



# Time periodic solutions to the 2D quasi-geostrophic equation with the supercritical dissipation

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*Abstract.* We consider the 2D dissipative quasi-geostrophic equation with the time periodic external force and prove the existence of a unique time periodic solution in the case of the supercritical dissipation. In this case, the smoothing effect of the semigroup generated by the dissipation term is too weak to control the nonlinearity in the Duhamel term of the corresponding integral equation. In this paper, we give a new approach which does not depend on the contraction mapping principle for the integral equation.

## 1. Introduction

We consider the 2D dissipative quasi-geostrophic equation with the time periodic external force:

$$\begin{cases} \partial_t \theta + (-\Delta)^{\frac{\alpha}{2}} \theta + u \cdot \nabla \theta = F, & t \in \mathbb{R}, x \in \mathbb{R}^2, \\ u = \mathcal{R}^\perp \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), & t \in \mathbb{R}, x \in \mathbb{R}^2, \end{cases} \quad (1.1)$$

where  $\theta = \theta(t, x)$  and  $u = (u_1(t, x), u_2(t, x))$  represent the unknown potential temperature of the fluid and the unknown velocity field of the fluid, respectively. The given external force  $F = F(t, x)$  is  $T$ -time periodic, that is  $F$  satisfies  $F(t+T) = F(t)$  ( $t \in \mathbb{R}$ ) for some  $T > 0$ . The two operators  $(-\Delta)^{\frac{\alpha}{2}}$  ( $0 < \alpha \leq 2$ ) and  $\mathcal{R}_k$  ( $k = 1, 2$ ) denote the nonlocal differential operators so-called the fractional Laplacian and the Riesz transforms on  $\mathbb{R}^2$ , respectively, and they are defined by

$$(-\Delta)^{\frac{\alpha}{2}} f = \mathcal{F}^{-1} [|\xi|^\alpha \widehat{f}(\xi)], \quad \mathcal{R}_k f = \partial_{x_k} (-\Delta)^{-\frac{1}{2}} f = \mathcal{F}^{-1} \left[ \frac{i\xi_k}{|\xi|} \widehat{f}(\xi) \right].$$

In this paper, we prove the existence of a unique  $T$ -time periodic solution of (1.1) with the supercritical dissipation if the given  $T$ -time periodic external force is sufficiently small.

Before we state the main result precisely, we recall some known results for the initial value problem of the 2D dissipative quasi-geostrophic equation with the case

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$F = 0$ :

$$\begin{cases} \partial_t \theta + (-\Delta)^{\frac{\alpha}{2}} \theta + u \cdot \nabla \theta = 0, & t > 0, x \in \mathbb{R}^2, \\ u = \mathcal{R}^\perp \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), & t \geq 0, x \in \mathbb{R}^2, \\ \theta(0, x) = \theta_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{1.2}$$

Based on the scaling transform and the  $L^\infty(\mathbb{R}^2)$ -conservation, the dissipative quasi-geostrophic equation is divided into the subcritical case  $1 < \alpha \leq 2$ , critical case  $\alpha = 1$  and supercritical case  $0 < \alpha < 1$ . In the subcritical case, Constantin-Wu [6] proved the existence of a weak solution and decay estimates with respect to  $L^2$  norm for the initial data  $\theta_0 \in L^2(\mathbb{R}^2)$ . Wu [14] proved the global well-posedness for small data in the scaling subcritical setting  $\theta_0 \in L^p(\mathbb{R}^2)$  ( $p > 2/(\alpha - 1)$ ) via the contraction mapping principle for the corresponding integral equation. In the critical case, the order of the spatial derivative in the dissipation term coincides with that in the nonlinear term. Zhang [16] used this property and proved the existence of the global in time mild solution in the scaling critical Besov space  $\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$  ( $1 \leq p \leq \infty$ ). Global well-posedness in the Triebel–Lizorkin spaces  $F_{p,q}^s(\mathbb{R}^2)$  ( $s > 2/p, 1 < p, q < \infty$ ) is proved by Chen-Zhang [5]. In the supercritical case, the order of the spatial derivative in the dissipation term is less than that in the nonlinear term. Therefore, the smoothing effect of the fractional heat kernel  $e^{-t(-\Delta)^{\frac{\alpha}{2}}}$  is too weak to control the spatial derivative in the nonlinear term. This implies that it seems to be impossible to construct a solution of (1.2). It is able to overcome this and the local well-posedness for large data and the global well-posedness for small data in the scaling critical Sobolev  $H^{2-\alpha}(\mathbb{R}^2)$  by Miura [12] and Besov spaces  $B_{p,q}^{1+\frac{2}{p}-\alpha}(\mathbb{R}^2)$  ( $2 \leq p < \infty, 1 \leq q < \infty$ ) by Chae-Lee [3] and Chen-Miao-Zhang [4]. Their method is based on the energy estimates for the iteration of the transport-diffusion-type equation, and they control the nonlinear term by the divergence free condition  $\nabla \cdot u = 0$  and the commutator estimates.

On the other hand, despite the large number of previous studies on the well-posedness of the initial value problem (1.2), the study on the existence of time periodic solutions to the 2D quasi-geostrophic equation is hardly known.

In this manuscript, we consider the supercritical case and prove the existence of a unique time periodic solution to (1.1) in the scaling critical Besov space if the given time periodic external force is sufficiently small. More precisely, our main result of this paper reads as follows:

**Theorem 1.1.** *Let  $T > 0$  and  $2/3 < \alpha < 1$ . Let exponents  $p, q$  and  $r$  satisfy*

$$\frac{2}{2\alpha - 1} < r \leq p < \frac{4}{\alpha}, \quad 1 \leq q < \infty. \tag{1.3}$$

*Then, there exist positive constants  $\delta = \delta(\alpha, p, q, r, T)$  and  $K = K(\alpha, p, q, r, T)$  such that if the given  $T$ -time periodic external force  $F \in BC(\mathbb{R}; \dot{B}_{r,\infty}^0(\mathbb{R}^2))$  satisfies*

$$\sup_{t \in \mathbb{R}} \|F(t)\|_{\dot{B}_{r,\infty}^0} \leq \delta,$$

then there exist a unique  $T$ -time periodic solution  $\theta$  to (1.1) satisfying

$$\theta \in BC(\mathbb{R}; B_{p,q}^{1+\frac{2}{p}-\alpha}(\mathbb{R}^2)), \quad \|\theta\|_{\tilde{L}^\infty(\mathbb{R}; \dot{B}_{p,q}^{1+\frac{2}{p}-\alpha})} \leq K. \tag{1.4}$$

**Remark 1.2.** (1) If  $\theta$  and  $F$  satisfy (1.1), then

$$\theta_\lambda(t, x) = \lambda^{\alpha-1} \theta(\lambda^\alpha t, \lambda x), \quad F_\lambda(t, x) = \lambda^{2\alpha-1} F(\lambda^\alpha t, \lambda x)$$

also satisfy (1.1) for all  $\lambda > 0$ . Since it holds

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|\theta_\lambda(t)\|_{\dot{B}_{p,q}^{1+\frac{2}{p}-\alpha}} &= \sup_{t \in \mathbb{R}} \|\theta(t)\|_{\dot{B}_{p,q}^{1+\frac{2}{p}-\alpha}}, \\ \sup_{t \in \mathbb{R}} \|F_\lambda(t)\|_{\dot{B}_{2/(2\alpha-1),\infty}^0} &= \sup_{t \in \mathbb{R}} \|F(t)\|_{\dot{B}_{2/(2\alpha-1),\infty}^0} \end{aligned}$$

for all dyadic numbers  $\lambda > 0$ , the function spaces  $BC(\mathbb{R}; B_{p,q}^{1+\frac{2}{p}-\alpha}(\mathbb{R}^2))$  and  $BC(\mathbb{R}; \dot{B}_{r,\infty}^0(\mathbb{R}^2))$  in Theorem 1.1 are scaling critical and subcritical setting, respectively.

(2) The assumption  $2/3 < \alpha$  in Theorem 1.1 ensures the existence of  $p$  and  $r$  satisfying (1.3).

Our approach can be applied to the time periodic problem on the half time line  $(0, \infty)$ :

$$\begin{cases} \partial_t \theta + (-\Delta)^{\frac{\alpha}{2}} \theta + u \cdot \nabla \theta = F, & t > 0, x \in \mathbb{R}^2, \\ u = \mathcal{R}^\perp \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), & t \geq 0, x \in \mathbb{R}^2, \\ \theta(0, x) = \theta_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.5}$$

Here, we note that  $F$  is a given  $T$ -time periodic external force and the initial data  $\theta_0$  is *unknown*. In this setting, we prove *the existence of a unique suitable initial data*  $\theta_0$  such that (1.5) possesses a unique  $T$ -time periodic solution  $\theta$ . More precisely, the following theorem is obtained:

**Theorem 1.3.** *Let  $T > 0$  and  $2/3 < \alpha < 1$ . Let exponents  $p, q$  and  $r$  satisfy*

$$\frac{2}{2\alpha - 1} < r \leq p < \frac{4}{\alpha}, \quad 1 \leq q < \infty. \tag{1.6}$$

*Then, there exist positive constants  $\delta = \delta(\alpha, p, q, r, T)$  and  $K = K(\alpha, p, q, r, T)$  such that if the given  $T$ -time periodic external force  $F \in BC((0, \infty); \dot{B}_{r,\infty}^0(\mathbb{R}^2))$  satisfies*

$$\sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \leq \delta,$$

*then there exist a unique initial data  $\theta_0 \in B_{p,q}^{1+\frac{2}{p}-\alpha}(\mathbb{R}^2)$  and a unique  $T$ -time periodic solution  $\theta$  to (1.5) satisfying*

$$\theta \in BC([0, \infty); B_{p,q}^{1+\frac{2}{p}-\alpha}(\mathbb{R}^2)), \quad \|\theta\|_{\tilde{L}^\infty(0, \infty; \dot{B}_{p,q}^{1+\frac{2}{p}-\alpha})} \leq K. \tag{1.7}$$

In the study of the Navier–Stokes equation, the existence of time periodic solutions is often proved by applying the contraction mapping principle to the corresponding integral equation. (It was in [10] that first used this idea.) However, in the case of our problem, the supercritical dissipation prevents us from using this scheme. Indeed, when we apply the idea of [10] to (1.1), we meet the difficulty that the smoothing effect of the fractional heat kernel  $e^{-t(-\Delta)^{\frac{\alpha}{2}}}$  is too weak to control the first-order spatial derivative of the nonlinear term and it is pretty difficult to find a Banach space  $X$  satisfying

$$\sup_{t \in \mathbb{R}} \left\| \int_{-\infty}^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} (\mathcal{R}^\perp \theta(\tau) \cdot \nabla \theta(\tau)) d\tau \right\|_X \leq C \left( \sup_{t \in \mathbb{R}} \|\theta(t)\|_X \right)^2.$$

As another approach, let us consider the successive approximation defined by the transport diffusion-type equation

$$\begin{cases} \partial_t \theta^{(n+1)} + (-\Delta)^{\frac{\alpha}{2}} \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} = F, & t \in \mathbb{R}, x \in \mathbb{R}^2, \\ u^{(n)} = \mathcal{R}^\perp \theta^{(n)}, & t \in \mathbb{R}, x \in \mathbb{R}^2. \end{cases} \tag{1.8}$$

Then, we can obtain the a priori estimates for the approximation solutions by the energy method, with some commutator estimates, which is the similar method to the analysis of the initial value problem. However, it seems to be difficult to construct a time periodic linear solution  $\theta^{(n+1)}$  of (1.8) when  $\theta^{(n)}$  is determined. Therefore, we are not able to proceed in parallel with the energy method of the initial value problem for the supercritical case.

We now introduce an idea to overcome these difficulties and get a time periodic solution. Once Theorem 1.3 is proved, then we can get the time periodic solution on  $\mathbb{R}$  by extending the time periodic solution on  $[0, \infty)$  to the other half time line  $(-\infty, 0)$  periodically. Thus, in the following, we only consider the time periodic problem (1.5) and Theorem 1.3.

Outline of our idea is to construct a specific initial data  $\theta_0$  and the corresponding local in time solution  $\theta$  on  $[0, T]$  satisfying  $\theta(T) = \theta(0) = \theta_0$ . Then, we get a  $T$ -time periodic solution by extending  $\theta$  periodically in time. In order to construct such a solution on  $[0, T]$ , we define a new approximation sequences.

Let us explain the new approximation system more precisely. If a solution  $\theta$  to (1.5) satisfies  $\theta(T) = \theta(0) = \theta_0$ , then we have

$$(1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})\theta_0 = \theta(T) - e^{-T(-\Delta)^{\frac{\alpha}{2}}}\theta_0. \tag{1.9}$$

Therefore, from this observation, we approximate (1.5) together with (1.9) successively and define the iteration scheme  $\{\theta_0^{(n)}\}_{n=0}^\infty$  and  $\{\theta^{(n)}\}_{n=0}^\infty$  as follows:

$$\begin{cases} \partial_t \theta^{(n+1)} + (-\Delta)^{\frac{\alpha}{2}} \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} = F, \\ u^{(n)} = \mathcal{R}^\perp \theta^{(n)}, \\ \theta^{(n+1)}(0, x) = \theta_0^{(n+1)}; \\ (1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})\theta_0^{(n+1)} = \theta^{(n)}(T) - e^{-T(-\Delta)^{\frac{\alpha}{2}}}\theta^{(n)}(0). \end{cases} \tag{1.10}$$

((1.10) is the formal definition. See Sect. 3 for the accurate definition of the approximation sequences.)

Finally, we remark the estimates for the approximation sequences. It is well known that the uniform boundedness is a key property of the convergence of sequences. The uniform boundedness of  $\{\theta_0^{(n)}\}_{n=0}^\infty$  can be obtained by the classical energy estimates (see, for instance, [1]). However, the new idea is required for the uniform boundedness of  $\{\theta_0^{(n)}\}_{n=0}^\infty$  since  $\{\theta_0^{(n)}\}_{n=0}^\infty$  is defined by solving the following time-independent equation:

$$(1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})u = f.$$

To obtain estimates for  $\{\theta_0^{(n)}\}_{n=0}^\infty$ , we establish Lemmas 2.6 and 2.7 in Sect. 2 which are inspired by the idea of [8, Theorem 2.4].

This paper is organized as follows. In Sect. 2, we summarize some notations and introduce lemmas which are used in the proof of the main results. In Sect. 3, we prove Theorem 1.1.

Throughout this paper, we denote by  $C$  the constant, which may differ in each line. In particular,  $C = C(a_1, \dots, a_n)$  means that  $C$  depends only on  $a_1, \dots, a_n$ . We define a commutator for two operators  $A$  and  $B$  as  $[A, B] = AB - BA$ .

## 2. Preliminaries

Let  $\mathcal{S}(\mathbb{R}^2)$  be the set of all Schwartz functions on  $\mathbb{R}^2$ , and let  $\mathcal{S}'(\mathbb{R}^2)$  be the set of all tempered distributions on  $\mathbb{R}^2$ . For  $f \in \mathcal{S}(\mathbb{R}^2)$ , we define the Fourier transform and the inverse Fourier transform of  $f$  by

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^2} e^{-i\xi \cdot x} f(x) dx, \quad \mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\xi \cdot x} f(\xi) d\xi,$$

respectively.  $\{\varphi_j\}_{j \in \mathbb{Z}}$  is called the homogeneous Littlewood–Paley decomposition if  $\varphi_0 \in \mathcal{S}(\mathbb{R}^2)$  satisfy  $\text{supp } \widehat{\varphi}_0 \subset \{2^{-1} \leq |\xi| \leq 2\}$ ,  $0 \leq \widehat{\varphi}_0 \leq 1$  and

$$\sum_{j \in \mathbb{Z}} \widehat{\varphi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^2 \setminus \{0\},$$

where  $\widehat{\varphi}_j(\xi) = \widehat{\varphi}_0(2^{-j}\xi)$ . Let us write

$$\Delta_j f := \varphi_j * f$$

for  $j \in \mathbb{Z}$  and  $f \in \mathcal{S}'(\mathbb{R}^2)$ . Using the homogeneous Littlewood–Paley decomposition, we define the Besov spaces. For  $1 \leq p, q \leq \infty$  and  $s \in \mathbb{R}$ , the homogeneous Besov space  $\dot{B}_{p,q}^s(\mathbb{R}^2)$  is defined by

$$\begin{aligned} \dot{B}_{p,q}^s(\mathbb{R}^2) &:= \left\{ f \in \mathcal{S}'(\mathbb{R}^2) ; \|f\|_{\dot{B}_{p,q}^s} < \infty \right\}, \\ \|f\|_{\dot{B}_{p,q}^s} &:= \left\| \left\{ 2^{js} \|\Delta_j f\|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})}, \end{aligned}$$

where  $\mathcal{S}'_0(\mathbb{R}^2)$  is the dual space of

$$\mathcal{S}'_0(\mathbb{R}^2) := \left\{ f \in \mathcal{S}'(\mathbb{R}^2) ; \int_{\mathbb{R}^2} x^\gamma f(x) dx = 0 \text{ for all } \gamma \in (\mathbb{N} \cup \{0\})^2 \right\}.$$

Note that  $\dot{B}^s_{p,q}(\mathbb{R}^2)$  is a Banach space with respect to the norm  $\|\cdot\|_{\dot{B}^s_{p,q}}$ . It is well known that if  $1 \leq p, q \leq \infty$  and  $s < 2/p$ , then we can identify  $\dot{B}^s_{p,q}(\mathbb{R}^2)$  as:

$$\left\{ f \in \mathcal{S}'(\mathbb{R}^2) ; f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}'(\mathbb{R}^2) \text{ and } \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}.$$

(See for the detail in [11] and [13].) For  $s > 0$  and  $1 \leq p, q \leq \infty$ , the inhomogeneous Besov space  $B^s_{p,q}(\mathbb{R}^2)$  is defined by

$$\begin{aligned} B^s_{p,q}(\mathbb{R}^2) &:= \dot{B}^s_{p,q}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2), \\ \|f\|_{B^s_{p,q}} &:= \|f\|_{\dot{B}^s_{p,q}} + \|f\|_{L^p}. \end{aligned}$$

In this paper, we also use the space-time Besov spaces defined by

$$\begin{aligned} \tilde{L}^r(0, T; \dot{B}^s_{p,q}(\mathbb{R}^2)) &:= \left\{ F : (0, T) \rightarrow \mathcal{S}'_0(\mathbb{R}^2) ; \|F\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q})} < \infty \right\}, \\ \|F\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q})} &:= \left\| \left\{ 2^{sj} \|\Delta_j F\|_{L^r(0,T;L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \end{aligned}$$

for  $1 \leq p, q, r \leq \infty, s \in \mathbb{R}$  and  $0 < T \leq \infty$ .

Next, we introduce the semigroup generated by the fractional Laplacian  $(-\Delta)^{\frac{\alpha}{2}}$ . It is given explicitly by using the Fourier transform:

$$e^{-t(-\Delta)^{\frac{\alpha}{2}}} f = \mathcal{F}^{-1} \left[ e^{-t|\xi|^\alpha} \widehat{f}(\xi) \right].$$

Then, this semigroup possesses the following properties:

**Lemma 2.1.** *Let  $\alpha > 0$  and  $1 \leq p, q \leq \infty$ . Then, the followings hold:*

(1) *There exists a positive constant  $C = C(\alpha)$  such that*

$$\left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} \Delta_j f \right\|_{L^p} \leq C e^{-C^{-1}2^{\alpha j} t} \|\Delta_j f\|_{L^p}$$

*holds for all  $t > 0, j \in \mathbb{Z}$  and  $f \in \mathcal{S}'_0(\mathbb{R}^2)$  with  $\Delta_j f \in L^p(\mathbb{R}^2)$ .*

(2) *Let  $s_1, s_2 \in \mathbb{R}$  satisfy  $s_1 \leq s_2$ . Then, there exists a positive constant  $C = C(\alpha, s_1, s_2)$  such that*

$$2^{s_2 j} \left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} \Delta_j f \right\|_{L^p} \leq C t^{-\frac{s_2-s_1}{\alpha}} 2^{s_1 j} \|\Delta_j f\|_{L^p}$$

holds for all  $t > 0$ ,  $j \in \mathbb{Z}$  and  $f \in \mathcal{S}'_0(\mathbb{R}^2)$  with  $\Delta_j f \in L^p(\mathbb{R}^2)$ . In particular, it holds

$$\left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} f \right\|_{\dot{B}_{p,q}^{s_2}} \leq C t^{-\frac{s_2-s_1}{\alpha}} \|f\|_{\dot{B}_{p,q}^{s_1}}$$

for all  $t > 0$  and  $f \in \dot{B}_{p,q}^{s_1}(\mathbb{R}^2)$ .

(3) Let  $s \in \mathbb{R}$ . Then, for each  $f \in \dot{B}_{p,q}^s(\mathbb{R}^2)$ , it holds

$$\lim_{t \rightarrow \infty} \left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} f \right\|_{\dot{B}_{p,q}^s} = 0.$$

*Proof.* (1) is proved in [9] and [16]. (2) is immediately obtained by (1) and

$$2^{s_2 j} e^{-C^{-1} 2^{\alpha j} t} \leq C t^{-\frac{s_2-s_1}{\alpha}} 2^{s_1 j}.$$

Let us prove (3). The density property yields that for any  $\varepsilon > 0$ , there exists a  $f_\varepsilon \in \mathcal{S}_0(\mathbb{R}^2)$  such that  $\|f_\varepsilon - f\|_{\dot{B}_{p,q}^s} < \varepsilon$ . Then, we see that

$$\begin{aligned} \left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} f \right\|_{\dot{B}_{p,q}^s} &\leq \left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} (f - f_\varepsilon) \right\|_{\dot{B}_{p,q}^s} + \left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} f_\varepsilon \right\|_{\dot{B}_{p,q}^s} \\ &\leq C \|f - f_\varepsilon\|_{\dot{B}_{p,q}^s} + C t^{-\frac{1}{\alpha}} \|f_\varepsilon\|_{\dot{B}_{p,q}^{s-1}} \\ &< C\varepsilon + C t^{-\frac{1}{\alpha}} \|f_\varepsilon\|_{\dot{B}_{p,q}^{s-1}}, \end{aligned}$$

which implies

$$\limsup_{t \rightarrow \infty} \left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} f \right\|_{\dot{B}_{p,q}^s} \leq C\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, the proof is completed. □

Next, we derive some bilinear estimates. We first recall the definition and basic properties of the Bony paraproduct formula. For  $f, g \in \mathcal{S}_0(\mathbb{R}^2)$ , we decompose the product  $fg$  as:

$$fg = T_f g + R(f, g) + T_g f,$$

where

$$T_f g := \sum_{l \in \mathbb{Z}} S_l f \Delta_l g, \quad R(f, g) := \sum_{l \in \mathbb{Z}} \sum_{|k-l| \leq 2} \Delta_k f \Delta_l g.$$

Here,  $S_l f$  is defined by

$$S_l f := \sum_{k \leq l-3} \Delta_k f, \quad l \in \mathbb{Z}.$$

Then, considering the supports of the functions of the Fourier side, we have

$$\Delta_j T_f g = \sum_{l:|l-j|\leq 3} \Delta_j (S_l f \Delta_l g), \quad \Delta_j R(f, g) = \sum_{(k,l): \substack{\max\{j,k\} \geq l-3, \\ |k-l| \leq 2}} \Delta_j (\Delta_k f \Delta_l g).$$

Using them, we have for  $T > 0$ ,  $1 \leq p, q \leq \infty$  and  $s_1, s_2 \in \mathbb{R}$  with  $s_1 < 0$  (if  $q = 1$ , then  $s_1 \leq 0$ ) that

$$2^{(s_1+s_2-\frac{2}{p})j} \|\Delta_j T_f g\|_{L^\infty(0,T;L^p)} \leq C \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \sum_{|l-j|\leq 3} 2^{s_2 l} \|\Delta_l g\|_{L^\infty(0,T;L^p)} \tag{2.1}$$

and it also holds for  $1 \leq p, q \leq \infty$  and  $s_1, s_2 \in \mathbb{R}$  with  $s_1 + s_2 > 0$

$$\|R(f, g)\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1+s_2})} \leq C \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \|g\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_2})}. \tag{2.2}$$

See [2] for the idea of the proof of these estimates. From easy applications of (2.1) and (2.2), we obtain the following lemma:

**Lemma 2.2.** *Let  $2 \leq p \leq \infty$  and  $1 \leq q \leq \infty$ . Let  $s_1, s_2 \in \mathbb{R}$  satisfy  $s_1 + s_2 > 0$  and  $s_1, s_2 < 2/p$ . Then, there exists a positive constant  $C = C(p, q, s_1, s_2)$  such that*

$$\|fg\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}})} \leq C \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \|g\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_2})}$$

holds for all  $T > 0$ ,  $f \in \tilde{L}^\infty(0, T; \dot{B}_{p,q}^{s_1}(\mathbb{R}^2))$  and  $g \in \tilde{L}^\infty(0, T; \dot{B}_{p,q}^{s_2}(\mathbb{R}^2))$ .

By the standard argument of the proof of commutator estimates (see, for instance, [2], [12]), we get the following lemma:

**Lemma 2.3.** *Let  $2 \leq p \leq \infty$  and  $1 \leq q \leq \infty$ . Let  $s_1, s_2 \in \mathbb{R}$  satisfy  $s_1 + s_2 > 0$ ,  $0 < s_1 < 1 + 2/p$  and  $s_2 < 2/p$ . Then, there exists a positive constant  $C = C(p, q, s_1, s_2)$  such that*

$$\left\| \left\{ 2^{(s_1+s_2-\frac{2}{p})j} \|[f, \Delta_j]g\|_{L^\infty(0,T;L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \leq C \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \|g\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_2})}$$

holds for all  $T > 0$ ,  $f \in \tilde{L}^\infty(0, T; \dot{B}_{p,q}^{s_1}(\mathbb{R}^2))$  and  $g \in \tilde{L}^\infty(0, T; \dot{B}_{p,q}^{s_2}(\mathbb{R}^2))$ .

The next lemma helps us to control the product term which will appear in equations of the perturbation such as (3.5), (3.22) and (3.32).

**Lemma 2.4.** *Let  $\lambda > 0$ ,  $\alpha > 0$ ,  $\beta \leq \alpha$ ,  $2 \leq p \leq \infty$  and  $1 \leq q \leq \infty$ . Let  $s_1, s_2 \in \mathbb{R}$  satisfy  $s_1 + s_2 > 0$ ,  $2/p < s_1 < 2/p + \alpha$  and  $s_2 < 2/p$ . Then, there exists a positive constant  $C = C(\lambda, \alpha, \beta, p, q, s_1, s_2)$  such that*

$$\begin{aligned} & \left\| \left\{ \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_1+s_2-\frac{2}{p})j} \|\Delta_j (f(\tau) e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g)\|_{L^p} d\tau \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ & \leq C T^{1-\frac{\beta}{\alpha}-\frac{1}{\alpha}(s_1-\frac{2}{p})} \left\{ \|f\|_{L^\infty(0,T;L^p)} + \left(1 + T^{\frac{1}{\alpha}(s_1-\frac{2}{p})}\right) \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \right\} \|g\|_{\dot{B}_{p,q}^{s_2}} \end{aligned} \tag{2.3}$$



holds for all  $T > 0$ ,  $f \in L^\infty(0, T; L^p(\mathbb{R}^2)) \cap \tilde{L}^\infty(0, T; \dot{B}_{p,q}^{s_1}(\mathbb{R}^2))$  and  $g \in \dot{B}_{p,q}^{s_2}(\mathbb{R}^2)$ .

In particular, if  $\beta < \alpha$ , then the following estimate holds:

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \|f(\tau) e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g\|_{\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}}} d\tau \\ & \leq CT^{1-\frac{\beta}{\alpha}-\frac{1}{\alpha}(s_1-\frac{2}{p})} \left\{ \|f\|_{L^\infty(0,T;L^p)} + \left(1 + T^{\frac{1}{\alpha}(s_1-\frac{2}{p})}\right) \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \right\} \|g\|_{\dot{B}_{p,q}^{s_2}}. \end{aligned} \tag{2.4}$$

*Remark 2.5.* Let  $1/2 < \alpha < 1$  and  $2 \leq p < 4/(2\alpha - 1)$ . Then, it immediately follows from (2.3) with  $s_1 = s_c := 1 + 2/p - \alpha$  and the continuous embedding  $\tilde{L}^\infty(0, T; \dot{B}_{p,1}^0(\mathbb{R}^2)) \hookrightarrow L^\infty(0, T; L^p(\mathbb{R}^2))$  that

$$\begin{aligned} & \left\| \left\{ \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_c+s_2-\frac{2}{p})j} \|\Delta_j(f(\tau) e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g)\|_{L^p} d\tau \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ & \leq CT^{1-\frac{\beta}{\alpha}} (T^{-\frac{1-\alpha}{\alpha}} + 1) \|f\|_{X_T^{p,q}} \|g\|_{\dot{B}_{p,q}^{s_2}}, \end{aligned} \tag{2.5}$$

where  $X_T^{p,q} := \tilde{L}^\infty(0, T; \dot{B}_{p,1}^0(\mathbb{R}^2)) \cap \tilde{L}^\infty(0, T; \dot{B}_{p,q}^{s_c}(\mathbb{R}^2))$ . If  $\beta < \alpha$ , then (2.4) yields that

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \|f(\tau) e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g\|_{\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}}} d\tau \\ & \leq CT^{1-\frac{\beta}{\alpha}} (T^{-\frac{1-\alpha}{\alpha}} + 1) \|f\|_{X_T^{p,q}} \|g\|_{\dot{B}_{p,q}^{s_2}}. \end{aligned} \tag{2.6}$$

*Proof of Lemma 2.4.* First, we prove (2.3). It follows from an inequality of (2.1) type and (2) of Lemma 2.1 that

$$\begin{aligned} & \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_1+s_2-\frac{2}{p})j} \|\Delta_j T_{f(\tau)} e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g\|_{L^p} d\tau \\ & \leq C \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \|f(\tau)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \sum_{|l-j| \leq 3} 2^{((s_1-\frac{2}{p})+s_2)l} \|e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} \Delta_l g\|_{L^p} d\tau \\ & \leq C \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \tau^{-\frac{1}{\alpha}(s_1-\frac{2}{p})} d\tau \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,1}^{\frac{2}{p}})} \sum_{|l-j| \leq 3} 2^{s_2 l} \|\Delta_l g\|_{L^p}. \end{aligned} \tag{2.7}$$

By virtue of  $s_1 - 2/p < \alpha$  and  $\beta \leq \alpha$ , it is easy to see that

$$\sup_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \tau^{-\frac{1}{\alpha}(s_1-\frac{2}{p})} d\tau \leq CT^{1-\frac{\beta}{\alpha}-\frac{1}{\alpha}(s_1-\frac{2}{p})}.$$

Hence, taking  $l^q(\mathbb{Z})$ -norm of (2.7) and using

$$\begin{aligned} \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,1}^{\frac{2}{p}})} & \leq \sum_{j \leq 0} 2^{\frac{2}{p}j} \|\Delta_j f\|_{L^\infty(0,T;L^p)} + \left\| \left\{ 2^{(\frac{2}{p}-s_1)j} \right\}_{l \in \mathbb{Z}} \right\|_{l^{\frac{q}{q-1}}(\mathbb{N})} \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \\ & \leq C \left( \|f\|_{L^\infty(0,T;L^p)} + \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \right), \end{aligned} \tag{2.8}$$

we obtain that

$$\begin{aligned} & \left\| \left\{ \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_1+s_2-\frac{2}{p})j} \|\Delta_j T f(\tau) e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g\|_{L^p} d\tau \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ & \leq CT^{1-\frac{\beta}{\alpha}-\frac{1}{\alpha}(s_1-\frac{2}{p})} \left( \|f\|_{L^\infty(0,T;L^p)} + \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \right) \|g\|_{\dot{B}_{p,q}^{s_2}}. \end{aligned} \tag{2.9}$$

Since it holds

$$\begin{aligned} & \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_1+s_2-\frac{2}{p})j} \left\| \Delta_j R(f(\tau), e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g) \right\|_{L^p} d\tau \\ & \leq \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau 2^{(s_1+s_2-\frac{2}{p})j} \left\| \Delta_j R(f, e^{-t(-\Delta)^{\frac{\alpha}{2}}} g) \right\|_{L_t^\infty(0,T;L^p)} \end{aligned} \tag{2.10}$$

taking  $l^q(\mathbb{Z})$ -norm of (2.10), we see by (2.2) and (2) of Lemma 2.1 that

$$\begin{aligned} & \left\| \left\{ \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_1+s_2-\frac{2}{p})j} \left\| \Delta_j R(f(\tau), e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g) \right\|_{L^p} d\tau \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ & \leq \sup_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau \left\| R(f, e^{-t(-\Delta)^{\frac{\alpha}{2}}} g) \right\|_{\tilde{L}_t^\infty(0,T;\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}})} \\ & \leq CT^{1-\frac{\beta}{\alpha}} \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \|g\|_{\dot{B}_{p,q}^{s_2}}. \end{aligned} \tag{2.11}$$

Here, we have used

$$\sup_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{\lambda 2^{\alpha j}(T-\tau)} d\tau \leq CT^{1-\frac{\beta}{\alpha}}, \quad \beta \leq \alpha. \tag{2.12}$$

Similarly, it follows from (2.1) and (2) of Lemma 2.1 that

$$\begin{aligned} & \left\| \left\{ \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_1+s_2-\frac{2}{p})j} \left\| \Delta_j T e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g f(\tau) \right\|_{L^p} d\tau \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ & \leq \sup_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau \left\| T e^{-t(-\Delta)^{\frac{\alpha}{2}}} g f(t) \right\|_{\tilde{L}_t^\infty(0,T;\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}})} \\ & \leq CT^{1-\frac{\beta}{\alpha}} \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \left\| e^{-t(-\Delta)^{\frac{\alpha}{2}}} g \right\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_2})} \\ & \leq CT^{1-\frac{\beta}{\alpha}} \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \|g\|_{\dot{B}_{p,q}^{s_2}}. \end{aligned} \tag{2.13}$$

Combining (2.9), (2.11) and (2.13), we complete the proof of (2.5). Next, we show (2.4). By similar inequality to (2.1) and (2) of Lemma 2.1, we see that

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \|T_{f(\tau)} e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g\|_{\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}}} d\tau \\ & \leq C \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \tau^{-\frac{1}{\alpha}(s_1-\frac{2}{p})} d\tau \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,1}^{\frac{2}{p}})} \|g\|_{\dot{B}_{p,q}^{s_2}} \\ & \leq CT^{1-\frac{\beta}{\alpha}-\frac{1}{\alpha}(s_1-\frac{2}{p})} \left( \|f\|_{L^\infty(0,T;L^p)} + \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \right) \|g\|_{\dot{B}_{p,q}^{s_2}}. \end{aligned} \tag{2.14}$$

We also obtain from (2.2) and (2) of Lemma 2.1 that

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \left\| R(f(\tau), e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g) \right\|_{\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}}} d\tau \\ & \leq \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau \left\| R(f, e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g) \right\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}})} \\ & \leq CT^{1-\frac{\beta}{\alpha}} \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \|g\|_{\dot{B}_{p,q}^{s_2}}. \end{aligned} \tag{2.15}$$

Here, we have used the following inequalities in (2.14) and (2.15):

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} \tau^{-\frac{\gamma}{\alpha}} d\tau \leq CT^{1-\frac{\beta}{\alpha}-\frac{\gamma}{\alpha}}, \quad \beta < \alpha, \quad \gamma < \alpha, \\ & \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau \leq CT^{1-\frac{\beta}{\alpha}}, \quad \beta < \alpha. \end{aligned} \tag{2.16}$$

Similarly, we have

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \int_0^T 2^{\beta j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_1+s_2-\frac{2}{p})j} \left\| T_{e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} g} f(\tau) \right\|_{\dot{B}_{p,q}^{s_1+s_2-\frac{2}{p}}} d\tau \\ & \leq CT^{1-\frac{\beta}{\alpha}} \|f\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_1})} \|g\|_{\dot{B}_{p,q}^{s_2}}. \end{aligned} \tag{2.17}$$

Hence, we complete the proof by combining (2.14), (2.15) and (2.17).

To derive some estimates for initial data related to a time periodic solution in the proof of the main results, we introduce the following two lemmas whose ideas are inspired by [8].

**Lemma 2.6.** *Let  $\alpha > 0$ ,  $1 \leq p \leq \infty$ ,  $1 \leq q < \infty$  and  $s \in \mathbb{R}$ . Then, for any  $f \in \dot{B}_{p,q}^s(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s-\alpha}(\mathbb{R}^2)$ , the series*

$$u = \sum_{k=0}^{\infty} e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} f \tag{2.18}$$

converges in  $\dot{B}_{p,q}^s(\mathbb{R}^2)$  and  $u$  satisfies

$$(1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})u = f \quad \text{in } \dot{B}_{p,q}^s(\mathbb{R}^2). \tag{2.19}$$

Moreover, there exists a positive constant  $C = C(\alpha)$  such that

$$\|u\|_{\dot{B}_{p,q}^s} \leq C \left( T^{-1} \|f\|_{\dot{B}_{p,q}^{s-\alpha}} + \|f\|_{\dot{B}_{p,q}^s} \right). \tag{2.20}$$

*Proof.* Let  $m, n \in \mathbb{N}$  satisfy  $m < n$ . Then, it follows from (1) of Lemma 2.1 that

$$\begin{aligned} \left\| \sum_{k=m}^n e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} f \right\|_{\dot{B}_{p,q}^s}^q &= \sum_{j \in \mathbb{Z}} \left( 2^{sj} \left\| \sum_{k=m}^n e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} \Delta_j f \right\|_{L^p} \right)^q \\ &\leq \sum_{j \in \mathbb{Z}} \left( 2^{sj} \|\Delta_j f\|_{L^p} \sum_{k=m}^n C e^{-C^{-1}T2^{\alpha j}k} \right)^q, \end{aligned} \tag{2.21}$$

where  $C$  is the same constant as in (1) of Lemma 2.1. Since it holds

$$\begin{aligned} &\sum_{j \in \mathbb{Z}} \left( 2^{sj} \|\Delta_j f\|_{L^p} \frac{C}{1 - e^{-C^{-1}2^{\alpha j}T}} \right)^q \\ &= \sum_{j: 2^{\alpha j}T < 1} \left( 2^{sj} \|\Delta_j f\|_{L^p} \frac{C^{-1}2^{\alpha j}T}{1 - e^{-C^{-1}2^{\alpha j}T}} \cdot C^2 2^{-\alpha j} T^{-1} \right)^q \\ &\quad + \sum_{j: 2^{\alpha j}T \geq 1} \left( 2^{sj} \|\Delta_j f\|_{L^p} \frac{C}{1 - e^{-C^{-1}2^{\alpha j}T}} \right)^q \\ &\leq \left( \frac{C}{1 - e^{-C^{-1}}} \right)^q \left( T^{-q} \|f\|_{\dot{B}_{p,q}^{s-\alpha}}^q + \|f\|_{\dot{B}_{p,q}^s}^q \right) \\ &\leq \left( \frac{2C}{1 - e^{-C^{-1}}} \right)^q \left( T^{-1} \|f\|_{\dot{B}_{p,q}^{s-\alpha}} + \|f\|_{\dot{B}_{p,q}^s} \right)^q < \infty, \end{aligned} \tag{2.22}$$

we have

$$\begin{aligned} 2^{sj} \|\Delta_j f\|_{L^p} \sum_{k=m}^n C e^{-C^{-1}T2^{\alpha j}k} &\leq 2^{sj} \|\Delta_j f\|_{L^p} \sum_{k=0}^{\infty} C e^{-C^{-1}T2^{\alpha j}k} \\ &= 2^{sj} \|\Delta_j f\|_{L^p} \frac{C}{1 - e^{-C^{-1}2^{\alpha j}T}} \in l^q(\mathbb{Z}). \end{aligned}$$

Hence, it follows from (2.21) and the dominated convergence theorem that

$$\limsup_{n, m \rightarrow \infty} \left\| \sum_{k=m}^n e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} f \right\|_{\dot{B}_{p,q}^s}^q \leq \sum_{j \in \mathbb{Z}} \left( 2^{sj} \|\Delta_j f\|_{L^p} \lim_{m, n \rightarrow \infty} \sum_{k=m}^n C e^{-C^{-1}T2^{\alpha j}k} \right)^q = 0.$$

Thus, the series (2.18) converges in  $\dot{B}_{p,q}^s(\mathbb{R}^2)$ , and we find that  $u$  satisfies (2.20) by

$$\begin{aligned} \|u\|_{\dot{B}_{p,q}^s}^q &= \sum_{j \in \mathbb{Z}} \left( 2^{sj} \left\| \sum_{k=0}^{\infty} e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} \Delta_j f \right\|_{L^p} \right)^q \\ &\leq \sum_{j \in \mathbb{Z}} \left( 2^{sj} \sum_{k=0}^{\infty} \left\| e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} \Delta_j f \right\|_{L^p} \right)^q \\ &\leq \sum_{j \in \mathbb{Z}} \left( 2^{sj} \|\Delta_j f\|_{L^p} \sum_{k=0}^{\infty} C e^{-C^{-1}T2^{\alpha j}k} \right)^q \\ &\leq \left( \frac{2C}{1 - e^{-C^{-1}}} \right)^q \left( T^{-1} \|f\|_{\dot{B}_{p,q}^{s-\alpha}} + \|f\|_{\dot{B}_{p,q}^s} \right)^q, \end{aligned}$$

where we have used (2.22). Finally, we show (2.19). Let

$$u_N := \sum_{k=0}^{N-1} e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} f, \quad N \in \mathbb{N}.$$

Note that  $u_N$  converges to  $u$  in  $\dot{B}_{p,q}^s(\mathbb{R}^2)$  as  $N \rightarrow \infty$ . By a simple calculation, we see that

$$(1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})u_N = f - e^{-TN(-\Delta)^{\frac{\alpha}{2}}} f. \tag{2.23}$$

Here, it follows from (2) of Lemma 2.1 that

$$\begin{aligned} &\left\| (1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})u_N - (1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})u \right\|_{\dot{B}_{p,q}^s} \\ &\leq \|u - u_N\|_{\dot{B}_{p,q}^s} + \left\| e^{-T(-\Delta)^{\frac{\alpha}{2}}} (u - u_N) \right\|_{\dot{B}_{p,q}^s} \\ &\leq C \|u - u_N\|_{\dot{B}_{p,q}^s} \\ &\rightarrow 0 \end{aligned} \tag{2.24}$$

as  $N \rightarrow \infty$  and it holds by (3) of Lemma 2.1 that

$$\left\| e^{-TN(-\Delta)^{\frac{\alpha}{2}}} f \right\|_{\dot{B}_{p,q}^s} \rightarrow 0 \tag{2.25}$$

as  $N \rightarrow \infty$ . Hence, letting  $N \rightarrow \infty$  in (2.23) by (2.24) and (2.25), we find that  $u$  satisfies (2.19). This completes the proof.  $\square$

**Lemma 2.7.** *Let  $T > 0, \alpha > 0, 1 \leq p \leq \infty, 1 \leq q < \infty$  and  $s < 2/p$ . Then, for any  $T$ -time periodic function  $F$  satisfying*

$$f(t) := \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} F(\tau) d\tau \in BC((0, \infty); \dot{B}_{p,q}^{s-\alpha}(\mathbb{R}^2) \cap \dot{B}_{p,q}^s(\mathbb{R}^2)),$$

there exists a unique element  $u_0 \in \dot{B}_{p,q}^s(\mathbb{R}^2)$  such that the function

$$u(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}} u_0 + f(t), \quad t \geq 0$$

is  $T$ -time periodic. Moreover,  $u_0$  satisfies

$$\|u_0\|_{\dot{B}_{p,q}^s} \leq C \left( T^{-1} \|f(T)\|_{\dot{B}_{p,q}^{s-\alpha}} + \|f(T)\|_{\dot{B}_{p,q}^s} \right), \tag{2.26}$$

where  $C$  is the same constant as in Lemma 2.6.

*Proof.* By Lemma 2.6, the series

$$u_0 := \sum_{k=0}^{\infty} e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} f(T)$$

converges in  $\dot{B}_{p,q}^s(\mathbb{R}^2)$  and  $u_0$  satisfies (2.26) and

$$(1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})u_0 = f(T). \tag{2.27}$$

By the periodicity of  $F$ , we have

$$f(t + T) = f(t) + e^{-t(-\Delta)^{\frac{\alpha}{2}}} f(T), \quad t > 0. \tag{2.28}$$

Therefore, it follows from (2.27) and (2.28) that

$$\begin{aligned} u(t + T) &= e^{-t(-\Delta)^{\frac{\alpha}{2}}} e^{-T(-\Delta)^{\frac{\alpha}{2}}} u_0 + f(t + T) \\ &= e^{-t(-\Delta)^{\frac{\alpha}{2}}} (u_0 - f(T)) + f(t) + e^{-t(-\Delta)^{\frac{\alpha}{2}}} f(T) \\ &= u(t) \end{aligned}$$

for all  $t > 0$ . Hence,  $u(t)$  is  $T$ -time periodic. Next, we prove the uniqueness. Let  $v_0$  be an arbitrary element of  $\dot{B}_{p,q}^s(\mathbb{R}^2)$  such that  $v(t) := e^{-t(-\Delta)^{\frac{\alpha}{2}}} v_0 + f(t)$  is a  $T$ -time periodic function. Then, since  $u_0 - v_0 = u(NT) - v(NT) = e^{-NT(-\Delta)^{\frac{\alpha}{2}}} (u_0 - v_0)$  holds for all  $N \in \mathbb{N}$  by the periodicity, we obtain by (3) of Lemma 2.1 that

$$\|u_0 - v_0\|_{\dot{B}_{p,q}^s} = \left\| e^{-NT(-\Delta)^{\frac{\alpha}{2}}} (u_0 - v_0) \right\|_{\dot{B}_{p,q}^s} \rightarrow 0$$

as  $N \rightarrow \infty$ . Therefore, we have  $u_0 = v_0$  in  $\dot{B}_{p,q}^s(\mathbb{R}^2)$  and this completes the proof.  $\square$

Finally, we recall a positivity lemma for the  $L^p$ -energy of the fractional dissipation:

**Lemma 2.8.** ([4], [15]) Let  $0 < \alpha \leq 2$  and  $2 \leq p < \infty$ . Then, there exists a positive constant  $\lambda = \lambda(\alpha, p)$  such that

$$\int_{\mathbb{R}^2} |\Delta_j f(x)|^{p-2} \Delta_j f(x) (-\Delta)^{\frac{\alpha}{2}} \Delta_j f(x) dx \geq \lambda 2^{\alpha j} \|\Delta_j f\|_{L^p}^p$$

for all  $j \in \mathbb{Z}$  and  $f \in \mathcal{S}'_0(\mathbb{R}^2)$  with  $\Delta_j f \in L^p(\mathbb{R}^2)$ .

### 3. Proof of main results

In this section, we prove Theorems 1.1 and 1.3. Once Theorem 1.3 is proved, then we can get the time periodic solution on  $\mathbb{R}$  by extending the solution on  $[0, \infty)$  to the other half time line  $(-\infty, 0)$  periodically. Therefore, we only prove Theorem 1.3.

Let  $T, \alpha, p, q$  and  $r$  satisfy the assumptions of Theorem 1.3 and let  $\sigma$  satisfy  $\alpha - 2/p < \sigma < 2/p$ . We use the following notation for simplicity in this section:

$$s_c := 1 + \frac{2}{p} - \alpha,$$

$$X_T^{p,q} := \tilde{L}^\infty(0, T; \dot{B}_{p,1}^0(\mathbb{R}^2)) \cap \tilde{L}^\infty(0, T; \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)).$$

We consider the successive approximation sequences  $\{\theta_0^{(n)}\}_{n=0}^\infty \subset \dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)$  of the initial data and  $\{\theta^{(n)}\}_{n=0}^\infty \subset X_T^{p,q}$  of solutions to (1.5) defined inductively as follows:

First, let  $\theta_0^{(0)}(x) = 0$  and  $\theta^{(0)}(t, x) = 0$ . Next, if  $\theta_0^{(n)}$  and  $\theta^{(n)}$  are determined, then we define  $\theta_0^{(n+1)}$  and  $\theta^{(n+1)}$  by the following linear equation:

$$\begin{cases} \partial_t \theta^{(n+1)} + (-\Delta)^{\frac{\alpha}{2}} \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} = S_{n+4} F, & 0 < t \leq T, x \in \mathbb{R}^2, \\ u^{(n)} = \mathcal{R}^\perp \theta^{(n)}, & 0 \leq t \leq T, x \in \mathbb{R}^2, \\ \theta^{(n+1)}(0, x) = S_{n+4} \theta_0^{(n+1)}, & x \in \mathbb{R}^2, \end{cases} \quad (3.1)$$

where  $\theta_0^{(n+1)}$  is given by

$$\theta_0^{(n+1)} := \sum_{k=0}^\infty e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} \left( \theta^{(n)}(T) - e^{-T(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0) \right). \quad (3.2)$$

For  $n \in \mathbb{N} \cup \{0\}$ , we put  $\psi^{(n)}(t) := \theta^{(n)}(t) - e^{-t(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0)$  and

$$A_n := \max \left\{ \|\theta_0^{(n)}\|_{\dot{B}_{p,1}^0 \cap \dot{B}_{p,q}^{s_c}}, \|\theta^{(n)}\|_{X_T^{p,q}} \right\},$$

$$B_n := \|\theta_0^{(n+1)} - \theta_0^{(n)}\|_{\dot{B}_{p,q}^\sigma} + \|\theta^{(n+1)} - \theta^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)}.$$

The well-definedness of the sequences is assured if the series in (3.2) converges in  $\dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)$ . In the following lemma, we check the convergence and derive some properties of the sequences.

**Lemma 3.1.** *Let  $n$  be an positive integer. Assume that  $\theta_0^{(n)} \in \dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)$  and  $\theta^{(n)} \in X_T^{p,q}$ . Then, for every  $F \in BC((0, \infty); \dot{B}_{r,\infty}^0(\mathbb{R}^2))$ , the series in (3.2) converges in  $\dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)$  and it holds*

$$(1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}}) \theta_0^{(n+1)} = \theta^{(n)}(T) - e^{-T(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0) \quad (3.3)$$

in  $\dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)$ . Moreover, there exist positive constants  $\delta_1 = \delta_1(\alpha, p, q, r, T)$  and  $C_1 = C_1(\alpha, p, q, r, T)$  such that if  $F$  satisfies

$$\sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \leq \delta_1,$$

then it holds

$$\begin{aligned} \sup_{m \in \mathbb{N} \cup \{0\}} \|\theta_0^{(m)}\|_{\dot{B}_{p,1}^0 \cap \dot{B}_{p,q}^{sc}} &\leq 2C_1 \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0}, \\ \sup_{m \in \mathbb{N} \cup \{0\}} \|\theta^{(m)}\|_{X_T^{p,q}} &\leq 2C_1 \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0}. \end{aligned} \tag{3.4}$$

*Proof.* To prove the convergence of the series in (3.2) in  $\dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{sc}(\mathbb{R}^2)$ , Lemma 2.6 yields that it suffices to check

$$\begin{aligned} \theta^{(n)}(T) - e^{-T(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0) &= \psi^{(n)}(T) \\ &\in \left( \dot{B}_{p,q}^{sc}(\mathbb{R}^2) \cap \dot{B}_{p,q}^{sc-\alpha}(\mathbb{R}^2) \right) \cap \left( \dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,1}^{-\alpha}(\mathbb{R}^2) \right). \end{aligned}$$

Since  $\psi^{(n)}$  satisfies

$$\begin{aligned} \partial_t \psi^{(n)} + (-\Delta)^{\frac{\alpha}{2}} \psi^{(n)} + u^{(n-1)} \cdot \nabla \psi^{(n)} \\ + u^{(n-1)} \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0) &= S_{n+4} F, \end{aligned} \tag{3.5}$$

applying  $\Delta_j$  to (3.5), we see that

$$\begin{aligned} \partial_t \Delta_j \psi^{(n)} + (-\Delta)^{\frac{\alpha}{2}} \Delta_j \psi^{(n)} \\ = S_{n+4} \Delta_j F + [u^{(n-1)}, \Delta_j] \cdot \nabla \psi^{(n)} \\ - u^{(n-1)} \cdot \nabla \Delta_j \psi^{(n)} - \Delta_j (u^{(n-1)} \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0)). \end{aligned} \tag{3.6}$$

Multiplying (3.6) by  $p|\Delta_j \psi^{(n)}|^{p-2} \Delta_j \psi^{(n)}$  and integrating over  $\mathbb{R}^2$ , we have by the Hölder inequality that

$$\begin{aligned} \frac{d}{dt} (\|\Delta_j \psi^{(n)}(t)\|_{L^p}^p) + p \int_{\mathbb{R}^2} |\Delta_j \psi^{(n)}(t, x)|^{p-2} \Delta_j \psi^{(n)}(t, x) (-\Delta)^{\frac{\alpha}{2}} \Delta_j \psi^{(n)}(t, x) dx \\ \leq Cp \|\Delta_j F(t)\|_{L^p} \|\Delta_j \psi^{(n)}(t)\|_{L^p}^{p-1} + p \|[u^{(n-1)}(t), \Delta_j] \cdot \nabla \psi^{(n)}(t)\|_{L^p} \|\Delta_j \psi^{(n)}(t)\|_{L^p}^{p-1} \\ - p \int_{\mathbb{R}^2} |\Delta_j \psi^{(n)}(t, x)|^{p-2} \Delta_j \psi^{(n)}(t, x) u^{(n-1)}(t, x) \cdot \nabla \Delta_j \psi^{(n)}(t, x) dx \\ + p \|\Delta_j (u^{(n-1)}(t) \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0))\|_{L^p} \|\Delta_j \psi^{(n)}(t)\|_{L^p}^{p-1}. \end{aligned} \tag{3.7}$$

By Lemma 2.8, we obtain that

$$\int_{\mathbb{R}^2} |\Delta_j \psi^{(n)}(t, x)|^{p-2} \Delta_j \psi^{(n)}(t, x) (-\Delta)^{\frac{\alpha}{2}} \Delta_j \psi^{(n)}(t, x) dx \geq \lambda 2^{\alpha j} \|\Delta_j \psi^{(n)}(t)\|_{L^p}^p \tag{3.8}$$

for some  $\lambda = \lambda(\alpha, p) > 0$ . On the other hand, it follows from the divergence free condition  $\nabla \cdot u^{(n-1)} = 0$  that

$$\int_{\mathbb{R}^2} |\Delta_j \psi^{(n)}(t, x)|^{p-2} \Delta_j \psi^{(n)}(t, x) u^{(n-1)}(t, x) \cdot \nabla \Delta_j \psi^{(n)}(t, x) dx = 0. \tag{3.9}$$



Substituting (3.8) and (3.9) in (3.7), we have

$$\begin{aligned} & \frac{d}{dt} \|\Delta_j \psi^{(n)}(t)\|_{L^p} + \lambda 2^{\alpha j} \|\Delta_j \psi^{(n)}(t)\|_{L^p} \\ & \leq C \|\Delta_j F(t)\|_{L^p} + \|[u^{(n-1)}(t), \Delta_j] \cdot \nabla \psi^{(n)}(t)\|_{L^p} \\ & \quad + \|\Delta_j(u^{(n-1)}(t) \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0))\|_{L^p}, \end{aligned}$$

which implies that

$$\begin{aligned} \|\Delta_j \psi^{(n)}(T)\|_{L^p} & \leq C \int_0^T e^{-\lambda 2^{\alpha j}(T-\tau)} \|\Delta_j F(\tau)\|_{L^p} d\tau \\ & \quad + \int_0^T e^{-\lambda 2^{\alpha j}(T-\tau)} \|[u^{(n-1)}(\tau), \Delta_j] \cdot \nabla \psi^{(n)}(\tau)\|_{L^p} d\tau \\ & \quad + \int_0^T e^{-\lambda 2^{\alpha j}(T-\tau)} \|\Delta_j(u^{(n-1)}(\tau) \cdot \nabla e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0))\|_{L^p} d\tau. \end{aligned} \tag{3.10}$$

Let  $s \in \{s_c, s_c - \alpha\}$ . Multiplying (3.10) by  $2^{sj}$ , we have

$$\begin{aligned} & 2^{sj} \|\Delta_j \psi^{(n)}(T)\|_{L^p} \\ & \leq C \int_0^T 2^{(s+\frac{2}{r}-\frac{2}{p})j} e^{-\lambda 2^{\alpha j}(T-\tau)} \|\Delta_j F(\tau)\|_{L^r} d\tau \\ & \quad + \int_0^T 2^{(\alpha+s-s_c)j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau \\ & \quad \quad \times 2^{(2s_c-1-\frac{2}{p})j} \|[u^{(n-1)}, \Delta_j] \cdot \nabla \psi^{(n)}\|_{L^\infty(0,T;L^p)} \\ & \quad + \int_0^T 2^{(\alpha+s-s_c)j} e^{-\lambda 2^{\alpha j}(T-\tau)} \\ & \quad \quad \times 2^{(2s_c-1-\frac{2}{p})j} \|\Delta_j(u^{(n-1)}(\tau) \cdot \nabla e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0))\|_{L^p} d\tau. \end{aligned} \tag{3.11}$$

Taking  $l^q(\mathbb{Z})$ -norm of (3.11), we see by (2.12) that

$$\begin{aligned} & \|\psi^{(n)}(T)\|_{\dot{B}_{p,q}^s} \\ & \leq C \sum_{j \in \mathbb{Z}} \int_0^T 2^{(s+\frac{2}{r}-\frac{2}{p})j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ & \quad + CT^{\frac{s_c-s}{\alpha}} \left\| \left\{ 2^{(2s_c-1-\frac{2}{p})j} \|[u^{(n-1)}, \Delta_j] \cdot \nabla \psi^{(n)}\|_{L^\infty(0,T;L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ & \quad + \left\| \left\{ \int_0^T 2^{(\alpha+s-s_c)j} e^{-\lambda 2^{\alpha j}(T-\tau)} \right. \right. \\ & \quad \quad \left. \left. \times 2^{(2s_c-1-\frac{2}{p})j} \|\Delta_j(u^{(n-1)}(\tau) \cdot \nabla e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} \theta^{(n)}(0))\|_{L^p} d\tau \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})}. \end{aligned}$$

Hence, it follows from Lemma 2.3 and (2.5) that

$$\begin{aligned} \|\psi^{(n)}(T)\|_{\dot{B}_{p,q}^s} &\leq CT^{1-\frac{1}{\alpha}(s+\frac{2}{r}-\frac{2}{p})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ &\quad + CT^{\frac{s_c-s}{\alpha}} \|u^{(n-1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} \|\psi^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} \\ &\quad + CT^{\frac{s_c-s}{\alpha}} (T^{-\frac{1-\alpha}{\alpha}} + 1) \|u^{(n-1)}\|_{X_T^{p,q}} \|\theta^{(n)}(0)\|_{\dot{B}_{p,q}^{s_c}}. \end{aligned}$$

Using

$$\begin{aligned} \|\psi^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} &\leq \|\theta^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} + \|e^{-t(-\Delta)^{\frac{\alpha}{2}}}\theta^{(n)}(0)\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} \\ &\leq \|\theta^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} + C\|\theta^{(n)}(0)\|_{\dot{B}_{p,q}^{s_c}} \\ &\leq \|\theta^{(n)}\|_{X_T^{p,q}} + C\|\theta_0^{(n)}\|_{\dot{B}_{p,q}^{s_c}} \\ &\leq CA_n \end{aligned}$$

and the boundedness of the Riesz transform on the homogeneous space-time Besov spaces, we obtain

$$\begin{aligned} \|\psi^{(n)}(T)\|_{\dot{B}_{p,q}^s} &\leq CT^{1-\frac{1}{\alpha}(s+\frac{2}{r}-\frac{2}{p})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ &\quad + CT^{\frac{s_c-s}{\alpha}} \|\theta^{(n-1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} A_n \\ &\quad + CT^{\frac{s_c-s}{\alpha}} (T^{-\frac{1-\alpha}{\alpha}} + 1) \|\theta^{(n-1)}\|_{X_T^{p,q}} \|\theta_0^{(n)}\|_{\dot{B}_{p,q}^{s_c}}. \end{aligned} \tag{3.12}$$

Hence, it follows from (3.12) and Lemma 2.6 that

$$\begin{aligned} \|\theta_0^{(n+1)}\|_{\dot{B}_{p,q}^{s_c}} &\leq C(T^{-1}\|\psi^{(n)}(T)\|_{\dot{B}_{p,q}^{s_c-\alpha}} + \|\psi^{(n)}(T)\|_{\dot{B}_{p,q}^{s_c}}) \\ &\leq CT^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} + C(1 + T^{-\frac{1-\alpha}{\alpha}}) A_{n-1} A_n. \end{aligned} \tag{3.13}$$

Let  $s' := s - s_c \in \{-\alpha, 0\}$ . Multiplying (3.10) by  $2^{s'j}$  and taking  $l^1(\mathbb{Z})$ -norm, we see that

$$\begin{aligned} &\|\psi^{(n)}(T)\|_{\dot{B}_{p,1}^{s'}} \\ &\leq C \sum_{j \in \mathbb{Z}} \int_0^T 2^{(s'+\frac{2}{r}-\frac{2}{p})j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ &\quad + \sum_{j \in \mathbb{Z}} \int_0^T 2^{(\alpha+s-2s_c)j} e^{-\lambda 2^{\alpha j}(T-\tau)} d\tau \\ &\quad \times \left\| \left\{ 2^{(2s_c-1-\frac{2}{p})j} \|[u^{(n-1)}, \Delta_j] \cdot \nabla \psi^{(n)}\|_{L^\infty(0,T;L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ &\quad + \sum_{j \in \mathbb{Z}} \int_0^T 2^{(\alpha+s-2s_c)j} e^{-\lambda 2^{\alpha j}(T-\tau)} \\ &\quad \times \|u^{(n-1)}(\tau) \cdot \nabla e^{-\tau(-\Delta)^{\frac{\alpha}{2}}}\theta^{(n)}(0)\|_{\dot{B}_{p,q}^{2s_c-1-\frac{2}{p}}} d\tau. \end{aligned}$$

It follows from Lemma 2.3 and (2.6) that

$$\begin{aligned} \|\psi^{(n)}(T)\|_{\dot{B}_{p,1}^{s'}} &\leq CT^{1-\frac{1}{\alpha}(s'+\frac{2}{r}-\frac{2}{p})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ &\quad + CT^{\frac{2s_c-s}{\alpha}} \|u^{(n-1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} \|\psi^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} \\ &\quad + CT^{\frac{2s_c-s}{\alpha}} (T^{-\frac{1-\alpha}{\alpha}} + 1) \|u^{(n-1)}\|_{X_T^{p,q}} \|\theta^{(n)}(0)\|_{\dot{B}_{p,q}^{s_c}}. \end{aligned}$$

Therefore, by the same argument as above, we have

$$\begin{aligned} \|\psi^{(n)}(T)\|_{\dot{B}_{p,q}^{s'}} &\leq CT^{\frac{s_c}{\alpha}} T^{\frac{s_c-s}{\alpha}} T^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ &\quad + CT^{\frac{s_c}{\alpha}} T^{\frac{s_c-s}{\alpha}} (1 + T^{-\frac{1-\alpha}{\alpha}}) A_{n-1} A_n \end{aligned}$$

and we see that the series in (3.2) converges in  $\dot{B}_{p,1}^0(\mathbb{R}^2)$  and

$$\begin{aligned} \|\theta_0^{(n+1)}\|_{\dot{B}_{p,1}^0} &\leq C(T^{-1} \|\psi^{(n)}(T)\|_{\dot{B}_{p,1}^{-\alpha}} + \|\psi^{(n)}(T)\|_{\dot{B}_{p,1}^0}) \\ &\leq CT^{\frac{s_c}{\alpha}} T^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} + CT^{\frac{s_c}{\alpha}} (1 + T^{-\frac{1-\alpha}{\alpha}}) A_{n-1} A_n. \end{aligned} \tag{3.14}$$

Combining (3.12) and (3.14), we find that Lemma 2.6 implies the series in (3.2) converges in  $\dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)$  and (3.3) also holds.

Next, we prove (3.4). Since  $\Delta_j \theta^{(n+1)}$  satisfies

$$\begin{aligned} \partial_t \Delta_j \theta^{(n+1)} + (-\Delta)^{\frac{\alpha}{2}} \Delta_j \theta^{(n+1)} \\ = S_{n+4} \Delta_j F + [u^{(n)}, \Delta_j] \cdot \nabla \theta^{(n+1)} - u^{(n)} \cdot \nabla \Delta_j \theta^{(n+1)}, \end{aligned}$$

the same argument as in the derivation of (3.10) yields that

$$\begin{aligned} \|\Delta_j \theta^{(n+1)}(t)\|_{L^p} &\leq e^{-\lambda 2^{\alpha j} t} \|\Delta_j \theta^{(n)}(0)\|_{L^p} + C \int_0^t e^{-\lambda 2^{\alpha j} (t-\tau)} \|\Delta_j F(\tau)\|_{L^p} d\tau \\ &\quad + \int_0^t e^{\lambda 2^{\alpha j} (t-\tau)} \|[u^{(n)}(\tau), \Delta_j] \cdot \nabla \theta^{(n+1)}(\tau)\|_{L^p} d\tau, \end{aligned} \tag{3.15}$$

which implies that

$$\begin{aligned} 2^{s_c j} \|\Delta_j \theta^{n+1}(t)\|_{L^p} \\ \leq C 2^{s_c j} \|\Delta_j \theta_0^{(n+1)}\|_{L^p} + C \int_0^t 2^{(1+\frac{2}{r}-\alpha)j} e^{-\lambda 2^{\alpha j} (t-\tau)} \|\Delta_j F(\tau)\|_{L^p} d\tau \\ + \int_0^t 2^{\alpha j} e^{-\lambda 2^{\alpha j} (t-\tau)} d\tau 2^{(2s_c-1-\frac{2}{p})j} \|[u^{(n)}, \Delta_j] \cdot \nabla \theta^{(n+1)}\|_{L^\infty(0,T;L^p)}. \end{aligned} \tag{3.16}$$

Taking  $L_r^\infty(0, T)$ -norm of (3.16) and then taking  $l^q(\mathbb{Z})$ -norm, we see by Lemma 2.3 that

$$\begin{aligned}
 & \|\theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{sc})} \\
 & \leq C\|\theta_0^{(n+1)}\|_{\dot{B}_{p,q}^{sc}} + CT^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\
 & \quad + C \left\| \left\{ 2^{(2s_c-1-\frac{2}{p})j} \|[u^{(n)}, \Delta_j] \cdot \nabla\theta^{(n+1)}\|_{L^\infty(0,T;L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\
 & \leq C\|\theta_0^{(n+1)}\|_{\dot{B}_{p,q}^{sc}} + CT^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\
 & \quad + C\|\theta^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{sc})} \|\theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{sc})}. \tag{3.17}
 \end{aligned}$$

On the other hand, by (3.15), we see that

$$\begin{aligned}
 & \|\Delta_j\theta^{n+1}(t)\|_{L^p} \\
 & \leq C\|\Delta_j\theta_0^{(n+1)}\|_{L^p} + C \int_0^t 2^{(\frac{2}{r}-\frac{2}{p})j} e^{-\lambda 2^{\alpha j}(t-\tau)} d\tau \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\
 & \quad + \int_0^t 2^{(\alpha-s_c)j} e^{-\lambda 2^{\alpha j}(t-\tau)} d\tau \\
 & \quad \times \left\| \left\{ 2^{(2s_c-1-\frac{2}{p})j} \|[u^{(n)}, \Delta_j] \cdot \nabla\theta^{(n+1)}\|_{L^\infty(0,T;L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})}. \tag{3.18}
 \end{aligned}$$

Taking  $L_r^\infty(0, T)$ -norm and then  $l^1(\mathbb{Z})$ -norm of (3.18), we have

$$\begin{aligned}
 & \|\theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,1}^0)} \\
 & \leq C\|\theta_0^{(n+1)}\|_{\dot{B}_{p,1}^0} + C \sum_{j \in \mathbb{Z}} \sup_{0 \leq t \leq T} \int_0^t 2^{(1+\frac{2}{r}-\alpha)j} e^{-\lambda 2^{\alpha j}(t-\tau)} d\tau \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\
 & \quad + \sum_{j \in \mathbb{Z}} \sup_{0 \leq t \leq T} \int_0^t 2^{(\alpha-s_c)j} e^{-\lambda 2^{\alpha j}(t-\tau)} d\tau \\
 & \quad \times \left\| \left\{ 2^{(2s_c-1-\frac{2}{p})j} \|[u^{(n)}, \Delta_j] \cdot \nabla\theta^{(n+1)}\|_{L^\infty(0,T;L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})}.
 \end{aligned}$$

Using Lemma 2.3 and the second inequality of (2.16), we obtain that

$$\begin{aligned}
 \|\theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,1}^0)} & \leq C\|\theta_0^{(n+1)}\|_{\dot{B}_{p,1}^0} + CT^{\frac{s_c}{\alpha}} T^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\
 & \quad + CT^{\frac{s_c}{\alpha}} \|\theta^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{sc})} \|\theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{sc})}. \tag{3.19}
 \end{aligned}$$

Hence, combining estimates (3.12), (3.14), (3.17) and (3.19), we obtain

$$A_{n+1} \leq C_1 \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} + C_1 A_{n-1} A_n + C_1 A_n A_{n+1} \tag{3.20}$$

for some  $C_1 = C_1(\alpha, p, q, r, T)$ .

On the other hand, since  $\theta^{(1)}$  satisfies

$$\begin{cases} \partial_t \theta^{(1)} + (-\Delta)^{\frac{\alpha}{2}} \theta^{(1)} = S_4 F, & t > 0, x \in \mathbb{R}^2, \\ \theta^{(1)}(0, x) = 0, & x \in \mathbb{R}^2, \end{cases}$$

the simpler argument than above yields that

$$A_1 \leq C_1 \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0}. \tag{3.21}$$

Hence, if  $F$  satisfies

$$\sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \leq \delta_1 := \frac{1}{8C_1^2},$$

then by (3.20), (3.21) and the inductive argument, we obtain

$$A_m \leq 2C_1 \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0}$$

for all  $m \in \mathbb{N} \cup \{0\}$ . This completes the proof. □

Next lemma ensures the convergence of the approximation sequences.

**Lemma 3.2.** *There exists a positive constant  $\delta_2 = \delta_2(\alpha, p, q, r, \sigma, T) \leq \delta_1$  such that if  $F \in BC((0, \infty); \dot{B}_{r,\infty}^0(\mathbb{R}^2))$  satisfies*

$$\sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \leq \delta_2,$$

then it holds

$$\sum_{n=0}^{\infty} \|\theta_0^{(n+1)} - \theta_0^{(n)}\|_{\dot{B}_{p,q}^{\sigma}} + \sum_{n=0}^{\infty} \|\theta^{(n+1)} - \theta^{(n)}\|_{\tilde{L}^{\infty}(0,T;\dot{B}_{p,q}^{\sigma})} < \infty.$$

*Proof.* Due to

$$\theta_0^{(n+2)} - \theta_0^{(n+1)} = \sum_{k=0}^{\infty} e^{-Tk(-\Delta)^{\frac{\alpha}{2}}} (\psi^{(n+1)}(T) - \psi^{(n)}(T))$$

and Lemma 2.6, we consider the estimates of  $\psi^{(n+1)}(T) - \psi^{(n)}(T)$  in  $\dot{B}_{p,q}^{\sigma-\alpha}(\mathbb{R}^2) \cap \dot{B}_{p,q}^{\sigma}(\mathbb{R}^2)$ . Since  $\psi^{(n+1)} - \psi^{(n)}$  satisfies

$$\begin{aligned} &\partial_t (\psi^{(n+1)} - \psi^{(n)}) + (-\Delta)^{\frac{\alpha}{2}} (\psi^{(n+1)} - \psi^{(n)}) + u^{(n)} \cdot \nabla (\psi^{(n+1)} - \psi^{(n)}) \\ &+ u^{(n)} \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}} (\theta^{(n+1)}(0) - \theta^{(n)}(0)) + (u^{(n)} - u^{(n-1)}) \cdot \nabla \theta^{(n)} = \Delta_{n+1} F, \end{aligned} \tag{3.22}$$

we see that

$$\begin{aligned} &\partial_t \Delta_j (\psi^{(n+1)} - \psi^{(n)}) + (-\Delta)^{\frac{\alpha}{2}} \Delta_j (\psi^{(n+1)} - \psi^{(n)}) \\ &= \Delta_{n+1} \Delta_j F + [u^{(n)}, \Delta_j] \cdot \nabla (\psi^{(n+1)} - \psi^{(n)}) - u^{(n)} \cdot \nabla \Delta_j (\psi^{(n+1)} - \psi^{(n)}) \\ &\quad - \Delta_j (u^{(n)} \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}} (\theta^{(n+1)}(0) - \theta^{(n)}(0))) - \Delta_j ((u^{(n)} - u^{(n-1)}) \cdot \nabla \theta^{(n)}). \end{aligned}$$

Thus, the similar energy calculation as in the proof of Lemma 3.1 yields that

$$\begin{aligned} & \|\Delta_j(\psi^{(n+1)}(T) - \psi^{(n)}(T))\|_{L^p} \\ & \leq C \int_0^T e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{2(\frac{1}{r}-\frac{1}{p})j} \|\Delta_{n+1} \Delta_j F(\tau)\|_{L^r} d\tau \\ & \quad + \int_0^T e^{-\lambda 2^{\alpha j}(T-\tau)} \|[u^{(n)}(\tau), \Delta_j] \cdot \nabla(\psi^{(n+1)}(\tau) - \psi^{(n)}(\tau))\|_{L^p} d\tau \\ & \quad + \int_0^T e^{-\lambda 2^{\alpha j}(T-\tau)} \|\Delta_j(u^{(n)}(\tau) \cdot \nabla e^{-\tau(-\Delta)^{\frac{\alpha}{2}}}(\theta^{(n+1)}(0) - \theta^{(n)}(0)))\|_{L^p} d\tau \\ & \quad + \int_0^T e^{-\lambda 2^{\alpha j}(T-\tau)} \|\Delta_j((u^{(n)}(\tau) - u^{(n-1)}(\tau)) \cdot \nabla \theta^{(n)}(\tau))\|_{L^p} d\tau. \end{aligned}$$

Let  $s \in \{\sigma, \sigma - \alpha\}$ . Multiplying this by  $2^{sj}$  and taking  $l^q(\mathbb{Z})$ -norm of this, we obtain that

$$\begin{aligned} & \|\psi^{(n+1)}(T) - \psi^{(n)}(T)\|_{\dot{B}_{p,q}^s} \\ & \leq CT^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} 2^{-(s_c-\sigma)n} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ & \quad + CT^{\frac{\sigma-s}{\alpha}} \left\| \left\{ 2^{(s_c+(\sigma-1)-\frac{2}{p})j} \|[u^{(n)}, \Delta_j] \cdot \nabla(\psi^{(n+1)} - \psi^{(n)})\|_{L^\infty(0,T;L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ & \quad + \left\| \left\{ \int_0^T 2^{(\alpha+s-\sigma)j} e^{-\lambda 2^{\alpha j}(T-\tau)} 2^{(s_c+(\sigma-1)-\frac{2}{p})j} \right. \right. \\ & \quad \quad \left. \left. \times \|\Delta_j(u^{(n)} \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}}(\theta^{(n+1)}(0) - \theta^{(n)}(0)))\|_{L^p} d\tau \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} \\ & \quad + CT^{\frac{\sigma-s}{\alpha}} \|(u^{(n)} - u^{(n-1)}) \cdot \nabla \theta^{(n)}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^{\sigma+(s_c-1)-\frac{2}{p}})}. \end{aligned}$$

Therefore, it follows from Lemmas 2.2, 2.3 and (2.5) that

$$\begin{aligned} & \|\psi^{(n+1)}(T) - \psi^{(n)}(T)\|_{\dot{B}_{p,q}^s} \\ & \leq CT^{\frac{\sigma-s}{\alpha}} T^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} 2^{-(s_c-\sigma)n} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ & \quad + CT^{\frac{\sigma-s}{\alpha}} \|\theta^{(n)}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^{s_c})} \|\psi^{(n+1)} - \psi^{(n)}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^\sigma)} \\ & \quad + CT^{\frac{\sigma-s}{\alpha}} (1 + T^{-\frac{1-\alpha}{\alpha}}) \|\theta^{(n)}\|_{X_T^{p,q}} \|\theta^{(n)}(0) - \theta^{(n-1)}(0)\|_{\dot{B}_{p,q}^\sigma} \\ & \quad + CT^{\frac{\sigma-s}{\alpha}} \|\theta^{(n)}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^{s_c})} \|\theta^{(n)} - \theta^{(n-1)}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^\sigma)}. \end{aligned} \tag{3.23}$$

This gives

$$\begin{aligned} & \|\theta_0^{(n+1)} - \theta_0^{(n)}\|_{\dot{B}_{p,q}^\sigma} \\ & \leq C(T^{-1} \|\psi^{(n+1)}(T) - \psi^{(n)}(T)\|_{\dot{B}_{p,q}^{\sigma-\alpha}} + \|\psi^{(n+1)}(T) - \psi^{(n)}(T)\|_{\dot{B}_{p,q}^\sigma}) \\ & \leq CT^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} 2^{-(s_c-\sigma)n} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ & \quad + CA_n B_n + C(1 + T^{-\frac{1-\alpha}{\alpha}}) A_n B_{n-1}. \end{aligned} \tag{3.24}$$

Since  $\theta^{(n+2)} - \theta^{(n+1)}$  satisfy

$$\begin{aligned} \partial_t(\theta^{(n+2)} - \theta^{(n+1)}) + (-\Delta)^{\frac{\alpha}{2}}(\theta^{(n+2)} - \theta^{(n+1)}) \\ + u^{(n+1)} \cdot \nabla(\theta^{(n+2)} - \theta^{(n+1)}) + (u^{(n+1)} - u^{(n)}) \cdot \nabla\theta^{(n+1)} = \Delta_{n+2}F, \end{aligned}$$

we see that

$$\begin{aligned} \partial_t \Delta_j(\theta^{(n+2)} - \theta^{(n+1)}) + (-\Delta)^{\frac{\alpha}{2}} \Delta_j(\theta^{(n+2)} - \theta^{(n+1)}) \\ = \Delta_j \Delta_{n+2}F + [u^{(n+1)}, \Delta_j] \cdot \nabla(\theta^{(n+2)} - \theta^{(n+1)}) \\ - u^{(n+1)} \cdot \nabla \Delta_j(\theta^{(n+2)} - \theta^{(n+1)}) - \Delta_j((u^{(n+1)} - u^{(n)}) \cdot \nabla\theta^{(n+1)}). \end{aligned}$$

By the similar energy calculation as in the proof of Lemma 3.1, we have

$$\begin{aligned} \|\Delta_j(\theta^{(n+2)}(t) - \theta^{(n+1)}(t))\|_{L^p} \\ \leq e^{-\lambda 2^{\alpha j} t} \|\Delta_j(\theta^{(n+2)}(0) - \theta^{(n+1)}(0))\|_{L^p} \\ + C \int_0^t 2^{(\frac{2}{r} - \frac{2}{p})j} e^{-\lambda 2^{\alpha j}(t-\tau)} \|\Delta_j \Delta_{n+2}F(\tau)\|_{L^r} d\tau \\ + \int_0^t e^{-\lambda 2^{\alpha j}(t-\tau)} \|[u^{(n+1)}(\tau), \Delta_j] \cdot \nabla(\theta^{(n+2)}(\tau) - \theta^{(n+1)}(\tau))\|_{L^p} d\tau \\ + \int_0^t e^{-\lambda 2^{\alpha j}(t-\tau)} \|\Delta_j((u^{(n+1)}(\tau) - u^{(n)}(\tau)) \cdot \nabla\theta^{(n+1)}(\tau))\|_{L^p} d\tau. \end{aligned} \tag{3.25}$$

Multiplying (3.25) by  $2^{\sigma j}$ , we see that

$$\begin{aligned} 2^{\sigma j} \|\Delta_j(\theta^{(n+2)}(t) - \theta^{(n+1)}(t))\|_{L^p} \\ \leq 2^{\sigma j} \|\Delta_j(\theta^{(n+2)}(0) - \theta^{(n+1)}(0))\|_{L^p} \\ + C \int_0^t 2^{(1+\frac{2}{r}-\alpha)j} e^{-\lambda 2^{\alpha j}(t-\tau)} d\tau \sup_{t>0} \|\Delta_{n+2}F(t)\|_{\dot{B}_{r,\infty}^{s_c-\sigma}} \\ + \int_0^t 2^{\alpha j} e^{-\lambda 2^{\alpha j}(t-\tau)} d\tau 2^{(s_c+(\sigma-1)-\frac{2}{p})j} \|[u^{(n+1)}, \Delta_j] \cdot \nabla(\theta^{(n+2)} - \theta^{(n+1)})\|_{L^\infty(0,T;L^p)} \\ + \int_0^t 2^{\alpha j} e^{-\lambda 2^{\alpha j}(t-\tau)} d\tau 2^{(\sigma+(s_c-1)-\frac{2}{p})j} \|\Delta_j((u^{(n+1)} - u^{(n)}) \cdot \nabla\theta^{(n+1)})\|_{L^\infty(0,T;L^p)}. \end{aligned}$$

By taking  $L_t^\infty(0, T)$  and then  $l^q(\mathbb{Z})$ -norm, it follows from (2.16) and Lemmas 2.2 and 2.3 that

$$\begin{aligned} \|\theta^{(n+2)} - \theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)} \\ \leq \|\theta^{(n+2)}(0) - \theta^{(n+1)}(0)\|_{\dot{B}_{p,q}^\sigma} \\ + C T^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} 2^{-(s_c-\sigma)n} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \\ + C \|\theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} \|\theta^{(n+2)} - \theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)} \\ + C \|\theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} \|\theta^{(n+1)} - \theta^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)}. \end{aligned} \tag{3.26}$$

Since it holds

$$\begin{aligned} \|\theta^{(n+2)}(0) - \theta^{(n+1)}(0)\|_{\dot{B}_{p,q}^\sigma} &\leq \|S_{n+5}(\theta_0^{(n+2)} - \theta_0^{(n+1)})\|_{\dot{B}_{p,q}^\sigma} + \|\Delta_{n+2}\theta^{(n+1)}\|_{\dot{B}_{p,q}^\sigma} \\ &\leq C\|\theta_0^{(n+2)} - \theta_0^{(n+1)}\|_{\dot{B}_{p,q}^\sigma} + C2^{-(s_c-\sigma)n}\|\theta_0^{(n+1)}\|_{\dot{B}_{p,q}^{s_c}} \\ &\leq C\|\theta_0^{(n+2)} - \theta_0^{(n+1)}\|_{\dot{B}_{p,q}^\sigma} + C2^{-(s_c-\sigma)n}A_{n+1}, \end{aligned}$$

we have by (3.26) that

$$\begin{aligned} &\|\theta^{(n+2)} - \theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)} \\ &\leq C\|\theta_0^{(n+2)} - \theta_0^{(n+1)}\|_{\dot{B}_{p,q}^\sigma} \\ &\quad + C2^{-(s_c-\sigma)n} \left( T^{\frac{1}{\alpha}(2\alpha-1-\frac{2}{r})} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} + A_{n+1} \right) \\ &\quad + C\|\theta^{(n+1)}\|_{X_T^{p,q}} \|\theta^{(n+1)} - \theta^{(n)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)} \\ &\quad + C\|\theta^{(n+1)}\|_{X_T^{p,q}} \|\theta^{(n+2)} - \theta^{(n+1)}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)}. \end{aligned} \tag{3.27}$$

Therefore, combining (3.24) and (3.27), we obtain

$$\begin{aligned} B_{n+1} &\leq C_2 2^{-(s_c-\sigma)n} \left( \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} + A_{n+1} \right) \\ &\quad + C_2 A_n B_{n-1} + C_2 (A_n + A_{n+1}) B_n + C_2 A_{n+1} B_{n+1} \end{aligned} \tag{3.28}$$

for some  $C_2 = C_2(\alpha, p, q, r, \sigma, T) > 0$ . Here, we assume that

$$\sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \leq \delta_2 =: \min \left\{ \delta_1, \frac{1}{16C_1C_2} \right\}.$$

Let  $N \in \mathbb{N}$  satisfy  $N \geq 2$ . Then, summing (3.28) over  $n = 1, \dots, N - 1$  and using Lemma 3.1, we have

$$\begin{aligned} \sum_{n=1}^{N-1} B_{n+1} &\leq C_T \delta_1 \sum_{n=1}^{N-1} 2^{-(s_c-\sigma)n} \\ &\quad + 2C_1C_2 \sum_{n=1}^{N-1} B_{n-1} + 4C_1C_2 \sum_{n=1}^{N-1} B_n + 2C_1C_2 \sum_{n=1}^{N-1} B_{n+1} \end{aligned}$$

for some constant  $C_T > 0$  depending on  $T$ . This implies

$$\begin{aligned} \sum_{n=2}^N B_n &\leq C_T \delta_1 \sum_{n=1}^{N-1} 2^{-(s_c-\sigma)n} + \frac{1}{8} \sum_{n=0}^{N-2} B_n + \frac{2}{8} \sum_{n=1}^{N-1} B_n + \frac{1}{8} \sum_{n=2}^N B_n \\ &\leq C_T \delta_1 \sum_{n=1}^\infty 2^{-(s_c-\sigma)n} + \frac{1}{2} \sum_{n=0}^N B_n. \end{aligned}$$



Hence, we have

$$\frac{1}{2} \sum_{n=0}^{\infty} B_n \leq C_T \delta_1 \sum_{n=1}^{\infty} 2^{-(s_c-\sigma)n} + B_0 + B_1 < \infty,$$

which completes the proof. □

**Lemma 3.3.** *There exists a positive constant  $C_3 = C_3(\alpha, p, q, r, T, \sigma)$  such that*

$$\begin{aligned} & \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^\sigma)} \\ & \leq C_3 \left( \|\theta\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^{s_c})} + \|\tilde{\theta}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^{s_c})} \right) \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^\sigma)} \end{aligned} \quad (3.29)$$

for all  $T$ -time periodic solutions  $\theta \in BC([0, \infty); B_{p,q}^{s_c}(\mathbb{R}^2)) \cap X_T^{p,q}$  and  $\tilde{\theta} \in BC([0, \infty); B_{p,q}^{s_c}(\mathbb{R}^2)) \cap \tilde{L}^\infty(0, \infty; \dot{B}_{p,q}^{s_c}(\mathbb{R}^2))$  to (1.5) with the same  $T$ -time periodic external force  $F$ .

*Proof.* Since  $\theta - \tilde{\theta}$  satisfies

$$\partial_t(\theta - \tilde{\theta}) + (-\Delta)^{\frac{\alpha}{2}}(\theta - \tilde{\theta}) + u \cdot \nabla(\theta - \tilde{\theta}) + (u - \tilde{u}) \cdot \nabla\tilde{\theta} = 0,$$

where  $u = \mathcal{R}^\perp \theta, \tilde{u} = \mathcal{R}^\perp \tilde{\theta}$ , we see that

$$\begin{aligned} & \partial_t \Delta_j(\theta - \tilde{\theta}) + (-\Delta)^{\frac{\alpha}{2}} \Delta_j(\theta - \tilde{\theta}) \\ & = [u, \Delta_j] \cdot \nabla(\theta - \tilde{\theta}) - u \cdot \nabla \Delta_j(\theta - \tilde{\theta}) - \Delta_j((u - \tilde{u}) \cdot \nabla\tilde{\theta}). \end{aligned}$$

Therefore, it follows from the similar energy calculation as in the derivation of (3.27) that

$$\begin{aligned} & \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^\sigma)} \\ & \leq \|\theta(0) - \tilde{\theta}(0)\|_{\dot{B}_{p,q}^\sigma} \\ & \quad + C \left( \|\theta\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^{s_c})} + \|\tilde{\theta}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^{s_c})} \right) \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T; \dot{B}_{p,q}^\sigma)}. \end{aligned} \quad (3.30)$$

Next, we derive the estimate for  $\theta(0) - \tilde{\theta}(0)$ . Since  $\theta - \tilde{\theta}$  is  $T$ -time periodic and the Duhamel principle gives

$$\begin{aligned} & \theta(t) - \tilde{\theta}(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}}(\theta(0) - \tilde{\theta}(0)) \\ & \quad - \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}(u(\tau) \cdot \nabla\theta(\tau) - \tilde{u}(\tau) \cdot \nabla\tilde{\theta}(\tau))d\tau, \end{aligned}$$

we have by Lemma 2.7 that

$$\|\theta(0) - \tilde{\theta}(0)\|_{\dot{B}_{p,q}^\sigma} \leq C(T^{-1}\|\psi(T) - \tilde{\psi}(T)\|_{\dot{B}_{p,q}^{\sigma-\alpha}} + \|\psi(T) - \tilde{\psi}(T)\|_{\dot{B}_{p,q}^\sigma}), \quad (3.31)$$

where  $\psi(t) := \theta(t) - e^{-t(-\Delta)^{\frac{\alpha}{2}}}\theta(0)$  and  $\tilde{\psi}(t) := \tilde{\theta}(t) - e^{-t(-\Delta)^{\frac{\alpha}{2}}}\tilde{\theta}(0)$ . Since it holds

$$\begin{aligned} & \partial_t(\psi - \tilde{\psi}) + (-\Delta)^{\frac{\alpha}{2}}(\psi - \tilde{\psi}) \\ & = -u \cdot \nabla(\psi - \tilde{\psi}) - u \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}}(\theta(0) - \tilde{\theta}(0)) - (u - \tilde{u}) \cdot \nabla\tilde{\theta}, \end{aligned} \quad (3.32)$$

applying  $\Delta_j$  to (3.32), we have

$$\begin{aligned} & \partial_t \Delta_j(\psi - \tilde{\psi}) + (-\Delta)^{\frac{\alpha}{2}} \Delta_j(\psi - \tilde{\psi}) \\ &= [u, \Delta_j] \cdot \nabla(\psi - \tilde{\psi}) - u \cdot \nabla \Delta_j(\psi - \tilde{\psi}) \\ & \quad - \Delta_j(u \cdot \nabla e^{-t(-\Delta)^{\frac{\alpha}{2}}}(\theta(0) - \tilde{\theta}(0))) - \Delta_j((u - \tilde{u}) \cdot \nabla \tilde{\theta}). \end{aligned}$$

Hence, by the similar energy calculation as in the derivation of (3.23), we obtain

$$\begin{aligned} & T^{-1} \|\psi(T) - \tilde{\psi}(T)\|_{\dot{B}_{p,q}^{\sigma-\alpha}} + \|\psi(T) - \tilde{\psi}(T)\|_{\dot{B}_{p,q}^{\sigma}} \\ & \leq C(1 + T^{-\frac{1-\alpha}{\alpha}}) \left( \|\theta\|_{X_T^{p,q}} + \|\tilde{\theta}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{sc})} \right) \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{\sigma})}. \end{aligned} \tag{3.33}$$

Here, we have used

$$\|\theta(0) - \tilde{\theta}(0)\|_{\dot{B}_{p,q}^{\sigma}} \leq \sup_{0 \leq t \leq T} \|\theta(t) - \tilde{\theta}(t)\|_{\dot{B}_{p,q}^{\sigma}} \leq \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{\sigma})}.$$

Combining (3.30), (3.31) and (3.33), we get (3.29). This completes the proof.  $\square$

Now we are in a position to prove Theorem 1.3.

*Proof of Theorem 1.3.* Let  $\alpha, p, q, r$  and  $T$  satisfy the assumptions of Theorem 1.3 and let  $\sigma := \alpha/2$ . Then,  $\sigma$  satisfies  $\alpha - 2/p < \sigma < 2/p$ . We put

$$\delta := \min \left\{ \delta_1, \delta_2, \frac{1}{8C_1C_3} \right\}$$

and let  $F \in BC((0, \infty); \dot{B}_{r,\infty}^0(\mathbb{R}^2))$  satisfy

$$\sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \leq \delta.$$

It follows from (3.4) that

$$\sup_{n \in \mathbb{N} \cup \{0\}} \|\theta^{(n)}\|_{X_T^{p,q}} \leq 2C_1 \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} \leq 2C_1\delta =: K. \tag{3.34}$$

From Lemma 3.2, there exist limits  $\theta_0 \in \dot{B}_{p,q}^{\sigma}(\mathbb{R}^2)$  and  $\theta \in L^\infty(0, T; \dot{B}_{p,q}^{\sigma}(\mathbb{R}^2))$  such that

$$\begin{aligned} \theta_0 &= \sum_{n=0}^{\infty} (\theta_0^{(n+1)} - \theta_0^{(n)}) = \lim_{n \rightarrow \infty} \theta_0^{(n)} \quad \text{in } \dot{B}_{p,q}^{\sigma}(\mathbb{R}^2), \\ \theta &= \sum_{n=0}^{\infty} (\theta^{(n+1)} - \theta^{(n)}) = \lim_{n \rightarrow \infty} \theta^{(n)} \quad \text{in } L^\infty(0, T; \dot{B}_{p,q}^{\sigma}(\mathbb{R}^2)). \end{aligned}$$

By Lemma 3.1 and (3.34), we see that  $\theta_0 \in \dot{B}_{p,q}^{sc}(\mathbb{R}^2)$ ,  $\theta \in X_T^{p,q}$  and

$$\|\theta\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{sc})} \leq \|\theta\|_{X_T^{p,q}} \leq K. \tag{3.35}$$

It is easy to check that  $\theta$  is a solution to (1.5) on  $[0, T]$ . Next, we show the continuity in time of the solution  $\theta$  by the idea in [7]. Let  $(s, \rho) \in \{(s_c, q), (0, 1)\}$ . Since it holds  $\partial_t \Delta_j \theta = \Delta_j F - (-\Delta)^{\frac{\alpha}{2}} \Delta_j \theta - \Delta_j (u \cdot \nabla \theta)$ , we have

$$\begin{aligned} & \|\partial_t \Delta_j \theta\|_{\dot{B}_{p,\rho}^s} \\ & \leq C 2^{sj} \|\partial_t \Delta_j \theta\|_{L^p} \\ & \leq C 2^{sj+2(\frac{1}{r}-\frac{1}{p})j} \|\Delta_j F(t)\|_{L^r} + C 2^{sj+\alpha j} \|\Delta_j \theta\|_{L^p} + C 2^{sj+j} \|u(t)\|_{L^p} \|\theta(t)\|_{L^\infty} \\ & \leq C 2^{sj+2(\frac{1}{r}-\frac{1}{p})j} \sup_{t>0} \|F(t)\|_{\dot{B}_{r,\infty}^0} + C 2^{sj+\alpha j} \|\theta\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,1}^0)} + C 2^{sj+j} \|\theta\|_{X_T^{p,q}}^2, \end{aligned}$$

which implies  $\partial_t \Delta_j \theta \in L^\infty(0, T; \dot{B}_{p,\rho}^s(\mathbb{R}^2))$ . Therefore, we have

$$\Theta_m := \sum_{|j| \leq m} \Delta_j \theta \in C([0, T]; \dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)), \quad m \in \mathbb{N}.$$

It follows from  $q < \infty$  and (3.35) that

$$\begin{aligned} & \|\Theta_m - \theta\|_{L^\infty(0,T;\dot{B}_{p,1}^0 \cap \dot{B}_{p,q}^{s_c})} \\ & \leq C \sum_{|j| \geq m} \|\Delta_j \theta\|_{L^\infty(0,T;L^p)} + C \left\| \left\{ 2^{s_c j} \|\Delta_j \theta\|_{L^\infty(0,T;L^p)} \right\}_{\{j:|j| \geq m\}} \right\|_{l^q} \\ & \rightarrow 0, \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Hence, we see that  $\theta \in C([0, T]; \dot{B}_{p,1}^0(\mathbb{R}^2) \cap \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)) \subset C([0, T]; B_{p,q}^{s_c}(\mathbb{R}^2))$ . Since

$$\begin{aligned} & \|\theta^{(n)}(0) - \theta_0\|_{\dot{B}_{p,q}^\sigma} \\ & \leq C \|S_{n+3}(\theta_0^{(n)} - \theta_0)\|_{\dot{B}_{p,q}^\sigma} + \|(1 - S_{n+3})\theta_0\|_{\dot{B}_{p,q}^\sigma} \\ & \leq C \|\theta_0^{(n)} - \theta_0\|_{\dot{B}_{p,q}^\sigma} + \|(1 - S_{n+3})\theta_0\|_{\dot{B}_{p,q}^\sigma} \rightarrow 0, \\ & \|(1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})\theta_0^{(n+1)} - (1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})\theta_0\|_{\dot{B}_{p,q}^\sigma} \\ & \leq \|\theta_0^{(n+1)} - \theta_0\|_{\dot{B}_{p,q}^\sigma} + \|e^{-T(-\Delta)^{\frac{\alpha}{2}}}(\theta_0^{(n+1)} - \theta_0)\|_{\dot{B}_{p,q}^\sigma} \\ & \leq C \|\theta_0^{(n+1)} - \theta_0\|_{\dot{B}_{p,q}^\sigma} \rightarrow 0, \\ & \|(\theta^{(n)}(T) - e^{-(-\Delta)^{\frac{\alpha}{2}}}\theta^n(0)) - (\theta(T) - e^{-(-\Delta)^{\frac{\alpha}{2}}}\theta_0)\|_{\dot{B}_{p,q}^\sigma} \\ & \leq \|\theta^{(n)}(T) - \theta(T)\|_{\dot{B}_{p,q}^\sigma} + \|e^{-T(-\Delta)^{\frac{\alpha}{2}}}(\theta^{(n)}(0) - \theta_0)\|_{\dot{B}_{p,q}^\sigma} \\ & \leq \|\theta^{(n)} - \theta\|_{L^\infty(0,T;\dot{B}_{p,q}^\sigma)} + C \|\theta^{(n)}(0) - \theta_0\|_{\dot{B}_{p,q}^\sigma} \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , we obtain by letting  $n \rightarrow \infty$  in (3.3) that

$$(1 - e^{-T(-\Delta)^{\frac{\alpha}{2}}})\theta_0 = \theta(T) - e^{-T(-\Delta)^{\frac{\alpha}{2}}}\theta_0, \quad \theta(0) = \theta_0,$$

which implies

$$\theta(T) = \theta(0) = \theta_0. \tag{3.36}$$

Let us extend  $\theta$  to the function on the interval  $[0, \infty)$  periodically as

$$\theta(t) = \theta(t - NT), \quad \text{for } NT < t \leq (N + 1)T, \quad N \in \mathbb{N}.$$

Then,  $\theta \in BC([0, \infty); B_{p,q}^{s_c}(\mathbb{R}^2))$  and  $\theta$  is a  $T$ -time periodic solution to (1.5) satisfying (1.4). Finally, we prove the uniqueness. Let  $\tilde{\theta}$  be arbitrary solution satisfying (1.7). Note that since  $0 < \sigma < s_c$ , we see that  $\theta, \tilde{\theta} \in L^\infty(0, T; L^p(\mathbb{R}^2)) \cap \tilde{L}^\infty(0, T; \dot{B}_{p,q}^{s_c}(\mathbb{R}^2)) \subset \tilde{L}^\infty(0, T; \dot{B}_{p,q}^\sigma(\mathbb{R}^2))$  holds by the similar calculation as (2.8). Then, it follows from (3.29) and (3.35) that

$$\begin{aligned} \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)} &\leq C_3 \left( \|\theta\|_{X_T^{p,q}} + \|\tilde{\theta}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^{s_c})} \right) \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)} \\ &\leq 2KC_3 \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)} \\ &\leq \frac{1}{2} \|\theta - \tilde{\theta}\|_{\tilde{L}^\infty(0,T;\dot{B}_{p,q}^\sigma)}. \end{aligned}$$

Thus, we see that  $\tilde{\theta} = \theta$  on  $[0, T]$ . The periodicity of  $\theta$  and  $\tilde{\theta}$  implies  $\theta = \tilde{\theta}$  on  $[0, \infty)$ . This completes the proof.

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**REFERENCES**

- [1] Bae, Hantaek, Global well-posedness of dissipative quasi-geostrophic equations in critical spaces, Proc. Amer. Math. Soc., 136, 2008
- [2] Bahouri, Hajer, Chemin, Jean-Yves, Danchin, Raphaël, Fourier analysis and nonlinear partial differential equations, 343, Springer, Heidelberg, 2011,
- [3] Chae, Dongho, Lee, Jihoon, Global well-posedness in the super-critical dissipative quasi-geostrophic equations, Comm. Math. Phys., 233, 2003, 297–311,
- [4] Chen, Qionglei, Miao, Changxing, Zhang, Zhifei, A new Bernstein’s inequality and the 2D dissipative quasi-geostrophic equation, Comm. Math. Phys., 271, 2007
- [5] Chen, Qionglei, Zhang, Zhifei, Global well-posedness of the 2D critical dissipative quasi-geostrophic equation in the Triebel-Lizorkin spaces, Nonlinear Anal., 67, 2007, 1715–1725,
- [6] Constantin, Peter, Wu, Jiahong, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J. Math. Anal., 30, 1999, 937–948,
- [7] Danchin, R., Global existence in critical spaces for compressible Navier-Stokes equations, Invent. Math., 141, 2000, 579–614,

- [8] Geissert, Matthias, Hieber, Matthias, Nguyen, Thieu Huy, A general approach to time periodic incompressible viscous fluid flow problems, *Arch. Ration. Mech. Anal.*, 220, 2016, 1095–1118,
- [9] Hmidi, Taoufik, Keraani, Sahbi, Global solutions of the super-critical 2D quasi-geostrophic equation in Besov spaces, *Adv. Math.*, 214, 2007, 618–638,
- [10] Kozono, Hideo, Nakao, Mitsuhiro, Periodic solutions of the Navier-Stokes equations in unbounded domains, *Tohoku Math. J. (2)*, 48, 1996, 33–50,
- [11] Kozono, Hideo, Yamazaki, Masao, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, *Comm. Partial Differential Equations*, 19, 1994, 959–1014,
- [12] Miura, Hideyuki, Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space, *Comm. Math. Phys.*, 267, 2006
- [13] Sawano, Yoshihiro, *Theory of Besov spaces*, *Developments in Mathematics*, 56, Springer, Singapore, 2018,
- [14] Wu, Jiahong, Dissipative quasi-geostrophic equations with  $L^p$  data, *Electron. J. Differential Equations*, 2001, No. 56, 13,
- [15] Wu, Jiahong, Lower bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces, *Comm. Math. Phys.*, 263, 2006, 803–831,
- [16] Zhang, Zhi-fei, Global well-posedness for the 2D critical dissipative quasi-geostrophic equation, *Sci. China Ser. A*, 50, 2007, 485–494,

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