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The Cauchy problem for the generalized Ostrovsky equation with negative dispersion

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Abstract. This paper is devoted to studying the Cauchy problem for the generalized Ostrovsky equation

$$u_t - \beta\partial_x^3 u - \gamma\partial_x^{-1}u + \frac{1}{k+1}\left(u^{k+1}\right)_x = 0, k \geq 5$$

with $\beta\gamma < 0, \gamma > 0$. Firstly, we prove that the Cauchy problem for the generalized Ostrovsky equation is locally well-posed in $H^s(\mathbf{R})\left(s > \frac{1}{2} - \frac{2}{k}\right)$. Then, we prove that the Cauchy problem for the generalized Ostrovsky equation is locally well-posed in $X_s(\mathbf{R}) := \|f\|_{H^s} + \left\|\mathscr{F}_x^{-1}\left(\frac{\mathscr{F}_x f(\xi)}{\xi}\right)\right\|_{H^s}\left(s > \frac{1}{2} - \frac{2}{k}\right)$. Finally, we show that the solution to the Cauchy problem for generalized Ostrovsky equation converges to the solution to the generalized KdV equation as the rotation parameter γ tends to zero for data belonging to $X_s(\mathbf{R})(s > \frac{3}{2})$. The main difficulty is that the phase function of Ostrovsky equation with negative dispersive $\beta\xi^3 + \frac{\gamma}{\xi}$ possesses the zero singular point.

1. Introduction

In this paper, we consider the Cauchy problem for the generalized Ostrovsky equation

$$u_t - \beta\partial_x^3 u + \frac{1}{k+1}(u^{k+1})_x - \gamma\partial_x^{-1}u = 0, \beta < 0, \gamma > 0, \tag{1.1}$$

$$u(x,0) = u_0(x). \tag{1.2}$$

Here ∂_x^{-1} is defined by

$$\partial_x^{-1}f(x) = \mathscr{F}^{-1}((i\xi)^{-1}\mathscr{F}f(\xi)) = \frac{1}{2}\left(\int_{-\infty}^{x} f(y)\mathrm{d}y - \int_{x}^{\infty} f(y)\mathrm{d}y\right).$$

This equation was introduced by Levandosky and Liu in [28]. When $k = 2$, (1.1) was Ostrovsky equation with negative dispersion, which was introduced by Ostrovsky in [33] as a model for weakly nonlinear long waves, by taking into account of the Coriolis force, to describe the propagation of surface waves in the ocean in a rotating frame of reference [1,11,12]. The Ostrovsky equation with negative dispersion has

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been investigated by some authors [15–20,29,31,36]. In the absence of rotation (that is, $\gamma = 0$), it becomes the generalized Korteweg–de Vries equation, which has been investigated by some authors [5–10,22,23]. Kenig et al. [23] established the small data global theory of generalized Korteweg–de Vries equation in the critical Sobolev space $\dot{H}^{s_k}(\mathbf{R})$ with $s_k = \frac{1}{2} - \frac{2}{k}$. Farah and Pastor [7] presented an alternative proof of the result of Kenig, Ponce and Vega [23].

To the best of our knowledge, the optimal regularity problem of the Cauchy problem for (1.1) in inhomogeneous Sobolev spaces has not been investigated. In this paper, we investigate the Cauchy problem for (1.1) in inhomogeneous Sobolev spaces. By using the Fourier restriction norm method introduced in [2,3] and developed in [24–26], firstly we establish three multilinear estimates. Then, by exploiting the multilinear estimates and the fixed point theorem, we prove that the Cauchy problem for the generalized Ostrovsky equation is locally well-posed in $H^s(\mathbf{R})\left(s > \frac{1}{2} - \frac{2}{k}\right)$ and in $X_s(\mathbf{R}) := \|f\|_{H^s} + \left\|\mathscr{F}_x^{-1}\left(\frac{\mathscr{F}_x f(\xi)}{\xi}\right)\right\|_{H^s}\left(s > \frac{1}{2} - \frac{2}{k}\right)$. Finally, we show that the solution to the Cauchy problem for generalized Ostrovsky equation converges to the solution to the generalized KdV equation as the rotation parameter γ tends to zero for data belonging to $X_s(\mathbf{R})(s > \frac{3}{2})$.

We give some notations before presenting the main results. $\chi_A(x) = 1$ if $x \in A$, otherwise $\chi_A(x) = 0$. $a \sim b$ means that there exists two positive constants C_1, C_2 which may depend on β, γ such that $C_1|a| \le |b| \le C_2|a|$. $a \gg b$ means that there exists positive constant C which may depend on β, γ such that $|a| \ge C|b|$. We define $A := \max\left\{1, \left|\frac{6\gamma}{7\beta}\right|^{\frac{1}{4}}, \left|\frac{\gamma}{3\beta}\right|^{\frac{1}{2}}, \left|\frac{\gamma}{\beta}\right|, 100|\beta|, 100|\gamma|\right\}$, $a := 2^{[A]}$, where $[A]$ denotes the largest integer which is smaller than A. We define $\langle\cdot\rangle = 1 + |\cdot|$. Let ψ be a smooth jump function, satisfying $0 \le \psi \le 1$, $\psi(t) = 1$ for $|t| \le 1$, $supp\psi \in [-2, 2]$ and $\psi(t) = 0$ for $|t| > 2$. For $\delta > 0$ define $\psi_\delta(t) = \psi(\frac{t}{\delta})$.

$$\phi(\xi) = \beta\xi^3 + \frac{\gamma}{\xi}, \sigma = \tau + \phi(\xi),$$

$$\mathscr{F}_x f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} f(x)\mathrm{d}x.$$

$$\mathscr{F}_x^{-1} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} f(x)\mathrm{d}x,$$

$$\mathscr{F} u(\xi, \tau) = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{-ix\xi - it\tau} u(x, t)\mathrm{d}x\mathrm{d}t,$$

$$\mathscr{F}^{-1} u(\xi, \tau) = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{ix\xi + it\tau} u(x, t)\mathrm{d}x\mathrm{d}t,$$

$$D_x^\alpha u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} |\xi|^\alpha e^{ix\xi} \mathscr{F}_x u_0(\xi)\mathrm{d}\xi,$$

$$J_x^\alpha u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \langle\xi\rangle^\alpha e^{ix\xi} \mathscr{F}_x u_0(\xi)\mathrm{d}\xi,$$

$$U^{\gamma,\beta}(t)u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi - it\phi(\xi)} \mathscr{F}_x u_0(\xi)\mathrm{d}\xi,$$

$$D_t^{\alpha}U^{\gamma,\beta}(t)u_0 = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} |\phi(\xi)|^{\alpha}\, e^{ix\xi - it\phi(\xi)}\mathscr{F}_x u_0(\xi)\mathrm{d}\xi,$$
$$J_t^{\alpha}U^{\gamma,\beta}(t)u_0 = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \langle\phi(\xi)\rangle^{\alpha}\, e^{ix\xi - it\phi(\xi)}\mathscr{F}_x u_0(\xi)\mathrm{d}\xi,$$
$$\|f\|_{L_t^p L_x^q} = \left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}} |f(x,t)|^q \mathrm{d}x\right)^{\frac{p}{q}} \mathrm{d}t\right)^{\frac{1}{p}},$$
$$P_N u = \frac{1}{\sqrt{2\pi}}\int_{|\xi|<N} e^{ix\xi}\mathscr{F}_x u(\xi)\mathrm{d}\xi,$$
$$P^N u = \frac{1}{\sqrt{2\pi}}\int_{|\xi|\geq N} e^{ix\xi}\mathscr{F}_x u(\xi)\mathrm{d}\xi,$$
$$\|f\|_{L_{xt}^p} = \|f\|_{L_x^p L_t^p}.$$

$H^s(\mathbf{R}) = \left\{ f \in \mathscr{S}'(\mathbf{R}) : \|f\|_{H^s(\mathbf{R})} = \|\langle\xi\rangle^s \mathscr{F}_x f\|_{L_\xi^2(\mathbf{R})} < \infty \right\}$. The space $X_{s,b}(\mathbf{R}^2)$ is defined as follows:

$$X_{s,b}(\mathbf{R}^2) = \left\{ u \in \mathscr{S}'(\mathbf{R}^2) : \|u\|_{X_{s,b}} = \left[\int_{\mathbf{R}^2} \langle\xi\rangle^{2s}\langle\sigma\rangle^{2b}|\mathscr{F}u(\xi,\tau)|^2 \mathrm{d}\xi\right]^{\frac{1}{2}} < \infty \right\}.$$

Here, $\langle\sigma\rangle = 1 + |\tau + \phi(\xi)|$. The space $X_s(\mathbf{R})$ is defined $X_s(\mathbf{R}) = \{f \in H^s(\mathbf{R}) : \mathscr{F}_x^{-1}(\frac{\mathscr{F}_x f(\xi)}{\xi}) \in H^s(R)\}$ with the norm $\|f\|_{X_s(\mathbf{R})} = \|f\|_{H^s} + \left\|\mathscr{F}_x^{-1}\left(\frac{\mathscr{F}_x f(\xi)}{\xi}\right)\right\|_{H^s}$. The space $\tilde{X}_{s,b}$ is defined as follows

$$\|u\|_{\tilde{X}_{s,b}} = \|u\|_{X_{s,b}} + \|\partial_x^{-1}u\|_{X_{s,b}}.$$

The main result of this paper are as follows:

Theorem 1.1 (1.1). *is locally well-posed for the initial data u_0 in $H^s(\mathbf{R})$ with $s > \frac{1}{2} - \frac{2}{k}$, $k \geq 5$ and $\beta < 0, \gamma > 0$.*

Remark 1. Theorem 1.1 is obtained by combining the multilinear estimate proved in Lemma 3.1, the linear estimate of Lemma 2.1, and a fixed point argument. Thus, Lemma 3.1 plays the crucial role in establishing Theorem 1.1. The structure of (1.1) is more complicated than that of the generalized KdV equation. More precisely, the phase function of (1.1) $\beta\xi^3 + \frac{\gamma}{\xi}$ possesses the zero singular point. For the generalized KdV equation, the following two important facts are valid.

$$\left\| \int_{\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2} \left|\xi_1^2 - \xi_2^2\right|^{\frac{1}{2}} \mathscr{F}u_1(\xi_1,\tau_1)\mathscr{F}u_2(\xi_2,\tau_2)\mathrm{d}\xi_1\mathrm{d}\tau_1 \right\|_{L_{\xi\tau}^2} \leq C\prod_{j=1}^{2}\left\|(1+|\tau_j - \xi_j^3|)^b\mathscr{F}u_j\right\|_{L_{\xi_j\tau_j}^2} \tag{1.3}$$

and

$$\|P_M u\|_{L_x^2 L_t^\infty} \leq C\left\|(1+|\tau - \xi^3|)^b \mathscr{F}u\right\|_{L_{\xi\tau}^2} \tag{1.4}$$

where $0 < M \leq 1$. For the proof of (1.3) and (1.4), we refer the readers to [14,24]. The two main ingredients in the proof of the multilinear estimate are as follows:

$$\left\|\int_{\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2} \left|\phi'(\xi_1)-\phi'(\xi_2)\right|^{\frac{1}{2}} \mathscr{F}u_1(\xi_1,\tau_1)\mathscr{F}u_2(\xi_2,\tau_2)\mathrm{d}\xi_1\mathrm{d}\tau_1\right\|_{L^2_{\xi\tau}} \leq C\prod_{j=1}^{2}\left\|\left(1+\left|\tau_j+\beta\xi_j^3+\tfrac{\gamma}{\xi_j}\right|\right)^b \mathscr{F}u_j\right\|_{L^2_{\xi_j\tau_j}} \tag{1.5}$$

and

$$\|P_M\psi(t)u\|_{L^2_xL^\infty_t} \leq C\left\|D_x^{-s_1}(1+\left|\tau+\beta\xi^3+\frac{\gamma}{\xi}\right|)^b\mathscr{F}u\right\|_{L^2_{\xi\tau}} \tag{1.6}$$

for $0 < M \leq 1, s_1 > \frac{1}{4}$. For the proof of (1.5) and (1.6), we refer the readers to the proof of Lemmas 2.2, 2.4, respectively.

For (1.5), only when

$$(\xi_1,\tau_1,\xi,\tau)\in\Omega := \left\{(\xi_1,\tau_1,\xi,\tau)\in\mathbf{R}^4 : \left|1+\frac{\gamma}{3\beta\xi_1^2\xi_2^2}\right| \geq \frac{1}{2}\right\}, \tag{1.7}$$

then we have

$$\left\|\int_{\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2} \left|\xi_1^2-\xi_2^2\right|^{\frac{1}{2}} \mathscr{F}u_1(\xi_1,\tau_1)\mathscr{F}u_2(\xi_2,\tau_2)\mathrm{d}\xi_1\mathrm{d}\tau_1\right\|_{L^2_{\xi\tau}} \leq C\prod_{j=1}^{2}\left\|\left(1+\left|\tau_j+\beta\xi_j^3+\tfrac{\gamma}{\xi_j}\right|\right)^b \mathscr{F}u_j\right\|_{L^2_{\xi_j\tau_j}}. \tag{1.8}$$

When

$$(\xi_1,\tau_1,\xi,\tau)\in\Omega^c := \left\{(\xi_1,\tau_1,\xi,\tau)\in\mathbf{R}^4 : \left|1+\frac{\gamma}{3\beta\xi_1^2\xi_2^2}\right| < \frac{1}{2}\right\}, \tag{1.9}$$

in most cases, we consider

$$\left|1+\frac{\gamma}{3\beta\xi^2\xi_{k+1}^2}\right| \geq \frac{1}{2}, \left|1+\frac{\gamma}{3\beta\xi^2\xi_{k+1}^2}\right| < \frac{1}{2}.$$

Moreover, the maximal function estimate (2.46) plays the important role in dealing with $(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega_5$.

Remark 2. From [31], we know that when $u_0 \in H^1(\mathbf{R})$, we cannot obtain the upper bound of $\|u\|_{H^1(\mathbf{R})}$; thus, we cannot obtain the global well-posedness of (1.1).

Remark 3. From [23] and [7], we know that $s = \frac{1}{2} - \frac{2}{k}$ is the critical regularity index in Sobolev spaces for (1.1) with $\gamma = 0$, which is just the generalized KdV equation.

Theorem 1.2 (1.1). *is locally well-posed for the initial data u_0 in $X_s(\mathbf{R})$ with $s > \frac{1}{2} - \frac{2}{k}, k \geq 5$ and $\beta < 0, \gamma > 0$.*

Theorem 1.3. *Let u^{γ} be the solution to (1.1) in $X_s(\mathbf{R})$ with $s > \frac{3}{2}, k \geq 5$ and $\beta < 0, \gamma > 0$. Then, u^{γ} converges in $H^s(\mathbf{R})$ with $s > \frac{3}{2}, k \geq 5$ to the solution to generalized KdV equation as $\gamma \to 0$ and u_0 tends to the initial data of the generalized KdV in $L^2(\mathbf{R})$. More precisely,*

$$\|u - v\|_{L^2} \leq e^{CT \sup\limits_{t\in[0,T]} [\|u\|_{X_s} + \|v\|_{X_s}]^k} \left[\|u_0 - v_0\|_{L^2} + C|\gamma| T \sup_{t\in[0,T]} \|u\|_{X_s}\right]. \tag{1.10}$$

Here, T is the time lifespan of the solution to (1.1) for data in $X_s(\mathbf{R})$ guaranteed by Theorem 1.2 and u is the solution to (1.1)–(1.2) and v is the solution to the Cauchy problem for the generalized KdV equation

$$\begin{aligned} & v_t - \beta \partial_x^3 v + \frac{1}{k+1}(v^{k+1})_x = 0, \\ & v(x, 0) = v_0(x). \end{aligned}$$

The rest of the paper is arranged as follows. In Sect. 2, we give some preliminaries. In Sect. 3, we show some multilinear estimates. In Sect. 4, we prove Theorem 1.1. In Sect. 5, we prove Theorem 1.2. In Sect. 6, we prove Theorem 1.3.

2. Preliminaries

Lemma 2.1. *Let $\delta \in (0, 1)$ and $s \in \mathbf{R}$ and $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$. Then, for $h \in X_{s,b'}$, we have*

$$\left\|\psi(t) U^{\gamma,\beta}(t) h\right\|_{X_{s,\frac{1}{2}+\epsilon}} \leq C\|h\|_{H^s}, \tag{2.1}$$

$$\left\|\psi\left(\frac{t}{\delta}\right) \int_0^t U^{\gamma,\beta}(t-\tau) h(\tau) \mathrm{d}\tau\right\|_{X_{s,b}} \leq C\delta^{1+b'-b} \|h\|_{X_{s,b'}}. \tag{2.2}$$

For the proof of Lemma 2.1, we refer the readers to [2,3,14].

Lemma 2.2. *Let $q \geq 8, 0 < M \leq 1$ and $s = \frac{1}{8} - \frac{1}{q}$, $s_1 = \frac{1}{4} + \epsilon$ and $b = \frac{1}{2} + \frac{\epsilon}{24}$ and $0 \leq \epsilon \leq 10^{-3}$. Then, we have*

$$\|U^{\gamma,\beta}(t) u_0\|_{L^8_{xt}} \leq C\|u_0\|_{L^2_x}, \tag{2.3}$$

$$\|u\|_{L^q_{xt}} \leq C\|u\|_{X_{4s,b}}, \tag{2.4}$$

$$\|D_x^{\frac{1}{6}} P^a u\|_{L^6_{xt}} \leq C\|u\|_{X_{0,b}}, \tag{2.5}$$

$$\|u\|_{L^8_{xt}} \leq C\|u\|_{X_{0,b}}, \tag{2.6}$$

$$\|u\|_{L_{xt}^{\frac{8}{1+\epsilon}}} \le C\|u\|_{X_{0,\frac{3-\epsilon}{3}b}} \le C\|u\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}, \tag{2.7}$$

$$\|D_x P^a u\|_{L_x^\infty L_t^2} \le C\|u\|_{X_{0,b}}, \tag{2.8}$$

$$\|D_x^{s_1}\psi(t) P_M u\|_{L_x^2 L_t^\infty} \le C\|u\|_{X_{0,b}}, \tag{2.9}$$

$$\|D_x^{1-2\epsilon} P^a u\|_{L_x^{\frac{1}{\epsilon}} L_t^2} \le C\|u\|_{X_{0,(1-2\epsilon)b}}, \tag{2.10}$$

$$\|D_x^{\frac{3-\epsilon}{18}} P^a u\|_{L_{xt}^{\frac{6}{1+\epsilon}}} \le C\|u\|_{X_{0,\frac{3-\epsilon}{3}b}}. \tag{2.11}$$

Proof. For (2.3), we refer the readers to Lemma 2.1 of [37]. By using the Sobolev embeddings theorem and (2.3), we derive

$$\begin{aligned}
\left\|U^{\gamma,\beta}(t)u_0\right\|_{L_{xt}^q} &\le C\left\|D_x^s D_t^s U^{\gamma,\beta}(t)u_0\right\|_{L_{xt}^8} \\
&= C\left\|\int_{\mathbf{R}} e^{ix\xi - it\phi}\left|\beta\xi^3 + \frac{\gamma}{\xi}\right|^s |\xi|^s \mathscr{F}_x u_0(\xi)\mathrm{d}\xi\right\|_{L_{xt}^8} \\
&\le C\left\||\xi^4 + \frac{\gamma}{\beta}|^s \mathscr{F}_x u_0\right\|_{L_\xi^2} \le C\left\||\xi^2 + a|^{2s}\mathscr{F}_x u_0\right\|_{L_\xi^2} \\
&\le C\left\|\xi^{4s}u_0\right\|_{L_\xi^2} + C\|u_0\|_{L_\xi^2} \le C\|u_0\|_{H^{4s}}.
\end{aligned} \tag{2.12}$$

By changing variable $\tau = \lambda - \phi(\xi)$, we derive

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{ix\xi + it\tau}\mathscr{F}u(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau \\
&= \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{ix\xi + it(\lambda - \phi(\xi))}\mathscr{F}u(\xi,\lambda - \phi(\xi))\mathrm{d}\xi\mathrm{d}\lambda \\
&= \frac{1}{2\pi}\int_{\mathbf{R}} e^{it\lambda}\left(\int_{\mathbf{R}} e^{ix\xi - it\phi(\xi)}\mathscr{F}u(\xi,\lambda - \phi(\xi))\mathrm{d}\xi\right)\mathrm{d}\lambda.
\end{aligned} \tag{2.13}$$

By using (2.12), (2.13) and Minkowski's inequality, for $b > \frac{1}{2}$, we derive

$$\begin{aligned}
\|u\|_{L_{xt}^q} &\le C\int_{\mathbf{R}}\left\|\left(\int_{\mathbf{R}} e^{ix\xi - it\phi(\xi)}\mathscr{F}u(\xi,\lambda - \phi(\xi))\mathrm{d}\xi\right)\right\|_{L_{xt}^q}\mathrm{d}\lambda \\
&\le C\int_{\mathbf{R}}\|\mathscr{F}u(\xi,\lambda - \phi(\xi))\|_{H^{4s}}\mathrm{d}\lambda \\
&\le C\left[\int_{\mathbf{R}}(1+|\lambda|)^{2b}\|\mathscr{F}u(\xi,\lambda - \phi(\xi))\|_{H^{4s}}^2\mathrm{d}\lambda\right]^{\frac{1}{2}}\left[\int_{\mathbf{R}}(1+|\lambda|)^{-2b}\mathrm{d}\lambda\right]^{\frac{1}{2}} \\
&\le C\left[\int_{\mathbf{R}}(1+|\tau + \phi(\xi)|)^{2b}\|\mathscr{F}u(\xi,\tau)\|_{H^{4s}}^2\mathrm{d}\tau\right]^{\frac{1}{2}} = \|u\|_{X_{4s,b}}.
\end{aligned} \tag{2.14}$$

Thus, we obtain (2.4).

$$\|D_x^{\frac{1}{6}}U^{\gamma,\beta}(t)P^a u_0\|_{L_{xt}^6} \le C\|u_0\|_{L^2} \tag{2.15}$$

can be seen in (2.3) of Lemma 2.1 of [15]. By using (2.15) and a proof similar to (2.4), we obtain (2.5). By using (2.3) and a proof similar to (2.4), we obtain (2.6). Interpolating (2.6) with

$$\|u\|_{L^2_{xt}} \leq C\,\|u\|_{X_{0,0}} \tag{2.16}$$

yields (2.7).

$$\|D_x U^{\gamma,\beta}(t)P^a u_0\|_{L^\infty_x L^2_t} \leq C\|u_0\|_{L^2_x} \tag{2.17}$$

can be seen in (2.1) of Lemma 2.1 of [15]. By using (2.17) and a proof similar to (2.4), we obtain (2.8). From (31) of Lemma 3.4 of [31], for $s_1 = \frac{1}{4}+\epsilon$, $s_2 > \frac{3}{4}$, $0 < M \leq 1$, we know that

$$\begin{aligned}\|U^{\gamma,\beta}(t)P_M\psi(t)u_0\|_{L^2_x L^\infty_t} &\leq C\|D_x^{-s_1}P_M u_0\|_{L^2} + C\|P_M u_0\|_{H^{s_2}}\\ &\leq C\|D_x^{-s_1}P_M u_0\|_{L^2}.\end{aligned} \tag{2.18}$$

By using (2.18) and a proof similar to (2.4), we obtain (2.9). Interpolating (2.8) with (2.16) leads to (2.10). Interpolating (2.5) with (2.16) leads to (2.11).

We have completed the proof of Lemma 2.2.

Lemma 2.3. *We assume that $\phi \in C^\infty(\mathbf{R})$ and $x_i (1 \leq i \leq n)$ are only simple zeros of $\phi(x)$ which means $\phi(x_i) = 0$ and $\phi'(x_i) \neq 0$. Then, we have*

$$\delta[\phi(x)] = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|\phi'(x_i)|}, \tag{2.19}$$

where δ is Dirac delta function.

Proof. For $f \in C_0(\mathbf{R})$ as a test function for the distribution. Since $f \in C_0(\mathbf{R})$ and $\phi \in C^\infty(\mathbf{R})$ and $\phi'(x_i) \neq 0$, $\forall \epsilon > 0 (< \frac{|\phi'(x_i)|}{2})$, there exists $\delta_1 > 0$ such that when $|x - x_i| < \delta_1$, we have

$$|f(x) - f(x_i)| < \epsilon, \tag{2.20}$$

$$|\phi'(x) - \phi'(x_i)| < \epsilon. \tag{2.21}$$

From (2.21), we know that $|\phi'(x)| \geq |\phi'(x_i)| - \epsilon \geq \frac{|\phi'(x_i)|}{2}$. Thus, when $x \in (x_i - \delta_1, x_i + \delta_1)$, we have

$$\begin{aligned}&\left|\frac{f(x)}{\phi'(x)} - \frac{f(x_i)}{\phi'(x_i)}\right| = \left|\frac{f(x)\phi'(x_i) - f(x_i)\phi'(x)}{\phi'(x_i)\phi'(x)}\right|\\ &\leq \left|\frac{|f(x) - f(x_i)|}{\phi'(x)}\right| + \left|\frac{f(x_i)(\phi'(x) - \phi'(x_i))}{\phi'(x)\phi'(x_i)}\right|\\ &\leq \frac{2\epsilon}{|\phi'(x_i)|} + \frac{2\epsilon|f(x_i)|}{|\phi'(x_i)|^2}.\end{aligned} \tag{2.22}$$

We define

$$F_i = \int_{x_i-\delta_1}^{x_i+\delta_1} f(x)\delta[\phi(x)]\mathrm{d}x. \tag{2.23}$$

We claim

$$F_i = \frac{f(x_i)}{|\phi'(x_i)|}. \tag{2.24}$$

By a change of the variable $u = \phi(x)$, $\mathrm{d}u = \phi'(x)\mathrm{d}x$. Then, by using (2.24) and $\int_{\phi(x_i-\delta_1)}^{\phi(x_i+\delta_1)} \delta(u)\mathrm{d}u = 1$, we have

$$\begin{aligned}
&\left|F_i - \frac{f(x_i)}{|\phi'(x_i)|}\right| = \left|\int_{\phi(x_i-\delta_1)}^{\phi(x_i+\delta_1)} f(x)\delta(u)\frac{\mathrm{d}u}{|\phi'(x)|} - \int_{\phi(x_i-\delta_1)}^{\phi(x_i+\delta_1)} f(x_i)\delta(u)\frac{\mathrm{d}u}{|\phi'(x_i)|}\right| \\
&= \left|\int_{\phi(x_i-\delta_1)}^{\phi(x_i+\delta_1)} \left|\frac{f(x)}{\phi'(x)} - \frac{f(x_i)}{\phi'(x_i)}\right| \delta(u)\mathrm{d}u\right| \\
&\leq \left[\frac{2\epsilon}{|\phi'(x_i)|} + \frac{2\epsilon|f(x_i)|}{|\phi'(x_i)|^2}\right] \int_{\phi(x_i-\delta_1)}^{\phi(x_i+\delta_1)} \delta(u)\mathrm{d}u = \left[\frac{2\epsilon}{|\phi'(x_i)|} + \frac{2\epsilon|f(x_i)|}{|\phi'(x_i)|^2}\right].
\end{aligned} \tag{2.25}$$

Since $\epsilon > 0$ is arbitrary small, from (2.25), we know that (2.24) is valid. Using (2.24), we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x)\delta[\phi(x)]\mathrm{d}x &= \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \frac{f(x_i)}{|\phi'(x_i)|} \\
&= \int_{-\infty}^{+\infty} f(x) \sum_{i=1}^{n} \left[\frac{\delta(x-x_i)}{|\phi'(x_i)|}\right] \mathrm{d}x.
\end{aligned} \tag{2.26}$$

Thus, since $f \in C_0(\mathbf{R})$, we obtain

$$\delta[\phi(x)] = \sum_{i=1}^{n} \left[\frac{\delta(x-x_i)}{|\phi'(x_i)|}\right]. \tag{2.27}$$

We have completed the proof of Lemma 2.3.

Remark 3. In page 184 of [11], Gel'fand and Shilov [11] presented the conclusion of Lemma 2.3, however, they did not give the strict proof.

Lemma 2.4. *Let $b > \frac{1}{2}$. Then, we have*

$$\left\| I^{\frac{1}{2}}(e^{t(\beta\partial_x^3+\partial_x^{-1}\gamma)} f_1, e^{t(\beta\partial_x^3+\partial_x^{-1}\gamma)} f_2)\right\|_{L_{xt}^2} \leq C\|f_1\|_{L_x^2}\|f_2\|_{L_x^2}. \tag{2.28}$$

Here

$$\mathscr{F}_x I^s(f_1, f_2)(\xi) = \int_{\xi=\xi_1+\xi_2} \left|\phi'(\xi_1) - \phi'(\xi_2)\right|^s \mathscr{F}_x f_1(\xi_1)\mathscr{F}_x f_2(\xi_2)\,\mathrm{d}\xi_1. \tag{2.29}$$

In particular, for $b > \frac{1}{2}$, we have

$$\left\| I^{\frac{1}{2}}(u_1, u_2) \right\|_{L^2_{xt}} \leq C \prod_{j=1}^{2} \|u_j\|_{X_{0,b}}. \tag{2.30}$$

Here

$$\mathscr{F} I^{\frac{1}{2}}(u_1, u_2)(\xi, \tau) = \int_{\tau=\tau_1+\tau_2, \xi=\xi_1+\xi_2} \left|\phi'(\xi_1) - \phi'(\xi_2)\right|^{\frac{1}{2}} \mathscr{F} u_1(\xi_1, \tau_1) \mathscr{F} u_2(\xi_2, \tau_2) \mathrm{d}\xi_1 \mathrm{d}\tau_1.$$

Proof. Following the idea of [13,14], we present the proof of Lemma 2.4. By using the Plancherel identity with respect to the space variable and

$$\int_{\mathbf{R}} e^{-it(\phi(\xi_1)+\phi(\xi_2)-\phi(\eta_1)-\phi(\eta_2))} \mathrm{d}t = C\delta(\phi(\xi_1) + \phi(\xi_2) - \phi(\eta_1) - \phi(\eta_2)),$$

we derive

$$\begin{aligned}
&\left\| I^{\frac{1}{2}} \left(e^{t(\beta\partial_x^3 + \partial_x^{-1}\gamma)} f_1, e^{t(\beta\partial_x^3 + \partial_x^{-1}\gamma)} f_2 \right) \right\|^2_{L^2_{xt}} \\
&= C \int_{\mathbf{R}^2} \left| \int_{\xi=\xi_1+\xi_2} \left|\phi'(\xi_1) - \phi'(\xi_2)\right|^{\frac{1}{2}} e^{-it(\phi(\xi_1)+\phi(\xi_2))} \mathscr{F}_x f_1(\xi_1) \mathscr{F}_x f_2(\xi_2) \mathrm{d}\xi_1 \right|^2 \mathrm{d}t \mathrm{d}\xi \\
&= C \int_{\mathbf{R}^2} \int_{\xi=\xi_1+\xi_2} \int_{\xi=\eta_1+\eta_2} e^{-it(\phi(\xi_1)+\phi(\xi_2)-\phi(\eta_1)-\phi(\eta_2))} \\
&\quad \left(\left|\phi'(\xi_1) - \phi'(\xi_2)\right| \left|\phi'(\eta_1) - \phi'(\eta_2)\right| \right)^{\frac{1}{2}} \prod_{i=1}^{2} \mathscr{F}_x f_i(\xi_i) \overline{\mathscr{F}_x f_i(\eta_i)} \mathrm{d}\xi_1 \mathrm{d}\eta_1 \mathrm{d}\xi \mathrm{d}t \\
&= C \int_{\mathbf{R}} \int_{\xi=\xi_1+\xi_2} \int_{\xi=\eta_1+\eta_2} \delta(\phi(\xi_1) + \phi(\xi_2) - \phi(\eta_1) - \phi(\eta_2)) \\
&\quad \left(\left|\phi'(\xi_1) - \phi'(\xi_2)\right| \left|\phi'(\eta_1) - \phi'(\eta_2)\right| \right)^{\frac{1}{2}} \prod_{i=1}^{2} \mathscr{F}_x f_i(\xi_i) \overline{\mathscr{F}_x f_i(\eta_i)} \mathrm{d}\xi_1 \mathrm{d}\eta_1 \mathrm{d}\xi. \quad (2.31)
\end{aligned}$$

From Lemma 2.3, we have $\delta[g(x)] = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|g'(x_i)|}$, where

$$g(x) = 3\xi\beta(x^2 - \xi_1^2 + \xi(\xi_1 - x)) - \frac{\gamma\xi}{x(\xi - x)} + \frac{\gamma\xi}{\xi_1(\xi - \xi_1)}, \tag{2.32}$$

and Lemma 2.3 is going to be applied with the variable $x = \eta_1$. Since $g'' \neq 0$, then g has only two simple zeros, $x_1 = \xi_1$ and $x_2 = \xi - \xi_1$.
Hence, $g'(x_1) = \phi'(\xi_2) - \phi'(\xi_1)$ and $g'(x_2) = \phi'(\xi_1) - \phi'(\xi_2)$. Thus, (2.31) can be rewritten as

$$C \int_{\mathbf{R}} \int_{\xi=\xi_1+\xi_2} \int_{\xi=\eta_1+\eta_2} \frac{1}{|\phi'(\xi_2) - \phi'(\xi_1)|} \delta(\eta_1 - \xi_1)$$

$$
\begin{aligned}
&\left(\left|\phi'(\xi_1)-\phi'(\xi_2)\right|\left|\phi'(\eta_1)-\phi'(\eta_2)\right|\right)^{\frac{1}{2}}\prod_{i=1}^{2}\mathscr{F}_x f_i(\xi_i)\overline{\mathscr{F}_x f_i(\eta_i)}\mathrm{d}\xi_1\mathrm{d}\eta_1\mathrm{d}\xi\\
&+C\int_{\mathbf{R}}\int_{\xi=\xi_1+\xi_2}\int_{\xi=\eta_1+\eta_2}\frac{1}{|\phi'(\xi_2)-\phi'(\xi_1)|}\delta(\eta_1-(\xi-\xi_1))\\
&\left(\left|\phi'(\xi_1)-\phi'(\xi_2)\right|\left|\phi'(\eta_1)-\phi'(\eta_2)\right|\right)^{\frac{1}{2}}\prod_{i=1}^{2}\mathscr{F}_x f_i(\xi_i)\overline{\mathscr{F}_x f_i(\eta_i)}\mathrm{d}\xi_1\mathrm{d}\eta_1\mathrm{d}\xi\\
=\;&C\int_{\mathbf{R}}\int_{\xi=\xi_1+\xi_2}\prod_{i=1}^{2}|\mathscr{F}_x f_i(\xi_i)|^2\mathrm{d}\xi_1\mathrm{d}\xi\\
&+C\int_{\mathbf{R}}\int_{\xi=\xi_1+\xi_2}\mathscr{F}_x f_1(\xi_1)\overline{\mathscr{F}_x f_1(\xi_2)}\mathscr{F}_x f_2(\xi_2)\overline{\mathscr{F}_x f_2(\xi_1)}\mathrm{d}\xi_1\mathrm{d}\xi\\
\leq\;&C\left(\prod_{i=1}^{2}\|f_i\|_{L_x^2}^2+\|\mathscr{F}_x f_1\mathscr{F}_x f_2\|_{L_\xi^1}^2\right)\leq C\prod_{i=1}^{2}\|f_i\|_{L_x^2}^2. \qquad (2.33)
\end{aligned}
$$

We define $\mathscr{F} v_{j\lambda}(\xi)=\mathscr{F} u_j(\xi,\lambda-\phi(\xi))(j=1,2)$, and

$$
\mathscr{F}_x I^{\frac{1}{2}}(u_1,u_2)(\xi,t)=\int_{\xi=\xi_1+\xi_2}\left|\phi'(\xi_1)-\phi'(\xi_2)\right|^{\frac{1}{2}}\mathscr{F}_x u_1(\xi_1,t)\mathscr{F}_x u_2(\xi_2,t)\mathrm{d}\xi_1. \qquad (2.34)
$$

Thus, by using (2.34), we have

$$
I^{\frac{1}{2}}(u_1,u_2)(x,t)=\int_{\mathbf{R}^2}I^{\frac{1}{2}}(e^{it\lambda_1}U^{\gamma,\beta}(t)v_{1\lambda_1},e^{it\lambda_2}U^{\gamma,\beta}(t)v_{2\lambda_2})(x,t)\mathrm{d}\lambda_1\mathrm{d}\lambda_2. \qquad (2.35)
$$

By using a direct computation, we have

$$
\mathscr{F}\left[I^{\frac{1}{2}}(e^{it\lambda_1}v_{1\lambda_1},e^{it\lambda_2}v_{2\lambda_2})\right](\xi,\tau)=\mathscr{F}\left[I^{\frac{1}{2}}(u_1,u_2)\right](\xi,\tau-\lambda_1-\lambda_2). \qquad (2.36)
$$

By using the Minkowski's inequality, the Plancherel identity, (2.28), (2.35) and (2.36), we have

$$
\begin{aligned}
\left\|I^{\frac{1}{2}}(u_1,u_2)\right\|_{L_{xt}^2}&\leq C\int_{\mathbf{R}^2}\left\|I^{\frac{1}{2}}(e^{it\lambda_1}U^{\gamma,\beta}(t)v_{1\lambda_1},e^{it\lambda_2}U^{\gamma,\beta}(t)v_{2\lambda_2})\right\|_{L_{xt}^2}\mathrm{d}\lambda_1\mathrm{d}\lambda_2\\
&\leq C\int_{\mathbf{R}^2}\left\|I^{\frac{1}{2}}(U^{\gamma,\beta}(t)v_{1\lambda_1},U^{\gamma,\beta}(t)v_{2\lambda_2})\right\|_{L_{xt}^2}\mathrm{d}\lambda_1\mathrm{d}\lambda_2\\
&\leq C\int_{\mathbf{R}^2}\|v_{1\lambda_1}\|_{L_x^2}\|v_{2\lambda_2}\|_{L_x^2}\mathrm{d}\lambda_1\mathrm{d}\lambda_2\\
&\leq C\int_{\mathbf{R}}\|\mathscr{F}_x v_{1\lambda_1}\|_{L_\xi^2}\mathrm{d}\lambda_1\int_{\mathbf{R}}\|\mathscr{F}_x v_{2\lambda_2}\|_{L_\xi^2}\mathrm{d}\lambda_2\\
&\leq C\prod_{j=1}^{2}\left\|\langle\lambda_j\rangle^b\mathscr{F}u_j(\xi_j,\lambda_j-\phi(\xi_j))\right\|_{L_{\xi\lambda_j}^2}\leq C\prod_{j=1}^{2}\|u_j\|_{X_{0,b}}.
\end{aligned}
$$

We have completed the proof of Lemma 2.4.

Lemma 2.5. *We assume that* $b > \frac{1}{2}$, $0 \le s \le \frac{1}{2}$. *Then, we have*

$$\left\| I^s(u_1, u_2) \right\|_{L^2_{xt}} \le C \prod_{j=1}^{2} \|u_j\|_{X_{0,\frac{2+2s}{3}b}}. \tag{2.37}$$

Lemma 2.5 can be proved similarly to Corollary 3.2 of [13] and Lemma 2.2 of [30] with the aid of Lemma 2.4.

Lemma 2.6. *Let* $\phi_j (j = 1, 2) \in C^\infty(\mathbf{R})$ *and* $supp\phi_2 \subset (a, b)$. *If* $\phi_1'(\xi) \neq 0$ *for all* $\xi \in [a, b]$, *then*

$$\left| \int_a^b e^{i\lambda\phi_1(\xi)} \phi_2(\xi) \mathrm{d}\xi \right| \le \frac{C}{|\lambda|^k}$$

for all $k \ge 0$, *where the constant* C *depends on* ϕ_1, ϕ_2 *and* k.

Lemma 2.6 can be seen in [35].

Lemma 2.7. *Let* $\phi_4(\xi) \in C^\infty(\mathbf{R})$ *and* $\phi_3(\xi) \in C^3(\mathbf{R})$ *and* $supp\phi_3(\xi) \subset (a, b)$ *and* $\left|\phi_3^{(3)}(\xi)\right| \ge 1$ *uniformly with respect to* ξ. *Then, we have*

$$\left| \int_a^b e^{i\lambda\phi_3(\xi)} \phi_4(\xi) \mathrm{d}\xi \right| \le \frac{C}{|\lambda|^{\frac{1}{3}}} \left(|\phi_4(b)| + \int_a^b |\phi_4'(\xi)| \mathrm{d}\xi \right).$$

Lemma 2.7 can be seen in [35].

Lemma 2.8. *Let*

$$K(x, t) = \int_{\mathbf{R}} e^{-it\phi(\xi) + ix\xi} \chi_{[N,4N]}(|\xi|) \mathrm{d}\xi.$$

Here, $N \in 2^Z$, $N \ge a$. *For* $\gamma \ge 7$, *we obtain*

$$\|K\|_{L_x^{\frac{\gamma}{2}} L_t^\infty} \le C N^{\frac{\gamma-2}{\gamma}}. \tag{2.38}$$

Proof. We define

$$\Omega_1 := \left\{ (x, t) \in \mathbf{R} \times \mathbf{R} : |x| \le \frac{1}{N} \right\},$$
$$\Omega_2 := \left\{ (x, t) \in \mathbf{R} \times \mathbf{R} : |x| > \frac{1}{N}, |x| \ge 4000aN^2 t \right\},$$
$$\Omega_3 := \left\{ (x, t) \in \mathbf{R} \times \mathbf{R} : |x| > \frac{1}{N}, |x| < 4000aN^2 t \right\}.$$

Obviously, $\mathbf{R}_x \times \mathbf{R}_t = \bigcup_{j=1}^3 \Omega_j$. We define $\Omega_{x,i} := \{t \in \mathbf{R} | (x, t) \in \Omega_i\}$ for a fixed $x \in \mathbf{R}$. Without loss of generality, we assume that $N \le \xi \le 4N$. Assume that $\eta = \frac{\xi}{N}$, then we have

$$K(x, t) = N \int_1^4 e^{-iN^3\eta^3 t - \frac{i\gamma t}{N\eta} + ixN\eta} \mathrm{d}\eta.$$

By using a direct computation and from the definition Ω_{x1}, we have

$$\left[\int_{|x|\leq\frac{1}{N}}\left[\sup_{t\in\Omega_{x,1}}|K(x,t)|\right]^{\frac{\gamma}{2}}\mathrm{d}x\right]^{\frac{2}{\gamma}}\leq CN\left[\int_{|x|\leq\frac{1}{N}}\mathrm{d}x\right]^{\frac{2}{\gamma}}\leq CN^{\frac{\gamma-2}{\gamma}}. \tag{2.39}$$

By using a direct computation, we have $-it\phi(\xi)+ix\xi=-i\beta N^3\eta^3t-\frac{i\gamma t}{N\eta}+ixN\eta=ixN(\eta-\frac{\beta N^2\eta^3t}{x}-\frac{\gamma t}{N^2x\eta}):=ixN\phi_5(\eta)$, where $\phi_5(\eta)=\eta-\frac{\beta N^2\eta^3t}{x}-\frac{\gamma t}{N^2x\eta}$. Obviously,

$$|\phi_5'(\eta)|=\left|1-\frac{3\beta N^2t\eta^2}{x}-\frac{\gamma t}{N^2x\eta^2}\right|\geq 1-48\left|\frac{t\beta N^2}{x}\right|-\left|\frac{\gamma t}{N^2x}\right|\geq 1-\frac{1}{8}-\frac{1}{8}\geq\frac{1}{2}$$

for any $(x,t)\in\Omega_2$. Therefore, $\phi_5'\neq 0$ in this region. From Lemma 2.6, we know that $|K(x,t)|\leq CN(N|x|)^{-2}=N^{-1}x^{-2}$. Thus, we derive

$$\left[\int_{|x|>\frac{1}{N}}\left[\sup_{t\in\Omega_{x,2}}|K(x,t)|\right]^{\frac{\gamma}{2}}\mathrm{d}x\right]^{\frac{2}{\gamma}}\leq CN^{-1}\left[\int_{|x|>\frac{1}{N}}|x|^{-\gamma}\mathrm{d}x\right]^{\frac{2}{\gamma}}\leq CN^{-1}N^{2-\frac{2}{\gamma}}=N^{\frac{\gamma-2}{\gamma}}. \tag{2.40}$$

We define $-i\beta N^3\eta^3t+ixN\eta-\frac{i\gamma t}{N\eta}=-it\beta N^3(\eta^3+\frac{\gamma}{\beta\eta N^4}-\frac{x\eta}{\beta N^2t}):=itN^3\beta\phi_6(\eta)$. Obviously, we have $|\phi_6^{(3)}(\eta)|\geq 1$. By using Lemma 2.7, we have

$$|K(x,t)|=N\left|\int_1^4 e^{-iN^3\eta^3t-\frac{i\gamma t}{N\eta}+ixN\eta}\mathrm{d}\eta\right|\leq CN(N^3t)^{-\frac{1}{3}}\leq\frac{C}{t^{\frac{1}{3}}}\leq C\frac{N^{\frac{2}{3}}}{|x|^{\frac{1}{3}}}. \tag{2.41}$$

Thus, by using (2.41), for $\gamma\geq 7$, we have

$$\left[\int_{|x|>\frac{1}{N}}\left[\sup_{t\in\Omega_{x,3}}|K(x,t)|\right]^{\frac{\gamma}{2}}\mathrm{d}x\right]^{\frac{2}{\gamma}}\leq C\left[\int_{|x|>\frac{1}{N}}N^{\frac{\gamma}{3}}|x|^{-\frac{\gamma}{6}}\mathrm{d}x\right]^{\frac{2}{\gamma}}\leq CN^{\frac{\gamma-2}{\gamma}}. \tag{2.42}$$

Putting together (2.39), (2.40) with (2.42), we derive

$$\|K\|_{L_x^{\frac{\gamma}{2}}L_t^\infty}\leq CN^{\frac{\gamma-2}{\gamma}}.$$

We have completed the proof of Lemma 2.8.

Remark 3. Inspired by the idea of Proposition 2.5 of [34], we show Lemma 2.8.

Lemma 2.9. *We assume that $\gamma\geq 4$ and $supp(\mathscr{F}_xf)\subseteq\{\xi|N\leq|\xi|\leq 4N\}$, where $N\in 2^Z$, $N\geq a$. For $f\in L_x^2(\mathbf{R})$, we have*

$$\|U^{\gamma,\beta}(t)f(x)\|_{L_x^\gamma L_t^\infty}\leq CN^{\frac{\gamma-2}{2\gamma}}\|f\|_{L_x^2}. \tag{2.43}$$

Here

$$U^{\gamma,\beta}(t)f(x) = \int_{\mathbf{R}} e^{ix\xi - it\phi(\xi)} \mathscr{F}_x f(\xi)\chi_{[N,4N]}(|\xi|)\mathrm{d}\xi. \tag{2.44}$$

We assume that $b > \frac{1}{2}$, $\gamma \geq 4$ *and* $supp(\mathscr{F}_x u) \subseteq \{\xi | N \leq |\xi| \leq 4N\}$, *where* $N \in 2^Z$, $N \geq a$. *Then, we derive*

$$\|u\|_{L_x^\gamma L_t^\infty} \leq CN^{\frac{\gamma-2}{2\gamma}} \|u\|_{X_{0,b}}. \tag{2.45}$$

We assume that $b > \frac{1}{2}$, $\gamma \geq 4$ *and* $supp(\mathscr{F}_x u) \subseteq \{\xi | a \leq |\xi| < \infty\}$ *for any* t. *Then, we have*

$$\|u\|_{L_x^\gamma L_t^\infty} \leq C\|u\|_{X_{s,b}}. \tag{2.46}$$

Here, $s = \frac{1}{2} - \frac{1}{\gamma} + \epsilon$.

Proof. From (2.1) of [15], we have

$$\|U^{\gamma,\beta}(t)f(x)\|_{L_x^4 L_t^\infty} \leq CN^{\frac{1}{4}} \|f\|_{L_x^2}. \tag{2.47}$$

We firstly show that (2.43) is valid for $\gamma \geq 7$. We define $Tf := U^{\gamma,\beta}(t)f$, where $T : L_x^2 \to L_x^\gamma L_t^\infty$. Obviously, we have $T^*F = \int_{\mathbf{R}} e^{t(\beta\partial_x^3 + \gamma\partial_x^{-1})} F \mathrm{d}t$. By Using the TT^* idea, we know that that (2.43) is equivalent to

$$\left\| \int_{\mathbf{R}} U^{\gamma,\beta}(t-s)F ds \right\|_{L_x^\gamma L_t^\infty} \leq CN^{\frac{\gamma-2}{\gamma}} \|F\|_{L_x^{\frac{\gamma}{\gamma-1}} L_t^1}. \tag{2.48}$$

Here, $F \in L_x^2 L_t^1(\mathbf{R} \times \mathbf{R})$ possesses the same frequency support as u. Thus, to obtain (2.43), it suffices to prove (2.48). By using a direct computation, we have

$$\begin{aligned} \mathscr{F}_x^{-1}\left(e^{-i(t-s)\phi(\xi)} \mathscr{F}_x F(\xi, s)\right) &= C \int_{\mathbf{R}} e^{-i(t-s)\phi(\xi) + ix\xi} \mathscr{F}_x F(\xi, s) \mathrm{d}\xi \\ &= \mathscr{F}_x^{-1}\left(e^{-i(t-s)\phi(\xi)} \chi_{[N,4N]}(|\xi|)\right) * F(x, s). \end{aligned}$$

Thus, the term on the left hand side of (2.48) can be rewritten as

$$\begin{aligned} &\int_{\mathbf{R}} \mathscr{F}_x^{-1}\left(e^{-i(t-s)\phi(\xi)} \chi_{[N,4N]}(|\xi|)\right) * F(x, s) ds \\ &= \mathscr{F}_x^{-1}\left(e^{-it\phi(\xi)} \chi_{[N,4N]}(|\xi|)\right) * F(x, t) = CK * F. \end{aligned}$$

Here, $*$ denotes the convolution with respect to variables x, t and

$$K(x, t) = \int_{\mathbf{R}} e^{-it\phi(\xi) + ix\xi} \chi_{[N,4N]}(|\xi|) \mathrm{d}\xi. \tag{2.49}$$

From Young inequality and Lemma 2.8, we derive

$$\|K * F\|_{L_x^\gamma L_t^\infty} \leq \|K\|_{L_x^{\frac{\gamma}{2}} L_t^\infty} \|F\|_{L_x^{\frac{\gamma}{\gamma-1}} L_t^1} \leq CN^{\frac{\gamma-2}{\gamma}} \|F\|_{L_x^{\frac{\gamma}{\gamma-1}} L_t^1}.$$

Consequently, when $\gamma \geq 7$, we derive

$$\|U^{\gamma,\beta}(t)f(x)\|_{L_x^\gamma L_t^\infty} \leq CN^{\frac{\gamma-2}{2\gamma}}\|f\|_{L_x^2}. \tag{2.50}$$

Interpolating (2.47) with (2.50) leads to

$$\|U^{\gamma,\beta}(t)f(x)\|_{L_x^\gamma L_t^\infty} \leq CN^{\frac{\gamma-2}{2\gamma}}\|f\|_{L_x^2}, 4 \leq \gamma \leq 7. \tag{2.51}$$

By changing variable $\tau = \lambda - \phi(\xi)$, we derive

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{ix\xi+it\tau}\mathscr{F}u(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{ix\xi+it(\lambda-\phi(\xi))}\mathscr{F}u(\xi,\lambda-\phi(\xi))\mathrm{d}\xi\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{\mathbf{R}} e^{it\lambda}\left(\int_{\mathbf{R}} e^{ix\xi-it\phi(\xi)}\mathscr{F}u(\xi,\lambda-\phi(\xi))\mathrm{d}\xi\right)\mathrm{d}\lambda. \end{aligned} \tag{2.52}$$

By using (2.51), (2.52) and Minkowski's inequality and the change of variable $\lambda = \tau + \phi$, for $b > \frac{1}{2}$, we have

$$\begin{aligned} \|u\|_{L_x^\gamma L_t^\infty} &\leq C\int_{\mathbf{R}}\left\|\left(\int_{\mathbf{R}} e^{ix\xi-it\phi(\xi)}\mathscr{F}u(\xi,\lambda-\phi(\xi))\mathrm{d}\xi\right)\right\|_{L_x^\gamma L_t^\infty}\mathrm{d}\lambda \\ &\leq CN^{\frac{\gamma-2}{2\gamma}}\int_{\mathbf{R}}\|\mathscr{F}u(\xi,\lambda-\phi(\xi))\|_{L^2}\,\mathrm{d}\lambda \\ &\leq CN^{\frac{\gamma-2}{2\gamma}}\left[\int_{\mathbf{R}}(1+|\lambda|)^{2b}\|\mathscr{F}u(\xi,\lambda-\phi(\xi))\|_{L^2}^2\,\mathrm{d}\lambda\right]^{\frac{1}{2}}\left[\int_{\mathbf{R}}(1+|\lambda|)^{-2b}\mathrm{d}\lambda\right]^{\frac{1}{2}} \\ &\leq CN^{\frac{\gamma-2}{2\gamma}}\left[\int_{\mathbf{R}}(1+|\tau+\phi(\xi)|)^{2b}\|\mathscr{F}u(\xi,\tau)\|_{L^2}^2\,\mathrm{d}\tau\right]^{\frac{1}{2}} = CN^{\frac{\gamma-2}{2\gamma}}\|u\|_{X_{0,b}}. \end{aligned} \tag{2.53}$$

Since

$$P^aU^{\gamma,\beta}u_0 = \sum_{N\geq a}\int_N^{4N} e^{ix\xi-it\phi}\mathscr{F}_x u_0\mathrm{d}\xi, \tag{2.54}$$

thus, by using the Minkowski equality and Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left\|P^aU^{\gamma,\beta}u_0\right\|_{L_x^\gamma L_t^\infty} &\leq \sum_{N\geq a}\left\|\int_N^{4N} e^{ix\xi-it\phi}\mathscr{F}_x u_0\mathrm{d}\xi\right\|_{L_x^\gamma L_t^\infty} \\ &\leq C\sum_{N\geq a}N^{\frac{1}{2}-\frac{1}{\gamma}}\|\chi_{[N,4N]}(|\xi|)\mathscr{F}_x u_0\|_{L_\xi^2} \\ &\leq C\sum_{N\geq a}\left[N^{2(\frac{1}{2}-\frac{1}{\gamma}+\epsilon)}\|\chi_{[N,4N]}(|\xi|)\mathscr{F}_x u_0\|_{L_\xi^2}^2\right]^{\frac{1}{2}} \\ &\leq C\|u_0\|_{H^s}, \end{aligned} \tag{2.55}$$

where $s=\frac{1}{2}-\frac{1}{\gamma}+\epsilon$. By using (2.55) and a proof similar to (2.45), we obtain that (2.46) is valid.

We have completed the proof of Lemma 2.9.

Lemma 2.10. *We assume that $\gamma\geq 4$ and $supp(\mathscr{F}_x u)\subseteq\{\xi|N\leq|\xi|\leq 4N\}$, $N\in 2^Z$. For $b>\frac{1}{2}$ and $f\in L_x^2(\mathbf{R})$, we have*

$$\|\psi(t)u\|_{L_{xt}^\infty}\leq CN^{\frac{1}{4}-\epsilon}\|u\|_{X_{0,b}}. \tag{2.56}$$

For $b>\frac{1}{2}$, and $supp(\mathscr{F}_x u)\subseteq\{\xi|0<|\xi|\leq a\}$ for any t, we have

$$\|\psi(t)u\|_{L_{xt}^\infty}\leq C\|D_x^s u\|_{X_{0,b}}. \tag{2.57}$$

Here, $s=\frac{1}{4}-\epsilon$. For $b>\frac{1}{2}$, and $supp(\mathscr{F}_x u)\subseteq\{\xi|0<|\xi|\leq a\}$ for any t, we have

$$\|\psi(t)u\|_{L_x^{\frac{2}{1-2\epsilon}}L_t^\infty}\leq C\|D_x^{-\frac{1}{4}}u\|_{X_{0,b}}. \tag{2.58}$$

Proof. Without loss of generality, we can assume that $supp(\mathscr{F}_x u)\subseteq[N,4N]$. By using the Cauchy–Schwarz inequality and the Minkowski's inequality, we have

$$\begin{aligned}|u(x)| &=\left|\int_N^{4N}e^{ix\xi}\mathscr{F}_x u(\xi,t)\mathrm{d}\xi\right|\leq\int_N^{4N}|\mathscr{F}_x u(\xi,t)|\,\mathrm{d}\xi\\ &\leq CN^{\frac{1}{2}}\left[\int_N^{4N}|\mathscr{F}_x u(\xi,t)|^2\,\mathrm{d}\xi\right]^{\frac{1}{2}}\\ &=CN^{\frac{1}{2}}\|\mathscr{F}_x u(\xi,t)\|_{L_\xi^2}=CN^{\frac{1}{2}}\|u\|_{L_x^2}.\end{aligned} \tag{2.59}$$

Thus, by using the Minkowski's inequality, from (2.59) and (2.9), we have

$$\|\psi(t)u\|_{L_{xt}^\infty}\leq CN^{\frac{1}{2}}\|u\|_{L_t^\infty L_x^2}\leq CN^{\frac{1}{2}}\|u\|_{L_x^2L_t^\infty}\leq CN^{\frac{1}{4}-\epsilon}\|u\|_{X_{0,b}}. \tag{2.60}$$

By using (2.60) and a proof similar to (2.46), we obtain that (2.57) is valid. Interpolating (2.9) with (2.57) yields (2.58).

This completes the proof of Lemma 2.10.

Lemma 2.11. *For $b>\frac{1}{2}$, we have*

$$\left\|D_x^{-\frac{1}{2}-4\epsilon}\mathscr{F}_x^{-1}\left(\chi_{|\xi|\geq a}(\xi)\mathscr{F}u(\xi)\right)\right\|_{L_{xt}^\infty}\leq C\,\|u\|_{X_{0,b}}\,. \tag{2.61}$$

Proof. Using the Sobolev embeddings theorem and (2.3),we have

$$\begin{aligned}&\left\|D_x^{-\frac{1}{2}-4\epsilon}\mathscr{F}_x^{-1}\left(\chi_{|\xi|\geq a}(\xi)\mathscr{F}U^{\gamma,\beta}(t)u_0\right)\right\|_{L_{xt}^\infty}\\ &\leq C\left\|D_x^{-\frac{1}{2}-4\epsilon}J_x^{\frac{1}{8}+\epsilon}J_t^{\frac{1}{8}+\epsilon}\mathscr{F}_x^{-1}\left(\chi_{|\xi|\geq a}(\xi)\mathscr{F}U^{\gamma,\beta}(t)u_0\right)\right\|_{L_{xt}^8}\end{aligned}$$

$$= C \left\| \int_{\mathbf{R}} e^{ix\xi - it\phi(\xi)} \chi_{|\xi|\geq a}(\xi) \, |\xi|^{-\frac{1}{2}-4\epsilon} \, \langle\xi\rangle^{\frac{1}{8}+\epsilon} \left(1 + \left|\beta\xi^3 + \frac{\gamma}{\xi}\right|\right)^{\frac{1}{8}+\epsilon} \mathscr{F}_x u_0 \mathrm{d}\xi \right\|_{L^8_{xt}}$$

$$\leq C \left\| |\xi|^{-\frac{1}{2}-4\epsilon} \, \langle\xi\rangle^{\frac{1}{8}+\epsilon} \left|\beta\xi^3 + \frac{\gamma}{\xi}\right|^{\frac{1}{8}+\epsilon} \chi_{|\xi|\geq a}(\xi) \mathscr{F}_x u_0 \right\|_{L^2_\xi}$$

$$\leq C \left\| \mathscr{F}_x u_0 \right\|_{L^2_\xi} = C \left\| u_0 \right\|_{L^2_x}. \tag{2.62}$$

By using (2.62) and a proof similar to (2.4), we obtain that (2.61) is valid.

This completes the proof of Lemma 2.11.

Lemma 2.12. *Let $s > 0$, $1 < p < \infty$. Then, we have*

$$\left\| J^s(fg) - f J^s g \right\|_{L^p} \leq C \left\| f_x \right\|_{L^\infty} \left\| J^{s-1} g \right\|_{L^p} + C \left\| J^s f \right\|_{L^p} \left\| g \right\|_{L^\infty}. \tag{2.63}$$

For the proof of Lemma 2.12, we refer the readers to Lemma XI of [21].

3. Multilinear estimates

In this section, we present some crucial multilinear estimates which play an important role in establishing Theorems 1.1 and 1.2.

Lemma 3.1. *Let $s \geq \frac{1}{2} - \frac{2}{k} + 2\epsilon$, $k \geq 5$ and $b = \frac{1}{2} + \frac{\epsilon}{24}$ and $g_j = \psi(t) u_j$. Then, we have*

$$\left\| \partial_x \Big(\prod_{j=1}^{k+1} g_j\Big) \right\|_{X_{s,-\frac{1}{2}+\frac{\epsilon}{12}}} \leq C \prod_{j=1}^{k+1} \|g_j\|_{X_{s,b}}. \tag{3.1}$$

Proof. To prove (3.1), by duality, it suffices to prove

$$\left| \int_{\mathbf{R}^2} J^s \partial_x \left(\prod_{j=1}^{k+1} g_j \right) \bar{h} \mathrm{d}x \mathrm{d}t \right| \leq C \|h\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}} \prod_{j=1}^{k+1} \|g_j\|_{X_{s,b}}. \tag{3.2}$$

We define

$$f(\xi,\tau) = \langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}} \mathscr{F} h(\xi,\tau), \; f_j(\xi_j,\tau_j) = \langle\xi_j\rangle^s \langle\sigma_j\rangle^b \mathscr{F} g_j(\xi_j,\tau_j) (1 \leq j \leq k+1).$$

By using the Plancherel identity, to prove (3.2), it suffices to prove

$$\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j} \int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j} \frac{|\xi|\langle\xi\rangle^s f \prod\limits_{j=1}^{k+1} f_j}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}} \prod\limits_{j=1}^{k+1} \langle\xi_j\rangle^s \langle\sigma_j\rangle^b} \mathrm{d}\delta \leq C \|f\|_{L^2} \prod_{j=1}^{k+1} \|f_j\|_{L^2}, \tag{3.3}$$

where $\mathrm{d}\delta = \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_k \mathrm{d}\xi \mathrm{d}\tau_1 \mathrm{d}\tau_2 \cdots \mathrm{d}\tau_k \mathrm{d}\tau$.

We define

$$K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)=\frac{|\xi|\langle\xi\rangle^s}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\xi_j\rangle^s\langle\sigma_j\rangle^b},$$

$$\mathscr{F}F=\frac{f}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}},\ \mathscr{F}F_j=\frac{f_j}{\langle\sigma_j\rangle^b}(1\leq j\leq k+1),$$

$$I_1=\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)f\prod_{j=1}^{k+1}f_j\mathrm{d}\delta.$$

Without loss of generality, we can assume that $|\xi_1|\geq|\xi_2|\geq\cdots\geq|\xi_{k+1}|$. Obviously,

$$\Omega:=\left\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\mathbf{R}^{2(k+1)}:|\xi_1|\geq|\xi_2|\geq\cdots\geq|\xi_{k+1}|\right\}\subset\bigcup_{j=0}^{7}\Omega_j.$$

Here,

$$\begin{aligned}
&\Omega_0=\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega,|\xi_1|\leq 80ka\},\\
&\Omega_1=\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega,|\xi_1|\geq 80ka,|\xi_1|\geq 80ka|\xi_2|\},\\
&\Omega_2=\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega,|\xi_1|\geq 80ka,|\xi_1|\sim|\xi_2|\geq 80ka|\xi_3|\},\\
&\Omega_3=\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega,|\xi_1|\geq 80ka,|\xi_1|\sim|\xi_3|\geq 80ka|\xi_4|\},\\
&\Omega_4=\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega,|\xi_1|\geq 80ka,|\xi_1|\sim|\xi_4|\geq 80ka|\xi_5|\},\\
&\Omega_5=\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega,|\xi_1|\geq 80ka,|\xi_1|\sim|\xi_5|\geq 80ka|\xi_6|\},\\
&\Omega_6=\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega,|\xi_1|\geq 80ka,|\xi_1|\sim|\xi_{l-1}|\\
&\qquad\geq 80ka|\xi_l|(7\leq l\leq k+1)\},\\
&\Omega_7=\{(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega,|\xi_1|\geq 80ka,|\xi_1|\sim|\xi_{k+1}|\}.
\end{aligned}$$

(1) Case $(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega_0$, by using the Plancherel identity, the Hölder inequality and (2.4), we have

$$\begin{aligned}
I_1&\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f\prod\limits_{j=1}^{k+1}f_j}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\xi_j\rangle^s\langle\sigma_j\rangle^b}\mathrm{d}\delta\\
&\leq C\|F\|_{L^2_{xt}}\prod_{j=1}^{k+1}\|F_j\|_{L^{2(k+1)}_{xt}}\leq C\|f\|_{L^2}\prod_{j=1}^{k+1}\|F_j\|_{X_{\frac{1}{2}-\frac{2}{k+1},b}}\leq C\|f\|_{L^2}\prod_{j=1}^{k+1}\|f_j\|_{L^2}.
\end{aligned}\tag{3.4}$$

(2) Case $(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \in \Omega_1$, we have $|\xi| \sim |\xi_1|$, then we consider

$$\left|1 + \frac{\gamma}{3\beta\xi_1^2\xi_2^2}\right| \geq \frac{1}{2}, \tag{3.5}$$

$$\left|1 + \frac{\gamma}{3\beta\xi_1^2\xi_2^2}\right| < \frac{1}{2}. \tag{3.6}$$

When (3.5) is valid, we have

$$\begin{aligned}
\left|\phi'(\xi_1) - \phi'(\xi_2)\right| &= \left|3\beta\xi_1^2 - \frac{\gamma}{\xi_1^2} + \frac{\gamma}{\xi_2^2} - 3\beta\xi_2^2\right| \\
&= \left|3\beta(\xi_1^2 - \xi_2^2) + \frac{\gamma(\xi_1^2 - \xi_2^2)}{\xi_1^2\xi_2^2}\right| = \left|3\beta + \frac{\gamma}{\xi_1^2\xi_2^2}\right|\left|\xi_1^2 - \xi_2^2\right| \\
&= |3\beta|\left|1 + \frac{\gamma}{3\beta\xi_1^2\xi_2^2}\right|\left|\xi_1^2 - \xi_2^2\right| \geq \frac{|3\beta|}{2}\left|\xi_1^2 - \xi_2^2\right|.
\end{aligned}$$

Thus,

$$\begin{aligned}
K(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) &\leq C\frac{|\xi_1^2 - \xi_2^2|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=2}^{k+1}\langle\xi_j\rangle^s\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b} \\
&\leq C\frac{|\phi'(\xi_1) - \phi'(\xi_2)|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=2}^{k+1}\langle\xi_j\rangle^s\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}.
\end{aligned} \tag{3.7}$$

By using (3.7), the Plancherel identity, the Hölder inequality, Lemma 2.5, (2.4) and (2.7), we have

$$\begin{aligned}
I_1 &\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f(\prod\limits_{j=1}^{k+1}f_j)|\phi'(\xi_1) - \phi'(\xi_2)|^{\frac{1}{2}}\prod\limits_{j=2}^{k+1}\langle\xi_j\rangle^{-s}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}\mathrm{d}\delta \\
&\leq C\left\|I^{1/2}(F_1, F_2)\right\|_{L_{xt}^2}\left(\prod_{j=3}^{k+1}\|J_x^{-\frac{ks}{k-1}}F_j\|_{L_{xt}^{\frac{8(k-1)}{3-\epsilon}}}\right)\|F\|_{L_{xt}^{\frac{8}{1+\epsilon}}} \\
&\leq C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}} \leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L_{\xi\tau}^2}\right)\|f\|_{L_{\xi\tau}^2}.
\end{aligned}$$

When (3.6) is valid, we have $|\xi_1| \sim |\xi_2|^{-1}$. In this case, we consider

$$\left|1 + \frac{\gamma}{3\beta\xi^2\xi_{k+1}^2}\right| \geq \frac{1}{2}, \tag{3.8}$$

$$\left|1+\frac{\gamma}{3\beta\xi^2\xi_{k+1}^2}\right|<\frac{1}{2}, \tag{3.9}$$

respectively.

When (3.8) is valid, since $|\xi|\sim|\xi_1|\sim|\xi_2|^{-1}$, $s\geq\frac{1}{2}-\frac{2}{k}+2\epsilon$, we have

$$K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\leq C\frac{|\xi_1||\xi_2|^{\frac{1}{4}+\epsilon}|\xi|^{1-4\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}. \tag{3.10}$$

By using (3.10), the Plancherel identity, the Hölder inequality, Lemma 2.5, (2.8), (2.9) and (2.57), we have

$$\begin{aligned}
I_1&\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f(\prod_{j=1}^{k+1}f_j)|\phi'(\xi)-\phi'(\xi_{k+1})|^{\frac{1}{2}-2\epsilon}|\xi_1||\xi_2|^{\frac{1}{4}+\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod_{j=1}^{k+1}\langle\sigma_j\rangle^b}\mathrm{d}\delta\\
&\leq C\left\|I^{\frac{1}{2}-2\epsilon}(F_{k+1},F)\right\|_{L^2_{xt}}\left(\prod_{j=3}^{k}\|F_j\|_{L^\infty_{xt}}\right)\|D_xF_1\|_{L^\infty_xL^2_t}\|D_x^{\frac{1}{4}+\epsilon}F_2\|_{L^2_xL^\infty_t}\\
&\leq C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}}\leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L^2_{\xi\tau}}\right)\|f\|_{L^2_{\xi\tau}}.
\end{aligned}$$

When (3.9) is valid, we have that $|\xi_2|^{-1}\sim|\xi_1|\sim|\xi|\sim|\xi_{k+1}|^{-1}$, since $s\geq\frac{1}{2}-\frac{2}{k}+2\epsilon$, we have

$$K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\leq C\frac{|\xi_1||\xi_2|^{\frac{1}{4}+\epsilon}|\xi_{k+1}|^{\frac{1}{4}}|\xi|^{1-2\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}. \tag{3.11}$$

By using (3.11), the Plancherel identity, the Hölder inequality, (2.8)–(2.10), (2.57), (2.58), we have

$$\begin{aligned}
I_1&\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f(\prod_{j=1}^{k+1}f_j)|\xi_1||\xi_2|^{\frac{1}{4}}|\xi_{k+1}|^{\frac{1}{4}}|\xi|^{1-2\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod_{j=1}^{k+1}\langle\sigma_j\rangle^b}\mathrm{d}\delta\\
&\leq C\|D_xF_1\|_{L^\infty_xL^2_t}\|D_x^{\frac{1}{4}+\epsilon}F_2\|_{L^2_xL^\infty_t}\left(\prod_{j=3}^{k}\|F_j\|_{L^\infty_{xt}}\right)\|D_x^{\frac{1}{4}}F_{k+1}\|_{L_x^{\frac{2}{1-2\epsilon}}L^\infty_t}\|D_x^{1-2\epsilon}F\|_{L_x^{\frac{1}{\epsilon}}L^2_t}\\
&\leq C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}}\leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L^2_{\xi\tau}}\right)\|f\|_{L^2_{\xi\tau}}.
\end{aligned}$$

(3) Case $(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega_2$, we have $|\xi_1|\sim|\xi_2|\geq 80ka|\xi_3|$, and then we consider

$$\left|1+\frac{\gamma}{3\beta\xi_2^2\xi_3^2}\right|\geq\frac{1}{2}, \tag{3.12}$$

$$\left|1+\frac{\gamma}{3\beta\xi_2^2\xi_3^2}\right|<\frac{1}{2}. \tag{3.13}$$

When (3.12) is valid, since $s\geq\frac{1}{2}-\frac{2}{k}+2\epsilon$, $|\xi_1|\sim|\xi_2|$, $|\xi|\leq C|\xi_1|$, we have

$$\begin{aligned}
K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)&\leq C\frac{|\xi_2^2-\xi_3^2|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod_{j=2}^{k+1}\langle\xi_j\rangle^s\prod_{j=1}^{k+1}\langle\sigma_j\rangle^b}\\
&\leq C\frac{|\xi_2^2-\xi_3^2|^{\frac{1}{2}}|\xi_1|^{\frac{1}{6}}\prod\limits_{j=4}^{k+1}\langle\xi_j\rangle^{-\frac{3k-11}{6(k-2)}-2k\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}\\
&\leq C\frac{|\phi'(\xi_2)-\phi'(\xi_3)|^{\frac{1}{2}}|\xi_1|^{\frac{1}{6}}\prod_{j=4}^{k+1}\langle\xi_j\rangle^{-\frac{3k-11}{6(k-2)}-2k\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod_{j=1}^{k+1}\langle\sigma_j\rangle^b}.
\end{aligned} \tag{3.14}$$

By using (3.14), the Plancherel identity, the Hölder inequality, Lemma 2.5, (2.4), (2.5), (2.7), we have

$$\begin{aligned}
I_1&\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f(\prod\limits_{j=1}^{k+1}f_j)|\phi'(\xi_2)-\phi'(\xi_3)|^{\frac{1}{2}}|\xi_1|^{\frac{1}{6}}\prod\limits_{j=4}^{k+1}\langle\xi_j\rangle^{-\frac{3k-11}{6(k-2)}-2\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}\mathrm{d}\delta\\
&\leq C\left\|I^{1/2}(F_2,F_3)\right\|_{L^2_{xt}}\left(\prod_{j=4}^{k+1}\|J_x^{-\frac{3k-11}{6(k-2)}-2\epsilon}F_j\|_{L_{xt}^{\frac{24(k-2)}{5-3\epsilon}}}\right)\|D_x^{\frac{1}{6}}F_1\|_{L^6_{xt}}\|F\|_{L_{xt}^{\frac{8}{1+\epsilon}}}\\
&\leq C\left(\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}}\right)\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L^2_{\xi\tau}}\right)\|f\|_{L^2_{\xi\tau}}.
\end{aligned}$$

When (3.13) is valid, we have $|\xi_1|\sim|\xi_2|\sim|\xi_3|^{-1}$, and we consider $|\xi|\leq a$, $|\xi|\geq a$, respectively.

When $|\xi|\leq a$, we have

$$K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\leq C\frac{\prod\limits_{j=1}^{k}\langle\xi_j\rangle^{-s}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}. \tag{3.15}$$

By using (3.15), the Plancherel identity, the Hölder inequality, (2.4) and (2.57), we have

$$I_1 \le C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{\prod\limits_{j=1}^{k}\langle\xi_j\rangle^{-s}f(\prod\limits_{j=1}^{k+1}f_j)}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}\mathrm{d}\delta$$
$$\le C\left(\prod_{j=1}^{k}\|J_x^{-s}F_j\|_{L^{2k}_{xt}}\right)\|F_{k+1}\|_{L^\infty_{xt}}\|F\|_{L^2_{xt}}$$
$$\le C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}} \le C\left(\prod_{j=1}^{k+1}\|f_j\|_{L^2_{\xi\tau}}\right)\|f\|_{L^2_{\xi\tau}}.$$

When $|\xi|\ge a$, we consider (3.8), (3.9), respectively.

When (3.8) is valid, since $|\xi_1|\sim|\xi_2|\sim|\xi_3|^{-1}$, $s\ge\frac{1}{2}-\frac{2}{k}+2\epsilon$, we have

$$K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\le C\frac{|\xi_2||\xi_1|^{-\frac{1}{2}-\epsilon}|\xi_3|^{\frac{1}{4}+\epsilon}|\xi|^{1-4\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}. \quad (3.16)$$

By using (3.16), the Plancherel identity, the Hölder inequality, Lemmas 2.5, 2.11, (2.8), (2.9) and (2.57), we have

$$I_1\le C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f(\prod\limits_{j=1}^{k+1}f_j)|\phi'(\xi)-\phi'(\xi_{k+1})|^{\frac{1}{2}-2\epsilon}|\xi_1|^{-\frac{1}{2}-4\epsilon}|\xi_2||\xi_3|^{\frac{1}{4}+\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}\mathrm{d}\delta$$
$$\le C\left\|I^{\frac{1}{2}-2\epsilon}(F_{k+1},F)\right\|_{L^2_{xt}}\left\|D_x^{-\frac{1}{2}-4\epsilon}F_1\right\|_{L^\infty_{xt}}\left(\prod_{j=4}^{k}\|F_j\|_{L^\infty_{xt}}\right)\|D_xF_2\|_{L^\infty_xL^2_t}\|D_x^{\frac{1}{4}+\epsilon}F_3\|_{L^2_xL^\infty_t}$$
$$\le C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}}\le C\left(\prod_{j=1}^{k+1}\|f_j\|_{L^2_{\xi\tau}}\right)\|f\|_{L^2_{\xi\tau}}.$$

When (3.9) is valid, since $|\xi|\sim|\xi_{k+1}|^{-1}$, $|\xi_1|\sim|\xi_2|\sim|\xi_3|^{-1}$, $s\ge\frac{1}{2}-\frac{2}{k}+2\epsilon$; thus, we have

$$K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\le C\frac{|\xi_1|^{-\frac{1}{2}-4\epsilon}|\xi_2||\xi_3|^{\frac{1}{4}+\epsilon}|\xi_{k+1}|^{\frac{1}{4}}|\xi|^{1-2\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}} \quad (3.17)$$

By using (3.17), the Plancherel identity, the Hölder inequality, Lemma 2.11, (2.8)–(2.10), (2.57), (2.58), we have

$$
\begin{aligned}
I_1 &\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f(\prod\limits_{j=1}^{k+1}f_j)|\xi_1|^{-\frac{1}{2}-4\epsilon}|\xi_2||\xi_3|^{\frac{1}{4}+\epsilon}|\xi_{k+1}|^{\frac{1}{4}}|\xi|^{1-2\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}\mathrm{d}\delta\\
&\leq C\|D_xF_2\|_{L_x^\infty L_t^2}\|D_x^{\frac{1}{4}+\epsilon}F_3\|_{L_x^2L_t^\infty}\|D_x^{-\frac{1}{2}-4\epsilon}F_1\|_{L_{xt}^\infty}\left(\prod_{j=4}^{k}\|F_j\|_{L_{xt}^\infty}\right)\\
&\quad\times\|D_x^{\frac{1}{4}}F_{k+1}\|_{L_x^{\frac{2}{1-2\epsilon}}L_t^\infty}\|D_x^{1-2\epsilon}F\|_{L_x^{\frac{1}{\epsilon}}L_t^2}\\
&\leq C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}}\leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L^2_{\xi\tau}}\right)\|f\|_{L^2_{\xi\tau}}.
\end{aligned}
$$

(4) When $(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega_3$, we have $|\xi_1|\sim|\xi_3|$, then we consider

$$|1+\frac{\gamma}{3\beta\xi_3^2\xi_4^2}|\geq\frac{1}{2}, \tag{3.18}$$

$$|1+\frac{\gamma}{3\beta\xi_3^2\xi_4^2}|<\frac{1}{2}, \tag{3.19}$$

respectively.

When (3.18) is valid, since $|\xi_1|\sim|\xi_3|$, $|\xi|\leq C|\xi_1|$, we have

$$
\begin{aligned}
K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau) &\leq C\frac{|\xi_3^2-\xi_4^2|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=2}^{k+1}\langle\xi_j\rangle^{s}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}\\
&\leq C\frac{|\phi'(\xi_3)-\phi'(\xi_4)|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=2}^{k+1}\langle\xi_j\rangle^{s}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}.
\end{aligned}
\tag{3.20}
$$

By using (3.20), the Plancherel identity, the Hölder inequality and (2.4), (2.6), (2.7) as well as Lemma 2.5, we have

$$
\begin{aligned}
I_1 &\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f|\phi'(\xi_3)-\phi'(\xi_4)|^{\frac{1}{2}}\prod\limits_{j=2}^{k+1}\langle\xi_j\rangle^{-s}(\prod\limits_{j=1}^{k+1}f_j)}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}\mathrm{d}\delta\\
&\leq C\,\|F_1\|_{L^8_{xt}}\,\|F_2\|_{L^8_{xt}}\,\|F\|_{L_{xt}^{\frac{8}{1+\epsilon}}}\|I^{\frac{1}{2}}(F_3,F_4)\|_{L^2_{xt}}\prod_{j=5}^{k+1}\|J_x^{-\frac{k-4}{2(k-3)}-\epsilon}F_j\|_{L_{xt}^{\frac{8(k-3)}{1-\epsilon}}}
\end{aligned}
$$

$$\leq C\left(\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}}\right)\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}} \leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L^2_{\xi\tau}}\right)\|f\|_{L^2_{\xi\tau}}.$$

When (3.19) is valid, we have $|\xi_1| \sim |\xi_3| \sim |\xi_4|^{-1}$.
We consider $|\xi| \leq a$, $|\xi| \geq a$, respectively.
When $|\xi| \leq a$, we have

$$K(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \leq C\frac{\prod\limits_{j=1}^{k}\langle\xi_j\rangle^{-s}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}. \tag{3.21}$$

By using (3.21), the Plancherel identity, the Hölder inequality, (2.4) and (2.57), we have

$$I_1 \leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{\prod\limits_{j=1}^{k}\langle\xi_j\rangle^{-s} f(\prod\limits_{j=1}^{k+1} f_j)}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}\mathrm{d}\delta$$

$$\leq C\left(\prod_{j=1}^{k}\|J_x^{-s}F_j\|_{L^{2k}_{xt}}\right)\|F_{k+1}\|_{L^{\infty}_{xt}}\|F\|_{L^2_{xt}}$$

$$\leq C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}} \leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L^2_{\xi\tau}}\right)\|f\|_{L^2_{\xi\tau}}.$$

When $|\xi| \geq a$, we consider

$$|1+\frac{\gamma}{3\beta\xi_3^2\xi_5^2}| \geq \frac{1}{2}, \tag{3.22}$$

$$|1+\frac{\gamma}{3\beta\xi_3^2\xi_5^2}| < \frac{1}{2}, \tag{3.23}$$

respectively.
When (3.22) is valid, this case can be proved similarly to (3.18).
When (3.23) is valid, we have that $|\xi_1| \sim |\xi_3| \sim |\xi_4|^{-1} \sim |\xi_5|^{-1}$, we consider (3.8), (3.9), respectively.
When (3.8) is valid, since $|\xi_1| \sim |\xi_3| \sim |\xi_4|^{-1} \sim |\xi_5|^{-1}$, $s \geq \frac{1}{2} - \frac{2}{k} + 2\epsilon$, we have

$$K(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \leq C\frac{|\xi|^{1-4\epsilon}|\xi_3||\xi_4|^{-\frac{1}{4}+\epsilon}|\xi_5|^{\frac{1}{4}+\epsilon}\prod\limits_{j=1}^{2}|\xi_j|^{-\frac{1}{2}-4\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}. \tag{3.24}$$

By using (3.24), the Plancherel identity, the Hölder inequality, Lemmas 2.5, 2.11, (2.8) and (2.9), (2.57), we have

$$I_1 \le C \int_{\xi=\sum_{j=1}^{k+1}\xi_j} \int_{\tau=\sum_{j=1}^{k+1}\tau_j} \frac{|\xi|^{1-4\epsilon}|\xi_3||\xi_4|^{\frac{1}{4}+\epsilon}|\xi_5|^{-\frac{1}{4}+\epsilon}\prod_{j=1}^{2}|\xi_j|^{-\frac{1}{2}-4\epsilon} f(\prod_{j=1}^{k+1} f_j)}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod_{j=1}^{k+1}\langle\sigma_j\rangle^b} \mathrm{d}\delta$$

$$\le C\left[\prod_{j=1}^{2}\left\|D_x^{-\frac{1}{2}-4\epsilon}F_j\right\|_{L_{xt}^\infty}\right]\|D_xF_3\|_{L_x^\infty L_t^2}\|D_x^{\frac{1}{4}+\epsilon}F_4\|_{L_x^2L_t^\infty}\|D_x^{-\frac{1}{4}+\epsilon}F_5\|_{L_{xt}^\infty}\|I^{\frac{1}{2}-2\epsilon}$$

$$(F, F_{k+1})\|_{L_{xt}^2}\prod_{j=6}^{k}\|F_j\|_{L_{xt}^\infty}$$

$$\le C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}} \le C\left(\prod_{j=1}^{k+1}\|f_j\|_{L_{\xi\tau}^2}\right)\|f\|_{L_{\xi\tau}^2}.$$

When (3.9) is valid, since $|\xi| \sim |\xi_{k+1}|^{-1}$, $|\xi_1| \sim |\xi_3| \sim |\xi_4|^{-1} \sim |\xi_5|^{-1}$, $s \ge \frac{1}{2} - \frac{2}{k} + 2\epsilon$, we have

$$K(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \le C\frac{|\xi_{k+1}|^{\frac{1}{4}}|\xi|^{1-2\epsilon}\prod_{j=1}^{3}|\xi_j|^{\frac{1}{6}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod_{j=1}^{k+1}\langle\sigma_j\rangle^b}. \tag{3.25}$$

By using (3.25), the Plancherel identity, the Hölder inequality, (2.5) and (2.10), (2.58), we have

$$I_1 \le C \int_{\xi=\sum_{j=1}^{k+1}\xi_j} \int_{\tau=\sum_{j=1}^{k+1}\tau_j} \frac{|\xi_{k+1}|^{\frac{1}{4}}|\xi|^{1-2\epsilon}\prod_{j=1}^{3}|\xi_j|^{\frac{1}{6}} f(\prod_{j=1}^{k+1} f_j)}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod_{j=1}^{k+1}\langle\sigma_j\rangle^b} \mathrm{d}\delta$$

$$\le C\left[\prod_{j=1}^{3}\left\|D_x^{\frac{1}{6}}F_j\right\|_{L_{xt}^6}\right]\left[\prod_{j=4}^{k}\|F_j\|_{L_{xt}^\infty}\right]\|D_x^{\frac{1}{4}}F_{k+1}\|_{L_x^{\frac{2}{1-2\epsilon}}L_t^\infty}\|D_x^{1-2\epsilon}F\|_{L_x^{\frac{1}{\epsilon}}L_t^2}$$

$$\le C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}} \le C\left(\prod_{j=1}^{k+1}\|f_j\|_{L_{\xi\tau}^2}\right)\|f\|_{L_{\xi\tau}^2}.$$

(5) Case $(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \in \Omega_4$, this case can be proved similarly to Case (4).

(6) Case $(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \in \Omega_5$, we have $|\xi_1| \sim |\xi_5|$, then we consider

$$|1 + \frac{\gamma}{3\beta\xi_5^2\xi_6^2}| \ge \frac{1}{2}, \tag{3.26}$$

$$|1+\frac{\gamma}{3\beta\xi_5^2\xi_6^2}|<\frac{1}{2}, \tag{3.27}$$

respectively.

When (3.26) is valid, we consider $k=5$, $k\geq 6$, respectively.

When $k=5$, since $|\xi_1|\sim|\xi_5|$, $s\geq\frac{1}{2}-\frac{2}{k}+2\epsilon$, we have

$$K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\leq C\frac{\langle\xi_6\rangle^{-\frac{1}{3}-3\epsilon}\prod\limits_{j=1}^{5}|\xi_j|^{\frac{1}{6}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{6}\langle\sigma_j\rangle^{b}}. \tag{3.28}$$

By using (3.28), the Plancherel identity, the Hölder inequality and (2.4), (2.5) and (2.7), we have

$$\begin{aligned}
I_1&\leq C\int_{\xi=\sum\limits_{j=1}^{6}\xi_j}\int_{\tau=\sum\limits_{j=1}^{6}\tau_j}\frac{ff_6(\prod\limits_{j=1}^{5}|\xi_j|^{\frac{1}{6}}f_j)\langle\xi_6\rangle^{-\frac{1}{3}-3\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{6}\langle\sigma_j\rangle^{b}}\mathrm{d}\delta\\
&\leq C\|F\|_{L_{xt}^{\frac{8}{1+\epsilon}}}\|J_x^{-\frac{1}{3}-3\epsilon}F_6\|_{L_{xt}^{\frac{24}{1-3\epsilon}}}\prod_{j=1}^{5}\left\|D_x^{\frac{1}{6}}F_j\right\|_{L_{xt}^{6}}\\
&\leq C\left(\prod_{j=1}^{6}\|f_j\|_{L_{xt}^2}\right)\|f\|_{L_{xt}^2}.
\end{aligned}$$

When $k\geq 6$, since $|\xi_1|\sim|\xi_5|$, $s\geq\frac{1}{2}-\frac{2}{k}+2\epsilon$, we have

$$\begin{aligned}
K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)&\leq C\frac{|\xi_5^2-\xi_6^2|^{\frac{1}{2}}\prod\limits_{j=1}^{4}\langle\xi_j\rangle^{-s}\prod\limits_{j=7}^{k+1}\langle\xi_j\rangle^{-\frac{(k-4)s}{(k-5)}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}\\
&\leq C\frac{|\phi'(\xi_5)-\phi'(\xi_6)|^{\frac{1}{2}}\prod\limits_{j=1}^{4}\langle\xi_j\rangle^{-s}\prod\limits_{j=7}^{k+1}\langle\xi_j\rangle^{-\frac{(k-4)s}{(k-5)}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}.
\end{aligned} \tag{3.29}$$

By using (3.29), the Plancherel identity, the Hölder inequality and (2.4), (2.7), Lemma 2.5, we have

$$I_1\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{|\phi'(\xi_5)-\phi'(\xi_6)|^{\frac{1}{2}}f(\prod\limits_{j=1}^{k+1}f_j)\prod\limits_{j=1}^{4}\langle\xi_j\rangle^{-s}\prod\limits_{j=7}^{k+1}\langle\xi_j\rangle^{-\frac{(k-4)s}{(k-5)}}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^{b}}d\delta$$

$$\leq C\|F\|_{L_{xt}^{\frac{8}{1+\epsilon}}}\left(\prod_{j=1}^{4}\|J_x^{-s}F_j\|_{L_{xt}^{2k}}\right)\|I^{\frac{1}{2}}(F_5,F_6)\|_{L_{xt}^2}\prod_{j=7}^{k+1}\|J_x^{-\frac{(k-4)s}{(k-5)}}F_j\|_{L_{xt}^{\frac{8k(k-5)}{3k-16-k\epsilon}}}$$
$$\leq C\left(\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}}\right)\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L_{xt}^2}\right)\|f\|_{L_{xt}^2}.$$

When (3.27) is valid, we have $|\xi_1|\sim|\xi_5|\sim|\xi_6|^{-1}$. If there exists some $k\in N, 7\leq k\leq k+1$ such that

$$|1+\frac{\gamma}{3\beta\xi_5^2\xi_k^2}|\geq\frac{1}{2},$$

we can use a proof similar to (3.26) to derive the result, otherwise we have that $|\xi_1|\sim|\xi_5|\sim|\xi_6|^{-1}\sim|\xi_{k+1}|^{-1}$.
In this case, since $|\xi_1|\sim|\xi_5|\sim|\xi_6|^{-1}\sim|\xi_{k+1}|^{-1}$, $s\geq\frac{1}{2}-\frac{2}{k}+2\epsilon$, we have

$$K(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\leq C\frac{|\xi|^{1-2\epsilon}|\xi_1|\prod\limits_{j=2}^{5}|\xi_j|^{-\frac{1}{4}-2\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}. \tag{3.30}$$

By using (3.30), the Plancherel identity, the Hölder inequality, (2.8), (2.10), (2.46) and (2.57), we have

$$I_1\leq C\int_{\xi=\sum\limits_{j=1}^{k+1}\xi_j}\int_{\tau=\sum\limits_{j=1}^{k+1}\tau_j}\frac{f(\prod\limits_{j=1}^{k+1}f_j)|\xi|^{1-2\epsilon}|\xi_1|\prod\limits_{j=2}^{5}|\xi_j|^{-\frac{1}{4}-2\epsilon}}{\langle\sigma\rangle^{\frac{1}{2}-\frac{\epsilon}{12}}\prod\limits_{j=1}^{k+1}\langle\sigma_j\rangle^b}d\delta$$
$$\leq C\|D_xF_1\|_{L_x^\infty L_t^2}\left(\prod_{j=2}^{5}\|D_x^{-\frac{1}{4}-2\epsilon}F_j\|_{L_x^{\frac{4}{1-\epsilon}}L_t^\infty}\right)\left(\prod_{j=6}^{k+1}\|F_j\|_{L_{xt}^\infty}\right)\|D_x^{1-2\epsilon}F\|_{L_x^{\frac{1}{\epsilon}}L_t^2}$$
$$\leq C\|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}}\prod_{j=1}^{k+1}\|F_j\|_{X_{0,b}}\leq C\left(\prod_{j=1}^{k+1}\|f_j\|_{L_{xt}^2}\right)\|f\|_{L_{xt}^2}.$$

(7) Case $(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega_6$, this case can be proved similarly to $(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega_5$.

(8) Case $(\xi_1,\xi_2,\ldots,\xi_k,\xi,\tau_1,\tau_2,\ldots,\tau_k,\tau)\in\Omega_7$, we have $|\xi_1|\sim|\xi_{k+1}|$, and then we consider $|\xi|\leq a$, $|\xi|\geq a$, respectively.

When $|\xi|\leq a$, we consider (3.8), (3.9), respectively.

When (3.8) is valid, we have

$$K(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \leq C \frac{|\xi_{k+1}|^{1-4\epsilon} \prod_{j=1}^{k+1} \langle \xi_j \rangle^{-s}}{\langle \sigma \rangle^{\frac{1}{2}-\frac{\epsilon}{12}} \prod_{j=1}^{k+1} \langle \sigma_j \rangle^{b}}. \quad (3.31)$$

By using (3.31), the Plancherel identity, the Hölder inequality and (2.4), Lemma 2.5, we have

$$\begin{aligned} I_1 &\leq C \int_{\xi=\sum_{j=1}^{k+1}\xi_j} \int_{\tau=\sum_{j=1}^{k+1}\tau_j} \frac{|\xi_{k+1}|^{1-4\epsilon} f(\prod_{j=1}^{k+1} f_j) \prod_{j=1}^{k+1} \langle \xi_j \rangle^{-s}}{\langle \sigma \rangle^{\frac{1}{2}-\frac{\epsilon}{12}} \prod_{j=1}^{k+1} \langle \sigma_j \rangle^{b}} d\delta \\ &\leq C \left[\prod_{j=1}^{k} \|D_x^{-s} F_j\|_{L_{xt}^{2k}} \right] \|I^{\frac{1}{2}-2\epsilon}(F_{k+1}, F)\|_{L_{xt}^2} \\ &\leq C \left(\prod_{j=1}^{k+1} \|F_j\|_{X_{0,b}} \right) \|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}} \leq C \left(\prod_{j=1}^{k+1} \|f_j\|_{L_{\xi\tau}^2} \right) \|f\|_{L_{\xi\tau}^2}. \end{aligned}$$

When (3.9) is valid, we have $|\xi| \sim |\xi_{k+1}|^{-1}$ we consider $|\sigma| \geq |\xi_1|, |\sigma| \leq |\xi_1|$, respectively.
When $|\sigma| \geq |\xi_1|$, we have

$$K(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \leq C \frac{|\xi_1|^{-\frac{1}{2}+4\epsilon} \prod_{j=1}^{k+1} \langle \xi_j \rangle^{-s}}{\prod_{j=1}^{k+1} \langle \sigma_j \rangle^{b}}. \quad (3.32)$$

By using (3.32), the Plancherel identity, the Hölder inequality and (2.4), we have

$$\begin{aligned} I_1 &\leq C \int_{\xi=\sum_{j=1}^{k+1}\xi_j} \int_{\tau=\sum_{j=1}^{k+1}\tau_j} \frac{f(\prod_{j=1}^{k+1} f_j) \prod_{j=1}^{k+1} \langle \xi_j \rangle^{-s-\frac{\frac{1}{2}-4\epsilon}{k+1}}}{\prod_{j=1}^{k+1} \langle \sigma_j \rangle^{b}} d\delta \\ &\leq C \left[\prod_{j=1}^{k+1} \|D_x^{-s-\frac{\frac{1}{2}-4\epsilon}{k+1}} F_j\|_{L_{xt}^{2(k+1)}} \right] \|\mathscr{F}^{-1} f\|_{L_{xt}^2} \\ &\leq C \left(\prod_{j=1}^{k+1} \|F_j\|_{X_{0,b}} \right) \|f\|_{L_{\xi\tau}^2} \leq C \left(\prod_{j=1}^{k+1} \|f_j\|_{L_{\xi\tau}^2} \right) \|f\|_{L_{\xi\tau}^2}. \end{aligned}$$

When $|\sigma| \leq |\xi_1|$, since $|\xi_1| \sim |\xi_{k+1}| \geq 80ka$, $|\xi| \sim |\xi_{k+1}|^{-1}$, $|\xi| \leq a$, we have

$$K(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \leq C \frac{|\xi_1|^{\frac{1}{2}} \prod_{j=1}^{k+1} \langle \xi_j \rangle^{-s}}{\langle \sigma \rangle^{\frac{1}{2}-\frac{\epsilon}{12}} \prod_{j=1}^{k+1} \langle \sigma_j \rangle^{b}}. \tag{3.33}$$

By using (3.33), the Plancherel identity, the Hölder inequality and (2.4), (2.10), (2.58), we have

$$\begin{aligned} I_1 &\leq C \int_{\xi=\sum_{j=1}^{k+1}\xi_j} \int_{\tau=\sum_{j=1}^{k+1}\tau_j} \frac{|\xi_1|^{\frac{1}{2}} f(\prod_{j=1}^{k+1} f_j) \prod_{j=1}^{k+1} \langle \xi_j \rangle^{-s}}{\langle \sigma \rangle^{\frac{1}{2}-\frac{\epsilon}{12}} \prod_{j=1}^{k+1} \langle \sigma_j \rangle^{b}} d\delta \\ &\leq C \left[\prod_{j=1}^{k} \|D_x^{-s} F_j\|_{L_{xt}^{2k}}\right] \|D_x^{1-2\epsilon} F_{k+1}\|_{L_x^{\frac{1}{\epsilon}} L_t^2} \|D_x^{\frac{1}{4}} F\|_{L_x^{\frac{2}{1-2\epsilon}} L_t^{\infty}} \\ &\leq C \left(\prod_{j=1}^{k+1} \|F_j\|_{X_{0,b}}\right) \|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}} \leq C \left(\prod_{j=1}^{k+1} \|f_j\|_{L_{xt}^2}\right) \|f\|_{L_{xt}^2}. \end{aligned}$$

When $|\xi| \geq a$, since $|\xi_1| \sim |\xi_{k+1}| \geq 80ka$, we have

$$K(\xi_1, \xi_2, \ldots, \xi_k, \xi, \tau_1, \tau_2, \ldots, \tau_k, \tau) \leq C \frac{|\xi|^{\frac{3-\epsilon}{18}} |\xi_1|^{\frac{1}{6}} |\xi_2|^{\frac{1}{6}} \prod_{j=3}^{k+1} \langle \xi_j \rangle^{\frac{1-2ks}{2(k-1)}}}{\langle \sigma \rangle^{\frac{1}{2}-\frac{\epsilon}{12}} \prod_{j=1}^{k+1} \langle \sigma_j \rangle^{b}}. \tag{3.34}$$

By using (3.34), the Plancherel identity, the Hölder inequality and (2.4), (2.5), (2.11), we have

$$\begin{aligned} I_1 &\leq C \int_{\xi=\sum_{j=1}^{k+1}\xi_j} \int_{\tau=\sum_{j=1}^{k+1}\tau_j} \frac{|\xi|^{\frac{3-\epsilon}{18}} |\xi_1|^{\frac{1}{6}} |\xi_2|^{\frac{1}{6}} \prod_{j=3}^{k+1} \langle \xi_j \rangle^{\frac{1-2ks}{2(k-1)}} f(\prod_{j=1}^{k+1} f_j)}{\langle \sigma \rangle^{\frac{1}{2}-\frac{\epsilon}{12}} \prod_{j=1}^{k+1} \langle \sigma_j \rangle^{b}} d\delta \\ &\leq C \|D_x^{\frac{3-\epsilon}{18}} F\|_{L_{xt}^{\frac{6}{1+\epsilon}}} \left[\prod_{j=1}^{2} \|D_x^{\frac{1}{6}} F_j\|_{L_{xt}^6}\right] \left[\prod_{j=3}^{k+1} \|D_x^{\frac{1-2ks}{2(k-1)}} F_j\|_{L_{xt}^{\frac{6(k-1)}{3-\epsilon}}}\right] \\ &\leq C \left(\prod_{j=1}^{k+1} \|F_j\|_{X_{0,b}}\right) \|F\|_{X_{0,\frac{1}{2}-\frac{\epsilon}{12}}} \leq C \left(\prod_{j=1}^{k+1} \|f_j\|_{L_{\xi\tau}^2}\right) \|f\|_{L_{\xi\tau}^2}. \end{aligned}$$

We have completed the proof of Lemma 3.1.

Lemma 3.2. *Let* $s \geq \frac{1}{2} - \frac{2}{k} + 2\epsilon, k \geq 5,\ b = \frac{1}{2} + \frac{\epsilon}{24}$ *and* $b' = -\frac{1}{2} + \frac{\epsilon}{12}$ *and* $g = \psi(t)u$. *Then, we have*

$$\left\| g^{k+1} \right\|_{X_{s,b'}} \leq C \|f\|_{X_{s,b}}^{k+1}. \tag{3.35}$$

Lemma 3.2 can be proved similarly to Lemma 3.1.

Lemma 3.3. *Let* $s \geq \frac{1}{2} - \frac{2}{k} + 2\epsilon, k \geq 5,\ b = \frac{1}{2} + \frac{\epsilon}{24}$ *and* $b' = -\frac{1}{2} + \frac{\epsilon}{12}$ *and* $g = \psi(t)u$. *Then, we have*

$$\left\| \partial_x (g^{k+1}) \right\|_{\tilde{X}_{s,b'}} \leq C \|g\|_{\tilde{X}_{s,b}}^{k+1}. \tag{3.36}$$

Proof. Since $\|g\|_{\tilde{X}_{s,b}} = \|g\|_{X_{s,b}} + \|\partial_x^{-1} g\|_{X_{s,b}}$, we have

$$\|\partial_x (g^{k+1})\|_{\tilde{X}_{s,b'}} = \|\partial_x (g^{k+1})\|_{X_{s,b'}} + \|g^{k+1}\|_{X_{s,b'}}, \tag{3.37}$$

using Lemmas 3.1 and 3.2, we have

$$\|\partial_x (g^{k+1})\|_{\tilde{X}_{s,b'}} \leq C \|g\|_{X_{s,b}}^{k+1} \leq C \|g\|_{\tilde{X}_{s,b}}^{k+1}.$$

We have completed the proof of Lemma 3.3.

4. Proof of Theorem 1.1

Proof. Obviously, (1.1)–(1.2) are equivalent to

$$u(t) = U^{\gamma,\beta}(t)u_0 - \frac{1}{k+1} \int_0^t U^{\gamma,\beta}(t-\tau)\partial_x[(\psi(\tau)u)^{k+1}]d\tau.$$

For $u_0 \in H^s(\mathbf{R})$ and $\delta \in (0, 1]$, we define

$$\Gamma(u) = \psi(t)U^{\gamma,\beta}(t)u_0 + \frac{1}{k+1}\psi_\delta(t) \int_0^t U^{\gamma,\beta}(t-\tau)\partial_x((\psi(\tau)u)^{k+1})d\tau. \tag{4.1}$$

We define $B(0, r) = \{u \in X_{s,b} \cap C([-\delta, \delta], H^s(\mathbf{R})), \|u\|_{X_{s,b}} \leq r := 2C\|u_0\|_{H^s(\mathbf{R})}\}$. By using Lemmas 2.1, 3.1 and choosing sufficiently small $\delta > 0$ such that

$$C\delta^{b'+1-b}(2C\|u_0\|_{H^s(\mathbf{R})})^{k+1} \leq C\|u_0\|_{H^s},$$

we have

$$\begin{aligned} \|\Gamma(u)\|_{X_{s,b}} &\leq C\|u_0\|_{H^s} + C\delta^{b'+1-b}\|\partial_x(u^{k+1})\|_{X_{s,b'}} \\ &\leq C\|u_0\|_{H^s} + C\delta^{b'+1-b}\|u\|_{X_{s,b}}^{k+1} \\ &\leq C\|u_0\|_{H^s} + C\delta^{b'+1-b}(2C\|u_0\|_{H^s(\mathbf{R})})^{k+1} \leq 2C\|u_0\|_{H^s}. \end{aligned} \tag{4.2}$$

By a similar calculation, we have

$$\|\Gamma(u_1)-\Gamma(u_2)\|_{X_{s,b}} \leq C\delta^{b'+1-b}\|u_1-u_2\|_{X_{s,b}}(\|u_1\|_{X_{s,b}}^k+\|u_2\|_{X_{s,b}}^k)$$
$$\leq \frac{1}{2}\|u_1-u_2\|_{X_{s,b}}. \tag{4.3}$$

Thus, Γ is a contraction mapping from the closed ball

$$B(0,r)=\{u\in X_{s,b}\cap C([-\delta,\delta],H^s(\mathbf{R})),\ \|u\|_{X_{s,b}}\leq r\}$$

into itself. From the fixed point theorem and (4.3), we have $\Gamma(v)=v$. The uniqueness of solution to (4.1) is easily derived from (4.3).

The rest of the local well-posedness results of Theorem 1.1 follow from a standard argument, for instance, see [24].

This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

By using Lemmas 2.1, 3.3 and the fixed point argument as well as a proof similar to Theorem 1.1, we obtain Theorem 1.2.

6. Proof of Theorem 1.3

Proof. In this section, inspired by [4,27,32,36], we study the relationship between the solution to (1.1)–(1.2) and the solution to

$$v_t-\beta\partial_x^3 v+\frac{1}{k+1}(v^{k+1})_x=0, \tag{6.1}$$
$$v(x,0)=v_0(x), \tag{6.2}$$

as $\gamma\to 0$.
From (1.1), we have

$$J_x^s u_t-\beta\partial_x^3 J_x^s u-\gamma\partial_x^{-1}J_x^s u+\frac{1}{k+1}J_x^s(u^{k+1})_x=0, k\geq 5 \tag{6.3}$$

Multiplying by $J_x^s u$ on both sides of (6.3) and integration by parts with respect to x on $\mathbf{R}$ as well as $H^{s-1}(\mathbf{R})\hookrightarrow L^\infty$ with $s>\frac{3}{2}$, by using Lemma 2.12, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2=\int_{\mathbf{R}}J_x^s uJ_x^s u_t\mathrm{d}x=\beta\int_{\mathbf{R}}J_x^s u\partial_x^3J_x^s u\mathrm{d}x+\gamma\int_{\mathbf{R}}J_x^s u\partial_x^{-1}J_x^s u\mathrm{d}x\\
&-\int_{\mathbf{R}}J_x^s uJ_x^s(u^k u_x)\mathrm{d}x=-\int_{\mathbf{R}}J_x^s uJ_x^s(u^k u_x)\mathrm{d}x\\
&=-\int_{\mathbf{R}}J_x^s u[J_x^s,u^k]u_x\mathrm{d}x-\int_{\mathbf{R}}(J_x^s u)(u^kJ_x^s u_x)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbf{R}} J_x^s u[J_x^s, u^k]u_x \mathrm{d}x + \frac{k}{2}\int_{\mathbf{R}} u^{k-1}u_x (J_x^s u_x)^2 \mathrm{d}x \\
&\le C \left\| J_x^s u \right\|_{L^2} \left[\|u^{k-1}u_x\|_{L^\infty} \|J_x^s u\|_{L^2} + \|J_x^s (u)^k\|_{L^2} \|u_x\|_{L^\infty} \right] \\
&\quad + C \left\| u^{k-1}u_x \right\|_{L^\infty} \left\| J_x^s u \right\|_{L^2}^2 \le C_0 \left\| u \right\|_{H^s}^{k+2} \le C_0 \left\| u \right\|_{X_s}^{k+2}. \qquad (6.4)
\end{aligned}
$$

Here, C_0 is a constant independent of γ. Similarly, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{-1}u\|_{H^s}^2 &= -2\int_{\mathbf{R}} J_x^s \partial_x^{-1} u J_x^s (u^{k+1}) \mathrm{d}x \\
&\le C_0 \|J_x^s \partial_x^{-1} u\|_{L^2} \|u^{k+1}\|_{H^s} \le C_0 \|u\|_{X_s}^{k+2}. \qquad (6.5)
\end{aligned}
$$

Then, using (6.4) and (6.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{X_s}^2 \le C_0 \|u\|_{X_s}^{k+2}, \qquad (6.6)$$

from (6.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{X_s} \le C_0 \|u\|_{X_s}^{k+1}. \qquad (6.7)$$

When $t < \min\left\{T, \frac{1}{Ck\|u_0\|_{X_s}^k}\right\}$, where T is the time lifespan of the solution to (1.1)–(1.2) for data in $X_s(\mathbf{R})$ with $s > \frac{3}{2}$ in Theorem 1.2, by using (6.7), we have

$$\|u\|_{X_s} \le \frac{k^{\frac{1}{k}} \|u_0\|_{X_s}}{\sqrt{1 - Ck\|u_0\|_{X_s}^k t}}. \qquad (6.8)$$

Let $u := u^\gamma$ and the solution to (1.1). Therefore, $w := u - v$ satisfies the equation

$$w_t - \beta \partial_x^3 w + \gamma \partial_x^{-1} u + \tfrac{1}{k+1}(w \sum_{j=0}^{k} (v+w)^j v^{k-j})_x = 0, k \ge 5, \qquad (6.9)$$

$$w(x,0) = u_0(x) - v_0(x). \qquad (6.10)$$

Multiplying by w on both sides of (6.9) and integrating by parts with respect to x on $\mathbf{R}$, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &= -\frac{1}{k+1}\int_{\mathbf{R}} w(w\sum_{j=0}^{k}(v+w)^j v^{k-j})_x \mathrm{d}x + \int_{\mathbf{R}} w\gamma \partial_x^{-1} u)\mathrm{d}x \\
&= -\frac{1}{k+1}\int_{\mathbf{R}} w(w\sum_{j=0}^{k} u^j v^{k-j})_x \mathrm{d}x + \gamma \int_{\mathbf{R}} w \partial_x^{-1} u \mathrm{d}x \\
&\le C \sum_{j=0}^{k} \|(u^j v^{k-j})_x\|_{L^\infty} \|w\|_{L^2}^2 + |\gamma| \|w\|_{L^2} \|\partial_x^{-1} u\|_{L^2}
\end{aligned}
$$

$$\leq C\left[\|u\|_{H^s}^k+\|v\|_{H^s}^k\right]\|w\|_{L^2}^2+C|\gamma|\|w\|_{L^2}\|u\|_{X_s}. \tag{6.11}$$

From (6.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}\leq C\sup_{t\in[0,T]}\left[\|u\|_{X_s}+\|v\|_{X_s}\right]^k\|w\|_{L^2}+C|\gamma|\sup_{t\in[0,T]}\|u\|_{X_s}. \tag{6.12}$$

By using the Gronwall's inequality and (6.12), we can get

$$\|w\|_{L^2}\leq e^{CT\sup\limits_{t\in[0,T]}[\|u\|_{X_s}+\|v\|_{X_s}]^k}\left[\|u_0-v_0\|_{L^2}+C|\gamma|T\sup_{t\in[0,T]}\|u\|_{X_s}\right]. \tag{6.13}$$

Thus, when $\gamma\to 0$ and $\|u_0-v_0\|_{L^2}\to 0$, we have $\|w\|_{L^2}=\|u-v\|_{L^2}\to 0$.

This completes the proof of Theorem 1.3.

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