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Regularity of solutions to Kolmogorov equation with Gilbarg–Serrin matrix

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Abstract. In $\mathbb{R}^d$, $d \geq 3$, consider the divergence and the non-divergence form operators

$$-\Delta - \nabla \cdot (a - I) \cdot \nabla + b \cdot \nabla,$$
$$-\Delta - (a - I) \cdot \nabla^2 + b \cdot \nabla,$$

where the second-order perturbations are given by the matrix

$$a - I = c|x|^{-2} x \otimes x, \quad c > -1.$$

The vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is form-bounded with form-bound $\delta > 0$. (This includes vector fields with entries in L^d, as well as vector fields having critical-order singularities.) We characterize quantitative dependence on c and δ of the $L^q \to W^{1,qd/(d-2)}$ regularity of solutions of the corresponding elliptic and parabolic equations in L^q, $q \geq 2 \vee (d-2)$.

Contents

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1. Introduction

1. In this paper, we are concerned with the second-order perturbations of $-\Delta$,

$$\begin{aligned} &-\Delta - \nabla \cdot (a - I) \cdot \nabla, \\ &-\Delta - (a - I) \cdot \nabla^2, \\ &a_{ij}(x) := \delta_{ij} + c|x|^{-2}x_i x_j, \quad c > -1. \end{aligned} \tag{1}$$

These are model examples of divergence/non-divergence form operators that are not accessible by classical means such as the parametrix [8], [19, Ch. IV]. Although the matrix a is discontinuous at the origin, it is uniformly elliptic, so, by the De Giorgi–Nash theory, solution $u \in W^{1,2}(\mathbb{R}^d)$ to the elliptic equation $(\mu - \nabla \cdot a \cdot \nabla)u = f$, $\mu > 0$, $f \in L^p \cap L^2$, $p \in]\frac{d}{2}, \infty[$, is in $C^{0,\gamma}$, where the Hölder continuity exponent $\gamma \in]0, 1[$ depends only on d and c. The operators (1) and their modifications have been studied by many authors in order to make more precise the relationship between the regularity properties of the solutions to the corresponding parabolic and elliptic equations and the continuity properties of the matrix, see [1,3,5], [18, Ch. 1.2], [7,9,20–26] and references therein. In fact, there is a quantitative dependence of the regularity properties of solutions on the value of c. In this sense, the matrix a has a critical-order discontinuity at the origin.

The critical-order perturbations of $-\Delta$ and its generalizations have been the subject of intensive study over the past few decades as they reveal otherwise inaccessible aspects of the theory of the unperturbed operator. For example, consider the Schrödinger operator $-\Delta - V_0$, $V_0(x) = \delta \frac{(d-2)^2}{4}|x|^{-2}$, on $\mathbb{R}^d$, $d \geq 3$. If $0 < \delta < 1$, then the self-adjoint operator realization H^- of $-\Delta - V_0$ on $L^2 \equiv L^2(\mathbb{R}^d)$ is defined as the (minus) generator of a C_0 semigroup $e^{-tH^-} = s\text{-}L^2\text{-}\lim_{\varepsilon \downarrow 0} e^{-tH^-(V_\varepsilon)}$, $V_\varepsilon(x) = \delta \frac{(d-2)^2}{4}|x|_\varepsilon^{-2}$, $|x|_\varepsilon^2 := |x|^2 + \varepsilon$, $\varepsilon > 0$. For $\delta > 1$, however, by the celebrated result of [4] (see also [10]),

$$\lim_{\varepsilon \downarrow 0} e^{-tH^-(V_\varepsilon)} u_0(x) = \infty, \quad t > 0, \quad x \in \mathbb{R}^d, \quad u_0 \geq 0,\ u_0 \not\equiv 0,$$

i.e., all positive solutions explode instantly at any point. This phenomenon is not observable for any $V_0 = \delta V$, $V \in L^{\frac{d}{2}}$, regardless of how large $\delta > 0$ is (in this sense, the class $L^{\frac{d}{2}}$ does not contain potentials having critical-order singularities). The perturbations $\nabla \cdot (a - I) \cdot \nabla$, $(a - I) \cdot \nabla^2$, $a - I = c|x|^{-2} x \otimes x$, of $-\Delta$ can be viewed as the second-order analogues of the critical potential $V_0(x) = \delta \frac{(d-2)^2}{4}|x|^{-2}$.

Our goal is to determine to what extent adding $\nabla \cdot (a - I) \cdot \nabla$, $(a - I) \cdot \nabla^2$ affects the perturbation-theoretic and the regularity properties of $-\Delta$. Our interest is motivated by applications to diffusion processes, and so we restrict our study to the first-order perturbations of (1).

2. The following result concerning the special case $a = I$ (i.e., $c = 0$) will serve as the point of departure. Consider on $\mathbb{R}^d$, $d \geq 3$, the Kolmogorov operator

$$-\Delta + b \cdot \nabla, \quad b : \mathbb{R}^d \to \mathbb{R}^d.$$

We will need the following

Definition. A measurable vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is said to be form-bounded (with respect to $-\Delta$) if $|b| \in L^2_{\mathrm{loc}}$ and there exist constants $\delta > 0$ and $\lambda = \lambda_\delta \geq 0$ such that

$$\||b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \leq \sqrt{\delta}$$

(write $b \in \mathbf{F}_\delta$).

Here and below, $\|\cdot\|_{p\to q}$ denotes the $\|\cdot\|_{L^p\to L^q}$ operator norm.

The condition $b \in \mathbf{F}_\delta$ is equivalent to the quadratic form inequality

$$\langle bf, bf\rangle \leq \delta\langle\nabla f, \nabla f\rangle + c_\delta\langle f, f\rangle, \quad \text{for all } f \in W^{1,2}$$

for a constant c_δ $(= \lambda\delta)$, where, from now on,

$$\langle h\rangle := \int_{\mathbb{R}^d} h(x)dx, \quad \langle h, g\rangle := \langle h\bar{g}\rangle.$$

The constant δ is called the form-bound of b. It measures the size of critical singularities of the vector field, see examples below.

It is clear that

$$b_1 \in \mathbf{F}_{\delta_1}, b_2 \in \mathbf{F}_{\delta_2} \quad \Rightarrow \quad b_1 + b_2 \in \mathbf{F}_\delta, \qquad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}.$$

Examples. The class of form-bounded vector fields $\mathbf{F}_\delta$ contains vector fields b with $|b| \in L^d + L^\infty$ (i.e., $b = b_1 + b_2$, where $|b_1| \in L^d$, $|b_2| \in L^\infty$), with $\delta > 0$ that can be chosen arbitrarily small (by Sobolev's inequality).

The class $\mathbf{F}_\delta$ also contains vector fields having critical-order singularities. For example, by Hardy's inequality, the vector field

$$b(x) := \sqrt{\delta}\frac{d-2}{2}|x|^{-2}x, \quad \delta > 0,$$

having a model critical-order singularity at the origin, is contained in $\mathbf{F}_\delta$ (with $\lambda = 0$). More generally, the class $\mathbf{F}_\delta$ contains vector fields b with $|b|$ in $L^{d,\infty} + L^\infty$ (the weak L^d class, by Strichartz' inequality [16]), the Campanato–Morrey class or the Chang–Wilson–Wolff class [6], with δ depending on the norm of $|b|$ in these classes.

For every $\varepsilon > 0$, one can find $b \in \mathbf{F}_\delta$ such that $|b| \notin L^{2+\varepsilon}_{\mathrm{loc}}$, e.g., consider a vector field b with

$$|b(x)|^2 = C\frac{\mathbf{1}_{B(0,1+\kappa)} - \mathbf{1}_{B(0,1-\kappa)}}{\big||x|-1\big|^{-1}(-\ln\big||x|-1\big|)^\alpha}, \quad \alpha > 1, \quad 0 < \kappa < 1.$$

See, e.g., [11, Sect. 4] for other examples and more detailed discussion concerning the class $\mathbf{F}_\delta$.

Here is our point of departure. By [15, Lemma 5], for $b \in \mathbf{F}_\delta$ with $\delta < 1 \wedge \left(\frac{2}{d-2}\right)^2$ and $q \in [2 \vee (d-2), \frac{2}{\sqrt{\delta}}[$ the solution u to the elliptic equation

$$\big(\mu + \Lambda_q(b)\big)u = f, \quad f \in L^q,$$

where $\Lambda_q(b)$ is an operator realization of $-\Delta + b\cdot\nabla$ in L^q as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup (see details in Sect. 3), satisfies

$$\begin{aligned} \|\nabla u\|_q &\leq K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q, \\ \|\nabla u\|_{\frac{qd}{d-2}} &\leq K_2(\mu-\mu_0)^{\frac{1}{q}-\frac{1}{2}}\|f\|_q \end{aligned} \qquad (*)$$

for all $\mu > \mu_0$, where constants $\mu_0 = \mu_0(d,q,\delta) > 0$, $K_i = K_i(d,q,\delta)$, $i = 1, 2$. In particular, if additionally $q > d-2$, then by the Sobolev embedding theorem u is in $C^{0,\gamma}$ (possibly after change on a measure zero set) with Hölder continuity exponent $\gamma = 1 - \frac{d-2}{q}$.

3. In our main result, Theorem 2, we show that the perturbation $-\nabla\cdot(a-I)\cdot\nabla$ of $-\Delta$ preserves, under appropriate assumptions on c, the properties of $-\Delta$ that allow to establish estimates (∗) for $u = \big(\mu + \Lambda_q(a,b)\big)^{-1} f$, where $\Lambda_q(a,b)$ is an operator realization of the formal operator

$$-\Delta - \nabla\cdot(a-I)\cdot\nabla + b\cdot\nabla, \qquad b \in \mathbf{F}_\delta$$

in L^q as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup, constructed as the limit of the semigroups corresponding to smooth approximations of a, b. The existing literature on $-\Delta - \nabla\cdot(a-I)\cdot\nabla + b\cdot\nabla$ dealing with discontinuous/locally unbounded coefficients, provides a detailed regularity theory of this operator in the case $a = I + c|x|^{-2}(x\otimes x)$ and $b(x) = c|x|^{-2}x$, see [5,7,20–24]. In the present paper, we are dealing with a substantially larger class of singular drifts b. Our results thus do not depend on the specific structure of b such as differentiability or symmetry, and, in fact, follow from the a priori estimates (∗) for solutions to the corresponding elliptic equations with smoothed out coefficients.

Now, define vector field ∇a by $(\nabla a)_k := \sum_{i=1}^d \nabla_i a_{ik}$, $1 \leq k \leq d$, where, from now on, $\nabla_i := \partial_{x_i}$. Then $\nabla a = c(d-1)|x|^{-2}x$, so by Hardy's inequality $\nabla a \in \mathbf{F}_\delta$, $\delta_a = \frac{4c^2(d-1)^2}{(d-2)^2}$. We construct an operator realization in L^q of the non-divergence form (formal) operator

$$-a\cdot\nabla^2 + b\cdot\nabla \equiv -\sum_{i,j=1}^d a_{ij}(x)\nabla_i\nabla_j + \sum_{k=1}^d b_k(x)\nabla_k, \quad b \in \mathbf{F}_{\delta_1}$$

as $\Lambda_q(a, \nabla a + b)$ (we have $-a\cdot\nabla^2 + b\cdot\nabla \equiv -\nabla\cdot a\cdot\nabla + (\nabla a + b)\cdot\nabla$). As a result, we can characterize the quantitative dependence of the regularity properties of $u = (\mu + \Lambda_q(a, \nabla a + b))^{-1} f$, $f \in L^q$, on c, d, q, μ and δ, see Corollary 3. In this regard, we note that the class of admissible first-order perturbations $b\cdot\nabla$, $b \in \mathbf{F}_\delta$ of

$-a \cdot \nabla^2$ cannot be achieved on the basis of the Krylov–Safonov a priori estimates [17, Ch. 4.2]. (We note that the operator $-a \cdot \nabla^2$ with $\partial_{x_k} a_{ij} \in L^{d,\infty}$ has been studied earlier in [2], see also [3].)

Concerning the application of ($*$) to establishing the $C^{0,\gamma}$ continuity of u, we note the following. Let $d \geq 4$. In the proof of Theorem 2, we establish a stronger than ($*$) estimate:

$$\|\nabla |\nabla u|^{\frac{q}{2}}\|_2^2 \leq K \|f\|_q^q$$

(and so $u \in C^{0,\gamma}$, $\gamma = 1 - \frac{d-2}{q}$). We do not appeal, for the purpose of establishing Hölder continuity of u, to $W^{2,r}$ estimates on u for a large r. In fact, the condition $b \in \mathbf{F}_\delta$, $\delta < 1$ yields only $u \in W^{2,2}$. The latter allows to conclude that u is Hölder continuous only in dimension $d = 3$.

In Theorem 2, we tried to find the least restrictive assumptions on c and δ (a measure of discontinuity of matrix a and a measure of singularity of vector field b, respectively), permitted by the method, such that the estimates ($*$) hold for a $b \in \mathbf{F}_\delta$. (We emphasize that our result is not of Cordes type.) The weaker result that there exist sufficiently small c and δ such that the estimates ($*$) are valid (still not accessible by the existing results prior to our work) can be obtained with considerably less effort by following the proof and discarding the corresponding multiples of c and δ.

The question of optimality of our assumptions on c and δ in Theorem 2 is difficult. Even in the case $c = 0$, it is not yet clear whether the corresponding assumption on δ (i.e., $\delta < 1 \wedge \frac{4}{(d-2)^2}$), although dictated by the method, is optimal. (We remark that the constant $\frac{4}{(d-2)^2}$, incidentally, coincides with the constant in Hardy's inequality for $d \geq 4$.) In this regard, we note the following:

1. We believe that the examples showing the optimality of the assumptions on c and δ in Theorem 2 (at least in the limiting cases discussed in the fourth remark after Theorem 2) could be obtained once one fully exploits the method, e.g., in the context of the problem of constructing the corresponding diffusion process. (In this regard, we note [12, Example 1].)
2. In [23,24], the authors constructed an operator realization Q_p of $-\Delta - \nabla \cdot (a - I) \cdot \nabla + b \cdot \nabla$ in L^p for the model vector field $b(x) = c|x|^{-2}x$ and characterized the domain of Q_p, establishing $W^{1,p}$ and $W^{2,p}$ regularity of any u in the domain of Q_p. The results in [23,24] do not follow as a special case of Theorem 2 if we take there $b(x) = c|x|^{-2}x$. In fact, for this vector field, one can modify our method to take into account the sign of the divergence of b (cf. [11, Corollaries 4.9–4.11]); however, this still does not allow to obtain as a special case the related result in [23,24].

We note that having a complete characterization of the domain of (an operator realization of) $-\nabla \cdot a \cdot \nabla$ in L^q for some q does not help on its own to characterize regularity of the domain of $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$, $b \in \mathbf{F}_\delta$ in L^q (as is already apparent in the case $a = I$ discussed above).

The method of this paper is suited to treat second-order perturbations $-\nabla\cdot(a-I)\cdot\nabla$, $-(a-I)\cdot\nabla^2$ of $-\Delta$ more general than $a-I=c|x|^{-2}x\otimes x$, for example, $a-I=v\otimes v$, where a bounded $v:\mathbb{R}^d\to\mathbb{R}^d$, $v\in W^{1,2}_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d)$ satisfies

$$\Big(\sum_k(\nabla v_k)^2\Big)^{\frac{1}{2}}\in\mathbf{F}_\delta \tag{2}$$

(although not distinguishing between positive and negative c). Our method also admits extension to

$$a_{ij}(x)=\delta_{ij}+\sum_l c_l\kappa_{ij}(x-y^l),\quad \kappa_{ij}(x)=|x|^{-2}x_ix_j,$$
$$c_+:=\sum_{c_l>0}c_l<\infty,\quad c_-:=\sum_{c_l<0}c_l>-1,$$

where $\{y^l\}\subset\mathbb{R}^d$. Let us also note that arguments in this paper can be transferred without significant changes from $\mathbb{R}^d$ to the ball $B(0,1)$.

After this paper was written [13], in subsequent paper [14] we constructed and investigated the diffusion process corresponding to $-a\cdot\nabla^2+b\cdot\nabla$ with a as in (2) using analogues of estimates ($*$), although valid, if restricted to $a=I+c|x|^{-2}x\otimes x$, under substantially more restrictive assumptions on c than in the present paper.

Outline of the method Let us give an informal outline of the proof of Theorem 2, i.e., estimates ($*$) for solution u to the elliptic equation $(\mu-\nabla\cdot a\cdot\nabla+b\cdot\nabla)u=f$, $\mu>0$, $f\in L^q$, $q>d-2$ (sufficiently close to $d-2$).

Step 1: The basic equality. We multiply the equation by carefully chosen "test function" and integrate to obtain the *basic equality*

$$\mu\|\nabla u\|_q^q+I_q+(q-1)J_q=[\dots]+\|f\|_q^q,\quad J_q:=\|\nabla|\nabla u|^{\frac{q}{2}}\|_2^2. \tag{BE}$$

The term I_q is greater than J_q, so if we replace it by J_q we arrive at

Step 2: The principal inequality. The terms $[\dots]$ in the RHS of (BE) are estimated from above by κJ_q with a sufficiently small coefficient $\kappa>0$ (using the structure of the matrix a and the condition $b\in\mathbf{F}_\delta$). Thus, we obtain from (BE)

$$\mu\|\nabla u\|_q^q+(q-1)J_q\leq\kappa J_q+C\|f\|_q^q,$$

and so if $\kappa<q-1$ ($\Leftrightarrow$ our assumptions on a, b are satisfied) then we obtain *the principal inequality*

$$\mu\|\nabla u\|_q^q+\eta J_q\leq C\|f\|_q^q,\quad \eta>0.$$

Step 3: The sought estimates ($*$) on u now follow from the previous inequality by applying the Sobolev embedding theorem to J_q.

Structure of the paper In Sect. 2, we state our results in detail. In Sect. 3, we illustrate the use of our method by applying it first to operator $-\Delta+b\cdot\nabla$, $b\in\mathbf{F}_\delta$. In Sect. 4, we prove our main result, Theorem 2 concerning the divergence form operator $-\nabla\cdot a\cdot\nabla+b\cdot\nabla$.

2. Main results

1. In what follows, we consider C^∞ smooth approximation of the matrix $a(x) = I + c|x|^{-2}x \otimes x$ by

$$a^\varepsilon = (a^\varepsilon_{ij}), \quad 1 \le i, j \le d,$$

where

$$a^\varepsilon_{ij} := \delta_{ij} + c|x|_\varepsilon^{-2} x_i x_j, \quad |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}, \quad \varepsilon > 0.$$

For a given vector field $b \in \mathbf{F}_\delta$, we consider its C^∞ smooth approximation defined by

$$b_n := c_n \gamma_{\epsilon_n} * \mathbf{1}_n b, \quad n \ge 1, \quad c_n \uparrow 1, \tag{3}$$

where $\mathbf{1}_n$ is the indicator of $\{x \in \mathbb{R}^d \mid |x| \le n, |b(x)| \le n\}$, γ_ϵ is the K. Friedrichs mollifier, for appropriate $\epsilon_n \downarrow 0$ and $c_n \uparrow 1$ so that $b_n \in \mathbf{F}_\delta$ for all $n \ge 1$ with $\lambda \neq \lambda(n)$, see Remark 2 for details.

In the course of the proof of Theorem 2, we will first prove the required regularity estimates ($*$) for solution $u^{\varepsilon,n}$ to the elliptic equation with smooth coefficients $(\mu - \nabla \cdot a^\varepsilon \cdot \nabla + b_n \cdot \nabla)u^{\varepsilon,n} = f$, $f \in C_c^\infty$, $\mu \ge \mu_0$ for constants $\mu_0 > 0$ and K_l, $l = 1, 2$ independent of ε, n. Taking $\varepsilon \downarrow 0$ and $n \to \infty$, we will establish estimates ($*$) for solution u to the equation $(\mu - \nabla \cdot a \cdot \nabla + b \cdot \nabla)u = f$. However, first we need to specify in what sense the operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ is defined; we will also need the corresponding convergence result; see Theorem 1.

Recall that there is a unique self-adjoint operator $A \equiv A_2 \ge 0$ in L^2 associated with the sesquilinear form $\mathfrak{t}[u, v] := \langle \nabla u \cdot a \cdot \nabla \bar{v} \rangle$, $D(\mathfrak{t}) = W^{1,2}$:

$$D(A) \subset D(\mathfrak{t}) \quad \text{and} \quad \langle Au, v \rangle = \mathfrak{t}[u, v], \quad u \in D(A), v \in D(\mathfrak{t}).$$

The operator $-A$ is the generator of a positivity preserving L^∞ contraction C_0 semigroup $T_2^t \equiv e^{-tA}$, $t \ge 0$, on L^2. Then, since T_2^t is a L^∞ contraction, $T_q^t := \left[T_2^t \upharpoonright_{L^q \cap L^2}\right]^{\text{clos}}_{L^q \to L^q}$ determines by interpolation a contraction C_0 semigroup in L^q for all $q \in [2, \infty[$ and hence, by self-adjointness, for all $q \in]1, \infty[$. The (minus) generator A_q of T_q^t $(\equiv e^{-tA_q})$ is an operator realization of the formal operator $\nabla \cdot a \cdot \nabla$ on L^q, $q \in]1, \infty[$.

In what follows, given a Banach space Y and a sequence of bounded linear operators $T_n, T : Y \to Y$, we write $T = s\text{-}Y\text{-}\lim_n T_n$ if $Tf = \lim_n T_n f$ in Y for every $f \in Y$.

We will need

Theorem 1. *Let $d \ge 3$. Let $b \in \mathbf{F}_\delta$ with $\delta_1 := [1 \vee (1+c)^{-2}]\,\delta < 4$. Let $q > \frac{2}{2-\sqrt{\delta_1}}$. The following is true.*

(i) There exists the limit

$$s\text{-}L^q\text{-}\lim_{n\to\infty} e^{-t\Lambda_q(a,b_n)} \quad (\textit{locally uniformly in } t \ge 0),$$

where $\Lambda_q(a,b_n) = A_q + b_n \cdot \nabla$, $D(\Lambda_q(a,b_n)) = D(A_q)$, *and determines a positivity preserving,* L^∞ *contraction, quasi-contraction* C_0 *semigroup* $e^{-t\Lambda_q(a,b)}$ *in* L^q. *Its (minus) generator* $\Lambda_q(a,b)$ *is an appropriate operator realization of the formal operator* $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ *in* L^q.

(ii)

$$\|e^{-t\Lambda_q(a,b)}\|_{q\to q} \leq e^{\omega_q t}, \quad t > 0, \quad \omega_q := \frac{\lambda\delta_1}{2(q-1)}.$$

(iii)

$$(\mu + \Lambda_q(a,b))^{-1} = s\text{-}L^q\text{-}\lim_{n\to\infty}\lim_{\varepsilon\downarrow 0}(\mu + \Lambda_q(a^\varepsilon, b_n))^{-1}, \quad \textit{for all } \mu > \omega_q,$$

where $\Lambda_q(a^\varepsilon, b_n) = -\nabla \cdot a^\varepsilon \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_q(a^\varepsilon, b_n)) = W^{2,q}$.

Proof. Assertions (i), (ii) are a special case of [11, Theorem 4.2] or of [11, Theorem 4.3] (both valid for an arbitrary uniformly elliptic matrix a).

Proof of (iii). By [11, Theorem 4.6], for every $n \geq 1$,

$$(\mu + \Lambda_q(a,b_n))^{-1} = s\text{-}L^q\text{-}\lim_{\varepsilon\downarrow 0}(\mu + \Lambda_q(a^\varepsilon, b_n))^{-1}.$$

It remains to apply (i). □

2. We are in position to state the main result of the paper. Let us introduce the following quantities:

$$L_1(c,\delta,d) := c\left[\frac{1}{2(d-1)} + \frac{\sqrt{\delta}}{2}(d-2)(d+4)\right] + \left[\frac{(d-2)^2\delta}{4} + \frac{(d-4)(d-2)}{2}\sqrt{\delta}\right],$$
$$L_2(c,\delta,d) := -c\left[2d - 4 + \frac{\sqrt{\delta}}{2}(d-2)(d+4)\right] + \left[\frac{(d-2)^2\delta}{4} + \frac{(d-4)(d-2)}{2}\sqrt{\delta}\right].$$

Clearly, L_1, L_2 are small if c, δ are small.

Theorem 2. (The operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$). *Let* $d \geq 3$, $a(x) = I + c|x|^{-2}x \otimes x$, $c > -1$, *and* $b \in \mathbf{F}_\delta$, $\delta > 0$.

(i) *Let* $d \geq 4$. *Assume that* c, δ *(i.e., a measure of discontinuity of matrix* a *and a measure of singularity of vector field* b*, respectively) satisfy* $[1 \vee (1+c)]\sqrt{\delta} < 2 - \frac{2}{d-2}$ *and one of the following conditions:*

(1) $c > 0$ *and* $1 + c\big(1 - \frac{1}{2(d-1)} - \frac{(d-2)\sqrt{\delta}}{4}\big) > 0$, *and*

$$L_1(c,\delta,d) < d-3.$$

(2) $-1 < c < 0$ *and* $1 + c\big(1 + \frac{(d-2)\sqrt{\delta}}{4}\big) > 0$, *and*

$$L_2(c,\delta,d) < d-3.$$

Then for every $q > d-2$ sufficiently close to $d-2$ there exist constants $\mu_0 = \mu_0(d, q, c, \delta)$ $(\geq \omega_q)$ and $K_l = K_l(d, q, c, \delta)$ $(l = 1, 2)$ such that, for all $\mu > \mu_0$ and $f \in L^q$, $u := (\mu + \Lambda_q(a, b))^{-1} f \in W^{1,q} \cap W^{1,\frac{qd}{d-2}}$ and satisfies

$$\begin{aligned} \|\nabla u\|_q &\leq K_1(\mu - \mu_0)^{-\frac{1}{2}} \|f\|_q, \\ \|\nabla u\|_{\frac{qd}{d-2}} &\leq K_2(\mu - \mu_0)^{\frac{1}{q}-\frac{1}{2}} \|f\|_q. \end{aligned} \tag{$\star$}$$

(ii) *Let $d \geq 3$. Assume that c, δ satisfy $\delta < 1 \wedge (1+c)^2$ and one of the following conditions holds:*

$$c > 0, \quad 1 - c\left[\frac{4}{(d-2)^2} + \sqrt{\delta}\left(2\frac{d+3}{d-2} + 1\right)\right] - \delta > 0,$$

or

$$-1 < c < 0, \quad 1 - |c|\left[1 + \frac{4(d-1)}{(d-2)^2} + \sqrt{\delta}\left(2\frac{d+3}{d-2} + 1\right)\right] - \delta > 0.$$

Then ($\star$) holds with $q = 2$ and moreover, $u \in W^{2,2}$.

Remarks. 1. In Theorem 2, if $c = 0$, then the assumptions on δ are reduced to $\delta < 1 \wedge \frac{4}{(d-2)^2}$, so we recover the result in [15, Lemma 5].

2. In assertion (i) of Theorem 2, we could also include $d = 3$, $q \geq 2$; however, for $d = 3$ assertion (ii) yields a stronger regularity result $u \in W^{2,2}$.
3. Following closely the proof of Theorem 2, one can obtain conditions on c, δ and $q > d-2$ that provide estimates ($\star$), not necessarily assuming that q is close to $d-2$. In this case, we would have to replace hypothesis 1) in Theorem 2, i.e., "$1 + c\big(1 - \frac{1}{2(d-1)} - \frac{(d-2)\sqrt{\delta}}{4}\big) > 0$ and $L_1(c, \delta, d) < d-3$," by "$1 + c(1 - \frac{1}{2(d-1)} - \frac{q\sqrt{\delta}}{4})$ and $(q-1)\frac{(d-2)^2}{q^2} - Ł_1(c, \delta, q, d) > 0$," where $Ł_1$ is defined in the proof of Theorem 2. Similarly for hypothesis 2). We opt to work with q close to $d-2$ to keep the assumptions of the theorem tractable.
4. In the assumptions of Theorem 2, the second estimate in ($\star$) and the Sobolev embedding theorem yield that u is Hölder continuous (possibly after a change on a measure zero set). For the illustration purposes, let us state the corresponding result in the case when either δ is small or c is small. Let $d \geq 3$, $a(x) = I + c|x|^{-2} x \otimes x$, $c > -1$, and $b \in \mathbf{F}_\delta$. Assume that

$$\begin{cases} c \in \left]-\frac{1}{2+\frac{2}{d-3}}, 2(d-1)(d-3)\right[, & d \geq 4, \\ c \in]-\frac{1}{9}, \frac{1}{4}[, & d = 3, \end{cases} \qquad \text{and } \delta > 0 \text{ is sufficiently small,}$$

or

$$|c| \text{ is sufficiently small and } \delta < 1 \wedge \frac{4}{(d-2)^2}.$$

Then, for all $d \geq 4$ and all $q > d-2$ sufficiently close to $d-2$

$$(\mu + \Lambda_q(a, b))^{-1} L^q \subset C^{0,\gamma}, \quad \gamma = 1 - \frac{d-2}{q};$$

and, for $d = 3$,

$$(\mu + \Lambda_2(a,b))^{-1} L^2 \subset C^{0,\gamma}, \quad \gamma = \frac{1}{2}.$$

3. We now consider the non-divergence form operator. By Hardy's inequality,

$$\nabla a \in \mathbf{F}_{\delta_a}, \quad \delta_a = \frac{4c^2(d-1)^2}{(d-2)^2}$$

(where, recall, $(\nabla a)_k := \sum_{i=1}^d \nabla_i a_{ik}$, $1 \leq k \leq d$), so

$$\nabla a + b \in \mathbf{F}_{\hat{\delta}}, \quad \sqrt{\hat{\delta}} := \sqrt{\delta_a} + \sqrt{\delta}.$$

We construct an operator realization of $-a \cdot \nabla^2 + b \cdot \nabla$ ($\equiv -\nabla \cdot a \cdot \nabla + (\nabla a + b) \cdot \nabla$) in L^q as $\Lambda_q(a, \nabla a + b)$ and obtain the following result as a consequence of Theorem 2:

Theorem 3. (The operator $-a \cdot \nabla^2 + b \cdot \nabla$). *Let $d \geq 3$, $a(x) = I + c|x|^{-2} x \otimes x$, $c > -1$, and $b \in \mathbf{F}_\delta$.*

- (i) *Let $d \geq 4$. Assume that c, δ satisfy the assumptions of Theorem 2(i) with δ there replaced by $\hat{\delta}$. Then for every $q > d-2$ sufficiently close to $d-2$ there exist constants $\mu_0 = \mu_0(d,q,c,\delta) > 0$ and $K_l = K_l(d,q,c,\delta)$ ($l = 1,2$) such that, for all $\mu > \mu_0$ and $f \in L^q$, $u := (\mu + \Lambda_q(a, \nabla a + b))^{-1} f \in W^{1,q} \cap W^{1,\frac{qd}{d-2}}$ and satisfies estimates ($\star$).*
- (ii) *Let $d \geq 3$. Assume that c, δ satisfy the assumptions of Theorem 2(ii) with δ there replaced by $\hat{\delta}$. Then ($\star$) holds with $q = 2$ and $u \in W^{2,2}$.*

Remark 1. One can prove Theorem 3 directly by carrying out the same analysis as in the proof of Theorem 2. This leads to somewhat less restrictive assumptions on c, δ, see [13] for details.

Remark 2. Let us show that the smooth vector fields b_n defined by (3) are in $\mathbf{F}_\delta$ with λ independent of n for appropriate $\varepsilon_n \downarrow 0$ and $c_n \uparrow 1$.

Indeed, let us define first $\tilde{b}_n = \gamma_{\epsilon_n} * \mathbf{1}_n b$ where $\varepsilon_n \downarrow 0$ is to be chosen. Since, clearly, $\mathbf{1}_n b \in \mathbf{F}_\delta$, we have for $f \in L^2$

$$\begin{aligned}\||\tilde{b}_n|(\lambda - \Delta)^{-\frac{1}{2}} f\|_2^2 &\leqslant \||\mathbf{1}_n b|(\lambda - \Delta)^{-\frac{1}{2}} f\|_2^2 + \||\tilde{b}_n - \mathbf{1}_n b|(\lambda - \Delta)^{-\frac{1}{2}} f\|_2^2 \\ &\leqslant \delta \|f\|_2^2 + \||\tilde{b}_n - \mathbf{1}_n b|(\lambda - \Delta)^{-\frac{1}{2}} f\|_2^2\end{aligned}$$

In turn, by Hölder's inequality and the Sobolev embedding theorem,

$$\||\tilde{b}_n - \mathbf{1}_n b|(\lambda - \Delta)^{-\frac{1}{2}} f\|_2^2 \leqslant \|\tilde{b}_n - \mathbf{1}_n b\|_{2d} \|(\lambda - \Delta)^{-\frac{1}{2}} f\|^2_{\frac{2d}{d-1}} \leqslant C_S \|\tilde{b}_n - \mathbf{1}_n b\|_{2d} \|f\|_2^2.$$

Since $\mathbf{1}_n b \in L^\infty$ and has compact support (and hence $\gamma_\varepsilon * \mathbf{1}_n b \to \mathbf{1}_n b$ in L^{2d} as $\varepsilon \downarrow 0$), for a given $\tilde{\delta}_n > \delta$, $\tilde{\delta}_n \downarrow \delta$, we can select ε_n, $n = 1, 2, \dots$ sufficiently small so that $\|\tilde{b}_n - \mathbf{1}_n b\|_{2d} < \frac{\tilde{\delta} - \delta}{C_d}$, and hence $\||\tilde{b}_n - \mathbf{1}_n b|(\lambda - \Delta)^{-\frac{1}{2}} f\|_2^2 < (\tilde{\delta}_n - \delta)\|f\|_2^2$. Therefore, $\||\tilde{b}_n|(\lambda - \Delta)^{-\frac{1}{2}} f\|_2^2 < \tilde{\delta}_n \|f\|_2^2$.

It is now clear that $b_n := c_n \tilde{b}_n$ as in (3) with $c_n := \sqrt{\delta \tilde{\delta}_n^{-1}}$ is in $\mathbf{F}_\delta$ with λ independent of n, as claimed.

3. Special case: operator $-\Delta + b\cdot\nabla$, $b \in \mathbf{F}_\delta$

For illustration purposes, we first prove Theorem 2 in the case $c = 0$.

Let $d \geq 3$. Assume that $b \in \mathbf{F}_\delta$, $\delta < 4$. We consider the approximating operators

$$\Lambda_p(b_n) := -\Delta + b_n \cdot \nabla, \quad D(\Lambda_p(b_n)) = W^{2,p}(\mathbb{R}^d), \quad 1 < p < \infty.$$

Recall that the resolvent set of operator $\Lambda_p(b_n)$ contains $\{\mu \mid \mu > \omega_p\}$, $\omega_p = \frac{\lambda\delta}{2(p-1)}$, and for every $p \in]\frac{2}{2-\sqrt{\delta}}, \infty[$ we have

$$\|e^{-t\Lambda_p(b_n)}\|_{p\to p} \leq e^{\omega_p t}, \tag{4}$$

cf. Theorem 1 with $a = I$.

The proof of our main result, Theorem 2, is modeled after the proof of the following

Theorem A. (see [15, Lemma 5], see also [11, Theorem 4.8]). *Let $d \geq 3$. Assume that $b \in \mathbf{F}_\delta$, $\delta < 1 \wedge (\frac{2}{d-2})^2$. Let $q \in [2 \vee (d-2), \frac{2}{\sqrt{\delta}}[$. The following is true.*

The limit

$$s\text{-}L^q\text{-}\lim_n (\mu + \Lambda_q(b_n))^{-1}, \quad \mu > \omega_q,$$

exists and determines the resolvent of the (minus) generator $\Lambda_q(b)$ of a positivity preserving L^∞ contraction C_0 semigroup in L^q. The operator $\Lambda_q(b)$ is an appropriate operator realization of the formal operator $-\Delta + b\cdot\nabla$ in L^q.

There exist constants $\mu_0 = \mu_0(d,q,\delta)$ $(\geq \omega_q)$, $K_l = K_l(d,q,\delta)$, $l = 1,2$, such that, for all $\mu > \mu_0$ and $f \in L^q$, $u := (\mu + \Lambda_q(b))^{-1} f \in W^{1,q} \cap W^{1,\frac{qd}{d-2}}$ and satisfies

$$\begin{aligned} \|\nabla u\|_q &\leq K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q, \\ \|\nabla u\|_{\frac{qd}{d-2}} &\leq K_2(\mu-\mu_0)^{\frac{1}{q}-\frac{1}{2}}\|f\|_q. \end{aligned} \tag{$\star\star$}$$

In particular,

$$(\mu + \Lambda_q(b))^{-1} : L^q \to C^{0,1-\frac{d-2}{q}}$$

whenever $d \geq 4$, $q \in \left]d-2, \frac{2}{\sqrt{\delta}}\right[$ and $\mu > \mu_0$. For $d = 3$, $(\mu + \Lambda_q(b))^{-1} : L^q \to C^{0,1-\frac{1}{q}}$ whenever $q \in \left[2, \frac{2}{\sqrt{\delta}}\right[$, $\mu > \mu_0$.

Proof of Theorem A. First, we show that the estimates ($\star\star$) hold for $0 \leq u_n := (\mu + \Lambda_q(b_n))^{-1} f$, $0 \leq f \in C_c^\infty$, with constants μ_0, K_l, $l = 1,2$, independent of n. Since b_n is smooth and bounded, we have $u_n \in W^{3,q}$. For brevity, write $u \equiv u_n$, $b \equiv b_n$. We will use the following notations:

$$w := \nabla u, \quad w_i := \nabla_i u, \quad w_{ik} := \nabla_i w_k,$$
$$\phi := -\nabla \cdot (w|w|^{q-2}) \equiv -\sum_{i=1}^d \nabla_i (w_i |w|^{q-2}).$$

Step 1 (The basic equality). We multiply the equation for u_n by the "test function" ϕ and integrate to obtain

$$\langle (\mu - \Delta)u, \phi\rangle = -\langle b \cdot \nabla u, \phi\rangle + \langle f, \phi\rangle, \tag{5}$$

where, recall,

$$\langle h\rangle := \int_{\mathbb{R}^d} h(x)dx, \quad \langle h, g\rangle := \langle h\bar{g}\rangle.$$

In the LHS of (5), we integrate by parts twice to obtain

$$\langle -\Delta u, \phi\rangle = \big\langle -\Delta w, w|w|^{q-2}\big\rangle = \sum_{i,k=1}^d \big\langle w_{ik}, w_{ik}|w|^{q-2} + (q-2)|w|^{q-3} w_k \nabla_i |w|\big\rangle,$$

thus arriving at the *basic equality*

$$\mu \|w\|_q^q + I_q + (q-2)J_q = \langle -b \cdot w\phi\rangle + \langle f, \phi\rangle, \tag{BE}$$

where

$$I_q := \sum_{i=1}^d \big\langle |\nabla w_i|^2, |w|^{q-2}\big\rangle, \qquad J_q := \frac{4}{q^2}\big\|\nabla |w|^{\frac{q}{2}}\big\|_2^2 = \big\langle |\nabla |w||^2, |w|^{q-2}\big\rangle \ \big(\le I_q\big).$$

Step 2 We bound the RHS of the basic equality (BE) in terms of J_q, $\|w\|_q^{q-2}$ and $\|f\|_q^2$. These bounds will give us the *principal inequality*

$$(\mu - \mu_0)\|w\|_q^q + \eta J_q \le C\|w\|_q^{q-2}\|f\|_q^2, \tag{PI}$$

for all $\mu > \mu_0$, for some constants $\mu_0 \ge \omega_q$, $\eta = \eta(q, d, \delta) > 0$ and $C = C(q, d, \delta) < \infty$, from which the required estimates ($\star\star$) will follow easily upon applying the Sobolev embedding theorem to J_q (Step 3 below).

We rewrite the "test function" ϕ as

$$\phi = -|w|^{q-2}\Delta u - (q-2)|w|^{q-3} w \cdot \nabla |w|,$$

where, using the equation for $u \equiv u_n$, we represent $-\Delta u = -\mu u - b \cdot w + f$. Thus, we obtain from (BE)

$$\mu\|w\|_q^q + I_q + (q-2)J_q = \big\langle b \cdot w - f, |w|^{q-2}(\mu u + b \cdot w - f) + (q-2)|w|^{q-3} w \cdot \nabla |w|\big\rangle.$$

Using $I_q \ge J_q$, we obtain

$$\mu\|w\|_q^q + (q-1)J_q \le \big\langle b \cdot w - f, |w|^{q-2}(\mu u + b \cdot w - f) + (q-2)|w|^{q-3} w \cdot \nabla |w|\big\rangle. \tag{6}$$

We now estimate the RHS of (6) in terms of J_q, $\|w\|_{q-2}^q$ and $\|f\|_2^q$. We will use (set $B_q := \langle |b \cdot w|^2 |w|^{q-2}\rangle$):

(1) $\langle b\cdot w, |w|^{q-2}\mu u\rangle \leq \frac{\mu}{\mu-\omega_q} B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q$. ($\frac{2}{2-\sqrt{\delta}} < q \Rightarrow \|u\|_q \leq (\mu-\omega_q)^{-1}\|f\|_q$, see (4)).

(2) $\langle b\cdot w, |w|^{q-2} b\cdot w\rangle = B_q$.

(3) $|\langle b\cdot w, |w|^{q-2}(-f)\rangle| \leq B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q$.

(4) $(q-2)\langle b\cdot w, |w|^{q-3} w\cdot\nabla|w|\rangle \leq (q-2) B_q^{\frac{1}{2}} J_q^{\frac{1}{2}}$.

(5) $\langle -f, |w|^{q-2}\mu u\rangle \leq 0$.

(6) $\langle -f, |w|^{q-2} b\cdot w\rangle \leq B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q$.

(7) $\langle f, |w|^{q-2} f\rangle \leq \|w\|_q^{q-2}\|f\|_q^2$.

(8) $(q-2)\langle -f, |w|^{q-3} w\cdot\nabla|w|\rangle \leq (q-2) J_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q$.

Using (1)–(8) and applying quadratic inequalities, we obtain ($\varepsilon, \kappa > 0$):

$$\text{RHS of (6)} \leq (q-2)\varepsilon J_q + (q-2)\big(\kappa J_q + \frac{1}{4\kappa} B_q\big) + (1+3\varepsilon)B_q + \left(1 + \frac{q}{4\varepsilon} + \frac{1}{4\varepsilon}\frac{\mu^2}{(\mu-\omega_q)^2}\right)\|w\|_q^{q-2}\|f\|_q^2. \tag{7}$$

In turn,

$$\begin{aligned} B_q &\leq \|b|w|^{\frac{q}{2}}\|_2^2 \\ &\quad \text{(we are using condition } b(\equiv b_n) \in \mathbf{F}_\delta) \\ &\leq \delta\|\nabla|w|^{\frac{q}{2}}\|_2^2 + \lambda\delta\|w\|_q^q \\ &= \frac{\delta q^2}{4} J_q + \lambda\delta\|w\|_q^q. \end{aligned}$$

Thus, one sees that the RHS of (6) can be estimated, by means of (7) and the above bound on B_q, in terms of J_q, $\|w\|_q$ and $\|f\|_q$ only. (Then we will re-group the resulting J_q terms in the LHS. Since the LHS of (6) already contains $(q-1)J_q$ with $q-1 \geq (1 \vee d-3) \geq 1$, it is clear that, by fixing $\varepsilon > 0$ sufficiently small, we can ignore in (7) the terms multiplied by ε.)

Select $\kappa = \frac{q\sqrt{\delta}}{4}$. We obtain:

$$\text{RHS of (6)} \leq \left[(q-2)\frac{q\sqrt{\delta}}{2} + \frac{\delta q^2}{4}\right] J_q + (q-2)\varepsilon J_q + 3\varepsilon\frac{q^2\delta}{4} J_q + \left(1 + \frac{q-2}{q\sqrt{\delta}} + 3\varepsilon\right)\lambda\delta\|w\|_q^q + \left(1 + \frac{q}{4\varepsilon} + \frac{1}{4\varepsilon}\frac{\mu^2}{(\mu-\omega_q)^2}\right)\|w\|_q^{q-2}\|f\|_q^2.$$

Thus,

$$\mu\|w\|_q^q + \left[q - 1 - (q-2)\frac{q\sqrt{\delta}}{2} - \frac{\delta q^2}{4} - (q-2)\varepsilon - 3\varepsilon\frac{q^2\delta}{4}\right] J_q \leq \left(1 + \frac{q-2}{q\sqrt{\delta}} + 3\varepsilon\right)\lambda\delta\|w\|_q^q + \left(1 + \frac{q}{4\varepsilon} + \frac{1}{4\varepsilon}\frac{\mu^2}{(\mu-\omega_q)^2}\right)\|w\|_q^{q-2}\|f\|_q^2.$$

In view of our assumptions on q and δ, the coefficient $q-1-(q-2)\frac{q\sqrt{\delta}}{2}-\frac{\delta q^2}{4}$ is strictly positive, so selecting $\varepsilon>0$ sufficiently small the *principal inequality* (**PI**).

Step 3. By the principal inequality (**PI**), $(\mu-\mu_0)\|w\|_q^q \leq C\|w\|_q^{q-2}\|f\|_q^2$, $w=\nabla u_n$, and so

$$\|\nabla u_n\|_q \leq K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q, \quad K_1:=C^{\frac{1}{2}}.$$

Again by (**PI**), $\eta J_q \leq C\|w\|_q^{q-2}\|f\|_q^2$, $J_q=\frac{4}{q^2}\|\nabla|w|^{\frac{q}{2}}\|_2^2$, so by the previous inequality $\eta\|\nabla|\nabla u_n|^{\frac{q}{2}}\|_2^2 \leq \frac{q^2}{4}CK_1^{q-2}(\mu-\mu_0)^{1-\frac{q}{2}}\|f\|_q^q$. The Sobolev embedding theorem now yields

$$\|\nabla u_n\|_{qj} \leq K_2(\mu-\mu_0)^{\frac{1}{q}-\frac{1}{2}}\|f\|_q, \quad j:=\frac{d}{d-2} \quad K_2:=C_S\eta^{-\frac{1}{q}}(q^2/4)^{\frac{1}{q}}C^{\frac{1}{q}}K_1^{\frac{q-2}{q}}.$$

It remains to pass to the limit $n\to\infty$. For this, we will use the first assertion of the theorem which is, in fact, the content of [15, Theorem 1]. (We could also refer to Theorem 1 with $a=I$.) Thus, we have $u_n\to u$ strongly in L^q, $u:=(\mu+\Lambda_q(b))^{-1}f$, where, recall, $0\leq f\in C_c^\infty$. Furthermore,

$$\|u-u_n\|_{qj}^{qj} \leq \|u-u_n\|_q^q\|u-u_n\|_\infty^{qj-q} \leq \|u-u_n\|_q^q(2\|f\|_\infty)^{qj-q}\to 0 \quad (n\to\infty)$$

since $\|u\|_\infty, \|u_n\|_\infty \leq \|f\|_\infty<\infty$. Since ∇ is weakly closed in L^q, L^{qj}, a standard weak compactness argument now yields ($\star\star$) for $f\in(C_c^\infty)_+$. Using a standard density argument, we obtain ($\star\star$) for all $f\in L^q_+$. The assertion of theorem follows for all $f\in L^q$ upon replacing K_l by $4K_l$, $l=1,2$. □

Remark 3. 1. In fact, the proof above yields a stronger variant of the principal inequality (**PI**)

$$(\mu-\mu_0)\|w\|_q^q+\epsilon I_q+(\eta-\epsilon)J_q \leq C\|w\|_q^{q-2}\|f\|_q^2, \quad \mu>\mu_0$$

for constants $\epsilon>0$, $\eta>0$, $C<\infty$, where, recall $I_q\geq J_q$. Indeed, it suffices to replace (6) in the proof above by

$$\mu\|w\|_q^q+\epsilon I_q+(q-1-\epsilon)J_q=\langle b\cdot w-f, |w|^{q-2}(\mu u+b\cdot w-f) \\ +(q-2)|w|^{q-3}w\cdot\nabla|w|\rangle.$$

Since our assumption on $\delta>0$ is a strict inequality, we take $\epsilon>0$ sufficiently small so that $q-1-\epsilon$ stays as close to $q-1$ as needed to repeat the rest of the proof while keeping the extra term ϵI_q.

2. In Theorem A, we could have chosen $b_n:=\frac{b}{|b|}|b|_n$, $|b|_n:=|b|\wedge n$. Although this would only give $(\mu+\Lambda_q(b_n))^{-1}C_c^\infty\subset W^{2,q}$ (rather than $W^{3,q}$), it is still possible to "differentiate" the equation $(\mu+\Lambda_q(b_n))u_n=f$, $f\in C_c^\infty$. Indeed,

define $\phi_m = e^{\frac{\Delta}{m}}\phi$, $m > 1$. We multiply the equation $(\mu - \Delta)u = -b_n \cdot \nabla u + f$ (where $u \equiv u_n$) by ϕ_m and integrate:

$$\langle(\mu - \Delta)u, \phi_m\rangle = \langle -b_n \cdot \nabla u + f, \phi_m\rangle.$$

We evaluate integrating by parts twice:

$$\begin{aligned}\langle -\Delta u, \phi_m\rangle &= \langle -\Delta e^{\frac{\Delta}{m}} u, \phi\rangle \\ &= -\sum_{k=1}^{d} \langle \Delta e^{\frac{\Delta}{m}} w_k, w_k |w|^{q-2}\rangle \\ &= \sum_{k,i=1}^{d} \langle e^{\frac{\Delta}{m}} \nabla_i w_k, \nabla_i w_k |w|^{q-2}\rangle + (q-2) \sum_{k,i=1}^{d} \langle e^{\frac{\Delta}{m}} \nabla_i w_k, w_k |w|^{q-3} \nabla_i |w|\rangle \\ &= I_{q,m} + (q-2) J_{q,m},\end{aligned}$$

where $I_{q,m} = \sum_{i,k=1}^{d} \langle e^{\frac{\Delta}{m}} w_{ik}, w_{ik} |w|^{q-2}\rangle$ and $J_{q,m} = \sum_{i,k=1}^{d} \langle e^{\frac{\Delta}{m}} w_{ik}, |w|^{q-3} w_k \nabla_i |w|\rangle$. Thus, we obtain

$$\mu\langle e^{\frac{\Delta}{m}} w, w|w|^{q-2}\rangle + I_{q,m} + (q-2) J_{q,m} = \langle -b_n \cdot w + f, \phi_m\rangle.$$

Using the fact that $w_k, w_{ik} \in L^q$, we can pass to the limit $m \to \infty$ in the LHS of the last equality appealing to Hölder's inequality and to the standard properties of mollifiers. Its RHS is

$$\langle -b_n \cdot w + f, \phi_m\rangle = \big\langle e^{\frac{\Delta}{m}}(b_n \cdot w - f), |w|^{q-2}(\mu u + b_n \cdot w - f) + (q-2)|w|^{q-3} w \cdot \nabla |w|\big\rangle,$$

so, using the inclusions $u, w_k, w_{ik} \in L^q$ and $f \in C_c^\infty$, $b_n \in L^\infty$ and appealing to Hölder's inequality, we can again pass to the limit in m. Thus, we arrive at the same *basic equality*:

$$\mu\langle |w|^q\rangle + I_q + (q-2) J_q = \langle -b_n \cdot w + f, \phi\rangle.$$

Now we continue as in the proof of Theorem A.

4. Proof of Theorem 2

Proof of assertion (i). In what follows, we will be working with a smooth approximation

$$a^\varepsilon(x) = I + c|x|_\varepsilon^{-2} x \otimes x \quad |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}, \quad \varepsilon > 0$$

of the matrix $a(x) = I + c|x|^{-2} x \otimes x$ rather than with the matrix a itself (a_ε, generally speaking, inherits the features of a). This is needed to ensure that the solutions to the corresponding elliptic equations are sufficiently regular so that all manipulations with the equations (such as integration by parts twice in Step 1 below) are justified.

In what follows, we follow the structure of the proof of Theorem A.

By the assumptions of the theorem, $[1 \vee (1+c)]\sqrt{\delta} < 2 - \frac{2}{d-2}$, i.e.,

$$\frac{2}{2-\sqrt{\delta_1}} < d-2, \quad \text{where, recall, } \delta_1 := [1 \vee (1+c)^{-2}]\delta.$$

Therefore, by Theorem 1, for every $q > d-2$ the set $\{\mu \mid \mu > \omega_q\}$ (where $\omega_q = \frac{\lambda\delta_1}{2(q-1)}$) is in the resolvent set of the operator

$$\Lambda_q(a^\varepsilon, b_n) = -\nabla \cdot a^\varepsilon \cdot \nabla + b_n \cdot \nabla, \quad D(\Lambda_q(a^\varepsilon, b_n)) = W^{2,q}$$

for all $\varepsilon > 0$, $n \geq 1$. Set

$$0 \leq u^{\varepsilon,n} := (\mu + \Lambda_q(a^\varepsilon, b_n))^{-1} f, \quad 0 \leq f \in C_c^\infty.$$

Since $a^\varepsilon, b_n \in C^\infty$, it is clear that $u^{\varepsilon,n} \in W^{3,q}$. For brevity, write

$$u \equiv u^{\varepsilon,n}, \quad w \equiv w^{\varepsilon,n} := \nabla u^{\varepsilon,n}.$$

Set

$$I_q := \sum_{r=1}^d \langle |\nabla_r w|^2 |w|^{q-2} \rangle, \qquad J_q := \frac{4}{q^2} \|\nabla |w|^{\frac{q}{2}}\|_2^2 = \big\langle |\nabla |w||^2, |w|^{q-2} \big\rangle.$$

We will use the equation for $u \equiv u^{\varepsilon,n}$ to obtain the *principal inequality*: for every $q > d-2$ sufficiently close to $d-2$

$$(\mu - \mu_0)\|w\|_q^q + \eta J_q \leq C\|w\|_q^{q-2}\|f\|_q^2, \quad \mu > \mu_0, \qquad (\mathbf{PI}_b)$$

for some constants $\eta = \eta(q,d,c,\delta) > 0$, $\mu_0 = \mu_0(d,q,c,\delta) > 0$, $C = C(q,d,c,\delta) < \infty$. We will obtain from ($\mathbf{PI}_b$), applying the Sobolev embedding theorem to J_q (Step 3 below), the estimates

$$\begin{aligned} \|\nabla u^{\varepsilon,n}\|_q &\leq K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q, \\ \|\nabla u^{\varepsilon,n}\|_{\frac{qd}{d-2}} &\leq K_2(\mu-\mu_0)^{\frac{1}{q}-\frac{1}{2}}\|f\|_q \end{aligned} \qquad (\star\star\star)$$

for all $\mu > \mu_0$ for constants $(\omega_q \leq)\ \mu_0$, K_l $(l = 1,2)$ independent of ε, n. Then the required estimates ($\star$) in Theorem 2 will follow upon taking $\varepsilon \downarrow 0$, $n \to \infty$ using Theorem 1, see details below.

We will also need the following auxiliary quantities:

$$\chi := |x|^2|x|_\varepsilon^{-2}, \quad x \cdot \nabla w \equiv \sum_{i=1}^d (x_i\nabla_i)w,$$

$$\bar{I}_{q,\chi} := \langle |x \cdot \nabla w|^2 \chi |x|^{-2} |w|^{q-2} \rangle,$$

$$\bar{J}_{q,\chi} := \langle (x \cdot \nabla |w|)^2 \chi |x|^{-2} |w|^{q-2} \rangle,$$

$$
\begin{aligned}
H_{q,\chi^i} &:= \langle \chi^i |x|^{-2} |w|^q \rangle, \quad i \geq 0, \\
G_{q,\chi^i} &:= \langle \chi^i |x|^{-4} (x \cdot w)^2 |w|^{q-2} \rangle, \quad i \geq 0, \\
H_q \equiv H_{q,\chi^0} &:= \langle |x|^{-2} |w|^q \rangle, \\
G_q \equiv G_{q,\chi^0} &:= \langle |x|^{-4} (x \cdot w)^2 |w|^{q-2} \rangle,
\end{aligned}
$$

We will need

Lemma 1.

$$
\begin{aligned}
&\chi \left(= |x|^2 |x|_\varepsilon^{-2} \right) \leq 1, \\
&I_q \geq \bar{I}_{q,\chi}, \quad J_q \geq \bar{J}_{q,\chi}, \\
&I_q \geq J_q, \quad \bar{I}_{q,\chi} \geq \bar{J}_{q,\chi}, \\
&H_{q,\chi^i} \geq H_{q,\chi^j}, \quad G_{q,\chi^i} \geq G_{q,\chi^j} \quad \textit{if } i \leq j, \\
&H_{q,\chi^i} \geq G_{q,\chi^i}, \quad i \geq 0, \\
&J_q \geq \frac{(d-2)^2}{q^2} H_q \quad (\textit{the Hardy inequality}).
\end{aligned}
$$

If we were to ignore the necessity to work with the smooth approximation of a, then we could take $\chi \equiv 1$ ($\Leftrightarrow \varepsilon = 0$), in which case we would have a more compact albeit formal proof.

We prove (**PI**$_b$) in two steps:
Step 1 (The basic equalities)

$$
\begin{aligned}
&\mu \|w\|_q^q + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\left(1 + (q-2)\frac{d}{q}\right) H_{q,\chi} + 2c(d-1) G_{q,\chi^2} \\
&+ 2c\frac{q-2}{q} H_{q,\chi^2} + 8c\varepsilon \langle |x|_\varepsilon^{-6} (x \cdot w)^2 |w|^{q-2} \rangle = \beta_1 + \langle -b_n \cdot w, \phi \rangle + \langle f, \phi \rangle,
\end{aligned} \tag{BE$_+$}
$$

$$
\begin{aligned}
&\mu \|w\|_q^q + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\left(1 + (q-2)\frac{d}{q}\right) H_{q,\chi} + cd G_{q,\chi^2} \\
&+ 2c\frac{q-2}{q} H_{q,\chi^2} + 4c\varepsilon \langle |x|_\varepsilon^{-6} (x \cdot w)^2 |w|^{q-2} \rangle = -\frac{1}{2}\beta_2 + \langle -b_n \cdot w, \phi \rangle + \langle f, \phi \rangle,
\end{aligned} \tag{BE$_-$}
$$

where

$$
\begin{aligned}
\phi &= -\nabla \cdot (w|w|^{q-2}) \equiv -\sum_{i=1}^d \nabla_i (w_i |w|^{q-2}) \quad (\text{“test function”}), \\
\beta_1 &:= -2c \langle |x|_\varepsilon^{-4} x \cdot w, x \cdot (x \cdot \nabla w) |w|^{q-2} \rangle, \\
\beta_2 &:= -2c(q-2) \langle |x|_\varepsilon^{-4} (x \cdot w)^2 x \cdot \nabla |w|, |w|^{q-3} \rangle.
\end{aligned}
$$

Remarks. 1. In comparison with the basic equality (BE) in the proof of Theorem A, here, in addition to terms I_q and J_q, we get other terms. However, we will be able to estimate them in terms of I_q and J_q using Hardy's inequality, see details below.

2. We will use equality (BE_-) to treat the case $c > 0$, and equality (BE_-) to treat the case $c < 0$. One can still use (BE_-) if $c < 0$ or (BE_-) if $c > 0$, but this would lead to more restrictive assumptions on c.

Proof of the basic equalities (BE_-), (BE_-). We multiply the equation $\mu u + A^\varepsilon_q u + b_n \cdot w = f$ by ϕ and integrate:

$$\mu \|w\|_q^q + \langle A^\varepsilon_q w, w|w|^{q-2}\rangle + \langle [\nabla, A^\varepsilon_q]_- u, w|w|^{q-2}\rangle = -\langle b_n \cdot w, \phi\rangle + \langle f, \phi\rangle,$$

where, recall, $A^\varepsilon_q = -\nabla \cdot a^\varepsilon \cdot \nabla = -\Delta - c\nabla \cdot |x|_\varepsilon^{-2}(x \otimes x) \cdot \nabla$, and we denote by $[F, G]_-$ the commutator of two operators,

$$[F, G]_- := FG - GF.$$

We evaluate $\langle A^\varepsilon_q w, w|w|^{q-2}\rangle$ by integrating by parts twice (cf. Step 1 in the proof of Theorem A):

$$\langle A^\varepsilon_q w, w|w|^{q-2}\rangle = I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}),$$

where, recall, $\bar{I}_{q,\chi} = \langle |x \cdot \nabla w|^2 \chi |x|^{-2}|w|^{q-2}\rangle$, $\bar{J}_{q,\chi} = \langle (x \cdot \nabla |w|)^2 \chi |x|^{-2}|w|^{q-2}\rangle$.

Thus, we have

$$\mu \|w\|_q^q + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) + \langle [\nabla, A^\varepsilon_q]_- u, w|w|^{q-2}\rangle = \langle f, \phi\rangle. \tag{8}$$

It remains to evaluate:

$$\langle [\nabla, A^\varepsilon_q]_- u, w|w|^{q-2}\rangle \equiv \langle [\nabla_r, A^\varepsilon_q]_- u, w_r|w|^{q-2}\rangle := \sum_{r=1}^d \langle [\nabla_r, A^\varepsilon_q]_- u, w_r|w|^{q-2}\rangle.$$

Remark. From now on, we omit the summation sign in repeated indices.

Note that

$$[\nabla_r, A^\varepsilon_q]_- = -\nabla \cdot (\nabla_r a^\varepsilon) \cdot \nabla, \qquad (\nabla_r a^\varepsilon)_{ik} = c|x|_\varepsilon^{-2}\delta_{ri}x_k + c(|x|_\varepsilon^{-2}\delta_{rk}x_i - 2|x|_\varepsilon^{-4}x_i x_k x_r),$$

so

$$\begin{aligned}\langle [\nabla_r, A^\varepsilon_q]_- u, w_r|w|^{q-2}\rangle &= -c\langle w_k \nabla_i(|x|_\varepsilon^{-2}\delta_{ri}x_k) + |x|_\varepsilon^{-2}\delta_{ri}x_k \nabla_i w_k, w_r|w|^{q-2}\rangle \\ &\quad + c\langle (|x|_\varepsilon^{-2}\delta_{rk}x_i - 2|x|_\varepsilon^{-4}x_i x_k x_r) w_k, \nabla_i(w_r|w|^{q-2})\rangle \\ &=: \alpha_1 + \alpha_2.\end{aligned}$$

We have

$$\alpha_1 = -c\langle(|x|_\varepsilon^{-2}\delta_{rk} - 2|x|_\varepsilon^{-4}\delta_{ri}x_k x_r)w_k + |x|_\varepsilon^{-2}x\cdot\nabla w_r, w_r|w|^{q-2}\rangle$$
$$= -c\langle|x|_\varepsilon^{-2}|w|^q\rangle + 2c\langle|x|_\varepsilon^{-4}(x\cdot w)^2|w|^{q-2}\rangle - c\langle|x|_\varepsilon^{-2}x\cdot\nabla|w|, |w|^{q-1}\rangle.$$

Then

$$\alpha_1 = -c\left(1 - \frac{d-2}{q}\right)H_{q,\chi} + 2cG_{q,\chi^2} + 2\frac{c}{q}\varepsilon\langle|x|_\varepsilon^{-4}|w|^q\rangle$$

due to

$$\langle|x|_\varepsilon^{-2}x\cdot\nabla|w|, |w|^{q-1}\rangle = \frac{1}{q}\langle|x|_\varepsilon^{-2}x\cdot\nabla|w|^q\rangle$$
$$= -\frac{1}{q}\langle|w|^q\nabla\cdot(x|x|_\varepsilon^{-2})\rangle = -\frac{d}{q}H_{q,\chi} + \frac{2}{q}\langle|x|^2|x|_\varepsilon^{-4}|w|^q\rangle$$
$$= -\frac{d-2}{q}H_{q,\chi} - \frac{2}{q}\varepsilon\langle|x|_\varepsilon^{-4}|w|^q\rangle.$$

Next,

$$\alpha_2 = c\langle|x|_\varepsilon^{-2}w, x\cdot\nabla(w|w|^{q-2})\rangle - 2c\langle|x|_\varepsilon^{-4}x\cdot w, x\cdot(x\cdot\nabla(w|w|^{q-2}))\rangle.$$

Then

$$\alpha_2 = \beta_1 + \beta_2 + c\langle|x|_\varepsilon^{-2}x\cdot\nabla|w|, |w|^{q-1}\rangle + c(q-2)\langle|x|_\varepsilon^{-2}x\cdot\nabla|w|, |w|^{q-1}\rangle$$
$$= \beta_1 + \beta_2 - c(q-1)\left(\frac{d-2}{q}H_{q,\chi} + \frac{2}{q}\varepsilon\langle|x|_\varepsilon^{-4}, |w|^q\rangle\right).$$

In view of

$$\beta_1 = -\frac{1}{2}\beta_2 + c(d-2)G_{q,\chi^2} + 4c\varepsilon\langle|x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle,$$

we rewrite $\alpha_1 + \alpha_2 = \langle[\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle$ in two ways:

$$\langle[\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle = -\beta_1 - c\left(1 + (q-2)\frac{d-2}{q}\right)H_{q,\chi} + 2c(d-1)G_{q,\chi^2}$$
$$- 2c\frac{q-2}{q}\varepsilon\langle|x|_\varepsilon^{-4}|w|^q\rangle + 8c\varepsilon\langle|x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle$$

and

$$\langle[\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle = \frac{1}{2}\beta_2 - c\left(1 + (q-2)\frac{d-2}{q}\right)H_{q,\chi} + cdG_{q,\chi^2}$$
$$- 2c\frac{q-2}{q}\varepsilon\langle|x|_\varepsilon^{-4}|w|^q\rangle + 4c\varepsilon\langle|x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle.$$

The last two identities applied in (8) yield (BE_-), (BE_-). □

Step 2 The principal inequality (**PI**$_b$) will follow once we estimate properly the terms $\langle -b_n\cdot w, \phi\rangle$, $\langle f, \phi\rangle$ and β_i $(i = 1, 2)$ in the RHS of the basic equalities (BE_-), (BE_-). For that, we will need the next three lemmas.

Lemma 2. *For every $\varepsilon_0 > 0$, there exist constants $C_i = C_i(\varepsilon_0)$ $(i = 1, 2)$* and k_1 such that

$$\langle -b_n \cdot w, \phi\rangle \leq |c|(d+3)\frac{q\sqrt{\delta}}{2} G_{q,\chi^2}^{\frac{1}{2}} J_q^{\frac{1}{2}} + |c|\frac{q\sqrt{\delta}}{2} \bar{I}_{q,\chi}^{\frac{1}{2}} J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right) J_q + k_1\varepsilon_0 I_q + C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.$$

Proof of Lemma 2. We follow the argument in Step 2 of the proof of Theorem A. For brevity, below we write $b \equiv b_n$. We have:

$$\langle -b\cdot w, \phi\rangle = \langle -\Delta u, |w|^{q-2}(-b\cdot w)\rangle - (q-2)\langle |w|^{q-3} w\cdot\nabla|w|, -b\cdot w\rangle =: F_1 + F_2.$$

Then, clearly,

$$F_2 \leq (q-2) B_q^{\frac{1}{2}} J_q^{\frac{1}{2}}, \qquad \text{where } B_q := \langle |b\cdot w|^2 |w|^{q-2}\rangle.$$

Next, we bound F_1. Using the equation for u $(\equiv u^{\varepsilon,n})$, we represent

$$-\Delta u = \nabla\cdot(a^\varepsilon - I)\cdot w - \mu u - b\cdot w + f,$$

and evaluate

$$\begin{aligned} F_1 &= \langle \nabla\cdot(a^\varepsilon - I)\cdot w, |w|^{q-2}(-b\cdot w)\rangle + \langle(-\mu u - b\cdot w + f), |w|^{q-2}(-b\cdot w)\rangle \\ &\quad \text{(we expand the first term using } \nabla a^\varepsilon = c(d+1)|x|_\varepsilon^{-2}x - 2c|x|^2|x|_\varepsilon^{-4}x) \\ &= c(d+1)\langle |x|_\varepsilon^{-2}x\cdot w, |w|^{q-2}(-b\cdot w)\rangle \\ &\quad - 2c\langle \chi|x|_\varepsilon^{-2}x\cdot w, |w|^{q-2}(-b\cdot w)\rangle \\ &\quad + c\langle |x|_\varepsilon^{-2}x\cdot(x\cdot\nabla w), |w|^{q-2}(-b\cdot w)\rangle \\ &\quad + \langle(-\mu u - b\cdot w + f), |w|^{q-2}(-b\cdot w)\rangle, \end{aligned}$$

where, recall, $x\cdot\nabla w \equiv \sum_{i=1}^d (x_i\nabla_i)w$. We bound F_1 from above by applying consecutively the following estimates:

(1°) $|\langle |x|_\varepsilon^{-2}x\cdot w, |w|^{q-2}(-b\cdot w)\rangle| \leq G_{q,\chi^2}^{\frac{1}{2}} B_q^{\frac{1}{2}}$ (where, recall, $G_{q,\chi^i} := \langle \chi^i |x|^{-4} (x\cdot w)^2 |w|^{q-2}\rangle$).

(2°) $|\langle \chi|x|_\varepsilon^{-2}x\cdot w, |w|^{q-2}(-b\cdot w)\rangle| \leq G_{q,\chi^4}^{\frac{1}{2}} B_q^{\frac{1}{2}} \leq G_{q,\chi^2}^{\frac{1}{2}} B_q^{\frac{1}{2}}$.

(3°) $|\langle |x|_\varepsilon^{-2}x\cdot(x\cdot\nabla w), |w|^{q-2}(-b\cdot w)\rangle| \leq \bar{I}_{q,\chi}^{\frac{1}{2}} B_q^{\frac{1}{2}}$ (recall $\bar{I}_{q,\chi} := \langle |x\cdot\nabla w|^2\chi|x|^{-2} |w|^{q-2}\rangle$).

(4°) $\langle(-\mu u), |w|^{q-2}(-b\cdot w)\rangle \leq \frac{\mu}{\mu-\omega_q} B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q$ (here $\|u_n\|_q \leq (\mu-\omega_q)^{-1} \|f\|_q$ by Theorem 1).

(5°) $\langle b\cdot w, |w|^{q-2} b\cdot w\rangle = B_q$.

(6°) $\langle f, |w|^{q-2}(-b \cdot w)\rangle| \leq B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q$.

In (4°) and (6°), we estimate $B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q \leq \varepsilon_0 B_q + \frac{1}{4\varepsilon_0} \|w\|_q^{q-2} \|f\|_q^2$ ($\varepsilon_0 > 0$). Therefore,

$$\begin{aligned}&\langle -b \cdot w, \phi\rangle \\ &\leq |c|(d+3)G_{q,\chi^2}^{\frac{1}{2}} B_q^{\frac{1}{2}} + |c|\bar{I}_{q,\chi}^{\frac{1}{2}} B_q^{\frac{1}{2}} + B_q + (q-2)B_q^{\frac{1}{2}} J_q^{\frac{1}{2}} + \varepsilon_0 B_q + C_2(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2.\end{aligned} \tag{9}$$

Since $b \in \mathbf{F}_\delta$ is equivalent to the inequality

$$\langle |b|^2|\varphi|^2\rangle \leq \delta\langle|\nabla\varphi|^2\rangle + \lambda\delta\langle|\varphi|^2\rangle, \quad \varphi \in W^{1,2},$$

we have

$$B_q \leq \|b|w|^{\frac{q}{2}}\|_2^2 \leq \delta\|\nabla|w|^{\frac{q}{2}}\|_2^2 + \lambda\delta\|w\|_q^q = \frac{q^2\delta}{4} J_q + \lambda\delta\|w\|_q^q,$$

and so

$$B_q^{\frac{1}{2}} \leq \frac{q\sqrt{\delta}}{2} J_q^{\frac{1}{2}} + \sqrt{\lambda\delta}\|w_q\|^{\frac{q}{2}}.$$

We apply the last two bounds in (9) and estimating the resulting terms that contain $\sqrt{\lambda\delta}\|w\|_q^{\frac{q}{2}}$ as

$$\begin{aligned}G_{q,\chi^2}^{\frac{1}{2}}\sqrt{\lambda\delta}\|w_q\|^{\frac{q}{2}} &\leq \varepsilon_0 G_{q,\chi^2} + \frac{\lambda\delta}{4\varepsilon_0}\|w_q\|^q,\\ \bar{I}_q^{\frac{1}{2}}\sqrt{\lambda\delta}\|w_q\|^{\frac{q}{2}} &\leq \varepsilon_0 \bar{I}_q + \frac{\lambda\delta}{4\varepsilon_0}\|w_q\|^q,\\ J_q^{\frac{1}{2}}\sqrt{\lambda\delta}\|w_q\|^{\frac{q}{2}} &\leq \varepsilon_0 J_q + \frac{\lambda\delta}{4\varepsilon_0}\|w\|_q^q.\end{aligned}$$

We use Lemma 1 to bound G_{q,χ^2}, $\bar{I}_q$, J_q in terms of I_q, and obtain that there exists a constant $k_1 = k_1(c, d, q, \delta) > 0$ such that

$$|c|(d+3)\varepsilon_0 G_{q,\chi^2} + |c|\varepsilon_0\bar{I}_q + (q-2)\varepsilon_0 J_q + \frac{q^2\delta}{4}\varepsilon_0 J_q \leq k_1\varepsilon_0 I_q.$$

The assertion of Lemma 2 now follows. □

Next, we estimate the term $\langle f, \phi\rangle$ in the RHS of (BE_-), (BE_-).

Lemma 3. *For each $\varepsilon_0 > 0$, there exist constants $C = C(\varepsilon_0)$ and k_2 such that*

$$\langle f, \phi\rangle \leq k_2\varepsilon_0 I_q + C(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2.$$

where, recall, $I_q := \sum_{r=1}^d \langle(\nabla_r w)^2|w|^{q-2}\rangle$.

Proof of Lemma 3. Clearly,

$$\langle f,\phi\rangle = \langle f,(-\Delta u)|w|^{q-2}\rangle - (q-2)\langle f,|w|^{q-3}w\cdot\nabla|w|\rangle =: F_1+F_2.$$

Since $-\Delta u = \nabla\cdot(a^\varepsilon - I)\cdot w - \mu u + f$, where $a^\varepsilon - I = c|x|_\varepsilon^{-2}x\otimes x$, and

$$\begin{aligned} F_1 &= \langle\nabla\cdot(a^\varepsilon - I)\cdot w,|w|^{q-2}f\rangle + \langle(-\mu u+f),|w|^{q-2}f\rangle \\ &\quad \text{(we expand the first term using } \nabla a^\varepsilon = c(d+1)x|x|_\varepsilon^{-2} - 2c|x|^2|x|_\varepsilon^{-4}x) \\ &= c(d+1)\langle|x|_\varepsilon^{-2}x\cdot w,|w|^{q-2}f\rangle - 2c\langle\chi|x|_\varepsilon^{-2}x\cdot w,|w|^{q-2}f\rangle \\ &\quad + c\langle|x|_\varepsilon^{-2}x\cdot(x\cdot\nabla w),|w|^{q-2}f\rangle + \langle(-\mu u+f),|w|^{q-2}f\rangle, \end{aligned}$$

where, recall, $x\cdot\nabla w := \sum_{i=1}^d(x_i\nabla_i)w$. We bound F_1 and F_2 from above by applying consecutively the following estimates:

(1) $\langle|x|_\varepsilon^{-2}x\cdot w,|w|^{q-2}f\rangle \leq H_q^{\frac12}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$.
(2) $\langle\chi|x|_\varepsilon^{-2}x\cdot w,|w|^{q-2}f\rangle \leq H_q^{\frac12}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$.
(3) $\langle|x|_\varepsilon^{-2}x\cdot(x\cdot\nabla w),|w|^{q-2}f\rangle \leq (\bar I_{q,\chi})^{\frac12}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$ (recall $\bar I_{q,\chi} := \langle|x\cdot\nabla w|^2\chi|x|^{-2}|w|^{q-2}\rangle$).
(4) $\langle -f,|w|^{q-2}\mu u\rangle \leq 0$.
(5) $\langle f,|w|^{q-2}f\rangle \leq \|w\|_q^{q-2}\|f\|_q^2$.
(6) $(q-2)\langle -f,|w|^{q-3}w\cdot\nabla|w|\rangle \leq (q-2)J_q^{\frac12}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$.

Now, (1)–(6), the quadratic inequality and Lemma 1 yield the lemma. □

It remains to estimate the terms β_1 and $-\frac12\beta_2$ in the RHS of the basic equalities (BE$_-$), (BE$_-$).

Lemma 4. *We have*

$$\beta_1 \leq c\theta\bar I_{q,\chi} + c\theta^{-1}G_{q,\chi^2},$$

and

$$|\beta_2| \leq 2|c|(q-2)\big(\theta\bar J_{q,\chi} + 4^{-1}\theta^{-1}G_{q,\chi^2}\big).$$

In both inequalities, $\theta>0$ will be chosen later.

Proof.

$$\begin{aligned} \beta_1 &\leq 2c\langle|x|_\varepsilon^{-4}|x\cdot w|^2|w|^{q-2}\rangle^{\frac12}\langle|x|_\varepsilon^{-4}|x|^2|x\cdot\nabla w|^2|w|^{q-2}\rangle^{\frac12} \\ &\leq 2cG_{q,\chi^2}^{\frac12}\bar I_{q,\chi}^{\frac12} \\ &\quad \text{(we are applying quadratic inequality)} \\ &\leq c\theta\bar I_{q,\chi} + c\theta^{-1}G_{q,\chi^2}. \\ |\beta_2| &= \big|2c(q-2)\langle|x|_\varepsilon^{-4}|x\cdot w|^2x\cdot\nabla|w|,|w|^{q-3}\rangle\big| \end{aligned}$$

(we apply quadratic inequality)

$$\leq 2|c|(q-2)\big(\theta \bar{J}_{q,\chi} + 4^{-1}\theta^{-1}G_{q,\chi^2}\big).$$

□

We are in position to complete the proof of the principal inequality (**PI**$_b$).

Proof of (**PI**$_b$) ***in the case*** $c > 0$. We will need

Lemma 5. *(*Hardy-type inequality*).*

$$\frac{d^2}{4}H_{q,\chi} - (d+2)H_{q,\chi^2} + 3H_{q,\chi^3} \leq \frac{q^2}{4}\bar{J}_{q,\chi} \qquad \textbf{(HI)}$$

Proof. Set $F := |x|_\varepsilon^{-1}|w|^{\frac{q}{2}}$. Then

$$\frac{q^2}{4}\bar{J}_{q,\chi} = \big\langle \big(|x|_\varepsilon^{-1}x\cdot\nabla|w|^{\frac{q}{2}}\big)^2\big\rangle = \big\langle (x\cdot\nabla F + \chi F)^2\big\rangle = \langle (x\cdot\nabla F)^2\rangle + \langle \chi^2F^2\rangle + 2\langle x\cdot\nabla F, \chi F\rangle.$$

Now (**HI**) follows from the inequality

$$\langle (x\cdot\nabla F)^2\rangle \equiv \|x\cdot\nabla F\|_2^2 \geq \frac{d^2}{4}\|F\|_2^2 \equiv \frac{d^2}{4}H_{q,\chi}$$

and the equalities

$$2\langle x\cdot\nabla F, \chi F\rangle = -d\langle \chi F^2\rangle - \langle F^2, x\cdot\nabla\chi\rangle, \quad x\cdot\nabla\chi = 2\Big(\frac{|x|^2}{|x|_\varepsilon^2} - \frac{|x|^4}{|x|_\varepsilon^4}\Big) = 2\chi(1-\chi).$$

□

Put

$$k := k_1 + k_2,$$

where k_1 and k_2 are the constants in Lemmas 2 and 3. Thus, applying Lemmas 2–4 in the RHS of (BE_-), we obtain

$$\begin{aligned}
&\mu\|w\|_q^q + (1-k\varepsilon_0)I_q + c(1-\theta)\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\left(1 + \frac{q-2}{q}d\right)H_{q,\chi} \\
&\quad + c\big(2(d-1) - \theta^{-1}\big)G_{q,\chi^2} + 2c\frac{q-2}{q}H_{q,\chi^2} \\
&\leq c(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac{1}{2}}J_q^{\frac{1}{2}} + c\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q \\
&\quad + \tilde{C}_1(\varepsilon_0)\|w\|_q^q + \tilde{C}_2(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2,
\end{aligned} \qquad (10)$$

where $\tilde{C}_1(\varepsilon_0) = C_1(\varepsilon_0)$, $\tilde{C}_2(\varepsilon_0) = C_2(\varepsilon_0) + C(\varepsilon_0)$.

We have to consider two sub-cases:

Case 1. Suppose that $1-\frac{cq\sqrt{\delta}}{4}>0$. Let $0<\theta<1$ (one can verify that the choice of $\theta \geq 1$ leads to more restrictive constraints on c). Using inequality $\bar{J}_{q,\chi} \leq \bar{I}_{q,\chi}$, we replace $c(1-\theta)\bar{I}_{q,\chi}$ by $c(1-\theta)\bar{J}_{q,\chi}$ in (12), arriving at

$$\begin{aligned}\mu\|w\|_q^q &+ (1-k\varepsilon_0)I_q + (q-2)J_q + c[q-1-\theta]\bar{J}_{q,\chi} \\ &- c\left(1+\frac{q-2}{q}d\right)H_{q,\chi} + c\big(2(d-1)-\theta^{-1}\big)G_{q,\chi^2} \\ &+ 2c\frac{q-2}{q}H_{q,\chi^2} \leq \text{RHS of (12)}.\end{aligned}$$

Next, we apply to $\bar{J}_{q,\chi}$ the inequality (**HI**) to obtain

$$\begin{aligned}\mu\|w\|_q^q &+ (1-k\varepsilon_0)I_q + (q-2)J_q + c\big[2(d-1)-\theta^{-1}\big]G_{q,\chi^2} \\ &+ c\langle M(\chi)|x|^{-2}|w|^q\rangle \leq \text{RHS of (12)},\end{aligned}$$

where

$$M(\chi) := \left[(q-1-\theta)\frac{4}{q^2}\left(\frac{d^2}{4}-(d+2)\chi+3\chi^2\right)-\left(1+\frac{q-2}{q}d\right)+2\frac{q-2}{q}\chi\right]\chi.$$

We take $\theta := \frac{1}{2(d-1)}$, so that

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q + (q-2)J_q + c\langle M(\chi)|x|^{-2}|w|^q\rangle \leq \text{RHS of (12)}.$$

Next, we claim that, for $q>d-2$ sufficiently close to $d-2$,

$$\min_{0\leq t\leq 1} M(t) = M(1) = \big(q-1-\frac{1}{2(d-1)}\big)\frac{(d-2)^2}{q^2} - \big(1+\frac{q-2}{q}(d-2)\big) < 0.$$

(It is easily seen that if $q \downarrow d-2$, then $M(1) \downarrow -\frac{1}{2(d-1)} < 0$. To show that the minimum is attained in $t=1$, we argue as follows. Put $C=\frac{4}{q^2}(q-1-\theta)$. Then $M(t)=3Ctf(t)$, where $f(t)=t^2+\frac{1}{3}(2\frac{q-2}{qC}-d-2)t+\frac{d^2}{12}-\frac{1}{3C}(1+\frac{q-2}{q}d)$. Since $f(1)<0$, $f(t)=0$ has real roots $t_1<t_2$. Clearly, it is enough to show that $\frac{t_1+t_2}{2}\geq 1$. One has $t_1+t_2=\frac{1}{3}(d+2-2\frac{q-2}{Cq})$, and so, since $q>d-2$ is assumed to be sufficiently close to $d-2$, we have $\frac{t_1+t_2}{2}\geq 1$ (equivalently $d \geq 4+\frac{q}{2}(1-\frac{1-\theta}{q-1-\theta})$) for $d\geq 5$. Another elementary calculation also gives the desired for $d=4$.)

Since $0<\chi<1$, we obtain

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q + (q-2)J_q + cM(1)H_q \leq \text{RHS of (12)}.$$

Thus, applying $H_q \geq G_{q,\chi^2}$ in the RHS of (12), we obtain

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q + (q-2)J_q + cM(1)H_q$$
$$\leq c(d+3)\frac{q\sqrt{\delta}}{2}H_q^{\frac{1}{2}}J_q^{\frac{1}{2}} + c\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q$$
$$+ \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.$$

Applying the quadratic inequality twice in the RHS, we obtain (let $\theta_2, \theta_3 > 0$)

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q + (q-2)J_q + cM(1)H_q$$
$$\leq c(d+3)\frac{q\sqrt{\delta}}{4}(\theta_2 J_q + \theta_2^{-1}H_q) + c\frac{q\sqrt{\delta}}{4}(\theta_3\bar{I}_{q,\chi} + \theta_3^{-1}J_q) + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q$$
$$+ \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.$$

We select $\theta_2 = \frac{q}{d-2}$, $\theta_3 = 1$. Then

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q + (q-2)J_q + cM(1)H_q$$
$$\leq c(d+3)\frac{q\sqrt{\delta}}{4}\left(\frac{q}{d-2}J_q + \frac{d-2}{q}H_q\right) + c\frac{q\sqrt{\delta}}{4}(\bar{I}_{q,\chi} + J_q) + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q$$
$$+ \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.$$

Since by our assumption $1 - \frac{cq\sqrt{\delta}}{4} > 0$, selecting $\varepsilon_0 > 0$ sufficiently small so that $1 - k\varepsilon_0 - \frac{cq\sqrt{\delta}}{4} > 0$, we can estimate, using $I_q \geq \bar{I}_{q,\chi}$ and $I_q \geq J_q$,

$$(1-k\varepsilon_0)I_q - \frac{cq\sqrt{\delta}}{4}\bar{I}_{q,\chi} \geq (1-k\varepsilon_0 - \frac{cq\sqrt{\delta}}{4})I_q \geq \big(1-k\varepsilon_0 - \frac{cq\sqrt{\delta}}{4}\big)J_q.$$

Thus, the previous inequality becomes

$$\mu\|w\|_q^q + \left(q-1-k\varepsilon_0 - \frac{cq\sqrt{\delta}}{4}\right)J_q + cM(1)H_q$$
$$\leq c(d+3)\frac{q\sqrt{\delta}}{4}\left(\frac{q}{d-2}J_q + \frac{d-2}{q}H_q\right) + c\frac{q\sqrt{\delta}}{4}J_q + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q$$
$$+ \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.$$

We now regroup the J_q terms together in the LHS. Then, applying Hardy's inequality $J_q \geq \frac{(d-2)^2}{q^2}H_q$ to the H_q terms (which enter the LHS with a negative coefficient), we obtain

$$\mu\|w\|_q^q + \left[\big(q-1-k\varepsilon_0\big)\frac{(d-2)^2}{q^2} - Ł_1(c,\delta,q,d)\right]\frac{q^2}{(d-2)^2}J_q$$
$$\leq \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2,$$

where

$$Ł_1(c,\delta,q,d):=c\left[1+\frac{q-2}{q}(d-2)-\left(q-1-\frac{1}{2(d-1)}\right)\frac{(d-2)^2}{q^2}\right.$$
$$\left.+\frac{\sqrt{\delta}}{2}\left(\frac{(d-2)^2}{q}+(d+3)(d-2)\right)\right]$$
$$+\left[\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right]\frac{(d-2)^2}{q^2}.$$

By the assumption of the theorem, $d-3-L_1(c,\delta,d)>0$. It is easily seen that the latter yields, for all $q>d-2$ sufficiently close to $d-2$, the inequality $(q-1)\frac{(d-2)^2}{q^2}-Ł_1(c,\delta,q,d)>0$. Thus, selecting $\varepsilon_0>0$ sufficiently small, we obtain the principal inequality (**PI**$_b$) (with $\mu_0:=\tilde{C}_1$, $C:=\tilde{C}_2$).

Case 2. Let $1-\frac{cq\sqrt{\delta}}{4}\leq 0$. Similar argument applied in (12) yields (the only difference with the case 1 is that we keep for a moment the term $\theta\bar{I}_{q,\chi}$, $\theta:=\frac{1}{2(d-1)}$ intact, and so we define $M(1)$ differently):

$$\mu\|w\|_q^q+(1-k\varepsilon_0)I_q+(q-2)J_q+c(1-\frac{1}{2(d-1)})\bar{I}_{q,\chi}+cM(1)H_q$$
$$\leq c(d+3)\frac{q\sqrt{\delta}}{4}\left(\frac{q}{d-2}J_q+\frac{d-2}{q}H_q\right)+c\frac{q\sqrt{\delta}}{4}(\bar{I}_{q,\chi}+J_q)+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right)J_q$$
$$+\tilde{C}_1\|w\|_q^q+\tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.$$

where $M(1):=\big(q-2\big)\frac{(d-2)^2}{q^2}-\big(1+\frac{q-2}{q}(d-2)\big)<0$.

If $1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}<0$, then, regrouping the terms $\bar{I}_{q,\chi}$ together, we have $c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\bar{I}_{q,\chi}\geq c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)I_q$ since $I_q\geq\bar{I}_{q,\chi}$. Hence

$$(1-k\varepsilon_0)I_q+(q-2)J_q+c(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4})\bar{I}_{q,\chi}$$
$$\geq\big[1-k\varepsilon_0+c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\big]I_q$$

(by the assumptions of the theorem $1+c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)>0$,

so we select ε_0 sufficiently small to have coefficient $[\dots]>0$)

$$\geq\big[1-k\varepsilon_0+c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\big]J_q.$$

Applying the latter in the previous inequality, we obtain

$$\mu\|w\|_q^q+\big[1-k\varepsilon_0+c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\big]J_q+(q-2)J_q+cM(1)H_q$$
$$\leq c(d+3)\frac{q\sqrt{\delta}}{4}\left(\frac{q}{d-2}J_q+\frac{d-2}{q}H_q\right)+c\frac{q\sqrt{\delta}}{4}J_q+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right)J_q$$
$$+\tilde{C}_1\|w\|_q^q+\tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.$$

We regroup the J_q and the H_q terms:

$$\mu\|w\|_q^q + \left[q-1-k\varepsilon_0 + c\left(1-\frac{1}{2(d-1)} - \frac{q\sqrt{\delta}}{4}\right) - c(d+3)\frac{q^2\sqrt{\delta}}{4(d-2)} - c\frac{q\sqrt{\delta}}{4} - \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)\right] J_q + \left[cM(1) - c(d+3)\frac{\sqrt{\delta}(d-2)}{4}\right] H_q \leq \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.$$

Applying Hardy's inequality $J_q \geq \frac{(d-2)^2}{q^2} H_q$ to the H_q term (which, clearly, enters the LHS with negative coefficient), we finally obtain

$$\mu\|w\|_q^q + \left[(q-1-k\varepsilon_0)\frac{(d-2)^2}{q^2} - Ł_1(c,\delta,q,d)\right]\frac{q^2}{(d-2)^2} J_q \leq \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2, \tag{11}$$

where, by the assumption $d-3-L_1(c,\delta,d) > 0$ of the theorem, $(q-1-k\varepsilon_0)\frac{(d-2)^2}{q^2} - Ł_1(c,\delta,q,d) > 0$ for all $q > d-2$ sufficiently close to $d-2$ and all $\varepsilon_0 > 0$ sufficiently small. The principal inequality (**PI**$_b$) follows (with $\mu_0 := \tilde{C}_1$, $C := \tilde{C}_2$).

If $1 - \frac{1}{2(d-1)} - \frac{q\sqrt{\delta}}{4} \geq 0$, then clearly $(1-k\varepsilon_0)I_q + c\big(1 - \frac{1}{2(d-1)} - \frac{q\sqrt{\delta}}{4}\big)\bar{I}_{q,\chi} \geq (1-k\varepsilon_0)J_q + c\big(1 - \frac{1}{2(d-1)} - \frac{q\sqrt{\delta}}{4}\big)\bar{J}_{q,\chi}$. Arguing as above, we obtain (11) and therefore (**PI**$_b$).

Proof of* (PI$_b$) *in the case $-1 < c < 0$. Set $s := -c > 0$ and

$$k := k_1 + k_2,$$

where k_1 and k_2 are the constants in Lemmas 2 and 3. Applying Lemmas 2–4 in the RHS of (BE$_-$), we obtain

$$\begin{aligned} &\mu\|w\|_q^q + (1-k\varepsilon_0)I_q - s\bar{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)\bar{J}_{q,\chi}) + s\left(1+(q-2)\frac{d}{q}\right)H_{q,\chi} \\ &\quad - 2s\frac{q-2}{q}H_{q,\chi^2} - sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^3} - s(q-2)\frac{1}{4\theta}G_{q,\chi^2} \\ &\leq s(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac{1}{2}}J_q^{\frac{1}{2}} + s\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q \\ &\quad + \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2, \end{aligned} \tag{12}$$

where $\tilde{C}_1(\varepsilon_0) = C_1(\varepsilon_0)$, $\tilde{C}_2(\varepsilon_0) = C_2(\varepsilon_0) + C(\varepsilon_0)$. Applying $H_{q,\chi} \geq H_{q,\chi^2}$ and $J_q \geq \bar{J}_{q,\chi}$ (recall that this and similar inequalities appearing below is the content of Lemma 1), we have

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q - s\bar{I}_{q,\chi} + (q-2)(1-s(1+\theta))J_q + s\left(1+(q-2)\frac{d-2}{q}\right)H_{q,\chi}$$
$$- sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^3} - s(q-2)\frac{1}{4\theta}G_{q,\chi^2} \leq \text{RHS of (12)}.$$

Since $s\big(1+(q-2)\frac{d-2}{q}\big) > 0$, we can apply $H_{q,\chi} \geq G_{q,\chi}$ to obtain

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q - s\bar{I}_{q,\chi} + (q-2)(1-s(1+\theta))J_q + s\left(1+(q-2)\frac{d-2}{q}\right)G_{q,\chi}$$
$$- sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^3} - s(q-2)\frac{1}{4\theta}G_{q,\chi^2} \leq \text{RHS of (12)},$$

that is,

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q - s\bar{I}_{q,\chi} + (q-2)(1-s(1+\theta))J_q$$
$$+ s\langle M(\chi)|x|^{-4}(x\cdot w)^2|w|^{q-2}\rangle \leq \text{RHS of (12)},$$

where

$$M(\chi) := \left[1+(q-2)\frac{d-2}{q} + \left(-d-4+4\chi-(q-2)\frac{1}{4\theta}\right)\chi\right]\chi.$$

Select $\theta := \frac{1}{2}\frac{q}{d-2}$. Next, we claim that, for $q > d-2$, $d \geq 4$,

$$\min_{0\leq t\leq 1} M(t) = M(1) = -d+1+\frac{1}{2}(q-2)\frac{d-2}{q} < 0.$$

Indeed, write $M(t) = 4tf(t)$, where $f(t) = t^2 - \frac{1}{4}(d+4+\frac{q-2}{4\theta})t + \frac{1}{4} + \frac{q-2}{4q}(d-2)$. Then $f(1) < 0$ and so $f(t) = 0$ has real roots $t_1 < t_2$. It suffices to note that $\frac{t_1+t_2}{2} \geq 1$. Indeed, $t_1 + t_2 = \frac{d+4}{4} + \frac{q-2}{16\theta} \geq 2$ or $d + \frac{q-2}{4\theta} \geq 4$ clearly holds for $d \geq 4$.

Since $0 < \chi < 1$, we obtain

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q - s\bar{I}_{q,\chi} + (q-2)(1-s(1+\theta))J_q$$
$$+ sM(1)G_q \leq \text{RHS of (12)},$$

and so, applying $G_q \geq G_{q,\chi^2}$ in the RHS of (12),

$$\mu\|w\|_q^q + (1-k\varepsilon_0)I_q - s\bar{I}_{q,\chi} + (q-2)(1-s(1+\theta))J_q + sM(1)G_q$$
$$\leq s(d+3)\frac{q\sqrt{\delta}}{2}G_q^{\frac{1}{2}}J_q^{\frac{1}{2}} + s\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q$$
$$+ \tilde{C}_1\|w\|_q^q + \tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.$$

Applying the quadratic inequality twice in the RHS of the last inequality, we obtain ($\theta_2, \theta_3 > 0$),

$$\begin{aligned}&\mu\|w\|_q^q+(1-k\varepsilon_0)I_q-s\bar{I}_{q,\chi}+(q-2)(1-s(1+\theta))J_q+sM(1)G_q\\&\le s(d+3)\frac{q\sqrt{\delta}}{4}(\theta_2J_q+\theta_2^{-1}G_q)+s\frac{q\sqrt{\delta}}{4}(\theta_3\bar{I}_{q,\chi}+\theta_3^{-1}J_q)+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right)J_q\\&\quad+\tilde{C}_1\|w\|_q^q+\tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

Selecting $\theta_2=\frac{q}{d-2}$, $\theta_3=1$, applying inequality $I_q\ge\bar{I}_{q,\chi}$ and regrouping the terms, we obtain

$$\begin{aligned}&\mu\|w\|_q^q+\left[1-k\varepsilon_0-s\left(1+\frac{q\sqrt{\delta}}{4}\right)\right]I_q\\&+\left[(q-2)\left(1-s\left(1+\frac{1}{2}\frac{q}{d-2}\right)\right)-s\frac{q\sqrt{\delta}}{4}-s(d+3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2}\right.\\&\left.-\frac{q^2\delta}{4}-(q-2)\frac{q\sqrt{\delta}}{2}\right]J_q\\&+s\left[M(1)-(d+3)\frac{q\sqrt{\delta}}{4}\frac{d-2}{q}\right]G_q\le\tilde{C}_1\|w\|_q^q+\tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

By the assumptions of the theorem, $1-s\big(1+\frac{q\sqrt{\delta}}{4}\big)>0$, so selecting ε_0 sufficiently small we may assume that the coefficient of I_q above is positive. Now, using inequalities $J_q\le I_q$ and $J_q\ge\frac{(d-2)^2}{q^2}H_q\ge\frac{(d-2)^2}{q^2}G_q$, we arrive at

$$\begin{aligned}&\mu\|w\|_q^q+\left[(q-1-k\varepsilon_0)\frac{(d-2)^2}{q^2}-Ł_2(-s,\delta,q,d)\right]\frac{q^2}{(d-2)^2}J_q\\&\le\tilde{C}_1\|w\|_q^q+\tilde{C}_2\|w\|_q^{q-2}\|f\|_q^2,\end{aligned}$$

where

$$\begin{aligned}Ł_2(-s,\delta,q,d):=\,&s\left[d-1+(q-1)\frac{(d-2)^2}{q^2}+\frac{\sqrt{\delta}}{2}\left(\frac{(d-2)^2}{q}+(d+3)(d-2)\right)\right]\\&+\left[\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right]\frac{(d-2)^2}{q^2}.\end{aligned}$$

By the assumption $d-3-L_2(c,\delta,d)>0$ of the theorem, $\big(q-1\big)\frac{(d-2)^2}{q^2}-Ł_2(-s,\delta,q,d)>0$ for all $q>d-2$ sufficiently close to $d-2$. Thus, selecting ε_0 even smaller, if needed, we obtain (**PI**$_b$) (with $\mu_0:=\tilde{C}_1$, $C:=\tilde{C}_2$).

Step 3. Repeating Step 3 in the proof of Theorem A, we obtain that the principal inequality (**PI**$_b$), the Young inequality and the Sobolev embedding theorem yield the estimates ($\star\star\star$):

$$\begin{aligned}\|\nabla u^{\varepsilon,n}\|_q&\le K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q,\\\|\nabla u^{\varepsilon,n}\|_{\frac{qd}{d-2}}&\le K_2(\mu-\mu_0)^{\frac{1}{q}-\frac{1}{2}}\|f\|_q\end{aligned}$$

with constants $K_1 := C^{\frac{1}{2}}$, $K_2 := C_S\eta^{-\frac{1}{q}}(q^2/4)^{\frac{1}{q}}C^{\frac{1}{q}}K_1^{\frac{q-2}{q}}$ and μ_0 independent of ε, n. Since the weak gradient in L^q, $L^{\frac{qd}{d-2}}$ is closed, Theorem 1 and a weak compactness argument in L^q, $L^{\frac{qd}{d-2}}$ allow us to pass to the limit in the above estimates in $\varepsilon \downarrow 0$ and then in $n \to \infty$, obtaining ($\star$):

$$
\begin{aligned}
\|\nabla u\|_q &\leq K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q,\\
\|\nabla u\|_{\frac{qd}{d-2}} &\leq K_2(\mu-\mu_0)^{\frac{1}{q}-\frac{1}{2}}\|f\|_q.
\end{aligned}
$$

for $u := (\mu+\Lambda_q(a,b))^{-1}f$, $0 \leq f \in C_c^\infty$. Now, a standard density argument allows to conclude that these bounds hold for all $0 \leq f \in L^q$. Finally, we note that these bounds hold for all $f \in L^q$ with K_l above replaced by $4K_l$, $l=1,2$.

The proof of assertion (i) of Theorem 2 is completed.

Proof of assertion (ii). The proof of the basic equalities (BE_-), (BE_-) works for $q=2$ as well. Let us write for brevity $w = \nabla u^{\varepsilon,n}$, where $0 \leq u^{\varepsilon,n} = (\mu+\Lambda_2(a^\varepsilon, b_n))^{-1}f$, $0 \leq f \in C_c^\infty$, $\varepsilon > 0$. We multiply the equation for $u^{\varepsilon,n}$ by the "test function" $\phi = -\nabla \cdot w \equiv -\sum_{i=1}^d \nabla_i w_i$, obtaining

$$
\mu\|w\|_2^2 + \langle A_2^\varepsilon w, w\rangle + \langle [\nabla, A_2^\varepsilon]_- u, w\rangle = \langle -b_n \cdot w, -\nabla\cdot w\rangle + \langle f, -\nabla \cdot w\rangle. \quad (13)
$$

For $c>0$, we evaluate in (13) (arguing as in the proof of (BE_-)):

$$
\langle [\nabla, A_2^\varepsilon]_- u, w\rangle = -\beta_1 - cH_{2,\chi} + 2c(d-1)G_{2,\chi^2} + 8c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2\rangle,
$$

so

$$
\begin{aligned}
\mu\|w\|_2^2 + I_2 + c\bar{I}_{2,\chi} - \beta_1 - cH_{2,\chi} + 2c(d-1)G_{2,\chi^2} + 8c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2\rangle \\
= \langle -b_n\cdot w, -\nabla\cdot w\rangle + \langle f, -\nabla\cdot w\rangle,
\end{aligned}
$$

where $\beta_1 = -2c\langle |x|_\varepsilon^{-4} x\cdot w, x\cdot(x\cdot\nabla w)\rangle$ and, recall,

$$
\begin{aligned}
& I_2 = \sum_{r=1}^d \langle |\nabla_r w|^2\rangle, \quad \bar{I}_{2,\chi} = \langle |x\cdot\nabla w|^2 \chi|x|^{-2}\rangle, \quad x\cdot\nabla w \equiv \sum_{i=1}^d (x_i\nabla_i)w,\\
& H_{2,\chi} := \langle \chi|x|^{-2}|w|^2\rangle, \quad G_{2,\chi^2} := \langle \chi^2|x|^{-4}(x\cdot w)^2\rangle.
\end{aligned}
$$

We estimate β_1 as in Lemma 4 (with $\theta=1$), arriving at

$$
\mu\|w\|_2^2 + I_2 - cH_{2,\chi} + c\big(2(d-1)-1\big)G_{2,\chi^2} \leq \langle -b_n\cdot w, -\nabla\cdot w\rangle + \langle f, -\nabla\cdot w\rangle. \quad (14)
$$

For $c<0$, we evaluate in (13) (arguing as in the proof of (BE_-)):

$$
\langle [\nabla, A_2^\varepsilon]_- u, w\rangle = \frac{1}{2}\beta_2 - cH_{2,\chi} + cdG_{2,\chi^2} + 4c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2\rangle, \quad \beta_2 = 0,
$$

so

$$
\mu\|w\|_2^2 + I_2 + c\bar{I}_{2,\chi} - cH_{2,\chi} + cdG_{2,\chi^2}
$$

$$+ 4c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2\rangle$$
$$= \langle -b_n \cdot w, -\nabla \cdot w\rangle + \langle f, -\nabla \cdot w\rangle,$$

and hence by $I_2 \geq \bar{I}_{2,\chi}$ and $H_{2,\chi} \geq G_{2,\chi}$

$$\mu\|w\|_2^2 + (1-|c|)I_2 + |c|G_{2,\chi} - |c|dG_{2,\chi^2} - 4|c|G_{2,\chi^2} + 4|c|G_{2,\chi^3}$$
$$\leq \langle -b_n \cdot w, -\nabla \cdot w\rangle + \langle f, -\nabla \cdot w\rangle,$$

i.e.,

$$\mu\|w\|_2^2 + (1-|c|)I_2 + |c|\langle M(\chi)|x|^{-4}(x\cdot w)^2\rangle \leq \langle -b_n \cdot w, -\nabla \cdot w\rangle + \langle f, -\nabla \cdot w\rangle,$$

where $M(\chi) = \big(1-(d+4)\chi + 4\chi^2\big)\chi$. Since $\min_{0\leq t\leq 1} M(t) = M(1) = 1-d < 0$, we arrive at

$$\mu\|w\|_2^2 + (1-|c|)I_2 + |c|(1-d)G_2 \leq \langle -b_n \cdot w, -\nabla \cdot w\rangle + \langle f, -\nabla \cdot w\rangle. \quad (15)$$

In the RHS of (14), (15), we estimate $\langle -b_n\cdot w, -\nabla\cdot w\rangle$ using Lemma 2, and $\langle f, -\nabla\cdot w\rangle$ using Lemma 3. All the terms that appear in these estimates are further bounded from above by I_2 using inequalities $I_2\,(\geq J_2 \geq \frac{(d-2)^2}{4}H_2 \geq)\ \frac{(d-2)^2}{4}H_{2,\chi^i}, \frac{(d-2)^2}{4}G_{2,\chi^i}$, $i \geq 0$ (Lemma 1).

We estimate the LHS in (14), (15) repeating the argument in the proof of (i) above.

In the resulting inequalities, taking into account our assumptions on c and δ, we arrive at $I_2(u^{\varepsilon,n}) \leq K\|f\|_2$ with K independent of ε, n. So, by passing to the limit in ε and then in n using Theorem 1, we arrive at $I_2(u) \leq K\|f\|_2$, $u = (\mu + \Lambda_2(a,b))^{-1} f$ $\Rightarrow u \in W^{2,2}$. By the density argument, the latter holds for all $f \in L^2$ (with K replaced by $4K$).

The proof of assertion (ii) is completed.

The proof of Theorem 2 is completed. □

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