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Harnack's inequality for singular parabolic equations with generalized Orlicz growth under the non-logarithmic Zhikov's condition

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In memory of DiBenedetto

Abstract. For a general class of divergence type quasi-linear singular parabolic equations with generalized Orlicz growth, we prove the intrinsic Harnack inequality for positive solutions. This class of singular equations includes new cases of equations with (p, q) nonlinearity and non-logarithmic growth.

1. Introduction and main results

In this paper, we are concerned with general divergence type singular parabolic equations with nonstandard growth conditions. Let Ω be a domain in $\mathbb{R}^n$, $T > 0$, $\Omega_T := \Omega \times (0, T)$, we study bounded solutions to the equation

$$u_t - \text{div}\mathbb{A}(x, t, \nabla u) = 0, \quad (x, t) \in \Omega_T. \tag{1}$$

Throughout the paper, we suppose that the functions $\mathbb{A} : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $\mathbb{A}(\cdot, \cdot, \xi)$ are Lebesgue measurable for all $\xi \in \mathbb{R}^n$, and $\mathbb{A}(x, t, \cdot)$ are continuous for almost all $(x, t) \in \Omega_T$. We also assume that the following structure conditions are satisfied

$$\begin{aligned} &\mathbb{A}(x, t, \xi)\xi \geqslant K_1\, g(x, t, |\xi|)\, |\xi|, \\ &|\mathbb{A}(x, t, \xi)| \leqslant K_2\, g(x, t, |\xi|), \end{aligned} \tag{2}$$

where K_1, K_2 are positive constants and the function $g(x, t, v) : \Omega_T \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

(g$_0$) for all $(x, t) \in \Omega_T$ the function $g(x, t, \cdot)$ is increasing, continuous and

$$\lim_{v \to +0} g(x, t, v) = 0, \quad \lim_{v \to +\infty} g(x, t, v) = +\infty;$$

(g$_1$) $c_0^{-1} \leqslant g(x, t, 1) \leqslant c_0$ for all $(x, t) \in \Omega_T$ and some $c_0 > 0$;

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(g_2) for all $(x, t) \in \Omega_T$ and for $w \geqslant v > 0$ there hold

$$\Big(\frac{w}{v}\Big)^{p-1} \leqslant \frac{g(x,t,w)}{g(x,t,v)} \leqslant \Big(\frac{w}{v}\Big)^{q-1}, \quad \frac{2n}{n+1} < p < \min\{2, q\},\ q \neq 2.$$

Fix $R_0 > 0$ such that $Q_{R_0,R_0}(x_0, t_0) \subset \Omega_T$, where

$$\begin{aligned} Q_{R_1,R_2}(x_0,t_0) &:= Q^-_{R_1,R_2}(x_0,t_0) \cup Q^+_{R_1,R_2}(x_0,t_0),\\ Q^-_{R_1,R_2}(x_0,t_0) &:= B_{R_1}(x_0) \times (t_0 - R_2, t_0),\\ Q^+_{R_1,R_2}(x_0,t_0) &:= B_{R_1}(x_0) \times (t_0, t_0 + R_2). \end{aligned}$$

In what follows, we assume that

(g_λ) there exists positive, continuous and non-decreasing function $\lambda(r)$ on the interval $(0, R_0)$, $0 < \lambda(r) \leqslant 1$, $\lim\limits_{r\to 0} r^{1-\overline{a}}/\lambda(r) = 0$, $\overline{a} \in (0, 1)$, such that for any K, $K_0 > 0$ and some $c_1(K, K_0) > 0$, $b \geqslant 0$ there holds

$$g(x,t,v/r) \leqslant c_1(K,K_0)\, g(y,\tau,v/r)$$

for all $br < v \leqslant K\lambda(r)$ and $(x, t), (y, \tau) \in Q_{r,rK_0}(x_0, t_0) \subset Q_{R_0,R_0}(x_0, t_0)$.

(g_μ) there exists positive, continuous and non-increasing function $\mu(r)$ on the interval $(0, R_0)$, $\mu(r) \geqslant 1$, $\lim\limits_{r\to 0} r^{1-\overline{a}}\mu(r) = 0$, $\overline{a} \in (0, 1)$ such that for any K, $K_0 > 0$ and some $c_2(K, K_0) > 0$, $b \geqslant 0$ there holds

$$g(x,t,v/r) \leqslant c_2(K,K_0)\mu(r)\, g(y,\tau,v/r),$$

for all $br < v \leqslant K$ and $(x, t), (y, \tau) \in Q_{r,rK_0}(x_0, t_0) \subset Q_{R_0,R_0}(x_0, t_0)$.

As one can easily see, for $\lambda(r) = \mu(r) = 1$, (g_λ) and (g_μ) reduce to the standard Zhikov's logarithmic condition. We will also need the following technical inequality

$$\frac{\lambda(\rho)}{\lambda(r)} + \frac{\mu(r)}{\mu(\rho)} \leqslant \Big(\frac{\rho}{r}\Big)^{c_3},$$

which we assume with some $c_3 > 0$ and for all $0 < r \leqslant \rho \leqslant R_0$.

We will establish that nonnegative bounded weak solutions of Eq. (1) satisfy an intrinsic form of the Harnack inequality in a neighborhood of (x_0, t_0) provided that this p.d.e. is singular, i.e., the function $\psi(x, t, v) := g(x, t, v)/v$ satisfies the following assumptions:

(ψ) there exist $a_0, b_0 \geqslant 0$, $0 < q_1 < 1$ and $R_0 = R_0(x_0, t_0) > 0$ such that

$$\frac{\psi(x,t,w)}{\psi(x,t,v)} \leqslant \Big(\frac{v}{w}\Big)^{1-q_1}$$

for all $(x, t) \in Q_{R_0,R_0}(x_0, t_0) \subset \Omega_T$ and for $w \geqslant v > b_0 R_0^{-a_0}$.

Condition (ψ) with $q_1 = q - 1$ and $b_0 = a_0 = 0$ is a consequence of (g_2) in the case $q < 2$.

Remark 1. We note that the continuity of solutions was proved in [34] under weaker conditions, namely it was assumed that $\psi(x_0, t_0, v)$ is non-increasing (”singular” case) or $\psi(x_0, t_0, v)$ is non-decreasing (”degenerate” case) for $v \geqslant b_0 R_0^{-a_0}$. Unfortunately, we were unable to prove Harnack's inequality under similar conditions. The following examples show that additional conditions on the function $g(x, t, v)$ arise naturally.

Example 1. Consider the function

$$g_1(x, t, v) := v^{p-1} + a(x, t)v^{q-1}, (x, t) \in \Omega_T, v > 0,$$

where $a(x, t) \geqslant 0$, $q > p$, $\operatorname{osc}_{Q_{r,r}(x_0,t_0)} a(x, t) \leqslant A r^{q-p} \mu_1(r)$, $0 < A < +\infty$,

$$\lim_{r \to 0} \mu_1(r) = +\infty, \lim_{r \to 0} r^{q-p} \mu_1(r) = 0.$$

The function g_1 satisfies condition (g_λ) with $\lambda(r) = [\mu_1(r)]^{-1/(q-p)}$ and $b = 0$. Indeed,

$$g_1(x, t, v/r) - g_1(y, \tau, v/r) \leqslant |a(x, t) - a(y, \tau)| \left(\frac{v}{r}\right)^{q-1}$$
$$\leqslant A\mu_1(r)\, v^{q-p} \left(\frac{v}{r}\right)^{p-1} \leqslant A K^{q-p} \left(\frac{v}{r}\right)^{p-1} \leqslant A K^{q-p} g(y, t, v/r)$$

if $0 < v \leqslant K\lambda(r)$ and $(x, t), (y, \tau) \in Q_{r,r}(x_0, t_0)$. Condition ($\mathrm{g}_\mu$) with $\mu(r) = \mu_1(r)$ and $b = 0$ is verified similarly.

Note that condition (ψ) can be violated in the case $q > 2$ and $a(x_0, t_0) = 0$.

Example 2. Consider the function

$$g_2(x, t, v) := v^{p-1} + a(x, t)v^{q-1}(1 + \ln(1 + v))^\beta, \quad (x, t) \in \Omega_T, \quad v > 0,$$

where $q > p$, $\beta > 0$, $a(x, t) \geqslant 0$, $\operatorname{osc}_{Q_{r,r}(x_0,t_0)} a(x, t) \leqslant A r^{q-p} \mu_2(r)$, $0 < A < +\infty$,

$$\lim_{r \to 0} \mu_2(r) = +\infty, \quad \lim_{r \to 0} r^{q-p} \mu_2(r)(\ln r^{-1})^\beta = 0.$$

Conditions (g_λ) and (g_μ) with $\lambda(r) = [\mu_2(r)]^{-\frac{1}{q-p}} (\ln r^{-1})^{-\frac{\beta}{q-p}}$, $\mu(r) = \mu_2(r) \times$ $\times (\ln r^{-1})^\beta$ and $b = 0$ are checked similarly to Example 1. Condition (ψ) is a consequence of condition (g_1) in the case $q + \beta < 2$. In the case $q + \beta > 2$ and $a(x_0, t_0) = 0$ condition (ψ) is violated. In the case $q > 2$ and $a(x_0, t_0) > 0$ condition (ψ) is also fails . Let us check condition (ψ) in the case $q < 2 < q + \beta$ and $a(x_0, t_0) > 0$. For this we note that

$$\frac{\psi'_v(x, t, v)v}{\psi(x, t, v)} \leqslant q - 2 + \frac{\beta}{1 + \ln(1 + v)} \leqslant \frac{q - 2}{2} \quad \text{if } v \geqslant e^{\frac{2\beta}{2-q}} - 1,$$

which implies (ψ) with $q_1 = q/2$, $a_0 = 0$ and $b_0 = e^{\frac{2\beta}{2-q}} - 1$.

Example 3. The function

$$g_3(x,t,v) := v^{p-1}\left(1 + L\ln(1 + a(x,t)v)\right),\ (x,t)\in\Omega_T,\ v>0$$

where $0 < L < 2-p$, $a(x,t)\geqslant 0$, $\operatorname{osc}_{Q_{r,r}(x_0,t_0)} a(x,t)\leqslant Br\mu_3(r)$, $0<B<+\infty$,

$$\lim_{r\to 0}\mu_3(r) = +\infty,\ \lim_{r\to 0} r\mu_3(r) = 0,$$

satisfies condition (g_λ) with $\lambda(r) = 1/\mu_3(r)$ and $b = 0$. Indeed,

$$\begin{aligned} &g_3(x,t,v/r) - g_3(y,\tau,v/r) \leqslant L\left(\frac{v}{r}\right)^{p-1}\ln\left(1 + |a(x,t)-a(y,\tau)|\frac{v}{r}\right)\\ &\leqslant L\left(\frac{v}{r}\right)^{p-1}\ln\left(1 + B\mu_3(r)v\right)\leqslant L\left(\frac{v}{r}\right)^{p-1}\ln(1+BK)\leqslant L\ln(1+BK)g_3(y,\tau,v/r)\end{aligned}$$

if $0 < v \leqslant K\lambda(r)$ and $(x,t),(y,\tau)\in Q_{r,r}(x_0,t_0)$. Condition ($\mathrm{g}_\mu$) with $\mu(r) = \ln\mu_3(r)$ and $b=0$ we obtain similarly. Condition (ψ) holds since $q = p+L<2$.

We note also that the function $\tilde g_3(x,t,v) := v^{p-1}(1+\ln(1+a(x,t)v))$ satisfies condition (ψ) in the case $p<2$ and $a(x_0,t_0)>0$. To check this we note that

$$\begin{aligned}\frac{\psi'_v(x,t,v)v}{\psi(x,t,v)} &= p-2+\frac{va(x,t)}{1+va(x,t)}\,\frac{1}{1+\ln(1+a(x,t)v)}\\ &\leqslant p-2+\frac{1}{1+\ln(1+a(x,t)v)}.\end{aligned}$$

Choose R_0 from the condition $BR_0\mu(R_0) = \frac{1}{2}a(x_0,t_0)$. This choice guarantees that

$$\frac{1}{2}a(x_0,t_0)\leqslant a(x,t)\leqslant\frac{3}{2}a(x_0,t_0),\quad (x,t)\in Q_{R_0,R_0}(x_0,t_0).$$

So, if $v\geqslant\frac{1}{4B}(e^{\frac{p}{2-p}}-1)R_0^{-1}$, from the previous we obtain

$$\frac{\psi'_v(x,t,v)v}{\psi(x,t,v)}\leqslant\frac{p-2}{2},\quad (x,t)\in Q_{R_0,R_0}(x_0,t_0),$$

which implies condition (ψ) with $b_0 = \frac{1}{4B}(e^{\frac{p}{2-p}}-1)$, $a_0 = 1$ and $q_1 = p/2 < 1$. Unfortunately, condition (ψ) is violated for the function $\tilde g_3$ in the case $a(x_0,t_0) = 0$ and $p<2$.

Example 4. Consider the functions

$$\begin{aligned} &g_4(x,t,v) := v^{p-1}(1 + a(x,t)\ln(1+v)),\\ &g_5(x,t,v) := v^{p(x,t)-1},\quad v>0,\quad (x,t)\in\Omega_T,\end{aligned}$$

where $p<2$, $p(x,t)\leqslant q<2$, $a(x,t)\geqslant 0$ and

$$\operatorname{osc}_{Q_{r,r}(x_0,t_0)} a(x,t) + \operatorname{osc}_{Q_{r,r}(x_0,t_0)} p(x,t)\leqslant\frac{L}{\ln r^{-1}}.$$

It is obvious that $g_4(x,t,v)$ satisfies conditions (g_λ),(g_μ) with $\lambda(r)=\mu(r)=1$ and $b=0$.
Similarly, the function $g_5(x,t,v)$ satisfies conditions (g_λ),(g_μ) with $\lambda(r)=\mu(r)=1$ and $b=1$. To check condition (ψ) for the function g_4, we note

$$\frac{\psi'_v(x,t,v)v}{\psi(x,t,v)} = p-2+\frac{a(x,t)v}{1+v}(1+a(x,t)\ln(1+v))^{-1}$$
$$\leqslant p-2+\ln^{-1}(1+v)\leqslant \frac{p-2}{2}$$

if $v\geqslant e^{\frac{2}{2-p}}-1$, which implies condition (ψ) with $b_0=e^{\frac{2}{2-p}}-1$, $a_0=0$ and $q_1=p/2$.

Before describing the main results, a few words concern the history of the problem. The study of regularity of minima of functionals with nonstandard growth has been initiated by Kolodij [25,26], Zhikov [44–47,49], Marcellini [30,31] and Lieberman [29], and in the last thirty years there has been growing interest and substantial development in the qualitative theory of quasi-linear elliptic and parabolic equations with so-called "log-conditions" (i.e., if $\lambda(r)=1$ and $\mu(r)=1$). We refer the reader to the papers [1,3–10,12–14,18,19,21–24,36–43,51] for the basic results, historical surveys and references.

The case when conditions (g_λ) or (g_μ) hold differ substantially from the logarithmic case. To our knowledge, there are few results in this direction. Zhikov [48] obtained a generalization of the logarithmic condition which guarantees the density of smooth functions in Sobolev space $W^{1,p(x)}(\Omega)$. Particularly, this result holds if $p(x)\geqslant p>1$, and for every $x,y\in\Omega$, $x\neq y$,

$$|p(x)-p(y)|\leqslant\frac{\ln\mu(|x-y|)}{\big|\ln|x-y|\big|},\quad \int_0[\mu(r)]^{-\frac{n}{p}}\frac{dr}{r}=+\infty.$$

We note that the function $\mu(r)=(\ln r^{-1})^L$, $0\leqslant L\leqslant p/n$ satisfies the above condition. Later Zhikov and Pastuchova [50] under the same condition proved higher integrability of the gradient of solutions to the $p(x)$-Laplace equation.

Interior continuity, continuity up to the boundary and Harnack's inequality to the $p(x)$-Laplace equation were proved by Alkhutov, Krasheninnikova [1], Alkhutov, Surnachev [2] and Surnachev [35] under the conditions

$$|p(x)-p(y)|\leqslant\frac{\ln\mu(|x-y|)}{|\ln|x-y||},\quad x,y\in\Omega,\quad x\neq y,\quad \int_0\exp(-\gamma\mu^c(r))\frac{dr}{r}=+\infty \tag{3}$$

with some $\gamma,\ c>1$. Particularly, the function $\mu(r)=(\ln\ln r^{-1})^L$, $0<L<1/c$ satisfies the above condition.

These results were generalized in [32,33] for a wide class of elliptic and parabolic equations with non-logarithmic Orlicz growth. Later, the results from [32,33] were substantially refined in [20,34]. For interior continuity, instead of condition (3), it was required that

$$\int_0 \lambda(r)\frac{dr}{r} = +\infty. \tag{4}$$

In addition, in [20] Harnack's inequality was proved for quasilinear elliptic equations under the condition

$$\int_0 [\mu(r)]^{-\frac{2n}{p-1}}\lambda(r)\frac{dr}{r} = +\infty. \tag{5}$$

We note that this condition is worse than condition (4), but at the same time it is much better then condition (3).

In this paper, we establish the intrinsic Harnack inequality for nonnegative solutions to Eq. (1) under the similar conditions as (5). To describe our results, let us introduce the definition of a weak solution to Eq. (1).

Definition 1. We say that u is a bounded weak sub (super) solution to Eq. (1) if

$$u \in C_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(\Omega)) \cap L^q_{\mathrm{loc}}(0, T; W^{1,q}_{\mathrm{loc}}(\Omega)) \cap L^\infty(\Omega_T),$$

and for any compact set $E \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0, T]$ the integral identity

$$\int_E u\eta\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_E \{-u\eta_\tau + \mathbb{A}(x, \tau, \nabla u)\nabla\eta\}\, dx d\tau \leqslant (\geqslant)\, 0 \tag{6}$$

holds true for any testing functions $\eta \in W^{1,2}(0, T; L^2(E)) \cap L^q(0, T; W_0^{1,q}(E))$, $\eta \geqslant 0$.

It would be technically convenient to have a formulation of weak solution that involves u_t. Let $\rho(x) \in C_0^\infty(\mathbb{R}^n)$, $\rho(x) \geqslant 0$ in $\mathbb{R}^n$, $\rho(x) \equiv 0$ for $|x| > 1$ and $\int_{\mathbb{R}^n}\rho(x)\, dx = 1$, and set

$$\rho_h(x) := h^{-n}\rho\,(x/h)\,, \quad u_h(x, t) := h^{-1}\int_t^{t+h}\int_{\mathbb{R}^n} u(y, \tau)\rho_h(x - y)\, dy d\tau.$$

Fix $t \in (0, T)$ and let $h > 0$ be so small that $0 < t < t + h < T$. In (1.6) take $t_1 = t$, $t_2 = t + h$ and replace η by $\int_{\mathbb{R}^n}\eta(y, t)\rho_h(x - y)\, dy$. Dividing by h, since the testing function does not depend on τ, we obtain

$$\int_{E\times\{t\}}\left(\frac{\partial u_h}{\partial t}\eta + [\mathbb{A}(x, t, \nabla u)]_h\nabla\eta\right) dx \leqslant (\geqslant)\, 0, \tag{7}$$

for all $t \in (0, T - h)$ and for all nonnegative $\eta \in W_0^{1,q}(E)$.

We refer to the parameters $M := \sup_{\Omega_T} u$, K_1, K_2, c_0, $c_1 = c_1(M, M)$, $c_2 = c_2(M, M)$, c_3, n, p, q and q_1 as our structural data, and we write γ if it can be quantitatively determined a priori only in terms of the above quantities. The generic constant γ may change from line to line. Note that the constants a_0, b and b_0 can be equal to zero, therefore, in the proof we keep an explicit track of the dependence of the various constants on a_0, b and b_0.

Our first main result concerns the Harnack inequality for the case of logarithmic growth (i.e., $\lambda(r) = \mu(r) = 1$). Fix a point $(x_0, t_0) \in \Omega_T$ such that $u(x_0, t_0) > 0$ and consider the cylinders

$$Q_{\rho,\theta}(x_0, t_0), \quad \theta := \rho^2/\psi(x_0, t_0, u(x_0, t_0)\rho^{-1}).$$

Theorem 1. *Let u be a positive bounded weak solution to Eq. (1). Let the conditions (2), (g_0)–(g_2), (ψ), (g_λ) and (g_μ) with $\lambda(r) = \mu(r) = 1$ be fulfilled in the cylinder $Q_{R_0,R_0}(x_0, t_0) \subset \Omega_T$. Then, there exist positive constants c, $\bar{c}$, δ_1 depending only upon the data such that for all cylinders $Q_{8\rho,8\theta}(x_0, t_0) \subset Q_{8\rho,8\bar{c}\rho}(x_0, t_0) \subset Q_{R_0,R_0}(x_0, t_0)$, $0 < \rho \leqslant R_0^{a_1}$, $a_1 = 1 + a_0/\overline{a}$, either*

$$u(x_0, t_0) \leqslant c(b + b_0)\rho^{1-a_0/a_1}, \tag{8}$$

or

$$u(x_0, t_0) \leqslant c \inf_{B_\rho(x_0)} u(\cdot, t), \tag{9}$$

for all times $|t - t_0| \leqslant \rho^2/\psi(x_0, t_0, \delta_1 u(x_0, t_0)\rho^{-1})$, and the numbers a_0, $\bar{a}$ are defined in conditions (g_λ), (g_μ) and (ψ).

Our next result is the Harnack inequality for the case of non-logarithmic growth. Having fixed $(x_0, t_0) \in \Omega_T$, construct the cylinder $Q_{\rho,\bar{\theta}}(x_0, t_0)$, where

$$\bar{\theta} = \frac{\rho^2}{\psi(x_0, t_0, u(x_0, t_0)\lambda_1(\rho)\rho^{-1})}, \quad \lambda_1(\rho) = \lambda(\rho)[\mu(\rho)]^{-\beta},$$
$$\beta = \frac{1}{2-p} + n\frac{1 + 2(2-p)}{p + n(p-2)}.$$

Theorem 2. *Let u be a positive bounded weak solution to Eq. (1). Let the conditions (2), (g_0)–(g_2), (g_λ), (g_μ) and (ψ) be fulfilled in the cylinder $Q_{R_0,R_0}(x_0, t_0) \subset \Omega_T$. Assume also that*

$$(\mathbb{A}(x, t, \xi) - \mathbb{A}(x, t, \eta))(\xi - \eta) > 0, \quad \xi, \eta \in \mathbb{R}^n, \quad \xi \neq \eta. \tag{10}$$

If additionally $\int_0 \lambda_1(r) r^{-1}\, dr = +\infty$ and $\lim_{r\to 0} r^{1-\bar{a}_1}/\lambda_1(r) = 0$ with some $\bar{a}_1 \in (0, 1)$, then there exist positive constants c, $\bar{c}$, δ_1 depending only upon the data such that for all cylinders

$$Q_{8\rho,8\bar{\theta}}(x_0, t_0) \subset Q_{8\rho,8\bar{c}\rho}(x_0, t_0) \subset Q_{R_0,R_0}(x_0, t_0), \ 0 < \rho \leqslant R_0^{a_1}, \ a_1 = 1 + a_0/\bar{a}_1,$$

either

$$u(x_0,t_0)\leqslant c(b+b_0)\frac{\rho^{1-a_0/a_1}}{\lambda_1(\rho)}, \tag{11}$$

or

$$u(x_0,t_0)\leqslant \frac{c}{\lambda_1(\rho)}\inf_{B_\rho(x_0)} u(\cdot,t) \tag{12}$$

for all times $|t-t_0|\leqslant \rho^2/\psi\left(x_0,t_0,\delta_1\rho^{-1}\lambda_1(\rho)u(x_0,t_0)\right)$.

Remark 2. We note that in the case $g_1(x,t,v)=v^{p-1}+a(x,t)v^{q-1}$,

$$\mu_1(\rho)=(\ln\rho^{-1})^{\alpha},\quad \lambda(\rho)=(\ln\rho^{-1})^{-\frac{\alpha}{q-p}},\quad 0\leqslant\alpha\leqslant\frac{q-p}{1+\beta(q-p)},$$

inequality (12) translates into

$$u(x_0,t_0)\leqslant c\ln\rho^{-1}\inf_{B_\rho(x_0)}u(\cdot,t),\quad |t-t_0|\leqslant\frac{\rho^2}{\psi\left(x_0,t_0,\delta_1\dfrac{u(x_0,t_0)}{\rho\ln\rho^{-1}}\right)}. \tag{13}$$

Similar result is also valid for the function $g_2(x,t,v)=v^{p-1}+a(x,t)v^{q-1}(1+\ln(1+v))^{\beta}$ and

$$\mu_2(\rho)=(\ln\rho^{-1})^{\alpha},\quad \lambda(\rho)=(\ln\rho^{-1})^{-\frac{\alpha+\beta}{q-p}},\quad 0\leqslant\alpha+\beta\leqslant\frac{q-p}{1+\beta(q-p)}.$$

In addition, note that in the case $g_3(x,t,v)=v^{p-1}(1+L\ln(1+a(x,t)))$ and

$$\mu_3(\rho)=(\ln\rho^{-1})^{\alpha},\quad \lambda(\rho)=(\ln\rho^{-1})^{-\alpha},\quad \mu(\rho)=\alpha\ln\ln\rho^{-1},\quad 0\leqslant\alpha<1,$$

inequality (12) can be rewritten as (13).

We would like to mention the approach taken in this paper. To prove our results, we use Di Benedetto's approach [15], who developed an innovative intrinsic scaling methods for degenerate and singular parabolic equations. For the p-Laplace evolution equation, the intrinsic Harnack inequality was proved in the famous papers [16,17]. The main stage in the proof of our results is the so-called $L^1_{\rm loc}-L^\infty_{\rm loc}$ Harnack inequality.

Theorem 3. *Let u be a nonnegative, bounded weak solution to Eq. (1) and let (2),* (g_0)–(g_2), (g_μ) *and* (ψ) *be fulfilled in the cylinder $Q_{R_0,R_0}(x_0,t_0)\subset\Omega_T$. Then for all cylinders*

$$Q^+_{2r,2(t-s)}(y,2s-t)\subset Q_{R_0,R_0}(x_0,t_0),\quad 0<t-s<\bar c r,\quad r\leqslant R_0^a,\quad a>a_0,\quad \bar c>0,$$

$$\sup_{Q^+_{\frac{r}{2},t-s}(y,s)}u\leqslant\gamma(t-s)^{\frac12}\varphi^{-1}_{Q^+_{2r,2(t-s)}(y,2s-t)}\left((t-s)^{-\frac{n+1}{2}}\inf_{2s-t<\tau<t}\int_{B_{2r}(y)}u(x,\tau)dx\right)$$

$$
+\gamma (t-s)^{\frac{1}{2}} \varphi^{-1}_{Q^+_{2r,2(t-s)}(y,2s-t)} \left(\left(\frac{r^2}{t-s}\right)^{\frac{n+1}{2}} \psi^{-1}_{Q^+_{2r,2(t-s)}(y,2s-t)} \left(\frac{r^2}{t-s}\right)\right)
$$

$$
+\gamma r \psi^{-1}_{Q^+_{2r,(t-s)}(y,2s-t)} \left(\frac{r^2}{t-s}\right), \tag{14}
$$

provided that

$$
\psi^{-1}_{Q^+_{2r,2(t-s)}(y,2s-t)} \left(\frac{r^2}{t-s}\right) \geqslant (b+b_0) r^{-a_0/a}, \tag{15}
$$

the constant γ depends only on the data, $\bar{c}$ and a. Here,

$$
\varphi_Q(v) := \frac{v^{n+1}}{[G_Q^{-1}(v^2)]^n}, \quad G_Q(v) := \inf_{(x,t)\in Q} G(x,t,v),
$$

$$
G(x,t,v) := \int_0^v g(x,t,z)\,dz, \quad \psi_Q(v) := \sup_{(x,t)\in Q} \psi(x,t,v)
$$

and $G_Q^{-1}(\cdot)$, $\psi_Q^{-1}(\cdot)$ and $\varphi_Q^{-1}(\cdot)$ are the inverse functions to $G_Q(\cdot)$, $\psi_Q(\cdot)$ and $\varphi_Q(\cdot)$ respectively.

Remark 3. Note that by our choices $r^{-a_0/a} \geqslant R_0^{-a_0}$, therefore, condition ($\psi$) is applicable and the left-hand side of inequality (15) makes sense. In addition, the function $\varphi_{Q^+_{2r,2(t-s)}(y,2s-t)}(\cdot)$ is strictly increasing (see Lemma 1 below), so, the right-hand side of inequality (14) also makes sense.

Estimate (14) coincides with the well-known L^1_{loc}–L^∞_{loc} form of Harnack's inequality (see [15]) in the case of p-Laplace evolution equation ($p < 2$) :

$$
\sup_{Q^+_{r,t-s}(y,s)} u \leqslant \gamma \left(\frac{r^p}{t-s}\right)^{n/\kappa} \left(r^{-n} \inf_{2s-t<\tau<t} \int_{B_{2r}(y)} u(x,\tau)dx \right)^{p/\kappa} + \gamma \left(\frac{t-s}{r^p}\right)^{1/(2-p)},
$$

where $\kappa = p + n(p-2) > 0$.

Main difficulty arising in the proof of our main results is related to the so-called theorem on the expansion of positivity. Roughly speaking, having information on the measure of the "positivity set" of u over the ball $B_r(\bar{x})$ for some time level $\bar{t}$:

$$
|\{x \in B_r(\bar{x}) : u(x,\bar{t}) \leqslant N\}| \leqslant (1-\alpha(r))\,|B_r(\bar{x})|,
$$

with some $r > 0$, $N > 0$ and $\alpha(r) \in (0,1)$, $\alpha(r) \to 0$, as $r \to 0$, and using the standard Di Benedetto's arguments, we inevitably arrive at the estimate

$$
u(x,t) \geqslant \gamma_1^{-1} e^{-\gamma_1 \alpha^{-\gamma_2}(r) \mu^{\gamma_3}(r)} N, \quad x \in B_{2r}(\bar{x}),
$$

for some time level $t > \bar{t}$ and with some $\gamma_1, \gamma_2, \gamma_3 > 1$. This estimate leads us to condition similar to that of (3) (see, e.g., [34]). To avoid this, we use a workaround that goes back to Landis's papers [27,28] and his so called "growth" lemma. So, in Sect. 3 we use the auxiliary solutions and prove the integral and pointwise estimates of these solutions.

The rest of the paper contains the proof of the above theorem. In Sect. 2, we give a proof of L^1_{loc}–L^∞_{loc} Harnack's inequality, Theorem 3. Section 3 contains the upper and lower bounds for the auxiliary solutions. Finally, in Sect. 4 we prove our main results, Theorems 1 and 2.

2. L^1_{loc}–L^∞_{loc} Harnack type inequality, Proof of Theorem 3

2.1. An auxiliary lemma

The following inequalities will be used in the sequel, they are simple consequences of the condition (g_2).

Lemma 1. *The following inequalities hold:*

$$g(x,t,w)v \leqslant \varepsilon g(x,t,w)w + \max\{\varepsilon^{p-1}, \varepsilon^{q-1}\} g(x,t,v)v \tag{16}$$

if $v, w, \varepsilon > 0,\ (x,t) \in \Omega_T$;

$$\frac{1}{q} g(x,t,v)v \leqslant G(x,t,v) \leqslant \frac{1}{p} g(x,t,v)v \ \ \textit{if}\ \ v > 0,\ (x,t) \in \Omega_T;$$
$$\left(\frac{w}{v}\right)^p \leqslant \frac{G(x,t,w)}{G(x,t,v)} \leqslant \left(\frac{w}{v}\right)^q \ \ \textit{if}\ \ w \geqslant v > 0,\ (x,t) \in \Omega_T;$$
$$\left(\frac{w}{v}\right)^{\varkappa(p)/p} \leqslant \frac{\varphi(x,t,w)}{\varphi(x,t,v)} \leqslant \left(\frac{w}{v}\right)^{\varkappa(q)/q} \ \ \textit{if}\ \ w \geqslant v > 0,\ (x,t) \in \Omega_T.$$

Here

$$\varphi(x,t,v) := \frac{v^{n+1}}{[G^{-1}(x,t,v^2)]^n},\ \ \varkappa(p) := p + n(p-2) > 0,\ \ \varkappa(q) := q + n(q-2) > 0.$$

Note that the third and fourth inequalities in Lemma 1 ensure that the functions $G(\cdot, v)$ and $\varphi(\cdot, v)$ are increasing, and therefore, the inverse functions $G^{-1}(\cdot, v)$ and $\varphi^{-1}(\cdot, v)$ are exist.

2.2. An L^1_{loc} form of Harnack's inequality

For fixed $(y,s) \in \Omega_T$, for $0 < r \leqslant R^a$, $a \geqslant 1 + a_0$ and for $0 < t - s < \bar{c}r$, we will assume later that inequality (15) holds, i.e.,

$$\varepsilon := r\psi^{-1}_{Q^+_{2r,2(t-s)}(y,2s-t)}\left(\frac{r^2}{t-s}\right) \geqslant (b+b_0) r^{1-a_0/a}. \tag{17}$$

Proposition 1. *Let u be a nonnegative bounded weak solution to Eq. (1) and let (2),* (g_0)–(g_2), (g_μ) *and* (ψ) *be fulfilled in the cylinder* $Q_{R_0,R_0}(x_0,t_0)$. *Then*

$$\sup_{s<\tau<t}\int_{B_r(y)} u(x,\tau)\,dx \leqslant \gamma \inf_{s<\tau<t}\int_{B_{2r}(y)} u(x,\tau)\,dx + \gamma r^{n+1}\psi^{-1}_{Q^+_{2r,2(t-s)}(y,2s-t)}\left(\frac{r^2}{t-s}\right). \tag{18}$$

To prove Proposition 2.1, we need the following lemma.

Lemma 2. *Let the conditions of Proposition 1 be fulfilled. Then, there exists $\gamma > 0$ depending only on the data, such that for all $\sigma, \delta \in (0,1)$ there holds*

$$\frac{1}{r}\int_s^t\int_{B_{\sigma r}(y)} g(x,\tau,|\nabla u|)\,dxd\tau \leqslant \delta J + \gamma\delta^{-\gamma}(1-\sigma)^{-\gamma}r^n\varepsilon, \tag{19}$$

where $J := \sup\limits_{s<\tau<t}\int_{B_r(y)} u(x,\tau)\,dx$.

Proof. Assume without loss that $s = 0$ and let $\alpha \in (0, \frac{1}{q-1})$, $\tilde{q} = q-1$ if $q < 2$ and $\tilde{q} = q_1$ if $q > 2$, $\beta \in (1, \min\{2, 1/\tilde{q}\})$. Fix $\sigma \in (0,1)$ and let $\zeta \in C_0^\infty(B_r(y))$, $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in $B_{\sigma r}(y)$, $|\nabla\zeta| \leqslant \frac{1}{(1-\sigma)r}$, and set $\varepsilon_1 := rt^{-\alpha}\varepsilon^{\beta-1}$.

Using inequality (16), we obtain

$$\frac{1}{r}\int_0^t\int_{B_{\sigma r}(y)} g(x,\tau,|\nabla u|)\,dxd\tau \leqslant \frac{\varepsilon_1}{r}\int_0^t\int_{B_r(y)}(u+\varepsilon)^{-\beta}\tau^\alpha g(x,\tau,|\nabla u|) \times |\nabla u|\,\zeta^q\,dxd\tau + \frac{1}{r}\int_0^t\int_{B_r(y)} g\left(x,\tau,\frac{(u+\varepsilon)^\beta}{\varepsilon_1\tau^\alpha}\right)dxd\tau = I_1 + I_2. \tag{20}$$

First we estimate I_2, by (g_2), (g_μ), (ψ) (note that by (17) condition (ψ) is applicable), we obtain

$$\begin{aligned}
&\frac{1}{r}\int_0^t\int_{B_r(y)} g\left(x,\tau,\frac{(u+\varepsilon)^\beta}{\varepsilon_1\tau^\alpha}\right)dxd\tau \\
&= \frac{1}{r}\int_0^t\int_{B_r(y)} g\left(x,\tau,\left(\frac{u+\varepsilon}{\varepsilon}\right)^\beta\frac{\varepsilon}{r}\left(\frac{t}{\tau}\right)^\alpha\right)dxd\tau \\
&\leqslant \frac{\gamma}{r}\int_0^t\int_{B_r(y)}\left(\frac{t}{\tau}\right)^{\alpha(q-1)}\left(\frac{u+\varepsilon}{\varepsilon}\right)^{\beta\tilde{q}} g(x,\tau,\varepsilon/r)\,dxd\tau \\
&\leqslant \frac{\gamma}{r}\sup_{Q_{r,t}(y,0)} g(\cdot,\cdot,\varepsilon/r)\int_0^t\int_{B_r(y)}\left(\frac{t}{\tau}\right)^{\alpha(q-1)}\left(\frac{u+\varepsilon}{\varepsilon}\right)^{\beta\tilde{q}}dxd\tau \\
&\leqslant \gamma\frac{\varepsilon}{t}\int_0^t\int_{B_r(y)}\left(\frac{t}{\tau}\right)^{\alpha(q-1)}\left(\frac{u+\varepsilon}{\varepsilon}\right)^{\beta\tilde{q}}dxd\tau \\
&\leqslant \gamma(\varepsilon r^n)^{1-b\tilde{q}}\left(\sup_{0<\tau<t}\int_{B_r(y)}(u+\varepsilon)\,dx\right)^{\beta\tilde{q}} \leqslant \frac{\delta}{4}J + \gamma\delta^{-\gamma}\varepsilon r^n.
\end{aligned} \tag{21}$$

To estimate I_1 we test identity (7) by $\eta = (u_h + \varepsilon)^{1-\beta}\tau^\alpha\zeta^q$, integrating over $(0, t)$ and letting $h \to 0$, we obtain that

$$\int_0^t \int_{B_r(y)} (u+\varepsilon)^{-\beta}\tau^\alpha g(x,\tau,|\nabla u|)\,|\nabla u|\,\zeta^q\,dxd\tau$$
$$\leqslant \gamma t^\alpha \int_{B_r(y)\times\{t\}} (u+\varepsilon)^{2-\beta}\zeta^q\,dx + \frac{\gamma}{(1-\sigma)r}\int_0^t \int_{B_r(y)} (u+\varepsilon)^{1-\beta}\tau^\alpha g(x,\tau,|\nabla u|)\,\zeta^{q-1}\,dxd\tau,$$

from this by inequality (16) we arrive at

$$\begin{aligned} &\frac{\varepsilon_1}{r}\int_0^t \int_{B_r(y)} (u+\varepsilon)^{-\beta}\tau^\alpha g(x,\tau,|\nabla u|)\,|\nabla u|\,\zeta^q\,dxd\tau \\ &\leqslant \gamma\frac{\varepsilon_1}{r}t^\alpha \int_{B_r(y)\times\{t\}} (u+\varepsilon)^{2-\beta}\,\zeta^q + dx \\ &+ \frac{\gamma}{(1-\sigma)^\gamma}\frac{\varepsilon_1}{r^2}\int_0^t \int_{B_r(y)} (u+\varepsilon)^{1-\beta}\tau^\alpha g\left(x,\tau,\frac{u+\varepsilon}{r}\right)dxd\tau = I_3 + I_4. \end{aligned} \tag{22}$$

Let us estimate the terms on the right-hand side of (22). By our choice of β and Hölder's inequality, we have

$$\begin{aligned} I_3 &= \gamma\varepsilon^{\beta-1}\int_{B_r(y)\times\{t\}} (u+\varepsilon)^{2-\beta}\,\zeta^q\,dx \\ &\leqslant \gamma(\varepsilon r^n)^{\beta-1}\left(\sup_{0<\tau<t}\int_{B_r(y)} (u+\varepsilon)\,dx\right)^{2-\beta} \leqslant \frac{\delta}{4}J + \gamma\delta^{-\gamma}\varepsilon r^n. \end{aligned} \tag{23}$$

The function $\psi(\cdot, v)$ is non-increasing for $v \geqslant b_0 R_0^{-a_0}$, so by (17) we estimate the second term on the right-hand side of (22) as follows:

$$\begin{aligned} I_4 &\leqslant \frac{\gamma}{(1-\sigma)^\gamma}\frac{\varepsilon^{\beta-1}}{r}\int_0^t \int_{B_r(y)} (u+\varepsilon)^{1-\beta} g\left(x,\tau,\frac{u+\varepsilon}{r}\right)dxd\tau \\ &\leqslant \frac{\gamma}{(1-\sigma)^\gamma}\frac{\varepsilon^{\beta-1}}{r^2}\int_0^t \int_{B_r(y)} (u+\varepsilon)^{2-\beta}\,\psi(x,\tau,\varepsilon/r)\,dxd\tau \\ &\leqslant \frac{\gamma}{(1-\sigma)^\gamma}\frac{t\varepsilon^{\beta-1}}{r^2}\,\psi_{Q^+_{r,t}(y,0)}(\varepsilon/r)\sup_{0<\tau<t}\int_{B_r(y)} (u+\varepsilon)^{2-\beta}\,dx \\ &\leqslant \frac{\gamma\varepsilon^{\beta-1}}{(1-\sigma)^\gamma}\sup_{0<\tau<t}\int_{B_r(y)} (u+\varepsilon)^{2-\beta}\,dx \leqslant \frac{\delta}{4}J + \gamma\delta^{-\gamma}(1-\sigma)^{-\gamma}\varepsilon r^n. \end{aligned} \tag{24}$$

Collecting estimates (21)–(24), we arrive at the required inequality (19). This completes the proof of Lemma 2. □

2.3. Proof of Proposition 1

Assume that $s = 0$, and for $j = 0, 1, 2, \ldots$ set

$$r_j := \sum_{i=1}^{j}\frac{r}{2^i}, \quad \bar r_j := \frac{r_j + r_{j+1}}{2}, \quad B_j := B_{r_j}(y), \quad \bar B_j := B_{\bar r_j}(y)$$

and let $\zeta_j \in C_0^\infty(\bar{B}_j)$, $0 \leqslant \xi_j \leqslant 1$, $\zeta_j = 1$ in B_j, $|\nabla \zeta_j| \leqslant 2^{j+2}/r$. Test (1.7) by $\eta = \zeta_j$, integrating over $(t_1, t_2) \subset (0, t)$ for any two time levels t_1 and t_2, and letting $h \to 0$, we obtain that

$$\int_{\bar{B}_j} u(x, t_2)\, \zeta_j\, dx \leqslant \int_{\bar{B}_j} u(x, t_1)\, \zeta_j\, dx + \frac{\gamma 2^j}{r} \int_{t_1}^{t_2} \int_{\bar{B}_j} g(x, \tau, |\nabla u|)\, dx d\tau.$$

Fix t_1 by the condition

$$\int_{B_{2r}(y)} u(x, t_1)\, dx = \inf_{0<\tau<t} \int_{B_{2r}(y)} u\, dx,$$

and set $J_j := \sup\limits_{0<\tau<t} \int_{B_{2r}(y)} u\, dx$. By Lemma 2, choosing δ from the condition $\delta = \delta_0 \gamma^{-1} 2^{-j}$, $\delta_0 \in (0, 1)$, from the previous we obtain

$$J_j \leqslant \delta_0 J_{j+1} + \gamma \delta_0^{-\gamma} 2^{j\gamma} \left(\inf_{0<\tau<t} \int_{B_{2r}(y)} u\, dx + r^n \varepsilon \right), \quad j = 0, 1, 2, \ldots$$

Iterating this inequality and choosing δ_0 sufficiently small we arrive at (18), which completes the proof of Proposition 1.

2.4. $L^1_{\mathrm{loc}} - L^\infty_{\mathrm{loc}}$ estimate of solution

Theorem 3 is a simple consequence of Proposition 1 and the following lemma.

Lemma 3. *Let u be a nonnegative bounded weak solution to Eq. (1) and let (2), (g_0)–(g_2), (g_μ) and (ψ) be fulfilled in the cylinder $Q_{R_0,R_0}(x_0, t_0) \subset \Omega_T$. Assume also that (17) holds, then there exists $\gamma > 0$ depending only upon the data, such that*

$$\begin{aligned} \sup_{Q^+_{\frac{r}{2}, t-s}(y,s)} u \leqslant & \gamma (t-s)^{1/2}\, \varphi^{-1}_{Q^+_{2r,2(t-s)}(y,2s-t)} \left((t-s)^{-\frac{n+1}{2}-1} \int_{2s-t}^{t} \int_{B_r(y)} u\, dx d\tau \right) \\ & + \gamma r \psi^{-1}_{Q^+_{2r,2(t-s)}(y,2s-t)} \left(\frac{r^2}{t-s} \right). \end{aligned} \tag{25}$$

Proof. Assume that $s = 0$ and for fixed $\sigma \in (0, 1)$ and $j = 0, 1, 2, \ldots$ set

$$r_j := \sigma r + \frac{1-\sigma}{2j} r, \quad t_j := -\sigma t - \frac{1-\sigma}{2j} t, \quad B_j := B_{r_j}(y), \quad Q_j := B_j \times (t_j, t)$$

and let $M_0 := \sup\limits_{Q_0} u$, $M_\sigma := \sup\limits_{Q_\infty} u$.

Next, let $\zeta = \zeta_1(x)\zeta_2(t)$, where $\zeta_1 \in C_0^\infty(B_j)$, $\zeta_2 \in C^\infty(-t, t)$,

$$0 \leqslant \zeta_1 \leqslant 1, \quad \zeta_1 = 1 \text{ in } B_{j+1}, \quad |\nabla \zeta_1| \leqslant \frac{2^{j+1}}{(1-\sigma)r},$$

$$0 \leqslant \zeta_2 \leqslant 1, \quad \zeta_2 = 1 \text{ for } t \geqslant t_{j+1}, \quad \zeta_2 = 0 \text{ for } t \leqslant t_j, \quad 0 \leqslant \frac{d\zeta_2}{dt} \leqslant \frac{2^{j+1}}{(1-\sigma)t}.$$

Define also the sequence $k_j := k - 2^{-j}k$, where $k > 0$ to be chosen later. Testing (7) by $\eta = (u_h - k_{j+1})_+ \zeta^q$, integrating over (t_j, t) and letting $h \to 0$, by inequality (16), we obtain

$$\sup_{t_j<\tau<t} \int_{B_j} (u - k_{j+1})_+^2 \zeta^q \, dx + \iint\limits_{Q_j} g(x, \tau, |\nabla(u - k_{j+1})_+|)\, |\nabla(u - k_{j+1})_+| \; \times$$
$$\times \zeta^q \, dx d\tau \leqslant \frac{\gamma 2^{j\gamma} t^{-1}}{(1-\sigma)^\gamma} \iint\limits_{Q_j} (u - k_{j+1})_+^2 \, dx d\tau \; +$$
$$\frac{\gamma 2^{j\gamma} r^{-1}}{(1-\sigma)^\gamma} \iint\limits_{Q_j} g\left(x, \tau, \frac{(u - k_{j+1})_+}{r}\right) (u - k_{j+1})_+ \, dx d\tau.$$

If

$$k \geqslant r \psi^{-1}_{Q^+_{r,s}(y,0)}(r^2/t) \;\text{ and }\; k \leqslant M_0, \tag{26}$$

then by (g_μ) and (ψ) we obtain

$$\frac{1}{r} \iint\limits_{Q_j} g\left(x, \tau, \frac{(u - k_{j+1})_+}{r}\right) (u - k_{j+1})_+ \, dx d\tau \leqslant$$
$$\frac{\gamma}{r} \sup_{Q^+_{r,s}(y,0)} g(\cdot, \cdot, M_0/r) \iint\limits_{Q_j} (u - k_{j+1})_+ \, dx d\tau$$
$$\leqslant \frac{\gamma M_0}{t} \iint\limits_{Q_j} (u - k_{j+1})_+ \, dx d\tau \leqslant \frac{\gamma 2^j}{t} \frac{M_0}{k} \iint\limits_{Q_j} (u - k_j)_+^2 \, dx d\tau.$$

Therefore, the previous inequalities yield

$$\begin{aligned} &\sup_{t_j<\tau<t} \int_{B_j} (u - k_{j+1})_+^2 \, \zeta^q \, dx + \iint\limits_{Q_j} g(x, \tau, |\nabla(u - k_{j+1})_+|) \\ &\times |\nabla(u - k_{j+1})_+| \zeta^q \, dx d\tau \leqslant \frac{\gamma 2^{j\gamma}}{(1-\sigma)^\gamma} \frac{M_0}{tk} \iint\limits_{Q_j} (u - k_j)_+^2 \, dx d\tau. \end{aligned} \tag{27}$$

Let us estimate the second term on the left-hand side of (27). By (16), for any $b > 0$, we have

$$\iint\limits_{Q_j} |\nabla(u - k_{j+1})_+| \, \zeta^q \, dx d\tau = \iint\limits_{Q_j} \frac{|\nabla(u - k_{j+1})_+|}{g^-_{Q_0}(b)} g^-_{Q_0}(b) \, \zeta^q \, dx d\tau$$
$$\leqslant \frac{1}{g^-_{Q_0}(b)} \iint\limits_{Q_j} g^-_{Q_0}(|\nabla(u - k_{j+1})_+|) \, |\nabla(u - k_{j+1})_+| \, \zeta^q \, dx d\tau$$

$$+b \iint_{Q_j} \chi[(u-k_{j+1})_+ > 0]\, dxd\tau$$
$$\leqslant \frac{1}{g^-_{Q_0}(b)} \iint_{Q_j} g(x,\tau,|\nabla(u-k_{j+1})_+|)\,|\nabla(u-k_{j+1})_+|\,\zeta^q\, dxd\tau$$
$$+\gamma \frac{2^{2j} b}{k^2} \iint_{Q_j} (u-k_j)^2_+\, dxd\tau,$$

here we used the notation $g^-_{Q_0}(v) := \inf\limits_{(x,t)\in Q_0} g(x,t,v)$, $v > 0$. By (27) from the previous, we obtain

$$\iint_{Q_j} |\nabla(u-k_{j+1})_+|\,\zeta^q\, dxd\tau \leqslant \frac{\gamma 2^{j\gamma}}{(1-\sigma)^\gamma}\left(\frac{M_0 t^{-1}}{k g^-_{Q_0}(b)} + \frac{b}{k^2}\right) \iint_{Q_j} (u-k_j)^2_+\, dxd\tau.$$

Choosing b from the condition $\dfrac{M_0 t^{-1}}{k g^-_{Q_0}(b)} = \dfrac{b}{k^2}$, i.e., $g^-_{Q_0}(b) b = \dfrac{M_0 k}{t} \leqslant \dfrac{M_0^2}{t}$, which by Lemma 2.1 implies

$$\iint_{Q_j} |\nabla(u-k_{j+1})_+|\,\zeta^q\, dxd\tau \leqslant \frac{\gamma 2^{j\gamma}}{(1-\sigma)^\gamma} k^{-2}\, G^{-1}_{Q_0}\left(\frac{M_0^2}{t}\right) \iint_{Q_j} (u-k_j)^2_+\, dxd\tau. \tag{28}$$

Using Hölder's inequality, Sobolev embedding theorem, (27) and (28), we arrive at

$$Y_{j+1} := \iint_{Q_j} (u-k_{j+1})^2_+\, dxd\tau \leqslant M_0 \left(\iint_{Q_j} [(u-k_{j+1})_+\zeta^q]^{\frac{n+2}{n}}\, dxd\tau\right)^{\frac{n}{n+2}}$$
$$\times \left(\iint_{Q_j} \chi[(u-k_{j+1})_+ > 0]\, dxd\tau\right)^{\frac{2}{n+2}} \leqslant \gamma M_0 \left(\sup_{t_j<\tau<t} \int_{B_j} (u-k_{j+1})^2_+\,\zeta^q\, dx\right)^{\frac{1}{n+2}}$$
$$\times \left(\iint_{Q_j} |\nabla[(u-k_{j+1})_+\zeta^q]|\, dxd\tau\right)^{\frac{n}{n+2}} \left(\iint_{Q_j} \chi[(u-k_{j+1})_+ > 0]\, dxd\tau\right)^{\frac{z}{n+2}}$$
$$\leqslant \frac{\gamma 2^{j\gamma}}{(1-\sigma)^\gamma} M_0 t^{-\frac{1}{n+2}} \left[G^{-1}_{Q_0}\left(M_0^2/t\right)\right]^{\frac{n}{n+2}} k^{-2}\, Y_j^{1+\frac{1}{n+2}}, \quad j = 0, 1, 2, \ldots$$

It follows that $Y_j \to 0$ as $j \to \infty$ provided k is chosen to satisfy

$$k^{2(n+2)} = \gamma(1-\sigma)^{-\gamma} M_0^{n+2} t^{-1} \left[G^{-1}_{Q_0}\left(M_0^2/t\right)\right]^n \iint_{Q_0} u^2 dxd\tau.$$

By this choice, we have

$$M_\sigma^{2(n+2)} \leqslant \gamma(1-\sigma)^{-\gamma} M_0^{n+2} t^{-1} \left[G_{Q_0}^{-1}(M_0^2/t)\right]^n \iint\limits_{Q_0} u^2 dx d\tau. \tag{29}$$

Since the function $\varphi_{Q_0}(v)$ is increasing

$$\left[G_{Q_0}^{-1}(M_0^2/t)\right]^n \leqslant \left(\frac{M_0}{M_\sigma}\right)^{n+1} \left[G_{Q_0}^{-1}(M_\sigma^2/t)\right]^n,$$

inequality (29) can be rewritten as

$$M_\sigma^{3n+5} \left[G_{Q_0}^{-1}(M_\sigma^2/t)\right]^{-n} \leqslant \gamma(1-\sigma)^{-\gamma} M_0^{2(n+2)} t^{-1} \iint\limits_{Q_0} u\, dx d\tau.$$

Set

$$f(v) := [\varphi_{Q_0}(v)]^{\frac{1}{2(n+2)}}, \quad A := t^{-\frac{n+1}{4(n+2)}} \left(t^{-1} \iint\limits_{Q_0} u\, dx d\tau\right)^{\frac{1}{2(n+2)}},$$

then from the last inequality we have

$$M_\sigma f(M_\sigma t^{-\frac{1}{2}}) \leqslant \gamma(1-\sigma)^{-\gamma} M_0 A.$$

Using inequality (16), we obtain for any $\delta \in (0, 1)$

$$\begin{aligned} f(M_\sigma t^{-\frac{1}{2}}) &\leqslant f(\delta M_0 t^{-\frac{1}{2}}) + \delta^{-1} f(M_\sigma t^{-\frac{1}{2}}) \frac{M_\sigma}{M_0} \\ &\leqslant \delta^{\varkappa_0} f(M_\sigma t^{-\frac{1}{2}}) + \gamma(1-\sigma)^{-\gamma} \delta^{-1} A, \quad \varkappa_0 = \frac{\varkappa(p)}{2p(n+2)}. \end{aligned}$$

By standard interpolation arguments, taking into account (26), from the previous we arrive at (25), which completes the proof of Lemma 3. □

3. Integral and pointwise estimates of auxiliary solutions

Fix $(x_0, t_0) \in \Omega_T$ such that $Q_{R_0,R_0}(x_0, t_0) \subset \Omega_T$ and assume that conditions (g_0), (g_2), (g_λ) and (ψ) be fulfilled in $Q_{R_0,R_0}(x_0, t_0)$. Let $r < \rho \leqslant R_0^a$ with some $a \geqslant 1 + a_0$, where a_0 is the constant from condition (ψ). Fix $(\bar{x}, \bar{t})$ such that $Q_{r,r}(\bar{x}, \bar{t}) \subset Q_{R_0,R_0}(x_0, t_0)$, $|\bar{t} - t_0| \leqslant \bar{c}\rho$ with some $\bar{c} > 0$, and $E \subset B_r(\bar{x})$, $|E| > 0$, $0 < N \leqslant \lambda(r)$.

We consider the function

$$v \in C\left(\bar{t}, \bar{t} + 8\tau_1(\xi); L^2(B_{8\rho}(\bar{x}))\right) \cap L^q\left(\bar{t}, \bar{t} + 8\tau_1(\xi); W_0^{1,q}(B_{8\rho}(\bar{x}))\right)$$

as the solution of the following problem

$$v_t - \operatorname{div}\mathbb{A}(x,t,\nabla v) = 0, \quad (x,t) \in Q_1 := B_{8\rho}(\bar{x}) \times (\bar{t}, \bar{t} + 8\tau_1(\xi)), \tag{30}$$

$$v(x,t) = 0, (x,t) \in \partial B_{8\rho}(\bar{x}) \times (\bar{t}, \bar{t} + 8\tau_1(\xi)), \tag{31}$$

where $\tau_1(\xi) := \frac{\rho^2}{\psi(\bar{x}, t_0, \xi N|E|\rho^{-n-1})}$, and

$$v(x,\bar{t}) = N\chi(E), \quad x \in B_{8\rho}(\bar{x}), \tag{32}$$

here $\xi \in (0,1)$ depends only on the data will be chosen later.

In addition, the integral identity

$$\int_{B_{8\rho}(\bar{x})\times\{t\}} \left(\frac{\partial v_h}{\partial t}\eta + [\mathbb{A}(x,t,\nabla v)]_h \nabla\eta\right) dx = 0, \tag{33}$$

holds for all $t \in (\bar{t}, \bar{t} + 8\tau_1(\xi) - h)$ and for all $\eta \in W_0^{1,q}(B_{8\rho}(\bar{x}))$. Here, v_h is defined similarly to (7). The existence of the solution v follows from the general theory of monotone operators. Testing (33) by $\eta = (v_h)_-$ and $\eta = (v_h - N)_+$, integrating over $(\bar{t}, t)$, $t \in (\bar{t}, \bar{t} + 8\tau_1(\xi))$ and letting $h \to 0$ we obtain that $0 \leqslant v \leqslant N \leqslant \lambda(r)$.

Lemma 4. *Let v be a solution of (30)–(32), then for any $\xi \in (0,1)$ either*

$$\xi N|E|\rho^{-n} \leqslant \gamma(b + b_0)\rho^{1-a_0/a}, \tag{34}$$

or

$$v(x,t) \leqslant \gamma\xi^{-\gamma} N|E|\rho^{-n}, \quad (x,t) \in B_{8\rho}(\bar{x}) \times (\bar{t} + \frac{1}{8}\tau_1(\xi), \bar{t} + 2\tau_1(\xi)), \tag{35}$$

and

$$v(x,t) \leqslant \gamma N|E|\rho^{-n}, \quad (x,t) \in \left(B_{4\rho}(\bar{x}) \setminus B_{\frac{3}{2}\rho}(\bar{x})\right) \times (\bar{t}, \bar{t} + 8\tau_1(\xi)), \tag{36}$$

with constant γ depending only on the known data and $\bar{c}$.

Proof. For fixed $\sigma \in (0,1)$, $\rho \leqslant s \leqslant s(1+\sigma) \leqslant 2\rho$, and $j = 0, 1, 2, \ldots$ set

$$s_{1,j} := \frac{1}{8}s(1+\sigma) + \frac{\sigma s}{2^{j+3}}, \; s_{2,j} := s(1+\sigma) - \frac{\sigma s}{2^j}, \quad k_j := k - 2^{-j}k, \quad k > 0,$$

$$\mathcal{D}_j := B_{8\rho}(\bar{x}) \times (\bar{t} + \frac{s_{1,j}^2}{\psi(\bar{x}, t_0, \xi N|E|\rho^{-n-1})}, \; \bar{t} + \frac{s_{2,j}^2}{\psi(\bar{x}, t_0, \xi N|E|\rho^{-n-1})}),$$

and let $M_0 := \sup\limits_{\mathcal{D}_0} v$, $M_\sigma := \sup\limits_{\mathcal{D}_\infty} v$, and consider the function $\zeta(t) \in C^\infty(\mathbb{R}^1)$, $0 \leqslant \zeta(t) \leqslant 1$,

$$\zeta(t) = 1 \text{ in } \mathcal{D}_{j+1}, \; \zeta(t) = 0 \text{ in } \mathbb{R}^1 \setminus \mathcal{D}_j, |\zeta_t| \leqslant \frac{2^{2(j+3)}}{(\sigma\, s)^2}\, \psi\left(\bar{x}, t_0, \frac{\xi N|E|}{\rho^{n+1}}\right).$$

Testing (33) by $\eta = (v_h - k_j)_+ \zeta^q(t)$, integrating over $(\bar{t}, t)$, $t \in (\bar{t}, \bar{t} + 8\tau_1(\xi))$ and letting $h \to 0$ and repeating the same arguments as in Sect. 2.4, similarly to (29), using condition (g_λ) and the fact that $v \leqslant N \leqslant \lambda(r)$ in Q_1, we arrive at

$$M_\sigma^{2n+5} \left[G_{Q_1}^{-1} \left(\frac{M_\sigma^2}{\tau_1(\xi)} \right) \right]^{-n} \leqslant \gamma\, \xi^{-\gamma} \sigma^{-\gamma} M_0^{2n+3} \tau_1(\xi)^{-1} \iint\limits_{\mathcal{D}_0} v^2 dx d\tau, \tag{37}$$

provided that

$$M_\sigma \geqslant \rho\, \psi_{Q_1}^{-1} \left(\frac{\rho^2}{\tau_1(\xi)} \right) \geqslant \gamma \xi N \frac{|E|}{\rho^n}, \tag{38}$$

and

$$\psi_{Q_1}^{-1} \left(\psi(\bar{x}, t_0, \xi N |E| \rho^{-n-1}) \right) \geqslant (b + b_0) \rho^{-a_0/a}.$$

Since

$$\xi N |E| \rho^{-n} \leqslant \xi N \leqslant \lambda(r),$$

by condition (g_λ) the last inequality holds if

$$\xi N \frac{|E|}{\rho^{n+1}} \geqslant \gamma(c_1)(b + b_0) \rho^{-a_0/a}. \tag{39}$$

To estimate the integral on the right-hand side of (37), we test (33) by $\eta = \min(v_h, M_0)$, integrating over $(\bar{t}, t)$, $t \in (\bar{t}, \bar{t} + 8\tau_1(\xi) - h)$ and letting $h \to 0$, we obtain for $v_{M_0} := \min(v, M_0)$

$$\sup_{\bar{t} < \tau < \bar{t} + 8\tau_1(\xi)} \int_{B_{8\rho}(\bar{x})} v_{M_0}^2 dx + \iint\limits_{Q_1} G(x, \tau, |\nabla v_{M_0}|) dx\, d\tau \leqslant \gamma M_0 N |E|. \tag{40}$$

From this and (37) since $v(x, t) = v_{M_0}$ for $(x, t) \in \mathcal{D}_0$, we obtain

$$M_\sigma^{2n+5} \left[G_{Q_1}^{-1} \left(\frac{M_\sigma^2}{\tau_1(\xi)} \right) \right]^{-n} \leqslant \gamma\, \xi^{-\gamma} \sigma^{-\gamma} M_0^{2(n+2)} N |E|.$$

Repeating the iterative arguments similar to that of Sect. 2.4, we arrive at

$$\sup_{\mathcal{D}_\rho(\xi)} v \leqslant \gamma \xi^{-\gamma} N \frac{|E|}{\rho^n},$$

taking into account (38), (39), we obtain inequality (35).

To prove (36) we set

$$s_{1,j} := \frac{3}{2} s(1+\sigma) + \frac{\sigma s}{2^j}, \quad s_{2,j} := 2s(1+\sigma) - \frac{\sigma s}{2^j}, \quad \rho \leqslant s \leqslant 2\rho,$$
$$k_j := k - 2^{-j} k, \quad k > 0, \quad \mathcal{D}_j := \left(B_{s_{2,j}}(\bar{x}) \setminus B_{s_{1,j}}(\bar{x}) \right) \times (\bar{t}, \bar{t} + 8\tau_1(\xi)),$$

and let $M_0 := \sup\limits_{\mathcal{D}_0} v$, $M_\sigma := \sup\limits_{\mathcal{D}_\infty} v$, and consider the function $\zeta(x) \in C^\infty(\mathbb{R}^n)$, $0 \leqslant \zeta(x) \leqslant 1$,

$$\zeta(x) = 1 \ \text{in}\ \mathcal{D}_{j+1}, \quad \zeta(x) = 0 \ \text{in}\ \mathbb{R}^n \setminus \mathcal{D}_j, |\nabla \zeta| \leqslant \frac{2^{2(j+1)}}{(\sigma s)^2}.$$

Testing (33) by $\eta = (v_h - k_j)_+ \zeta^q(x)$, integrating over $(\bar{t}, t)$, $t \in (\bar{t}, \bar{t} + 8\tau_1(\xi))$ and letting $h \to 0$ and repeating the same arguments as in Sect. 2.4, similarly to (29), using condition (g_λ) and the fact that $v \leqslant N \leqslant \lambda(r)$ in Q_1, we arrive at

$$\sup_{\bar{t}<\tau<\bar{t}+8\tau_1(\xi)} \int_{B_{8\rho}(\bar{x})} (v - k_{j+1})_+^2 \, \zeta^q \, dx + \iint\limits_{\mathcal{D}_j} g(x, \tau, |\nabla(v - k_{j+1})_+|) \, |\nabla(v - k_{j+1})_+| \times \zeta^q \, dx d\tau$$
$$\leqslant \gamma 2^{j\gamma} \sigma^{-\gamma} \rho^{-1} \iint\limits_{\mathcal{D}_j} g\left(x, \tau, \frac{(v - k_j)_+}{\rho}\right) (v - k_j)_+ \, dx d\tau,$$

which by (16) and condition (g_λ) yields

$$\iint\limits_{\mathcal{D}_j} |\nabla(v - k_{j+1})_+| \, \zeta^q \, dx d\tau \leqslant \gamma 2^{j\gamma} \sigma^{-\gamma} \rho^{-1} \iint\limits_{\mathcal{D}_j} (v - k_j)_+ \, dx d\tau.$$

From this by the Sobolev embedding theorem, we arrive at

$$M_\sigma^2 \leqslant \gamma \sigma^{-\gamma} \rho^{-n-1} g^-_{\mathcal{D}_0}\left(\frac{M_0}{\rho}\right) \iint\limits_{\mathcal{D}_0} v dx d\tau,$$

here we used the notation $g^-_{\mathcal{D}_0}(v) = \min\limits_{\mathcal{D}_0} g(x, \tau, v)$. By inequality (16) , Poincare inequality and (40) from the previous we obtain for any $\varepsilon \in (0, 1)$

$$M_\sigma^2 \leqslant \varepsilon M_0 \frac{g^-_{\mathcal{D}_0}(M_0 \rho^{-1})}{\psi(\bar{x}, t_0, \xi N |E| \rho^{-n-1})} + \gamma \sigma^{-\gamma} \varepsilon^{-\gamma} \rho^{-n} \iint\limits_{\mathcal{D}_0} g^-_{\mathcal{D}_0}\left(\frac{v}{\rho}\right) \frac{v}{\rho} dx \, dt$$
$$\leqslant \varepsilon M_0 \frac{g^-_{\mathcal{D}_0}(M_0 \rho^{-1})}{\psi(\bar{x}, t_0, \xi N |E| \rho^{-n-1})} + \gamma \sigma^{-\gamma} \varepsilon^{-\gamma} \rho^{-n} \iint\limits_{\mathcal{D}_0} g^-_{\mathcal{D}_0}(|\nabla v|) |\nabla v| dx \, dt$$
$$\leqslant \varepsilon M_0 \frac{g^-_{\mathcal{D}_0}(M_0 \rho^{-1})}{\psi(\bar{x}, t_0, \xi N |E| \rho^{-n-1})} + \gamma \sigma^{-\gamma} \varepsilon^{-\gamma} M_0 N \frac{|E|}{\rho^n}.$$

If inequality (36) is violated then the last inequality implies

$$M_\sigma^2 \leqslant \varepsilon M_0^2 + \gamma \sigma^{-\gamma} \varepsilon^{-\gamma} M_0 N \frac{|E|}{\rho^n} \leqslant \varepsilon M_0^2 + \gamma \sigma^{-\gamma} \varepsilon^{-\gamma} \left(N \frac{|E|}{\rho^n}\right)^2.$$

Repeating the iterative arguments similar to that of Sect. 2, we arrive at the required (36),which completes the proof of the lemma. □

The following proposition is the main result of this Section.

Proposition 2. *There exist numbers* $\varepsilon_1, \alpha_1 \in (0, 1)$ *depending only on the data,* $\bar{c}$ *and* ξ *such that either (34) holds, or*

$$|\{x \in B_{4\rho}(\bar{x}) : v(x, t) \leqslant \varepsilon_1 N|E|\rho^{-n}\}| \leqslant (1 - \alpha_1)\, |B_{4\rho}(\bar{x})|, \tag{41}$$

for all time levels $t \in (\bar{t} + \frac{1}{8}\tau_1(\xi),\ \bar{t} + 2\tau_1(\xi))$.

Proof. Let $\zeta_1 \in C_0^\infty(B_{3\rho}(\bar{x}))$, $0 \leqslant \zeta_1 \leqslant 1$, $\zeta_1 = 1$ in $B_{2\rho}(\bar{x})$, $|\nabla\zeta_1| \leqslant 1/\rho$. Testing (33) by $\eta = v_h - N\zeta_1^q$, integrating over $(\bar{t} + \frac{1}{8}\tau_1(\xi), t)$, $t \in (\bar{t} + \frac{1}{8}\tau_1(\xi), \bar{t} + 2\tau_1(\xi))$ and letting $h \to 0$ we obtain

$$\begin{aligned} &\frac{N^2}{2}|E| + \frac{1}{2}\int_{B_{8\rho}(\bar{x})} v^2(x,\ t)\, dx + \gamma^{-1}\int_{\bar{t}}^{t}\int_{B_{8\rho}(\bar{x})} g(x, t, |\nabla v|)\, |\nabla v|\, dxdt \\ &\leqslant N\int_{B_{3\rho}(\bar{x})} v(x,\ t)\ dx + \gamma\frac{N}{\rho}\int_{\bar{t}}^{t}\int_{B_{3\rho}(\bar{x})\setminus B_{2\rho}(\bar{x})} g(x, t, |\nabla v|)\, \zeta_1^{q-1}\, dxdt \\ &= I_1 + I_2. \end{aligned} \tag{42}$$

Let us estimate the terms on the right-hand side of (42). Further we will assume that (34) is violated, i.e.,

$$\xi N|E|\rho^{-n} \geqslant \gamma(b + b_0)\rho^{1-a_0/a}. \tag{43}$$

Set $\tilde{Q}_1 = B_{4\rho}(\bar{x}) \times (\bar{t} + \frac{1}{8}\tau_1(\xi), \bar{t} + 2\tau_1(\xi))$, by Proposition 1 we obtain

$$\begin{aligned} I_1 &\leqslant \gamma N \inf_{\bar{t}+\frac{1}{8}\tau_1(\xi)<t<\bar{t}+2\tau_1(\xi)} \int_{B_{4\rho}(\bar{x})} v(x, t)\, dx + \gamma\rho^{n+1}\psi^{-1}_{\tilde{Q}_1}\left(\frac{\rho^2}{\tau_1(\xi)}\right) \\ &\leqslant \gamma N \inf_{\bar{t}+\frac{1}{8}\tau_1(\xi)<t<\bar{t}+2\tau_1(\xi)} \int_{B_{4\rho}(\bar{x})} v(x, t)\, dx + \gamma\xi N^2|E|, \end{aligned}$$

which by Lemma 4 yields for all $t \in (\bar{t} + \frac{1}{8}\tau_1(\xi),\ \bar{t} + 2\tau_1(\xi))$

$$I_1 \leqslant (\varepsilon_1\ +\ \xi)\gamma N^2|E| + \gamma\,\xi^{-\gamma}\,N^2\,\frac{|E|}{\rho^n}|\{B_{4\rho}(\bar{x}) : v(\cdot,\ t) \geqslant \varepsilon_1 N|E|\rho^{-n}\}|. \tag{44}$$

Let $\zeta_2 \in C^\infty(\mathbb{R}^n)$, $0 \leqslant \zeta_2 \leqslant 1$, $\zeta_2 = 1$ in $B_{3\rho}(\bar{x}) \setminus B_{2\rho}(\bar{x})$, $\zeta_2 = 0$ in $B_{3\rho/2}(\bar{x}) \cup (\mathbb{R}^n \setminus B_{4\rho}(\bar{x}))$ and $|\nabla\zeta_2| \leqslant \gamma\rho^{-1}$. Using inequality (16) with $\varepsilon = N|E|\rho^{-n-1}$,we obtain

$$I_2 \leqslant \frac{\gamma N}{\varepsilon\rho} \int\limits_{\bar t}^{\bar t+2\tau_1(\xi)} \int_{B_{4\rho}(\bar x)\setminus B_{3\rho/2}(\bar x)} g(x,t,|\nabla v|)\,|\nabla v|\,\zeta_2^q\,dxdt$$
$$+\frac{\gamma N}{\rho} \int\limits_{\bar t}^{\bar t+2\tau_1(\xi)} \int_{B_{4\rho}(\bar x)} g(x,t,\varepsilon)\,dxdt = I_3 + I_4.$$

By (43) conditions (g_λ) and (ψ) are applicable, so by our choice of N we have

$$\begin{aligned} I_4 &\leqslant \frac{\gamma N}{\rho} \int\limits_{\bar t}^{\bar t+2\tau_1(\xi)} \int_{B_{4\rho}(\bar x)} g(x,t,N|E|\rho^{-n-1})\,dxdt \\ &\leqslant \frac{\gamma N}{\rho}\, g(\bar x,\bar t,N|E|\rho^{-n-1})\,|B_\rho(\bar x)|\tau_1(\xi) \leqslant \gamma\xi^{1-q_1}N^2|E|. \end{aligned} \tag{45}$$

To estimate I_3 we test (33) by $\eta = v_h\zeta_2^q$, integrating over $(\bar t, \bar t+\tau_1(\xi))$ and letting $h\to 0$, we arrive at

$$I_3 \leqslant \frac{\gamma N}{\varepsilon\rho} \int\limits_{\bar t}^{\bar t+2\tau_1(\xi)} \int_{B_{4\rho}(\bar x)\setminus B_{3\rho/2}(\bar x)} g\left(x,t,\frac{v}{\rho}\right)\frac{v}{\rho}\,dxdt.$$

From this, by condition (g_λ) and Lemma 4 we obtain

$$\begin{aligned} I_3 &\leqslant \frac{\gamma N}{\varepsilon\rho} \int\limits_{\bar t}^{\bar t+2\tau_1(\xi)} \int_{B_{4\rho}(\bar x)\setminus B_{3\rho/2}(\bar x)} g\left(\bar x,\bar t,\frac{v}{\rho}\right)\frac{v}{\rho}\,dxdt \\ &\leqslant \gamma\frac{N}{\varepsilon}|E|\frac{g(\bar x,t_0,N|E|\rho^{-n-1})}{\psi(\bar x,t_0,\xi N|E|\rho^{-n-1})} \leqslant \gamma\xi^{1-q_1}N^2|E|. \end{aligned} \tag{46}$$

Collecting estimates (42)–(46), we obtain for all $t\in(\bar t+\frac{1}{8}\tau_1(\xi),\ \bar t+2\tau_1(\xi))$

$$\frac{1}{2}N^2|E| \leqslant \left(\varepsilon_1+\xi+\xi^{1-q_1}\right)\gamma N^2|E| +$$
$$+\gamma\xi^{-\gamma}N^2|E|\rho^{-n}\,|\{B_{4\rho}(\bar x): v(\cdot,t)\geqslant \varepsilon_1 N|E|\rho^{-n}\}|,$$

choosing ε_1 and ξ so small that $\left(\varepsilon_1+\xi+\xi^{1-q_1}\right)\gamma = \frac{1}{4}$, we arrive at

$$|\{B_{4\rho}(\bar x): v(\cdot,t)\geqslant \varepsilon_1 N|E|\rho^{-n}\}| \geqslant \gamma^{-1}\xi^{\gamma}\rho^n,$$

for all $t\in(\bar t+\frac{1}{8}\tau_1(\xi),\ \bar t+2\tau_1(\xi))$,which completes the proof of Proposition 2. □

4. Harnack's inequality, proof of Theorems 1, 2

4.1. Expansion of positivity

The following lemma can be found in [34]. In the case of singular p-Laplace evolution equation, this result was proved by Chen and Di Benedetto [11] (see also [15,17]); in the case when g is independent of (x,t) this lemma was proved in [23,24].

Lemma 5. *Let u be a nonnegative, bounded weak solution to Eq. (1) and let (2),* (g_0)–(g_2), (g_λ) *be fulfilled in the cylinder $Q_{R_0,R_0}(x_0,t_0)\subset\Omega_T$. Assume also that for some $0<r\leqslant R_0^a$, $a\geqslant 1+a_0$, $0<N\leqslant\lambda(r)$ and some $\alpha,\delta_0\in(0,1)$, $\bar c>0$ there holds*

$$Q^-_{8r,8\bar c r}(y,s)\subset Q_{R_0,R_0}(x_0,t_0),$$

$$|\{x\in B_r(y):u(x,t)\leqslant N\}|\leqslant(1-\alpha)|B_r(y)| \tag{47}$$

for all $t\in(s-\theta,s)$, $\theta=r^2/\psi(y,s,\delta_0 N/r)$.

Then, there exists $\sigma_0\in(0,1)$ depending only upon the data and α,δ_0 such that either

$$N\leqslant(b+b_0)r^{1-a_0/a},\quad a>a_0 \tag{48}$$

or

$$u(x,t)\geqslant\sigma_0 N\ \textit{for all}\ (x,t)\in B_{2r}(y)\times\left(s-\frac{7}{8}\theta,s\right). \tag{49}$$

Here, b, a_0 and b_0 are the numbers from conditions (g_λ) *and* (ψ).

4.2. Proof of Theorems 1, 2

Having fixed $(x_0,t_0)\in\Omega_T$ such that $Q_{R_0,R_0}(x_0,t_0)\subset\Omega_T$, we assume that conditions (g_2), (g_λ), (g_μ) and (ψ) hold in the cylinder $Q_{R_0,R_0}(x_0,t_0)$. We will prove Theorems 1 and 2 simultaneously, making the necessary remarks during the proof. Assume without loss that $(x_0,t_0)=(0,0)$ and set $u_0:=u(0,0)$. Further we will assume that inequalities (8) and (11) are violated, i.e.,

$$u_0\geqslant C(b+b_0)\rho^{1-a_0/a_1}\lambda^{-1}(\rho)\mu^{\beta}(\rho), \tag{50}$$

taking into account that in the logarithmic case (i.e., under the conditions of Theorem 1) $\lambda(\rho)=\mu(\rho)=1$.

For $\tau\in(0,1)$ set $M_\tau:=\sup\limits_{B_{\tau\rho}(0)}u(\cdot,0)$ and

$$N_\tau:=(1-\tau)^{-m}\frac{\lambda(\rho)}{\lambda((1-\tau)\rho)}\,\frac{\mu^{\beta}((1-\tau)\rho)}{\mu^{\beta}(\rho)}\,u_0,$$

where $m > 1$ positive number to be fixed later. As $\tau \to 1$, $N_\tau \to +\infty$, whereas $M_\tau < +\infty$. So, the equation $M_\tau = N_\tau$ has roots, let τ_0 be the largest root of the above equation $M_{\tau_0} = N_{\tau_0}$ and $M_\tau \leqslant N_\tau$ for $\tau > \tau_0$.

The conditions of Theorems 1 and 2 imply the continuity of the solution u, therefore M_{τ_0} is achieved at some $\bar{x} \in B_{\tau_0\rho}(0)$. Choose τ_1 from the condition $(1-\tau_1)^{-m} = 4(1-\tau_0)^{-m}$, i.e., $\tau_1 = 1-4^{-\frac{1}{m}}(1-\tau_0)$, and set $2r = (\tau_1-\tau_0)\rho = (1-4^{-\frac{1}{m}})(1-\tau_0)\rho$. For these choices $B_{2r}(\bar{x}) \subset B_{\tau_1\rho}(0)$, $M_{\tau_1} \leqslant N_{\tau_1}$, and

$$\begin{aligned}
&\sup_{B_{\tau_0\rho}(0)} u(\cdot,0) \leqslant \sup_{B_{\tau_1\rho}(0)} u(\cdot,0) \leqslant N_{\tau_1}\\
&\leqslant 4(1-\tau_0)^{-m}\frac{\lambda(\rho)}{\lambda(4^{-\frac{1}{m}}(1-\tau_0)\rho)}\frac{\mu^\beta\left(4^{-\frac{1}{m}}(1-\tau_0)\rho\right)}{\mu^\beta(\rho)}u_0\\
&\leqslant 4^{1+c_3\frac{1+\beta}{m}}(1-\tau_0)^{-m}\frac{\lambda(\rho)}{\lambda((1-\tau_0)\rho)}\frac{\mu^\beta((1-\tau_0)\rho)}{\mu^\beta(\rho)}u_0 \leqslant 4^2 N_{\tau_0},
\end{aligned}$$

provided that $m \geqslant c_3(1+\beta)$. Here, c_3 is the constant from the technical condition of Sect. 1.

Construct the cylinders $Q^-_{2r,\theta}(\bar{x},0)$ and $Q^-_{2r,\theta_0}(\bar{x},0)$, where

$$\theta := \frac{(2r)^2}{\psi\left(\bar{x},0,\frac{N_{\tau_0}}{2r}\right)},\quad \theta_0 := \frac{(2r)^2}{\psi_{Q^-_{2r,\theta}}(\bar{x},0)\left(\frac{N_{\tau_0}}{2r}\right)},$$
$$\psi_{Q^-_{2r,\theta}}(\bar{x},0)\left(\frac{N_{\tau_0}}{2r}\right) := \sup_{Q^-_{2r,\theta}(\bar{x},0)} \psi\left(\cdot,\cdot,\frac{N_{\tau_0}}{2r}\right).$$

We have an inclusion

$$Q^-_{2r,\theta_0}(\bar{x},0) \subset Q^-_{2r,\theta}(\bar{x},0) \subset Q^-_{2r,2\bar{c}r}(\bar{x},0) \subset Q^-_{\rho,\bar{c}\rho}(0,0),$$

where $\bar{c} = c_0 \max(M^{2-p}, M^{2-q})$. Indeed, by (50) we have

$$N_{\tau_0} \geqslant Cb_0(1-\tau_0)^{-m}[\lambda(2r)]^{-1}\mu^{\beta_2}(2r)\rho^{1-\frac{a_0}{a_1}} \geqslant Cb_0\rho^{1-\frac{a_0}{a_1}} \geqslant 2b_0 r^{1-\frac{a_0}{a_1}}, \quad (51)$$

provided that $C \geqslant 2^{a_0/a}$. So, using condition (g_2), we obtain

$$\theta_0 \leqslant \theta \leqslant \frac{(2r)^2}{\psi\left(\bar{x},0,\frac{M}{2r}\right)} \leqslant \frac{2^q\max(M^{2-p},M^{2-q})r}{g(\bar{x},0,1)} \leqslant 2\bar{c}r \leqslant \bar{c}\rho.$$

Lemma 6. *The following inequality holds:*

$$\sup_{Q^-_{r,\theta_0/2}(\bar{x},0)} u \leqslant \gamma N_{\tau_0}\mu(2r)^{m_1}, \quad (52)$$

where $m_1 = n/\varkappa(p)$, $\varkappa(p) = p + n(p-2) > 0$.

Proof. We use Theorem 3 over the pair of cylinder $Q^-_{r,\theta_0/2}(\bar{x},0)$ and $Q^-_{2r,\theta_0}(\bar{x},0)$ two times, first for the choices $s=0$, $t=\theta_0/2$ and second for the choices $s=-\theta_0/2$, $t=0$. By condition (g_μ)

$$\psi_{Q^-_{2r,\theta}(\bar{x},0)}\left(\frac{N_{\tau_0}}{2r}\right)\leqslant c_2\,\mu(2r)\,\psi_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{N_{\tau_0}}{2r}\right),$$

which by (g_2) and (51) implies

$$\begin{aligned}
&\psi^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\psi_{Q^-_{2r,\theta}(\bar{x},0)}\left(\frac{N_{\tau_0}}{2r}\right)\right)\\
&\quad\geqslant (c_2\mu(2r))^{-\frac{1}{2-p}}\,\psi^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{c_2^{-1}}{\mu(2r)}\,\psi_{Q^-_{2r,\theta}(\bar{x},0)}\left(\frac{N_{\tau_0}}{2r}\right)\right)\\
&\quad\geqslant (c_2\mu(2r))^{-\frac{1}{2-p}}\,\frac{N_{\tau_0}}{2r}\geqslant b_0c^{-\frac{1}{2-p}}\,C\lambda^{-1}(2r)\,\mu(2r)^{\beta_2-\frac{1}{2-p}}\,r^{-a_0/a_1}\geqslant b_0R_0^{-a_0},
\end{aligned}$$

provided that $C>c_2^{\frac{1}{2-p}}$. Therefore, Theorem 3 yields

$$\begin{aligned}
\sup_{Q^-_{r,\frac{\theta_0}{2}}(\bar{x},0)} u &\leqslant \gamma\theta_0^{\frac12}\varphi^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\theta_0^{-\frac{n+1}{2}}\int_{B_r(\bar{x})}u(x,0)dx\right)\\
&+\gamma\theta_0^{\frac12}\varphi^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\left(\frac{(2r)^2}{\theta_0}\right)^{\frac{n+1}{2}}\psi^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{(2r)^2}{\theta_0}\right)\right)+\gamma r\psi^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{(2r)^2}{\theta_0}\right)\\
&\leqslant \gamma\theta_0^{\frac12}\varphi^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\theta_0^{-\frac{n+1}{2}}r^nN_{\tau_0}\right)+\gamma N_{\tau_0}.
\end{aligned}$$

To estimate the first term on the right-hand side of the previous inequality, note

$$\begin{aligned}
G^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{N^2_{\tau_0}}{\theta_0}\right)&=G^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{N^2_{\tau_0}}{(2r)^2}\,\psi_{Q^-_{2r,\theta}(\bar{x},0)}\left(\frac{N_{\tau_0}}{2r}\right)\right)\\
&\leqslant G^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{N^2_{\tau_0}}{(2r)^2}\,\mu(2r)\,\psi_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{N_{\tau_0}}{2r}\right)\right)\\
&\leqslant \gamma\mu(2r)^{1/p}\,G^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(G_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\frac{N_{\tau_0}}{2r}\right)\right)=\gamma\mu(2r)^{1/p}\,\frac{N_{\tau_0}}{r},
\end{aligned}\tag{53}$$

here we used condition (g_μ) and Lemma 1.

By Lemma 1 and inequality (53), we have

$$\begin{aligned}
&\varphi_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(\mu^{m_1}(2r)N_{\tau_0}\theta_0^{-\frac12}\right)\geqslant\mu(2r)^{m_1\varkappa(p)/p}\varphi_{Q^-_{2r,\theta_0}(\bar{x},0)}\left(N_{\tau_0}\theta_0^{-\frac12}\right)\\
&=\mu(2r)^{n/p}\,\frac{\theta_0^{-\frac{n+1}{2}}N^{n+1}_{\tau_0}}{\left[G^{-1}_{Q^-_{2r,\theta_0}(\bar{x},0)}(N^2_{\tau_0}/\theta_0)\right]^n}\geqslant\gamma^{-1}\theta_0^{-\frac{n+1}{2}}r^nN_{\tau_0}.
\end{aligned}\tag{54}$$

From which the required inequality (52) follows, which completes the proof of the lemma. □

Lemma 7. *There exist numbers* δ, ε, $\alpha \in (0, 1)$ *depending only on the data such that*

$$|\{x \in B_r(\bar{x}) : u(x,t) \geqslant \varepsilon \mu(r)^{-m_2} N_{\tau_0}\}| \geqslant \alpha \mu(r)^{-m_1-m_2} |B_r(\bar{x})| \tag{55}$$

for all

$$|t| \leqslant \bar{\theta}_0 := \frac{r^2}{\psi_{Q^-_{2r,\theta}(\bar{x},0)}\left(\dfrac{\delta N_{\tau_0}}{r\mu^{m_2}(r)}\right)},$$

where $m_1 = n/\varkappa(p)$, $m_2 = (2-p)n/\varkappa(p)$.

Proof. We use Theorem 3 over the pair of cylinder $Q^-_{r/2,\bar{\theta}_0/2}(\bar{x},0)$ and $Q^-_{r,\bar{\theta}_0}(\bar{x},0)$ for the choices $s = 0$ and $t = \bar{\theta}_0$. By (51) condition (15) holds, therefore for all $|t - t_0| \leqslant \bar{\theta}_0$

$$\begin{aligned} N_{\tau_0} = u(\bar{x},0) &\leqslant \gamma \bar{\theta}_0^{\frac{1}{2}} \varphi^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\bar{\theta}_0^{-\frac{n+1}{2}} \int_{B_r(\bar{x})} u(x,t)\,dx\right) \\ &\quad + \gamma \bar{\theta}_0^{\frac{1}{2}} \varphi^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\left(\frac{r^2}{\bar{\theta}_0}\right)^{\frac{n+1}{2}} \psi^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\psi_{Q^-_{2r,\theta}(\bar{x},0)}\left(\frac{\delta N_{\tau_0}}{r\mu^{m_2}(r)}\right)\right)\right) \\ &\quad + \gamma r \psi^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\psi_{Q^-_{2r,\theta}(\bar{x},0)}\left(\frac{\delta N_{\tau_0}}{r\mu^{m_2}(r)}\right)\right) \\ &\leqslant \gamma \delta N_{\tau_0} + \gamma \bar{\theta}_0^{\frac{1}{2}} \varphi^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\delta r^n \bar{\theta}_0^{-\frac{n+1}{2}} \mu^{-m_2}(r) N_{\tau_0}\right) \\ &\quad + \gamma \bar{\theta}_0^{\frac{1}{2}} \varphi^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\bar{\theta}_0^{-\frac{n+1}{2}} \int_{B_r(\bar{x})} u(x,t)\,dx\right) = \gamma\delta N_{\tau_0} + I_1 + I_2. \end{aligned}$$

Let us estimate the terms on the right-hand side of the previous inequality. Similarly to (53), (54), we obtain

$$\begin{aligned} G^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\frac{N^2_{\tau_0}}{\bar{\theta}_0}\right) &\leqslant \gamma G^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\delta^{p-2}\mu(r)^{(2-p)m_2} \frac{N^2_{\tau_0}}{r^2} \psi_{Q^-_{2r,\theta}(\bar{x},0)}\left(\frac{N_{\tau_0}}{r}\right)\right) \\ &\leqslant \gamma \delta^{\frac{p-2}{p}} \mu(r)^{\frac{2-p}{p} m_2} G^{-1}_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\mu(r)\frac{N^2_{\tau_0}}{r^2} \psi_{Q^-_{r,\bar{\theta}_0}(\bar{x},0)}\left(\frac{N_{\tau_0}}{r}\right)\right) \\ &\leqslant \gamma \delta^{\frac{p-2}{p}} \mu(r)^{\frac{2-p}{p}(m_2+1)} \frac{N_{\tau_0}}{r}, \end{aligned}$$

and by this and Lemma 1

$$\varphi_{Q^-_{r,\bar\theta_0}(\bar x,0)}\left(\delta^{\frac{\varkappa(p)}{p(n+1)}} N_{\tau_0}\bar\theta_0^{-\frac{1}{2}}\right) = \frac{\delta^{\frac{\varkappa(p)}{p}} N_{\tau_0}^{n+1}\bar\theta^{-\frac{n+1}{2}}}{\left[G^{-1}_{Q^-_{r,\bar\theta_0}(\bar x,0)}\left(\delta^{\frac{\varkappa(p)}{p(n+1)}2} N^2_{\tau_0}\bar\theta_0^{-1}\right)\right]^n}$$
$$\geqslant \frac{\delta^{\frac{\varkappa(p)}{p}} N_{\tau_0}^{n+1}\bar\theta^{-\frac{n+1}{2}}}{\left[G^{-1}_{Q^-_{r,\bar\theta_0}(\bar x,0)}\left(N^2_{\tau_0}\bar\theta_0^{-1}\right)\right]^n} \geqslant \gamma\delta r^n\,\bar\theta_0^{-\frac{n+1}{2}}\mu(r)^{-m_2}N_{\tau_0},$$

which implies

$I_1 \leqslant \gamma\delta^{\frac{\varkappa(p)}{p(n+1)}} N_{\tau_0}.$

Choose δ from the condition $\gamma(\delta+\delta^{\frac{\varkappa(p)}{p(n+1)}}) \leqslant 1/4$, from the previous and Lemma 6 we obtain for all $|t| \leqslant \bar\theta_0$

$$\frac{3}{4}N_{\tau_0} \leqslant \gamma I_2 \leqslant \gamma(\delta)\,\bar\theta_0^{\frac{1}{2}}\varphi^{-1}_{Q^-_{r,\bar\theta_0}(\bar x,0)}(\varepsilon\bar\theta_0^{-\frac{n+1}{2}}\mu(r)^{-m_2}r^nN_{\tau_0}) + \gamma(\delta)\,\bar\theta_0^{\frac{1}{2}}\varphi^{-1}_{Q^-_{r,\bar\theta_0}(\bar x,0)}$$
$$\times\left(\bar\theta_0^{-\frac{n+1}{2}}\mu(r)^{m_1}N_{\tau_0}|\{B_r(\bar x): u(\cdot,t)\geqslant\varepsilon\mu(r)^{-m_2}N_{\tau_0}\}|\right) = I_3 + I_4. \tag{56}$$

First term on the right-hand side of (56), we estimate similarly to I_1

$$I_3 \leqslant \varepsilon^{\frac{\varkappa(p)}{p(n+1)}}\gamma(\delta)N_{\tau_0}.$$

Choosing ε from the condition $\varepsilon^{\frac{\varkappa(p)}{p(n+1)}}\gamma(\delta) = 1/4$, from (56) we have

$$\bar\theta_0^{-\frac{n+1}{2}}\mu^{m_1}(r)N_{\tau_0}|\{x\in B_r(\bar x): u(x,t)\geqslant\varepsilon\mu(r)^{-m_2}N_{\tau_0}\}|$$
$$\geqslant \gamma^{-1}(\delta)\varphi_{Q^-_{r,\bar\theta_0}(\bar x,0)}(N_{\tau_0}\bar\theta_0^{-1/2}),$$

which similarly to I_1 we estimate as follows

$$\varphi_{Q^-_{r,\bar\theta_0}(\bar x,0)}(N_{\tau_0}\bar\theta_0^{-1/2}) \geqslant \gamma^{-1}(\delta)\,\bar\theta_0^{-\frac{n+1}{2}}\mu(r)^{-m_2}N_{\tau_0}|B_r(\bar x)|.$$

Collecting the last two inequalities, we arrive at

$$|\{x\in B_r(\bar x): u(x,t)\geqslant\varepsilon\mu(r)^{-m_2}N_{\tau_0}\}| \geqslant \gamma^{-1}(\delta)\,[\mu(r)]^{-m_1-m_2}|B_r(\bar x)|,$$

for all $|t|\leqslant\bar\theta_0$, which proves Lemma 7. □

4.3. Proof of Theorem 1

First note that under the conditions of Theorem 1, inequality (55) can be rewritten as

$$|\{x\in B_r(\bar x): u(x,t)\leqslant\varepsilon N_{\tau_0}\}| \leqslant (1-\alpha)\,|B_r(\bar x)| \tag{57}$$

for all $|t| \leqslant \bar{\theta}_0 := \dfrac{r^2}{\psi_{Q^-_{2r,\theta}(\bar{x},0)}(\delta N_{\tau_0}/r)}$, and with some fixed $\varepsilon, \delta, \alpha \in (0,1)$ depending only on the data, $N_{\tau_0} = (1-\tau_0)^{-m} u_0$.

Apply Lemma 5 with $\rho = r$ and $N = \varepsilon N_{\tau_0}$, we obtain that

$$u(x,t) \geqslant \sigma_0 \varepsilon N_{\tau_0}$$

for all $x \in B_{2r}(\bar{x})$ and for all times $-\bar{\theta}_0 + \frac{1}{8}\bar{\theta}^{(0)}_{0,\varepsilon} \leqslant t \leqslant \bar{\theta}_0$, $\bar{\theta}^{(0)}_{0,\varepsilon} := r^2/\psi(\bar{x}, 0, \delta \varepsilon N_{\tau_0} r^{-1})$.

After j iterations

$$u(x,t) \geqslant \sigma_0^j \varepsilon N_{\tau_0}, \tag{58}$$

for all $x \in B_{2^{j+1}r}(\bar{x})$ and for all times

$$-\bar{\theta}_0 + \frac{1}{8}\sum_{i=0}^{j} \bar{\theta}^{(j)}_{0,\varepsilon} \leqslant t \leqslant \bar{\theta}_0, \quad \bar{\theta}^{(i)}_{0,\varepsilon} := \frac{(2^i r)^2}{\psi\left(\bar{x}, 0, \dfrac{\delta\varepsilon\sigma_0^i N_{\tau_0}}{2^i r}\right)}.$$

By our choices and condition (ψ)

$$\frac{\psi\left(\bar{x}, 0, \dfrac{\delta N_{\tau_0}}{r}\right)}{\psi\left(\bar{x}, 0, \dfrac{\delta\varepsilon\sigma_0^i N_{\tau_0}}{2^i r}\right)} \leqslant \left(\frac{\varepsilon\sigma_0^i}{2^i}\right)^{1-q_1}, \quad i = 1, 2, \dots, j,$$

provided $C \geqslant \left(\frac{2}{\sigma_0}\right)^j \frac{(1-\tau_0)^m}{\delta\varepsilon} \frac{r}{\rho}$. By our assumptions

$$\bar{\theta}_0 \geqslant \theta' := \frac{c_2^{-1} r^2}{\psi(\bar{x}, 0, \delta N_{\tau_0}/r)}.$$

Therefore, inequality (58) holds for all times

$$-\theta'\left(1 - \varepsilon^{1-q_1} c_2 \gamma(\sigma_0)\right) \leqslant t \leqslant \theta'.$$

Choose j from the condition $\rho \leqslant 2^j r \leqslant 2\rho$ and m from the condition $\sigma_0 2^m = 1$, and then choose ε smaller if necessary, we arrive at the required (9) for all times $|t| \leqslant \theta'/2$, provided

$$C \geqslant \frac{2^{2m+1}}{\delta\varepsilon(1-4^{-\frac{1}{m}})^m} \geqslant \left(\frac{2}{\sigma_0}\right)^j \frac{(1-\tau_0)^m}{\delta\varepsilon} \frac{r}{\rho}.$$

By conditions (ψ) and (g_μ) with $\mu(r) = 1$

$$\theta' = \frac{c_2^{-1} r^2}{\psi(\bar{x}, 0, \delta N_{\tau_0}/r)} \geqslant \frac{c_2^{-1} r^2}{\psi(\bar{x}, 0, \delta u_0/\rho)} \left(\frac{\rho N_{\tau_0}}{r u_0}\right)^{1-q_1}$$

$$\geqslant \frac{c_2^{-2}\rho^2}{\psi(0,0,\delta u_0/\rho)}\left(\frac{r}{\rho}\right)^{1+q_1}(1-\tau_0)^{-m(1-q_1)}.$$

From the definition of r and choosing $m \geqslant \frac{1+q_1}{1-q_1}$, which we may assume taking σ_0 smaller if necessary, we obtain

$$\theta' \geqslant \frac{1}{\gamma}\left(\frac{1-4^{-\frac{1}{m}}}{2}\right)^{1+q_1}\frac{\rho^2}{\psi(0,0,\delta u_0/\rho)},$$

which completes the proof of Theorem 1.

4.4. Proof of Theorem 2

We note that under the conditions of Theorem 2, inequality (55) implies

$$|\{x \in B_r(\bar{x}) : u(x,t) \geqslant N_1\}| \geqslant \alpha(r)|B_r(\bar{x})|, \quad \alpha(r) = \alpha\mu(r)^{-m_1-m_2}, \tag{59}$$

for all $|t| \leqslant \bar{\theta}_0' := \dfrac{r^2}{\psi\left(\bar{x},0,\delta\frac{N_1}{r}\right)}$, where

$$N_1 := \varepsilon\lambda_1(r)N_{\tau_0}, \quad N_{\tau_0} = (1-\tau_0)^{-m}\frac{\lambda(\rho)}{\lambda((1-\tau_0)\rho)}\frac{\mu^{\beta}((1-\tau_0)\rho)}{\mu^{\beta}(\rho)}u_0,$$
$$\lambda_1(r) = \lambda(r)\mu(r)^{-m_2-\frac{1}{2-p}}, \quad m_1 = \frac{n}{\varkappa(p)}, \quad m_2 = \frac{(2-p)n}{\varkappa(p)}.$$

As was already mentioned, direct application of inequality (59) leads us to condition (3). We will use auxiliary solutions, defined in Sect. 3.

Set $E(N,t) := \{B_r(\bar{x}) : u(\cdot,t) \geqslant N\}$ and for $t \in [-\bar{\theta}_0', -\bar{\theta}_1']$, $\bar{\theta}_1' = \dfrac{r^2}{\psi\left(\bar{x},0,\delta\alpha(r)\frac{N_1}{r}\right)}$ consider the equation

$$f(t) = t + \frac{r^2}{\psi\left(\bar{x},0,\delta N_1\frac{|E(N_1,t)|}{r^{n+1}}\right)} = 0.$$

By our choices $f(-\bar{\theta}_0') \leqslant 0$ and $f(-\bar{\theta}_1') \geqslant 0$. By the continuity of u and by the continuity of the Lebesgue measure, $f(t)$ is continuous, so the previous equation has roots. Let $\bar{t}$ be the largest root of the above equation, i.e.,

$$\bar{t} = -\frac{r^2}{\psi\left(\bar{x},0,\delta N_1\frac{|E(N_1,\bar{t})|}{r^{n+1}}\right)}.$$

Taking δ smaller if necessary, we construct the auxiliary solution v with $\rho = r$ and $\xi = \delta$ in the cylinder $Q_1{=}B_{8r}(\bar{x}) \times (\bar{t}, \bar{t} + 8\bar{\tau}_1)$, $\bar{\tau}_1 = \dfrac{r^2}{\psi\left(\bar{x}, 0, \delta N_1 \dfrac{|E(N_1, \bar{t})|}{r^{n+1}}\right)}$.

If

$$\sigma_0 \delta N_1 \frac{|E(N_1, \bar{t})|}{r^n} \geqslant \gamma (b + b_0) r^{1 - a_0/a_1},$$

then by Proposition 2 with some $\varepsilon_1, \alpha_1 \in (0, 1)$ depending only on the data

$$|\{B_{4r}(\bar{x}) v(\cdot, t) \leqslant \varepsilon_1 N_1 \frac{|E(N_1, \bar{t})|}{r^n}\}| \leqslant (1 - \alpha_1)|B_{4r}(\bar{x})|,$$

for all $t \in (\bar{t} + \frac{1}{8}\bar{\tau}_1, \bar{t} + 2\bar{\tau}_1)$, which by Lemma 5 implies

$$v(x, t) \geqslant \sigma_0 \varepsilon_1 N_1 \frac{|E(N_1, \bar{t})|}{r^n}, \tag{60}$$

for all $(x, t) \in B_{2r}(\bar{x}) \times (\bar{t} + \frac{1}{4}\bar{\tau}_1,\ \bar{t} + 2\bar{\tau}_1)$, and the constant $\sigma_0 \in (0, 1)$ depends only on the data. By our choice of $\bar{t}$ and by (55)

$$B_{2r}(\bar{x}) \times \left(-\frac{3}{4}\frac{r^2}{\psi(\bar{x}, 0, \delta\alpha(r)\frac{N_1}{r})},\ \frac{r^2}{\psi(\bar{x}, 0, \delta\alpha(r)\frac{N_1}{r})}\right) \subset B_{2r}(\bar{x}) \times (\bar{t} + \frac{1}{4}\bar{\tau}_1,\ \bar{t} + 2\bar{\tau}_1),$$

moreover, by (55)

$$\frac{|E(N_1, \bar{t})|}{r^n} \geqslant \alpha(r).$$

We note also that since $u \geqslant v$ on the parabolic boundary of the cylinder $B_{8r}(\bar{x}) \times (\bar{t}, \bar{t} + 8\bar{\tau}_1)$, inequality (60) and the monotonicity condition (10) yield

$$u(x, t) \geqslant v(x, t) \geqslant \sigma_0 \varepsilon_1 \alpha(r) N_1, \quad x \in B_{2r}(\bar{x}), \tag{61}$$

for all $t \in (-\frac{3}{4}\theta_0^{(1)},\quad \theta_0^{(1)}),\quad \theta_0^{(1)} = \dfrac{r^2}{\psi(\bar{x}, 0, \delta\alpha(r)\frac{N_1}{r})}$.

Now we can use Lemma 5, which implies that if

$$\sigma_0^2 \varepsilon_1 \alpha(r) N_1 \geqslant \gamma (b + b_0)(2r)^{1 - a_0/a_1},$$

then

$$u(x, t) \geqslant \sigma_0^2 \varepsilon_1 \alpha(r) N_1, \quad x \in B_{2^2 r}(\bar{x}), \tag{62}$$

for all $t \in (-\frac{3}{4}\theta_0^{(1)} + \frac{1}{8}\theta_0^{(2)},\quad \theta_0^{(1)}),\quad \theta_0^{(2)} = \dfrac{(2r)^2}{\psi(\bar{x}, 0, \sigma_0 \delta \varepsilon_1 \alpha(r)\frac{N_1}{2r})}$.

After j iterations

$$u(x,t) \geqslant \sigma_0^j \varepsilon_1 \alpha(r) N_1, \quad x \in B_{2^j r}(\bar{x}), \tag{63}$$

and for all times

$$-\theta_0^{(1)} + \frac{1}{4}\sum_{i=0}^{j-1} \theta_0^{(i)} \leqslant t \leqslant \theta_0^{(1)}, \quad \theta_0^{(i)} := \frac{(2^i r)^2}{\psi\left(\bar{x}, 0, \sigma_0^i \delta \varepsilon_1 \alpha(r) \frac{N_1}{2^i r}\right)},$$

provided that

$$\sigma_0^j \varepsilon_1 \alpha(r) N_1 \geqslant \gamma(b + b_0)(2^j r)^{1-a_0/a_1}. \tag{64}$$

By our choices and condition (ψ)

$$\frac{\psi\left(\bar{x}, 0, \varepsilon\varepsilon_1 \delta \alpha(r) \lambda_1(r) \frac{N_{\tau_0}}{r}\right)}{\psi\left(\bar{x}, 0, \sigma_0^i \varepsilon_1 \delta \alpha(r) \frac{N_1}{2^i r}\right)} \leqslant \left(\frac{\sigma_0}{2}\right)^{i(1-q_1)},$$

provided that inequality (64) holds.

Therefore, inequality (63) holds for all times

$$-\theta_0^{(1)} + \frac{1}{4}\theta_0^{(1)} \sum_{i=0}^{j-1} 2^{i+1} \sigma_0^{i(1-q_1)} \leqslant -\frac{1}{2}\theta_0^{(1)} \leqslant \quad t \quad \leqslant \theta_0^{(1)}.$$

Choose j from the condition $\rho \leqslant 2^j r \leqslant 2\rho$, and m from the condition $\sigma_0 2^m = 1$, then we obtain

$$u(x,t) \geqslant \gamma^{-1}(\varepsilon, \varepsilon_1, \delta, m) \frac{\lambda(\rho)}{\lambda((1-\tau_0)\rho)} \frac{\mu^\beta((1-\tau_0)\rho)}{\mu^\beta(\rho)} \lambda_1(r) \alpha(r)\, u_0 = \tilde{u}_0$$

for all $x \in B_{2\rho}(\bar{x})$ and for all times $|t| \leqslant \frac{1}{2}\theta_0^{(1)}$ provided that (64) holds.

Let us estimate the term on the right-hand side of the previous inequality. By our choices, we have

$$\tilde{u_0} \geqslant \gamma^{-1}(\varepsilon, \varepsilon_1, \delta, m) \lambda(\rho) \frac{\mu^\beta(r)}{\mu^\beta(\rho)} \mu(r)^{-m_1 - 2m_2 - \frac{1}{2-p}} u_0.$$

Choosing

$$\beta = m_1 + 2m_2 + \frac{1}{2-p},$$

we arrive at the required (12) for all times $|t| \leqslant \frac{1}{2}\theta_0^{(1)}$, provided that (64) holds.

Inequality (64) holds, if

$$C \geqslant \frac{2^{2m+1+3n}}{\varepsilon\varepsilon_1(1-4^{-\frac{1}{m}})^m} \geqslant \frac{8^n}{\varepsilon\varepsilon_1}\left(\frac{2}{\sigma_0}\right)^j (1-\tau_0)^m\,\frac{r}{\rho},$$

and moreover, by conditions (ψ) and (g_λ)

$$\begin{aligned}\theta_0^{(1)} &= \frac{r^2}{\psi(\bar{x}, 0, \varepsilon\varepsilon_1\lambda(r)\alpha(r)\frac{N_{\tau_0}}{r})}\\ &\geqslant \frac{\rho^2}{c_1\psi(0, 0, \varepsilon\varepsilon_1\dfrac{\lambda(\rho)}{\mu^\beta(\rho)}\dfrac{u_0}{\rho})}\left(\frac{r}{\rho}\right)^{1+q_1}(1-\tau_0)^{-m(1-q_1)}.\end{aligned}$$

From the definition of r and choosing $m \geqslant \frac{1+q_1}{1-q_1}$, we obtain

$$\theta_0^{(1)} \geqslant \gamma^{-1}\frac{\rho^2}{\psi(0, 0, \varepsilon\varepsilon_1\dfrac{\lambda(\rho)}{\mu^\beta(\rho)}\dfrac{u_0}{\rho})},$$

which completes the proof of Theorem 2.

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