J. Evol. Equ. 21 (2021), 2449–2470

1424-3199/21/022449-22, *published online* April 8, 2021
https://doi.org/10.1007/s00028-021-00689-z

Journal of Evolution Equations

Large time behavior of solutions to a Stokes-Magneto equations in three dimensions

Yingdan Ji and Wen Tan

Abstract. This paper is devoted to the large time decay of solutions of a three-dimensional Stokes-Magneto equations. It is shown that, when initial data belong to L^2, weak solutions of the equations decay to zero in $L^{3/2,\infty} \times L^2$ without a uniform rate, and this decay estimate is optimal. Furthermore, the optimal temporal decay estimates for weak solutions are established when initial data belongs to $L^1 \cap L^2$.

1. Introduction

In this paper, we study the following equations

$$-\nu\Delta u + \nabla p_* = b \cdot \nabla b \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^3, \tag{1.1}$$

$$\partial_t b + u \cdot \nabla b - \eta\Delta b = b \cdot \nabla u \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^3, \tag{1.2}$$

$$\nabla \cdot u = 0, \quad \nabla \cdot b = 0, \tag{1.3}$$

$$b|_{t=0} = b_0. \tag{1.4}$$

Here u is the velocity field, b is the magnetic field, $p_* = p + \frac{1}{2}|b|^2$ is the total pressure, p is the pressure, $\nu > 0$ is the viscosity coefficient and $\eta > 0$ is the magnetic resistivity coefficient.

Equations (1.1)–(1.3) is obtained by removing the advective terms $(\partial_t + u \cdot \nabla)u$ from the u equation of the magnetohydrodynamics (MHD) equations. It is well-known that MHD equations, which was first derived by Alfvén, govern the motion of the electrically conducting fluids arising from plasmas, liquid metals, and electrolytes, etc (see [12]). It is also known that MHD equations are one of the most important equations in the study of phenomena arising from geophysics, astrophysics, cosmology and engineering (see, e.g., [2,5]).

Equation (1.1)–(1.3) is closely connected with the method of *magnetic relaxation* (see [14]). When $\eta = 0$, Moffatt [16] argued that (1.1)–(1.3) on a smooth bounded domain Ω should produce a magnetostatic equilibrium $b^E(x)$ that satisfies

Mathematics Subject Classification: 35B40, 35M11, 35Q35, 76D07, 76W05
Keywords: 3D Stokes-Magneto equations, Non-uniform decay estimate, Optimal rate of temporal decay.

$$j^E \times b^E = \nabla p^E, \quad j^E = \nabla \times b^E, \quad \nabla \cdot b^E = 0 \text{ in } \Omega, \qquad b^E \cdot n = 0 \text{ on } \partial\Omega, \tag{1.5}$$

if the topology of the magnetic field is non-trivial. Note that (1.5) almost shares the same form with the steady Euler equation:

$$u^E \times \omega^E = \nabla h^E, \quad \omega^E = \nabla \times u^E, \quad \nabla \cdot u^E = 0 \text{ in } \Omega, \qquad u^E \cdot n = 0 \text{ on } \partial\Omega, \tag{1.6}$$

if one "identifies" b^E with velocity field u^E, here $\nabla h^E = \nabla(p_e + \frac{1}{2}|u^E|^2)$ and p_e denotes the pressure of the Euler equation. This indicates that the study of (1.1)–(1.3) might be helpful to understand the unstable Euler flows. Moffatt also argued that the steady state of some non-resistive MHD equations should also obey (1.5) (see [15]). However, there is no rigorous proof that the magnetic relaxation will yield a steady Euler flow. One of the reasons is that the global well-posedness of 3D MHD equations remains open (see [14] and references therein).

From a limiting state point of view, the dynamical model used to obtain the above steady state is not particularly important (see [8,14,16]). In fact, it was argued by Moffatt that dropping the acceleration terms from the u equation and working with a "Stokes" model might prove more mathematically amenable (see [8,14,16]). In recent years, the well-posedness of (1.1)–(1.4) and related models have attracted great attention. McCormick et al. proved the existence of weak solutions of the equations in [14], where the uniqueness of weak solutions for two-dimensional case is also shown. Furthermore, they proved that weak solutions of the 2D equations become regular if b_0 is smooth (see [14]). We refer readers to [4,8] for the local-in-time existence of regular solutions of 3D non-resistive MHD equations. Recently, we established an optimal regularity criterion for (1.1)–(1.4), and studied the global-in-time existence of strong solutions when initial data is small in critical Sobolev spaces or critical Besov spaces (see [22]). We also established global-in-time existence of strong solutions of the equations with arbitrary initial data when $-\Delta$ in (1.2) is replaced by $(-\Delta)^\alpha$ with $\alpha \geq 3/2$ (see [10]).

The purpose of this paper is to investigate the decay of weak solutions of (1.1)–(1.4). The analysis of decay of solutions of fluid flow motions originally goes back to Leray [13], in which he asked whether or not weak solutions of 3D Navier–Stokes equations decay to zero in L^2 as time tends to infinity. Since then, this kind of problem has been extensively studied, see [3,17–20] for Navier–Stokes equations and [1,6,7,21] for MHD equations. In this paper, motivated by the work of [1,20], we show that the $L^{3/2,\infty}$ norm of velocity u and L^2 norm of magnetic field b are decay to zero without a uniform rate when initial data belong to L^2. It stated as follows:

Theorem 1.1. *Let $b_0 \in L^2$ with $\nabla \cdot b_0 = 0$. Assume that (u, b) a weak solution of the initial value problem (1.1)–(1.4). Then (u, b) satisfies*

$$\lim_{t\to\infty} (\|u(t)\|_{L^{3/2,\infty}} + \|b(t)\|_{L^2}) = 0. \tag{1.7}$$

For the proof, it should be pointed out that in contrast with Navier–Stokes equations or MHD, there is no a priori bound for u in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3))$ because of the absence of $\partial_t u$ in (1.1). Note that this estimate for u is crucial to study the corresponding part for Navier–Stokes or MHD (see [1,17,20]). However, by the energy estimate of solutions, we would overcome this difficulty by some proper interpolation inequalities, see Sect. 3 below.

Furthermore, we show that the decay result obtained in Theorem 1.1 is optimal in the sense that for any sphere with radius α in $L^2(\mathbb{R}^3)$, there exists a b_0 on the sphere such that the corresponding solutions should decay arbitrarily slow. The result is stated as follows:

Theorem 1.2. *For any $T > 0$, $\alpha > 0$ and $0 < \epsilon < 1$, there exists $b_0 \in L^2$ with $\nabla \cdot b_0 = 0$ and $\|b_0\|_{L^2} = \alpha$, such that if (u, b) is a weak solution of* (1.1)–(1.4) *corresponding to the initial data b_0, then $\frac{\|b(T)\|_{L^2}}{\|b_0\|_{L^2}} \geq 1 - \epsilon$.*

Motivated by [1,20], we prove this result by choosing a suitable scaling transform δ_λ on L^2 that preserves L^2-norm. Since this scaling does not preserve the semi-norm in $\dot{H}^1$, we assume without loss of generality that b_0 belongs to a more regular space, say H^1. Taking $\delta_\lambda b_0$ as the initial data, we establish a global-in-time bound of solutions of (1.1)–(1.4) when λ is small, then we show Theorem 1.2, see Sect. 4.

The non-uniform decay of weak solutions derived in Theorem 1.1 can be improved if initial data satisfies some additional assumptions. More precisely, it is shown that when b_0 belongs to $L^1 \cap L^2$, the L^2 norm of $b(t)$ will decay like $O(t^{-3/4})$ as $t \to \infty$. This indicates that, on the one hand, the temporal decay of $\|b(t)\|_{L^2}$ can be uniformly dominated by $t^{-3/4}$ (in the sense of '$\lesssim$'). On the other hand, this decay rate is optimal in the sense that the lower bound for rate of decay is proportional to $t^{-3/4}$. The two results are stated as follows:

Theorem 1.3. *Let $b_0 \in L^1 \cap L^2$ with $\nabla \cdot b_0 = 0$. Then there exists a Leray-Hopf weak solution of the initial value problem* (1.1)–(1.4)*, which satisfies*

$$\|u(t)\|_{L^{3/2,\infty}} \leq c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{2}}, \qquad \|b(t)\|_{L^2} \leq c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{4}}. \tag{1.8}$$

Theorem 1.4. *Let $R_\beta = \{f \in L^1 : \inf_{|\xi|\leq\beta} |\hat{f}(\xi)| \geq \beta\}$ for $\beta > 0$ and let $b_0 \in L^1 \cap L^2 \cap R_\beta$ with $\nabla \cdot b_0 = 0$. Then there exists a Leray-Hopf weak solution of* (1.1)–(1.4) *such that*

$$\|b(t)\|_{L^2} \geq c(\nu, \eta, \|b_0\|_{L^1\cap L^2}, \beta)(1+t)^{-\frac{3}{4}}. \tag{1.9}$$

Remark 1.5. The space R_β plays a crucial role in the proof of the optimal decay rate of solutions. When $b_0 \in L^2 \cap R_\beta$, the weak solution of the heat equation

$$\begin{cases} \partial_t b' - \eta\Delta b' = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ b'(0) = b_0, \end{cases} \tag{1.10}$$

decays at most as $\|b'(t)\|_{L^2} \geq c(\eta, \beta)(1+t)^{-3/4}$ (see Lemma 2.5 below). Compared with the linear equation (1.10), the system (1.1)–(1.4) contains complicated nonlinearities, but they do not make the decay of solutions worse (see Sect. 6). Thus, if $b_0 \in L^2 \cap R_\beta$, the component b of the weak solutions (u, b) of the system in general cannot decay faster than $(1+t)^{-3/4}$.

Remark 1.6. The space R_β is strictly contained in L^1. This means that there are functions that contained in L^1 but they do not belong to R_β. In fact, let $\chi : \mathbb{R}^3 \to [0, 1]$ be a smooth function that satisfies $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Let $\phi(x) = \chi(x) - \chi(2x)$. Then both χ and ϕ are Schwartz functions on $\mathbb{R}^3$. It is clear that $\mathcal{F}^{-1}\chi \in R_\beta$, and $\mathcal{F}^{-1}\phi \in L^1$ whereas $\mathcal{F}^{-1}\phi \notin R_\beta$, here $\mathcal{F}^{-1}$ denotes the Fourier inverse transform.

The proof of Theorems 1.3 and 1.4 are based on the Fourier splitting method [19]. The main task is to estimate $\hat{b}$ for its lower frequency part. We point out that there is a difficulty similar as that has been stated above. That is, the absence of $\partial_t u$ in (1.1) leads to the absence of a priori bound of u in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3))$, and this is much unlike the Navier–Stokes equations [17,19,20] or MHD equations [1,21]. However, we overcome the difficulty by energy estimate of solutions and applications of interpolation inequalities.

2. Preliminaries

Throughout the paper, c represents a positive constant (depending only on ν, η) whose value may change at each occurrence. $A \lesssim B$ denotes the inequality $A \leq cB$. $c(\alpha_1, \alpha_2, \ldots)$ stands for a positive constant that depends on $\alpha_1, \alpha_2, \ldots$etc. We denote by $\hat{f}$ the Fourier transform of f, while the inverse Fourier transform of f is denoted by $\mathcal{F}^{-1}f$. We consider function spaces on $\mathbb{R}^3$, for instance, $C_c^\infty := C_c^\infty(\mathbb{R}^3)$, $L^p := L^p(\mathbb{R}^3)$, $H^s := H^s(\mathbb{R}^3)$. $L^{p,\infty}$ denotes the weak L^p space. We will use $\int := \int_{\mathbb{R}^3}$, $\|\cdot\|_p := \|\cdot\|_{L^p}$ and $\|\cdot\| := \|\cdot\|_2$ for convenient. We define $\mathcal{D}_\sigma = \{\mathbf{f} \in C_c^\infty : \nabla\cdot\mathbf{f} = 0\}$. Let L^2_σ and H^1_σ be the closure of $\mathcal{D}_\sigma$ in the L^2 and H^1 norm, respectively.

Definition 2.1. [14] Let $T > 0$ and let $b_0 \in L^2_\sigma$. A function (u, b) is called a weak solution of the equation (1.1)–(1.4) on $(0, T)$, if

(i) $u \in L^\infty(0, T; L^{3/2,\infty}) \cap L^2(0, T; H^1_\sigma)$ and $b \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1_\sigma)$,
(ii) (u, b) verifies:

$$\int \nu\nabla u : \nabla\phi_1 + (b\cdot\nabla)\phi_1 \cdot b\mathrm{d}x = 0,$$

$$\int b_0\cdot\phi_2(0)\mathrm{d}x - \int_0^T\int b\cdot\partial_t\phi_2 - \eta\nabla b : \nabla\phi_2 + (u\cdot\nabla)\phi_2\cdot b - (b\cdot\nabla)\phi_2\cdot u\mathrm{d}x\mathrm{d}t = 0,$$

for all test functions $\phi_1, \phi_2 \in C_c^\infty([0, T); \mathcal{D}_\sigma)$.

Motivated by Ogawa, Rajopadhye and Schonbek [17] about the decay of weak solutions of forced Navier–Stokes equations or Agapito and Schonbek [1] about the analysis of decay of MHD equations, we formulate the following technical lemma:

Lemma 2.2. *Let $b_0 \in L^2_\sigma$. Assume that (u, b) is a weak solution of (1.1)–(1.4). Then for $E(t) \in C^1(\mathbb{R}; \mathbb{R}_+)$ with $E(t) \geq 0$ and $\psi \in C^1(\mathbb{R}; C^1 \cap L^2)$ such that $\psi(t)$ is radial on $\mathbb{R}^3$, the solution (u, b) satisfies the following equations:*

$$0 = -2\nu \int_s^t E(\tau)\|\nabla\psi * u(\tau)\|^2 \mathrm{d}\tau + 2\int_s^t E(\tau)\langle b\cdot\nabla b(\tau), \psi*\psi*u(\tau)\rangle \mathrm{d}\tau, \quad (2.1)$$

and

$$\begin{aligned} E(t)\|\psi * b(t)\|^2 = E(s)\|\psi * b(s)\|^2 + \int_s^t E'(\tau)\|\psi * b(\tau)\|^2 \mathrm{d}\tau \\ + 2\int_s^t E(\tau)(\langle\psi' * b(\tau), \psi * b(\tau)\rangle - \eta\|\nabla\psi * b(\tau)\|^2)\mathrm{d}\tau \\ - 2\int_s^t E(\tau)(\langle u\cdot\nabla b(\tau), \psi*\psi*b(\tau)\rangle \\ - \langle b\cdot\nabla u(\tau), \psi*\psi*b(\tau)\rangle)\mathrm{d}\tau, \end{aligned} \quad (2.2)$$

for all $0 \leq s \leq t \leq \infty$.

Proof. We first give the proof of (2.2). Taking the inner product of (1.2) with $2E(t)\psi * \psi * b(t)$, then integrating in $[s, t] \times \mathbb{R}^3$, we obtain that

$$\begin{aligned} &2\int_s^t \int \partial_\tau b(\tau)\cdot E(\tau)\psi*\psi*b(\tau)\mathrm{d}x\mathrm{d}\tau \\ &\quad -2\eta\int_s^t\int \Delta b(\tau)\cdot E(\tau)\psi*\psi*b(\tau)\mathrm{d}x\mathrm{d}\tau \\ &= -2\int_s^t E(\tau)\int (u\cdot\nabla b(\tau) - b\cdot\nabla u(\tau))\cdot\psi*\psi*b(\tau)\mathrm{d}x\mathrm{d}\tau. \end{aligned} \quad (2.3)$$

The main task is to deal with the terms on the left-hand side of (2.3). For the first term, by integration by parts, we have

$$\begin{aligned} &\int \partial_\tau b(\tau)\cdot E(\tau)\psi*\psi*b(\tau)\mathrm{d}x \\ &= \int \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(b(\tau)\cdot E(\tau)\psi*\psi*b(\tau)\Big) - b(\tau)\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(E(\tau)\psi*\psi*b(\tau)\Big)\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\int b(\tau)\cdot E(\tau)\psi*\psi*b(\tau)\mathrm{d}x \\ &\quad - \int b(\tau)\cdot\Big(E'(\tau)\psi*\psi*b(\tau) + 2E(\tau)\psi'*\psi*b(\tau) \\ &\quad + E(\tau)\psi*\psi*\partial_\tau b(\tau)\Big)\mathrm{d}x, \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\int b(\tau)\cdot E(\tau)\psi * \psi * b(\tau)\mathrm{d}x \\
&\quad - \int b(\tau)\cdot \Big(E'(\tau)\psi * \psi * b(\tau) + 2E(\tau)\psi' * \psi * b(\tau)\Big)\mathrm{d}x \\
&\quad - \int \partial_\tau b(\tau)\cdot E(\tau)\psi * \psi * b(\tau)\mathrm{d}x,
\end{aligned} \tag{2.4}
$$

here ψ is radial has been used to derive the third equation. Thus, we apply Parseval's relation to obtain that

$$
\begin{aligned}
&2\int_s^t \int \partial_\tau b(\tau)\cdot E(\tau)\psi * \psi * b(\tau)\mathrm{d}x\mathrm{d}\tau \\
&\quad = E(t)\int b(t)\cdot \psi * \psi * b(t)\mathrm{d}x - E(s)\int b(s)\cdot \psi * \psi * b(s)\mathrm{d}x \\
&\quad\quad - \int_s^t \int b(\tau)\cdot (E'(\tau)\psi * \psi * b(\tau) + 2E(\tau)\psi' * \psi * b(\tau))\mathrm{d}x\mathrm{d}\tau \\
&\quad = E(t)\int \hat{b}(t)\cdot (\bar{\hat{\psi}})^2\bar{\hat{b}}(t)\mathrm{d}\xi - E(s)\int \hat{b}(s)\cdot (\bar{\hat{\psi}})^2\bar{\hat{b}}(s)\mathrm{d}\xi \\
&\quad\quad - \int_s^t \Big(E'(\tau)\int \hat{b}(\tau)\cdot (\bar{\hat{\psi}})^2\bar{\hat{b}}(\tau)\mathrm{d}\xi + 2E(\tau)\int \hat{b}(\tau)\cdot \bar{\hat{\psi}}'\bar{\hat{\psi}}\bar{\hat{b}}(\tau)\mathrm{d}\xi\Big)\mathrm{d}\tau.
\end{aligned} \tag{2.5}
$$

Since ψ is radial, it follows that $\bar{\hat{\psi}} = \hat{\psi}$ and $(\bar{\hat{\psi}})^2 = |\hat{\psi}|^2$. Hence the previous equation turns to the following:

$$
\begin{aligned}
&2\int_s^t \int \partial_\tau b(\tau)\cdot E(\tau)\psi * \psi * b(\tau)\mathrm{d}x\mathrm{d}\tau \\
&\quad = E(t)\int |\hat{\psi}|^2|\hat{b}(t)|^2\mathrm{d}\xi - E(s)\int |\hat{\psi}|^2|\hat{b}(s)|^2\mathrm{d}\xi \\
&\quad\quad - \int_s^t \Big(E'(\tau)\int |\hat{\psi}|^2|\hat{b}(\tau)|^2\mathrm{d}\xi + 2E(\tau)\int \hat{\psi}\hat{b}(\tau)\cdot \bar{\hat{\psi}}'\bar{\hat{b}}(\tau)\mathrm{d}\xi\Big)\mathrm{d}\tau \\
&\quad = E(t)\|\psi * b(t)\|^2 - E(s)\|\psi * b(s)\|^2 - \int_s^t E'(\tau)\|\psi * b(\tau)\|^2 \\
&\quad\quad + 2E(\tau)\langle \psi' * b(\tau), \psi * b(\tau)\rangle\mathrm{d}\tau,
\end{aligned} \tag{2.6}
$$

here Plancherel's theorem has been used to deduce the second equation.

For the second term, a similar computation gives that

$$
\begin{aligned}
-\int_s^t \int \Delta b(\tau)\cdot E(\tau)\psi * \psi * b(\tau)\mathrm{d}x\mathrm{d}\tau &= \int_s^t E(\tau)\int |\xi|^2\hat{b}(\tau)\cdot (\bar{\hat{\psi}})^2\bar{\hat{b}}(\tau)\mathrm{d}\xi\mathrm{d}\tau \\
&= \int_s^t E(\tau)\int |\xi|^2|\hat{\psi}|^2|\hat{b}(\tau)|^2\mathrm{d}\xi\mathrm{d}\tau \\
&= \int_s^t E(\tau)\|\nabla\psi * b(\tau)\|^2\mathrm{d}\tau.
\end{aligned} \tag{2.7}
$$

Substituting (2.6) and (2.7) into (2.3), we conclude that (2.2) holds.

The proof of (2.1) is slightly simpler and can be shown in a way similar to that of (2.2). Taking the inner product of (1.1) with $2E(t)\psi * \psi * u(t)$, then integrating in $[s,t] \times \mathbb{R}^3$, we obtain that

$$-2\nu \int_s^t \int \Delta u(\tau)\cdot E(\tau)\psi*\psi*u(\tau)\mathrm{d}x\mathrm{d}\tau = 2\int_s^t E(\tau)\int b\cdot\nabla b(\tau)\cdot\psi*\psi*u(\tau)\mathrm{d}x\mathrm{d}\tau. \tag{2.8}$$

By repeating the manipulation of the derivation of (2.7), we know that

$$-\int_s^t \int \Delta u(\tau)\cdot E(\tau)\psi * \psi * u(\tau)\mathrm{d}x\mathrm{d}\tau = \int_s^t E(\tau)\|\nabla\psi * u(\tau)\|^2\mathrm{d}\tau. \tag{2.9}$$

Substituting (2.9) into (2.8), it follows that (2.1) holds. The proof of Lemma 2.2 is completed. □

The following result is a straightforward of Lemma 2.2.

Corollary 2.3. *Let* $b_0 \in L^2_\sigma$. *Assume that* (u, b) *is a weak solution of* (1.1)–(1.4). *Then for a radial function* $\varphi \in L^2$, (u, b) *satisfies*

$$\begin{aligned}\|\mathcal{F}^{-1}\varphi * b(t)\|^2 &\le \|e^{\eta(t-s)\Delta}\mathcal{F}^{-1}\varphi * b(s)\|^2\\ &+2\int_s^t |\langle u\cdot\nabla b(\tau), e^{2\eta(t-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\rangle|\\ &+|\langle b\cdot\nabla u(\tau), e^{2\eta(t-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\rangle|\mathrm{d}\tau,\end{aligned} \tag{2.10}$$

for all $0 \le s \le t \le \infty$.

Proof. For any $\epsilon > 0$, set $\psi(\tau) = \mathcal{F}^{-1}(e^{-\eta|\xi|^2(t+\epsilon-\tau)}\varphi(\xi))$ and $E(t) = 1$ in (2.2), we deduce that

$$\begin{aligned}\|e^{\epsilon\eta\Delta}\mathcal{F}^{-1}\varphi * b(t)\|^2 &= \|e^{\eta(t+\epsilon-s)\Delta}\mathcal{F}^{-1}\varphi * b(s)\|^2\\ &+2\int_s^t (\langle\eta(-\Delta)e^{\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi * b(\tau), e^{\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi * b(\tau)\rangle\\ &-\eta\|\nabla e^{\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi * b(\tau)\|^2)\mathrm{d}\tau\\ &-2\int_s^t (\langle u\cdot\nabla b(\tau), e^{2\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\rangle\\ &-\langle b\cdot\nabla u(\tau), e^{2\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\rangle)\mathrm{d}\tau.\end{aligned} \tag{2.11}$$

By integration by parts, it is seen that

$$\begin{aligned}&\langle(-\Delta)e^{\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi * b(\tau), e^{\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi * b(\tau)\rangle\\ &\quad= \|\nabla e^{\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi * b(\tau)\|^2.\end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.11), we have

$$\|e^{\epsilon\eta\Delta}\mathcal{F}^{-1}\varphi * b(t)\|^2 \le \|e^{\eta(t+\epsilon-s)\Delta}\mathcal{F}^{-1}\varphi * b(s)\|^2$$
$$+2\int_s^t |\langle u\cdot\nabla b(\tau), e^{2\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\rangle|$$
$$+|\langle b\cdot\nabla u(\tau), e^{2\eta(t+\epsilon-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\rangle|\mathrm{d}\tau. \tag{2.13}$$

By passing to the limit as $\epsilon \to 0$ in (2.13), we finally conclude that (2.10) holds true. This completes the proof of Corollary 2.3. □

The following L^p-L^q estimate for heat operator will be frequently used in the rest of the paper.

Lemma 2.4 [11]. *Let $\mu > 0$, $1 \le p \le q \le \infty$, $f \in L^p$ and let $m \ge 0$. Then the following L^p-L^q estimate holds*

$$\|\nabla^m e^{\mu t\Delta} f\|_q \le c(\mu)t^{-\frac{m}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_p, \quad \textit{for any} \quad t > 0. \tag{2.14}$$

Lemma 2.5 [20]. *Let $\mu > 0$, $f \in L^2 \cap R_\beta$ for some $\beta > 0$. Let $e^{\mu t\Delta} f = \int K_t(x-y)f(y)dy$ with $K_t(x) = \frac{1}{(4\pi\mu t)^{3/2}}e^{-\frac{|x|^2}{4\mu t}}$. Then there exists $c(\mu,\beta) > 0$ such that $\|e^{\mu t\Delta} f\| \ge c(\mu,\beta)(1+t)^{-\frac{3}{4}}$.*

3. Proof of Theorem 1.1

We begin with the $L^{3/2,\infty} \times L^2$ estimate of (u, b). Taking the inner product of (1.1) and (1.2) with u and b, respectively, then integrating in $\mathbb{R}^3$ and summing the resultant equations, we use integration by parts and (1.3) to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b\|^2 + 2\nu\|\nabla u\|^2 + 2\eta\|\nabla b\|^2 = 0. \tag{3.1}$$

Integrating with respect to t, we deduce that for any $t \ge 0$,

$$\|b(t)\|^2 + 2\int_0^t \big(\nu\|\nabla u(\tau)\|^2 + \eta\|\nabla b(\tau)\|^2\big)\mathrm{d}\tau \le \|b_0\|^2. \tag{3.2}$$

Based on this L^2 estimate of b, we can deduce that $u(t)$ is bounded in $L^{3/2,\infty}$ (see [10,14,22]). In fact, consider the following nonhomogeneous Stokes equation

$$\begin{cases} -\nu\Delta u + \nabla p_* = b\cdot\nabla b, \\ \nabla\cdot u = 0, \end{cases} \tag{3.3}$$

we know that (u, p_*) is solved by

$$u(t,x) = \int \mathbf{U}(x-y)\cdot(b\cdot\nabla b)(t,y)dy \quad \text{and} \quad p_*(t,x) = \int \mathbf{q}(x-y)\cdot(b\cdot\nabla b)(t,y)dy,$$

here $(\mathbf{U}(\cdot), \mathbf{q}(\cdot))$ is the fundamental solution of Stokes equations and $\mathbf{U}(x) = O(|x|^{-1})$ as either $|x| \to 0$ or $|x| \to \infty$, see Section IV.2 in [9] for details. Thus, $\nabla\mathbf{U} \in L^{3/2,\infty}$.

Moreover, by $\nabla \cdot b = 0$ and Young inequality in weak L^p spaces, we deduce that for any $t \geq 0$,

$$\begin{aligned}\|u(t)\|_{L^{3/2,\infty}} &= \left\| \int \nabla \mathbf{U}(x-y)(b\otimes b)(t,y)dy \right\|_{L^{3/2,\infty}} \\ &\lesssim \|\nabla \mathbf{U}\|_{L^{3/2,\infty}} \|(b\otimes b)(t)\|_1 \\ &\lesssim \|b(t)\|^2. \end{aligned} \tag{3.4}$$

Let $\varphi : \mathbb{R}^3 \to [0,1]$ be a smooth, radial cutoff function such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$, $\varphi(\xi) = 0$ for $|\xi| \geq 2$. By Plancherel's theorem, $\|b(t)\| = \|\hat{b}(t)\| \leq \|\varphi\hat{b}(t)\| + \|(1-\varphi)\hat{b}(t)\|$. It suffices to show that $\lim_{t\to\infty} \|\varphi\hat{b}(t)\| = 0$ and $\lim_{t\to\infty} \|(1-\varphi)\hat{b}(t)\| = 0$, respectively.

We apply (2.10) and Plancherel's theorem to obtain

$$\begin{aligned}\|\varphi\hat{b}(t)\|^2 \leq \|e^{-\eta(t-s)|\xi|^2}\varphi\hat{b}(s)\|^2 &\\ + 2\int_s^t &|\langle u\cdot\nabla b(\tau), e^{2\eta(t-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\rangle| \\ &+ |\langle b\cdot\nabla u(\tau), e^{2\eta(t-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\rangle| \mathrm{d}\tau. \end{aligned} \tag{3.5}$$

By Young inequality, the second term on the right-hand side of (3.5) is bounded by (in the sense of '$\lesssim$')

$$\begin{aligned}&\int_s^t (\|u\cdot\nabla b(\tau)\cdot e^{2\eta(t-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\|_1 \\ &\quad + \|b\cdot\nabla u(\tau)\cdot e^{2\eta(t-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\|_1)\mathrm{d}\tau \\ &\lesssim \int_s^t (\|u\cdot\nabla b(\tau)\|_{\frac{3}{2}} + \|b\cdot\nabla u(\tau)\|_{\frac{3}{2}})\|e^{2\eta(t-\tau)\Delta}\mathcal{F}^{-1}\varphi^2 * b(\tau)\|_3 \mathrm{d}\tau \\ &\lesssim \int_s^t (\|u(\tau)\|_6\|\nabla b(\tau)\| + \|b(\tau)\|_6\|\nabla u(\tau)\|)\|e^{2\eta(t-\tau)\Delta}\mathcal{F}^{-1}\varphi^2\|_{\frac{6}{5}}\|b(\tau)\|\mathrm{d}\tau \\ &\lesssim \|b_0\|\int_s^t (\|\nabla u(\tau)\|^2 + \|\nabla b(\tau)\|^2)\mathrm{d}\tau. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), and passing to the limit as $t\to\infty$, we use Lebesgue's dominated convergence theorem and (3.2) to deduce that

$$\begin{aligned}\lim_{t\to\infty}\|\varphi\hat{b}(t)\| &\lesssim \lim_{t\to\infty}\|e^{-\eta(t-s)|\xi|^2}\varphi\hat{b}(s)\|^2 + \|b_0\|\int_s^\infty (\|\nabla u(\tau)\|^2 + \|\nabla b(\tau)\|^2)\mathrm{d}\tau \\ &\lesssim \|b_0\|\int_s^\infty (\|\nabla u(\tau)\|^2 + \|\nabla b(\tau)\|^2)\mathrm{d}\tau \to 0 \quad \text{as} \quad s\to\infty. \end{aligned} \tag{3.7}$$

This implies that $\lim_{t\to\infty}\|\varphi\hat{b}(t)\| = 0$.

Next, we prove that $\lim_{t\to\infty}\|(1-\varphi)\hat{b}(t)\| = 0$. Let $\gamma > 0$. Setting $E(t) = (1+t)^\gamma$ and $\psi = \mathcal{F}^{-1}(1-\varphi)$ in (2.1) and (2.2), then summing the resultant equations, we apply Plancherel's theorem to deduce that

$$
\begin{aligned}
E(t)\|(1-\varphi)\hat{b}(t)\|^2 \le{}& E(s)\|(1-\varphi)\hat{b}(s)\|^2 + \int_s^t E'(\tau)\|(1-\varphi)\hat{b}(\tau)\|^2 \mathrm{d}\tau \\
&- 2\eta \int_s^t E(\tau)\|\xi\cdot(1-\varphi)\hat{b}(\tau)\|^2 \mathrm{d}\tau \\
&+ 2\int_s^t E(\tau)\langle \widehat{b\cdot\nabla b}(\tau), (1-\varphi)^2\hat{u}(\tau)\rangle \mathrm{d}\tau \\
&- 2\int_s^t E(\tau)(\langle \widehat{u\cdot\nabla b}(\tau), (1-\varphi)^2\hat{b}(\tau)\rangle \mathrm{d}\tau \\
&+ 2\int_s^t E(\tau)\langle \widehat{b\cdot\nabla u}(\tau), (1-\varphi)^2\hat{b}(\tau)\rangle) \mathrm{d}\tau \\
:={}& \sum_{j=1}^{6} I_j. \qquad (3.8)
\end{aligned}
$$

The bound of I_2+I_3 is obtained by Fourier splitting method (see [19,20]). In fact, for each $t>0$, we define $r(t)=\sqrt{\frac{\gamma}{2\eta(1+t)}}$, here $\gamma>0$ is given in the previous paragraph. Thus, there holds

$$
\begin{aligned}
I_2+I_3 ={}& \int_s^t E'(\tau)\Big(\int_{|\xi|<r(\tau)} + \int_{|\xi|\ge r(\tau)}\Big)|(1-\varphi)\hat{b}(\tau)|^2 \mathrm{d}\xi\mathrm{d}\tau \\
&- 2\eta\int_s^t E(\tau)\int_{|\xi|\ge r(\tau)} |\xi\cdot(1-\varphi)\hat{b}(\tau)|^2 \mathrm{d}\xi\mathrm{d}\tau \\
\le{}& \int_s^t E'(\tau)\int_{|\xi|<r(\tau)} |(1-\varphi)\hat{b}(\tau)|^2 \mathrm{d}\xi d\tau + \int_s^t (E'(\tau) \\
&- 2\eta E(\tau)r^2(\tau))\int_{|\xi|\ge r(\tau)} |(1-\varphi)\hat{b}(\tau)|^2 \mathrm{d}\xi\mathrm{d}\tau.
\end{aligned}
$$

Since $E'(\tau)-2\eta E(\tau)r^2(\tau)=0$ for all $\tau\in[s,t]$, we deduce that

$$
\begin{aligned}
I_2+I_3 \le{}& \int_s^t E'(\tau)\int_{|\xi|<r(\tau)} |(1-\varphi)\hat{b}(\tau)|^2 \mathrm{d}\xi\mathrm{d}\tau \\
\le{}& \sup_{\tau\in[s,t]}\Big(\int_{|\xi|<r(\tau)} |(1-\varphi)\hat{b}(\tau)|^2 \mathrm{d}\xi\Big)\int_s^t E'(\tau)\mathrm{d}\tau \\
\le{}& E(t)\int_{|\xi|<r(s)} |(1-\varphi)\hat{b}(\tau)|^2 \mathrm{d}\xi. \qquad (3.9)
\end{aligned}
$$

Now we estimate $I_4+I_5+I_6$. Let $\zeta=-2\varphi+\varphi^2$. Then $\zeta\in C_c^\infty$ and $\mathcal{F}^{-1}\zeta$ belongs to Schwartz space. By the divergence free condition (1.3), there hold $\langle\widehat{u\cdot\nabla b},\hat{b}\rangle=0$, $\langle\widehat{b\cdot\nabla b},\hat{u}\rangle+\langle\widehat{b\cdot\nabla u},\hat{b}\rangle=0$. Thus, we use Plancherel's theorem to deduce that

$$
\begin{aligned}
|I_4+I_5+I_6| \lesssim{}& \int_s^t E(\tau)(|\langle b\cdot\nabla b(\tau), \mathcal{F}^{-1}\zeta * u(\tau)\rangle| + |\langle u\cdot\nabla b(\tau), \mathcal{F}^{-1}\zeta * b(\tau)\rangle| \\
&+ |\langle b\cdot\nabla u(\tau), \mathcal{F}^{-1}\zeta * b(\tau)\rangle|)\mathrm{d}\tau
\end{aligned}
$$

$$\begin{aligned}
&\lesssim E(t)\int_s^t \|b\cdot\nabla b(\tau)\|_1\|\mathcal{F}^{-1}\zeta * u(\tau)\|_\infty \\
&\quad+(\|u\cdot\nabla b(\tau)\|_{\frac{3}{2}}+\|b\cdot\nabla u(\tau)\|_{\frac{3}{2}})\|\mathcal{F}^{-1}\zeta * b(\tau)\|_3\mathrm{d}\tau \\
&\lesssim E(t)\int_s^t \|b\|\|\nabla b(\tau)\|\|\mathcal{F}^{-1}\zeta\|_{\frac{6}{5}}\|u(\tau)\|_6+(\|u(\tau)\|_6\|\nabla b(\tau)\| \\
&\quad+\|b(\tau)\|_6\|\nabla u(\tau)\|)\|\mathcal{F}^{-1}\zeta\|_{\frac{6}{5}}\|b(\tau)\|\mathrm{d}\tau \\
&\lesssim E(t)\|b_0\|\int_s^t(\|\nabla u(\tau)\|^2+\|\nabla b(\tau)\|^2)\mathrm{d}\tau. \qquad (3.10)
\end{aligned}$$

Substituting (3.9) and (3.10) into (3.8), then multiplying the resultant equation by $E(t)^{-1}$, we obtain

$$\begin{aligned}
\|(1-\varphi)\hat{b}(t)\|^2 &\lesssim \frac{E(s)}{E(t)}\|(1-\varphi)\hat{b}(s)\|^2 \\
&+\int_{|\xi|<r(s)}|(1-\varphi)\hat{b}(\tau)|^2\mathrm{d}\xi+\|b_0\|\int_s^t(\|\nabla u(\tau)\|^2+\|\nabla b(\tau)\|^2)\mathrm{d}\tau. \qquad (3.11)
\end{aligned}$$

Passing to the limit as $t, s\to\infty$, we have

$$\begin{aligned}
\lim_{t\to\infty}\|(1-\varphi)\hat{b}(t)\|^2 &= \lim_{s\to\infty}\lim_{t\to\infty}\|(1-\varphi)\hat{b}(t)\|^2 \\
&\lesssim \lim_{s\to\infty}\int_{|\xi|<r(s)}|(1-\varphi)\hat{b}(\tau)|^2\mathrm{d}\xi+\lim_{s\to\infty}\|b_0\|\int_s^\infty(\|\nabla u(\tau)\|^2 \\
&\quad+\|\nabla b(\tau)\|^2)\mathrm{d}\tau=0. \qquad (3.12)
\end{aligned}$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We assume that b_0 belongs to some Sobolev spaces, that is, $b_0\in H^1$ with $\|b_0\|=\alpha$. For $\lambda>0$, consider the scaling transformation $\delta_\lambda: L^2\to L^2$ via $f(\cdot)\mapsto\lambda^{3/2}(\lambda\cdot)$. It's clear that it preserves the L^2 norm, that is, $\|\delta_\lambda f\|=\|f\|$. Let (u^λ, b^λ) be a solution of (1.1)–(1.4) with the initial data $\delta_\lambda b_0$, then (u^λ, b^λ) satisfies following equations:

$$-\nu\Delta u^\lambda+\nabla p_*^\lambda=b^\lambda\cdot\nabla b^\lambda, \qquad (4.1)$$

$$\partial_t b^\lambda+u^\lambda\cdot\nabla b^\lambda-\eta\Delta b^\lambda=b^\lambda\cdot\nabla u^\lambda, \qquad (4.2)$$

$$\nabla\cdot u^\lambda=0,\quad \nabla\cdot b^\lambda=0, \qquad (4.3)$$

$$b^\lambda|_{t=0}=\delta_\lambda b_0, \qquad (4.4)$$

here $p_*^\lambda=p^\lambda+\frac{1}{2}|b^\lambda|^2$.

Lemma 4.1. (i) *For any* $\lambda>0$, *there exists a* $T_\lambda>0$, *such that* (4.1)–(4.4) *admits a unique strong solution* $(u^\lambda, b^\lambda)\in C([0,T_\lambda);H^1)\cap L^2(0,T_\lambda;H^2)$. (ii) *There* *exist* $\lambda_0 = \lambda_0(\nu, \eta, \|b_0\|_{H^1}) > 0$ *and* $c(\nu, \eta, \|b_0\|) > 0$*, such that for any* $0 < \lambda < \lambda_0$*,* (4.1)–(4.4) *admits a unique global solution* $(u^\lambda, b^\lambda) \in C([0, \infty); H^1) \cap L^2(0, \infty; H^2)$*, and there holds*

$$\|\nabla u^\lambda(t)\| \le c(\nu, \eta, \|b_0\|)\|\nabla \delta_\lambda b_0\|^{\frac{3}{2}}, \quad \|\nabla b^\lambda(t)\| \le c(\nu, \eta, \|b_0\|)\|\nabla \delta_\lambda b_0\|, \tag{4.5}$$

for all $t \ge 0$.

Proof of Lemma 4.1.. (i) The local-in-time existence of strong solutions are ensured by the global existence of weak solutions (u^λ, b^λ) (see [14, page 521]) together with a priori H^1 estimate near the initial time (see (4.7)–(4.9) below). This a priori estimate also leads to the uniqueness of strong solutions, as well as the continuity with respect to t or initial data. The proof is similar to that of Navier–Stokes equations and thus it is omitted here.

(ii) By repeating the manipulation of (3.1)–(3.4), we have the following bounds:

$$\|u^\lambda(t)\|_{L^{3/2,\infty}} \lesssim \|b^\lambda(t)\|^2, \quad \|b^\lambda(t)\|^2 + 2\int_0^t \left(\nu\|\nabla u^\lambda(\tau)\|^2 + \eta\|\nabla b^\lambda(\tau)\|^2\right)\mathrm{d}\tau \le \|b_0\|^2, \tag{4.6}$$

for all $t \ge 0$.

Taking the inner product of (4.1) with u^λ and integrating in $\mathbb{R}^3$, we use interpolation inequality and (4.6) to obtain $\|\nabla u^\lambda(t)\| \le c(\nu, \eta, \|b_0\|)\|\nabla b^\lambda(t)\|^{3/2}$. Thus, it sufficient to show $\|\nabla b^\lambda(t)\| \le c(\nu, \eta, \|b_0\|)\|\nabla \delta_\lambda b_0\|$.

Taking the inner product of (4.1) and (4.2) with $-\Delta u^\lambda$ and $-\Delta b^\lambda$ respectively, integrating in $\mathbb{R}^3$ and then summing the resultant equations, we use $\nabla \cdot u = 0$ to obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla b^\lambda\|^2 + \nu\|\Delta u^\lambda\|^2 + \eta\|\Delta b^\lambda\|^2 \\
&\quad = \int \nabla(b^\lambda \cdot \nabla b^\lambda) : \nabla u^\lambda \mathrm{d}x - \int \nabla(u^\lambda \cdot \nabla b^\lambda) : \nabla b^\lambda \mathrm{d}x + \int \nabla(b^\lambda \cdot \nabla u^\lambda) : \nabla b^\lambda \mathrm{d}x.
\end{aligned} \tag{4.7}$$

By (4.3) and integration by parts, the absolute value of the right-hand side of (4.7) is bounded (in the sense of '$\lesssim$') by

$$\begin{aligned}
\int |\nabla u^\lambda||\nabla b^\lambda|^2 \mathrm{d}x &\le c\|\nabla u^\lambda\|_6 \|\nabla b^\lambda\|^2_{\frac{12}{5}} \\
&\le \frac{\nu}{2}\|\Delta u^\lambda\|^2 + c\|\nabla b^\lambda\|^3\|\Delta b^\lambda\| \\
&\le \frac{\nu}{2}\|\Delta u^\lambda\|^2 + \frac{\eta}{2}\|\Delta b^\lambda\|^2 + c\|\nabla b^\lambda\|^6 \\
&\le \frac{\nu}{2}\|\Delta u^\lambda\|^2 + \frac{\eta}{2}\|\Delta b^\lambda\|^2 + c\|\nabla b^\lambda\|^2\|b^\lambda\|^2\|\Delta b^\lambda\|^2,
\end{aligned} \tag{4.8}$$

where we have used interpolation inequality to derive the second and the fourth inequalities. Substituting (4.8) into (4.7), we apply (4.6) to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla b^\lambda\|^2 + \nu\|\Delta u^\lambda\|^2 + \eta\|\Delta b^\lambda\|^2 \le c(\nu, \eta, \|b_0\|)\|\nabla b^\lambda\|^2\|\Delta b^\lambda\|^2. \tag{4.9}$$

Since $\|\nabla\delta_\lambda b_0\| = \lambda\|\nabla b_0\|$, we choose λ_0 small enough such that $c(\nu, \eta, \|b_0\|)\|\nabla\delta_{\lambda_0} b_0\|^2 < \eta$. For any $\lambda \in (0, \lambda_0)$, by the fact that $b^\lambda \in C([0, T_\lambda); H^1)$, we deduce that there exists a $\tilde{T}_\lambda \in (0, T_\lambda)$ such that $c(\nu, \eta, \|b_0\|)\|\nabla\delta_\lambda b_0(t)\|^2 < \eta$ for $t \in [0, \tilde{T}_\lambda]$. Thus, it follows from (4.9) that $\|\nabla b^\lambda(t)\|^2 \le \|\nabla\delta_\lambda b_0\|^2$ for $t \in [0, \tilde{T}_\lambda]$. By induction, we deduce that (4.5) holds for all $t \ge 0$. □

Assume that $\lambda < \lambda_0$. By Lemma 4.1 (ii), (4.1)–(4.4) admits a unique global strong solution, which is denoted by (u^λ, b^λ). By Fourier transformation, b^λ is solved as follows:

$$b^\lambda(t, x) = e^{\eta t\Delta}\delta_\lambda b_0(x) + \int_0^t e^{\eta(t-s)\Delta}(-u^\lambda \cdot \nabla b^\lambda(s) + b^\lambda \cdot \nabla u^\lambda(s))\mathrm{d}s. \tag{4.10}$$

Hence

$$\|b^\lambda(t)\| \ge \|e^{\eta t\Delta}\delta_\lambda b_0\| - \left\| \int_0^t e^{\eta(t-s)\Delta}(-u^\lambda \cdot \nabla b^\lambda(s) + b^\lambda \cdot \nabla u^\lambda(s))\mathrm{d}s \right\|. \tag{4.11}$$

The main task is to calculate the limit (as $\lambda \to 0^+$) of terms on the right-hand side of (4.11). For the first term, by Plancherel's theorem and Lebesgue's dominated convergence theorem, we know that

$$\begin{aligned}
\lim_{\lambda\to 0^+} \|e^{\eta t\Delta}\delta_\lambda b_0\|^2 &= \lim_{\lambda\to 0^+} \int e^{-2\eta t|\xi|^2}|\widehat{\delta_\lambda b_0}(\xi)|^2\mathrm{d}x \\
&= \lim_{\lambda\to 0^+} \lambda^3 \int e^{-2\eta t|\xi|^2}|\widehat{b_0(\lambda\cdot)}(\xi)|^2\mathrm{d}x \\
&= \lim_{\lambda\to 0^+} \lambda^{-3} \int e^{-2\eta t|\xi|^2}|\widehat{b_0}(\lambda^{-1}\xi)|^2\mathrm{d}x \\
&= \lim_{\lambda\to 0^+} \int e^{-2\eta t\lambda^2|\xi|^2}|\widehat{b_0}(\xi)|^2\mathrm{d}x \\
&= \|b_0\|^2.
\end{aligned} \tag{4.12}$$

While for the second term, we claim that

$$\lim_{\lambda\to 0^+} \left\| \int_0^t e^{\eta(t-s)\Delta}(-u^\lambda \cdot \nabla b^\lambda(s) + b^\lambda \cdot \nabla u^\lambda(s))\mathrm{d}s \right\| = 0. \tag{4.13}$$

In fact, for small enough $\lambda > 0$, one applies Lemma 2.4 and (4.5) to deduce that

$$\begin{aligned}
&\left\| \int_0^t e^{\eta(t-s)\Delta}(-u^\lambda \cdot \nabla b^\lambda(s) + b^\lambda \cdot \nabla u^\lambda(s))\mathrm{d}s \right\| \\
&\le \int_0^t \|e^{\eta(t-s)\Delta}(-u^\lambda \cdot \nabla b^\lambda(s) + b^\lambda \cdot \nabla u^\lambda(s))\|\mathrm{d}s \\
&\le c(\eta) \int_0^t (t-s)^{-\frac{1}{4}}(\|u^\lambda \cdot \nabla b^\lambda(s)\|_{\frac{3}{2}} + \|b^\lambda \cdot \nabla u^\lambda(s)\|_{\frac{3}{2}})\mathrm{d}s \\
&\le c(\eta) \int_0^t (t-s)^{-\frac{1}{4}}\|\nabla u^\lambda(s)\|\|\nabla b^\lambda(s)\|\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}&\leq c(\nu,\eta,\|b_0\|)\|\nabla\delta_\lambda b_0\|^{\frac{5}{2}}t^{\frac{3}{4}}\\&\leq c(\nu,\eta,\|b_0\|_{H^1})\lambda^{\frac{5}{2}}t^{\frac{3}{4}}.\end{aligned}\tag{4.14}$$

Passing to the limit as $\lambda\to 0^+$, it follows that (4.13) holds true.

Multiplying (4.11) by $\|b^\lambda(0)\|^{-1}$ and passing to the limit as $\lambda\to 0^+$, we use $\|b^\lambda(0)\|=\|b_0\|$, (4.12) and (4.13) to deduce that

$$\begin{aligned}\lim_{\lambda\to 0^+}\frac{\|b^\lambda(t)\|}{\|b^\lambda(0)\|}&=\frac{1}{\|b_0\|}\lim_{\lambda\to 0^+}\|b^\lambda(t)\|\\&=\frac{1}{\|b_0\|}\lim_{\lambda\to 0^+}\|e^{\eta t\Delta}\delta_\lambda b_0\|\\&\quad-\frac{1}{\|b_0\|}\lim_{\lambda\to 0^+}\left\|\int_0^t e^{\eta(t-s)\Delta}(-u^\lambda\cdot\nabla b^\lambda(s)+b^\lambda\cdot\nabla u^\lambda(s))\mathrm{d}s\right\|=1.\end{aligned}\tag{4.15}$$

This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

Step 1. A formal proof of $L^{3/2,\infty}\times L^2$ decay. Assume that (u,b) is a smooth solution of (1.1)–(1.4). Removing $2\nu\|\nabla u\|^2$ from (3.1) and using Plancherel's theorem, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{b}\|^2+2\eta\int|\xi|^2|\hat{b}|^2\mathrm{d}\xi\leq 0.\tag{5.1}$$

Let $r(t)=\sqrt{\frac{3}{2\eta(1+t)}}$, we deduce that

$$\begin{aligned}\int|\xi|^2|\hat{b}|^2\mathrm{d}\xi&=\int_{|\xi|<r(t)}|\xi|^2|\hat{b}|^2\mathrm{d}\xi+\int_{|\xi|\geq r(t)}|\xi|^2|\hat{b}|^2\mathrm{d}\xi\\&\geq\frac{3}{2\eta(1+t)}\int_{|\xi|\geq r(t)}|\hat{b}|^2\mathrm{d}\xi\\&\geq\frac{3}{2\eta(1+t)}\int|\hat{b}|^2\mathrm{d}\xi-\frac{3}{2\eta(1+t)}\int_{|\xi|<r(t)}|\hat{b}|^2\mathrm{d}\xi.\end{aligned}\tag{5.2}$$

Substituting (5.2) into (5.1), and multiplying the resultant equation by $(1+t)^3$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big((1+t)^3\|\hat{b}\|^2\Big)\leq 3(1+t)^2\int_{|\xi|<r(t)}|\hat{b}|^2\mathrm{d}\xi.\tag{5.3}$$

Now we prove that for all $|\xi|<r(t)$, there holds

$$|\hat{b}(\xi)|\leq c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+\xi^{-1}).\tag{5.4}$$

In fact, by taking Fourier transformation to (1.2), $\hat{b}$ is solved as

$$\hat{b}(\xi) = e^{-\eta t|\xi|^2}\hat{b}(0) + \int_0^t e^{-\eta(t-s)|\xi|^2}(\widehat{b\cdot\nabla u} - \widehat{u\cdot\nabla b})(s)\mathrm{d}s. \tag{5.5}$$

Since $b_0 \in L^1$, we deduce from (5.5) that

$$|\hat{b}(\xi)| \le e^{-\eta t|\xi|^2}\|b_0\|_1 + \int_0^t e^{-\eta(t-s)|\xi|^2}|\xi|(\|b\otimes u\|_1 + \|u\otimes b\|_1)(s)\mathrm{d}s. \tag{5.6}$$

By interpolation inequality and (3.2), the second term on the right-hand side of (5.6) is bounded by

$$\begin{aligned}
&|\xi|\int_0^t e^{-\eta(t-s)|\xi|^2}\|u(s)\|\|b(s)\|\mathrm{d}s\\
&\quad\lesssim |\xi|\|b_0\|\int_0^t e^{-\eta(t-s)|\xi|^2}\|u(s)\|_{L^{3/2,\infty}}^{\frac{2}{3}}\|\nabla u(s)\|^{\frac{1}{3}}\mathrm{d}s\\
&\quad\lesssim |\xi|\|b_0\|\int_0^t e^{-\eta(t-s)|\xi|^2}(\|u(s)\|_{L^{3/2,\infty}} + \|\nabla u(s)\|)\mathrm{d}s\\
&\quad\lesssim |\xi|\|b_0\|\Big[\|b_0\|^2\int_0^t e^{-\eta(t-s)|\xi|^2}\mathrm{d}s\\
&\qquad+\Big(\int_0^t e^{-\eta(t-s)|\xi|^2}\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^t \|\nabla u(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big]\\
&\quad\lesssim \|b_0\|^3|\xi|^{-1} + \|b_0\|^2,
\end{aligned} \tag{5.7}$$

Substituting (5.7) into (5.6), this proves (5.4).

Integrating (5.3) with respect to t and using (5.4), we deduce that

$$\begin{aligned}
\|\hat{b}(t)\|^2 &\le (1+t)^{-3}\|b_0\|^2 + c(\nu,\eta,\|b_0\|_{L^1\cap L^2})[(1+t)^{-\frac{3}{2}} + (1+t)^{-\frac{1}{2}}]\\
&\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{1}{2}},
\end{aligned} \tag{5.8}$$

for all $t \ge 0$. Hence, we have

$$\begin{aligned}
\|b(t)\| &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{1}{4}} \quad \text{and} \quad \|u(t)\|_{L^{3/2,\infty}}\\
&\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{5.9}$$

Now we prove that, by a iteration process, the decay rate for $b(t)$ in (5.9) can be improved to a much faster decay rate, which is proportional to that of solutions of heat equation. More precisely, we show that $\|b(t)\| \le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+t)^{-3/4}$ for all $t \ge 0$.

By a calculation similar to that of (5.7), we apply (5.9) to deduce that

$$\begin{aligned}
&\Big|\int_0^t e^{-\eta(t-s)|\xi|^2}(\widehat{b\cdot\nabla u} - \widehat{u\cdot\nabla b})(s)\mathrm{d}s\Big|\\
&\quad\le c|\xi|\int_0^t e^{-\eta(t-s)|\xi|^2}(\|u(s)\|_{L^{3/2,\infty}} + \|\nabla u(s)\|)\|b(s)\|\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})|\xi|\Big\{\int_0^t e^{-\eta(t-s)|\xi|^2}(1+s)^{-\frac{3}{4}}\mathrm{d}s \\
&\quad+\Big(\int_0^t e^{-2\eta(t-s)|\xi|^2}(1+s)^{-\frac{1}{2}}\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^t \|\nabla u(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big\} \\
&\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})|\xi|\int_0^t e^{-\eta(t-s)|\xi|^2}(1+s)^{-\frac{3}{4}}\mathrm{d}s \\
&\quad+c(\nu,\eta,\|b_0\|_{L^1\cap L^2})|\xi|\Big(\int_0^t e^{-2\eta(t-s)|\xi|^2}(1+s)^{-\frac{1}{2}}\mathrm{d}s\Big)^{\frac{1}{2}} \\
&:= I_1+I_2.
\end{aligned}
\tag{5.10}
$$

The bound of I_1 is obtained by dividing the integration interval $[0,t]$ into $[0,t/2]$ and $[t/2,t]$.

$$
\begin{aligned}
I_1 &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})|\xi|\Big(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\Big)e^{-\eta(t-s)|\xi|^2}(1+s)^{-\frac{3}{4}}\mathrm{d}s \\
&\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})|\xi| \\
&\quad\Big(e^{-\frac{\eta t|\xi|^2}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{3}{4}}\mathrm{d}s+(1+t)^{-\frac{3}{4}}\int_{\frac{t}{2}}^t e^{-\eta(t-s)|\xi|^2}\mathrm{d}s\Big) \\
&\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(|\xi|e^{-\frac{\eta t|\xi|^2}{2}}(1+t)^{\frac{1}{4}}+|\xi|^{-1}(1+t)^{-\frac{3}{4}}).
\end{aligned}
\tag{5.11}
$$

Similarly, I_2 is bounded by

$$
I_2 \le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(|\xi|e^{-\frac{\eta t|\xi|^2}{2}}(1+t)^{\frac{1}{4}}+(1+t)^{-\frac{1}{4}}).
\tag{5.12}
$$

Substituting (5.11) and (5.12) into (5.10), we deduce from (5.5) that

$$
|\hat{b}(\xi)| \le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+|\xi|e^{-\frac{\eta t|\xi|^2}{2}}(1+t)^{\frac{1}{4}}+(1+t)^{-\frac{1}{4}}+|\xi|^{-1}(1+t)^{-\frac{3}{4}}).
\tag{5.13}
$$

This implies that

$$
\begin{aligned}
\int_{|\xi|<r(t)}|\hat{b}|^2\mathrm{d}\xi &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2}) \\
&\quad\int_{|\xi|<r(t)}(1+|\xi|^2e^{-\eta t|\xi|^2}(1+t)^{\frac{1}{2}}+(1+t)^{-\frac{1}{2}} \\
&\quad+|\xi|^{-2}(1+t)^{-\frac{3}{2}})\mathrm{d}\xi \\
&\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})((1+t)^{-\frac{3}{2}}+(1+t)^{-2}) \\
&\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{2}}
\end{aligned}
\tag{5.14}
$$

Integrating (5.3) with respect to t and using (5.14), we obtain the desired decay estimate of $b(t)$:

$$
\|b(t)\|^2=\|\hat{b}(t)\|^2\le(1+t)^{-3}\|b_0\|^2+c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{2}}
$$

$$\leq c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{2}}. \tag{5.15}$$

Furthermore, the equation (5.15), together with (3.4), implies that $\|u(t)\| \leq c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-3/2}$.

Step 2. The decay of weak solutions. Let $\epsilon > 0$ be arbitrary. For $1 \leq p \leq \infty$, we define a mollification $\mathcal{J}_\epsilon : L^p \to L^p$ via $f \mapsto \rho_\epsilon * f$, where $\rho_\epsilon(\cdot) = \frac{1}{\epsilon^3}\rho(\frac{\cdot}{\epsilon})$ and $\rho \in C_0^\infty$ is a nonnegative, radial function that satisfies $\int \rho(x)\mathrm{d}x = 1$.

Consider the following equations:

$$-\nu\Delta u^\epsilon + \nabla p_*^\epsilon = b^\epsilon \cdot \nabla b^\epsilon, \tag{5.16}$$

$$\partial_t b^\epsilon + u^\epsilon \cdot \nabla b^\epsilon - \eta\Delta b^\epsilon + \epsilon(-\Delta)^{3/2} b^\epsilon = b^\epsilon \cdot \nabla u^\epsilon, \tag{5.17}$$

$$\nabla \cdot u^\epsilon = 0, \quad \nabla \cdot b^\epsilon = 0, \tag{5.18}$$

$$b^\epsilon|_{t=0} = \mathcal{J}_\epsilon b_0. \tag{5.19}$$

This system is obtained by adding an artificial diffusion term $\epsilon(-\Delta)^{3/2}$ to the b-equation of (1.1)–(1.4), then replacing the initial datum b_0 by a smooth function $\mathcal{J}_\epsilon b_0$. The fractional Laplacian $(-\Delta)^{3/2}$ is defined by the Fourier transform, namely, $(-\Delta)^{3/2} = \mathcal{F}^{-1}(|\xi|^3\mathcal{F})$.

We say that (u^ϵ, b^ϵ) is a global-in-time strong solution of the equations (5.16)–(5.19) if it is a weak solution of the system (this means that (u^ϵ, b^ϵ) belongs to a proper integrable space and it solves the system in the sense of distribution) and it satisfies

$$\begin{aligned} u^\epsilon &\in C([0,\infty; H_\sigma^{3/2}) \cap L^2_{loc}(0,\infty; H^2) \quad \text{and} \\ b^\epsilon &\in C([0,\infty; H_\sigma^1) \cap L^2_{loc}(0,\infty; H^{5/2}). \end{aligned} \tag{5.20}$$

It can be proved that (5.16)–(5.19) admits a unique global-in-time strong solution (u^ϵ, b^ϵ). In fact, it was recently shown by the authors in [10] that (5.16)–(5.19) admits a unique global-in-time strong solution when $\eta = 0$. This global well-posedness result is absolutely true for the same equations when $\eta > 0$.

Thus, for (5.16)–(5.19), we repeat the manipulation of *Step 1* to deduce that for all $\epsilon > 0$ and $t \geq 0$,

$$\|u^\epsilon(t)\|_{L^{3/2,\infty}} \leq c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{2}}, \tag{5.21}$$

$$\|b^\epsilon(t)\| \leq c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{4}}. \tag{5.22}$$

Note that the presence of $\epsilon(-\Delta)^{3/2}b^\epsilon$ does not affect the decay of (u^ϵ, b^ϵ) in $L^{3/2,\infty} \times L^2$.

It remains to show that $\{(u^\epsilon, b^\epsilon)\}_{\epsilon>0}$ (or a subsequence) converges in some sense to (u, b) as ϵ tends to zero, and (u, b) is a weak solution of (1.1)–(1.4) as well as (u, b) satisfies (1.8).

By the energy method, it is clear that

$$u^\epsilon \quad \text{is uniformly bounded in} \quad L^\infty(0,\infty; L^{3/2,\infty}) \cap L^2_{loc}(0,\infty; H_\sigma^1), \tag{5.23}$$

$$b^\epsilon \quad \text{is uniformly bounded in} \quad L^\infty(0,\infty;L^2)\cap L^2_{loc}(0,\infty;H^1_\sigma), \tag{5.24}$$

$$\partial_t b^\epsilon \quad \text{is uniformly bounded in} \quad L^{24/19}_{loc}(0,\infty;H^{-1}), \tag{5.25}$$

here H^{-1} denotes the dual space of H^1_σ. Thus, it follows from Banach-Alaoglu theorem that there exists a subsequence which is still denoted by $\{(u^\epsilon, b^\epsilon, \partial_t b^\epsilon)\}_{\epsilon>0}$ and an element $(u, b, \partial_t b)$, such that

$$\begin{aligned} &(u^\epsilon, b^\epsilon, \partial_t b^\epsilon) \to (u, b, \partial_t b) \text{ weakly star in } L^\infty(0,\infty;L^{3/2,\infty}) \times L^\infty(0,\infty;L^2) \\ &\quad \times L^{24/19}_{loc}(0,\infty;H^{-1}), \end{aligned} \tag{5.26}$$

$$\begin{aligned} &(u^\epsilon, b^\epsilon) \to (u, b) \qquad \text{weakly in} \quad L^2_{loc}(0,\infty;H^1_\sigma) \\ &\quad \times L^2_{loc}(0,\infty;H^1_\sigma) \quad \text{as} \quad \epsilon \to 0. \end{aligned} \tag{5.27}$$

Furthermore, for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, since $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it follows from Aubin-Lions theorem, (5.26) and (5.27) that there exists a subsequence which is still denoted by $\{b^\epsilon\}$ such that

$$b^\epsilon \to b \quad \text{strongly in} \quad L^2_{loc}(0,\infty;L^2(\Omega)) \quad \text{as} \quad \epsilon \to 0. \tag{5.28}$$

This convergence, together with the $L^{3/2,\infty}$-estimate of u^ϵ (see (3.4)), implies that

$$u^\epsilon \to u \quad \text{strongly in} \quad L^2_{loc}(0,\infty;L^{3/2,\infty}(\Omega)) \quad \text{as} \quad \epsilon \to 0. \tag{5.29}$$

Hence, by (5.28), (5.29) and the uniform boundedness of $\{(u^\epsilon, b^\epsilon)\}_{\epsilon>0}$, we deduce that

$$b^\epsilon \cdot \nabla b^\epsilon \to b \cdot \nabla b \quad \text{weakly star in} \quad L^{4/3}_{loc}(0,\infty;H^{-1}(\Omega)), \tag{5.30}$$

$$u^\epsilon \cdot \nabla b^\epsilon \to u \cdot \nabla b \quad \text{weakly star in} \quad L^{24/19}_{loc}(0,\infty;H^{-1}(\Omega)), \tag{5.31}$$

$$b^\epsilon \cdot \nabla u^\epsilon \to b \cdot \nabla u \quad \text{weakly star in} \quad L^{24/19}_{loc}(0,\infty;H^{-1}(\Omega)) \quad \text{as} \quad \epsilon \to 0. \tag{5.32}$$

By passing to the limit as $\epsilon \to 0$, we conclude that (u, b) is a weak solution of (1.1)–(1.4).

Moreover, by (5.28) and Fatou's lemma, we deduce from (5.22) that

$$\|b(t)\| \le c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{4}} \quad \text{for all} \quad t \ge 0. \tag{5.33}$$

This equation, together with (3.4), implies that

$$\|u(t)\|_{L^{3/2,\infty}} \le c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{3}{2}} \quad \text{for all} \quad t \ge 0. \tag{5.34}$$

This completes the proof of Theorem 1.3.

6. Proof of Theorem 1.4

Let $b_1(t) = e^{\eta t\Delta} b_0$, and let $w = b - b_1$. A direct computation yields $\|b(t)\| \ge \|b_1(t)\| - \|w(t)\|$. The lower bound of decay ratio for $b(t)$ is obtained by establishing the lower bound and upper bound for $b_1(t)$ and $w(t)$, respectively. By Lemma 2.5, it is seen that $\|b_1(t)\| \geq c(\eta, \beta)(1+t)^{-3/4}$. Thus, (1.9) holds if one shows that there exists a $\rho > 3/4$ such that $\|w(t)\| \leq c(1+t)^{-\rho}$.

Consider the equations that satisfied by (u, w):

$$-\nu\Delta u + \nabla p_* = b\cdot\nabla b, \tag{6.1}$$

$$\partial_t w - \eta\Delta w = -u\cdot\nabla b + b\cdot\nabla u, \tag{6.2}$$

$$\nabla\cdot u = 0, \quad \nabla\cdot w = 0, \tag{6.3}$$

$$w|_{t=0} = 0, \tag{6.4}$$

Taking the inner product of (6.1) and (6.2) with u and w, respectively, then integrating in $\mathbb{R}^3$ and summing the resultant equations, we use integration by parts and $\nabla\cdot u = \nabla\cdot b = 0$ to obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + 2\nu\|\nabla u\|^2 + 2\eta\|\nabla w\|^2 &= \int \big(b\cdot\nabla b\big)\cdot u\mathrm{d}x \\ &\quad - \int \big(u\cdot\nabla b\big)\cdot w\mathrm{d}x + \int \big(b\cdot\nabla u\big)\cdot w\mathrm{d}x \\ &= \int \big(u\cdot\nabla b\big)\cdot b_1\mathrm{d}x - \int \big(b\cdot\nabla u\big)\cdot b_1\mathrm{d}x \\ &= \int \big(b\otimes u - u\otimes b\big)\cdot\nabla b_1\mathrm{d}x. \end{aligned}\tag{6.5}$$

Let $r(t) = \sqrt{\frac{3}{2\eta(1+t)}}$. By repeating the manipulation of the derivation of (5.3), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big((1+t)^3\|\hat{w}\|^2\Big) \leq 3(1+t)^2\int_{|\xi|<r(t)}|\hat{w}|^2\mathrm{d}\xi + (1+t)^3\Big|\int \big(b\otimes u - u\otimes b\big)\cdot\nabla b_1\mathrm{d}x\Big|. \tag{6.6}$$

Applying interpolation inequality, Lemma 2.4 and (1.8), we bound the integration of the second term on the right-hand side of (6.6) as follows:

$$\begin{aligned}\Big|\int \big(b\otimes u - u\otimes b\big)(t)\cdot\nabla b_1(t)\mathrm{d}x\Big| &\leq c\|\nabla b_1(t)\|_\infty\|u(t)\|\|b(t)\| \\ &\leq c(\nu, \eta, \|b_0\|_{L^1\cap L^2})t^{-\frac{5}{4}}(1+t)^{-\frac{3}{4}}\|u(t)\|_{L^{3/2,\infty}}^{\frac{2}{3}}\|\nabla u(t)\|^{\frac{1}{3}} \\ &\leq c(\nu, \eta, \|b_0\|_{L^1\cap L^2})(1+t)^{-3}\|\nabla u(t)\|^{\frac{1}{3}}. \end{aligned}\tag{6.7}$$

Substituting (6.7) into (6.6), we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big((1+t)^3\|\hat{w}\|^2\Big) \leq 3(1+t)^2\int_{|\xi|<r(t)}|\hat{w}|^2\mathrm{d}\xi + c(\nu, \eta, \|b_0\|_{L^1\cap L^2})\|\nabla u(t)\|^{\frac{1}{3}}. \tag{6.8}$$

Now we estimate $|\hat{w}|$ on the ball $\{\xi\in\mathbb{R}^3 : |\xi| < r(t)\}$. Taking Fourier transformation to (6.2), we use (6.4) to deduce that

$$\hat{w}(\xi) = \int_0^t e^{-\eta(t-s)|\xi|^2}(\widehat{b\cdot\nabla u} - \widehat{u\cdot\nabla b})(s)\mathrm{d}s. \tag{6.9}$$

Thus, by repeating the manipulation of (5.7) or (5.10), we apply (1.8) to obtain

$$\begin{aligned}|\hat{w}(\xi)| &\le c|\xi|\int_0^t e^{-\eta(t-s)|\xi|^2}(\|u(s)\|_{L^{3/2,\infty}}+\|\nabla u(s)\|)\|b(s)\|\mathrm{d}s\\ &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})|\xi|\Big[\int_0^t e^{-\eta(t-s)|\xi|^2}(1+s)^{-\frac{9}{4}}\mathrm{d}s\\ &\quad+\Big(\int_0^t e^{-2\eta(t-s)|\xi|^2}(1+s)^{-\frac{3}{2}}\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^t\|\nabla u(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big]\\ &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})|\xi|\Big[\int_0^t(1+s)^{-\frac{9}{4}}\mathrm{d}s+\Big(\int_0^t(1+s)^{-\frac{3}{2}}\mathrm{d}s\Big)^{\frac{1}{2}}\Big]\\ &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})|\xi|.\end{aligned}\tag{6.10}$$

Substituting (6.10) into (6.8) and integrating the resultant equation in $[0,t]$, we see that

$$\begin{aligned}(1+t)^3\|\hat{w}(t)\|^2 &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})\Big[\int_0^t(1+s)^2\int_{|\xi|<r(s)}|\xi|^2\mathrm{d}\xi\mathrm{d}s+\int_0^t\|\nabla u(s)\|^{\frac{1}{3}}\mathrm{d}s\Big]\\ &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})\Big[\int_0^t(1+s)^{-\frac{1}{2}}\mathrm{d}s\\ &\quad+t^{\frac{5}{6}}\Big(\int_0^t\|\nabla u(s)\|^2\mathrm{d}s\Big)^{\frac{1}{6}}\Big]\\ &\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+t)^{\frac{5}{6}}.\end{aligned}\tag{6.11}$$

Thus, we finally obtain the desired upper decay rate for $w(t)$:

$$\|w(t)\|\le c(\nu,\eta,\|b_0\|_{L^1\cap L^2})(1+t)^{-\frac{13}{12}}.\tag{6.12}$$

This completes the proof of Theorem 1.4.

Acknowledgements

The authors would like to thank the referee for the carefully reading of the earlier version of the manuscript and insightful comments which greatly improved the scope of this paper. The first author was partially supported by NSFC Grant (#11701099), the second author was partially supported by Guangdong Basic and Applied Basic Research Foundation (#2020A1515110299).

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

REFERENCES

[1] R. Agapito, M. Schonbek, *Non-uniform decay of MHD equations with and without magnetic diffusion*, Commun. Part. Differ. Equ. 32 (2007) 1791-1812.

[2] D. Biskamp, *Nonlinear Magnetohydrodynamics*. Cambridge University Press, Cambridge, 1993.

[3] L. Brandolese, M. Schonbek, *Large time behavior of the Navier-Stokes flow*. Handbook of mathematical analysis in mechanics of viscous fluids, 579-645, Springer, 2018.

[4] J. Chemin, D. McCormick, J. Robinson, J. Rodrigo, *Local existence for the non-resistive MHD equations in Besov spaces*, Adv. Math. 286 (2016) 1-31.

[5] P. Davidson, *An Introduction to Magnetohydrodynamics*. Cambridge University Press, Cambridge, 2001.

[6] B. Dong, Y. Jia, J. Li, J. Wu, *Global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion*, J. Math. Fluid Mech. 20 (2018) 1541-1565.

[7] B. Dong, J. Li, J. Wu, *Global regularity for the 2D MHD equations with partial hyper-resistivity*, Int. Math. Res. Not. 14 (2019) 4261-4280.

[8] C. Fefferman, D. McCormick, J. Robinson, J. Rodrigo, *Higher order commutator estimates and local existence for the non-resistive MHD equations and related models*, J. Funct. Anal. 267 (2014) 1035-1056.

[9] G. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*. 2nd ed. Springer, Berlin, 2011.

[10] Y. Ji, W. Tan, *Global well-posedness of a 3D Stokes-Magneto equations with fractional magnetic diffusion*, Discrete Contin. Dyn. Syst. Ser. B. 26(6) (2021) 3271-3278.

[11] T. Kato, *Strong L^p-solutions of the Navier-Stokes equations in R^m, with applications to weak solutions*, Math. Z. 187 (1984) 471-480.

[12] L. Laudau, E. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed. Pergamon, New York, 1984.

[13] J. Leray, *Sur le mouvement dun liquide visqueux emplissant lespace*, Acta Math. 63 (1934) 193-248.

[14] D. McCormick, J. Robinson, J. Rodrigo, *Existence and uniqueness for a coupled parabolic-elliptic model with applications to magnetic relaxation*, Arch. Ration. Mech. Anal. 214 (2014) 503-523.

[15] H. Moffatt, *Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 1. Fundamentals*, J. Fluid Mech. 159 (1985) 359-378.

[16] H. Moffatt, *Relaxation routes to steady Euler flows of complex topology (2009)*, http://www2.warwick.ac.uk/fac/sci/maths/research/miraw/days/t3_d5_he/keith.pdf Slides of talk given during MIRaW Day, Weak Solutions of the 3D Euler Equations, University of Warwick, 8th June 2009

[17] T. Ogawa, S. Rajopadhye, M. Schonbek, *Energy decay for a weak solution of the Navier-Stokes equation with slowly varying external forces*, J. Funct. Anal. 144 (1997) 325-358.

[18] M. Oliver, E. Titi, *Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in R^n*, J. Funct. Anal. 172 (2000) 1-18.

[19] M. Schonbek, *L^2 decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. 88 (1985) 209-222.

[20] M. Schonbek, *Large time behaviour of solutions to the Navier-Stokes equations*, Commun. Part. Differ. Equ. 11 (1986) 733-763.

[21] M. Schonbek, T. Schonbek, E. Süli *Large-time behaviour of solutions to the magnetohydrodynamics equations*, Math. Ann. 304 (1996) 717-756.

[22] W. Tan, *On the global existence for a coupled parabolic-elliptic equations in three dimensions*, submitted.

Yingdan Ji and Wen Tan
School of Applied Mathematics
Guangdong University of Technology
Guangzhou 510520
China
E-mail: jiyingdan157@163.com

Wen Tan
E-mail: tanw@gdut.edu.cn

Accepted: 11 March 2021