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Global existence of a nonlinear Schrödinger equation with viscous damping

DAISUKE HIRATA

Abstract. In this note, we consider a Schrödinger evolution equation with a power nonlinearity $i|u|^\alpha u$ and a viscous damping term $\nu\Delta u$. Then, we demonstrate that the Cauchy problem always admits the global existence of classical solutions with finite mass. Moreover, we can also observe that our proof is applicable for a nonlinear complex Ginzburg–Landau equation.

1. Introduction

In this note, we consider the nonlinear Schrödinger equation with viscous damping

$$\partial_t u = i(\Delta u + |u|^\alpha u) + \nu\Delta u, \qquad u|_{t=0} = u_0 \tag{1.1}$$

on $\mathbb{R}^d$, where $\alpha, \nu > 0$ and u_0 is a prescribed $\mathbb{C}$-valued function on $\mathbb{R}^d$. We shall seek for an unknown function $u = u(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ governed by the Cauchy problem (1.1). Here, it should be noted that equation (1.1) is a particular case of the complex Ginzburg–Landau equation

$$\partial_t u = (\nu + i\kappa)\Delta u + (\lambda + i\mu)|u|^\alpha u + \gamma u, \tag{1.2}$$

where $\kappa, \lambda, \mu, \gamma \in \mathbb{R}$ (see e.g. [1,13]).

In the case of $\nu = 0$ in (1.1), some solutions of the purely nonlinear Schrödinger equation blow up in finite time under suitable assumptions on α and u_0. As is well known, if $\alpha < 4/(d-2)_+$, then the Cauchy problem is locally well-posed in the energy space $H^1(\mathbb{R}^d)$. In particular, if $\alpha < 4/d$, then we obtain the maximal existence time $T_* = +\infty$ for every $u_0 \in H^1(\mathbb{R}^d)$. On the other hand, if $4/d \le \alpha < 4/(d-2)_+$, then $T_* = T_*(u_0) < \infty$ for every initial value u_0 satisfying $|x|u_0 \in L^2(\mathbb{R}^d)$ and

$$E(u_0) := \frac{1}{2}||\nabla u_0||_{L^2}^2 - \frac{1}{\alpha+2}\int_{\mathbb{R}^d} |u_0|^{\alpha+2}\mathrm{d}x < 0.$$

That is, we have that $\limsup_{t\uparrow T_*} ||u(t)||_{H^1} = +\infty$ (see for instance [6]). Later on, Cazenave et. al. [3,4] extended the blowup result to the Ginzburg–Landau equation

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(1.2) in the case when $\kappa = \mu$ and $\nu = \lambda$:

$$\partial_t u = (\nu + i\kappa)(\Delta u + |u|^\alpha u) + \gamma u. \tag{1.3}$$

Our purpose of this note is to show that the Cauchy problem (1.1) always admits the global existence of classical solutions with finite mass due to the viscous damping term. More precisely, we shall establish the following result.

Theorem 1.1. *Given any $u_0 \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, there exists a unique function $u \in C_w([0,\infty), L^\infty(\mathbb{R}^d)) \cap C([0,\infty), L^2(\mathbb{R}^d))$ with $u|_{t=0} = u_0$,which is a classical solution of equation* (1.1) *on* $(0,\infty) \times \mathbb{R}^d$.

It seems that the standard strategy of compactness or monotonicity method is not available for deriving the global existence of such strong solutions at least when $\alpha > 2d/(d-2)_+$ and ν is sufficiently small (cf. [7–9,11,12]). Therefore, we need another approach to prove Theorem 1.1.

The outline of this note is as follows. In Sect. 2, we collect some preliminary results for demonstrating Theorem 1.1. In Sect. 3, we complete the proof of Theorem 1.1.

2. Preliminaries

Let $A_\nu \varphi := (\nu + i)\Delta\varphi$ with domain $H^2(\mathbb{R}^d)$. Then, it is well known that A_ν generates an analytic semigroup $(e^{tA_\nu})_{t\geq 0}$ on $L^2(\mathbb{R}^d)$, which is given by the explicit representation

$$e^{tA_\nu}\psi = \frac{1}{(4\pi(\nu+i)t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{(i-\nu)\frac{|x-y|^2}{4(1+\nu^2)t}} \psi(y)\,\mathrm{d}y. \tag{2.1}$$

In addition, $(e^{tA_\nu})_{t\geq 0}$ is a strongly continuous semigroup on $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$. Moreover, there is a constant $C = C(d,\nu) > 0$ such that

$$||e^{tA_\nu}\psi||_{L^r} \leq \frac{C}{t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}} ||\psi||_{L^p} \tag{2.2}$$

and

$$||\nabla e^{tA_\nu}\psi||_{L^r} \leq \frac{C}{t^{\frac{1}{2}+\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}} ||\psi||_{L^p} \tag{2.3}$$

for all $t > 0$, where $1 \leq p \leq r \leq \infty$.

Our problem (1.1) can be converted into the integral equation

$$u(t) = e^{tA_\nu}u_0 + i\int_0^t e^{(t-s)A_\nu}|u(s)|^\alpha u(s)\,\mathrm{d}s \tag{2.4}$$

for an unknown function $u = u(t) = u(t,\cdot) : [0,\infty) \times \mathbb{R}^d \to \mathbb{C}$. By applying the standard fixed point argument to (2.4), we can immediately obtain the local-in-time solvability of (1.1) on $L^\infty(\mathbb{R}^d)$. More precisely, we have the following.

Proposition 2.1. *There is a constant* $\varepsilon_0 = \varepsilon_0(d, \nu, \alpha) > 0$ *such that for every* $u_0 \in L^\infty(\mathbb{R}^d)$*, there exist a unique function* $u \in C_w([0, T], L^\infty(\mathbb{R}^d))$ *for a time* $T \geq \varepsilon_0/||u_0||^\alpha_{L^\infty}$ *with* $u(0) = u_0$*, which is a classical solution of equation* (1.1) *on* $(0, T) \times \mathbb{R}^d$.

In particular, for any $T_0 \leq T/2$*, there is a constant* $C = C(d, \nu, \alpha, T_0, M(T)) > 0$ *with* $M(T) := \sup_{0\leq t\leq T} ||u(t)||_{L^\infty}$ *such that*

$$\sup_{T-T_0\leq t<T} ||\nabla u(t)||_{L^\infty} \leq C. \tag{2.5}$$

Moreover, if in addition $u_0 \in L^2(\mathbb{R}^d)$*, then* $u \in C([0, T], L^2(\mathbb{R}^d))$ *and*

$$||u(t)||^2_{L^2} + 2\nu \int_0^t ||\nabla u(s)||^2_{L^2} \mathrm{d}s = ||u_0||^2_{L^2} \tag{2.6}$$

for all $t \in [0, T]$.

Remark 2.1. By standard parabolic regularity theory, we know that the resulting solution u is a classical solution in $C((0, T), C^3(\mathbb{R}^d)) \cap C^1((0, T), C(\mathbb{R}^d))$ with $\nabla\partial_t u \in C((0, T) \times \mathbb{R}^d)$. However, for small $\alpha > 0$, say, $0 < \alpha < 1$, one cannot expect these solutions to be in $C((0, T), C^4(\mathbb{R}^d))$ (cf. [5, Theorem 3.2 and A.1]).

Proof. For the uniqueness part, see [2, Lemma 9]. The estimate (2.5) is derived by applying the L^∞ smoothing estimate in (2.3) to the Cauchy problem (1.1) with the initial condition $u|_{t=T-2T_0} = u(T-2T_0)$ via a singular Gronwall's lemma [6, Lemma 8.1.1]. We omit the details of the proof of Proposition 2.1 □

Let us denote the Newtonian potential on $\mathbb{R}^d$ by

$$\Delta^{-1} f := (E * f)(x) = \int_{\mathbb{R}^d} E(x-y) f(y) \mathrm{d}y \tag{2.7}$$

for a function $f : \mathbb{R}^d \to \mathbb{R}$, where $E(x)$ is the fundamental solution of the Laplacian Δ on $\mathbb{R}^d$, i.e.,

$$E(x) = \begin{cases} -\dfrac{1}{(d-2)|\mathbb{S}^{d-1}|} |x|^{2-d} & (\text{for } d \neq 2), \\ \dfrac{1}{2\pi} \log |x| & (\text{for } d = 2). \end{cases}$$

Here, $|\mathbb{S}^{d-1}|$ denotes the surface area of the unit ball in $\mathbb{R}^d$. It is well known that there are constants $C = C(d, r, \rho) > 0$ and $C' = C'(d, r, \rho) > 0$ such that

$$||\nabla\Delta^{-1} f||_{L^\infty} \leq C||f||_{L^r(B(z;\rho))} \quad (\text{for } d < r \leq \infty) \tag{2.8}$$

and

$$||\Delta^{-1} f||_{L^\infty} \leq C'||f||_{L^r(B(z;\rho))} \quad (\text{for } d/2 < r \leq \infty) \tag{2.9}$$

for any $f \in L^r(\mathbb{R}^d)$ supported on $B(z;\rho)$ (cf. for instance [10, Theorem 10.2]). Here, $B(z;\rho)$ denotes the open ball centered at $z \in \mathbb{R}^d$ of radius $\rho > 0$.

Let us recall a dyadic partition of unity of Littlewood–Paley type: there exists a nonnegative, smooth, spherically symmetric function $\varphi : \mathbb{R}^d \to \mathbb{R}$ supported on the annulus $\{2^{-1} \le |x| \le 2\}$ such that

$$1 = \sum_{n\in\mathbb{Z}} \varphi(2^n x) \qquad (\forall x \neq 0).$$

Let $\phi = \{\phi_z^n(x)\}_{z\in\mathbb{R}^d}^{n\in\mathbb{Z}}$ denote a family of localization functions on $\mathbb{R}^d$ given by

$$\phi_z^n(x) := \sum_{k=n}^{\infty} \varphi(2^k(x-z)).$$

Note that $\phi_z^n = 1$ on $B(z;2^{-n})$ and $\phi_z^n = 0$ on $\mathbb{R}^d \backslash B(z;2^{-n+1})$. Moreover, we have the generalization $\phi = \{\phi_z^n(x)\}_{z\in\mathbb{R}^d}^{n\in\mathbb{R}}$ via interpolation. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, let us introduce the operator

$$f_z^{(n)} = f_z^{(n)}(x) := \big(\Delta^{-1}(\phi_z^n \Delta f)\big)(x) \tag{2.10}$$

for $x \in \mathbb{R}^d$. Then, we have the following identities

$$f_z^{(n)} = \phi_z^n f - 2\nabla\Delta^{-1}(\nabla\phi_z^n f) + \Delta^{-1}(\Delta\phi_z^n f) \tag{2.11}$$

and

$$f_z^{(n)} = \nabla\Delta^{-1}(\phi_z^n \nabla f) - \Delta^{-1}(\nabla\phi_z^n \cdot \nabla f) \tag{2.12}$$

for $f \in C^2(\mathbb{R}^d)$. Moreover, we can observe from integration by substitution the following identities

$$||\nabla\Delta^{-1}(\phi_z^n \nabla f)||_{L^\infty} = ||\nabla\Delta^{-1}(\phi_0^0 \nabla f_z^{-n})||_{L^\infty} \tag{2.13}$$

and

$$||\Delta^{-1}(\nabla\phi_z^n \cdot \nabla f)||_{L^\infty} = ||\Delta^{-1}(\nabla\phi_0^0 \cdot \nabla f_z^{-n})||_{L^\infty} \tag{2.14}$$

with $f_z^{-n}(x) := f(z + 2^{-n}x)$ for all $f \in C^1(\mathbb{R}^d)$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1

Proof of Theorem 1.1. We argue by contradiction. Suppose that there exist some initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution u obtained in Proposition 2.1 develops singularities at $t = T_* < \infty$, that is,

$$\lim_{t\uparrow T_*} ||u(t)||_{L^\infty} = +\infty. \tag{3.1}$$

Thus we can find a sequence of $(t_k)_{k\geq 1} \subset (0, T_*)$ with $t_k \uparrow T_*$ such that

$$2^{m_k-1} < ||u(t_k)||_{L^\infty} = \sup_{0\leq t\leq t_k} ||u(t)||_{L^\infty} \leq 2^{m_k} \uparrow +\infty \tag{3.2}$$

with an associated sequence $(m_k)_{k\geq 1} \in \mathbb{N}$.

Let us introduce the function

$$q(t) = q(t,x) := (1+|u(t,x)|^2)^\theta \qquad (0 < \theta < 1/d). \tag{3.3}$$

Since the identity (2.11) gives

$$\phi_z^n q(t) = q_z^{(n)}(t) + 2\big(\nabla\Delta^{-1}(\nabla\phi_z^n q(t))\big) - \big(\Delta^{-1}(\Delta\phi_z^n q(t))\big)$$

with $q_z^{(n)}(t) := \Delta^{-1}(\phi_z^n \Delta q(t))$, we deduce that

$$\begin{aligned} ||q(t)||_{L^\infty(B(z;2^{-n}))} &\leq ||\phi_z^n q(t)||_{L^\infty(B(z;2^{-n+1}))} \\ &\leq ||q_z^{(n)}(t)||_{L^\infty} + 2||\nabla\Delta^{-1}(\nabla\phi_z^n q(t))||_{L^\infty} \\ &\quad + ||\Delta^{-1}(\Delta\phi_z^n q(t))||_{L^\infty}. \end{aligned} \tag{3.4}$$

We thus estimate

$$\begin{aligned} ||\nabla\Delta^{-1}(\nabla\phi_z^n q(t))||_{L^\infty} &= ||\nabla\Delta^{-1}(\nabla\phi_0^0 q_z^{-n}(t))||_{L^\infty} \\ &\quad \big(\text{by}(2.13)\text{with } q_z^{-n}(t,x) := q(t, z+2^{-n}x)\big) \\ &\leq C||q(t)||_{L^{1/\theta}(B(z;2^{-n+1}))} \\ &\quad \big(\text{by } (2.8) \text{ with } r = 1/\theta > d\big) \\ &= C||1+|u(t)|^2||^\theta_{L^1(B(z;2^{-n+1}))} \\ &= C\big(|B(z;2^{-n+1})| + ||u(t)||^2_{L^2}\big)^\theta \\ &\leq C\big(1+||u_0||^{2\theta}_{L^2}\big) \end{aligned}$$

for all $(n,z,t) \in \mathbb{R}^+ \times \mathbb{R}^d \times (0,T_*)$, where we used the bound $||u(t)||_{L^2} \leq ||u_0||_{L^2}$ from (2.6) in the last inequality. Similarly, by using (2.14) and (2.9) with $r = 1/\theta$, we get

$$\begin{aligned} ||\Delta^{-1}(\Delta\phi_z^n q(t))||_{L^\infty} &= ||\Delta^{-1}(\Delta\phi_0^0 q_z^{-n}(t))||_{L^\infty} \\ &\leq C||q(t)||_{L^{1/\theta}(B(z;2^{-n+1}))} \\ &\leq C\big(1+||u_0||^{2\theta}_{L^2}\big) \end{aligned}$$

for every $(n,z,t) \in \mathbb{R}^+ \times \mathbb{R}^d \times (0,T_*)$. Hence, it follows from (3.4) that

$$||q(t)||_{L^\infty(B(z;2^{-n}))} \leq ||q_z^{(n)}(t)||_{L^\infty} + C\big(1+||u_0||^{2\theta}_{L^2}\big) \tag{3.5}$$

for any $(n,z,t) \in \mathbb{R}^+ \times \mathbb{R}^d \times (0,T_*)$.

For the time sequence $(t_k)_{k\geq 1}$ in (3.2), let us choose a point $x_k \in \mathbb{R}^d$ such that $|u(t_k, x_k)| > 2^{m_k-1}$ for each $k \geq 1$. Then for any sequence $(n_k)_{k\geq 1} \uparrow +\infty$, we have that

$$\lim_{k\to\infty} ||q(t_k)||_{L^\infty(B(x_k;2^{-n_k}))} = +\infty. \tag{3.6}$$

Let $M_k := 2^{m_k\alpha/2}$ and let us rescale the blowup solution u at $(t_k, x_k) \in (0, T_*) \times \mathbb{R}^d$ by

$$U_{(k)} = U_{(k)}(t, x) := M_k^{-\frac{2}{\alpha}} u(t_k + M_k^{-2}t, x_k + M_k^{-1}x) \tag{3.7}$$

By our construction, $U_{(k)}$ is a function defined on the rescaled interval $[-M_k^2 t_k, 0]$ such that

$$\sup_{-M_k^2 t_k\leq t\leq 0} ||U_{(k)}(t)||_{L^\infty} \leq 1. \tag{3.8}$$

Furthermore, by the scaling invariance of equation (1.1), the function $U_{(k)}$ is a solution of the Cauchy problem (1.1) on $[-M_k^2 t_k, 0] \times \mathbb{R}^d$ for any $k \geq 1$. Then, we can deduce from the L^∞ estimate (2.5) that

$$\sup_{-1\leq t<0} ||\nabla U_{(k)}(t)||_{L^\infty} \leq C_\alpha \tag{3.9}$$

for all $k \gg 1$ with a constant $C_\alpha = C(d, \nu, \alpha) > 0$.

Define the function

$$w_{(k)}(t, x) := u(t_k + M_k^{-2}t, x_k + M_k^{-(\frac{2}{\alpha}+1)}x) = M_k^{\frac{2}{\alpha}} U_{(k)}(t, M_k^{-\frac{2}{\alpha}}x).$$

By differentiation, we have

$$\begin{aligned}\sup_{-1\leq t<0} ||\nabla w_{(k)}(t)||_{L^\infty} &= \sup_{-1\leq t<0} ||\nabla U_{(k)}(t)||_{L^\infty} \\ &\leq C_\alpha\end{aligned}$$

for all $k \gg 1$. Furthermore, we see that the function

$$\begin{aligned}Q_{(k)}(t, x) &:= q(t_k + M_k^{-2}t, x_k + M_k^{-(\frac{2}{\alpha}+1)}x) \\ &= \Big(1 + |w_{(k)}(t, x)|^2\Big)^\theta \end{aligned} \tag{3.10}$$

(cf. (3.3)) satisfies

$$\begin{aligned}\sup_{-1\leq t<0} ||\nabla Q_{(k)}(t)||_{L^\infty} &= \sup_{-1\leq t<0} 2\theta|| \frac{w_{(k)}(t)\cdot\nabla w_{(k)}(t)}{(1 + |w_{(k)}(t)|^2)^{1-\theta}} ||_{L^\infty} \\ &\leq 2 \sup_{-1\leq t<0} ||\nabla w_{(k)}(t)||_{L^\infty} \\ &\leq C_\alpha,\end{aligned} \tag{3.11}$$

since $0 < \theta < 1/d$ and $2(1-\theta) > 1$.

We are now a position to complete the proof of Theorem 1.1. Since (3.6) implies

$$||q(t_k)||_{L^\infty(B(x_k;2^{-n_k}))} \uparrow +\infty \qquad (\text{as } k \to \infty)$$

with $n_k := m_k(\alpha+2)/2$, we deduce from the inequality (3.5) that

$$\lim_{k\to\infty} ||q_{x_k}^{(n_k)}(t_k)||_{L^\infty} = +\infty. \tag{3.12}$$

On the other hand, we observe that

$$\begin{aligned}
||q_{x_k}^{(n_k)}(t_k)||_{L^\infty} &\le ||\nabla\Delta^{-1}(\phi_{x_k}^{n_k}\nabla q(t_k))||_{L^\infty} \\
&\quad + ||\Delta^{-1}(\nabla\phi_{x_k}^{n_k}\cdot\nabla q(t_k))||_{L^\infty} \qquad \big(\text{since } 2.12\big) \\
&\le \sup_{-1/M_k^2 \le t<0} ||\nabla\Delta^{-1}(\phi_{x_k}^{n_k}\nabla q(t_k+t))||_{L^\infty} \\
&\quad + \sup_{-1/M_k^2 \le t<0} ||\Delta^{-1}(\nabla\phi_{x_k}^{n_k}\cdot\nabla q(t_k+t))||_{L^\infty} \\
&= \sup_{-1\le t<0} ||\nabla\Delta^{-1}(\phi_0^0\nabla Q_{(k)}(t))||_{L^\infty} \\
&\quad + \sup_{-1\le t<0} ||\Delta^{-1}(\nabla\phi_0^0\cdot\nabla Q_{(k)}(t))||_{L^\infty} \\
&\qquad \big(\text{since } 2.13\text{–}2.14 \text{ and } 2^{-n_k} = M_k^{-(\frac{2}{\alpha}+1)}\big) \\
&\le C \sup_{-1\le t<0} ||\nabla Q_{(k)}(t)||_{L^\infty(B(0;2))} \\
&\qquad \big(\text{since } 2.8 \textit{ and } 2.9 \text{ with } r=\infty\big) \\
&\le C \sup_{-1\le t<0} ||\nabla Q_{(k)}(t)||_{L^\infty} \\
&\le C_\alpha
\end{aligned}$$

for all $k \gg 1$, since (3.11). This contradicts (3.12). Hence, we have proved Theorem 1.1. □

We finish this note to state that our argument is available to establish the global existence of classical solutions for the following complex Ginzburg–Landau equation

$$\partial_t u = (\nu + i\kappa)\Delta u + (\lambda + i\mu)|u|^\alpha u \qquad (\lambda < 0), \tag{3.13}$$

and furthermore, for the derivative-type nonlinear Schrödinger equation with viscous damping

$$\partial_t u = i(\Delta u + |\nabla u|^\beta u) + \nu\Delta u \qquad (1 \le \beta < 2). \tag{3.14}$$

Theorem 3.1. *Consider the Cauchy problem for the equation* (3.13) *(resp.* (3.14)*) on* $\mathbb{R}^d$*. Given any* $u_0 \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$*, there exists a unique function* $u \in C_w([0,\infty), L^\infty(\mathbb{R}^d)) \cap C([0,\infty), L^2(\mathbb{R}^d))$ *with* $u|_{t=0} = u_0$*, which is a classical solution of equation* (3.13) *(resp.* (3.14)*) on* $(0,\infty)\times\mathbb{R}^d$.

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Daisuke Hirata
Institute for Mathematics and Computer Science
Tsuda University
Tsuda-chou
Kodaira-shi, Tokyo 187-8577
Japan
E-mail: hiradice@gmail.com

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