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Averaging of nonlinear Schrödinger equations with time-oscillatory coefficients

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Abstract. In this paper, the limit behavior of solutions for the nonlinear Schrödinger equation $i\partial_t u + \gamma(\omega t)\Delta u + \theta(\omega t)|u|^\alpha u = 0$ in $\mathbb{R}^N$ $(N = 1, 2, 3)$ is studied. Here α is an H^1- subcritical exponent and the coefficients γ, θ are periodic functions. The coefficient γ is further assumed to be one sign, bounded, and bounded away from zero. We prove local and global well-posedness results in H^1 and that the solution u_ω converges as $|\omega| \to \infty$ to the solution of the limiting equation with the same initial condition. Furthermore, we also prove that if the limiting solution is global and has a certain decay property, then u_ω is also global for $|\omega|$ sufficiently large.

1. Introduction

The interest in nonlinear Schrödinger equations with variable coefficients is found in a large number of physical models and their descriptions, for example, see [5,10,12,13] and the references therein. In the paper, we consider the nonlinear Schrödinger equation with time periodic coefficients

$$\begin{cases} i\partial_t u + \gamma(\omega t)\Delta u + \theta(\omega t)|u|^\alpha u = 0, \\ u(0) = \varphi, \end{cases} \tag{1}$$

in $\mathbb{R}^N$, $N = 1, 2, 3$, where

$$\begin{cases} 0 < \alpha < \infty & N = 1, 2, \\ 0 < \alpha < 4 & N = 3, \end{cases} \tag{2}$$

$\omega \in \mathbb{R}$ and γ, θ are τ–periodic functions for some $\tau > 0$. Moreover, we assume that $\theta \in C^1(\mathbb{R})$ and the function γ is one sign, bounded and bounded away from zero on $[0, \tau]$.

As usual, we consider the integral form via Duhamel's formula:

$$u(t) = e^{i\Gamma_\omega(t,0)\Delta}\varphi + i\int_0^t e^{i\Gamma_\omega(t,s)\Delta}\theta(\omega s)|u(s)|^\alpha u(s)\, ds, \tag{3}$$

where $e^{i\Gamma_\omega(t,s)\Delta}$ is the unitary group determined by the associated linear Schrödinger equation, i.e., when $\theta = 0$; see Sect. 2.1 for more details.

Keywords: Nonlinear Schrödinger equation, Global existence.

It is well known that the Cauchy problem (1) when $\gamma = 1$ and $\theta \in L^\infty(\mathbb{R})$ is well-posed in H^1, see [3] for the subcritical and [6] for the critical cases. The standard techniques they used also give us the following fundamental result for our case.

Proposition 1. *Given any $\varphi \in H^1(\mathbb{R}^N)$ and $\omega \in \mathbb{R}$, there exists a unique H^1–solution u of (3) defined on the maximal interval $[0, T_{\max})$ with $0 < T_{\max} \le \infty$. Moreover, the following properties hold:*

- (i) $u \in C([0, T_{\max}), H^1(\mathbb{R}^N)) \cap L^q_{\mathrm{loc}}((0, T_{\max}), W^{1,r}(\mathbb{R}^N))$ *for all admissible pair* (q, r).
- (ii) *(Blow-up alternative) If $T_{\max} < \infty$, then $\|u(t)\|_{H^1(\mathbb{R}^N)} \to \infty$ as $t \uparrow T_{\max}$.*
- (iii) *If $\alpha < 4/N$, then the solution u is global, i.e., $T_{\max} = \infty$.*

The main purpose is to study the behavior of solutions u_ω for (1) as $|\omega| \to \infty$. Since γ and θ are periodic, we expect it to be close to the solution of the limiting equation

$$\begin{cases} i\partial_t U + I(\gamma)\Delta U + I(\theta)|U|^\alpha U = 0, \\ U(0) = \varphi, \end{cases} \tag{4}$$

or its equivalent integral form

$$U(t) = e^{iI(\gamma)t\Delta}\varphi + i\int_0^t e^{iI(\gamma)(t-s)\Delta} I(\theta)|U(s)|^\alpha U(s)\, ds, \tag{5}$$

where $I(\gamma)$ and $I(\theta)$ are averages of γ and θ, respectively, i.e.,

$$I(\gamma) = \frac{1}{\tau}\int_0^\tau \gamma(s)\, ds \quad \text{and} \quad I(\theta) = \frac{1}{\tau}\int_0^\tau \theta(s)\, ds. \tag{6}$$

The existence of the maximal solution U for the Cauchy problem (4) or (5) has been extensively studied, e.g., [2]. So we investigate that our expectation is true on the maximal interval in which solution U exists. In the following theorem, we state our main consequences.

Theorem 1. *Fix an initial value $\varphi \in H^1(\mathbb{R}^N)$. Given $\omega \in \mathbb{R}$, denote by u_ω the maximal solution of (3). Let U be the solution of (5) defined on the maximal interval $[0, S_{\max})$.*

- (i) *For each $0 < S < S_{\max}$, the solution u_ω exists on $[0, S]$ provided that $|\omega|$ is sufficiently large.*
- (ii) *u_ω converges to U in $L^\infty((0, S), H^1(\mathbb{R}^N))$ as $|\omega| \to \infty$.*

Remark 1. The averaging theorem of NLS has widely been studied considering various forms of the time-dependent coefficients. In [1], the authors consider in the case of $\theta = 1$ and the fast dispersion management γ of the form $\gamma(t/\varepsilon)$, where γ is given by 2–periodic and piecewise constant, a typical example being $\gamma = 1$ on the interval $[0, 1)$ and $\gamma = -1$ on the interval $[1, 2)$. Moreover, they proved the scaling limit of fast dispersion management and the convergence in H^2 to an effective model with averaged dispersion. In [5,13] an Eq. (1) with the strong dispersion management γ of the form $\varepsilon^{-1}\gamma(t/\varepsilon)$ and lumped amplification was studied in dimension $N = 1$, which is closely related to a physical phenomenon. In contrast, the averaging theorem for $\gamma = 1$ were obtained by Cazenave and Scialom [3] .

If $\alpha \geq 4/N$ and $S_{\max} = \infty$, one may question whether u_ω is also global for $|\omega|$ sufficiently large. The following theorem gives us an affirmative answer under the condition that U has suitable decay as $t \to \infty$. Moreover, the convergence holds globally in time.

Theorem 2. *Assume (2) and further that $\alpha \geq 4/N$. Let r and a be defined by*

$$r = \alpha + 2 \quad \text{and} \quad a = \frac{2\alpha(\alpha+2)}{4-(N-2)\alpha}.$$

Fix an initial value $\varphi \in H^1(\mathbb{R}^N)$. Given $\omega \in \mathbb{R}$, denote by u_ω the maximal solution of (3). Let U be the solution of (5) defined on the maximal interval $[0, S_{\max})$. Suppose that

$$S_{\max} = \infty \quad \text{and} \quad U \in L^a((0,\infty), L^r(\mathbb{R}^N)). \tag{7}$$

Then u_ω is global for $|\omega|$ sufficiently large. Moreover, u_ω converges to U in $L^\infty((0,\infty), H^1(\mathbb{R}^N))$ as $|\omega| \to \infty$.

The existence of solutions satisfying (7) is guaranteed by the scattering theory (the details can be referred in [2,7,11]). Thus by applying Theorem 2, we obtain the following.

Corollary 1. *Assume (2). Fix an initial value $\varphi \in H^1(\mathbb{R}^N)$, let U be the maximal solution of (5). Given $\omega \in \mathbb{R}$, denote by u_ω the maximal solution of (3). If one of the following conditions is satisfied,*

(i) $I(\gamma)I(\theta) < 0$ *and* $\alpha > 4/N$
(ii) $I(\theta) = 0$ *and* $\alpha \geq 4/N$
(iii) $I(\gamma)I(\theta) > 0$, $\alpha \geq 4/N$ *and* $\|\varphi\|_{\dot{H}^s}$ *is sufficiently small, where* $s = (N\alpha - 4)/2\alpha \in [0, 1)$,

then it follows that the solution u_ω of (3) is global for $|\omega|$ sufficiently large. Moreover, u_ω converges to U in $L^\infty((0,\infty), H^1(\mathbb{R}^N))$ as $|\omega| \to \infty$.

Note that in case $I(\theta) = 0$, i.e., linear equation, $U(t) = e^{iI(\gamma)t\Delta}\varphi$. Using the change of variables $V(t,x) = U(t/I(\gamma), x)$, V solves

$$i\partial_t V + \Delta V + \frac{I(\theta)}{I(\gamma)}|V|^\alpha V = 0 \tag{8}$$

with the initial value $V(0) = \varphi$. The behavior of (8) is focusing or defocusing which depend only the sign of $I(\theta)/I(\gamma)$. Thus, we refer to defocusing equation when $I(\gamma)I(\theta) < 0$, otherwise we refer to focusing equation.

Notation. *We use $C > 0$ to denote various constants. For $1 \le r, q \le \infty$, the norm of mixed space $L^r(I, L^q(\mathbb{R}^N))$ is denoted by $\|\cdot\|_{L^r(I,L^q)}$.*

The paper is organized as follows: In Sect. 2, we establish some preliminaries and lemmas and derive the well-posedness results. In Sect. 3, we give the proof of Theorem 1. Finally, the proof of Theorem 2 is devoted to Sect. 4.

2. Preliminaries and well-posedness results

2.1. The linear propagator

Before proving Proposition 1, we collect some properties for the propagator associated with the linear Schrödinger equation

$$\begin{cases} i\partial_t u_{\mathrm{lin}} + \gamma(\omega t)\Delta u_{\mathrm{lin}} = 0, \\ u_{\mathrm{lin}}(0) = f, \end{cases} \tag{9}$$

for all $\omega \in \mathbb{R}$, where the τ–periodic function γ satisfies our assumptions. Here and below, we denote by

$$\Gamma_\omega(t,s) := \int_s^t \gamma(\omega t')\,dt' = \frac{1}{\omega}\int_{\omega s}^{\omega t} \gamma(t')\,dt' \tag{10}$$

for all $s, t \in \mathbb{R}$. One can express the associated propagator $e^{i\Gamma_\omega(t,0)\Delta}$ that describes the solution $u_{\mathrm{lin}}(x,t)$ for (9) as

$$e^{i\Gamma_\omega(t,0)\Delta} f(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i|\xi|^2\Gamma_\omega(t,0)} e^{ix\cdot\xi} \widehat{f}(\xi)\,d\xi \tag{11}$$

for $f \in L^2(\mathbb{R}^N)$, where $\widehat{f}$ denotes the Fourier transform of $f \in L^2(\mathbb{R}^N)$. We now define the operator $e^{i\Gamma_\omega(t,s)\Delta}$ by

$$e^{i\Gamma_\omega(t,s)\Delta} := e^{i\Gamma_\omega(t,0)\Delta}\, e^{-i\Gamma_\omega(s,0)\Delta}$$

on $L^2(\mathbb{R}^N)$. Then, fixed $s \in \mathbb{R}$, it is a unitary operator on $L^2(\mathbb{R}^N)$ also on $H^1(\mathbb{R}^N)$ satisfying

$$\|e^{i\Gamma_\omega(t,s)\Delta} f\|_{L^2} = \|f\|_{L^2} \quad \text{and} \quad \|e^{i\Gamma_\omega(t,s)\Delta} f\|_{H^1} = \|f\|_{H^1}$$

for every $\omega \in \mathbb{R}$. Moreover, fixed $s \in \mathbb{R}$, it follows from (11) that the mapping $t \mapsto e^{i\Gamma_\omega(t,s)\Delta} f$ is continuous for every $f \in L^2(\mathbb{R}^N)$.

From our assumption of γ, it follows that for any $s, t \in \mathbb{R}$, there exists $C > 0$ such that

$$\left| \int_s^t \gamma(\tau)\,d\tau \right| \ge C|t-s|,$$

which allows us to obtain the following result.

Lemma 1. *Let $\omega \in \mathbb{R}$. There exists a constant C independent of ω such that if $s \neq t$, then*

$$\|e^{i\Gamma_\omega(t,s)\Delta} f\|_{L^\infty} \le \frac{C}{|t-s|^{N/2}} \|f\|_{L^1}$$

for any $f \in L^1(\mathbb{R}^N)$.

Proof. Using the explicit form of the solution operator for the free Schrödinger equation

$$e^{it\Delta} f(x) = \frac{1}{(4i\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{i\frac{|x-y|^2}{4t}} f(y)\,dy, \quad t \neq 0,$$

we obtain that

$$\|e^{i\Gamma_\omega(t,s)\Delta} f\|_{L^\infty} \le \frac{1}{(4\pi|\Gamma_\omega(t,s)|)^{N/2}} \|f\|_{L^1}. \tag{12}$$

Note that since γ is one sign and bounded away from zero, we have

$$|\Gamma_\omega(t,s)| = \left|\frac{1}{\omega}\int_{\omega s}^{\omega t} \gamma(t')\,dt'\right| \ge C|t-s|.$$

This together with (12) completes the proof of Lemma 1. □

Observe that the usual Strichartz estimates hold for the semigroup $e^{i\Gamma_\omega(t,0)\Delta}$. To this end, for any $1 \le p \le \infty$, let p' be the Hölder conjugate, that is, $1/p + 1/p' = 1$, and a pair of exponents (q, r) is said to be admissible if

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r} \quad \text{and} \quad \begin{cases} 2 \le r \le \infty & N = 1, \\ 2 \le r < \infty & N = 2, \\ 2 \le r \le 6 & N = 3. \end{cases}$$

Using Lemma 1, we can show the following standard Strichartz estimates with an argument similar to that of, for example, [2] and [9]. So we omit the details of the proof.

Lemma 2. *(Strichartz's estimates) Let (q, r) and (q_0, r_0) be admissible pairs. For any $\omega \in \mathbb{R}$, the following properties hold:*

(i) *For every $f \in L^2(\mathbb{R}^N)$, the map $t \mapsto e^{i\Gamma_\omega(t,0)\Delta} f$ belongs to $L^q(\mathbb{R}, L^r(\mathbb{R}^N)) \cap C(\mathbb{R}, L^2(\mathbb{R}^N))$. Furthermore, there exists a constant C independent of ω such that*

$$\|e^{i\Gamma_\omega(\cdot,0)\Delta} f\|_{L^q(\mathbb{R},L^r)} \le C\|f\|_{L^2}.$$

(ii) *Let I be an interval of $\mathbb{R}$. For every $F \in L^{q_0'}(I, L^{r_0'}(\mathbb{R}^N))$, the map*

$$t \mapsto \int_I e^{i\Gamma_\omega(t,\tau)\Delta} F(\cdot,\tau)\,d\tau \quad \textit{for} \quad t \in I,$$

belongs to $L^q(I, L^r(\mathbb{R}^N)) \cap C(\overline{I}, L^2(\mathbb{R}^N))$. Furthermore, there exists a constant C independent of ω such that

$$\left\|\int_I e^{i\Gamma_\omega(\cdot,\tau)\Delta} F(\cdot,\tau)\,d\tau\right\|_{L^q(I,L^r)} \le C\|F\|_{L^{q_0'}(I,L^{r_0'})}.$$

2.2. Well-posedness results

This subsection concentrates on proving the existence and uniqueness of solutions for (1), i.e., Proposition 1. For any $\omega \in \mathbb{R}$, we consider the integral equation

$$u_\omega(t) = e^{i\Gamma_\omega(t,0)\Delta}\varphi + i\int_0^t e^{i\Gamma_\omega(t,s)\Delta}\theta(\omega s)|u_\omega(s)|^\alpha u_\omega(s)\,ds. \tag{13}$$

Recall that $\theta \in C^1(\mathbb{R})$ and $\Gamma_\omega(t, s)$ is given by (10). For this subsection, we only need to assume $\theta \in L^\infty(\mathbb{R})$ which is slightly more general that (3).

We start with the local well-posedness of (13). Based on Strichartz's estimate mentioned in (2), the well-posedness results are quite standard, see, for example, [2,8]. In fact, the proof of in the case $\gamma = 1$ can be found in [3]. For brevity we only state the results without detailed proofs.

Proposition 2. *Assume* (2).

(i) *Given* $A, M > 0$, *there exists* $T = T(A, M) > 0$ *such that if* $\|\theta\|_{L^\infty} \le A$ *and if* $\varphi \in H^1(\mathbb{R}^N)$ *satisfying* $\|\varphi\|_{H^1} \le M$, *then for any* $\omega \in \mathbb{R}$, *there exists a unique local solution* $u_\omega \in C([0, T], H^1(\mathbb{R}^N))$ *of* (13). *In addition,*

$$\|u_\omega\|_{L^q((0,T),W^{1,r})} \le 2C\|\varphi\|_{H^1}$$

for all admissible pair (q, r).

(ii) *Assume further that* $\alpha < 4/N$. *Given* $A, M' > 0$, *there exists* $T' = T'(A, M') > 0$ *such that if* $\|\theta\|_{L^\infty} \le A$ *and if* $\varphi \in L^2(\mathbb{R}^N)$ *satisfying* $\|\varphi\|_{L^2} \le M'$, *then for any* $\omega \in \mathbb{R}$, *there exists a unique local solution* $u_\omega \in C([0, T'], L^2(\mathbb{R}^N))$ *of* (13).

Remark 2. (i) Fix an initial value $\varphi \in H^1(\mathbb{R}^N)$. Given $\omega \in \mathbb{R}$, the solution u_ω of (13) obtained in Proposition 2 can be extended to a maximal interval $[0, T_{\max}(\omega))$. Moreover, we have the blowup alternative holds: If $T_{\max}(\omega) < \infty$, then

$$\lim_{t \to T_{\max}(\omega)} \|u_\omega(t)\|_{H^1} = \infty.$$

(ii) Arguing as in the case of constant coefficients, one can show that the mass is conserved, that is,

$$\|u_\omega(t)\|_{L^2} = \|\varphi\|_{L^2}$$

for all $0 \le t < T_{\max}(\omega)$. However, in our case, the energy is neither conserved nor decreasing.

(iii) Suppose $\alpha < 4/N$. From Proposition 2 (ii), we know that the local existence time T' depends on the L^2 norm of the initial value. It follows from the conservation of mass that the L^2–solution u_ω is globally defined for each $\omega \in \mathbb{R}$.

Proof of Proposition 1. The existence and uniqueness of the local H^1–solution of (13) follow from Proposition 2 (i). The maximal existence time and the blowup alternative are a consequence of Remark 2 (i), moreover u is in $L^q_{\text{loc}}((0, T_{\max}), W^{1,r}(\mathbb{R}^N))$ for all admissible pair. If $\alpha < 4/N$, then we can establish H^1 regularity of the global L^2–solution, see Theorem 5.2.2 in [2] for details. Thus, we obtain $u \in C([0,\infty), H^1(\mathbb{R}^N))$. □

We have the following results, which are the same as [3, Proposition 2.3] and [3, Corollary 2.4]. For proofs, the reader can consult, for example, [3, Proposition 2.3 and Corollary 2.4] and [4, Propositions 2.3 and 2.4].

Proposition 3. *Assume* (2) *and suppose further that* $\alpha \geq 4/N$. *Let* r, q, *and* a *be defined by*

$$r = \alpha + 2, \quad q = \frac{4(\alpha+2)}{N\alpha}, \quad a = \frac{2\alpha(\alpha+2)}{4-(N-2)\alpha}. \tag{14}$$

Given any $A > 0$, *there exists* $\varepsilon = \varepsilon(A)$ *and* Λ *such that for any* $\omega \in \mathbb{R}$, *if* $\|\theta\|_{L^\infty} < A$ *and if* $\varphi \in H^1(\mathbb{R}^N)$ *satisfies*

$$\|e^{i\Gamma_\omega(\cdot,0)\Delta}\varphi\|_{L^a((0,\infty),L^r)} \leq \varepsilon,$$

then the corresponding solution u_ω *of* (13) *is global and satisfies*

$$\|u_\omega\|_{L^a((0,\infty),L^r)} \leq 2\|e^{i\Gamma_\omega(\cdot,0)\Delta}\varphi\|_{L^a((0,\infty),L^r)}$$

and

$$\|u_\omega\|_{L^q((0,\infty),W^{1,r})} + \|u_\omega\|_{L^\infty((0,\infty),H^1)} \leq \Lambda\|\varphi\|_{H^1}.$$

Conversely, if the solution u_ω *of* (13) *is global and satisfies*

$$\|u_\omega\|_{L^a((0,\infty),L^r)} \leq \varepsilon,$$

then

$$\|e^{i\Gamma_\omega(\cdot,0)\Delta}\varphi\|_{L^a((0,\infty),L^r)} \leq 2\|u_\omega\|_{L^a((0,\infty),L^r)}.$$

Corollary 2. *Assume* (2) *and* $\alpha \geq 4/N$. *Let* r, q, *and* a *be defined by* (14). *Let* $A > 0$ *and consider* $\varepsilon = \varepsilon(A)$ *and* Λ *as in Proposition 3. Given* $\varphi \in H^1(\mathbb{R}^N)$ *and* $\|\theta\|_{L^\infty} \leq A$, *let* u_ω *be the corresponding solution of* (13) *defined on the maximal interval* $[0, T_{\max})$. *If there exists* $0 < T < T_{\max}$ *such that*

$$\|e^{i\Gamma_\omega(0,\cdot)\Delta}u_\omega(T)\|_{L^a((0,\infty),L^r)} \leq \varepsilon,$$

then the solution u_ω *is global, i.e.,* $T_{\max} = \infty$. *Moreover,*

$$\|u_\omega\|_{L^a((T,\infty),L^r)} \leq 2\varepsilon \quad \textit{and} \quad \|u_\omega\|_{L^q((T,\infty),W^{1,r})} \leq \Lambda\|\varphi\|_{H^1}.$$

3. Proof of Theorem 1

The following lemmas below play a key role in our proof of the convergence result stated in Theorem 1. Similar results are considered also in [1,3].

Lemma 3. *If* $g \in L^1((0,L), H^1(\mathbb{R}^N))$ *for some* $0 < L \leq \infty$*, then*

$$\int_0^t \theta(\omega s) e^{i\Gamma_\omega(t,s)\Delta} g(s)\, ds \underset{|\omega|\to\infty}{\longrightarrow} I(\theta) \int_0^t e^{i\Gamma_\omega(t,s)\Delta} g(s)\, ds \tag{15}$$

in $L^\infty((0,L), H^1(\mathbb{R}^N))$.

Proof. Set

$$\psi(t) = \theta(t) - I(\theta) \quad \text{and} \quad \Psi(t) = \int_0^t \psi(t')\, dt'.$$

Since θ is τ–periodic, Ψ is also τ–periodic, therefore, $\|\Psi\|_{L^\infty} < \infty$. Using Minkowski's inequality and the fact that the operator $e^{i\Gamma_\omega(\cdot,\cdot)\Delta}$ is unitary, it follows that

$$\left\| \int_0^\cdot \psi(\omega s) e^{i\Gamma_\omega(\cdot,s)\Delta} f(s)\, ds \right\|_{L^\infty((0,L),H^1)} \leq C \|\psi\|_{L^\infty} \|g\|_{L^1((0,L),H^1)}$$

for every $g \in L^1((0,L), H^1(\mathbb{R}^N))$. Therefore, by density, we only need to prove (15) for $g \in C_c^1((0,L), \mathcal{S}(\mathbb{R}^N))$. Since $\frac{d}{ds}\Psi(\omega s) = \omega\psi(\omega s)$, an integration by parts shows that

$$\begin{aligned}\int_0^t \psi(\omega s) e^{i\Gamma_\omega(t,s)\Delta} g(s)\, ds &= \frac{1}{\omega}\Psi(\omega t) g(t) \\ &\quad - \frac{1}{\omega}\int_0^t \Psi(\omega s) e^{i\Gamma_\omega(t,s)\Delta}\Big[g_t(s) - i\gamma(\omega s)\Delta g(s)\Big]\, ds.\end{aligned}$$

Since γ is bounded, we see that

$$\begin{aligned}&\left\| \frac{1}{\omega}\int_0^\cdot \Psi(\omega s) e^{i\Gamma_\omega(\cdot,s)\Delta}\Big[g_t(s) - i\gamma(\omega s)\Delta g(s)\Big]\, ds \right\|_{L^\infty((0,L),H^1)} \\ &\quad\leq \frac{1}{|\omega|}\|\Psi\|_{L^\infty}\|g_t(s) - i\gamma(\omega s)\Delta g(s)\|_{L^1((0,L),H^1)} \\ &\quad\leq \frac{C}{|\omega|}\|\Psi\|_{L^\infty}\big(\|g_t\|_{L^1((0,L),H^1)} + \|\Delta g\|_{L^1((0,L),H^1)}\big),\end{aligned}$$

where the constant C is independent of ω. This yields

$$\begin{aligned}&\left\| \int_0^\cdot \psi(\omega s) e^{i\Gamma_\omega(\cdot,s)\Delta} g(s)\, ds \right\|_{L^\infty((0,L),H^1)} \\ &\quad\leq \frac{C}{|\omega|}\|\Psi\|_{L^\infty}\left(\sup_{t\in(0,L)}\|g(t)\|_{H^1} + \|g_t\|_{L^1((0,L),H^1)} + \|\Delta g\|_{L^1((0,L),H^1)}\right).\end{aligned}$$

Letting $|\omega| \to \infty$, we obtain the desired convergence, which completes the proof of Lemma 3. □

Lemma 4. *If $f \in H^1(\mathbb{R}^N)$ for some $0 < L \le \infty$, then for a fixed $s \in [0, L)$, we have*

$$\sup_{t\in(0,L)} \left\| \left(e^{i\Gamma_\omega(t,s)\Delta} - e^{iI(\gamma)(t-s)\Delta} \right) f \right\|_{H^1} \underset{|\omega|\to\infty}{\longrightarrow} 0. \tag{16}$$

Proof. Since γ is τ–periodic, we can decompose Γ_ω as

$$\Gamma_\omega(t,s) = I(\gamma)(t-s) + \frac{1}{\omega}\int_{\omega s}^{\omega t} \gamma_0(t')\,dt'$$

for every $s, t \in \mathbb{R}$, where $I(\gamma) \in \mathbb{R} - \{0\}$ denotes the average defined by (6) and γ_0 is a τ–periodic function with mean zero. Denote by

$$\vartheta_\omega(t,s) = \int_{\omega s}^{\omega t} \gamma_0(t')\,dt',$$

since

$$\left| \int_s^t \gamma_0(t')\,dt' \right| \le \tau(M - I(\gamma))$$

we obtain that $\vartheta_\omega \in L^\infty(\mathbb{R}^2)$ uniformly. Hence, using Plancherel's identity and Minkowski's inequality, we have

$$\begin{aligned}
&\sup_{t\in(0,L)} \left\| \left(e^{i\Gamma_\omega(t,s)\Delta} - e^{iI(\gamma)(t-s)\Delta} \right) f \right\|_{H^1}^2 \\
&\quad = \sup_{t\in(0,L)} \int_{\mathbb{R}^N} (1+|\xi|^2) \left| e^{iI(\gamma)(t-s)|\xi|^2} (e^{i\frac{1}{\omega}\vartheta_\omega(t,s)|\xi|^2} - 1) \right|^2 |\widehat{f}(\xi)|^2\,d\xi \\
&\quad \le \int_{\mathbb{R}^N} (1+|\xi|^2)|\widehat{f}(\xi)|^2 \sup_{t\in(0,L)} \left| e^{i\frac{1}{\omega}\vartheta_\omega(t,s)|\xi|^2} - 1 \right|^2 d\xi
\end{aligned}$$

for a fixed $s \in [0, L)$. Thus (16) follows from the Lebesgue dominated convergence theorem. □

Lemma 5. *If $g \in L^1((0, L), H^1(\mathbb{R}^N))$ for some $0 < L \le \infty$, then*

$$\sup_{t\in(0,L)} \left\| \int_0^t \left(e^{i\Gamma_\omega(t,s)\Delta} - e^{iI(\gamma)(t-s)\Delta} \right) g(s)\,ds \right\|_{H^1} \underset{|\omega|\to\infty}{\longrightarrow} 0.$$

Proof. Since $g(s) \in H^1(\mathbb{R}^N)$, it follows from Lemma 4 that

$$h_\omega(s) := \sup_{t\in(0,L)} \left\| \left(e^{i\Gamma_\omega(t,s)\Delta} - e^{I(\gamma)(t-s)\Delta} \right) g(s) \right\|_{H^1} \underset{|\omega|\to\infty}{\longrightarrow} 0.$$

Using Minkowski's inequality, we get

$$\sup_{t\in(0,L)} \left\| \int_0^t \left(e^{i\Gamma_\omega(t,s)\Delta} - e^{iI(\gamma)(t-s)\Delta} \right) g(s)\,ds \right\|_{H^1} \le \int_0^L h_\omega(s)\,ds \underset{|\omega|\to\infty}{\longrightarrow} 0$$

because the Lebegsue dominated convergence theorem with the fact that

$$h_\omega \le C\|g(\cdot)\|_{H^1} \in L^1(0, L).$$

Recall the following Gronwall-type estimate whose proof can be found in [3, Lemma A.1] □

Lemma 6. *Assume that* $0 < T < \infty$, $1 \le p < q \le \infty$, *and* $A, B \ge 0$. *If* $f \in L^q(0, T)$ *satisfies*

$$\|f\|_{L^q(0,t)} \le A + B\|f\|_{L^p(0,t)}$$

for all $0 < t < T$, *then there exists a constant* $K = K(B, p, q, T)$ *such that*

$$\|f\|_{L^q(0,T)} \le AK.$$

For the proof of Theorem 1, we introduce the special admissible pairs (q, r) such that

$$\begin{cases} q = \alpha + 4, \quad r = \dfrac{2N(\alpha+4)}{N(\alpha+4)-4} & \text{if } N = 1, 2 \\ q = \dfrac{\alpha+4}{2}, \quad r = \dfrac{6(\alpha+4)}{3(\alpha+4)-8} & \text{if } N = 3. \end{cases} \tag{17}$$

Then since $\alpha < q$ and $N < r$, it follows from the Sobolev embedding theorem that

$$L^q((0, L), W^{1,r}(\mathbb{R}^N)) \hookrightarrow L^q((0, L), L^\infty(\mathbb{R}^N)). \tag{18}$$

Key for our proof of Theorem 1 is the following lemma.

Lemma 7. *Assume* (2). *Fix an initial value* $\varphi \in H^1(\mathbb{R}^N)$, *and given* $\omega \in \mathbb{R}$, *denote by* u_ω *the maximal solution of* (3). *Let* U *be the maximal solution of* (5) *defined on the interval* $[0, S_{\max})$. *For* $0 < L < S_{\max}$, *we assume that* u_ω *exists on* $[0, L]$ *for* $|\omega|$ *sufficiently large and*

$$\limsup_{|\omega|\to\infty} \|u_\omega\|_{L^\infty((0,L),H^1)} < \infty \tag{19}$$

and

$$\limsup_{|\omega|\to\infty} \|u_\omega\|_{L^q((0,L),W^{1,r})} < \infty$$

where (q, r) *is given by* (17). *Then it follows that*

$$\|u_\omega - U\|_{L^\infty((0,L),H^1)} \underset{|\omega|\to\infty}{\longrightarrow} 0.$$

Proof. From (3) and (5), we have

$$u_\omega(t) - U(t) = \left(e^{i\Gamma_\omega(t,0)\Delta} - e^{iI(\gamma)t\Delta}\right)\varphi + i\left(\mathcal{I}_1(t) + \mathcal{I}_2(t) + \mathcal{I}_3(t)\right), \tag{20}$$

where

$$\mathcal{I}_1(t) = \int_0^t e^{i\Gamma_\omega(t,s)\Delta}\theta(\omega s)\Big(|u_\omega(s)|^\alpha u_\omega(s) - |U(s)|^\alpha U(s)\Big)\,ds,$$
$$\mathcal{I}_2(t) = \int_0^t e^{i\Gamma_\omega(t,s)\Delta}\Big(\theta(\omega s) - I(\theta)\Big)|U(s)|^\alpha U(s)\,ds,$$
$$\mathcal{I}_3(t) = \int_0^t \Big(e^{i\Gamma_\omega(t,s)\Delta} - e^{iI(\gamma)(t-s)\Delta}\Big)I(\theta)|U(s)|^\alpha U(s)\,ds.$$

For the first term on the right hand side of (20), it follows from Lemma 4 that

$$\sup_{t\in(0,L)} \left\| \Big(e^{i\Gamma_\omega(t,0)\Delta} - e^{iI(\gamma)t\Delta}\Big)\varphi \right\|_{H^1} \underset{|\omega|\to\infty}{\longrightarrow} 0. \tag{21}$$

Observe that $|U|^\alpha U \in L^1((0,L), H^1(\mathbb{R}^N))$. Indeed, using Hölder's inequality and (18), we see that

$$\begin{aligned}\int_0^L \||U(s)|^\alpha U(s)\|_{H^1}\,ds &\le \int_0^L \|U(s)\|_{L^\infty}^\alpha \|U(s)\|_{H^1}\,ds\\ &\le \|U\|_{L^q((0,L),L^\infty)}^\alpha \|U\|_{L^{\frac{q}{q-\alpha}}((0,L),H^1)}\\ &\le C\|U\|_{L^q((0,L),W^{1,r})}^\alpha \|U\|_{L^\infty((0,L),H^1)}.\end{aligned}$$

Thus Lemmas 3 and 5 imply that

$$\|\mathcal{I}_2\|_{L^\infty((0,L),H^1)} + \|\mathcal{I}_3\|_{L^\infty((0,L),H^1)} \underset{|\omega|\to\infty}{\longrightarrow} 0. \tag{22}$$

We now estimate $\mathcal{I}_1$ to show $L^\infty L^2$–convergence. Denote the nonlinearity by $g(u) = |u|^\alpha u$ for simplicity. Recall that for all $u, v \in \mathbb{C}$, it holds

$$|g(u) - g(v)| \le C\left(|u|^\alpha + |v|^\alpha\right)|u - v|.$$

Applying the Hölder inequality in both space and time together with the Sobolev embedding (18), we see that

$$\begin{aligned}&\|g(u_\omega) - g(U)\|_{L^1((0,t),L^2)}\\ &\quad\le C\left(\|u_\omega\|_{L^q((0,t),L^\infty)}^\alpha + \|U\|_{L^q((0,t),L^\infty)}^\alpha\right)\|u_\omega - U\|_{L^{\frac{q}{q-\alpha}}((0,t),L^2)}\\ &\quad\le C\left(\|u_\omega\|_{L^q((0,t),W^{1,r})}^\alpha + \|U\|_{L^q((0,t),W^{1,r})}^\alpha\right)\|u_\omega - U\|_{L^{\frac{q}{q-\alpha}}((0,t),L^2)}\end{aligned}$$

for all $0 < t \le L$. With this we can estimate $\mathcal{I}_1$, using Strichartz's estimate, via

$$\begin{aligned}\|\mathcal{I}_1\|_{L^\infty((0,t),L^2)} &\le C\|g(u_\omega) - g(U)\|_{L^1((0,t),L^2)}\\ &\le C\big(\|u_\omega\|_{L^q((0,t),W^{1,r})}^\alpha + \|U\|_{L^q((0,t),W^{1,r})}^\alpha\big)\|u_\omega\\ &\quad - U\|_{L^{\frac{q}{q-\alpha}}((0,t),L^2)}\end{aligned} \tag{23}$$

for all $0 < t \le L$. From (21), (23), and (22) there exists a $\varepsilon_\omega > 0$ and a constant $C > 0$ independent of ω such that we have

$$\|u_\omega - U\|_{L^\infty((0,t),L^2)} \le \varepsilon_\omega + C\|u_\omega - U\|_{L^{\frac{q}{q-\alpha}}((0,t),L^2)}$$

for all $0 < t \le L$, which implies from Lemma 6 that

$$\|u_\omega - U\|_{L^\infty((0,L),L^2)} \le C\varepsilon_\omega \underset{|\omega|\to\infty}{\longrightarrow} 0. \tag{24}$$

We next prove convergence in $L^\infty((0,L),H^1(\mathbb{R}^N))$. For this, we use an argument of Kato [8]. Observe that by (20)

$$\nabla u_\omega(t) - \nabla U(t) = \Big(e^{i\Gamma_\omega(t,0)\Delta} - e^{iI(\gamma)t\Delta}\Big)\nabla\varphi + i\Big(\nabla\mathcal{I}_1(t) + \nabla\mathcal{I}_2(t) + \nabla\mathcal{I}_3(t)\Big).$$

Here $\nabla\mathcal{I}_1(t)$ can be rewritten as

$$\nabla\mathcal{I}_1(t) = \mathcal{J}_1(t) + \mathcal{J}_2(t),$$

where

$$\mathcal{J}_1(t) = \int_0^t e^{i\Gamma_\omega(t,s)\Delta}\theta(\omega s)g'(u_\omega(s))\cdot\Big(Du_\omega(s) - DU(s)\Big)\,ds,$$
$$\mathcal{J}_2(t) = \int_0^t e^{i\Gamma_\omega(t,s)\Delta}\theta(\omega s)\Big(g'(u_\omega(s)) - g'(U(s))\Big)\cdot DU(s)\,ds,$$

with

$$g'(u) = \begin{pmatrix} \frac{\alpha+2}{2}|u|^\alpha \\ \frac{\alpha}{2}|u|^{\alpha-2}u^2 \end{pmatrix} \quad\text{and}\quad Du = \begin{pmatrix} \nabla u \\ \nabla\overline{u} \end{pmatrix}.$$

Since $|g'(u_\omega)| \le C|u_\omega|^\alpha$, using Strichartz's estimate, Hölder's inequality in time and (18), we obtain

$$\begin{aligned}\|\mathcal{J}_1\|_{L^\infty((0,L),L^2)} &\le C\|g'(u_\omega)\cdot(Du_\omega - DU)\|_{L^1((0,L),L^2)}\\ &\le C\|u_\omega\|^\alpha_{L^q((0,L),W^{1,r})}\|\nabla u_\omega - \nabla U\|_{L^{\frac{q}{q-\alpha}}((0,L),L^2)}\\ &\le C\|\nabla u_\omega - \nabla U\|_{L^{\frac{q}{q-\alpha}}((0,L),L^2)}.\end{aligned} \tag{25}$$

Again, applying Strichartz's estimate and Hölder's inequality, we see that

$$\begin{aligned}\|\mathcal{J}_2\|_{L^\infty((0,L),L^2)} &\le C\|(g'(u_\omega) - g'(U))\cdot DU\|_{L^{\rho'}((0,L),L^{(\alpha+2)'})}\\ &\le C\|\nabla U\|_{L^\rho((0,L),L^{\alpha+2})}\|g'(u_\omega) - g'(U)\|_{L^{\frac{\rho}{\rho-2}}((0,L),L^{\frac{\alpha+2}{\alpha}})},\end{aligned}$$

where $(\rho, \alpha+2)$ is an admissible pair, i.e., $\rho = 4(\alpha+2)/N\alpha$.
If we assume

$$\|g'(u_\omega) - g'(U)\|_{L^\infty((0,L),L^{\frac{\alpha+2}{\alpha}})} \underset{|\omega|\to\infty}{\longrightarrow} 0, \tag{26}$$

we can obtain

$$\|\mathcal{J}_2\|_{L^\infty((0,L),L^2)} \underset{|\omega|\to\infty}{\longrightarrow} 0, \tag{27}$$

which, by (21), (22), (25), and (27), and virtue of Lemma 6, implies that

$$\|\nabla u_\omega - \nabla U\|_{L^\infty((0,L),L^2)} \underset{|\omega|\to\infty}{\longrightarrow} 0.$$

Hence to completes the proof, it suffices to show (26). It follows from (19) and (24) that $u_\omega \to U$ in $C([0,L], H^s(\mathbb{R}^N))$ as $|\omega| \to \infty$ for all $0 \le s < 1$. Choosing $s < 1$ sufficiently close to 1 so that $H^s(\mathbb{R}^N) \hookrightarrow L^{\alpha+2}(\mathbb{R}^N)$, we deduce that $u_\omega \to U$ in $C([0,L], L^{\alpha+2}(\mathbb{R}^N))$. From the well-known fact

$$|g'(u) - g'(v)| \le \begin{cases} C|u-v|^\alpha & \text{if } 0 < \alpha \le 1 \\ C(|u|^{\alpha-1} + |v|^{\alpha-1})|u-v| & \text{if } \alpha > 1, \end{cases}$$

we obtain the mapping $u \mapsto g'(u)$ is continuous $L^{\alpha+2}(\mathbb{R}^N) \to L^{(\alpha+2)/\alpha}(\mathbb{R}^N)$, which yields (26). This completes the proof of Lemma 7. □

Now, we are ready to complete the proof of Theorem 1.

Proof of Theorem 1. From Lemma 7, we only show that the conditions of Lemma 7 hold under the assumptions of Theorem 1. Fix $0 < S < S_{\max}$ and set

$$M = 2 \sup_{0\le t\le S} \|U(t)\|_{H^1}.$$

It follows from Proposition 2 that for $\|\varphi\|_{H^1} \le M$ there exists $T = T(A, M) > 0$, where $A = \|\theta\|_{L^\infty}$, such that u_ω exists on $[0, T]$ for all $\omega \in \mathbb{R}$, moreover,

$$\sup_{\omega\in\mathbb{R}} \|u_\omega\|_{L^\infty((0,T),H^1)} \le C\|\varphi\|_{H^1}$$

and

$$\sup_{\omega\in\mathbb{R}} \|u_\omega\|_{L^q((0,T),W^{1,r})} \le C\|\varphi\|_{H^1}$$

where (q, r) is given by (17). Next, let $0 < L \le S$ be such that u_ω exists on $[0, L]$ for $|\omega|$ sufficiently large,

$$\limsup_{|\omega|\to\infty} \|u_\omega\|_{L^\infty((0,L),H^1)} < \infty, \tag{28}$$

and

$$\limsup_{|\omega|\to\infty} \|u_\omega\|_{L^q((0,L),W^{1,r})} < \infty. \tag{29}$$

Note that $L = T$ is always a possible choice. Then by Lemma 7, we have that

$$\|u_\omega - U\|_{L^\infty((0,L),H^1)} \underset{|\omega|\to\infty}{\longrightarrow} 0$$

and, since $u_\omega - U \in C([0, L], H^1(\mathbb{R}^N))$, it follows that

$$\|u_\omega(L) - U(L)\|_{H^1} \underset{|\omega|\to\infty}{\longrightarrow} 0.$$

Hence $\|u_\omega(L)\|_{H^1} \leq M$ for $|\omega|$ sufficiently large. Applying Proposition 2 to (3) translated by L, we deduce that for $|\omega|$ sufficiently large, u_ω exists on $[0, L + T]$, moreover, applying (28) and (29) yields

$$\limsup_{|\omega|\to\infty} \|u_\omega\|_{L^\infty((0,L+T),H^1)} < \infty,$$

and

$$\limsup_{|\omega|\to\infty} \|u_\omega\|_{L^q((0,L+T),W^{1,r})} < \infty.$$

This means that the estimates (28) and (29) hold with L replaced by $L + T$, provided $L + T \leq S$. Iterating this argument, we see that the estimates (28) and (29) hold L replaced by S, which proves Theorem 1. □

4. Proof of Theorem 2

We give the proof of Theorem 2 at the end of this section after some lemmas.

Lemma 8. *Assume* (2) *and* $\alpha \geq 4/N$. *Let* r *and* a *be defined by* (14). *Then there exists a constant* $C > 0$ *such that*

$$\|e^{i\Gamma_\omega(\cdot,0)\Delta} f\|_{L^a(\mathbb{R},L^r)} \leq C\|\nabla f\|_{L^2}^{\frac{N\alpha-4}{2\alpha}} \|f\|_{L^2}^{\frac{4-(N-2)\alpha}{2\alpha}}$$

for all $f \in H^1(\mathbb{R}^N)$.

Proof. Using the Strichartz estimates in Lemma 2, the proof is virtually identical to the proof of [3, Lemma 3.2]. □

Lemma 9. *Assume* (2) *and* $\alpha \geq 4/N$. *Let* r *and* a *be defined by* (14). *If* $f \in H^1(\mathbb{R}^N)$, *then we have*

$$\left\| \left(e^{i\Gamma_\omega(\cdot,0)\Delta} - e^{iI(\gamma)\cdot\Delta}\right) f \right\|_{L^a((0,\infty),L^r)} \underset{|\omega|\to\infty}{\longrightarrow} 0. \tag{30}$$

Proof. In the following, we denote the operator by $A(t) := e^{i\Gamma_\omega(t,0)\Delta} - e^{iI(\gamma)t\Delta}$ for simplicity. First, we consider the case $\alpha > 4/N$. Then we have $a > q$, where q is given by (14). The Hölder inequality yields

$$\|A(\cdot)f\|_{L^a((0,\infty),L^r)} \leq \|A(\cdot)f\|_{L^\infty((0,\infty),L^r)}^{\frac{(\alpha+2)(N\alpha-4)}{N\alpha^2}} \|A(\cdot)f\|_{L^q((0,\infty),L^r)}^{\frac{8-2(N-2)\alpha}{N\alpha^2}}.$$

Since (q, r) is an admissible pair, we use the triangle inequality and the Strichartz estimate to see that there exists a constant $C > 0$, independent of ω, such that

$$\begin{aligned}\|A(\cdot)f\|_{L^q((0,\infty),L^r)} &\le C\left(\|e^{i\Gamma_\omega(\cdot,0)\Delta}f\|_{L^q((0,\infty),L^r)} + \|e^{iI(\gamma)\cdot\Delta}f\|_{L^q((0,\infty),L^r)}\right)\\ &\le C\|f\|_{L^2}. \end{aligned}\tag{31}$$

From Gagliardo–Nirenberg's inequality, we also obtain

$$\begin{aligned}\|A(t)f\|_{L^r} &\le C\|\nabla(A(t)f)\|_{L^2}^{\frac{N\alpha}{2(\alpha+2)}}\|A(t)f\|_{L^2}^{\frac{4-(N-2)\alpha}{2(\alpha+2)}}\\ &\le C\|A(t)f\|_{H^1}^{\frac{N\alpha}{2(\alpha+2)}}\left(\|e^{i\Gamma_\omega(t,0)\Delta}f\|_{L^2} + \|e^{iI(\gamma)t\Delta}f\|_{L^2}\right)^{\frac{4-(N-2)\alpha}{2(\alpha+2)}}\\ &\le C\|f\|_{L^2}^{\frac{4-(N-2)\alpha}{2(\alpha+2)}}\|A(t)f\|_{H^1}^{\frac{N\alpha}{2(\alpha+2)}},\end{aligned}\tag{32}$$

where we used the fact that $e^{i\Gamma_\omega(t,0)\Delta}$ and $e^{iI(\gamma)t\Delta}$ are unitary operators in $L^2(\mathbb{R}^N)$. Collecting (31) and (32), if follows that

$$\|A(\cdot)f\|_{L^a((0,\infty),L^r)} \le C\|f\|_{L^2}^{\frac{4-(N-2)\alpha}{2\alpha}}\|A(\cdot)f\|_{L^\infty((0,\infty),H^1)}^{\frac{N\alpha-4}{2\alpha}}.$$

Applying Lemma 4 to the second factor of the right-hand side above, we conclude (30).

Next, in the case of $\alpha = 4/N$, since $(a, r) = (\alpha+2, \alpha+2)$ is an admissible pair, it follows from Strichartz's estimate that

$$\begin{aligned}\|A(\cdot)f\|_{L^a((0,\infty),L^r)} &\le C\left(\|e^{i\Gamma_\omega(\cdot,0)\Delta}f\|_{L^a((0,\infty),L^r)} + \|e^{i\cdot I(\gamma)\Delta}f\|_{L^a((0,\infty),L^r)}\right)\\ &\le C\|f\|_{L^2}.\end{aligned}$$

Given any $\varepsilon > 0$, therefore, we can choose $0 < \widetilde{T} = \widetilde{T}(\varepsilon) < \infty$ such that

$$\|A(\cdot)f\|_{L^a((\widetilde{T},\infty),L^r)} \le \frac{\varepsilon}{2}\tag{33}$$

for every $\omega \in \mathbb{R}$. Note from the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ that

$$\|A(\cdot)f\|_{L^a((0,\widetilde{T}),L^r)} \le \|A(\cdot)f\|_{L^a((0,\widetilde{T}),H^1)} \le \widetilde{T}^{1/r}\|A(\cdot)f\|_{L^\infty((0,\widetilde{T}),H^1)}.$$

Thus, applying Lemma 4 together with (33), we have

$$\|A(\cdot)f\|_{L^a((0,\infty),L^r)} \le \varepsilon\tag{34}$$

for $|\omega|$ sufficiently large, which finishes the proof of Lemma 9. □

Now we are ready to give

Proof of Theorem 2. By Theorem 1, we know that the existence time S of u_ω goes to infinity as $|\omega| \to \infty$ and that

$$\|u_\omega - U\|_{L^\infty((0,S),H^1)} \underset{|\omega|\to\infty}{\longrightarrow} 0$$

for all $S < \infty$. In particular,

$$\|u_\omega(S) - U(S)\|_{H^1} \underset{|\omega|\to\infty}{\longrightarrow} 0. \tag{35}$$

To prove the global existence of u_ω for $|\omega|$ sufficiently large, let $\varepsilon > 0$ such that $\varepsilon \leq \varepsilon(A)$, where $A = \|\theta\|_{L^\infty}$ and $\varepsilon(A)$ is defined in Proposition 3. Since $U \in L^a((0,\infty), L^r(\mathbb{R}^N))$, we can choose S sufficiently large so that

$$\|U\|_{L^a((S,\infty),L^r)} \leq \frac{\varepsilon}{6}.$$

Moreover, it follows from Proposition 3 with $\Gamma_\omega(t,0)$ replaced by $I(\gamma)t$, see also [3, Proposition 2.4] or [4], that

$$\|e^{iI(\gamma)\cdot\Delta}U(S)\|_{L^a((0,\infty),L^r)} \leq 2\|U\|_{L^a((S,\infty),L^r)} \leq \frac{\varepsilon}{3}. \tag{36}$$

Notice that

$$\begin{aligned}\|e^{i\Gamma_\omega(\cdot,0)\Delta}u_\omega(S)\|_{L^a((0,\infty),L^r)} \leq\; & \|e^{i\Gamma_\omega(\cdot,0)\Delta}(u_\omega(S) - U(S))\|_{L^a((0,\infty),L^r)} \\ & + \|\left(e^{i\Gamma_\omega(\cdot,0)\Delta} - e^{iI(\gamma)\cdot\Delta}\right)U(S)\|_{L^a((0,\infty),L^r)} \\ & + \|e^{iI(\gamma)\cdot\Delta}U(S)\|_{L^a((0,\infty),L^r)}.\end{aligned}$$

By Lemma 8 and (35), we infer

$$\|e^{i\Gamma_\omega(\cdot,0)\Delta}(u_\omega(S) - U(S))\|_{L^a((0,\infty),L^r)} \leq C\|u_\omega(S) - U(S)\|_{H^1} \leq \frac{\varepsilon}{3}. \tag{37}$$

Combining (36), (37), and Lemma 9, we conclude

$$\|e^{i\Gamma_\omega(\cdot,0)\Delta}u_\omega(S)\|_{L^a((0,\infty),L^r)} \leq \varepsilon.$$

for $|\omega|$ sufficiently large. By virtue of Corollary 2, we see that u_ω is global and that

$$\|u_\omega\|_{L^a((S,\infty),L^r)} \leq 2\varepsilon$$

and

$$\|u_\omega\|_{L^q((S,\infty),W^{1,r})} + \|u_\omega\|_{L^\infty((S,\infty),H^1)} \leq \Lambda\|u_\omega(S)\|_{H^1} \tag{38}$$

provided $|\omega|$ is sufficiently large. In the same way, we also obtain

$$\|U\|_{L^a((S,\infty),L^r)} \leq 2\varepsilon$$

and

$$\|U\|_{L^q((S,\infty),W^{1,r})} + \|U\|_{L^\infty((S,\infty),H^1)} \leq \Lambda\|U(S)\|_{H^1}. \tag{39}$$

Hence there exits a constant M such that for L sufficiently large,

$$\sup_{\omega\geq L}\sup_{t\geq 0}\|u_\omega(t)\|_{H^1} + \sup_{t\geq 0}\|U(t)\|_{H^1} \leq M < \infty. \tag{40}$$

We now prove $u_\omega \to U$ in $L^\infty((0,\infty), H^1(\mathbb{R}^N))$ as $|\omega| \to \infty$. Observe that

$$\|u_\omega - U\|_{L^\infty((0,\infty),H^1)} \leq \|u_\omega - U\|_{L^\infty((0,S),H^1)} + \|u_\omega - U\|_{L^\infty((S,\infty),H^1)},$$

where $S > 0$ to be chosen later. Theorem 1 implies that

$$\|u_\omega - U\|_{L^\infty((0,S),H^1)} \underset{|\omega|\to\infty}{\longrightarrow} 0.$$

We claim that for every $\eta > 0$, there exists $S > 0$ such that

$$\|u_\omega - U\|_{L^\infty((S,\infty),H^1)} \leq \eta \tag{41}$$

for $|\omega|$ sufficiently large. To prove this, note that

$$\begin{aligned} u_\omega(S+t) - U(S+t) = e^{i\Gamma_\omega(t,0)\Delta}(u_\omega(S) - U(S)) + \left(e^{i\Gamma_\omega(t,0)\Delta} - e^{iI(\gamma)t\Delta}\right)U(S) \\ + i(a(t) - b(t)), \end{aligned}$$

where

$$a(t) := \int_0^t e^{i\Gamma_\omega(t,s)\Delta}\theta(\omega(S+s))|u_\omega(S+s)|^\alpha u_\omega(S+s)\,ds$$

and

$$b(t) := \int_0^t e^{iI(\gamma)(t-s)\Delta}I(\theta)|U(S+s)|^\alpha U(S+s)\,ds.$$

Using Strichartz's estimate and Hölder's inequality in time, there exists a constant $C > 0$, independent of S, such that

$$\begin{aligned} \|a\|_{L^\infty((0,\infty),H^1)} &\leq C\||u_\omega|^\alpha u_\omega\|_{L^{q'}((S,\infty),W^{1,r'})} \\ &\leq C\|u_\omega\|^\alpha_{L^a((S,\infty),L^r)}\|u_\omega\|_{L^q((S,\infty),W^{1,r})}, \end{aligned}$$

and similarly,

$$\begin{aligned} \|b\|_{L^\infty((0,\infty),H^1)} &\leq C\||U|^\alpha U\|_{L^{q'}((S,\infty),W^{1,r'})} \\ &\leq C\|U\|^\alpha_{L^a((S,\infty),L^r)}\|U\|_{L^q((S,\infty),W^{1,r})}. \end{aligned}$$

Given now $\eta > 0$, we choose $\varepsilon > 0$ sufficiently small so that $2^{\alpha+1}\varepsilon^\alpha CM \leq \eta/2$. We then fix S sufficiently large so that

$$\|U\|_{L^a((S,\infty),L^r)} \leq \frac{\varepsilon}{4}.$$

Then it follows from (38), (39) and (40) that

$$\|a\|_{L^\infty((0,\infty),H^1)} + \|b\|_{L^\infty((0,\infty),H^1)} \le 2^{\alpha+1}\varepsilon^\alpha CM \le \frac{\eta}{2}$$

for $|\omega|$ sufficiently large. It follows from (35) and Lemma 4 that

$$\sup_{t\ge 0}\|e^{i\Gamma_\omega(t,0)\Delta}(u_\omega(S) - U(S))\|_{H^1} + \sup_{t\ge 0}\|\big(e^{i\Gamma_\omega(t,0)\Delta} - e^{iI(\gamma)t\Delta}\big)U(S)\|_{H^1} \le \frac{\eta}{2}$$

for $|\omega|$ sufficiently large, which proves (41). This completes the proof of Theorem 2. □

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