



# A minimising movement scheme for the $p$ -elastic energy of curves

SIMON BLATT , CHRISTOPHER P. HOPPER AND NICOLE VORDEROBERMEIER

*Abstract.* We prove short-time existence for the negative  $L^2$ -gradient flow of the  $p$ -elastic energy of curves via a minimising movement scheme. In order to account for the degeneracy caused by the energy's invariance under curve reparametrisations, we write the evolving curves as approximate normal graphs over a fixed smooth curve. This enables us to establish short-time existence and give a lower bound on the solution's lifetime that depends only on the  $W^{2,p}$ -Sobolev norm of the initial data.

## 1. Introduction

For closed curves  $\gamma: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^n$  in the  $W^{2,p}$ -Sobolev class, we shall consider the energy

$$E(\gamma) = \frac{1}{p} \int_{\mathbb{R}/\mathbb{Z}} |\kappa|^p ds + \lambda \int_{\mathbb{R}/\mathbb{Z}} ds, \quad (1.1)$$

i.e. the sum of the  $p$ -elastic energy  $E^{(p)}(\gamma) = \frac{1}{p} \int_{\mathbb{R}/\mathbb{Z}} |\kappa|^p ds$  and a positive multiple  $\lambda > 0$  of the length of the curve. A family of regular curves  $\gamma = \gamma(t, x): [0, T) \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^n$  in the class

$$L^\infty([0, T), W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cap W^{1,2}([0, T), L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$$

is said to be a weak solution of the negative  $L^2$ -gradient flow of  $E$  if one has

$$\int_0^T \int_{\mathbb{R}/\mathbb{Z}} \langle \partial_t \gamma, \psi \rangle ds dt = \int_0^T \delta_{\psi_t} E(\gamma_t) dt \quad (1.2)$$

for all test functions  $\psi \in C_c^\infty(\mathbb{R}/\mathbb{Z} \times (0, T), \mathbb{R}^n)$ , i.e. the curve  $\gamma$  satisfies  $\partial_t \gamma = -\nabla_{L^2} E(\gamma)$  weakly, where  $\delta_{\psi_t} E(\gamma_t) = \frac{d}{d\varepsilon} E(\gamma_t + \varepsilon \psi_t)|_{\varepsilon=0}$  is the first variation of the functional  $E$  at the curve  $\gamma_t = \gamma(t, \cdot)$  in the direction of the test function  $\psi_t = \psi(t, \cdot)$ .

While the  $L^2$ -gradient flow of (1.1) has been extensively studied when  $p = 2$ , both in the Euclidean (cf. [4, 5, 9, 15]) and manifold constrained (cf. [3, 15, 20]) settings, very

---

*Mathematics Subject Classification:* 53C44, 53A04

*Keywords:* Minimising movement,  $p$ -elastic energy for curves, Gradient flow, Approximate normal graphs.

All three authors acknowledge support by the Austrian Science Fund (FWF), Grant P29487. Nicole Vorderobermeier also acknowledges funding by the Austrian Marshall Plan Foundation.

little is known in the degenerate  $p \neq 2$  case. For example, a second-order evolution equation has been considered for closed curves and planar networks (cf. [21, 22]) and the asymptotic of the flow has been studied away from degenerate points (cf. [24]); however, short-time existence for the Eq. (1.2) has yet to be established when  $p \neq 2$ . The aim of this article is to address both short and long time existence in the case  $p > 2$  for the geometric evolution (1.2) with initial data in the  $W^{2,p}$ -Sobolev class. Our approach is to rewrite the evolving curves as approximate normal graphs in order to utilise de Giorgi’s method of minimising movements (cf. [6]).

It is well-known that the invariance of the energy (1.1) under reparametrisations of the curve  $\gamma$  leads to an evolution equation (1.2) that fails to be strongly parabolic (even in the  $p = 2$  case). This characteristic is in common with many other geometric evolution equations. For example, the failure of the strong ellipticity of the Ricci tensor is principally due to the second Bianchi identities.<sup>1</sup> For this reason, short-time existence for the Ricci flow was originally established in [13] by appealing to the Nash–Moser implicit-function theorem (and the earlier exposition in [12]). DeTurck [7] subsequently showed that the Ricci flow is equivalent to an initial value problem for a parabolic system modulo the action of the diffeomorphism group of the underlying manifold. Thus, in a dramatic simplification that bypassed the Nash–Moser argument, one can pass from a weakly parabolic to a strongly parabolic system of equations by an appropriate choice of a 1-parameter family of diffeomorphisms. Perelman [23] also exploited the same diffeomorphism invariance in his gradient flow formalism for the Ricci flow. Versions of the DeTurck trick have also been used to obtain short-time existence for the mean curvature flow (cf. [2, 14]), the Willmore flow (cf. [16]) and the gradient flow of the elastic energy in both the Euclidean and manifold constrained cases.

In seeking to pass from the degenerate flow (1.1) to a strongly parabolic system, one can consider a time-dependent family of curves  $\gamma_t = \gamma(t, \cdot)$  that are written as normal graphs over a given fixed smooth curve  $\tilde{\gamma}$ , i.e. a family of curve of the form  $\gamma_t = \tilde{\gamma} + \phi_t$  where  $\phi_t = \phi(t, \cdot)$  is a perturbation normal to the fixed curve  $\tilde{\gamma}$ . In this way, we obtain an evolution equation of the form

$$\int_0^T \int_{\mathbb{R}/\mathbb{Z}} \langle \gamma, \partial_t^\perp \psi \rangle \, ds dt = \int_0^T \delta_{\psi_t} E(\gamma_t) \, dt \tag{1.3}$$

for all test functions  $\psi \in C_c^\infty((0, T) \times \mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ , i.e. the curve  $\gamma$  satisfies  $\partial_t^\perp \gamma = -\nabla_{L^2} E(\gamma)$  weakly, where the normal velocity  $\partial_t^\perp \gamma$  is the vector component of  $\partial_t \gamma$  normal to the fixed curve  $\tilde{\gamma}$ . Then in order to obtain a solution of (1.2) from a solution of (1.3), one can consider solutions  $\Theta_t = \Theta(t, \cdot)$  of the ordinary differential equation

$$\begin{aligned} \partial_t \Theta(t, x) &= F(t, \Theta(t, x)) \\ \Theta(0, x) &= x, \end{aligned} \tag{1.4}$$

---

<sup>1</sup>In fact, the diffeomorphism invariance of the Riemannian curvature tensor naturally yields the Bianchi identities (cf. [17]). Thus, the strongly ellipticity failure of the Ricci tensor is due entirely to this geometric invariance.

where  $F(t, y) = -\frac{\langle \partial_t \gamma(t, y), \gamma'(t, y) \rangle}{|\gamma'(t, y)|^2}$  and  $\gamma$  is a solution of (1.3). The existence of ODE solutions can thus be established on a time interval  $0 \leq t < \varepsilon$  for some  $\varepsilon > 0$  independent of the initial point  $x \in \mathbb{R}/\mathbb{Z}$ . Therefore, if  $\Theta_t = \Theta(t, \cdot)$  is a solution of (1.4) and  $\gamma_t = \gamma(t, \cdot)$  is a solution of (1.3), the composition  $\gamma_t \circ \Theta_t$  is a solution of (1.2). By taking this approach, one can thus establish the existence of solutions for geometric flows with initial data in the  $C^{2,\alpha}$ -Hölder class even though the original equations may be ill-defined (see, e.g. [10, 19, 26]). In fact, a recent paper by LeCrone, Shao and Simonett [18] showed how to reduce the regularity of the initial data to the  $C^{1,\alpha}$ -Hölder class.

In order to carry out the aforementioned programme, one has to guarantee that a given initial curve  $\Gamma$  can be written as a normal graph over a fixed smooth curve  $\tilde{\gamma}$ . Since it is not possible to write every curve  $\Gamma$  in the  $W^{2,p}$ -Sobolev class as a normal graph over a smooth curve, we are spurred on to introduce the notion of a unit quasi-tangent  $\tau$  (cf. Definition 2.4) which then defines an *approximate tangential projection*  $P_\tau^T$  and an *approximate normal projection*  $P_\tau^\perp = I - P_\tau^T$  (cf. Definition 2.6). In which case one can write the curve  $\Gamma$  as equal to  $\tilde{\gamma} + \Phi$  up to a reparametrisation, i.e. as an approximate normal graph over a smooth curve  $\tilde{\gamma}$  with some perturbation  $\Phi$  orthogonal to  $\tau$  (cf. Lemma 2.12). Then by applying a minimising movements scheme, it is possible to establish the existence of a family of curves of the form  $\gamma_t = \tilde{\gamma} + \phi_t$ , for a suitable perturbation  $\phi_t$  orthogonal to  $\tau$ , that satisfies  $\partial_t^\perp \gamma = -\nabla_{L^2} E(\gamma)$  weakly. Indeed, we have:

**Theorem 1.1.** (Existence) *For any given initial curve  $\Gamma \in W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length there exists a smooth curve  $\tilde{\gamma} \in C^\infty(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length, a quasi-tangent  $\tau$  to the curve  $\tilde{\gamma}$ , a finite time  $T = T(p, \lambda, E(\Gamma)) > 0$  and a family of perturbations  $\phi$  in the class*

$$L^\infty([0, T], W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)) \cap (W^{1,2} \cap C^{1/2})([0, T], L^2(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n))$$

which are orthogonal to  $\tau$  such that the family of curves

$$\gamma(t, s) = \tilde{\gamma}(s) + \phi(t, s), \quad 0 \leq t < T,$$

satisfies the initial condition  $\gamma(0, \cdot) = \Gamma \circ \sigma$  for some reparametrisation  $\sigma$  of  $\mathbb{R}/L\mathbb{Z}$  and

$$\int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_t^\perp \gamma, \psi \rangle \, ds dt = - \int_0^T \delta_{\psi_t} E(\gamma_t) dt \tag{1.5}$$

for all test functions  $\psi \in C_c^\infty((0, T) \times \mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  orthogonal to  $\tau$ .

Note that the time of existence only depends on the energy of the initial curve. So we are very close to restarting the flow and deduce long time existence. We discuss in the final section, why this is not as straightforward as it might seem.

By assuming the solution has some additional regularity, one can show that Eq. (1.5) holds for all test functions (i.e. our solution solves the original weak form of the desired evolution equation).

**Corollary 1.2.** *If the solution  $\gamma(t, \cdot)$  of Theorem 1.1 belongs to the  $W^{3,p}$ -Sobolev class for almost all  $0 \leq t < T$ , then*

$$\int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_t^\perp \gamma, \psi \rangle ds dt = - \int_0^T \delta_{\psi_t} E(\gamma) dt$$

for all test functions  $\psi \in C_c^\infty((0, T) \times \mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ .

## 2. Minimising movements scheme

It is remarked by De Giorgi [6] that a generalised minimising movements scheme could provide a formalism for the existence of steepest descent curves of a functional in a metric space. In order to establish the existence of weak solutions for (1.2), we need to take care of the twofold degeneracies arising from the invariance of (1.1) under curve reparametrisation and the fact that  $p > 2$ . We tackle this issue by writing the evolving curve as an approximate normal graph over a fixed smooth curve so that we can work with the normal velocity (rather than the time derivative) of the evolving curve.

### 2.1. Tubular neighbourhoods

For an embedded  $C^k$ -submanifold  $\mathcal{M}$  of  $\mathbb{R}^n$  without boundary, the normal bundle  $(T\mathcal{M})^\perp \rightarrow \mathcal{M}$  is only of the class  $C^{k-1}$ . If we define the ‘endpoint’ map  $E: (T\mathcal{M})^\perp \rightarrow \mathbb{R}^n$  by sending

$$(x, v) \mapsto x + v$$

and assume  $k \geq 2$ , one can use the inverse function theorem to show that there exists a tubular neighbourhood  $U$  of  $\mathcal{M}$  in  $\mathbb{R}^n$  that is the diffeomorphic image under the  $C^{k-1}$ -map  $E$  of an open neighbourhood of the zero section of  $(T\mathcal{M})^\perp$ . Moreover, the squared distance function  $\zeta(x) = \frac{1}{2} \text{dist}(x, \mathcal{M})^2$  is a function in  $C^k(U)$  (cf. [11]) and the Hessian matrix  $\nabla^2 \zeta(x)$  represents the orthogonal projection on the normal space to  $\mathcal{M}$  at a point  $x$  (cf. [1, p. 704]). Of course, these results no longer hold in the case  $k = 1$ , i.e. when the inverse function theorem is not applicable.

### 2.2. Approximate normal graphs

As the normal bundle of an embedded  $W^{2,p}$ -curve in  $\mathbb{R}^n$  is only of the class  $W^{1,p}$ , one cannot directly apply the standard methods of Sect. 2.1. In particular, we need to overcome the loss of regularity on the level of the tangent space in order to write the solution of our equation locally as a graph over a fixed smooth curve. This problem can be resolved by regularising the tangent using Friedrichs mollifiers (whilst taking into consideration the size of the constructed tubular neighbourhood). We will call this smoothed tangent *quasi-tangent*.

**Definition 2.1.** A function  $\eta \in C^\infty(\mathbb{R})$  is called a *mollifier* if it satisfies the conditions: (i)  $\eta \geq 0$  on  $\mathbb{R}$ , (ii)  $\eta(x) = 0$  for all  $|x| \geq 1$ , and (iii)  $\int_{\mathbb{R}} \eta(x) dx = 1$ . The associated *rescaled mollifier* is the function  $\eta_\varepsilon(x) = \frac{1}{\varepsilon} \eta(\frac{x}{\varepsilon})$  for any  $\varepsilon > 0$ .

Now consider a curve  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length. The *mollification* of  $\gamma$  is defined to be the function

$$\gamma_\varepsilon(x) = (\gamma * \eta_\varepsilon)(x) = \int_{\mathbb{R}} \gamma(x - y) \eta_\varepsilon(y) dy,$$

i.e. the convolution of the given curve  $\gamma$  and the rescaled mollifier  $\eta_\varepsilon$ .

For the mollified curve  $\gamma_\varepsilon$ , we derive the following well-known estimates. Firstly, from the mean value theorem and the Sobolev embeddings, we find that

$$\begin{aligned} |\gamma_\varepsilon(x) - \gamma(x)| &= \left| \int_{\mathbb{R}} (\gamma(x - y) - \gamma(x)) \eta_\varepsilon(y) dy \right| \\ &\leq \varepsilon \|\gamma'\|_{L^\infty} \\ &\leq C\varepsilon \|\gamma'\|_{W^{1,p}}. \end{aligned} \tag{2.1}$$

Likewise, we find that

$$\begin{aligned} |\gamma'_\varepsilon(x) - \gamma'(x)| &= \left| \int_{\mathbb{R}} \eta_\varepsilon(y) (\gamma'(x - y) - \gamma'(x)) dy \right| \\ &\leq \sqrt{\varepsilon} \|\gamma'\|_{C^{1/2}} \\ &\leq C\sqrt{\varepsilon} \|\gamma'\|_{W^{1,p}}. \end{aligned} \tag{2.2}$$

For higher derivatives, we can use the Sobolev embeddings, Hölder’s inequality and integration by parts to obtain the  $L^\infty$ -bound

$$\begin{aligned} |\gamma_\varepsilon^{(k+2)}(x)| &= \left| \int_{\mathbb{R}} \eta_\varepsilon^{(k)}(y) \gamma''(x - y) dy \right| \\ &\leq \|\eta_\varepsilon^{(k)}\|_{L^q} \|\gamma''\|_{L^p} \\ &\leq C\varepsilon^{-k-1+\frac{1}{q}} \|\gamma''\|_{L^p} \\ &= C\varepsilon^{-k-\frac{1}{p}} \|\gamma''\|_{L^p} \end{aligned} \tag{2.3}$$

for integers  $k \geq 0$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .

We will use the next lemma to fix the smoothing parameter  $\varepsilon$ .

**Lemma 2.2.** *If for an  $M > 0$ , we have a curve  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length which satisfies  $\|\gamma'\|_{W^{1,p}} \leq M$ , then there exists an  $\varepsilon = \varepsilon(p, M) > 0$  such that the unit tangent  $\tau = \frac{\gamma'}{|\gamma'|}$  satisfies*

$$\|\tau - \gamma'\|_{L^\infty} \leq \frac{1}{4}. \tag{2.4}$$

*Proof.* Using (2.2), we get  $\|\gamma'_\varepsilon - \gamma'\|_{L^\infty} \leq C\sqrt{\varepsilon}\|\gamma'\|_{W^{1,p}}$ . As the retraction map  $\Pi: x \mapsto \frac{x}{|x|}$  is locally Lipschitz on  $\mathbb{R}^n \setminus \{0\}$ , the tangent  $\tau = \Pi(\gamma'_\varepsilon) = \frac{\gamma'_\varepsilon}{|\gamma'_\varepsilon|}$  to the mollified curve  $\gamma_\varepsilon$  satisfies (2.4) for some  $\varepsilon > 0$  sufficiently small.  $\square$

**Corollary 2.3.** *If for an  $M > 0$ , we have a curve  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length which satisfies  $\|\gamma'\|_{W^{1,p}} \leq M$ , then for an  $\varepsilon = \varepsilon(p, M) > 0$  as in Lemma 2.2 the mollified curve  $\gamma_\varepsilon$  has a unit tangent map  $\tau: \mathbb{R}/\mathbb{Z} \rightarrow S^{n-1}$  that is smooth and satisfies*

$$\|\tau'\|_{L^\infty}, \|\tau''\|_{L^\infty} \leq C \tag{2.5}$$

for a constant  $C = C(p, M) > 0$ .

**Definition 2.4.** We say  $\tau$  is *unit quasi-tangent* to the  $W^{2,p}$ -curve  $\gamma$  if it is the unit tangent to the mollified curve  $\gamma_\varepsilon$  for some  $\varepsilon = \varepsilon(p, M) > 0$  as in Lemma 2.2.

**Definition 2.5.** We denote by  $P_v^T w = \langle w, \frac{v}{|v|} \rangle \frac{v}{|v|}$  the orthogonal projection of  $w$  onto the line  $\mathbb{R}v$  for any vectors  $v, w \in \mathbb{R}^n$ . Likewise, we denote by  $P_v^\perp w = w - P_v^T w$  the orthogonal projection of  $w$  onto the orthogonal complement  $(\mathbb{R}v)^\perp$  of the line  $\mathbb{R}v$ .

**Definition 2.6.** If  $\tau$  is unit quasi-tangent to a  $W^{2,p}$ -curve  $\gamma$ , we denote by  $(W^{2,p})_\tau^T$  (resp.  $(W^{2,p})_\tau^\perp$ ) the set of all  $w \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  such that  $P_\tau^\perp w = 0$  a.e. (resp.  $P_\tau^T w = 0$  a.e.).

We will now prove the following statement that gives a lower bound on the thickness of the set of regular curves around  $\gamma$ .

**Lemma 2.7.** *If for an  $M > 0$  we have a curve  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length which satisfies  $\|\gamma'\|_{W^{1,p}} \leq M$ , then there exists a constant  $K = K(p, M) > 0$  and a unit quasi-tangent  $\tau$  to the curve  $\gamma$  such that the curve  $\gamma + \phi$  satisfies*

$$\inf_{x \in \mathbb{R}/\mathbb{Z}} \langle \gamma' + \phi', \tau \rangle \geq \frac{1}{2}$$

and hence

$$\inf_{x \in \mathbb{R}/\mathbb{Z}} |\gamma'(x) + \phi'(x)| \geq \frac{1}{2}$$

for each  $\phi \in (W^{2,p})_\tau^\perp$  with  $\|\phi\|_{L^\infty} \leq K$ . In particular,  $\gamma + \phi$  is a regular curve.

*Proof.* We first note that  $\langle \gamma', \tau \rangle = |\gamma'|^2 + \langle \gamma', \tau - \gamma' \rangle \geq 1 - |\tau - \gamma'| \geq \frac{3}{4}$  by Lemma 2.2 and the fact that  $|\gamma'| = 1$ . Upon differentiating the orthogonality condition  $\langle \phi, \tau \rangle = 0$ , we get  $\langle \phi', \tau \rangle = -\langle \phi, \tau' \rangle$ . In which case the estimate (2.5) implies that  $|\langle \phi', \tau \rangle| = |\langle \phi, \tau' \rangle| \leq C\|\phi\|_{L^\infty} \leq \frac{1}{4}$  whenever  $\|\phi\|_{L^\infty} \leq \frac{1}{4C} = K$ . Thus,

$$\langle \gamma' + \phi', \tau \rangle \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

whenever  $\|\phi\|_{L^\infty} \leq K$ , i.e.  $\gamma + \phi$  is a regular curve. As  $\tau$  is of unit length, we also have  $|\gamma' + \phi'| \geq \frac{1}{2}$  on  $\mathbb{R}/\mathbb{Z}$ .  $\square$

We will now deduce the following lower bound for the  $L^p$ -norm of the curvature of a curve  $\tilde{\gamma} + \phi$  in terms of the  $L^p$ -norm of the second derivative of  $\phi$ . This bound extends to our situation the well-known analogous result for the case of a real normal graph over a smooth curve.

**Lemma 2.8.** *If for an  $M > 0$  we have a curve  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length which satisfies  $\|\gamma'\|_{W^{1,p}} \leq M$  and a unit quasi-tangent  $\tau$  to the curve  $\gamma$ , then for the constant  $K = K(p, M) > 0$  from Lemma 2.7 we have for each  $\phi \in (W^{2,p})^\perp_\tau$  with  $\|\phi\|_{L^\infty} \leq K$*

$$|v| \leq C|P_{\gamma'+\phi'}^\perp v|$$

for all  $v \in \mathbb{R}^n$  pointing in an approximate normal direction and

$$\int_{\mathbb{R}/\mathbb{Z}} |\phi''|^p ds \leq C \left( 1 + \int_{\mathbb{R}/\mathbb{Z}} |\kappa_{\gamma+\phi}|^p ds \right)$$

for some  $C = C(M, p)$ .

*Proof.* Since  $\langle \gamma' + \phi', \tau \rangle \geq \frac{1}{2}$  from Lemma 2.7 and  $|\gamma' + \phi'| \leq |\gamma'| + |\phi'| \leq 1 + \Lambda$ , we see that  $\langle \frac{\gamma'+\phi'}{|\gamma'+\phi'|}, \tau \rangle \geq \frac{1}{2} \frac{1}{1+\Lambda}$ . Hence, the angle between  $\gamma' + \phi'$  and  $\tau$  is bounded strictly away from  $\frac{\pi}{2}$ . In which case we have

$$|v| \leq C|P_{\gamma'+\phi'}^\perp v|$$

for all  $v \in \mathbb{R}^n$  pointing in an approximate normal direction.

For the second estimate, we recall the curvature formula given by

$$\kappa_{\gamma+\phi} = \frac{P_{\gamma'+\phi'}^\perp(\gamma'' + \phi'')}{|\gamma' + \phi'|^2}.$$

Now by the triangle inequality, we see that

$$\left| P_{\gamma'+\phi'}^\perp(\phi'') \right| \leq \left| P_{\gamma'+\phi'}^\perp(\gamma'' + \phi'') \right| + \left| P_{\gamma'+\phi'}^\perp(\gamma'') \right| \leq C(|\kappa_{\gamma+\phi}| + |\gamma''|),$$

since  $|\gamma' + \phi'| \leq |\gamma'| + |\phi'| \leq 1 + \Lambda$ . To control the tangential part  $P_\tau^T \phi''$ , we differentiate the equation  $\langle \phi, \tau \rangle = 0$  twice to get  $\langle \phi'', \tau \rangle = -2\langle \phi', \tau' \rangle - \langle \phi, \tau'' \rangle$ . It then follows that

$$|P_\tau^T \phi''| = |\langle \phi'', \tau \rangle| \leq |\langle \phi, \tau'' \rangle| + 2|\langle \phi', \tau' \rangle| \leq C(K + \Lambda),$$

since both  $\tau'$  and  $\tau''$  are bounded by Corollary 2.3. In combining both the tangential and normal parts of  $\phi''$  and using the fact that the angle between  $\gamma' + \phi'$  and  $\tau$  is bounded strictly away from  $\frac{\pi}{2}$ , we find that

$$|\phi''| \leq C(|P_{\gamma'+\phi'}^\perp \phi''| + |P_\tau^T \phi''|) \leq C(1 + |\gamma''| + |\kappa_{\gamma+\phi}|)$$

from which the desired integral estimate follows (since  $\|\gamma''\|_{L^p} \leq M$  by Lemma 2.12). □

Next we show that there exists a good substitute for the nearest neighbourhood projection which yields a local tubular neighbourhood. We also obtain a lower bound on thickness of the tubular neighbourhood that only depends on the  $W^{2,p}$ -norm of the curve.

**Definition 2.9.** If  $\tau$  is a unit quasi-tangent to a  $W^{2,p}$ -curve  $\gamma$ , the  $(n - 1)$ -dimensional subspace

$$\mathcal{N}_{x_0} = \{v \in \mathbb{R}^n : P_{\tau(x_0)}^T v = 0\}$$

is called an *approximate normal space* to  $\gamma$  at a given fixed point  $x_0 \in \mathbb{R}/\mathbb{Z}$ .

By considering the map  $H_{x_0} : B_\delta(x_0) \times \mathcal{N}_{x_0} \rightarrow \mathbb{R}^n$  given by

$$(x, v) \mapsto \gamma(x) + P_{\tau(x)}^\perp v \tag{2.6}$$

for some  $0 < \delta < 1$ , we obtain the following:

**Lemma 2.10.** *If for an  $M > 0$ , we have a curve  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length which satisfies  $\|\gamma'\|_{W^{1,p}} \leq M$  and a unit quasi-tangent  $\tau$  to the curve  $\gamma$ , then there exists a sufficiently small constant  $\delta = \delta(p, M) > 0$  such that (2.6) maps  $B_\delta(x_0) \times B_\delta(0)$  diffeomorphically onto its image and*

$$B_{\delta/4}(\gamma(B_{\delta/4}(x_0))) \subset H_{x_0}(B_\delta(x_0) \times B_\delta(0)). \tag{2.7}$$

*Proof.* We first show that  $H_{x_0}$  is a local diffeomorphism by way of the inverse function theorem. To do so, we calculate the partial derivatives

$$\begin{aligned} \frac{\partial H_{x_0}}{\partial x} &= \gamma'(x) - \langle v, \tau'(x) \rangle \tau(x) - \langle v, \tau(x) \rangle \tau'(x) \\ \frac{\partial H_{x_0}}{\partial v} &= P_{\tau(x)}^\perp v = v + P_{\tau(x)}^\perp v - P_{\tau(x_0)}^\perp v, \end{aligned}$$

since  $v \in \mathcal{N}_{x_0}$  and hence  $P_{\tau(x_0)}^\perp v = v$ . Then from the estimates (2.4) and (2.5) together with the Sobolev embedding  $W^{2,p}(B_\delta(x_0), \mathbb{R}^n) \hookrightarrow C^{1,1-\frac{1}{p}}(B_\delta(x_0), \mathbb{R}^n)$ , we find that

$$\begin{aligned} \left| \frac{\partial H_{x_0}}{\partial x} - \tau(x_0) \right| &\leq |\gamma'(x_0) - \tau(x_0)| + C|v| + |\gamma'(x) - \gamma'(x_0)| \\ &\leq \frac{1}{4} + C|v| + C\delta^{1-\frac{1}{p}} \end{aligned}$$

for some constant  $C = C(p, M) > 0$ . By taking some  $\delta > 0$  sufficiently small (depending only on  $p$  and  $M$ ), we have

$$\left| \frac{\partial H_{x_0}}{\partial x} - \tau(x_0) \right| \leq \frac{1}{2}$$

for all  $x \in B_\delta(x_0) \subset \mathbb{R}/\mathbb{Z}$  and  $v \in B_\delta(0) \subset \mathcal{N}_{x_0}$ . Likewise, whenever  $\delta > 0$  is sufficiently small, we also have

$$\left| \frac{\partial H_{x_0}}{\partial v} - v \right| \leq \frac{1}{2}$$



for all  $(x, v) \in B_\delta(x_0) \times B_\delta(0)$ .

Let us now assume that  $\tau(x_0) = e_1$  without loss of generality. From the above estimates, we see that the Jacobi matrix  $DH_{x_0}$  satisfies

$$\|DH_{x_0} - I\| \leq \frac{1}{2}, \tag{2.8}$$

where  $\|\cdot\|$  denotes the operator norm. Therefore,  $DH_{x_0}$  is invertible and so  $H_{x_0}$  maps  $B_\delta(x_0) \times B_\delta(0)$  diffeomorphically onto its image by the inverse function theorem. Moreover, (2.8) implies that

$$\begin{aligned} |H_{x_0}(z_1) - H_{x_0}(z_2)| &= \left| \int_0^1 DH_{x_0}(z_2 + \theta(z_1 - z_2))(z_1 - z_2) d\theta \right| \\ &\geq |z_1 - z_2| - \frac{1}{2}|z_1 - z_2| \\ &= \frac{1}{2}|z_1 - z_2|. \end{aligned}$$

In which case the map  $H_{x_0}$  is bi-Lipschitz and hence injective on  $B_\delta(x_0) \times B_\delta(0)$ . From the fact that  $\text{dist}(\partial(B_\delta(x_0) \times B_\delta(0)), B_{\delta/2}(x_0) \times \{0\}) \geq \frac{\delta}{2}$  and the latter bi-Lipschitz estimate, we have

$$\text{dist}\left(H_{x_0}(\partial(B_\delta(x_0) \times B_\delta(0))), \gamma(B_{\delta/2}(x_0))\right) \geq \frac{\delta}{4}$$

which then gives (2.7). □

We can now use Lemma 2.10 to show that any  $W^{2,p}$ -curve  $\gamma$  can be written as an approximate normal graph over a given  $W^{2,p}$ -curve  $\tilde{\gamma}$  whenever the curves are  $C^1$ -close to each other.

**Lemma 2.11.** *If for an  $M > 0$ , we have a curve  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length that satisfies  $\|\gamma'\|_{W^{1,p}} \leq M$  and a unit quasi-tangent  $\tau$  to the curve  $\gamma$ , then there exists a sufficiently small constant  $\rho = \rho(p, M) > 0$  such that for each curve  $\tilde{\gamma} \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  satisfying  $\|\gamma - \tilde{\gamma}\|_{C^1} \leq \rho$  we have some  $\phi \in (W^{2,p})^\perp_\tau$  and a reparametrisation  $\sigma$  of  $\mathbb{R}/\mathbb{Z}$  for which  $\tilde{\gamma} \circ \sigma = \gamma + \phi$ .*

*Proof.* Firstly, choose  $\delta > 0$  as in Lemma 2.10. Since  $\mathbb{R}/\mathbb{Z}$  is compact, there exist points  $x_1, \dots, x_\ell$  in  $\mathbb{R}/\mathbb{Z}$  such that the balls  $B_{\delta/4}(x_1), \dots, B_{\delta/4}(x_\ell)$  cover  $\mathbb{R}/\mathbb{Z}$ . Let the mappings  $H_{x_j}$  for  $j = 1, \dots, \ell$  be defined by (2.6) and let  $\Pi_{x_j} : B_{\delta/4}(\gamma(B_{\delta/4}(x_j))) \rightarrow \mathbb{R}/\mathbb{Z}$  be the corresponding retraction maps given by

$$\Pi_{x_j} = \pi \circ H_{x_j}^{-1}$$

where  $\pi : B_\delta(x_j) \times \mathcal{N}_{x_j} \rightarrow \mathbb{R}/\mathbb{Z}$  sends  $(x, v) \mapsto x$ , i.e. the projection onto the first coordinate. We can then set

$$\sigma(x) = \Pi_{x_j}(\tilde{\gamma}(x)) \tag{2.9}$$

for any  $x \in B_\delta(x_j)$  in order to get a well-defined  $C^1$ -mapping. Note that the affine subspaces  $\gamma(x) + \mathcal{N}_x$  and not their parametrisations determine the projections  $H_{x_j}$ .

Hence, they agree for different  $x_j$  if the domains of definition overlap. Furthermore, from the inverse function theorem applied to  $\Pi_{x_j}$  and the estimate (2.8) we see that  $\sigma'(x) > 0$  whenever  $\rho > 0$  is sufficiently small (i.e.  $\sigma$  is bi-Lipschitz). In addition, by setting  $\tilde{\phi} = \tilde{\gamma} - \gamma \circ \sigma$  we see from (2.6) that  $\tilde{\phi}$  belongs to  $(W^{2,p})_{\tau \circ \sigma}^\perp$ . Therefore,  $\gamma \circ \sigma$  is a regular curve equal to  $\tilde{\gamma} - \tilde{\phi}$ . In order to change the roles of  $\gamma$  and  $\tilde{\gamma}$ , we apply the inverse function theorem to  $\sigma$  to justify the reparametrisation  $\tilde{\gamma} \circ \sigma^{-1} = \gamma \circ \sigma \circ \sigma^{-1} + \tilde{\phi} \circ \sigma^{-1} = \gamma + \phi$ , where we set  $\phi = \tilde{\phi} \circ \sigma^{-1} \in (W^{2,p})_\tau^\perp$ . □

Using the above lemma, we can write every  $W^{2,p}$ -curve  $\gamma$  as an approximate normal graph over a smooth curve  $\tilde{\gamma}$ . Be aware that from now on till the end of this article we consider normal graphs over the curve  $\tilde{\gamma}$  instead of  $\gamma$ .

**Lemma 2.12.** *Let  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  be a curve parametrised by arc-length. For every  $\varepsilon_0 > 0$ , there exists a smooth curve  $\tilde{\gamma} \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length with  $\|\gamma - \tilde{\gamma}\|_{W^{2,p}} \leq \varepsilon_0$ , a unit quasi-tangent  $\tau$  to the curve  $\tilde{\gamma}$  and some  $\phi \in (W^{2,p})_\tau^\perp$  such that*

$$\gamma \circ \sigma = \tilde{\gamma} + \phi \tag{2.10}$$

for a reparametrisation  $\sigma$  of  $\mathbb{R}/\mathbb{Z}$ .

*Proof.* Firstly, there exists a smooth curve  $\tilde{\gamma} \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  parametrised by arc-length such that

$$\|\tilde{\gamma} - \gamma\|_{W^{2,p}} \leq \varepsilon_0$$

by the density of  $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  in  $W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ . Moreover, we have

$$\|\gamma - \tilde{\gamma}\|_{C^1} \leq C\varepsilon_0 = \rho$$

by the Sobolev embeddings. Thus, by taking some  $\varepsilon_0 > 0$  sufficiently small, Lemma 2.11 implies that there exists some  $\phi \in (W^{2,p})_\tau^\perp$  and a reparametrisation  $\sigma$  of  $\mathbb{R}/\mathbb{Z}$  such that  $\gamma \circ \sigma = \tilde{\gamma} + \phi$ . □

The representation of  $\gamma$  by a normal graph  $\phi$  over  $\tilde{\gamma}$  we obtain from Lemma 2.12 satisfies the following  $C^1$  estimates. These enable us to control the second derivative of  $\phi$  by the curvature of  $\gamma$  using Lemma 2.8.

**Corollary 2.13.** *For the decomposition (2.10), there exists a constant  $C > 0$  depending on an upper bound  $M$  on  $\|\gamma'\|_{W^{1,p}}$  and  $p$  such that*

$$\|\phi\|_{L^\infty} \leq C\|\gamma - \tilde{\gamma}\|_{L^\infty} \quad \text{and} \quad \|\phi'\|_{L^\infty} \leq C(1 + \|\gamma' - \tilde{\gamma}'\|_{L^\infty}).$$

*Proof.* From the construction of  $\sigma$  given by (2.9), we see that

$$|\sigma(x) - x| = |\sigma(x) - \sigma \circ \sigma^{-1}(x)| \leq \|\sigma'\|_{L^\infty} |x - \sigma^{-1}(x)|$$

and

$$|x - \sigma^{-1}(x)| = |\Pi_{x_j}(\tilde{\gamma}(x)) - \Pi_{x_j}(\gamma(x))| \leq \left(\max_j \|D\Pi_{x_j}\|_{L^\infty}\right) |\tilde{\gamma}(x) - \gamma(x)|,$$

since there exists some ball  $B_\delta(x_j)$  such that  $x = \Pi_{x_j}(\tilde{\gamma}(x))$ . As we have

$$|\phi(x)| = |\gamma(\sigma(x)) - \tilde{\gamma}(x)| \leq |\gamma(\sigma(x)) - \gamma(x)| + |\gamma(x) - \tilde{\gamma}(x)|$$

and  $|\gamma(\sigma(x)) - \gamma(x)| \leq \|\gamma'\|_{L^\infty} |\sigma(x) - x|$ , it follows that

$$\|\phi\|_{L^\infty} \leq (1 + C\|\gamma'\|_{W^{1,p}})\|\gamma - \tilde{\gamma}\|_{L^\infty}$$

by the Sobolev embeddings. In addition, we have

$$\begin{aligned} \|\phi'\|_{L^\infty} &= \|(\gamma \circ \sigma)' - \tilde{\gamma}'\|_{L^\infty} \leq \|\gamma' - \tilde{\gamma}'\|_{L^\infty} + \|\gamma'\|_{L^\infty} + \|(\gamma \circ \sigma)'\|_{L^\infty} \\ &\leq \|\gamma' - \tilde{\gamma}'\|_{L^\infty} + C\|\gamma'\|_{W^{1,p}} \end{aligned}$$

from the uniform bi-Lipschitz property of  $\sigma$  and the Sobolev embeddings. □

### 2.3. Existence of discrete-time approximations

After breaking the reparametrisation invariance of (1.1) by way of the approximate normal graphs, it is now a straight forward matter to prove the short-time existence of solutions for the minimising movement scheme.

Let us first consider an initial curve  $\Gamma \in W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  of length  $L$  parametrised by arc-length. In the following, it will be essential that all estimates only depend on an upper bound on the energy of this curve.

We first note that an upper bound on the energy also implies a lower bound on the length, since by Fenchel's theorem together with Hölder's inequality we have

$$2\pi \leq \int_{\mathbb{R}/L\mathbb{Z}} |\kappa| ds \leq L^{1-\frac{1}{p}} \left( \int_{\mathbb{R}/L\mathbb{Z}} |\kappa|^p ds \right)^{\frac{1}{p}}$$

so that

$$L^{p-1} \geq \frac{(2\pi)^p}{p E^{(p)}(\Gamma)}.$$

By scaling the results of Sect. 2.2, we can drop the assumption that the curve is of unit length and recover all the previous estimates concerning approximate normal graphs (with proviso that the relevant constants now depend on  $\lambda$  and the energy bound). In particular, we say that the unit vector field  $\tau$  is quasi-tangent to a  $W^{2,p}$ -curve  $\gamma$  of length  $L$  whenever  $\tau(\frac{\cdot}{L})$  is quasi-tangent to the curve  $\gamma(\frac{\cdot}{L})$ .

Now for the initial curve, the result of Lemma 2.12 implies that there exists a smooth curve  $\tilde{\gamma}$  parametrised by arc-length, a unit quasi-tangent  $\tau$  to the curve  $\tilde{\gamma}$  and a perturbation  $\Phi \in (W^{2,p})^\perp_\tau$  such that  $\Gamma \circ \sigma = \tilde{\gamma} + \Phi$ . Moreover, by combining the

norm bounds of Lemma 2.12 with Corollary 2.13 and the Sobolev embeddings, we see that

$$\|\Phi\|_{L^\infty} \leq \mu \quad \text{and} \quad \|\Phi'\|_{L^\infty} \leq W$$

for some sufficiently small constant  $\mu = \mu(p, \lambda, E(\Gamma)) > 0$  and some constant  $W = W(p, \lambda, E(\Gamma)) > 2$ .

For a series of discrete time steps  $0 = t_0 < t_1 < t_2 < \dots$ , we seek to define the curves

$$\gamma_{t_j} = \tilde{\gamma} + \phi_{t_j} \tag{2.11}$$

with the initial case  $\gamma_{t_0} = \tilde{\gamma} + \Phi$ . The time differences  $t_{j+1} - t_j = h$  are set to be equal to a fixed parameter  $h > 0$  (that we shall ultimately send to zero). We want to recursively define  $\phi_{t_{j+1}}$  for the next time step as the minimiser

$$\phi_{t_{j+1}} = \operatorname{argmin}_{\phi \in \mathcal{V}} \left\{ E(\tilde{\gamma} + \phi) + \frac{1}{2h} \int_{\mathbb{R}/L\mathbb{Z}} |P_{\gamma'_{t_j}}^\perp(\tilde{\gamma} + \phi - \gamma_{t_j})|^2 |\gamma'_{t_j}| dx \right\},$$

where the class of admissible perturbations is given by

$$\mathcal{V} = \mathcal{V}(\mu, W) = \{ \phi \in (W^{2,p})^\perp_\tau : \|\phi\|_{L^\infty} < 3\mu, \|\phi'\|_{L^\infty} < 3W \}.$$

The following lemma states that these discrete-time solutions can be constructed for at least a short time.

**Lemma 2.14.** *There exists a finite time  $T > 0$  depending only on  $p, \lambda$  and  $E(\Gamma)$  such that the solutions  $\gamma_{t_j} = \tilde{\gamma} + \phi_{t_j}$  exist for the series of discrete times  $0 = t_0 < t_1 < t_2 < \dots < t_N \leq T$  where  $N = \lfloor \frac{T}{h} \rfloor$ .*

*Proof.* We seek to establish the existence of the perturbations  $\phi_{t_{j+1}}$  that are minimisers of the functionals

$$\mathcal{F}_j(\phi) = E(\tilde{\gamma} + \phi) + \frac{1}{2h} \int_{\mathbb{R}/L\mathbb{Z}} |P_{\gamma'_{t_j}}^\perp(\tilde{\gamma} + \phi - \gamma_{t_j})|^2 |\gamma'_{t_j}| dx$$

over the admissible class  $\mathcal{V}$ . To do so, we proceed by an induction argument with an initial base case  $\phi_{t_0} = \Phi$  given by the decomposition of the initial curve  $\Gamma$ . Indeed, let us assume there exist minimisers  $\phi_{t_{i+1}}$  of  $\mathcal{F}_i$  over the class  $\mathcal{V}$  for  $i = 0, 1, \dots, j - 1$ .

Now as  $\mathcal{F}_i(\phi_{t_{i+1}}) \leq \mathcal{F}_i(\phi_{t_i})$  for  $i = 0, 1, \dots, j - 1$  (i.e.  $\phi_{t_i}$  is a competitor), we note that

$$E(\gamma_{t_j}) \leq E(\gamma_{t_0}) = E(\Gamma)$$

and

$$\frac{1}{2h} \int_{\mathbb{R}/L\mathbb{Z}} |P_{\gamma'_{t_i}}^\perp(\gamma_{t_{i+1}} - \gamma_{t_i})|^2 |\gamma'_{t_i}| dx \leq E(\gamma_{t_i}) - E(\gamma_{t_{i+1}}).$$

In which case Lemma 2.8 implies that the  $L^p$ -norm of  $\gamma''_{t_j}$  is uniformly bounded by a constant which depends only on  $p$  and  $E(\Gamma)$ . In addition, we have

$$\frac{1}{h} \int_{\mathbb{R}/L\mathbb{Z}} |\gamma_{t_{i+1}} - \gamma_{t_i}|^2 dx \leq C(E(\gamma_{t_i}) - E(\gamma_{t_{i+1}})).$$

Then by summing up the latter inequalities, we get the a priori estimate

$$\sum_{i=0}^{j-1} \frac{1}{h} \int_{\mathbb{R}/L\mathbb{Z}} |\gamma_{t_{i+1}} - \gamma_{t_i}|^2 dx \leq C(E(\gamma_{t_0}) - E(\gamma_{t_j})). \tag{2.12}$$

We also recall from Hölder’s inequality that

$$\begin{aligned} \|\gamma_{t_0} - \gamma_{t_j}\|_{L^2} &\leq \sum_{i=0}^{j-1} \frac{\|\gamma_{t_{i+1}} - \gamma_{t_i}\|_{L^2}}{\sqrt{h}} \sqrt{h} \\ &\leq \left( \sum_{i=0}^{j-1} \frac{1}{h} \int_{\mathbb{R}/L\mathbb{Z}} |\gamma_{t_{i+1}} - \gamma_{t_i}|^2 dx \right)^{\frac{1}{2}} \left( \sum_{i=0}^{j-1} h \right)^{\frac{1}{2}} \\ &\leq C \sqrt{E(\gamma_{t_0})} \sqrt{t_j} \end{aligned} \tag{2.13}$$

and from the Gagliardo–Nirenberg interpolation inequality we get

$$\|\gamma'_{t_0} - \gamma'_{t_j}\|_{L^\infty} \leq C \|\gamma''_{t_0} - \gamma''_{t_j}\|_{L^p}^\alpha \|\gamma_{t_0} - \gamma_{t_j}\|_{L^2}^{1-\alpha}$$

with  $\alpha = \frac{3p}{5p-2}$ . Since Lemma 2.8 implies that the  $L^p$ -norm of the second derivatives of  $\gamma_0$  and  $\gamma_{t_j}$  are uniformly bounded, we conclude that

$$\|\gamma'_{t_0} - \gamma'_{t_j}\|_{L^\infty} \leq C(\sqrt{t_j})^{1-\alpha}$$

for a constant  $C > 0$  depending on  $p, \lambda$  and  $E(\Gamma)$ . Furthermore, there exists a sufficiently small  $T > 0$  depending on  $p, \lambda$  and  $E(\Gamma)$  such that

$$\begin{aligned} \|\phi'_{t_j}\|_{L^\infty} &\leq \|\phi'_{t_0}\|_{L^\infty} + \|\gamma'_{t_j} - \gamma'_{t_0}\|_{L^\infty} \\ &\leq W + C(\sqrt{t_j})^{1-\alpha} \\ &< 2W \end{aligned} \tag{2.14}$$

whenever  $0 < t_j < T$ . Since

$$\|\gamma_{t_0} - \gamma_{t_j}\|_{L^\infty} \leq C \|\gamma''_{t_0} - \gamma''_{t_j}\|_{L^p}^\beta \|\gamma_{t_0} - \gamma_{t_j}\|_{L^2}^{1-\beta}$$

with  $\beta = \frac{p}{5p-2}$  by the Gagliardo–Nirenberg interpolation inequality, we also have

$$\|\phi_{t_j}\|_{L^\infty} < 2\mu \tag{2.15}$$

whenever  $0 < t_j < T$ .

In fact, we can show that the same estimates hold for a suitably chosen minimising sequence. Let us assume that  $(\phi_n)$  is a minimising sequence for the functional  $\mathcal{F}_j$  in the class  $\mathcal{V}$ , i.e.  $\mathcal{F}_j(\phi_n) \rightarrow \inf_{\phi \in \mathcal{V}} \mathcal{F}_j(\phi)$  and note that  $\mathcal{F}_j$  is bounded from below by construction. As  $\phi_{t_j}$  is still a competitor, we can assume without loss of generality that

$$\mathcal{F}_j(\phi_n) \leq \mathcal{F}_j(\phi_{t_j}) = E(\gamma_{t_j}) \leq E(\gamma_{t_0})$$

for all  $n \in \mathbb{N}$ . In which case we can repeat the argument from the above to obtain the bound

$$\|\gamma'_{t_0} - \gamma'_n\|_{L^\infty} \leq C(\sqrt{t_{j+1}})^{1-\alpha}$$

with  $\gamma_n = \tilde{\gamma} + \phi_n$ . It then follows that

$$\|\phi'_{t_n}\|_{L^\infty} < 2W \tag{2.16}$$

for all  $0 < t_{j+1} < T$ . We then use the Gagliardo–Nirenberg interpolation inequality to obtain as above

$$\|\phi_{t_n}\|_{L^\infty} < 2\mu. \tag{2.17}$$

*Compactness.* As a consequence of Lemma 2.8, the minimising sequence  $(\phi_n)$  is uniformly bounded in  $W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ . It then follows that there exists a weakly converging subsequence in  $W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  which we also denoted by  $(\phi_n)$ . In addition, the Rellich–Kondrašov compactness theorem implies that the subsequence  $(\phi_n)$  is strongly convergent in  $C^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ . Let us denote the limit of this sequence by  $\phi$ . Since we have already established that  $\|\phi_n\|_{L^\infty} < 2\mu$  and  $\|\phi'_n\|_{L^\infty} < 2W$ , it follows that  $\|\phi\|_{L^\infty} \leq 2\mu$  and  $\|\phi'\|_{L^\infty} \leq 2W$ . Therefore, the limit  $\phi$  also belongs to  $\mathcal{V}$ .

*Lower semi-continuity.* Let us finally prove that

$$\mathcal{F}_j(\phi) \leq \liminf_{n \rightarrow \infty} \mathcal{F}_j(\phi_n).$$

As the  $L^2$ -term in the functional  $\mathcal{F}_j$  converges by the theorem of Rellich–Kondrašov and the angle between  $\tau$  and  $\gamma'_{t_j}$  is uniformly bounded strictly away from  $\frac{\pi}{2}$ , it suffices to show that

$$E^{(p)}(\tilde{\gamma} + \phi) \leq \liminf_{n \rightarrow \infty} E^{(p)}(\tilde{\gamma} + \phi_n). \tag{2.18}$$

Note that the length term  $\lambda \int_{\mathbb{R}/L\mathbb{Z}} ds$  appearing in the considered energy  $E$ , cf. (1.1), can be dropped as well due to the convergence of the sequence  $(\phi_n)$  in  $C^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ . In order to prove (2.18), we use the curvature formula for  $\kappa_{\tilde{\gamma}+\phi_n}$  to rewrite

$$E^{(p)}(\tilde{\gamma} + \phi_n) = \int_{\mathbb{R}/L\mathbb{Z}} \frac{\left| P_{\tilde{\gamma}'+\phi'_n}^\perp(\tilde{\gamma}'' + \phi_n'') \right|^p}{|\tilde{\gamma}' + \phi'_n|^{2p}} |\tilde{\gamma}' + \phi'_n| ds$$

as the expression

$$E^{(p)}(\gamma_n) = \int_{\mathbb{R}/L\mathbb{Z}} \frac{|P_{\tilde{\gamma}'+\phi'}^\perp(\tilde{\gamma}'' + \phi_n'')|^p}{|\tilde{\gamma}' + \phi'|^{2p}} |\tilde{\gamma}' + \phi'| ds + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where

$$\begin{aligned} \mathcal{I}_1 &= \int_{\mathbb{R}/L\mathbb{Z}} \left( |P_{\tilde{\gamma}'+\phi_n'}^\perp(\tilde{\gamma}'' + \phi_n'')|^p - |P_{\tilde{\gamma}'+\phi'}^\perp(\tilde{\gamma}'' + \phi_n'')|^p \right) \frac{|\tilde{\gamma}' + \phi_n'|}{|\tilde{\gamma}' + \phi_n'|^{2p}} ds, \\ \mathcal{I}_2 &= \int_{\mathbb{R}/L\mathbb{Z}} \left( \frac{1}{|\tilde{\gamma}' + \phi_n'|^{2p}} - \frac{1}{|\tilde{\gamma}' + \phi'|^{2p}} \right) |P_{\tilde{\gamma}'+\phi'}^\perp(\tilde{\gamma}'' + \phi_n'')|^p |\tilde{\gamma}' + \phi_n'| ds, \\ \mathcal{I}_3 &= \int_{\mathbb{R}/L\mathbb{Z}} \frac{|P_{\tilde{\gamma}'+\phi'}^\perp(\tilde{\gamma}'' + \phi_n'')|^p}{|\tilde{\gamma}' + \phi'|^{2p}} \left( |\tilde{\gamma}' + \phi_n'| - |\tilde{\gamma}' + \phi'| \right) ds. \end{aligned}$$

The terms  $\mathcal{I}_1$ ,  $\mathcal{I}_2$  and  $\mathcal{I}_3$  vanish in the limit due to the convergence of the sequence  $(\phi_n)$  in  $C^1(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  and the uniform bound on the  $W^{2,p}$ -norm of  $\phi_n$ . Moreover, the expression

$$\mathcal{I}(\tilde{\gamma} + \phi_n) = \int_{\mathbb{R}/L\mathbb{Z}} \frac{|P_{\tilde{\gamma}'+\phi'}^\perp(\tilde{\gamma}'' + \phi_n'')|^p}{|\tilde{\gamma}' + \phi'|^{2p}} |\tilde{\gamma}' + \phi'| ds$$

is convex and continuous on  $W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  and hence lower semi-continuous by the following standard argument: Mazur’s lemma [25, Theorem 3.13] gives for every  $n_0 \in \mathbb{N}$  a sequence of convex combinations

$$P^l = \sum_{n=n_0}^l \alpha_n^l \phi_n, \quad 0 \leq \alpha_n^l \leq 1, \quad \sum_{n=n_0}^l \alpha_n^l = 1$$

such that  $P^l \rightarrow \phi$  strongly in  $W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ . The convexity of  $\mathcal{I}$  now implies

$$\mathcal{I}(\tilde{\gamma} + P^l) = \mathcal{I}\left(\sum_{n=n_0}^l \alpha_n^l (\tilde{\gamma} + \phi_n)\right) \leq \sum_{n=n_0}^l \alpha_n^l \mathcal{I}(\tilde{\gamma} + \phi_n) \leq \sup_{n \geq n_0} \mathcal{I}(\tilde{\gamma} + \phi_n).$$

Passing to the limit  $l \rightarrow \infty$  on the left-hand side, we get from the continuity of  $\mathcal{I}$  and  $P^l \rightarrow \phi$  in  $W^{2,p}$  that

$$\mathcal{I}(\tilde{\gamma} + \phi) \leq \sup_{n \geq n_0} \mathcal{I}(\tilde{\gamma} + \phi_n)$$

for all  $n_0 \in \mathbb{N}$ . This yields

$$\mathcal{I}(\tilde{\gamma} + \phi) \leq \inf_{n_0 \in \mathbb{N}} \sup_{n \geq n_0} \mathcal{I}(\tilde{\gamma} + \phi_n) = \liminf_{n \rightarrow \infty} \mathcal{I}(\tilde{\gamma} + \phi_n).$$

□

For later reference, let us also state the following a priori estimate for the piecewise linear interpolations that results from (2.12) and (2.13).

**Corollary 2.15.** *The piecewise linear interpolations*

$$\phi^{(h)}(t, \cdot) = \phi_{t_j} + \frac{t - t_j}{h}(\phi_{t_{j+1}} - \phi_{t_j}), \quad t_j \leq t \leq t_{j+1},$$

satisfies the estimates

$$\|\phi_{t''}^{(h)} - \phi_{t'}^{(h)}\|_{L^2} \leq C\sqrt{t'' - t'}$$

and

$$\int_{t'}^{t''} \int_{\mathbb{R}/L\mathbb{Z}} |\partial_t \phi^{(h)}(t, s)|^2 ds dt \leq C(E(\gamma_{t'}) - E(\gamma_{t''}))$$

for any  $0 \leq t' < t'' < T < \infty$ .

*Remark 2.16.* We thus obtain a piecewise linearly interpolated solution

$$\gamma_t^{(h)} = \tilde{\gamma} + \phi_t^{(h)}, \quad 0 \leq t < T, \tag{2.19}$$

for the minimising movements scheme.

### 3. Weak solutions

#### 3.1. Euler–Lagrange equations for the approximations

In order to improve the regularity of the approximations, we derive the Euler–Lagrange equations related to the minimising movement scheme.

We recall the following expression (cf. [9, Lemma 2.1]) for the first variation of the  $p$ -elastic energy, namely

$$\delta_\psi E^{(p)}(\gamma) = \int_{\mathbb{R}/L\mathbb{Z}} |\kappa|^{p-2} \langle \kappa, \delta_\psi \kappa \rangle ds + \frac{1}{p} \int_{\mathbb{R}/L\mathbb{Z}} |\kappa|^p \langle \partial_s \gamma, \partial_s \psi \rangle ds \tag{3.1}$$

where  $\delta_\psi \kappa = (\partial_s^2 \psi)^\perp - \langle \kappa, \partial_s \psi \rangle \partial_s \gamma - 2 \langle \partial_s \gamma, \partial_s \psi \rangle \kappa$ , cf. Proposition A.1. The first variation of the length term appearing in the definition of the energy  $E$ , cf. (1.1), is given by

$$\delta_\psi \left( \lambda \int_{\mathbb{R}/L\mathbb{Z}} ds \right) = \lambda \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_s \gamma, \partial_s \psi \rangle ds. \tag{3.2}$$

Combining (3.1) and (3.2) with the fact that

$$\partial_s \psi = \frac{1}{|\gamma'|} \partial_x \psi,$$



where  $|\gamma'| = |\partial_x \gamma|$ , we get

$$\partial_s^2 \psi = \frac{1}{|\gamma'|} \partial_x \left( \frac{1}{|\gamma'|} \partial_x \psi \right) = \frac{1}{|\gamma'|^2} \partial_x^2 \psi - \frac{1}{|\gamma'|^3} \left\langle \frac{\gamma'}{|\gamma'|}, \gamma'' \right\rangle \partial_x \psi$$

so that

$$\delta_\psi E(\gamma) = \int_{\mathbb{R}/L\mathbb{Z}} \frac{|\kappa|^{p-2}}{|\gamma'|} \langle \kappa, \partial_x^2 \psi \rangle dx + R(\psi),$$

where  $R(\psi)$  has the form

$$R(\psi) = \int_{\mathbb{R}/L\mathbb{Z}} \langle b, \partial_x \psi \rangle dx$$

for

$$b = \frac{|\kappa|^{p-2}}{|\gamma'|^3} \langle \gamma', \gamma'' \rangle \kappa - |\kappa|^{p-2} \langle \kappa, \gamma' \rangle \kappa - \left( 2 - \frac{1}{p} \right) |\kappa|^p \gamma'.$$

Note that  $b \in L^\infty L^1$  for any time-dependent family of curves  $\gamma \in L^\infty W^{2,p}$ , whereas a notational shorthand  $L^\infty L^1$  stands for  $L^\infty([0, T], L^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ .

On the other hand, solutions of the minimising movement scheme solve

$$\langle \partial_t \gamma, P_\tau^\perp \psi \rangle = -\delta_\psi E(\gamma) \tag{3.3}$$

for all  $\psi \in (W^{2,p})_\tau^\perp$ . Therefore, we conclude that

$$\int_{\mathbb{R}/L\mathbb{Z}} \frac{|\kappa|^{p-2}}{|\gamma'|} \langle \kappa, \partial_x^2 \psi \rangle dx + \tilde{R}(\psi) = 0, \tag{3.4}$$

where

$$\tilde{R}(\psi) = \int_{\mathbb{R}/L\mathbb{Z}} \langle b, \partial_x \psi \rangle dx + \int_{\mathbb{R}/L\mathbb{Z}} \langle P_\tau^\perp(\partial_t \gamma), \psi \rangle dx.$$

### 3.2. Higher regularity for the approximations

To deduce regularity from the equation above, we consider a smooth local orthonormal basis  $v_1, \dots, v_{n-1}$  for our approximate normal spaces. If  $\psi$  is a test function that is decomposed into the form

$$\psi = \sum_{i=1}^{n-1} \psi_i v_i$$

such that the scalar functions  $\psi_i$  vanish away from the neighbourhood, we find that

$$\partial_x^2 \psi = \sum_{i=1}^{n-1} \left( \partial_x^2 \psi_i v_i + 2\partial_x \psi_i \partial_x v_i + \psi_i \partial_x^2 v_i \right).$$

Therefore, the evolution equation for the approximation yields

$$\sum_{i=1}^{n-1} \int_{\mathbb{R}/L\mathbb{Z}} \frac{|\kappa|^{p-2}}{|\gamma'|} \partial_x^2 \psi_i \langle \kappa, P_\tau^\perp v_i \rangle dx = Q(h), \tag{3.5}$$

where

$$Q(h) = \int_{\mathbb{R}/L\mathbb{Z}} \langle b_t, \partial_x \psi \rangle + \langle c_t, \psi \rangle + \langle P_\tau^\perp (\partial_t \gamma(t, \cdot)), \psi \rangle dx.$$

The following lemma helps us to deduce regularity from this form of the equation.

**Lemma 3.1.** (*L<sup>1</sup>-estimates*) *Let  $I = (a, b)$  be an open subset of  $\mathbb{R}$ . If there exist functions  $u, f$  and  $F$  in  $L^1(I)$  such that*

$$\int_I (u \partial_x^2 \varphi + F \partial_x \varphi) dx = \int_I f \varphi dx$$

for all  $\varphi \in C_c^\infty(I)$ , then

$$u(x) = \int_a^x \left( F(y) + \int_a^y f(z) dz \right) dy + m(x - a) + d$$

with  $d = \lim_{x \searrow a} u(x)$  and

$$m(b - a) = \lim_{x \nearrow b} u(x) - \left( \int_I \left( F(y) + \int_a^y f(z) dz \right) dy + d \right).$$

Moreover, the function  $u \in W^{1,1}(I)$  with

$$\|u\|_{W^{1,1}} \leq C(\|f\|_{L^1} + \|F\|_{L^1}).$$

*Proof.* Let us first set

$$\begin{aligned} w(x) &= F(x) + \int_a^x f(y) dy \\ v(x) &= \int_a^x w(y) dy \end{aligned}$$

and note that  $v \in W^{1,1}(I)$  with  $v' = w$ . Then integration by parts implies that

$$\begin{aligned} \int_I v(x) \partial_x^2 \varphi(x) dx &= - \int_I v'(x) \partial_x \varphi(x) dx \\ &= - \int_I \left( F(x) \partial_x \varphi(x) + \left( \int_a^x f(y) dy \right) \partial_x \varphi(x) \right) dx \\ &= - \int_I (F(x) \partial_x \varphi(x) - f(x) \varphi(x)) dx. \end{aligned}$$

Therefore,

$$\int_I (u - v) \partial_x^2 \varphi dx = 0$$

for all  $\varphi \in C_c^\infty(I)$ . In which case  $u - v$  is an affine function from which the conclusion easily follows. □

We can now use the latter lemma to establish:

**Theorem 3.2.** (Higher regularity) *If  $\gamma_t^{(h)}$  is a solution to the minimising movements scheme given by (2.19), there exists a constant  $C > 0$  independent of  $h$  such that*

$$\left\| |\kappa|^{p-2} P_\tau^\perp \kappa \right\|_{L^2([0,T], W^{1,1})} \leq C. \tag{3.6}$$

*In particular, we have  $\kappa$  uniformly bounded in  $L^2 L^q$  and  $\gamma'$  uniformly bounded in  $L^2([0, T], W^{1,q})$  for all  $1 \leq q < \infty$ .*

*Proof.* This higher regularity result directly follows from the application of Lemma 3.1 to our evolution equation for the minimising movement scheme approximations. In particular, from Corollary 2.15 we see that  $\gamma_t^{(h)}$  satisfies

$$\int_0^T \int_{\mathbb{R}/L\mathbb{Z}} |\partial_t \gamma^{(h)}(t, s)|^2 ds dt \leq CE(\gamma_0).$$

Applying Lemma 2.8 to (3.5) together with a covering argument hence yields

$$\left\| \frac{|\kappa|^{p-2}}{|(\gamma^{(h)})'|} P_\tau^\perp \kappa \right\|_{L^2([0,T], W^{1,1})} < C.$$

Since  $(\gamma^{(h)})'$  is uniformly bounded in  $W^{1,1}$  and  $W^{1,1}$  is a Banach algebra, this implies

$$\| |\kappa|^{p-2} P_\tau^\perp \kappa \|_{L^2([0,T], W^{1,1})} < C.$$

□

### 3.3. Convergence to weak solutions

We will use the following result in order to obtain the convergence of solutions. This result is crucial for the control of the terms involving the energy.

**Theorem 3.3.** *Let  $\gamma_n = \gamma + \phi_n$  be a sequence bounded in  $L^\infty W^{2,p} \cap C^{\frac{1}{2}} L^2$  such that  $|\kappa_n|^{p-2} \kappa_n$  is uniformly bounded in  $L^2 W^{1,1}$ . Then there exists a subsequence  $\gamma_{n_j}$  such that the curvatures  $\kappa_{n_j}$  converge in  $L^2 W^{2,p}$ .*

The proof of this theorem relies on the following interpolation estimate.

**Lemma 3.4.** *There exists a constant  $C_0 > 0$  depending on  $p$  such that for any  $W^{2,p}$ -curves  $\gamma_1$  and  $\gamma_2$  with curvatures  $\kappa_1$  and  $\kappa_2$  we have*

$$\|\kappa_1 - \kappa_2\|_{L^p} \leq C_0 (\| |\kappa_1|^{p-2} \kappa_1 \|_{L^2 W^{1,1}} + \| |\kappa_2|^{p-2} \kappa_2 \|_{L^2 W^{1,1}}) \|\gamma_1' - \gamma_2'\|_{L^2 L^\infty}.$$

*If these curves are furthermore approximate normal graphs over  $\tilde{\gamma}$  as for the solutions to the minimising movement scheme, we get*

$$\begin{aligned} & \| \kappa_1 - \kappa_2 \|_{L^p} \\ & \leq C_0 (\| |\kappa_1|^{p-2} P_\tau^\perp \kappa_1 \|_{L^2 W^{1,1}} + \| |\kappa_2|^{p-2} P_\tau^\perp \kappa_2 \|_{L^2 W^{1,1}}) \|\gamma_1' - \gamma_2'\|_{L^2 L^\infty} \end{aligned}$$

where now  $C = C(\lambda, p, E(\Gamma))$ .

*Proof.* First note that

$$\int |\kappa_1 - \kappa_2|^p ds \leq C_0 \int \left( |\kappa_1|^{p-2} \kappa_1 - |\kappa_2|^{p-2} \kappa_2 \right) (\kappa_1 - \kappa_2) ds$$

(cf. [8, §1, Lemma 4.4]). Then integration by parts and Hölder’s inequality implies that

$$\begin{aligned} \int |\kappa_1 - \kappa_2|^p ds &\leq -C_0 \int \partial_s \left( |\kappa_1|^{p-2} \kappa_1 - |\kappa_2|^{p-2} \kappa_2 \right) (\partial_s \gamma_1 - \partial_s \gamma_2) ds \\ &\leq C_0 \left( \left\| |\kappa_1|^{p-2} \kappa_1 \right\|_{W^{1,1}} + \left\| |\kappa_2|^{p-2} \kappa_2 \right\|_{W^{1,1}} \right) \|\gamma'_1 - \gamma'_2\|_{L^\infty}. \end{aligned}$$

So by integrating over time and using Hölder’s inequality again, we get

$$\begin{aligned} \iint |\kappa_1 - \kappa_2|^p ds dt &\leq C_0 \left( \left\| |\kappa_1|^{p-2} \kappa_1 \right\|_{L^2 W^{1,1}} + \left\| |\kappa_2|^{p-2} \kappa_2 \right\|_{L^2 W^{1,1}} \right) \|\gamma'_1 - \gamma'_2\|_{L^2 L^\infty}. \end{aligned}$$

For the second estimate, we proceed in a similar way. We apply Lemma 2.8 to improve the first inequality to

$$\int |\kappa_1 - \kappa_2|^p ds \leq C_0 \int \left( |\kappa_1|^{p-2} P_\tau^\perp \kappa_1 - |\kappa_2|^{p-2} P_\tau^\perp \kappa_2 \right) (\kappa_1 - \kappa_2) ds.$$

Integrating by parts then yields

$$\begin{aligned} \int |\kappa_1 - \kappa_2|^p ds &\leq -C_0 \int \partial_s \left( |\kappa_1|^{p-2} P_\tau^\perp \kappa_1 - |\kappa_2|^{p-2} P_\tau^\perp \kappa_2 \right) (\partial_s \gamma_1 - \partial_s \gamma_2) ds \\ &\leq C_0 \left( \left\| |\kappa_1|^{p-2} P_\tau^\perp \kappa_1 \right\|_{W^{1,1}} + \left\| |\kappa_2|^{p-2} P_\tau^\perp \kappa_2 \right\|_{W^{1,1}} \right) \|\gamma'_1 - \gamma'_2\|_{L^\infty}. \end{aligned}$$

□

*Proof of Theorem 3.3.* Using a diagonal argument and the compact embedding  $W^{2,p} \hookrightarrow L^2$ , we get a subsequence  $\gamma_{n_j}$  converging in  $L^2$  for all times  $t \in \mathbb{Q} \cap [0, T)$  (and hence for all  $0 \leq t < T$  due to the uniform bound in  $C^{\frac{1}{2}} L^2$ ). This result, together with the uniform bound on the  $W^{2,p}$ -Sobolev norm and interpolation estimates, implies that  $\gamma_{n_j} \rightarrow \gamma \in C^\alpha([0, T), W^{1,\infty})$  with  $\alpha = \frac{p-1}{5p-2}$ . Thus,  $\gamma_{n_j}$  converge to  $\gamma$  in  $L^2 W^{2,p}$  by Lemma 3.4. □

*Proof of Theorem 1.1.* From the construction in Sect. 2.3, there exists a solution  $\gamma_t^{(h)}$  to the minimising movement scheme given by (2.19) for all  $0 \leq t < T$  up to some positive final time  $T$  that depends only on  $p, \lambda$  and the energy  $E(\Gamma)$  of the initial data. We think of this solution as solving a discrete version of the negative  $L^2$ -gradient flow of  $E$ . Theorem 3.3 and Corollary 2.15 can then be applied to get a subsequence that converges in  $L^2 W^{2,p}$  such that  $\partial_t \gamma^{(h)}$  weakly converges in  $L^2$ . Now in order to show

that the limit satisfies the desired evolution equations, we use the fact that the solutions of the minimising movement scheme satisfy

$$\int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_t^\perp \gamma_t^{(h)}, \psi \rangle ds dt = \int_{\mathbb{R}/L\mathbb{Z}} \delta_{\psi_t} E(\tilde{\gamma}_t^{(h)}) dt \tag{3.7}$$

for all test functions  $\psi \in C_c^\infty((0, T) \times \mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ . Here,  $\tilde{\gamma}_t^{(h)} = \gamma_{nh}^{(h)}$  for  $t \in [nh, (n+1)h)$ ,  $n \in \mathbb{N}$ , denotes the piecewise constant interpolation of the minimising movement scheme.

Let us now take a sequence  $h_n \rightarrow 0$  for which the piecewise linear interpolations of the minimising movement scheme  $\gamma^{(h_n)}$  converge to a family of curves  $\gamma$  in  $L^2 W^{2,p}$  such that  $\partial_t \gamma^{(h_n)}$  converges to  $\partial_t \gamma$  weakly in  $L^2([0, T], \mathbb{R}/L\mathbb{Z})$ . Then also the piecewise constant interpolations  $\tilde{\gamma}^{(h_n)}$  converge to  $\gamma$  in  $L^2 W^{2,p}$ . As  $\gamma^{(h_n)}$  converges strongly to  $\gamma'$  in  $L^2$ , we see that the weak convergence of  $\partial_t \gamma^{(h_n)}$  to  $\partial_t \gamma$  in  $L^2$  implies

$$\int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_t^\perp \gamma_t^{(h_n)}, \psi \rangle ds dt \rightarrow \int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_t^\perp \gamma_t, \psi \rangle ds dt. \tag{3.8}$$

Convergence for the right-hand side of (3.7) is also straight forward. If we denote by  $\kappa_n$  the curvature of  $\tilde{\gamma}_t^{(h_n)}$  and integrate (3.1), we find that

$$\begin{aligned} \int_0^T \delta_{\psi_t} E(\tilde{\gamma}_t^{(h_n)}) dt &= \int_0^T \int_{\mathbb{R}/L\mathbb{Z}} |\kappa_n|^{p-2} \langle \kappa_n, \delta_\psi \kappa_n \rangle ds dt \\ &\quad + \frac{1}{p} \int_0^T \int_{\mathbb{R}/L\mathbb{Z}} |\kappa_n|^p \langle \partial_s \tilde{\gamma}^{(h_n)}, \partial_s \psi \rangle ds dt \\ &\quad + \lambda \int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_s \tilde{\gamma}^{(h_n)}, \partial_s \psi \rangle ds dt. \end{aligned}$$

Since  $\kappa_n$  converges to  $\kappa$  in  $L^2([0, T], L^p(\mathbb{R}/L\mathbb{Z}))$  and  $\partial_s \tilde{\gamma}^{(h_n)}$  converges to  $\partial_s \gamma$  uniformly, the second term on the right-hand side of the latter equation converges to the corresponding term for  $\gamma$  in lieu of  $\tilde{\gamma}^{(h_n)}$ . One can deduce the same fact for the first term via the formula

$$\delta_\psi \kappa_n = \left( \partial_s^2 \psi \right)^\perp - \langle \kappa_n, \partial_s \psi \rangle \tau_n - \langle \partial_s \psi, \tau_n \rangle \kappa_n,$$

since it implies that  $\delta_\psi \kappa_n$  converges to  $\delta_\psi \kappa$  in  $L^2([0, T], L^p(\mathbb{R}/L\mathbb{Z}))$ . Therefore, we get

$$\int_0^T \delta_{\psi_t} E(\tilde{\gamma}^{(h_n)}) dt \rightarrow \int_0^T \delta_{\psi_t} E(\gamma) dt. \tag{3.9}$$

In which case Eqs. (3.7), (3.8) and (3.9) imply that

$$\int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \gamma_t, \partial_t^\perp \psi \rangle ds dt = - \int_0^T \delta_{\psi_t} E(\gamma_t) dt.$$

□

### 3.4. Flow in the direction of the normal velocity

Using the fact that the unit tangent belongs to  $W^{2,p}$ , we can finally prove Corollary 1.2 under the conditions of Theorem 1.1.

*Proof of Corollary 1.2.* In abuse of notation, let  $\tau = \frac{\gamma'}{|\gamma'|} \in W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  be the unit tangent and the vectors  $v_1, \dots, v_{n-1}$  be a smooth local orthonormal basis of our approximate normal space. Due to the fact that any  $\psi \in C_c^\infty(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$  can be written as

$$\psi = \psi_0 \tau + \sum_{i=1}^{n-1} \psi_i v_i$$

with functions  $\psi_i \in W^{2,p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n)$ , we find that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_t^\perp \gamma, \psi \rangle ds dt &= \sum_{i=1}^{n-1} \int_0^T \int_{\mathbb{R}/L\mathbb{Z}} \langle \partial_t^\perp \gamma, \psi_i v_i \rangle ds dt \\ &= - \int_0^T \delta_{\psi_i} E(\gamma_t) dt, \end{aligned}$$

since both  $\delta_{\psi_0 \tau} E(\gamma) = 0$  and  $\langle \partial_t^\perp \psi_0, \tau \rangle = 0$ . □

## 4. Epilogue

Although the minimising movement scheme leads in a rather straight forward way to the short-time existence of weak solution for our gradient flow, there are three key questions one would like to resolve, namely:

- (1) Are weak solutions unique and do they have long-time existence for  $0 \leq t < \infty$ ?
- (2) Can one use test functions for the gradient flow that are not orthogonal to a quasi-tangent?
- (3) Does our notion of solution depend on the choice of the reference curve and the approximate normal directions?

For long-time existence, it looks as if one could, in principle, restart the flow and the above short-time existence result to get an eternal solution. However, one should be aware that this solution might have kinks which our methods cannot rule out. If one has uniqueness and some way of modifying the approximate normal, long-time existence would be possible. Our Corollary 1.2 is a first indication that a more fastidious regularity theory is needed in order to resolve the above issues.

The question of uniqueness seems to be completely open. For the more standard non-homogeneous evolution equations involving the  $p$ -Laplace operator, papers discussing

uniqueness have only appeared rather recently. In particular, the method used to prove uniqueness in [3] breaks down for our curvature equations.

**Funding Information** Open access funding provided by Paris Lodron University of Salzburg.

**Open Access.** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**Appendix A: First variation for the  $p$ -elastic energy**

Recall that for closed curves  $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^n$  in the  $W^{2,p}$ -Sobolev class the  $p$ -elastic energy is given by

$$E^{(p)}(\gamma) = \frac{1}{p} \int_{\mathbb{R}/\mathbb{Z}} |\kappa|^p ds.$$

For the convenience of the reader, we give further details on the derivation of its first variation. The upcoming statement is proven along the line of [9, Lemma 2.1], for which we identify the arclength element by  $ds = |\partial_x \gamma| dx$  and the arclength derivative by  $\partial_s = |\partial_x \gamma|^{-1} \partial_x$ .

**Proposition A.1.** *The first variation of the  $p$ -elastic energy  $E^{(p)}$  for  $\gamma \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  in direction of  $\psi \in W^{2,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  is given by*

$$\delta_\psi E^{(p)}(\gamma) = \int_{\mathbb{R}/\mathbb{Z}} |\kappa|^{p-2} \langle \kappa, \delta_\psi \kappa \rangle ds + \frac{1}{p} \int_{\mathbb{R}/\mathbb{Z}} |\kappa|^p \langle \partial_s \gamma, \partial_s \psi \rangle ds.$$

where  $\delta_\psi \kappa = (\partial_s^2 \psi)^\perp - \langle \kappa, \partial_s \psi \rangle \partial_s \gamma - 2 \langle \partial_s \gamma, \partial_s \psi \rangle \kappa$ .

*Proof.* We first observe

$$\delta_\psi E^{(p)}(\gamma) = \frac{1}{p} \int_{\mathbb{R}/\mathbb{Z}} \delta_\psi (|\kappa|^p) ds + \frac{1}{p} \int_{\mathbb{R}/\mathbb{Z}} |\kappa|^p \delta_\psi (ds).$$

By applying the notation from above and the chain rule, we get

$$\begin{aligned} \delta_\psi (|\kappa|^p) &= \frac{d}{d\varepsilon} \left[ |\partial_s^2 (\gamma + \varepsilon \psi)|^p \right]_{\varepsilon=0} \\ &= \left[ p |\partial_s^2 (\gamma + \varepsilon \psi)|^{p-2} \langle \frac{d}{d\varepsilon} \partial_s^2 (\gamma + \varepsilon \psi), \partial_s^2 (\gamma + \varepsilon \psi) \rangle \right]_{\varepsilon=0} \\ &= p |\kappa|^{p-2} \langle \delta_\psi (\kappa), \kappa \rangle \end{aligned}$$

and

$$\begin{aligned} \delta\psi(ds) &= \frac{d}{d\varepsilon} [|\partial_x(\gamma + \varepsilon\psi)|dx]_{\varepsilon=0} \\ &= \left[ \frac{1}{|\partial_x(\gamma + \varepsilon\psi)|} \langle \partial_x(\gamma + \varepsilon\psi), \partial_x\psi \rangle dx \right]_{\varepsilon=0} \\ &= \left\langle \frac{\partial_x\gamma}{|\partial_x\gamma|}, \frac{\partial_x\psi}{|\partial_x\psi|} \right\rangle |\partial_x\gamma| dx \\ &= \langle \partial_s\gamma, \partial_s\psi \rangle ds. \end{aligned}$$

Similarly, we achieve

$$\begin{aligned} \delta\psi(\kappa) &= \frac{d}{d\varepsilon} \left[ \frac{1}{|\partial_x(\gamma + \varepsilon\psi)|} \partial_x \left( \frac{1}{|\partial_x(\gamma + \varepsilon\psi)|} \partial_x(\gamma + \varepsilon\psi) \right) \right]_{\varepsilon=0} \\ &= \left[ -\frac{1}{|\partial_x(\gamma + \varepsilon\psi)|^3} \langle \partial_x(\gamma + \varepsilon\psi), \partial_x\psi \rangle \partial_x \left( \frac{1}{|\partial_x(\gamma + \varepsilon\psi)|} \partial_x(\gamma + \varepsilon\psi) \right) \right]_{\varepsilon=0} \\ &\quad + \left[ \frac{1}{|\partial_x(\gamma + \varepsilon\psi)|} \partial_x \left( -\frac{1}{|\partial_x(\gamma + \varepsilon\psi)|^3} \langle \partial_x(\gamma + \varepsilon\psi), \partial_x h \rangle \partial_x(\gamma + \varepsilon\psi) + \frac{1}{|\partial_x(\gamma + \varepsilon\psi)|} \partial_x h \right) \right]_{\varepsilon=0} \\ &= -\left\langle \frac{\partial_x\gamma}{|\partial_x\gamma|}, \frac{\partial_x\psi}{|\partial_x\psi|} \right\rangle \frac{1}{|\partial_x\gamma|} \partial_x \left( \frac{\partial_x\gamma}{|\partial_x\gamma|} \right) - \frac{1}{|\partial_x\gamma|} \partial_x \left( \left\langle \frac{\partial_x\gamma}{|\partial_x\gamma|}, \frac{\partial_x\psi}{|\partial_x\psi|} \right\rangle \frac{\partial_x\gamma}{|\partial_x\gamma|} \right) + \frac{1}{|\partial_x\gamma|} \partial_x \left( \frac{\partial_x\psi}{|\partial_x\psi|} \right) \\ &= -\langle \partial_s\gamma, \partial_s\psi \rangle \kappa - \partial_s \langle \partial_s\gamma, \partial_s\psi \rangle \partial_s\gamma + \partial_s^2\psi \end{aligned}$$

and hence by rearranging and the Leibniz rule

$$\begin{aligned} \delta\psi(\kappa) &= \partial_s^2\psi - \langle \partial_s\gamma, \partial_s^2\psi \rangle \partial_s\gamma - \langle \partial_s^2\gamma, \partial_s\psi \rangle \partial_s\gamma - 2\langle \partial_s\gamma, \partial_s\psi \rangle \partial_s^2\gamma \\ &= P_{\partial_s\gamma}^\perp (\partial_s^2\psi) - 2\langle \partial_s\gamma, \partial_s\psi \rangle \kappa - \langle \kappa, \partial_s\psi \rangle \partial_s\gamma. \end{aligned}$$

□

**REFERENCES**

[1] L. Ambrosio and H. M. Soner, Level set approach to mean curvature flow in arbitrary codimension, *J. Differential Geom.* 43 (1996), no. 4, 693–737.

[2] C. Baker, The mean curvature flow of submanifolds of high codimension, 2011. [arXiv:1104.4409](https://arxiv.org/abs/1104.4409) [math.DG].

[3] A. Dall’Acqua, T. Laux, C.-C. Lin, P. Pozzi, and A. Spener, The elastic flow of curves on the sphere, *Geom. Flows* 3 (2018), 1–13.

[4] A. Dall’Acqua, C.-C. Lin, and P. Pozzi, A gradient flow for open elastic curves with fixed length and clamped ends, *Ann. Sc. Norm. Super. Cl. Sci.* (5) 17 (2017), no. 3, 1031–1066.

[5] A. Dall’Acqua, C.-C. Lin, and P. Pozzi, Elastic flow of networks: long-time existence result, *Geom. Flows* 4 (2019), 83–136.

[6] E. De Giorgi, New problems on minimizing movements, *Boundary value problems for partial differential equations and applications*. Dedicated to Enrico Magenes on the occasion of his 70th birthday, 1993, pp. 81–98. (repr. in *Ennio De Giorgi: Selected papers*, Springer, 2006, pp. 699–713).

[7] D. M. DeTurck, Deforming metrics in the direction of their Ricci tensors, *J. Differ. Geom.* 18 (1983), 157–162.

[8] E. DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, 1993.

[9] G. Dziuk, E. Kuwert, and R. Schatzle, Evolution of elastic curves in  $\mathbb{R}^n$ : Existence and computation, *SIAM J. Math. Anal.* 33 (2002), no. 5, 1228–1245.

[10] J. Escher, U. F. Mayer, and G. Simonett, The surface diffusion flow for immersed hypersurfaces, *SIAM Journal on Mathematical Analysis* 29 (1998), no. 6, 1419–1433.

[11] R. L. Foote, Shorter notes: Regularity of the distance function, *Proceedings of the American Mathematical Society* 92 (1984), no. 1, 153–155.



- [12] R. S. Hamilton, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc. (N.S.)* 7 (1982), no. 1, 65–222.
- [13] R. S. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differential Geom.* 17 (1982), no. 2, 255–306.
- [14] G. Huisken, Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.* 20 (1984), no. 1, 237–266.
- [15] G. Huisken and A. Polden, Geometric evolution equations for hypersurfaces, *Calculus of variations and geometric evolution problems (Cetraro, 1996)*, vol. 1713 of *Lecture Notes in Math.*, 1999, pp. 45–84.
- [16] R. Jakob, Short-time existence of the Möbius-invariant Willmore flow, *J. Geom. Anal.* 28 (2018), no. 2, 1151–1181.
- [17] J. L. Kazdan, Another proof of Bianchi’s identity in Riemannian geometry, *Proc. Amer. Math. Soc.* 81 (1981), 341–342.
- [18] J. LeCrone, Y. Shao, and G. Simonett, The surface diffusion and the Willmore flow for uniformly regular hypersurfaces, *Discrete and Continuous Dynamical Systems. Series S* 13 (2020), no. 12, 3503–3524.
- [19] U. F. Mayer and G. Simonett, Self-intersections for the surface diffusion and the volume-preserving mean curvature flow, *Differ. Integral Equ.* 13 (2000), no. 7-9, 1189–1199.
- [20] M. Müller and A. Spener, On the convergence of the elastic flow in the hyperbolic plane, *Geom. Flows* 5 (2020), 40–77.
- [21] M. Novaga and P. Pozzi, A second order gradient flow of  $p$ -elastic planar networks, *SIAM Journal on Mathematical Analysis* 52 (2020), no. 1, 682–708.
- [22] S. Okabe, P. Pozzi, and G. Wheeler, A gradient flow for the  $p$ -elastic energy defined on closed planar curves, *Math. Ann.* 378 (2020), no. 1-2, 777–828.
- [23] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, 2002. [arXiv:math/0211159](https://arxiv.org/abs/math/0211159) [math.DG].
- [24] M. Pozzetta, Convergence of elastic flows of curves into manifolds, *Nonlinear Analysis* (2022).
- [25] W. Rudin, *Functional analysis, Second*, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991. MR1157815
- [26] G. Simonett, The Willmore flow near spheres, *Differential Integral Equations* 14 (2001), no. 8, 1005–1014

*Simon Blatt, Christopher P. Hopper and  
Nicole Vorderobermeier  
Departement of Mathematics  
Paris Lodron Universität Salzburg  
Hellbrunner Strasse 34  
5020 Salzburg  
Austria  
E-mail: [simon.blatt@sbg.ac.at](mailto:simon.blatt@sbg.ac.at)*

*Christopher P. Hopper  
E-mail: [christopher.hopper@sbg.ac.at](mailto:christopher.hopper@sbg.ac.at)*

*Nicole Vorderobermeier  
E-mail: [nicole.vorderobermeier@sbg.ac.at](mailto:nicole.vorderobermeier@sbg.ac.at)*

Accepted: 19 February 2022