J. Evol. Equ. 21 (2021), 2807–2814

1424-3199/21/032807-8, *published online* April 19, 2021
https://doi.org/10.1007/s00028-021-00696-0

Journal of Evolution Equations

A note on the long-time behavior of dissipative solutions to the Euler system

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Dedicated to Matthias Hieber on the occasion of his 60-th birthday

Abstract. We show that the Reynolds defect measure for a dissipative weak solution of the compressible Euler system vanishes for large time. This may be seen as a piece of evidence that the dissipative solutions are asymptotically close to weak solutions in the turbulent regime, whence suitable for describing compressible fluid flows in the long run.

1. Introduction

In [2], we proposed the concept of *dissipative weak (DW) solution* to the compressible (isentropic) Euler system:

$$\begin{aligned} &\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \\ &\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ \ p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1, \end{aligned} \tag{1.1}$$

considered on a bounded domain $\Omega \subset R^d$, $d = 1, 2, 3$, with impermeable boundary

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.2}$$

and the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \ \ \mathbf{m}(0, \cdot) = \mathbf{m}_0. \tag{1.3}$$

A dissipative solution is a trio $[\varrho, \mathbf{m}, E]$, where ϱ, $\mathbf{m}$ satisfy (in the sense of distributions) the augmented system

$$\begin{aligned} &\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \\ &\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}, \end{aligned} \tag{1.4}$$

Keywords: Isentropic Euler system, Dissipative solutions, Long-time behavior.
The research of E.F. leading to these results has received funding from the Czech Sciences Foundation (GAČR), Grant Agreement 18-05974S. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

with the "turbulent" total energy $E = E(t)$—a non-increasing function of t—satisfying

$$
\begin{aligned}
E(\tau\pm) &\leq E_0 = \int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x, \\
E(\tau\pm) &\geq \int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x \quad \text{for any } \tau > 0, \\
P(\varrho) &\equiv \frac{a}{\gamma - 1} \varrho^{\gamma}.
\end{aligned}
\tag{1.5}
$$

Note that the total energy is defined as a convex l.s.c. function of $[\varrho, \mathbf{m}] \in R^{d+1}$,

$$
\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) = \begin{cases} \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) & \text{if } \varrho > 0, \\ 0 \quad \text{if } \varrho = 0, \ \mathbf{m} = 0, \\ \infty \quad \text{otherwise.} \end{cases}
$$

The quantity $\mathfrak{R}$ is a matrix-valued measure, specifically,

$$
\mathfrak{R} \in L^{\infty}([0, \infty); \mathcal{M}^{+}(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}})) \tag{1.6}
$$

called *Reynolds defect*. Here, the symbol $\mathcal{M}^{+}(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}})$ denotes the cone of positively semi-definite symmetric matrix-valued measures on $\overline{\Omega}$, specifically,

$$
\langle \mathfrak{R} : [\xi \otimes \xi]; g \rangle \geq 0 \quad \text{for any } g \in C(\overline{\Omega}), \ g \geq 0, \xi \in R^d.
$$

The crucial property of (DW) solutions is the *compatibility condition*

$$
\begin{aligned}
E(\tau+) &- \int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x \\
&\geq \underline{d} \int_{\overline{\Omega}} \mathrm{d} \ (\mathrm{trace}[\mathfrak{R}])(\tau) \quad \text{for any } \tau \in [0, \infty)
\end{aligned}
\tag{1.7}
$$

for a certain constant $\underline{d} > 0$. A detailed definition is given in Sect. 2 below.

Relation (1.7) can be interpreted in the way that the energy defect dominates the Reynolds defect. As shown in [2], the (DW) solutions exist globally in time for any finite energy initial data. Moreover, they can be identified as limits of consistent approximations arising in numerical analysis (see [11,12]) or as vanishing viscosity limits of solutions to the Navier–Stokes system (see [10]). Note that, despite the large number of ill-posedness results (see, e.g., Chiodaroli et al. [3–6]), the standard (admissible) weak solutions that correspond to the case $\mathfrak{R} = 0$ are not known to exist globally in time for arbitrary initial data. (DW) solutions share many important properties with the standard (admissible) weak solutions:

- **Compatibility.** Any (DW) solution, for which $[\varrho, \mathbf{m}]$ are continuously differentiable functions, is a classical solution. In particular, $\mathfrak{R} = 0$.
- **Weak–strong uniqueness.** A (DW) solution coincides with the strong solution starting from the same initial data as long as the latter exists.

Moreover, we have shown in [2] that the class of all (DW) solutions admits a semiflow selection. In particular, the selected solutions are minimal with respect to the relation "$\prec$":

$$[\varrho_1, \mathbf{m}_1, E_1] \prec [\varrho_2, \mathbf{m}_2, E_2] \Leftrightarrow E_1(\tau\pm) \leq E_2(\tau\pm) \quad \text{for any } \tau > 0.$$

The minimal solutions dissipate the maximal amount of the total energy, which is in agreement with the commonly accepted *maximal dissipation principle*, see, e.g., Dafermos [7–9].

In this note, we show another interesting property of minimal (DW) solutions, namely

$$E(\tau) - \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x \to 0 \text{ as } \tau \to \infty. \tag{1.8}$$

In view of (1.6), (1.7), the Reynolds defect $\mathfrak{R}$ vanishes in the asymptotic limit for large times. This fact may be seen as another piece of evidence supporting physical relevance of (minimal) (DW) solutions.

The paper is organized as follows. In Sect. 2, we introduce the necessary preliminary material and state the main result. In Sect. 3, we prove (1.8).

2. Preliminaries and main result

We recall the concept of dissipative weak solution introduced in [2, Definition 2.1].

Definition 2.1. *(Dissipative weak (DW) solution)* Let $\Omega \subset R^d$, $d = 1, 2, 3$ be a bounded domain. We say that $[\varrho, \mathbf{m}, E]$ is a *dissipative weak (DW) solution* of the Euler system (1.1)–(1.4) in $[0, \infty) \times \Omega$ if the following holds:

- $\varrho \geq 0$, and

$$\varrho \in C_{\text{weak,loc}}([0, \infty); L^\gamma(\Omega)), \ \mathbf{m} \in C_{\text{weak,loc}}([0, \infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)),$$
$$E \in BV[0, \infty), \ E \geq 0;$$

- $$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \mathrm{d}x \, \mathrm{d}t$$

  for any $\tau \geq 0$, $\varphi \in C_c^1([0, \infty) \times \overline{\Omega})$;

- $$\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_0^\tau \left(\int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t) \right) \mathrm{d}t$$

for any $\tau \geq 0$, $\varphi \in C^1_c([0,\infty) \times \overline{\Omega}; R^d)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$, where

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d\times d}_{\rm sym}))$$

is called *Reynolds defect*;

- $E : [0,\infty) \to [0,\infty)$ is a non-decreasing function,

$$\begin{aligned} E(0-) &\equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x, \\ E(\tau+) &- \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x \\ &\geq \underline{d} \int_{\overline{\Omega}} \mathrm{d}\ (\mathrm{trace}[\mathfrak{R}])(\tau) \quad \text{for a certain constant } \underline{d} > 0 \end{aligned} \tag{2.1}$$

for any $\tau \geq 0$.

As a matter of fact, the (DW) solutions introduced in [2] are defined as a barycenter of a Young measure $\{\nu_{t,x}\}_{t>0,x\in\Omega}$, specifically

$$\varrho(t,x) = \left\langle \nu_{t,x}; \tilde{\varrho} \right\rangle, \ \mathbf{m}(t,x) = \left\langle \nu_{t,x}; \tilde{\mathbf{m}} \right\rangle,$$

with the associated total energy

$$E = \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2} \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) \right\rangle \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E},$$

where $\mathfrak{E}$ is the so-called energy concentration defect. As observed in [10], the two definitions are equivalent.

Following [2], we introduce the relation $\prec$ for two (DW) solutions $[\varrho_1, \mathbf{m}_1, E_1]$ and $[\varrho_2, \mathbf{m}_2, E_2]$ starting from the same initial data $[\varrho_0, \mathbf{m}_0]$,

$$[\varrho_1, \mathbf{m}_1, E_1] \prec [\varrho_2, \mathbf{m}_2, E_2] \ \Leftrightarrow E_1(\tau\pm) \leq E_2(\tau\pm) \text{ for all } \tau > 0.$$

Finally, we introduce the admissible (DW) solution, cf. [2, Definition 2.3].

Definition 2.2. *(Admissible (DW) solutions)* A dissipative weak solution $[\varrho, \mathbf{m}, E]$ is called *admissible* if it is minimal with respect to the relation $\prec$. Specifically, if $[\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{E}]$ is another dissipative solution starting from the same initial data and such that

$$[\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{E}] \prec [\varrho, \mathbf{m}, E],$$

then

$$E(\tau\pm) = \widetilde{E}(\tau\pm) \quad \text{for any } \tau > 0.$$

We are ready to state our main result.

Theorem 2.3. *Let $\Omega \subset R^d$ be a bounded Lipschitz domain. Let $[\varrho, \mathbf{m}, E]$ be an admissible (DW) solution of the isentropic Euler system in the sense of Definition 2.2.*

Then,

$$\lim_{\tau\to\infty} E(\tau) = \lim_{\tau\to\infty} \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x, \tag{2.2}$$

in particular,

$$\operatorname*{ess\,sup}_{t>\tau} \|\mathfrak{R}(t)\|_{\mathcal{M}(\overline{\Omega};\, R^{d\times d}_{\mathrm{sym}})} \to 0 \textit{ as } \tau \to \infty.$$

The rest of the paper is devoted to the proof of Theorem 2.3.

3. Asymptotic behavior: proof of Theorem 2.3

The analysis leans on the following two results proved in [2].

Proposition 3.1. ([2, Proposition 3.2])
Let $T \geq 0$ and the initial data ϱ_T, $\mathbf{m}_T$,

$$\varrho_T \geq 0,\ E_T = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_T|^2}{\varrho_T} + P(\varrho_T)\right]\ \mathrm{d}x < \infty,$$

be given.

Then, the Euler system admits a global in time dissipative solution $[\varrho, \mathbf{m}, E]$ in $[T;\infty)$ in the sense of Definition 2.1, specifically,

$$\varrho \in C_{\mathrm{weak,loc}}([T,\infty); L^\gamma(\Omega)),\ \mathbf{m} \in C_{\mathrm{weak,loc}}([T,\infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)),$$
$$E \in BV[T,\infty) \textit{ non-increasing},$$

such that

$$0 \leq E(\tau\pm) \leq E_T,$$
$$E(\tau+) - \int_\Omega \frac{1}{2}\frac{|\mathbf{m}(\tau,\cdot)|^2}{\varrho(\tau,\cdot)} P(\varrho(\tau,\cdot))\ \mathrm{d}x \quad \geq \min\left\{\frac{1}{2}, \frac{1}{\gamma-1}\right\} \int_{\overline{\Omega}} \mathrm{d}\,(\mathrm{trace}[\mathfrak{R}])$$

for all $\tau > T$.

Proposition 3.2. ([2, Theorem 2.5]) *Given the initial data ϱ_0, $\mathbf{m}_0$,*

$$\varrho_0 \geq 0,\ E_0 = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0)\right]\ \mathrm{d}x < \infty,$$

the Euler system admits a global in time admissible *(DW) solution*

$$\varrho \in C_{\mathrm{weak,loc}}([0,\infty); L^\gamma(\mathbb{T}^d)),\ \mathbf{m} \in C_{\mathrm{weak,loc}}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^d)),\ E \in BV[0,\infty)$$

in the sense of Definition 2.2.

We are ready to prove Theorem 2.3. Let $[\varrho, \mathbf{m}, E]$ be an admissible (DW) solution of the Euler system in $[0, \infty) \times \Omega$, the existence of which is guaranteed by Proposition 3.2. As E is a non-increasing function, it admits a limit

$$E_\infty = \lim_{\tau \to \infty} E(\tau) \geq 0.$$

Moreover, in view of (2.1),

$$E_\infty \geq \limsup_{\tau \to \infty} \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x. \tag{3.1}$$

Next, we claim the following result.

Lemma 3.3. *Let $T > 0$ be arbitrary and denote*

$$E_T = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (T, \cdot) \ \mathrm{d}x.$$

Then,

$$E_\infty \leq E_T.$$

Proof. Supposing the contrary, meaning

$$E_T > E_\infty, \tag{3.2}$$

we may use Proposition 3.1 to construct a solution $\tilde{\varrho}, \tilde{\mathbf{m}}$ defined on the interval $[T, \infty)$, with the initial data

$$\tilde{\varrho}(T, \cdot) = \varrho(T, \cdot), \ \ \tilde{\mathbf{m}}(T, \cdot) = \mathbf{m}(T, \cdot),$$

with the non-decreasing total energy $\widetilde{E}$ such that

$$\widetilde{E}(\tau\pm) \leq E_T \text{ for all } \tau \in (T, \infty).$$

Finally, set

$$\widehat{\varrho} = \begin{cases} \varrho & \text{for } t \in [0, T), \\ \tilde{\varrho} & \text{for } t \in [T, \infty), \end{cases} \qquad \widehat{\mathbf{m}} = \begin{cases} \mathbf{m} \text{ for } t \in [0, T), \\ \tilde{\mathbf{m}} \text{ for } t \in [T, \infty), \end{cases}$$

and

$$\widehat{E}(t) = \begin{cases} E & \text{for } t \in [0, T), \\ E(T-)(\geq E_T \geq)\widetilde{E}(T+), \ t = T, \\ \widetilde{E} & \text{for } t \in (T, \infty). \end{cases}$$

Obviously, $[\widehat{\varrho}, \widehat{\mathbf{m}}]$ with the energy $\widehat{E}$ is a dissipative solutions (cf. [2, Proposition 5.1 - continuation property]), and, in view of (3.2),

$$[\widehat{\varrho}, \widehat{\mathbf{m}}, \widehat{E}] \prec [\varrho, \mathbf{m}, E] \text{ and } \lim_{\tau \to \infty} \widehat{E}(\tau) \leq E_T < E_\infty$$

in contrast with maximality of $[\varrho, \mathbf{m}, E]$. □

In view of Lemma 3.3, any maximal (DW) solution satisfies

$$E_\infty = \lim_{\tau\to\infty} E(\tau) \leq \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](T,\cdot)\ \mathrm{d}x$$

for any $T > 0$, in particular,

$$E_\infty \leq \liminf_{\tau\to\infty} \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x,$$

which, together with (3.1), yields (2.2). We have proved Theorem 2.3.

4. Conclusion

We have shown that the "turbulent" energy E and the "intrinsic" energy

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right]\ \mathrm{d}x$$

of any admissible (DW) solution $[\varrho, \mathbf{m}]$ of the compressible Euler system coincide in the asymptotic limit as $\tau \to \infty$, in particular, the limit

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x \to E_\infty \quad \text{as } \tau \to \infty$$

exists. Accordingly, the Reynolds defect measure $\mathfrak{R}$ in the momentum equation (1.4) vanishes for $\tau \to \infty$, and the (DW) solutions behave asymptotically as the standard weak solutions. As turbulent phenomena are usually attributed to the properties of the system in the long run, this may be seen as a positive argument concerning physical relevance of the (DW) solutions. We expect similar properties to hold for the (DW) solutions of the complete Euler system introduced in [1].

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Accepted: 22 March 2021