89) from t to τ^* and, after this, adding the term

$$\frac{3\delta}{4} \int_t^{\tau^*} \int_{B_1} p^2 v_t^2 \, dx \, d\tau + \frac{\delta}{8} \int_t^{\tau^*} M(\lambda(\tau)) \int_{B_1} |\nabla v|^2 \, dx \, d\tau$$

in both sides of the resultant equation, we have

$$\frac{1}{2} \Psi(t) + \frac{3\delta}{4} \int_t^{\tau^*} \int_{B_1} p^2 v_t^2 \, dx \, d\tau + \frac{\delta}{8} \int_t^{\tau^*} M(\lambda(\tau)) \int_{B_1} |\nabla v|^2 \, dx \, d\tau$$
$$= \frac{1}{2} \Psi(\tau^*) + \frac{3\delta}{2} \int_t^{\tau^*} \int_{B_1} p^2 v_t^2 \, dx \, d\tau + \frac{3\delta}{8} \int_t^{\tau^*} M(\lambda(t)) \int_{B_1} |\nabla v|^2 \, dx \, d\tau$$

$$+\frac{\delta\beta_0}{4}\int_t^{\tau^*} M(\lambda(\tau))\int_{B_1} p^2v^2\,dx\,d\tau + \int_t^{\tau^*} M(\lambda(\tau))\Big\{\int_{B_1}\Big[\Big(\beta_0-\beta_1|x|^2\Big)p^2vv_t$$
$$+(N-2)(\langle\nabla v, xpv_t\rangle + \langle\nabla v_t, xpv\rangle)\Big]dx\Big\}\,d\tau. \tag{90}$$

Analogously to (62), we have

$$\frac{1}{2}\Psi(t)+\frac{1}{\delta}\int_t^{\tau^*}\left(\frac{\delta^2}{2}-\frac{2C_2^2|\lambda(t)|^{2q}}{m_0}\right)\int_{B_1} p^2v_t^2\,dx\,d\tau + \frac{\delta m_0\beta_0}{4}\int_t^{\tau^*}\int_{B_1} p^2v^2\,dx\,d\tau$$
$$\leq \frac{1}{2}\Psi(\tau^*)+\frac{3\delta}{2}\int_t^{\tau^*}\int_{B_1} p^2v_t^2\,dx\,d\tau + \frac{\delta\beta_0}{4}\int_t^{\tau^*} M(\lambda(\tau))\int_{B_1} p^2v^2\,dx\,d\tau$$
$$+\frac{3\delta}{8}\int_t^{\tau^*} M(\lambda(\tau))\int_{B_1}|\nabla v|^2\,dx\,d\tau. \tag{91}$$

Combining (75), (79), (83), (84), (88), and (91), we conclude that

$$\left[\frac{1}{2}-\varepsilon C\right]\Psi(t)\leq CF^2(t), \tag{92}$$

for all $t \geq 0$. Taking $\varepsilon > 0$ small enough, we conclude that (74) holds. Therefore, from Nakao's lemma we obtain that Ψ decay exponentially, i.e., Theorem 1 is proved. □

4. Proof of the existence and uniqueness theorem

We use the Faedo–Galerkin method. Let $(w_j)_{j\in\mathbb{N}}$ be a bases in $H_0^1(B_1)\cap H^2(\Omega)$. For each $m \in \mathbb{N}$, we denote U_m the m-dimensional subspaces spanned by the first m vectors of $(w_j)_{j\in\mathbb{N}}$. Let $T > 0$ be any fixed positive number. For each $m \in \mathbb{N}$, we are looking for a $0 < T_m \leq T$ and $v_m : B_1 \times [0, T_m] \to \mathbb{R}$ such that

$$v_m(x,t)=\sum_{i=1}^m \alpha_{im}(t)w_i(x),$$

and it satisfies the approximate problem

$$(p^2v_m''(t), w_j) + M(\lambda_m(t))\Big[(\nabla v_m(t), \nabla w_j) + \beta_0(p^2v_m(t), w_j)\Big]$$
$$+(\delta p^2v_m'(t), w_j) = 0, \tag{93}$$
$$v_m(0) = v_{0m} = \sum_{i=1}^m v_0^i w_i \to v_0 \text{ in } H_0^1(B_1)\cap H^2(B_1), \tag{94}$$
$$v_m'(0) = v_{1m} = \sum_{i=1}^m v_1^i w_i \to v_1 \text{ in } H_0^1(\Omega), \tag{95}$$

where $' = \frac{d}{dt}$, $1 \leq j \leq m$, v_0^i, v_1^i, $i = 1, \ldots, m$, are known scalars, and

$$\lambda_m(t) = \int_{B_1} \left(|\nabla v_m|^2 - (N-2)\langle \nabla v_m, x p v_m \rangle + \beta_1 p^2 v_m^2 |x|^2 \right) dx,$$

where $\beta_1 = \left(\frac{N-2}{2}\right)^2$. From Ordinary Differential Equations Theory (for instance, see [13]), it is possible to prove that (93)–(95) has a local solution.

From (93) we have the following approximate equation

$$(p^2 v_m''(t), w) + M(\lambda_m(t))[(\nabla v_m(t), \nabla w) + \beta_0 (p^2 v_m(t), w)] \\ +(\delta p^2 v_m'(t), w) = 0, \tag{96}$$

for all $w \in U_m$.

Estimate 1. Initially, it is necessary to observe that

$$\frac{1}{2}\frac{d}{dt}\Psi_m(t) + \frac{\delta}{4}\int_{B_1} p^2 |v_m'|^2\, dx + \frac{\beta_0 m_0 \delta}{4}\int_{B_1} p^2 v_m^2\, dx \leq 0, \tag{97}$$

for all $t \leq T_m$, where

$$\Psi_m(t) = \int_{B_1} p^2 |v_m'|^2\, dV + \overline{M}\,(\lambda_m(t)) + \frac{\delta}{2}\int_{B_1} p^2 v_m' v_m\, dx + \frac{\delta^2}{4}\int_{B_1} p^2 v_m^2\, dx. \tag{98}$$

Indeed, taking in (96) $w = v_m'$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{B_1} p^2 |v_m'|^2\, dx + \frac{1}{2}M(\lambda_m(t))\frac{d}{dt}\left[\int_{B_1} \left(|\nabla v_m|^2 + \beta_0 p^2 v_m^2 \right) dx\right] \\ +\delta \int_{B_1} p^2 |v_m'|^2\, dx = 0. \tag{99}$$

We observe that (99) is similar to (48) with v replaced by v_m. Moreover, (97) is similar to (47). Therefore, to prove that (97) holds, it is enough to follow the steps of the proof of Lemma 3.

On the other hand, observing (64), we have,

$$\Psi_m(t) \geq \frac{1}{2}\int_{B_1} p^2 |v_m'|^2\, dV + \overline{M}\,(\lambda_m(t)) + \frac{\delta^2}{8}\int_{B_1} p^2 v_m^2\, dx. \tag{100}$$

Thus, integrating (97) from 0 to t and observing (100), we have

$$\frac{1}{2}\int_{B_1} p^2 |v_m'|^2\, dV + \overline{M}\,(\lambda_m(t)) + \frac{\delta^2}{8}\int_{B_1} p^2 v_m^2\, dx \leq \Psi_m(0). \tag{101}$$

Therefore,

$$\int_{B_1} p^2 |v_m'|^2\, dV + \overline{M}\,(\lambda_m(t)) + \int_{B_1} p^2 v_m^2\, dx \leq L_1, \tag{102}$$

for all $t \in [0, T_m)$, where $L_1 = \frac{\Psi(0)}{\min\left\{\frac{1}{2}, \frac{\delta^2}{8}\right\}}$, which is the Estimate 1. This estimate is enough to extend the approximate solution to whole $t \geq 0$. This gives us that (102) holds with T_m replaced by $T < \infty$. Thus, we have that

$$(pv_m)_{m\in\mathbb{N}} \quad \text{is bounded in } L^\infty(0, T; L^2(B_1))$$

and

$$(pv_m')_{m\in\mathbb{N}} \quad \text{is bounded in } L^\infty(0, T; L^2(B_1)).$$

Estimate 2. From (102) we can estimate $\|\nabla v_m(t)\|_2$. Indeed, we observe that

$$\begin{aligned}\int_{B_1} |\nabla v_m|^2\, dx &= \lambda_m(t) + \int_{B_1} \Big((N-2)\langle \nabla v_m, xpv_m\rangle - \beta_1 p^2 v_m^2 |x|^2\Big)\, dx \\ &\leq \frac{\overline{M}(\lambda_m(t))}{m_0} + \frac{1}{2}\int_{B_1} |\nabla v_m|^2\, dx + 3\beta_1 \int_{B_1} p^2 v_m^2\, dx. \end{aligned} \tag{103}$$

From (102) and (103), we infer

$$\int_{B_1} |\nabla v_m|^2\, dx \leq 2\left(\frac{1}{m_0} + 3\beta_1\right) L_1 := L_2, \tag{104}$$

for all $t \in [0, T]$. Therefore,

$$(v_m)_{m\in\mathbb{N}} \quad \text{is bounded in } L^\infty(0, T; H_0^1(B_1)).$$

Estimate 3. Multiplying (96) by $\frac{1}{M(\lambda_m(t))}$, differentiating the resultant equation with respect to t, and taking $w = v_m''$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx + \int_{B_1} |\nabla v_m'|^2\, dx + \beta_0 \int_{B_1} p^2 |v_m'|^2\right] \\ &\quad + \delta \int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx \\ &\quad = \frac{1}{2}\int_{B_1} \frac{p^2|v_m''|^2 M'(\lambda_m(t))\lambda_m'(t)}{M^2(\lambda_m(t))}\, dx \\ &\quad + \delta \int_{B_1} \frac{p^2 v_m' v_m'' M'(\lambda_m(t))\lambda_m'(t)}{M^2(\lambda_m(t))}\, dx. \end{aligned} \tag{105}$$

Using Hölder's inequality and Lemma 1, we have

$$\begin{aligned}|\lambda_m'(t)| &= \Big| \int_{B_1} \Big(2\langle \nabla v_m, \nabla v_m'\rangle - (N-2)(\langle xpv_m, \nabla v_m'\rangle \\ &\qquad + \langle xpv_m', \nabla v_m\rangle) + \beta_1 |x|^2 p^2 v_m v_m'\Big)\, dx \Big| \\ &\leq C_7 \left(\int_{B_1} |\nabla v_m|^2\, dx\right)^{\frac{1}{2}} \left(\int_{B_1} |\nabla v_m'|^2\, dx\right)^{\frac{1}{2}}, \end{aligned} \tag{106}$$

where

$$C_7 = 2 + 2C_H(n-2) + C_H^2\beta_1.$$

From this and using Estimate 2, we obtain

$$|\lambda_m'(t)| \le C_7\sqrt{L_2}\left(\int_{B_1} |\nabla v_m'|^2\, dx\right)^{\frac{1}{2}}. \tag{107}$$

Similarly to (68), it is possible to verify that

$$0 \le \lambda_m(t) \le \left(\frac{\delta^2 m_0}{8C_1^2 C_7^2}\right)^{\frac{1}{2q}}, \tag{108}$$

for all $t \in [0, T]$. Thus, observing (108) and the definition of M_1 in (36), we obtain

$$\begin{aligned} &\frac{1}{2}\int_{B_1} \frac{p^2|v_m''|^2 M'(\lambda_m(t))\lambda_m'(t)}{M^2(\lambda_m(t))}\, dx \\ &\quad \le \frac{M_1 C_7\sqrt{L_2}}{2m_0}\left(\int_{B_1} |\nabla v_m'|^2\, dx\right)^{\frac{1}{2}} \int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx \end{aligned} \tag{109}$$

and

$$\begin{aligned} &\delta\int_{B_1} \frac{p^2 v_m' v_m'' M'(\lambda_m(t))\lambda_m'(t)}{M^2(\lambda_m(t))}\, dx \\ &\le \frac{\delta}{8}\int_{B_1} p^2|v_m'|^2\, dx + \frac{2\delta L_2 M_1^2 C_7^2}{m_0^3}\int_{B_1} |\nabla v_m'|^2\, dx \int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx. \end{aligned} \tag{110}$$

Substituting (109) and (110) into (105), we infer

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx + \int_{B_1} |\nabla v_m'|^2\, dx + \beta_0\int_{B_1} p^2|v_m'|^2\right] \\ &\quad +\delta\int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx \\ &\le +\frac{\delta}{8}\int_{B_1} p^2|v_m'|^2\, dx + C_8\left(\int_{B_1} |\nabla v_m'|^2\, dx\right)^{\frac{1}{2}}\int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx \\ &\quad +\delta C_9\int_{B_1} |\nabla v_m'|^2\, dx \int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx, \end{aligned} \tag{111}$$

where

$$C_8 = \frac{M_1\sqrt{L_2}C_6}{2m_0} \quad \text{and} \quad C_9 = \frac{2M_1^2 L_2 C_6^2}{m_0^3}. \tag{112}$$

Adding (97) with (111), we have

$$\frac{1}{2}\frac{d}{dt}\widetilde{\Psi}_m(t) + \frac{\delta}{8}\int_{B_1} p^2|v_m'|^2\, dx + \left(\frac{\delta}{2} - \Lambda_m(t)\right)\int_{B_1} \frac{p^2|v_m''|^2}{M(\lambda_m(t))}\, dx \le 0, \tag{113}$$

where

$$\widetilde{\Psi}_m(t) = \Psi_m(t) + \int_{B_1} \frac{p^2|v''_m|^2}{M(\lambda_m(t))}\,dx + \int_{B_1} |\nabla v'_m|^2\,dx + \beta_0 \int_{B_1} p^2|v'_m|^2\,dx$$

and

$$\Lambda_m(t) = C_8\left(\int_{B_1} |\nabla v'_m|^2\,dx\right)^{\frac{1}{2}} + \delta C_9 \int_{B_1} |\nabla v'_m|^2\,dx.$$

We are going to prove that

$$\Lambda_m(t) < \frac{\delta}{8}, \tag{114}$$

for all $t \in [0, T]$.

We have

$$\Lambda_m(t) \le C_8 \widetilde{\Psi}_m^{\frac{1}{2}}(t) + \delta C_9 \widetilde{\Psi}_m(t), \tag{115}$$

for all $t \in [0, T]$. Using (115) and the Assumption 3, we infer

$$\Lambda_m(0) \le C_8 \widetilde{\Psi}_m^{\frac{1}{2}}(0) + \delta C_9 \widetilde{\Psi}_m(0) < \frac{\delta}{8}. \tag{116}$$

We suppose that (114) does not hold. From (116) and the continuity of the function $t \mapsto \Lambda_m(t)$, there exists $t^* > 0$ such that

$$\Lambda_m(t) < \frac{\delta}{8}, \tag{117}$$

for all $t \in [0, t^*)$ and

$$\Lambda_m(t^*) = \frac{\delta}{8}. \tag{118}$$

Integrating (113) from 0 to t^* and observing (117), we have

$$\widetilde{\Psi}_m(t^*) \le \widetilde{\Psi}_m(0). \tag{119}$$

The estimate (115), (119), and the Assumption 3 give us that

$$\Lambda_m(t^*) \le C_8 \widetilde{\Psi}_m^{\frac{1}{2}}(0) + \delta C_9 \widetilde{\Psi}_m(0) < \frac{\delta}{8}, \tag{120}$$

which is a contradiction with (118). Thus, (114) holds.

Therefore, integrating (113) from 0 to $t < T$, and observing (114), we conclude that

$$\widetilde{\Psi}_m(t) \le \widetilde{\Psi}_m(0) \le L_3, \tag{121}$$

for all $t \in [0, T]$, which is the Estimate 3. Thus,

$$(v'_m)_{m\in\mathbb{N}} \quad \text{is bounded in } L^\infty(0, T; H_0^1(B_1))$$

and

$$(pv''_m)_{m\in\mathbb{N}} \quad \text{is bounded in } L^\infty(0, T; L^2(B_1)).$$

The presence of singularities does not allow to estimate the norm of $(v_m)_{m\in\mathbb{N}}$ in $L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$ as in $\mathbb{R}^N$ case. Thus, the Estimates 1, 2, and 3 are not enough to pass to the limit in approximate equation. To overcome this difficulty it is necessary to make one more estimate.

Estimate 4. Multiplying (96) by $\frac{1}{M(\lambda_m(t))}$, we obtain

$$\frac{(p^2 v''_m(t), w) + (\delta p^2 v'_m(t), w)}{M(\lambda_m(t))} + (\nabla v_m(t), \nabla w) + \beta_0 (p^2 v_m(t), w) = 0. \quad (122)$$

Let m and n be natural numbers such that $m > n$. Defining $z_m = v_m - v_n$, we have

$$\begin{aligned} &\frac{(p^2 z''_m(t), w) + (\delta p^2 z'_m(t), w)}{M(\lambda_m(t))} + (\nabla z_m(t), \nabla w) + \beta_0 (p^2 z_m(t), w) \\ &\quad = \frac{M(\lambda_m(t)) - M(\lambda_n(t))}{M(\lambda_m(t)) M(\lambda_n(t))} \Big[(p^2 v''_m(t), w) + (\delta p^2 v'_m(t), w) \Big]. \end{aligned} \quad (123)$$

Taking $w = z'_m$, we infer

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\int_{B_1} \frac{p^2 |z'_m|^2}{M(\lambda_m(t))}\, dx + \int_{B_1} |\nabla z_m|^2\, dx + \beta_0 \int_{B_1} p^2 |z_m|^2\, dx \right] \\ &\quad + \delta \int_{B_1} \frac{p^2 |z'_m|^2}{M(\lambda_m(t))}\, dx \\ &= -\frac{M'(\lambda_m(t)) \lambda'_m(t)}{2M(\lambda_m(t))} \int_{B_1} \frac{p^2 |z'_m|^2}{M(\lambda_m(t))}\, dx \\ &\quad + \frac{M(\lambda_m(t)) - M(\lambda_n(t))}{M(\lambda_m(t)) M(\lambda_n(t))} \Big[(p^2 v''_m(t), z'_m) + (\delta p^2 v'_m(t), z'_m) \Big]. \end{aligned} \quad (124)$$

Observing the calculus of $\lambda'_m(t)$ in (107) and the Estimates 1, 2, and 3, we have

$$\frac{M'(\lambda_m(t)) \lambda'_m(t)}{2M(\lambda_m(t))} \leq C, \quad (125)$$

for all $m \in \mathbb{N}$ and for all $t \in [0, T]$.

We also observe that

$$
\begin{aligned}
|M(\lambda_m(t)) - M(\lambda_n(t))| &= \left| \int_{\lambda_m(t)}^{\lambda_n(t)} M'(s)\, ds \right| \\
&\leq \max_{0 \leq s \leq 2(L_1+L_2)} \{|M'(s)|\} |\lambda_m(t) - \lambda_n(t)|. \qquad (126)
\end{aligned}
$$

Observing the definition of $\lambda_m(t)$, we have

$$
\begin{aligned}
&|\lambda_m(t) - \lambda_n(t)| \leq C \Big[\int_{B_1} (|\nabla v_m| + |\nabla v_n|) |\nabla z_m|\, dx \\
&+ \beta_1 \int_{B_1} (p|x||v_m| + p|x||v_n|) p|x||z_m|\, dx \\
&+ (N-2) \int_{B_1} |\nabla z_m| p|x||v_m|\, dx + (N-2) \int_{B_1} |\nabla v_n| p|x||z_m|\, dx \Big].
\end{aligned}
$$

From this, using the Hölder inequality, Lemma 1, and the Estimates 1, 2, and 3, we obtain

$$
|\lambda_m(t) - \lambda_n(t)| \leq C \left(\int_{B_1} |\nabla z_m|^2\, dx \right)^{\frac{1}{2}}. \qquad (127)
$$

Combining (126) with (127), we infer

$$
|M(\lambda_m(t)) - M(\lambda_n(t))| \leq C \left(\int_{B_1} |\nabla z_m|^2\, dx \right)^{\frac{1}{2}}. \qquad (128)
$$

From (124), (125), (128), and using Hölder inequality and Lemma 1, and the Estimates 1, 2, and 3, we conclude

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left[\int_{B_1} \frac{p^2 |z_m'|^2}{M(\lambda_m(t))}\, dx + \int_{B_1} |\nabla z_m|^2\, dx + \beta_0 \int_{B_1} p^2 |z_m|^2 \right] \\
&\quad + \delta \int_{B_1} \frac{p^2 |z_m'|^2}{M(\lambda_m(t))}\, dx \\
&\quad \leq C \Big[\int_{B_1} \frac{p^2 |z_m'|^2}{M(\lambda_m(t))}\, dx + \int_{B_1} |\nabla z_m|^2\, dx \\
&\quad + \frac{N(N-2)}{4} \int_{B_1} \frac{p^2 |z_m|^2}{M(\lambda_m(t))}\, dx \Big]. \qquad (129)
\end{aligned}
$$

Using Lemma 1 and Gronwall's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left[\int_{B_1} \frac{p^2 |z_m'|^2}{M(\lambda_m(t))}\, dx + \int_{B_1} |\nabla z_m|^2\, dx + \beta_0 \int_{B_1} p^2 |z_m|^2\, dx \right] \\
&\leq C(T) \left[\int_{B_1} |\nabla z_m'(0)|^2\, dx + \int_{B_1} |\nabla z_m(0)|^2\, dx \right]. \qquad (130)
\end{aligned}
$$

which is the Estimate 4. Therefore, (94), (95), and (130) give us that

$$
\begin{aligned}
&(v_m)_{m\in\mathbb{N}} \quad \text{is a Cauchy sequence in } C^0([0,T]; H_0^1(B_1)),\\
&(pv_m)_{m\in\mathbb{N}} \quad \text{is a Cauchy sequence in } C^0([0,T]; L^2(B_1)),
\end{aligned}
$$

and

$$
(pv'_m)_{m\in\mathbb{N}} \quad \text{is a Cauchy sequence in } C^0([0,T]; L^2(B_1)).
$$

Pass to the limit. Estimates 1–4 yield subsequences, that we still denote in the same way, and a function v such that

$$
v_m \to v \text{ in } C^0([0,T]; H_0^1(B_1)),\ v'_m \overset{*}{\rightharpoonup} v' \text{ in } L^\infty(0,T; H_0^1(B_1)), \quad (131)
$$

$$
pv_m \overset{*}{\rightharpoonup} pv \text{ in } C^0([0,T]; L^2(B_1)),\ pv'_m \to pv' \text{ in } C^0([0,T]; L^2(B_1)), \quad (132)
$$

$$
pv''_m \overset{*}{\rightharpoonup} pv'' \text{ in } L^\infty(0,T; L^2(B_1)). \quad (133)
$$

From (131), (132), and observing the definition of $\lambda_m(t)$, we infer

$$
\lambda_m(\cdot) \to \lambda(\cdot) \text{ in } C^0([0,T]). \quad (134)
$$

This convergence and the continuity of M allow us to conclude that

$$
M(\lambda_m(\cdot)) \to M(\lambda(\cdot)) \text{ in } C^0([0,T]). \quad (135)
$$

The convergences (131)–(133), and (135) are enough to pass to the limit in the approximate equation (96) and to conclude that v is a unique solution of (22)–(24).

□

Summarizing the results of Theorems 1 and 2, we have the following result:

Corollary 1. *Assume that Assumptions 1, 2, and 3 are in place, then there exist a solution v of* (22)–(24) *in the class* (43) *which decay exponentially.*

Data availability statement My manuscript has no associated data.

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Accepted: 5 July 2022