J. Evol. Equ. 21 (2021), 2377–2411

1424-3199/21/022377-35, *published online* April 2, 2021
https://doi.org/10.1007/s00028-021-00687-1

Journal of Evolution Equations

Global existence of solutions of the time fractional Cahn–Hilliard equation in $\mathbb{R}^3$

HAILONG YE, QIANG LIU AND ZHI- MIN CHEN

Abstract. Cauchy problem for the Caputo-type time fractional Cahn–Hilliard equation in $\mathbb{R}^3$ is examined. The local existence and uniqueness of mild solutions and strong solutions are obtained for the initial data u_0 satisfying $u_0 - \bar{u} \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, where $\bar{u}$ is an equilibrium constant. The local solutions are extended globally if $u_0 - \bar{u}$ is small in $L^1(\mathbb{R}^3)$. These results are consistent with those of the traditional Cahn–Hilliard equation such as the property of mass conservation. However, extra difficulties arise in dealing with the singularity of Mittag-Leffler operators and non-Markovian property in the Caputo-type time fractional problem.

1. Introduction

In recent years, partial differential equations involving time fractional derivatives have attracted much attention since the time fractional derivatives can provide a nice instrument for the description of memory and hereditary properties of various materials and processes. These advantages of fractional models in comparison with classical integer-order models have offered strong motivation on many applications, such as dispersive anomalous diffusion problems [25,26], control engineering investigations [6,28], biological and medical systems [11,13,16,17], financial study [31] and image processing researches [14,20].

In present paper, we are interested in the following Cauchy problem for the time fractional Cahn–Hilliard equation in $\mathbb{R}^3$,

$$\begin{cases} {}_cD_t^\alpha u + \Delta^2 u + \Delta\varphi(u) = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(0) = u_0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, ${}_cD_t^\alpha$ denotes the Caputo fractional derivative operator of order α, Δ^2 is the biharmonic operator and φ is a smooth nonlinear function satisfying $\varphi(u) = O(1)|u - \bar{u}|^\sigma$ as $u \to \bar{u}$ with equilibrium constant $\bar{u} > 0$ and $\sigma = \frac{5}{3}$.

Mathematics Subject Classification: 35G25, 35K90
Keywords: Caputo fractional derivative, Cahn–Hilliard Equation, Mild solution, Strong solutions, Existence and uniqueness.
This work partially supported by research Grants: NSFC (Nos. 11701384, 11671155), NSF of GD (2020A1515010554) and Natural Science Foundation of SZU (No. 2017057).

Recall the classical Cahn–Hilliard equation

$$\partial_t u + \Delta^2 u + \Delta\varphi(u) = 0, \qquad \text{in } \mathbb{R}^3 \times (0, \infty) \tag{1.2}$$

arising in the study of phase separation on cooling binary solutions such as glasses, alloys and polymer mixtures [5]. Here, u is the relative concentration between two phases. The nonlinearity of the problem is defined by the function

$$\Phi(u) = -\int_0^u \varphi(s)\mathrm{d}s,$$

a double-well potential with two equal minima at $u^- \le u^+$ corresponding to the two pure phases. A typical example of such potential is $\Phi(u) = (1-u^2)^2$, and $u^- = -1$, $u^+ = 1$.

For the Cauchy problem of the classical Cahn–Hilliard equation (1.2), Caffarelli and Muler [4] assumed that the function $B(u) = \varphi(u) - u$ is Lipschitz continuous and equals to a constant outside a bounded interval with respect to u. They obtained the L^∞ bound of solutions with initial data $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Bricmont et al. [3] considered nonlinear stability and long-time asymptotic behaviors of solutions in $\mathbb{R}^1$ with the initial data close to the stationary solution $\tilde{u} = \tanh(\frac{x}{2})$. Liu et al. [23] obtained global solution existence and its asymptotic behaviors under the assumption that the nonlinear function $\varphi(u)$ satisfies a certain local growth condition at a fixed point u^- and $\|u_0 - u^-\|_{L^1}$ is small.

On the other hand, the theory of time fractional partial differential equations has been developed by many authors. Eidelman and Kochubei [9] constructed a fundamental solution of an equation with time fractional derivative and a uniformly elliptic operator with variable coefficients acting in the spatial variables. Wang et al. [37] obtained the existence of mild solutions and classical solutions to an equation with almost sectorial operators by constructing a pair of families of operators in terms of the generalized Mittag-Leffler-type functions and the properties of resolvent operators. Andrade et al. [1] studied several questions concerning to abstract fractional Cauchy problems, including the existence of local mild solutions for the problem, and its possible continuation to a maximal interval of existence. Carvalho-Neto and Planas [7] shown the existence and uniqueness of mild solutions to time fractional Navier–Stokes equations. Li et al. [21] investigated nonlinear fractional time-space generalized Keller–Segel equation. They obtain the existence and uniqueness of mild solutions and some other properties, such as the nonnegativity preservation, mass conservation and blowup behaviors. A general review on mild solutions of time fractional partial differential equations is presented in [8].

In the understanding of nonlinear phenomena for the anomalously subdiffusive transport behaviors in heterogeneous porous materials or memory effects of certain materials, numerical simulations for the time fractional Cahn–Hilliard equation have been obtained in [12,22,32,35,36,38,39]. For example, the study of numerical solutions was ever given by Tripathi et al. [36] by using the homotopy analysis method to a time fractional Cahn–Hilliard equation involving an advection and reaction term. Tang et al. [35] proved for the first time that the time-fractional phase-field models indeed admit an energy dissipation law of an integral type. In addition to the numerical study of time fractional Cahn–Hilliard equation, a theoretical study was given by Jan et al. [30] on the well-posedness and long-time behaviour for the non-isothermal Cahn–Hilliard equation with memory in a bounded domain of $\mathbb{R}^3$.

To the best of our knowledge, the well-posedness for the solutions of the time fractional Cahn–Hilliard equation in $\mathbb{R}^3$ in L^p spaces is not clear at present. The main purpose of this paper is to show some rigorous analytical theory on the problem (1.1), including the existence and uniqueness of mild solutions and strong solutions. Due to the observation that the biharmonic operator in the linear part of problem (1.1) can be regarded as a sectorial operator on some spaces, we follow some ideas in [8], properly adapted to our problem. It's worth mentioning that the singularity together with strong nonlinearity arising from the time fractional derivative ${}_cD_t^\alpha u$ and the nonlinear term $\Delta\varphi(u)$ make its mathematical analysis more difficult in comparison with the classical Cahn–Hilliard equation (1.2). For example, the Mittag-Leffler operators $E_\alpha(-t^\alpha\mathcal{A})$ and $E_{\alpha,\alpha}(-t^\alpha\mathcal{A})$ (see (2.7) below) do not satisfy the semigroup properties. So we need to overcome these essential difficulties to get some a priori estimates and extend the local solution to a global one. For more details, one can refer to Sects. 4 and 5.

This paper is organized as follows. In the next section, we introduce some elementary properties of Mittag-Leffler operators, which are essential throughout the whole paper and give the main results of this paper. In Sect. 3, $L^p - L^q$ estimates are given for the associated linear homogeneous problem. In Sect. 4, we first establish the existence and uniqueness of local mild solutions of the Cauchy problem (1.1) by the Banach contraction mapping principle, and then extend the local solution globally. Section 5 is dedicated to show that the global mild solution obtained in Sect. 4 is a strong solution.

2. Preliminaries and main results

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions and $\mathring{L}^p(\mathbb{R}^3)$ be the subspace of $L^p(\mathbb{R}^3)$ so that $\mathcal{S}(\mathbb{R}^3)$ is dense in $\mathring{L}^p(\mathbb{R}^3)$ for $1 \le p \le \infty$. It is readily seen that $\mathring{L}^p(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ and $\mathring{L}^\infty(\mathbb{R}^3) = C_0(\mathbb{R}^3)$ which denote the space of all continuous functions decaying to zero at infinity. For convenience, we always assume $1 \le p \le \infty$ for the symbol $\mathring{L}^p(\mathbb{R}^3)$. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ and inverse Fourier transform $\mathcal{F}^{-1}f = \check{f}$ are defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x \quad \text{and} \quad \check{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} f(x)e^{ix\cdot\xi}\,\mathrm{d}x.$$

The following lemma about the continuity for the shift functions is followed by Theorem 4.26 in [2].

Lemma 2.1. *For any fixed $f \in \mathring{L}^p(\mathbb{R}^3)$, $\|f(x+h)\|_{L^p(\mathbb{R}^3)}$ is uniformly continuous with respect to $h \in \mathbb{R}^3$.*

Then we get the uniform continuity of a convolution product.

Lemma 2.2. *Let* $g \in \mathcal{S}(\mathbb{R}^3)$ *and* $f \in \mathring{L}^p(\mathbb{R}^3)$. *Then* $g * f$ *is uniformly continuous. Moreover,* $\lim_{|x|\to\infty} g * f(x) = 0$.

Proof. Take $f \in \mathring{L}^p(\mathbb{R}^3)$. Applying Hölder inequality yields

$$\begin{aligned} |g * f(x+h) - g * f(x)| &\le \int_{\mathbb{R}^3} g(y)|f(x-y+h) - f(x-y)|\,\mathrm{d}y \\ &\le \|g\|_{L^q(\mathbb{R}^3)} \cdot \|f(x+h) - f(x)\|_{L^p(\mathbb{R}^3)}, \end{aligned}$$

where $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then Lemma 2.1 implies the uniform continuity of $g * f$. Since $g \in \mathcal{S}(\mathbb{R}^3)$, it is easy to check that $g * f$ vanishes at infinity. □

It is clear that the solution of the following linear problem

$$\begin{cases} \partial_t u + \Delta^2 u = 0, & (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u_0(x) \end{cases} \tag{2.1}$$

can be written as

$$u(x,t) = G_t * u_0,$$

where $G_t(x) = \mathcal{F}^{-1}(e^{-|\xi|^4 t})$ is the fundamental solution of $\partial_t u + \Delta^2 u = 0$. For any fixed $t > 0$, we define the operator $T(t)$ on $L^p(\mathbb{R}^3)$ by

$$T(t)f := G_t * f.$$

By a simple calculation ([23]), we could get $\|G_t\|_{L^1(\mathbb{R}^3)} = C$ for $t > 0$, where $C > 0$ is independent of t. Then the Young's inequality yields

$$\|T(t)f\|_{L^p(\mathbb{R}^3)} \le \|G_t\|_{L^1(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)} = C\|f\|_{L^p(\mathbb{R}^3)}. \tag{2.2}$$

So the operator $T(t) : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$ are well defined for any fixed $t > 0$. Moreover, we have the following result.

Proposition 2.1. ([10], Proposition 2.13) *The operators* $\{T(t); t > 0\}$ *with* $T(0) = I$ *form a strongly continuous semigroup on* $\mathring{L}^p(\mathbb{R}^3)$, *and its generator* $\mathcal{A}$ *coincides with the closure of the biharmonic operator*

$$(-\Delta)^2 f(x) = \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i^2} \frac{\partial^2}{\partial x_j^2} f(x)$$

defined for every $f \in \mathcal{S}(\mathbb{R}^3)$.

Let $T > 0$ and X be a Banach space. We introduce the definitions of fractional integration and derivation as follows.

Definition 2.1. Let $0 < \alpha < 1$ and $v \in L^1(0, T; X)$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is given by

$$J_t^\alpha v(t) := (g_\alpha * v)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s,$$

where Γ is the Gamma function and

$$g_\alpha(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Based on the Riemann-Liouville fractional integral operator, we present the Caputo fractional differential operator.

Definition 2.2. Let $0 < \alpha < 1$ and $v \in C([0, T]; X)$ such that the convolution $g_{1-\alpha} * v \in W^{1,1}(0, T; X)$. The Caputo fractional derivative of order α of v is defined by

$$ {}_cD_t^\alpha v(t) := \frac{\mathrm{d}}{\mathrm{d}t} J_t^{1-\alpha}(v - v(0))(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha}(v(s) - v(0))\,\mathrm{d}s. \tag{2.3}$$

Let's recall some properties of Mittag-Leffler operators. For $\alpha \in (0, 1)$, we denote the entire function $M_\alpha : \mathbb{C} \to \mathbb{C}$ the Mainardi function by

$$M_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1 - \alpha(1+n))},$$

which is a particular case of the Wright type function introduced by Mainardi in [24] in order to characterize the fundamental solutions for some standard boundary value problems in physics. The following classical result gives some essential relations used in this paper to obtain the main estimates.

Proposition 2.2. ([7], Proposition 2) *Let $0 < \alpha < 1$ and $-1 < \gamma < \infty$. If we restrict M_α to the positive real line, then it holds that*

$$M_\alpha(t) \geq 0 \ \textit{for all}\ t \geq 0 \ \ \textit{and} \ \int_0^\infty t^\gamma M_\alpha(t)\,\mathrm{d}t = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha\gamma+1)}.$$

For each $\alpha \in (0, 1)$, the Mittag-Leffler families is given by

$$E_\alpha(-t^\alpha \mathcal{A}) := \int_0^\infty M_\alpha(s) T(st^\alpha)\,\mathrm{d}s,$$

and

$$E_{\alpha,\alpha}(-t^\alpha \mathcal{A}) := \int_0^\infty \alpha s M_\alpha(s) T(st^\alpha)\,\mathrm{d}s.$$

It is interesting to notice that the Mainardi functions act as a bridge between the fractional and the classical abstract theories. Indeed we have

$$E_\alpha(-t^\alpha \mathcal{A}) = \frac{1}{2\pi i}\int_{H_a} e^{\lambda t}\lambda^{\alpha-1}\left(\lambda^\alpha + \mathcal{A}\right)^{-1} d\lambda, \tag{2.4}$$

and

$$E_{\alpha,\alpha}(-t^\alpha \mathcal{A}) = \frac{t^{1-\alpha}}{2\pi i}\int_{H_a} e^{\lambda t}\left(\lambda^\alpha + \mathcal{A}\right)^{-1} d\lambda, \tag{2.5}$$

where H_a is any Hankel's path, i.e. a loop which starts and ends at $-\infty$ and encircles the circular disk $|\lambda| \le |z|^{1/\alpha}$. For more details see [8,37].

Similarly to [37], we have the following continuity property.

Proposition 2.3. *$E_\alpha(-t^\alpha \mathcal{A})$ and $E_{\alpha,\alpha}(-t^\alpha \mathcal{A})$ are well defined from $\mathring{L}^p(\mathbb{R}^3)$ to $\mathring{L}^p(\mathbb{R}^3)$. For $t > 0$, $E_\alpha(-t^\alpha \mathcal{A})$ and $E_{\alpha,\alpha}(-t^\alpha \mathcal{A})$ are continuous in the uniform operator topology on $\mathring{L}^p(\mathbb{R}^3)$. Moreover, for every $r > 0$, the continuity is uniform on $[r,\infty)$.*

Proof. Taking $v \in \mathring{L}^p(\mathbb{R}^3)$, we have

$$\|E_\alpha(-t_2^\alpha \mathcal{A})v - E_\alpha(-t_1^\alpha \mathcal{A})v\|_{L^p(\mathbb{R}^3)} \le \int_0^\infty M_\alpha(s)\|T(st_2^\alpha)v - T(st_1^\alpha)v\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}s,$$

where $1 \le p \le \infty$. Let $\varepsilon > 0$ and $r > 0$. By the properties of the Mainardi function given in Proposition 2.2, we may choose $0 < \delta_1 < \delta_2$ such that

$$2\|v\|_{L^p(\mathbb{R}^3)}\int_0^{\delta_1} M_\alpha(s)\,\mathrm{d}s < \frac{\varepsilon}{3} \quad \text{and} \quad 2\|v\|_{L^p(\mathbb{R}^3)}\int_{\delta_2}^\infty M_\alpha(s)\,\mathrm{d}s < \frac{\varepsilon}{3}.$$

In addition, by the strongly continuity and the uniform boundedness of $T(t)$, we deduce that there exists a positive constant $\gamma > 0$ such that

$$\int_{\delta_1}^{\delta_2} M_\alpha(s)\|T(st_1^\alpha) - T(st_2^\alpha)\|\|v\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}s < \frac{\varepsilon}{3}$$

for any $t_1, t_2 \ge r$ and $|t_2 - t_1| < \gamma$. Therefore, combining with Proposition 3.1 below, we obtain that

$$\|E_\alpha(-t_2^\alpha \mathcal{A}) - E_\alpha(-t_1^\alpha \mathcal{A})\| < \varepsilon,$$

which implies that $E_\alpha(-t^\alpha \mathcal{A})$ is uniformly continuous on $[r,\infty)$ in the uniform operator topology on $\mathring{L}^p(\mathbb{R}^3)$. By the arbitrariness of $r > 0$, $E_\alpha(-t^\alpha \mathcal{A})$ is continuous in the uniform operator topology for $t > 0$. A similar argument enables us to give the characterization of continuity on $E_{\alpha,\alpha}(-t^\alpha \mathcal{A})$. □

We end this section with the concept of the solution of (1.1) and our main results. For a given function f and a smooth function $u : \mathbb{R}^3 \times \mathbb{R}^+ \to X$, we formally give a mild solution for the Cauchy problem

$$\begin{cases} {}_cD_t^\alpha u = -\mathcal{A}u + f(x,t), & t > 0 \\ u(x,0) = u_0(x) \in X, \end{cases} \tag{2.6}$$

where the rigorous deduction could be found in [8,15,18,37]. Denote $\mathcal{L}$ the operator of Laplace transform. By the convolution property of Laplace transform, we have

$$\mathcal{L}({}_cD_t^\alpha u) = \lambda^\alpha \mathcal{L}(u) - \lambda^{\alpha-1}u_0,$$

as in [18, Proposition 3.13]. So taking the Laplace transform to (2.6) gives

$$\mathcal{L}(u) = \lambda^{\alpha-1}(\lambda^\alpha + \mathcal{A})^{-1}u_0 + (\lambda^\alpha + \mathcal{A})^{-1}\mathcal{L}(f).$$

Application of Laplace inversion ([8, Proposition 2.43]) implies

$$u(t) = E_\alpha\left(-t^\alpha \mathcal{A}\right)u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(t-s)^\alpha\mathcal{A}\right)f(s)\mathrm{d}s.$$

This formal computation then motivates the definition of the mild solution of (1.1) as follows:

Definition 2.3. Let $0 < \alpha < 1$ and $T > 0$.

(i) A function u such that $u - \bar{u} \in C([0, T]; L^1(\mathbb{R}^3))$ defined by

$$u(x,t) = E_\alpha(-t^\alpha\mathcal{A})u_0 - \int_0^t (t-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\,\mathrm{d}s \tag{2.7}$$

is called a local mild solution of the Cauchy problem (1.1).

(ii) If $T = \infty$, we say that u is a global mild solution of the Cauchy problem (1.1).

Now we are the position to give the first result in this paper.

Theorem 2.1. (Global existence of mild solution) *Assume that $u_0 - \bar{u} \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ with sufficiently small $\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}$ and the nonlinear function $\varphi(u) \in C^2(\mathbb{R})$ satisfies $\varphi(u) = O(1)|u-\bar{u}|^\sigma$ as $u \to \bar{u}$ with $\sigma = \frac{5}{3}$. Then the Cauchy problem (1.1) admits a unique global mild solution u satisfying*

$$\|u(t) - \bar{u}\|_{L^\infty(\mathbb{R}^3)} \le 2\|u_0 - \bar{u}\|_{L^\infty(\mathbb{R}^3)}, \quad t \ge 0$$

and the integral preservation property,

$$\int_{\mathbb{R}^3}(u(x,t) - \bar{u})\,\mathrm{d}x = \int_{\mathbb{R}^3}(u_0(x) - \bar{u})\,\mathrm{d}x.$$

The second result of this paper is concerned with the global existence of the strong solutions, of which we give the definition as follows.

Definition 2.4. Let $0 < \alpha < 1$ and $T > 0$. By a strong solution to problem (1.1), we mean that a function u such that $u - \bar{u} \in C([0, T]; L^1(\mathbb{R}^3)) \cap C((0, T]; W^{4,1}(\mathbb{R}^3))$ and ${}_cD_t^\alpha u \in C((0, T]; L^1(\mathbb{R}^3))$ satisfies (1.1) almost everywhere. If $T = \infty$, we call u a global strong solution.

If the unique mild solution u satisfies $u - \bar{u} \in C((0, T]; W^{4,1}(\mathbb{R}^3))$, it's easy to verify that $u - \bar{u} \in C^\alpha((0, T]; L^1(\mathbb{R}^3))$ by [19, Proposition 3.3]. Naturally, we expect that

$$ {}_cD_t^\alpha u(t) = {}_cD_t^\alpha (u - \bar{u})(t) = \frac{\mathrm{d}}{\mathrm{d}t} J_t^{1-\alpha}(u - u(0))(t) $$

exists at least almost everywhere with respect to $t \in (0, \infty)$. However, this expectation may not be true (see [33]). That is to say, ${}_cD_t^\alpha u$ may be just a distribution. This disadvantage is another difficulty arises from the singularity of the time fractional derivative. In order to prove that the mild solution u is a strong solution, we have to employ other methods to show that

$$ {}_cD_t^\alpha u \in C((0, T]; L^1(\mathbb{R}^3)). $$

Theorem 2.2. (Global existence of strong solution) *Assume that the assumptions in Theorem 2.1 are satisfied and $\varphi(u) \in C^3(\mathbb{R})$, then the global mild solution u of the Cauchy problem (1.1) is in fact a global strong solution and satisfies $u - \bar{u} \in C((0, \infty); W^{4,p}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3))$ for each $1 \le p < \infty$.*

Remark 2.1. Although the L^1-norm of $u(x, t) - \bar{u}$ is small in our results, the L^∞-norm of $u(x, t) - \bar{u}$ is not necessarily small.

Remark 2.2. The existence and uniqueness of the solution obtained this paper can be easily extended to the case of $\mathbb{R}^N$ with $N = 1, 2$. While for larger N, due to the limitation of $L^p - L^q$ estimates of Mittag-Leffler operators, we can't get the same results now.

Remark 2.3. If we take the usual Cahn–Hilliard potential $\Phi(u) = (1 - u^2)^2$ and set $\bar{u} = 1$, then we have $\varphi(u) = u - u^3$. As mentioned in [23], the equation (1.1) could be rewritten as

$$ {}_cD_t^\alpha u(x, t) + \Delta^2 u(x, t) - 2\Delta u + \Delta(1 - u)^2(u + 2) = 0. $$

By taking $\mathcal{A} = \Delta^2 - 2\Delta$, we could obtain the same results as Theorem 2.1 and Theorem 2.2 since $(1 - u)^2(u + 2) = O(1)|u - 1|^{\frac{5}{3}}$ as $u \to 1$.

3. $L^p - L^q$ estimates

We first recall the $L^p - L^q$ estimates for the operator $T(t)$ in the following result. For convenience, we will usually use C to denote a generic positive constant which may vary from line to line.

Proposition 3.1. ([27], Lemma 3.1) *Let $1 \le q \le p \le \infty$ and $\nu \ge 0$. For any $u \in L^q(\mathbb{R}^3)$, we have*

$$ \|(-\Delta)^{\frac{\nu}{2}} T(t) u\|_{L^p(\mathbb{R}^3)} \le C t^{-\frac{3}{4}(\frac{1}{q} - \frac{1}{p}) - \frac{\nu}{4}} \|u\|_{L^q(\mathbb{R}^3)}, \quad t > 0, $$

where $(-\Delta)^{\frac{\nu}{2}} = \mathcal{F}^{-1}(|\xi|^\nu)$ denote the differential operator of ν-th order.

Now we show the similar $L^p - L^q$ estimates for both families of Mittag-Leffler operators.

Proposition 3.2. *Let* $1 \leq q \leq p \leq \infty$ *and* $\frac{1}{q} - \frac{1}{p} < \frac{4-\nu}{3}$ *with* $\nu \in [0, 4)$. *For* $u \in L^q(\mathbb{R}^3)$*, we have*

$$\|(-\Delta)^{\frac{\nu}{2}} E_\alpha(-t^\alpha \mathcal{A})u\|_{L^p(\mathbb{R}^3)} \leq C t^{-\frac{3\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{\alpha\nu}{4}} \|u\|_{L^q(\mathbb{R}^3)}, \quad t > 0.$$

Moreover, if $\nu = 4$*, for* $u \in \mathring{L}^q(\mathbb{R}^3)$*, we have*

$$\|(-\Delta)^2 E_\alpha(-t^\alpha \mathcal{A})u\|_{L^q(\mathbb{R}^3)} \leq C t^{-\alpha} \|u\|_{L^q(\mathbb{R}^3)}, \quad t > 0.$$

Proof. By Proposition 2.2 and 3.1, we obtain

$$\begin{aligned}
&\|(-\Delta)^{\frac{\nu}{2}} E_\alpha(-t^\alpha \mathcal{A})u\|_{L^p(\mathbb{R}^3)} \\
&\leq \int_0^\infty M_\alpha(s) \|(-\Delta)^{\frac{\nu}{2}} T(st^\alpha)u\|_{L^p(\mathbb{R}^3)} \, \mathrm{d}s \\
&\leq C \left(\int_0^\infty M_\alpha(s) s^{-\frac{3}{4}(\frac{1}{q}-\frac{1}{p})-\frac{\nu}{4}} \, \mathrm{d}s \right) \left(t^{-\frac{3\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{\alpha\nu}{4}} \|u\|_{L^q(\mathbb{R}^3)} \right) \\
&\leq C t^{-\frac{3\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{\alpha\nu}{4}} \|u\|_{L^q(\mathbb{R}^3)}, \quad \nu \in [0, 4).
\end{aligned}$$

If $\nu = 4$, taking the ν-th derivative of $E_\alpha(-t^\alpha \mathcal{A})u$, we have

$$\begin{aligned}
(-\Delta)^{\frac{\nu}{2}} E_\alpha(-t^\alpha \mathcal{A})u &= (-\Delta)^{\frac{\nu}{2}} \int_0^\infty M_\alpha(s) T(st^\alpha)u \, \mathrm{d}s \\
&= (-\Delta)^{\frac{\nu}{2}} \int_0^1 (M_\alpha(s) - M_\alpha(0)) \, T(st^\alpha)u \, \mathrm{d}s \\
&\quad + M_\alpha(0)(-\Delta)^{\frac{\nu}{2}} \int_0^1 T(st^\alpha)u \, \mathrm{d}s + (-\Delta)^{\frac{\nu}{2}} \int_1^\infty M_\alpha(s) T(st^\alpha)u \, \mathrm{d}s \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.1}$$

By Proposition 2.2 and 3.1, it's easy to see that

$$\begin{aligned}
\|I_1\|_{L^q} &\leq \int_0^1 |M_\alpha(s) - M_\alpha(0)| \cdot \|(-\Delta)^{\frac{\nu}{2}} T(st^\alpha)u\|_{L^q(\mathbb{R}^3)} \, \mathrm{d}s \\
&\leq C \left(\int_0^1 |M_\alpha(s) - M_\alpha(0)| \, s^{-1} \, \mathrm{d}s \right) \left(t^{-\alpha} \|u\|_{L^q(\mathbb{R}^3)} \right) \\
&\leq C t^{-\alpha} \|u\|_{L^q(\mathbb{R}^3)},
\end{aligned} \tag{3.2}$$

where we use the fact that

$$\sup_{0<s\leq 1} |M_\alpha(s) - M_\alpha(0)| s^{-1} \leq C.$$

Similarly, we have

$$\begin{aligned}\|I_3\|_{L^q} &\le \int_1^\infty M_\alpha(s)\|(-\Delta)^{\frac{\nu}{2}}T(st^\alpha)u\|_{L^q(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le C\left(\int_1^\infty M_\alpha(s)s^{-1}\,\mathrm{d}s\right)t^{-\alpha}\|u\|_{L^q(\mathbb{R}^3)}\\ &\le C\,t^{-\alpha}\|u\|_{L^q(\mathbb{R}^3)}.\end{aligned} \tag{3.3}$$

To estimate I_2, we see that for any $u\in \mathring{L}^q(\mathbb{R}^3)$,

$$\frac{\mathrm{d}}{\mathrm{d}s}T(s)u=-\mathcal{A}T(s)u,\quad s>0.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}s}T(st^\alpha)u=-t^\alpha\mathcal{A}T(st^\alpha)u,\quad s,t>0.$$

Integrating the above equality with respect to s over $[0, 1]$, we arrive at

$$T(t^\alpha)u-u=-t^\alpha\int_0^1\mathcal{A}T(st^\alpha)u\,\mathrm{d}s.$$

Then, we obtain

$$\|I_2\|_{L^q(\mathbb{R}^3)}\le |M_\alpha(0)|\left\|\int_0^1\mathcal{A}T(st^\alpha)u\,\mathrm{d}s\right\|_{L^q(\mathbb{R}^3)}\le C\,t^{-\alpha}\|u\|_{L^q(\mathbb{R}^3)}. \tag{3.4}$$

Inserting estimates (3.2)–(3.4) into (3.1), we complete the proof of Proposition 3.2. □

Proposition 3.3. *Let* $1\le q\le p\le\infty$ *and* $\frac{1}{q}-\frac{1}{p}<\frac{8-\nu}{3}$ *with* $\nu\in[0,8)$. *Then for* $u\in L^q(\mathbb{R}^3)$*, we have*

$$\|(-\Delta)^{\frac{\nu}{2}}E_{\alpha,\alpha}(-t^\alpha\mathcal{A})u\|_{L^p(\mathbb{R}^3)}\le C\,t^{-\frac{3\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{\alpha\nu}{4}}\|u\|_{L^q(\mathbb{R}^3)},\quad t>0.$$

Moreover, if $\nu=8$*, for* $u\in\mathring{L}^q(\mathbb{R}^3)$*, we have*

$$\|(-\Delta)^4E_{\alpha,\alpha}(-t^\alpha\mathcal{A})u\|_{L^q(\mathbb{R}^3)}\le C\,t^{-2\alpha}\|u\|_{L^q(\mathbb{R}^3)},\quad t>0.$$

Proof. The proof is similar with that of Proposition 3.2. By Proposition 2.2 and 3.1, we obtain

$$\begin{aligned}&\|(-\Delta)^{\frac{\nu}{2}}E_{\alpha,\alpha}(-t^\alpha\mathcal{A})u\|_{L^p(\mathbb{R}^3)}\\ &\quad\le \int_0^\infty \alpha s M_\alpha(s)\|(-\Delta)^{\frac{\nu}{2}}T(st^\alpha)u\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}s\\ &\quad\le C\left(\int_0^\infty M_\alpha(s)s^{1-\frac{3}{4}(\frac{1}{q}-\frac{1}{p})-\frac{\nu}{4}}\,\mathrm{d}s\right)\left(t^{-\frac{3\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{\alpha\nu}{4}}\|u\|_{L^q(\mathbb{R}^3)}\right)\\ &\quad\le C\,t^{-\frac{3\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{\alpha\nu}{4}}\|u\|_{L^q(\mathbb{R}^3)},\quad \nu\in[0,8).\end{aligned}$$

If $\nu = 8$, taking the ν-th derivative of $E_{\alpha,\alpha}(-t^\alpha \mathcal{A})u$, we have

$$
\begin{aligned}
(-\Delta)^{\frac{\nu}{2}} E_{\alpha,\alpha}(-t^\alpha \mathcal{A})u &= (-\Delta)^{\frac{\nu}{2}} \int_0^\infty \alpha s M_\alpha(s) T(st^\alpha)u \, \mathrm{d}s \\
&= (-\Delta)^{\frac{\nu}{2}} \int_0^1 \alpha s \left(M_\alpha(s) - M_\alpha(0)\right) T(st^\alpha)u \, \mathrm{d}s \\
&\quad + M_\alpha(0)(-\Delta)^{\frac{\nu}{2}} \int_0^1 \alpha s T(st^\alpha)u \, \mathrm{d}s \\
&\quad + (-\Delta)^{\frac{\nu}{2}} \int_1^\infty \alpha s M_\alpha(s) T(st^\alpha)u \, \mathrm{d}s \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.5}
$$

The estimates for I_1 and I_3 can be derived similarly as (3.2) and (3.3). To estimate I_2, we see that for any $u \in \mathring{L}^q(\mathbb{R}^3)$,

$$
st^\alpha \mathcal{A} \frac{\mathrm{d}}{\mathrm{d}s} T(st^\alpha)u = -st^{2\alpha} \mathcal{A}^2 \, T(st^\alpha)u, \quad s, t > 0.
$$

By the theory of analytic semigroups (see [10,29]), it is not difficult to verify that $\{s\mathcal{A}T(s);\ s > 0\}$ is strongly continuous on $\mathring{L}^q(\mathbb{R}^3)$ with respect to $s \in [0, \infty)$. Integrating the above equality with respect to s over $[0, 1]$ and integrating by parts, we arrive at

$$
\begin{aligned}
-t^{2\alpha} \int_0^1 s\mathcal{A}^2 \, T(st^\alpha)u \, \mathrm{d}s &= \int_0^1 st^\alpha \mathcal{A} \frac{\mathrm{d}}{\mathrm{d}s} T(st^\alpha)u \, \mathrm{d}s \\
&= t^\alpha \mathcal{A}T(t^\alpha)u - u - \int_0^1 t^\alpha \mathcal{A}T(st^\alpha)u \, \mathrm{d}s
\end{aligned}
$$

Then, by the estimate (3.4) and Proposition 3.1, we obtain

$$
\begin{aligned}
\|I_2\|_{L^q(\mathbb{R}^3)} &= \left\| \alpha M_\alpha(0) \int_0^1 s\mathcal{A}^2 \, T(st^\alpha)u \, \mathrm{d}s \right\|_{L^q(\mathbb{R}^3)} \\
&\le C\, t^{-2\alpha} \|t^\alpha \mathcal{A}T(t^\alpha)u\|_{L^q(\mathbb{R}^3)} + C\, t^{-2\alpha} \|u\|_{L^q(\mathbb{R}^3)} \\
&\quad + C\, t^{-2\alpha} \left\| \int_0^1 t^\alpha \mathcal{A}T(st^\alpha)u \, \mathrm{d}s \right\|_{L^q(\mathbb{R}^3)} \\
&\le C\, t^{-2\alpha} \|u\|_{L^q(\mathbb{R}^3)}.
\end{aligned} \tag{3.6}
$$

Inserting the estimates for I_1, I_2, I_3 into (3.5), we complete the proof of Proposition 3.3. □

Remark 3.1. It is a known fact that for unbounded generators of resolvent families, the solution becomes smoother at most by one unit of regularity in terms of the generator, see the monograph by Jan Pruss [29], which prevents us from getting more smooth solutions of the problem (1.1). In addition, the non-integrability of Mainardi function M_α with $\gamma \le -1$ implies that we can't obtain the estimate for $(-\Delta)^{\frac{\nu}{2}} E_\alpha(-t^\alpha \mathcal{A})u$ with $\nu > 4$, which also implies this disadvantage.

4. Mild solution

In this section, we shall construct a local mild solution u for the Cauchy problem (1.1) by employing Banach contracting mapping principle, and then extend u globally by virtue of the auxiliary estimate (4.6) below.

Proposition 4.1. ([10,29]) *For any fixed $\nu > 0$, $\{(-\Delta)^{\frac{\nu}{2}}T(t);\ t > 0\}$ is strongly continuous on $\mathring{L}^p(\mathbb{R}^3)$ with respect to $t \in (0, \infty)$.*

Based on Proposition 4.1, we can obtain the strong continuity of both families of Mittag-Leffler operators.

Proposition 4.2. *$(-\Delta)^{\frac{\nu_1}{2}} E_\alpha(-t^\alpha \mathcal{A})$ and $(-\Delta)^{\frac{\nu_2}{2}} E_{\alpha,\alpha}(-t^\alpha \mathcal{A})$ are strongly continuous on $\mathring{L}^p(\mathbb{R}^3)$ with respect to $t \in (0, \infty)$, where $\nu_1 \in (0, 4]$ and $\nu_2 \in (0, 8]$.*

Proof. Taking $v \in \mathring{L}^p(\mathbb{R}^3)$ and $t_0 \in (0, \infty)$, we have

$$\begin{aligned}
&\|(-\Delta)^{\frac{\nu_1}{2}} E_\alpha(-t^\alpha \mathcal{A})v - (-\Delta)^{\frac{\nu_1}{2}} E_\alpha(-t_0^\alpha \mathcal{A})v\|_{L^p(\mathbb{R}^3)} \\
&\leq \int_0^\infty M_\alpha(s)\|(-\Delta)^{\frac{\nu_1}{2}} T(st^\alpha)v - (-\Delta)^{\frac{\nu_1}{2}} T(st_0^\alpha)v\|_{L^p(\mathbb{R}^3)} \mathrm{d}s,
\end{aligned}$$

where $1 \leq p \leq \infty$ and $\nu_1 \in (0, 4]$. For any $\varepsilon > 0$ and $\lambda \in (0, t_0)$, by Proposition 3.1 and the properties of the Mainardi function given in Proposition 2.2, we may choose $0 < \delta_1 < \delta_2$ such that

$$2C\lambda^{-\frac{\alpha\nu_1}{4}}\|v\|_{L^p(\mathbb{R}^3)} \int_0^{\delta_1} M_\alpha(s)s^{-\frac{\nu_1}{4}}\,\mathrm{d}s < \frac{\varepsilon}{3}$$

and

$$2C\lambda^{-\frac{\alpha\nu_1}{4}}\|v\|_{L^p(\mathbb{R}^3)} \int_{\delta_2}^{\infty} M_\alpha(s)s^{-\frac{\nu_1}{4}}\,\mathrm{d}s < \frac{\varepsilon}{3},$$

where the positive constant C comes from Proposition 3.1 which is independent of λ and ε. Then by the strong continuity of $(-\Delta)^{\frac{\nu_1}{2}}T(t)$, we deduce that there exists a positive constant $\gamma > 0$ such that

$$\int_{\delta_1}^{\delta_2} M_\alpha(s)\|(-\Delta)^{\frac{\nu_1}{2}} T(st^\alpha)v - (-\Delta)^{\frac{\nu_1}{2}} T(st_0^\alpha)v\|_{L^p(\mathbb{R}^3)} \mathrm{d}s < \frac{\varepsilon}{3}$$

for $|t - t_0| < \gamma$ and $t, t_0 > \lambda$. Therefore, we obtain that

$$\lim_{t\to t_0} \|(-\Delta)^{\frac{\nu_1}{2}} E_\alpha(-t^\alpha \mathcal{A})v - (-\Delta)^{\frac{\nu_1}{2}} E_\alpha(-t_0^\alpha \mathcal{A})v\|_{L^p(\mathbb{R}^3)} = 0,$$

which implies that $(-\Delta)^{\frac{\nu_1}{2}} E_\alpha(-t^\alpha \mathcal{A})$ is strongly continuous in $(0, \infty)$. A similar argument enables us to give the strong continuity of $(-\Delta)^{\frac{\nu_2}{2}} E_{\alpha,\alpha}(-t^\alpha \mathcal{A})$ in $(0, \infty)$. $\square$

Now we are ready to show the local existence of mild solutions for the time fractional Cahn–Hilliard equation.

Theorem 4.1. *Assume that* $u_0 - \bar{u} \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ *with* $\|u_0 - \bar{u}\|_{L^\infty(\mathbb{R}^3)} \le r$ *for some* $r > 0$ *and* $\varphi(u) \in C^2(\mathbb{R})$*. Then there exists* $t_1 > 0$ *such that the Cauchy problem* (1.1) *admits a unique mild solution* u *satisfying* $u - \bar{u} \in C([0, t_1]; L^1(\mathbb{R}^3))$ *and*

$$\sup_{0\le t\le t_1} \|u(t) - \bar{u}\|_{L^1(\mathbb{R}^3)} \le 2\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}, \quad \|u(t) - \bar{u}\|_{L^\infty(\mathbb{R}^3)} \le 2r, \quad 0 \le t \le t_1. \tag{4.1}$$

Furthermore, the integral is preserved, that is,

$$\int_{\mathbb{R}^3} (u(x,t) - \bar{u})\,\mathrm{d}x = \int_{\mathbb{R}^3} (u_0(x) - \bar{u})\,\mathrm{d}x. \tag{4.2}$$

Proof. Noting that

$$T(t)1 = G_t * 1 = \int_{\mathbb{R}^3} G_t(x)\,\mathrm{d}x = (e^{|\xi|^4 t})|_{\xi=0} = 1, \quad t > 0,$$

by Proposition 2.2, we obtain

$$E_\alpha(-t^\alpha \mathcal{A})\bar{u} = \int_0^\infty M_\alpha(s) T(st^\alpha)\bar{u}\,\mathrm{d}s = \bar{u}.$$

Hence, the integro-differential equation (2.7) could be rewritten as

$$u(x,t) - \bar{u} = E_\alpha(-t^\alpha \mathcal{A})(u_0 - \bar{u}) - \int_0^t (t-s)^{\alpha-1} \Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\,\mathrm{d}s.$$

To employ the Banach contraction mapping principle, for some $t_1 > 0$, we define the set

$$\mathcal{E}_{t_1} = \Big\{ v \in C([0,t_1]; L^1(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}^3 \times (0,t_1)); \\ \sup_{0\le t\le t_1} \|v(t)\|_{L^1(\mathbb{R}^3)} \le 2\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)} \text{ and } \|v(t)\|_{L^\infty(\mathbb{R}^3\times(0,t_1))} \le 2r \Big\}$$

and a map on this set

$$S_{t_1} v(x,t) = E_\alpha(-t^\alpha \mathcal{A})(u_0 - \bar{u}) - \int_0^t (t-s)^{\alpha-1} \Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(v + \bar{u})\,\mathrm{d}s. \tag{4.3}$$

Indeed, we shall prove there exists only one fixed point $v(x,t)$ of the map S_{t_1}. Then $v(x,t) + \bar{u}$ is the mild solution. For this purpose, we need to show that S_{t_1} is a contraction mapping from $\mathcal{E}_{t_1}$ to itself for some small $t_1 > 0$. The following proof is divided into two steps.

(i). In the first step, we show that $S_{t_1}:\mathcal{E}_{t_1}\to\mathcal{E}_{t_1}$. By Propositions 3.2 and 3.3, we have

$$\begin{aligned}\|S_{t_1}v(t)\|_{L^\infty(\mathbb{R}^3)} &\le \|E_\alpha(-t^\alpha\mathcal{A})(u_0-\bar{u})\|_{L^\infty(\mathbb{R}^3)}\\ &\quad+\int_0^t (t-s)^{\alpha-1}\|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\left(\varphi(v(s)+\bar{u})-\varphi(\bar{u})\right)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le \|u_0-\bar{u}\|_{L^\infty}+Cb\int_0^t (t-s)^{\frac{\alpha}{2}-1}\|v(s)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le r+\frac{4r}{\alpha}Cbt_1^{\frac{\alpha}{2}},\quad 0<t\le t_1,\end{aligned}$$

where $b=\max_{u\in\bar{B}(\bar{u},2r)}\sum_{k=1}^{2}|D^k\varphi(u)|$. Obviously, if $t_1\le(\frac{\alpha}{8Cb})^{\frac{2}{\alpha}}$, then

$$\|S_{t_1}v\|_{L^\infty(\mathbb{R}^3\times(0,t_1))}\le 2r.$$

Similarly, for $0<t\le t_1$, we have

$$\begin{aligned}\|S_{t_1}v(t)\|_{L^1(\mathbb{R}^3)} &\le \|E_\alpha(-t^\alpha\mathcal{A})(u_0-\bar{u})\|_{L^1(\mathbb{R}^3)}\\ &\quad+\int_0^t (t-s)^{\alpha-1}\|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(v+\bar{u})\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le \|u_0-\bar{u}\|_{L^1(\mathbb{R}^3)}+Cb\int_0^t (t-s)^{\frac{\alpha}{2}-1}\|v(s)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le \|u_0-\bar{u}\|_{L^1(\mathbb{R}^3)}+\frac{4}{\alpha}Cbt_1^{\frac{\alpha}{2}}\|u_0-\bar{u}\|_{L^1(\mathbb{R}^3)}\\ &\le 2\|u_0-\bar{u}\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Now we show the continuity of $S_{t_1}v$ for $t\in[0,t_1]$. Fixing $t_0\in(0,t_1)$, we have

$$\begin{aligned}\|S_{t_1}v(t)-S_{t_1}v(t_0)\|_{L^1(\mathbb{R}^3)} &\le \|(E_\alpha(-t^\alpha\mathcal{A})-E_\alpha(-t_0^\alpha\mathcal{A}))(u_0-\bar{u})\|_{L^1(\mathbb{R}^3)}\\ &\quad+\int_{t_0}^t (t-s)^{\alpha-1}\left\|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(v(s)+\bar{u})\right\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\ &\quad+\int_0^{t_0}\Big\|(t-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(v(s)+\bar{u})\\ &\quad-(t_0-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t_0-s)^\alpha\mathcal{A})\varphi(v(s)+\bar{u})\Big\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\ &= I_1+I_2+I_3,\end{aligned}\tag{4.4}$$

where $t_0<t<t_1$. Using the strong continuity of $E_\alpha(-t^\alpha\mathcal{A})$ on $L^1(\mathbb{R}^3)$ for $t\in[0,\infty)$, we deduce easily that the first term I_1 goes to zero as $t\to t_0^+$. By Proposition 3.3, the estimate of the second term yields

$$\begin{aligned}I_2 &\le C\int_{t_0}^t (t-s)^{\frac{\alpha}{2}-1}\|\varphi(v(s)+\bar{u})-\varphi(\bar{u})\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le Cb\int_{t_0}^t (t-s)^{\frac{\alpha}{2}-1}\|v(s)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le 2Cb(t-t_0)^{\frac{\alpha}{2}}\|u_0-\bar{u}\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Obviously, this term vanishes as $t \to t_0^+$. For the estimate of the third term, we first denote

$$f(t,s) = \Big\| (t-s)^{\alpha-1} \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(v(s)+\bar{u}) \\ -(t_0-s)^{\alpha-1} \Delta E_{\alpha,\alpha}(-(t_0-s)^{\alpha}\mathcal{A})\varphi(v(s)+\bar{u}) \Big\|_{L^1(\mathbb{R}^3)},$$

where $0 < s < t_0 < t$. It is easy to see that

$$\begin{aligned} f(t,s) \le & \left|(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right| \cdot \left\| \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(v(s)+\bar{u}) \right\|_{L^1(\mathbb{R}^3)} \\ & + (t_0-s)^{\alpha-1} \big\| \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(v(s)+\bar{u}) \\ & \quad - \Delta E_{\alpha,\alpha}(-(t_0-s)^{\alpha}\mathcal{A})\varphi(v(s)+\bar{u}) \big\|_{L^1(\mathbb{R}^3)} \\ \le & \, Cb \left|(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right| \cdot (t-s)^{-\frac{\alpha}{2}} \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)} \\ & + (t_0-s)^{\alpha-1} \big\| \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(v(s)+\bar{u}) \\ & \quad - \Delta E_{\alpha,\alpha}(-(t_0-s)^{\alpha}\mathcal{A})\varphi(v(s)+\bar{u}) \big\|_{L^1(\mathbb{R}^3)}. \end{aligned}$$

Then combining with the strong continuity of $\Delta E_{\alpha,\alpha}(-t^{\alpha}\mathcal{A})$ on $L^1(\mathbb{R}^3)$ for $t \in (0,\infty)$, we have

$$\lim_{t\to t_0^+} f(t,s) = 0$$

for any fixed $s \in (0,t_0)$. In addition, for $0 < s < t_0 < t$, we have

$$\begin{aligned} I_3 &= \int_0^{t_0} f(t,s)\,\mathrm{d}s \\ &\le Cb \int_0^{t_0} (t_0-s)^{\frac{\alpha}{2}-1} \|v(s)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s \\ &\le Cb \int_0^{t_0} (t_0-s)^{\frac{\alpha}{2}-1}\,\mathrm{d}s \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}, \\ &\le Cbt_0^{\frac{\alpha}{2}} \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}. \end{aligned}$$

Applying dominated convergence theorem yields the third term I_3 also tends to zero as $t \to t_0^+$. Similarly, we can obtain the same limit as $t \to t_0^-$ with $t_0 \in (0,t_1]$. Thus,

$$\lim_{t\to t_0} \|S_{t_1}v(t) - S_{t_1}v(t_0)\|_{L^1(\mathbb{R}^3)} = 0, \quad t_0 \in (0,t_1].$$

Continuity up to $t = 0$ of $S_{t_1}v$ follows from the following estimate and the strong continuity of $E_\alpha(-t^\alpha\mathcal{A})$, we have

$$\begin{aligned}
&\|S_{t_1}v(t) - S_{t_1}v(0)\|_{L^1(\mathbb{R}^3)}\\
&\le \|(E_\alpha(-t^\alpha\mathcal{A}) - I)(u_0 - \bar{u})\|_{L^1(\mathbb{R}^3)}\\
&\quad + \int_0^t (t-s)^{\alpha-1} \left\|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(v(s)+\bar{u})\right\|_{L^1(\mathbb{R}^3)} \mathrm{d}s\\
&\le \|(E_\alpha(-t^\alpha\mathcal{A}) - I)(u_0 - \bar{u})\|_{L^1(\mathbb{R}^3)} + 2Cb\int_0^t (t-s)^{\frac{\alpha}{2}-1}\|v(s)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\
&\le \|(E_\alpha(-t^\alpha\mathcal{A}) - I)(u_0 - \bar{u})\|_{L^1(\mathbb{R}^3)} + \frac{4}{\alpha}Cbt^{\frac{\alpha}{2}}\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}.
\end{aligned} \tag{4.5}$$

Then $S_{t_1}v \in C([0, t_1]; L^1(\mathbb{R}^3))$.

(ii) In the second step, we need to show that S_{t_1} is a contraction mapping on $\mathcal{E}_{t_1}$. Letting $v_1 \in \mathcal{E}_{t_1}$ and $v_2 \in \mathcal{E}_{t_1}$, we have

$$\begin{aligned}
&\|S_{t_1}v_1(t) - S_{t_1}v_2(t)\|_{L^1(\mathbb{R}^3)}\\
&\le \int_0^t (t-s)^{\alpha-1}\|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})(\varphi(v_1+\bar{u}) - \varphi(v_2+\bar{u}))\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\
&\le Cb\int_0^t (t-s)^{\frac{\alpha}{2}-1}\mathrm{d}s \sup_{0\le t\le t_1}\|v_1(t) - v_2(t)\|_{L^1(\mathbb{R}^3)}\\
&\le \frac{2}{\alpha}Cbt_1^{\frac{\alpha}{2}} \sup_{0\le t\le t_1}\|v_1(t) - v_2(t)\|_{L^1(\mathbb{R}^3)}\\
&\le \frac{1}{4}\sup_{0\le t\le t_1}\|v_1(t) - v_2(t)\|_{L^1(\mathbb{R}^3)}.
\end{aligned}$$

That is,

$$\sup_{0\le t\le t_1}\|S_{t_1}v_1(t) - S_{t_1}v_2(t)\|_{L^1(\mathbb{R}^3)} \le \frac{1}{4}\sup_{0\le t\le t_1}\|v_1(t) - v_2(t)\|_{L^1(\mathbb{R}^3)}.$$

Therefore, S_{t_1} has a fixed point in $\mathcal{E}_{t_1}$.

In addition, subtracting $\bar{u}$ from both sides of (2.7), integrating the result equation with respect to $x \in \mathbb{R}^3$ and noting that $E_\alpha(-t^\alpha\mathcal{A})1 = 1$, we obtain

$$\int_{\mathbb{R}^3}(u(x,t) - \bar{u})\,\mathrm{d}x = \int_{\mathbb{R}^3}(u_0(x) - \bar{u})\,\mathrm{d}x,$$

which completes the proof of Theorem 4.1. □

Remark 4.1. It is readily seen that the proof of Theorem 4.1 also implies

$$u - \bar{u} \in C([0, t_1]; \mathring{L}^p(\mathbb{R}^3)), \quad 1 \le p \le \infty.$$

The following key lemma plays an essential role to extend the local solutions globally.

Lemma 4.1. *Suppose that the mild solution u obtained in Theorem 4.1 has been extended up to some time $T(T \geq t_1 > 0)$ and (4.1) are kept unchanged. In addition, assume that $\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}$ is sufficiently small and $\varphi(u) = O(1)|u - \bar{u}|^\sigma$ as $u \to \bar{u}$, where $\sigma = \frac{5}{3}$. Then there exists a positive constant C depending only on $\|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)}$ such that*

$$\|u(t) - \bar{u}\|_{L^1(\mathbb{R}^3)} + t^{\frac{\alpha}{2\sigma}} \|u(t) - \bar{u}\|_{L^\sigma(\mathbb{R}^3)} \leq C\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}, \quad 0 \leq t \leq T. \tag{4.6}$$

Proof. Let

$$X_T = \left\{u(t,x); u - \bar{u} \in C([0,T]; L^1(\mathbb{R}^3)), t^{\frac{\alpha}{2\sigma}}(u - \bar{u}) \in C([0,T]; L^\sigma(\mathbb{R}^3))\right\}$$

equipped with the norm

$$\|v\|_{X_T} = \sup_{0\leq t\leq T} \{\|v(t)\|_{L^1(\mathbb{R}^3)} + t^{\frac{\alpha}{2\sigma}} \|v(t)\|_{L^\sigma(\mathbb{R}^3)}\}.$$

Upon the observation

$$u(x,t) - \bar{u} = E_\alpha(-t^\alpha \mathcal{A})(u_0 - \bar{u}) - \int_0^t (t-s)^{\alpha-1} \Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\,\mathrm{d}s,$$

it then follows that

$$\begin{aligned}
\|u - \bar{u}\|_{X_T} &\leq \|E_\alpha(-t^\alpha \mathcal{A})(u_0 - \bar{u})\|_{X_T} \\
&\quad + \left\| \int_0^t (t-s)^{\alpha-1} \Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\,\mathrm{d}s \right\|_{X_T} \\
&\leq \|E_\alpha(-t^\alpha \mathcal{A})(u_0 - \bar{u})\|_{X_T} \\
&\quad + \sup_{0\leq t\leq T} \int_0^t (t-s)^{\alpha-1} \left\| \Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u) \right\|_{L^1(\mathbb{R}^3)} \mathrm{d}s \\
&\quad + \sup_{0\leq t\leq T} t^{\frac{\alpha}{2\sigma}} \int_0^t (t-s)^{\alpha-1} \left\| \Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u) \right\|_{L^\sigma(\mathbb{R}^3)} \mathrm{d}s \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For the estimate of the first term, we have

$$\begin{aligned}
I_1 &= \sup_{0\leq t\leq T} \left\{ \|E_\alpha(-t^\alpha \mathcal{A})(u_0 - \bar{u})\|_{L^1(\mathbb{R}^3)} + t^{\frac{\alpha}{2\sigma}} \|E_\alpha(-t^\alpha \mathcal{A})(u_0 - \bar{u})\|_{L^\sigma(\mathbb{R}^3)} \right\} \\
&\leq \sup_{0\leq t\leq T} \{\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)} + Ct^{\frac{\alpha}{2\sigma}} t^{-\frac{3\alpha}{4}(1-\frac{1}{\sigma})} \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}\} \\
&\leq (1 + C)\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}.
\end{aligned}$$

In addition, by employing a similar argument, we can deduce from $\varphi(u) = O(1)|u - \bar{u}|^\sigma$ that

$$
\begin{aligned}
I_2 &\le \sup_{0\le t\le T} \int_0^t (t-s)^{\alpha-1} \|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C \sup_{0\le t\le T} \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|\varphi(u(s))\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C \sup_{0\le t\le T} \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|u(s)-\bar{u}\|^\sigma_{L^\sigma(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C \sup_{0\le t\le T} t^{\frac{\alpha}{2}} \|u(t)-\bar{u}\|^\sigma_{L^\sigma(\mathbb{R}^3)} \\
&\le C\|u-\bar{u}\|^\sigma_{X_T},
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &\le \sup_{0\le t\le T} t^{\frac{\alpha}{2\sigma}} \int_0^t (t-s)^{\alpha-1} \|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\|_{L^\sigma(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C \sup_{0\le t\le T} t^{\frac{\alpha}{2\sigma}} \int_0^t (t-s)^{\frac{\alpha}{5}-1} \|\varphi(u(s))\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C \sup_{0\le t\le T} t^{\frac{\alpha}{2\sigma}} \int_0^t (t-s)^{\frac{\alpha}{5}-1} \|u(s)-\bar{u}\|^\sigma_{L^\sigma(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C \sup_{0\le t\le T} t^{\frac{\alpha}{2}} \|u(t)-\bar{u}\|^\sigma_{L^\sigma(\mathbb{R}^3)} \\
&\le C\|u-\bar{u}\|^\sigma_{X_T}.
\end{aligned}
$$

Summing up, we immediately conclude

$$
\|u-\bar{u}\|_{X_T} \le (1+C)\|u_0-\bar{u}\|_{L^1(\mathbb{R}^3)} + C\|u-\bar{u}\|^\sigma_{X_T},
$$

where C is independent of T. Thus, if $\|u-\bar{u}\|_{L^1(\mathbb{R}^3)}$ is sufficiently small, by Strauss's inequality (cf. [23,34]), we can obtain (4.6) immediately. The proof of Lemma 4.1 is completed. □

It is worth mentioning that (4.6) holds in $[0, T]$ even if $T = +\infty$, since the constant C in the estimate (4.1) is independent of T.

Now we are going to extend the local mild solution u globally. In comparison with that of classical integer-order Cahn–Hilliard equation in [23, Theorem 1.1], the proof given here is more concise since it doesn't rely on the estimates of high order derivatives of u.

Proof of Theorem 2.1. Let u defined on $[0, t_1]$ be the local mild solution obtained in the previous sections. We first extend u from $[0, t_1]$ to $[0, t_1+t_2]$ for some $t_2 \in (0, t_1)$.

Define the complete metric space

$$\mathcal{E}_{t_1+t_2} = \Big\{ v \in C([t_1, t_1+t_2]; L^1(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}^3 \times [t_1, t_1+t_2]); \\ v(x,t_1) = u(x,t_1) - \bar{u}, \\ \sup_{t_1 \le t \le t_1+t_2} \|v(t)\|_{L^1(\mathbb{R}^3)} \le 2\|u_0-\bar{u}\|_{L^1(\mathbb{R}^3)} \text{ and } \|v(t)\|_{L^\infty(\mathbb{R}^3\times[t_1,t_1+t_2])} \\ \le 2\|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)} \Big\}$$

and the operator

$$S_{t_1+t_2}v(x,t) = E_\alpha(-t^\alpha\mathcal{A})(u_0-\bar{u}) - \int_0^{t_1} (t-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\,\mathrm{d}s \\ - \int_{t_1}^{t} (t-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(v+\bar{u})\,\mathrm{d}s \quad \text{for } t \ge t_1.$$

By Proposition 3.2, 3.3, Lemma 4.1 and recalling that $\varphi(u) = O(1)|u-\bar{u}|^\sigma$ as $u \to \bar{u}$, we obtain

$$\|S_{t_1+t_2}v(t)\|_{L^\infty(\mathbb{R}^3)} \\ \le \|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)} + \int_0^{t_1-\frac{t_2}{2}} (t-s)^{\alpha-1}\|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ + \int_{t_1-\frac{t_2}{2}}^{t_1} (t-s)^{\alpha-1}\|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ + \int_{t_1}^{t} (t-s)^{\alpha-1}\|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(v+\bar{u})\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ \le \|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)} + C\int_0^{t_1-\frac{t_2}{2}} (t-s)^{-\frac{\alpha}{4}-1}\|\varphi(u)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s \\ + C\int_{t_1-\frac{t_2}{2}}^{t_1} (t-s)^{\frac{\alpha}{2}-1}\|\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s + C\int_{t_1}^{t} (t-s)^{\frac{\alpha}{2}-1}\|\varphi(v+\bar{u})\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ \le \|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)} + C\Big(t-\big(t_1-\frac{t_2}{2}\big)\Big)^{-\frac{3\alpha}{4}} \int_0^{t_1-\frac{t_2}{2}} (t-s)^{\frac{\alpha}{2}-1}s^{-\frac{\alpha}{2}}\left(s^{\frac{\alpha}{2}}\|u-\bar{u}\|^\sigma_{L^\sigma(\mathbb{R}^3)}\right)\mathrm{d}s \\ + C\int_{t_1-\frac{t_2}{2}}^{t_1} (t-s)^{\frac{\alpha}{2}-1}\|u-\bar{u}\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s + C\int_{t_1}^{t} (t-s)^{\frac{\alpha}{2}-1}\|v\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ \le \|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)} + C\Big(t-\big(t_1-\frac{t_2}{2}\big)\Big)^{-\frac{3\alpha}{4}}\|u_0-\bar{u}\|^\sigma_{L^1(\mathbb{R}^3)} + C(t-t_1)^{\frac{\alpha}{2}}\|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)}$$

for $v \in \mathcal{E}_{t_1+t_2}$ and $t \in [t_1, t_1+t_2]$, where C is independent of t and t_1. Choose small t_2 such that

$$C(t-t_1)^{\frac{\alpha}{2}} \le Ct_2^{\frac{\alpha}{2}} < \frac{1}{2}, \quad t \in [t_1, t_1+t_2].$$

Then for sufficiently small $\|u_0-\bar{u}\|_{L^1(\mathbb{R}^3)}$, we can obtain that

$$\|S_{t_1+t_2}v(t)\|_{L^\infty(\mathbb{R}^3)} \le 2\|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)}, \quad t \in [t_1, t_1+t_2].$$

Similarly, for $t \in [t_1, t_1 + t_2]$, we have

$$\begin{aligned}
\|S_{t_1+t_2} v(t)\|_{L^1(\mathbb{R}^3)} &\le \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)} \\
&\quad + \int_0^{t_1} (t-s)^{\alpha-1} \|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u(s))\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\quad + \int_{t_1}^{t} (t-s)^{\alpha-1} \|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(v+\bar{u})(s)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)} + C\int_0^{t_1} (t-s)^{\frac{\alpha}{2}-1} \|\varphi(u(s))\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\quad + C\int_{t_1}^{t} (t-s)^{\frac{\alpha}{2}-1} \|\varphi(v+\bar{u})\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)} + C\int_0^{t_1} (t_1-s)^{\frac{\alpha}{2}-1} s^{-\frac{\alpha}{2}} \left(s^{\frac{\alpha}{2}} \|u(s) - \bar{u}\|^{\sigma}_{L^\sigma(\mathbb{R}^3)} \right) \mathrm{d}s \\
&\quad + C\int_{t_1}^{t} (t-s)^{\frac{\alpha}{2}-1} \|v(s)\|^{\sigma}_{L^\sigma(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)} + C\|u_0 - \bar{u}\|^{\sigma}_{L^1(\mathbb{R}^3)} + C(t-t_1)^{\frac{\alpha}{2}} \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}, \\
&\le 2\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}
\end{aligned}$$

with sufficiently small $\|u_0 - \bar{u}\|_{L^1(\mathbb{R}^3)}$ and small t_2. The continuity of $\|S_{t_1+t_2} v(t)\|_{L^1(\mathbb{R}^3)}$ for $t \in [t_1, t_1 + t_2]$ follows the proof Theorem 4.1. Therefore, we obtain $S_{t_1+t_2} v \in \mathcal{E}_{t_1+t_2}$.

On the other hand, for $v_1, v_2 \in \mathcal{E}_{t_1+t_2}$ and $t_1 \le t \le t_1 + t_2$, we have

$$\begin{aligned}
&\|S_{t_1+t_2} v_1(t) - S_{t_1+t_2} v_2(t)\|_{L^1(\mathbb{R}^3)} \\
&\le \int_{t_1}^{t} (t-s)^{\alpha-1} \|\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})(\varphi(v_1+\bar{u}) - \varphi(v_2+\bar{u}))\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C\int_{t_1}^{t} (t-s)^{\frac{\alpha}{2}-1} \mathrm{d}s \sup_{t_1 \le t \le t_1+t_2} \|v_1(t) - v_2(t)\|_{L^1(\mathbb{R}^3)} \\
&\le C(t-t_1)^{\frac{\alpha}{2}} \sup_{t_1 \le t \le t_1+t_2} \|v_1(t) - v_2(t)\|_{L^1(\mathbb{R}^3)} \\
&\le \frac{1}{2} \sup_{t_1 \le t \le t_1+t_2} \|v_1(t) - v_2(t)\|_{L^1(\mathbb{R}^3)}.
\end{aligned}$$

That is,

$$\sup_{t_1 \le t \le t_1+t_2} \|S_{t_1+t_2} v_1(t) - S_{t_1+t_2} v_2(t)\|_{L^1(\mathbb{R}^3)} \le \frac{1}{2} \sup_{t_1 \le t \le t_1+t_2} \|v_1(t) - v_2(t)\|_{L^1(\mathbb{R}^3)}.$$

According to Banach contracting mapping principle, there exists a unique solution in $\mathcal{E}_{t_1+t_2}$, which is also denoted by $u(x, t)$. Therefore, the solution $u(x, t)$ can be extended up to time $t_1 + t_2$ and

$$\|u(t) - \bar{u}\|_{L^\infty(\mathbb{R}^3)} \le 2\|u_0 - \bar{u}\|_{L^\infty(\mathbb{R}^3)}, \quad 0 \le t \le t_1 + t_2.$$

Repeating above procedure, we can extend the solution $u(x,t)$ to time t_1+mt_2 for any $m\in\mathbb{Z}_+$ such that

$$\|u(t)-\bar{u}\|_{L^\infty(\mathbb{R}^3)}\le 2\|u_0-\bar{u}\|_{L^\infty(\mathbb{R}^3)},\quad 0\le t\le t_1+mt_2.$$

We thus establish the existence of the solution u in all $t\ge 0$. The integral preservation property of u follows directly from Theorem 4.1. The proof of Theorem 2.1 is completed. □

5. Strong solution

By Remark 3.1, the mild solution u obtained in Theorem 2.1 may not be a smooth solution. However, we will prove that u is in fact a strong solution of the problem (1.1) in this section. For this purpose, we first give some a priori estimates to show $u-\bar{u}\in C((0,\infty);W^{4,1}(\mathbb{R}^3))$. For simplicity, we assume $\bar{u}=0$ and take $r=\|u_0\|_{L^\infty(\mathbb{R}^3)}$ in the rest of this paper.

It is worth mentioning that the equation in (1.1) is non-Markovian and $E_\alpha(-t^\alpha\mathcal{A})$ and $E_{\alpha,\alpha}(-t^\alpha\mathcal{A})$ with $0<\alpha<1$ are not semigroup operators, which means that for $t>\bar{t}>0$,

$$\begin{aligned}u(x,t)&=E_\alpha(-t^\alpha\mathcal{A})u_0-\int_0^t(t-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\,\mathrm{d}s\\&\ne E_\alpha(-(t-\bar{t})^\alpha\mathcal{A})u(\bar{t})-\int_{\bar{t}}^t(t-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\,\mathrm{d}s.\end{aligned}$$

Thus the way to obtaining the regularity of solution as given in [23] for the traditional Cahn–Hilliard equation is no longer valid. However, by dividing the integro-differential equation (2.7) into the following three terms,

$$\begin{aligned}u(x,t)&=E_\alpha(-t^\alpha\mathcal{A})u_0-\int_{\bar{t}}^t(t-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\,\mathrm{d}s\\&\quad-\int_0^{\bar{t}}(t-s)^{\alpha-1}\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\,\mathrm{d}s,\quad t\ge\bar{t}>0.\end{aligned}\tag{5.1}$$

we see that the first two terms on the right hand side of (5.1) can also be estimated similarly. So it suffices to find how to deal with the third term on the right hand side of (5.1).

Lemma 5.1. *Let the assumptions of Lemma 4.1 be satisfied. Then for any $0<s_1<s_2<t$, we have*

$$\|Du(t)\|_{L^\infty(\mathbb{R}^3)}\le(t-s_1)^{-\frac{\alpha}{4}}C_1(r,s_2-s_1,t-s_1),\tag{5.2}$$

where C_1 is a continuous increasing function of $t-s_1$.

Proof. Applying the derivative operator D to (5.1) and taking $\bar{t} = s_1$, we get

$$\begin{aligned}\|Du(t)\|_{L^\infty} \leq\ & \|DE_\alpha(-t^\alpha \mathcal{A})u_0\|_{L^\infty(\mathbb{R}^3)} \\ & + \int_{s_1}^{t} (t-s)^{\alpha-1} \|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ & + \int_{0}^{s_1} (t-s)^{\alpha-1} \|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ =\ & I + II.\end{aligned}$$

By Propositions 3.2, 3.3 and Theorem 4.1, we have

$$\begin{aligned} I &= \|DE_\alpha(-t^\alpha \mathcal{A})u_0\|_{L^\infty(\mathbb{R}^3)} + \int_{s_1}^{t} (t-s)^{\alpha-1} \|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ &\leq C(r)(t-s_1)^{-\frac{\alpha}{4}} + C\int_{s_1}^{t} (t-s)^{\frac{\alpha}{40}-1} \|\varphi(u)\|_{L^{2\sigma}(\mathbb{R}^3)}\mathrm{d}s \\ &\leq C(r)(t-s_1)^{-\frac{\alpha}{4}} + C(r)\int_{s_1}^{t} (t-s)^{\frac{\alpha}{40}-1} \|u\|^{\frac{1}{2}}_{L^{\sigma}(\mathbb{R}^3)}\mathrm{d}s \\ &\leq C(r)(t-s_1)^{-\frac{\alpha}{4}} + C(r)\int_{s_1}^{t} (t-s)^{\frac{\alpha}{40}-1} s^{-\frac{\alpha}{4\sigma}} \left(s^{\frac{\alpha}{2\sigma}} \|u\|_{L^{\sigma}(\mathbb{R}^3)}\right)^{\frac{1}{2}}\mathrm{d}s \\ &\leq C(r)(t-s_1)^{-\frac{\alpha}{4}} + C(r)(t-s_1)^{-\frac{\alpha}{8}}. \end{aligned} \tag{5.3}$$

Similarly, we could deduce that

$$\begin{aligned} II &= \int_{0}^{s_1} (t-s)^{\alpha-1} \|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s \\ &\leq C\int_0^{s_1} (t-s)^{-\frac{\alpha}{2}-1} \|\varphi(u)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s \\ &\leq C\int_0^{s_1} (t-s)^{-\frac{\alpha}{2}-1} s^{-\frac{\alpha}{2}} \left(s^{\frac{\alpha}{2}}\|u\|^{\sigma}_{L^{\sigma}(\mathbb{R}^3)}\right)\mathrm{d}s \\ &\leq C(t-s_1)^{-\alpha}\int_0^{s_1} (s_1-s)^{\frac{\alpha}{2}-1} s^{-\frac{\alpha}{2}}\,\mathrm{d}s \\ &\leq C(t-s_1)^{-\frac{\alpha}{4}}(s_2-s_1)^{-\frac{3\alpha}{4}}. \end{aligned} \tag{5.4}$$

Combining (5.3) with (5.4), we obtain

$$\|Du(t)\|_{L^\infty(\mathbb{R}^3)} \leq (t-s_1)^{-\frac{\alpha}{4}} C_1(r, s_2-s_1, t-s_1),$$

which completes the proof of Lemma 5.1. □

Next, we give the estimates of $\|D^k u(t)\|_{L^1(\mathbb{R}^3)}$ with $k = 1, 2, 3$.

Lemma 5.2. *Let the assumptions of Lemma 4.1 be satisfied. Then*

$$\|Du(t)\|_{L^1(\mathbb{R}^3)} \leq M_1(r)t^{-\frac{\alpha}{4}}\|u_0\|_{L^1(\mathbb{R}^3)}, \quad t > 0, \tag{5.5}$$

where $M_1(r)$ is a positive constant depending on r.

Proof. According to (2.7) and (4.6), we have

$$\begin{aligned} t^{\frac{\alpha}{4}}\|Du(t)\|_{L^1(\mathbb{R}^3)} \le\ & t^{\frac{\alpha}{4}}\|DE_\alpha(-t^\alpha\mathcal{A})(u_0)\|_{L^1(\mathbb{R}^3)} \\ & + t^{\frac{\alpha}{4}}\int_0^t (t-s)^{\alpha-1}\|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s \\ \le\ & \|u_0\|_{L^1(\mathbb{R}^3)} + Ct^{\frac{\alpha}{4}}\int_0^t (t-s)^{\frac{\alpha}{4}-1}\|\varphi(u(s))\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s \\ \le\ & \|u_0\|_{L^1(\mathbb{R}^3)} + Ct^{\frac{\alpha}{4}}\int_0^t (t-s)^{\frac{\alpha}{4}-1}\|u(s)\|^\sigma_{L^\sigma(\mathbb{R}^3)}\,\mathrm{d}s \\ \le\ & \|u_0\|_{L^1(\mathbb{R}^3)} + Ct^{\frac{\alpha}{4}}\int_0^t (t-s)^{\frac{\alpha}{4}-1}s^{-\frac{\alpha}{2}}(s^{\frac{\alpha}{2\sigma}}\|u(s)\|_{L^\sigma(\mathbb{R}^3)})^\sigma\,\mathrm{d}s \\ \le\ & \|u_0\|_{L^1(\mathbb{R}^3)} + Cc_1^\sigma(r)\|u_0\|^\sigma_{L^1(\mathbb{R}^3)}. \end{aligned}$$

The proof is completed. □

Lemma 5.3. *Let the assumptions of Lemma 4.1 be satisfied. Then for any* $0 < s_1 < s_2 < \bar{s}_2 < \bar{s}_3 < t$*, we have*

$$\|D^2u(t)\|_{L^1(\mathbb{R}^3)} \le (t-\bar{s}_2)^{-\frac{\alpha}{2}}M_2(r, t-\bar{s}_2)\|u_0\|_{L^1(\mathbb{R}^3)} \tag{5.6}$$

and

$$\|D^3u(t)\|_{L^1(\mathbb{R}^3)} \le (t-\bar{s}_3)^{-\frac{3\alpha}{4}}M_3(r, s_2-s_1, \bar{s}_3-\bar{s}_2, t-\bar{s}_3)\|u_0\|_{L^1(\mathbb{R}^3)}, \tag{5.7}$$

where M_k *denote continuous increasing functions of* $t-\bar{s}_k$*,* $k = 2, 3$.

Proof. We first prove (5.6). For $t > \bar{s}_2$, we have

$$\begin{aligned} \|D^2u(t)\|_{L^1(\mathbb{R}^3)} \le\ & \|D^2E_\alpha(-t^\alpha\mathcal{A})u_0\|_{L^1(\mathbb{R}^3)} \\ & + \int_{\bar{s}_2}^t (t-s)^{\alpha-1}\|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})D\varphi(u)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s \\ & + \int_0^{\bar{s}_2} (t-s)^{\alpha-1}\|D^2\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s \\ =\ & I + II. \end{aligned}$$

By Propositions 3.2, 3.3 and Lemma 5.2, we obtain

$$\begin{aligned} I &\le C(t-\bar{s}_2)^{-\frac{\alpha}{2}}\|u_0\|_{L^1(\mathbb{R}^3)} + C(r)\int_{\bar{s}_2}^t (t-s)^{\frac{\alpha}{4}-1}\|Du\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s \\ &\le C(t-\bar{s}_2)^{-\frac{\alpha}{2}}\|u_0\|_{L^1(\mathbb{R}^3)} + C(r)\int_{\bar{s}_2}^t (t-s)^{\frac{\alpha}{4}-1}s^{-\frac{\alpha}{4}}\left(s^{\frac{\alpha}{4}}\|Du\|_{L^1(\mathbb{R}^3)}\right)\mathrm{d}s \\ &\le C(t-\bar{s}_2)^{-\frac{\alpha}{2}}\|u_0\|_{L^1(\mathbb{R}^3)} + C(r)\|u_0\|_{L^1(\mathbb{R}^3)}. \end{aligned} \tag{5.8}$$

By virtue of $\varphi(u) = O(1)|u|^{\sigma}$, we have

$$\begin{aligned} II &= \int_0^{\bar{s}_2} (t-s)^{\alpha-1} \|D^2 \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\ &\le C \int_0^{\bar{s}_2} (t-s)^{-1} \|\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\ &\le C \int_0^{\bar{s}_2} (\bar{s}_2 - s)^{\frac{\alpha}{2}-1} (t-\bar{s}_2)^{-\frac{\alpha}{2}} \|u\|^{\sigma}_{L^{\sigma}(\mathbb{R}^3)}\, \mathrm{d}s \\ &\le C(t-\bar{s}_2)^{-\frac{\alpha}{2}} \int_0^{\bar{s}_2} (\bar{s}_2 - s)^{\frac{\alpha}{2}-1} s^{-\frac{\alpha}{2}} \left(s^{\frac{\alpha}{2}} \|u\|^{\sigma}_{L^{\sigma}(\mathbb{R}^3)} \right) \mathrm{d}s \\ &\le (t-\bar{s}_2)^{-\frac{\alpha}{2}} C(r) \|u_0\|_{L^1(\mathbb{R}^3)}. \end{aligned} \tag{5.9}$$

Combining (5.8) with (5.9), we immediately conclude the estimate (5.6).

Next, we prove (5.7). Similarly, for $t > \bar{s}_3$, we have

$$\begin{aligned} \|D^3 u(t)\|_{L^1} \le\ & \|D^3 E_{\alpha}(-t^{\alpha}\mathcal{A})u_0\|_{L^1(\mathbb{R}^3)} \\ & + \int_{\bar{s}_3}^{t} (t-s)^{\alpha-1} \|D\Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A}) D^2\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\ & + \int_0^{\bar{s}_3} (t-s)^{\alpha-1} \|D^3 \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\ =\ & I' + II'. \end{aligned}$$

By Propositions 3.2 and 3.3, we have

$$\begin{aligned} I' &= \|D^3 E_{\alpha}(-t^{\alpha}\mathcal{A})u_0\|_{L^1(\mathbb{R}^3)} \\ &\quad + \int_{\bar{s}_3}^{t} (t-s)^{\alpha-1} \|D\Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A}) D^2\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\ &\le C(t-\bar{s}_3)^{-\frac{3\alpha}{4}} \|u_0\|_{L^1(\mathbb{R}^3)} + C \int_{\bar{s}_3}^{t} (t-s)^{\frac{\alpha}{4}-1} \|D^2\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\ &\le C(t-\bar{s}_3)^{-\frac{3\alpha}{4}} \|u_0\|_{L^1(\mathbb{R}^3)} + C \int_{\bar{s}_3}^{t} (t-s)^{\frac{\alpha}{4}-1} \left(\|Du\|^2_{L^2} + \|D^2 u\|_{L^1(\mathbb{R}^3)} \right) \mathrm{d}s \\ &\le C(t-\bar{s}_3)^{-\frac{3\alpha}{4}} \|u_0\|_{L^1(\mathbb{R}^3)} \\ &\quad + C \int_{\bar{s}_3}^{t} (t-s)^{\frac{\alpha}{4}-1} \left(\|Du\|_{L^{\infty}(\mathbb{R}^3)} \|Du\|_{L^1(\mathbb{R}^3)} + \|D^2 u\|_{L^1(\mathbb{R}^3)} \right) \mathrm{d}s. \end{aligned} \tag{5.10}$$

It should be notice that by virtue of (5.3) and (5.4) in the proof of Lemma 5.1, the function in the right hand side of (5.2) is the continuous decreasing function with respect to $t \in (s_1, \infty)$. So, for $0 < s_1 < s_2 < \bar{s}_2 < \bar{s}_3 < s < t$, we have

$$\begin{aligned} \|Du(s)\|_{L^{\infty}} &\le (s-s_1)^{-\frac{\alpha}{4}} C_1(r, s_2 - s_1, s - s_1) \\ &\le (s_2 - s_1)^{-\frac{\alpha}{4}} C_1(r, s_2 - s_1, s_2 - s_1) \\ &= C(r, s_2 - s_1). \end{aligned} \tag{5.11}$$

Then combining with Lemma 5.2, we have

$$\begin{aligned}
&\int_{\bar{s}_3}^{t}(t-s)^{\frac{\alpha}{4}-1}\|Du\|_{L^\infty(\mathbb{R}^3)}\|Du\|_{L^1(\mathbb{R}^3)}\mathrm{d}s\\
&\quad\le\ C(r,s_2-s_1)\|u_0\|_{L^1(\mathbb{R}^3)}\int_{\bar{s}_3}^{t}(t-s)^{\frac{\alpha}{4}-1}s^{-\frac{\alpha}{4}}\mathrm{d}s\\
&\quad\le\ C(r,s_2-s_1)\|u_0\|_{L^1(\mathbb{R}^3)}. \qquad (5.12)
\end{aligned}$$

Similarly, by virtue of (5.6), for $0<\bar{s}_2<\bar{s}_3<s<t$, we have

$$\|D^2u(s)\|_{L^1(\mathbb{R}^3)}\le(\bar{s}_3-\bar{s}_2)^{-\frac{\alpha}{2}}M_2(r,\bar{s}_3-\bar{s}_2)\|u_0\|_{L^1(\mathbb{R}^3)},$$

and then

$$\begin{aligned}
&\int_{\bar{s}_3}^{t}(t-s)^{\frac{\alpha}{4}-1}\|D^2u\|_{L^1(\mathbb{R}^3)}\mathrm{d}s\\
&\quad\le\ (\bar{s}_3-\bar{s}_2)^{-\frac{\alpha}{2}}M_2(r,\bar{s}_3-\bar{s}_2)\|u_0\|_{L^1(\mathbb{R}^3)}\int_{\bar{s}_3}^{t}(t-s)^{\frac{\alpha}{4}-1}\mathrm{d}s\\
&\quad\le\ (\bar{s}_3-\bar{s}_2)^{-\frac{\alpha}{2}}M_2(r,\bar{s}_3-\bar{s}_2)\|u_0\|_{L^1(\mathbb{R}^3)}\int_{\bar{s}_3}^{t}(t-s)^{\frac{\alpha}{4}-1}\mathrm{d}s\\
&\quad\le\ (t-\bar{s}_3)^{-\frac{3\alpha}{4}}M_3(r,\bar{s}_3-\bar{s}_2,t-\bar{s}_3)\|u_0\|_{L^1(\mathbb{R}^3)}. \qquad (5.13)
\end{aligned}$$

Inserting (5.12) and (5.13) into (5.10), we obtain

$$I'\le(t-\bar{s}_3)^{-\frac{3\alpha}{4}}M_3(r,s_2-s_1,\bar{s}_3-\bar{s}_2,t-\bar{s}_3)\|u_0\|_{L^1}.$$

By the similar method, we have

$$\begin{aligned}
II'&=\int_0^{\bar{s}_3}(t-s)^{\alpha-1}\|D^3\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\
&\le C\int_0^{\bar{s}_3}(t-s)^{-\frac{\alpha}{4}-1}\|\varphi(u)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\
&\le C\int_0^{\bar{s}_3}(t-s)^{-\frac{3\alpha}{4}}(t-s)^{\frac{\alpha}{2}-1}\|u\|^\sigma_{L^\sigma(\mathbb{R}^3)}\,\mathrm{d}s\\
&\le C(t-\bar{s}_3)^{-\frac{3\alpha}{4}}\int_0^{\bar{s}_3}(\bar{s}_3-s)^{\frac{\alpha}{2}-1}s^{-\frac{\alpha}{2}}\left(s^{\frac{\alpha}{2}}\|u\|^\sigma_{L^\sigma(\mathbb{R}^3)}\right)\mathrm{d}s\\
&\le(t-\bar{s}_3)^{-\frac{3\alpha}{4}}C(r)\|u_0\|_{L^1(\mathbb{R}^3)}.
\end{aligned}$$

Summing up, we immediately conclude the estimate (5.7). □

Lemma 5.4. *Let the assumptions of Lemma 4.1 be satisfied. Then we have*

$$\|D^ku(t)\|_{L^\infty(\mathbb{R}^3)}\le(t-s_k)^{-\frac{\alpha k}{4}}C_k(r,s_2-s_1,\ldots,s_{k+1}-s_k,t-s_k) \qquad (5.14)$$

for any $0<s_1<s_2<\cdots<s_k<s_{k+1}<t$*, where* C_k *are the continuous increasing functions of* $t-s_k$*,* $k=2,3$.

Proof. We first prove (5.14) for $k = 2$. Letting $t > s_3$, we have

$$
\begin{aligned}
\|D^2 u(t)\|_{L^\infty(\mathbb{R}^3)} \le\ & \|D^2 E_\alpha(-t^\alpha \mathcal{A})u_0\|_{L^\infty(\mathbb{R}^3)} \\
& + \int_{s_2}^{t} (t-s)^{\alpha-1} \|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) D\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}s \\
& + \int_{0}^{s_2} (t-s)^{\alpha-1} \|D^2\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) \varphi(u)\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}s \\
=\ & I + II.
\end{aligned}
$$

By Proposition 3.2 and 3.3, we obtain

$$
I \le (t-s_2)^{-\frac{\alpha}{2}} Cr + C\int_{s_2}^{t} (t-s)^{\frac{\alpha}{40}-1} \|D\varphi(u)\|_{L^{2\sigma}(\mathbb{R}^3)}\, \mathrm{d}s. \tag{5.15}
$$

Combining (5.5) with (5.11), we have

$$
\begin{aligned}
\|D\varphi(u(s))\|_{L^{2\sigma}(\mathbb{R}^3)} &\le \|Du(s)\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2\sigma}} \|Du(s)\|_{L^\infty(\mathbb{R}^3)}^{1-\frac{1}{2\sigma}} \\
&\le C(r, s_2 - s_1) s^{-\frac{3\alpha}{40}}, \quad s_2 < s < t.
\end{aligned}
$$

Inserting this inequality into (5.15), we get

$$
\begin{aligned}
I &\le (t-s_2)^{-\frac{\alpha}{2}} Cr + C\int_{s_2}^{t} (t-s)^{\frac{\alpha}{40}-1} s^{-\frac{3\alpha}{40}} C(r, s_2 - s_1)\, \mathrm{d}s \\
&\le (t-s_2)^{-\frac{\alpha}{2}} Cr + (t-s_2)^{-\frac{\alpha}{20}} C(r, s_2 - s_1).
\end{aligned} \tag{5.16}
$$

In addition, we have

$$
\begin{aligned}
II &= \int_0^{s_2} (t-s)^{\alpha-1} \|D^2\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) \varphi(u)\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C\int_0^{s_2} (t-s)^{-\frac{3\alpha}{4}-1} \|\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \\
&\le C\int_0^{s_2} (t-s)^{-\frac{3\alpha}{4}-1} s^{-\frac{\alpha}{2}}\, \mathrm{d}s \sup_t \left(t^{\frac{\alpha}{2}} \|u\|_{L^\sigma(\mathbb{R}^3)}^{\sigma} \right) \\
&\le (t-s_2)^{-\frac{\alpha}{2}-\frac{3\alpha}{4}} \int_0^{s_2} (s_2 - s)^{\frac{\alpha}{2}-1} s^{-\frac{\alpha}{2}}\, \mathrm{d}s\, C(r)\|u_0\|_{L^1(\mathbb{R}^3)} \\
&\le (t-s_2)^{-\frac{\alpha}{2}} (s_3 - s_2)^{-\frac{3\alpha}{4}} C(r)\|u_0\|_{L^1(\mathbb{R}^3)} \\
&\le (t-s_2)^{-\frac{\alpha}{2}}\, C_2(r, s_3 - s_2).
\end{aligned} \tag{5.17}
$$

Here we used the fact that $s_2 < s_3 < t$. Combining (5.16) with (5.17), we immediately conclude the estimate (5.14) for $k = 2$. It is also worth mentioning that the term on the right hand side of (5.14) for $k = 2$ is a continuous decreasing function of $t - s_2$, even though $C_2(r, s_2 - s_1, s_3 - s_2, t - s_2)$ is continuous increasing with respect to $t - s_2$.

Next, we prove (5.14) for $k = 3$. Similarly, for $t > s_4$, we have

$$\begin{aligned}\|D^3u(t)\|_{L^\infty(\mathbb{R}^3)} \le\ & \|D^3E_\alpha(-t^\alpha A)u_0\|_{L^\infty(\mathbb{R}^3)}\\ &+\int_{s_3}^{t}(t-s)^{\alpha-1}\|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha A)D^2\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\\ &+\int_0^{s_3}(t-s)^{\alpha-1}\|D^3\Delta E_{\alpha,\alpha}(-(t-s)^\alpha A)\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\\ =\ & I'+II'.\end{aligned}$$

By Proposition 3.2 and 3.3, we get

$$\begin{aligned}I' &\le (t-s_3)^{-\frac{3\alpha}{4}}Cr + C\int_{s_3}^{t}(t-s)^{\frac{\alpha}{4}-1}\|D^2\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le (t-s_3)^{-\frac{3\alpha}{4}}Cr + C\int_{s_3}^{t}(t-s)^{\frac{\alpha}{4}-1}\left(\|Du\|^2_{L^\infty(\mathbb{R}^3)}+\|D^2u\|_{L^\infty(\mathbb{R}^3)}\right)\mathrm{d}s.\end{aligned}\tag{5.18}$$

For $0 < s_1 < s_2 < s_3 < s < t$, we have

$$\begin{aligned}\|Du(s)\|_{L^\infty} &\le (s-s_1)^{-\frac{\alpha}{4}}C_1(r, s_2-s_1, s-s_1)\\ &\le (s_2-s_1)^{-\frac{\alpha}{4}}C_1(r, s_2-s_1, s_2-s_1)\\ &= C(r, s_2-s_1)\end{aligned}$$

and

$$\begin{aligned}\|D^2u(s)\|_{L^\infty} &\le (s-s_2)^{-\frac{\alpha}{2}}C_2(r, s_2-s_1, s_3-s_2, s-s_2)\\ &\le (s_3-s_2)^{-\frac{\alpha}{2}}C_2(r, s_2-s_1, s_3-s_2, s_3-s_2)\\ &= C(r, s_2-s_1, s_3-s_2).\end{aligned}$$

Inserting these two estimates into (5.18), we obtain

$$\begin{aligned}I' &\le (t-s_3)^{-\frac{3\alpha}{4}}Cr + C(r, s_2-s_1, s_3-s_2)\int_{s_3}^{t}(t-s)^{\frac{\alpha}{4}-1}\,\mathrm{d}s\\ &\le (t-s_3)^{-\frac{3\alpha}{4}}Cr + (t-s_3)^{\frac{\alpha}{4}}C(r, s_2-s_1, s_3-s_2)\\ &\le (t-s_3)^{-\frac{3\alpha}{4}}C_3(r, s_2-s_1, s_3-s_2, t-s_3).\end{aligned}\tag{5.19}$$

In addition, we have

$$\begin{aligned}II' &= \int_0^{s_3}(t-s)^{\alpha-1}\|D^3\Delta E_{\alpha,\alpha}(-(t-s)^\alpha A)\varphi(u)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le C\int_0^{s_3}(t-s)^{-\frac{3\alpha}{4}-1}\|D\varphi(u)\|_{L^1(\mathbb{R}^3)}\,\mathrm{d}s\\ &\le C\int_0^{s_3}(t-s)^{-\alpha}(t-s)^{\frac{\alpha}{4}-1}s^{-\frac{\alpha}{4}}\mathrm{d}s\,\sup_t\left(t^{\frac{\alpha}{4}}\|Du\|_{L^1(\mathbb{R}^3)}\right)\end{aligned}$$

$$\le C(t-s_3)^{-\alpha}\int_0^{s_3}(s_3-s)^{\frac{\alpha}{2}-1}s^{-\frac{\alpha}{2}}\mathrm{d}s$$
$$\le C(t-s_3)^{-\frac{3\alpha}{4}}(s_4-s_3)^{-\frac{\alpha}{4}}. \tag{5.20}$$

Here we used the fact that $s_3 < s_4 < t$. Combining (5.19) and (5.20), we immediately obtain the estimate (5.14) for $k=3$. □

Lemma 5.5. *Let the assumptions of Lemma 4.1 be satisfied and $\varphi(u) \in C^3(\mathbb{R})$. Then we have*

$$\|D^4u(t)\|_{L^p(\mathbb{R}^3)} \le (t-\bar{s}_4)^{-\alpha}C_4(r, s_4-s_1, \bar{s}_4-s_1, t-\bar{s}_4)\|u_0\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}} \tag{5.21}$$

for any $p \in [1,\infty)$ and $0 < s_1 < \bar{s}_1 < s_2 < \bar{s}_2 < s_3 < \bar{s}_3 < s_4 < \bar{s}_4 < t$. Here, C_4 is a continuous increasing function of $t-\bar{s}_4$.

Proof. For $t > \bar{s}_4$, we have

$$\begin{aligned}\|D^4u(t)\|_{L^p(\mathbb{R}^3)} \le\ & \|D^4E_\alpha(-t^\alpha\mathcal{A})u_0\|_{L^p(\mathbb{R}^3)}\\ &+\int_{\bar{s}_4}^t(t-s)^{\alpha-1}\|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})D^3\varphi(u)\|_{L^p(\mathbb{R}^3)}\mathrm{d}s\\ &+\int_0^{\bar{s}_4}(t-s)^{\alpha-1}\|D^4\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u)\|_{L^p(\mathbb{R}^3)}\mathrm{d}s\\ =\ & I+II+III.\end{aligned}$$

Since $p\in[1,\infty)$, by Proposition 3.2, we have

$$I \le (t-\bar{s}_4)^{-\alpha}C\|u_0\|_{L^p(\mathbb{R}^3)} \le (t-\bar{s}_4)^{-\alpha}C(r)\|u_0\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}}. \tag{5.22}$$

Combining Proposition 3.3 with Lemma 5.1–5.4, we obtain

$$\begin{aligned}II =\ & \int_{\bar{s}_4}^t(t-s)^{\alpha-1}\|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})D^3\varphi(u)\|_{L^p(\mathbb{R}^3)}\mathrm{d}s\\ \le\ & C\int_{\bar{s}_4}^t(t-s)^{\frac{\alpha}{4}-1}\|D^3\varphi(u)\|_{L^p(\mathbb{R}^3)}\mathrm{d}s\\ \le\ & C\int_{\bar{s}_4}^t(t-s)^{\frac{\alpha}{4}-1}\left(\|Du\|^3_{L^{3p}(\mathbb{R}^3)}+\|Du\cdot D^2u\|_{L^p(\mathbb{R}^3)}+\|D^3u\|_{L^p(\mathbb{R}^3)}\right)\mathrm{d}s\\ \le\ & C\int_{\bar{s}_4}^t(t-s)^{\frac{\alpha}{4}-1}\Big(\|Du\|_{L^\infty(\mathbb{R}^3)}^{3-\frac{1}{p}}\|Du\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}}\\ &+\|Du\|_{L^\infty(\mathbb{R}^3)}^{1-\frac{1}{p}}\|Du\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}}\|D^2u\|_{L^\infty(\mathbb{R}^3)}+\|D^3u\|_{L^\infty(\mathbb{R}^3)}^{1-\frac{1}{p}}\|D^3u\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}}\Big)\mathrm{d}s\\ \le\ & \int_{\bar{s}_4}^t(t-s)^{\frac{\alpha}{4}-1}(s-\bar{s}_3)^{-\frac{3\alpha}{4}}\|u_0\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}}\Big(C_1^{3-\frac{1}{p}}(r,s_2-s_1,s-s_1)\\ &+C_1^{1-\frac{1}{p}}(r,s_2-s_1,s-s_1)C_2(r,s_2-s_1,s_3-s_2,s-s_2)M_1^{\frac{1}{p}}(r)\end{aligned}$$

$$
\begin{aligned}
&+C_3^{1-\frac{1}{p}}(r, s_2-s_1, s_3-s_2, s_4-s_3, s-s_3)M_3^{\frac{1}{p}}(r, s_2-s_1, \bar{s}_3-\bar{s}_2, s-\bar{s}_3)\Big)\mathrm{d}s\\
&\le (t-\bar{s}_4)^{-\alpha}C_4(r, s_4-s_1, \bar{s}_4-s_1, t-\bar{s}_4)\|u_0\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}}, \qquad (5.23)
\end{aligned}
$$

where we also used the monotonicity of C_1, C_2, C_3 and M_3 with respect to t. By the similar method, we have

$$
\begin{aligned}
III &= \int_0^{\bar{s}_4}(t-s)^{\alpha-1}\|D^4\Delta E_{\alpha,\alpha}(-(t-s)^\alpha A)\varphi(u)\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}s\\
&\le C\int_0^{\bar{s}_4}(t-s)^{-\frac{\alpha}{2}-1}\|\varphi(u)\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}s\\
&\le C(r)\int_0^{\bar{s}_4}(t-s)^{-\frac{\alpha}{2}(1+\frac{1}{p})}(t-s)^{\frac{\alpha}{2p}-1}\|u\|_{L^\sigma(\mathbb{R}^3)}^{\frac{\sigma}{p}}\,\mathrm{d}s\\
&\le C(r)(t-\bar{s}_4)^{-\frac{\alpha}{2}(1+\frac{1}{p})}\int_0^{\bar{s}_4}(\bar{s}_4-s)^{\frac{\alpha}{2p}-1}s^{-\frac{\alpha}{2p}}\,\mathrm{d}s\,\sup_t\left(t^{\frac{\alpha}{2}}\|u\|_{L^\sigma(\mathbb{R}^3)}^{\sigma}\right)^{\frac{1}{p}}\\
&\le (t-\bar{s}_4)^{-\alpha}C(r, t-\bar{s}_4)\|u_0\|_{L^1}^{\frac{1}{p}}. \qquad (5.24)
\end{aligned}
$$

Summing up, we immediately conclude the estimate (5.21). □

Theorem 5.1. *Let the assumptions of Theorem 4.1 be satisfied and* $\varphi(u) \in C^3(\mathbb{R})$*. If* $\|u_0\|_{L^1(\mathbb{R}^3)}$ *is sufficiently small and* $\varphi(u) = O(1)|u|^\sigma$ *as* $u \to 0$*, where* $\sigma = \frac{5}{3}$*, then the mild solution of the problem (1.1)* $u \in C((0,\infty); W^{4,p}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3))$ *for* $1 \le p < \infty$*.*

Proof. Let $1 \le p < \infty$. By Lemma 5.5, we see that $u(t) : (0,\infty) \mapsto W^{4,p}(\mathbb{R}^3)$. Remark 4.1 shows that $u \in C([0,\infty); L^p(\mathbb{R}^3) \cap C_0(\mathbb{R}^3))$. So it suffices to show the continuity of $\|u(t)\|_{W^{4,p}(\mathbb{R}^3)}$ with respect to $t \in (0,\infty)$. For simplicity, we only prove the continuity of $\|D^4u(t)\|_{L^p(\mathbb{R}^3)}$, since the proofs of the other cases ($k = 1, 2, 3$) are similar.

Fixing $t_0 \in (0,\infty)$, we have the following estimate

$$
\begin{aligned}
\|D^4u(t) - D^4u(t_0)\|_{L^p(\mathbb{R}^3)} \le\ & \|(D^4E_\alpha(-t^\alpha\mathcal{A}) - D^4E_\alpha(-t_0^\alpha\mathcal{A}))u_0\|_{L^p(\mathbb{R}^3)}\\
&+\int_{t_0}^t(t-s)^{\alpha-1}\left\|D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})D^3\varphi(u(s))\right\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}s\\
&+\int_0^{t_0}\Big\|(t-s)^{\alpha-1}D^4\Delta E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\varphi(u(s))\\
&\quad-(t_0-s)^{\alpha-1}D^4\Delta E_{\alpha,\alpha}(-(t_0-s)^\alpha\mathcal{A})\varphi(u(s))\Big\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}s\\
\le\ & I_1 + I_2 + I_3, \quad t > t_0.
\end{aligned}
$$

Using the strong continuity of $D^4E_\alpha(-t^\alpha\mathcal{A})$ on $L^p(\mathbb{R}^3)$ for $t \in (0,\infty)$, we deduce that the first term I_1 goes to zero as $t \to t_0^+$. Take

$$
0 < s_1 < \bar{s}_1 < s_2 < \bar{s}_2 < s_3 < \bar{s}_3 < s_4 < \bar{s}_4 = \frac{t_0}{2} < t_0.
$$

Similarly with the estimate (5.23), we can obtain that

$$
\begin{aligned}
I_2 &= \int_{t_0}^{t} \left\| (t-s)^{\alpha-1} D\Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A}) D^3\varphi(u(s)) \right\|_{L^p(\mathbb{R}^3)} \mathrm{d}s \\
&\le C \int_{t_0}^{t} (t-s)^{\frac{\alpha}{4}-1} \left\| D^3\varphi(u(s)) \right\|_{L^p(\mathbb{R}^3)} \mathrm{d}s \\
&\le C \int_{t_0}^{t} (t-s)^{\frac{\alpha}{4}-1} \Big(\|Du\|_{L^\infty(\mathbb{R}^3)}^{3-\frac{1}{p}} \|Du\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}} \\
&\quad + \|Du\|_{L^\infty(\mathbb{R}^3)}^{1-\frac{1}{p}} \|Du\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}} \|D^2u\|_{L^\infty} + \|D^3u\|_{L^\infty(\mathbb{R}^3)}^{1-\frac{1}{p}} \|D^3u\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}} \Big) \mathrm{d}s \\
&\le \int_{t_0}^{t} (t-s)^{\frac{\alpha}{4}-1} (s-\bar{s}_3)^{-\frac{3\alpha}{4}} \|u_0\|_{L^1(\mathbb{R}^3)}^{\frac{1}{p}} \Big(C_1^{3-\frac{1}{p}}(r, s_2-s_1, s-s_1) \\
&\quad + C_1^{1-\frac{1}{p}}(r, s_2-s_1, s-s_1) C_2(r, s_2-s_1, s_3-s_2, s-s_2) \\
&\quad + C_3^{1-\frac{1}{p}}(r, s_2-s_1, s_3-s_2, s_4-s_3, s-s_3) M_3^{\frac{1}{p}}(r, s_2-s_1, \bar{s}_3-\bar{s}_2, s-\bar{s}_3) \Big) \mathrm{d}s \\
&\le \|u_0\|_{L^1}^{\frac{1}{p}} C(r, s_2-s_1, s_3-s_2, s_4-s_3, \bar{s}_3-\bar{s}_2, t-s_1)(t_0-\bar{s}_3)^{-\frac{3\alpha}{4}} \int_{t_0}^{t} (t-s)^{\frac{\alpha}{4}-1} \mathrm{d}s \\
&\le \|u_0\|_{L^1}^{\frac{1}{p}} C(r, s_2-s_1, s_3-s_2, s_4-s_3, \bar{s}_3-\bar{s}_2, t-s_1)(t_0-\bar{s}_3)^{-\frac{3\alpha}{4}} (t-t_0)^{\frac{\alpha}{4}},
\end{aligned}
$$

where we used the monotonicity of C_1, C_2, C_3 and M_3 with respect to t. Obviously, this term vanishes as $t \to t_0^+$.

For the third term I_3, we denote

$$
\begin{aligned}
f(t,s) = \Big\| &(t-s)^{\alpha-1} D^4 \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(u(s)) \\
&- (t_0-s)^{\alpha-1} D^4 \Delta E_{\alpha,\alpha}(-(t_0-s)^{\alpha}\mathcal{A})\varphi(u(s)) \Big\|_{L^p(\mathbb{R}^3)}
\end{aligned}
$$

and it is easy to see that

$$
\begin{aligned}
f(t,s) &\le \left| (t-s)^{\alpha-1} - (t_0-s)^{\alpha-1} \right| \cdot \left\| D^4 \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(u(s)) \right\|_{L^p(\mathbb{R}^3)} \\
&\quad + (t_0-s)^{\alpha-1} \Big\| D^4 \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(u(s)) \\
&\qquad - D^4 \Delta E_{\alpha,\alpha}(-(t_0-s)^{\alpha}\mathcal{A})\varphi(u(s)) \Big\|_{L^p(\mathbb{R}^3)} \\
&\le C \left| (t-s)^{\alpha-1} - (t_0-s)^{\alpha-1} \right| \cdot (t-s)^{-\frac{\alpha}{2}} \|u_0 - \bar{u}\|_{L^p(\mathbb{R}^3)} \\
&\quad + (t_0-s)^{\alpha-1} \Big\| D^4 \Delta E_{\alpha,\alpha}(-(t-s)^{\alpha}\mathcal{A})\varphi(u(s)) \\
&\qquad - D^4 \Delta E_{\alpha,\alpha}(-(t_0-s)^{\alpha}\mathcal{A})\varphi(u(s)) \Big\|_{L^p(\mathbb{R}^3)}.
\end{aligned}
$$

Combining with the strong continuity of $D^4 \Delta E_{\alpha,\alpha}(-t^{\alpha}\mathcal{A})$ on $L^p(\mathbb{R}^3)$, we have

$$
\lim_{t \to t_0^+} f(t,s) = 0
$$

for any fixed $s \in (0, t_0)$. In addition, we can control I_3 by the following two inequalities,

$$\begin{aligned}
I_3' &= \int_{\frac{t_0}{2}}^{t_0} \Big\| (t-s)^{\alpha-1} D\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) D^3\varphi(u(s)) \\
&\quad -(t_0-s)^{\alpha-1} D\Delta E_{\alpha,\alpha}(-(t_0-s)^\alpha \mathcal{A}) D^3\varphi(u(s)) \Big\|_{L^p(\mathbb{R}^3)} \,\mathrm{d}s \\
&\le C \int_{\frac{t_0}{2}}^{t_0} (t_0-s)^{\frac{\alpha}{4}-1} \Big\| D^3\varphi(u(s)) \Big\|_{L^p(\mathbb{R}^3)} \,\mathrm{d}s \\
&\le C \int_{\frac{t_0}{2}}^{t_0} (t_0-s)^{\frac{\alpha}{4}-1} \Big(\|Du\|_{L^\infty}^{3-\frac{1}{p}} \|Du\|_{L^1}^{\frac{1}{p}} \\
&\quad + \|Du\|_{L^\infty}^{1-\frac{1}{p}} \|Du\|_{L^1}^{\frac{1}{p}} \|D^2u\|_{L^\infty} + \|D^3u\|_{L^\infty}^{1-\frac{1}{p}} \|D^3u\|_{L^1}^{\frac{1}{p}} \Big) \mathrm{d}s \\
&\le \int_{\frac{t_0}{2}}^{t_0} (t_0-s)^{\frac{\alpha}{4}-1} (s-\bar{s}_3)^{-\frac{3\alpha}{4}} \|u_0\|_{L^1}^{\frac{1}{p}} \Big(C_1^{3-\frac{1}{p}}(r, s_2-s_1, s-s_1) \\
&\quad + C_1^{1-\frac{1}{p}}(r, s_2-s_1, s-s_1) C_2(r, s_2-s_1, s_3-s_2, s-s_2) \\
&\quad + C_3^{1-\frac{1}{p}}(r, s_2-s_1, s_3-s_2, s_4-s_3, s-s_3) M_3^{\frac{1}{p}}(r, s_2-s_1, \bar{s}_3-\bar{s}_2, s-\bar{s}_3) \Big) \mathrm{d}s \\
&\le \|u_0\|_{L^1}^{\frac{1}{p}} C(r, s_2-s_1, s_3-s_2, s_4-s_3, \bar{s}_3-\bar{s}_2, t_0-s_1)(t_0-\bar{s}_3)^{-\frac{3\alpha}{4}} \Big(\frac{t_0}{2}\Big)^{\frac{\alpha}{4}},
\end{aligned}$$

and

$$\begin{aligned}
I_3'' &= \int_0^{\frac{t_0}{2}} \Big\| (t-s)^{\alpha-1} D^4\Delta E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) \varphi(u(s)) \\
&\quad -(t_0-s)^{\alpha-1} D^4\Delta E_{\alpha,\alpha}(-(t_0-s)^\alpha \mathcal{A}) \varphi(u(s)) \Big\|_{L^p(\mathbb{R}^3)} \,\mathrm{d}s \\
&\le C \int_0^{\frac{t_0}{2}} (t_0-s)^{-\frac{\alpha}{2}-1} \|\varphi(u(s))\|_{L^p(\mathbb{R}^3)} \,\mathrm{d}s \\
&\le C(r, \|u_0\|_{L^1}) \Big(\frac{t_0}{2}\Big)^{-\frac{\alpha}{2}},
\end{aligned}$$

which implies

$$I_3 = I_3' + I_3'' \le \int_0^{t_0} g(s)\,\mathrm{d}s = C(r, t_0, \|u_0\|_{L^1}),$$

for some integrable function g and constant C independent of t. Then the application of dominated convergence theorem to I_3 shows that $I_3 \to 0$ as $t \to t_0^+$.

Analogously, we can also obtain the same limit of $\|D^4u(t) - D^4u(t_0)\|_{L^p(\mathbb{R}^3)}$ as $t \to t_0^-$ with $t_0 \in (0, \infty)$. Therefore,

$$\lim_{t\to t_0} \|D^4u(t) - D^4u(t_0)\|_{L^p(\mathbb{R}^3)} = 0, \quad t_0 \in (0, \infty),$$

which completes the proof. □

Next, we are going to prove Theorem 2.2. Several additional properties of both families of Mittag-Leffler operators are necessary in our proof which can be found in [8,37].

Proposition 5.1. *Let* $0 < \alpha < 1$ *and* $v \in \mathring{L}^p(\mathbb{R}^3)$. *Then we have*

(i) ${}_cD_t^\alpha E_\alpha\left(-t^\alpha \mathcal{A}\right) v = -\mathcal{A} E_\alpha\left(-t^\alpha \mathcal{A}\right) v,\ t > 0;$
(ii) $\frac{\mathrm{d}}{\mathrm{d}t} E_\alpha\left(-t^\alpha \mathcal{A}\right) v = -t^{\alpha-1} \mathcal{A}\, E_{\alpha,\alpha}\left(-t^\alpha \mathcal{A}\right) v,\ t > 0;$
(iii) $E_\alpha(-t^\alpha \mathcal{A}) v = J_t^{1-\alpha}(t^{\alpha-1} E_{\alpha,\alpha}\left(-t^\alpha \mathcal{A}\right) v),\ t > 0.$

Proof of Theorem 2.2. Let $u \in C([0,\infty); L^1(\mathbb{R}^3))$ be the global mild solution of the problem (1.1). By Theorem 5.1, we have $u \in C((0,\infty); W^{4,1}(\mathbb{R}^3))$. Denote

$$u = u_1 - u_2,$$

where

$$u_1 = E_\alpha(-t^\alpha \mathcal{A}) u_0 \quad \text{and} \quad u_2 = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) \Delta\varphi(u)\, \mathrm{d}s.$$

By Proposition 5.1 (i) and Theorem 5.1, it's to see that

$${}_cD_t^\alpha u_1 = -\mathcal{A}\, E_\alpha\left(-t^\alpha \mathcal{A}\right) u_0 = -\mathcal{A}\, u_1 \in C((0,\infty); L^1(\mathbb{R}^3)).$$

Next we show ${}_cD_t^\alpha u_2 \in C((0,\infty); L^1(\mathbb{R}^3))$. Noting $u_2(0) = 0$ and combining with Proposition 5.1 (iii), we derive that

$$\begin{aligned} {}_cD_t^\alpha u_2(t) &= \frac{\mathrm{d}}{\mathrm{d}t} J_t^{1-\alpha} u_2(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} J_t^{1-\alpha} \left(\mathcal{Q}_\alpha * \Delta\varphi(u)\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \left(J_t^{1-\alpha} \mathcal{Q}_\alpha\right) * \Delta\varphi(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} E_\alpha(-t^\alpha \mathcal{A}) * \Delta\varphi(u), \end{aligned}$$

where $\mathcal{Q}_\alpha(t) := t^{1-\alpha} E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})$. So it suffices to prove that

$$w(t) := (E_\alpha(-t^\alpha \mathcal{A}) * \Delta\varphi(u))(t) \in C^1((0,\infty); L^1(\mathbb{R}^3)).$$

Fix $t_0 > 0$ and take $0 < h < 1$. Then we can easily to obtain that

$$\begin{aligned} \frac{w(t_0+h) - w(t_0)}{h} &= \frac{1}{h} \int_0^{t_0} \Big(E_\alpha(-(t_0+h-s)^\alpha \mathcal{A}) - E_\alpha(-(t_0-s)^\alpha \mathcal{A}) \Big) \Delta\varphi(u)\, \mathrm{d}s \\ &\quad + \frac{1}{h} \int_{t_0}^{t_0+h} E_\alpha(-(t_0+h-s)^\alpha \mathcal{A}) \Delta\varphi(u)\, \mathrm{d}s \\ &:= I_1 + I_2. \end{aligned} \tag{5.25}$$

Similarly with (5.23) and (5.24), we obtain there exists $s_0 \in (0, t_0)$ such that

$$\int_0^{t_0} (t_0 - s)^{\alpha-1} \|\mathcal{A} E_{\alpha,\alpha}\left(-t_0^\alpha \mathcal{A}\right) \Delta\varphi(u)\|_{L^1(\mathbb{R}^3)}\, \mathrm{d}s \le C\|u_0\|_{L^1(\mathbb{R}^3)},$$

where C is dependent of $t_0 - s_0$. Therefore, applying dominated convergence theorem and Proposition 5.1 (ii) to the first term I_1, we get that

$$\begin{aligned}\lim_{h\to 0^+} I_1 &= \lim_{h\to 0^+} \frac{1}{h}\int_0^{t_0} \Big(E_\alpha(-(t_0+h-s)^\alpha \mathcal{A}) - E_\alpha(-(t_0-s)^\alpha \mathcal{A})\Big)\Delta\varphi(u)\, \mathrm{d}s \\ &= -\int_0^{t_0} (t_0-s)^{\alpha-1}\mathcal{A} E_{\alpha,\alpha}\left(-(t_0-s)^\alpha \mathcal{A}\right) \Delta\varphi(u)\, \mathrm{d}s \\ &= -\mathcal{A} u_2(t_0). \end{aligned} \tag{5.26}$$

Furthermore, note that

$$\begin{aligned} I_2 &= \frac{1}{h}\int_{t_0}^{t_0+h} E_\alpha(-(t_0+h-s)^\alpha \mathcal{A})\Delta\varphi(u(s))\, \mathrm{d}s \\ &= \frac{1}{h}\int_{t_0}^{t_0+h} \Big(E_\alpha(-(t_0+h-s)^\alpha \mathcal{A}) - I\Big)\Delta\varphi(u(s))\, \mathrm{d}s \\ &\quad + \frac{1}{h}\int_{t_0}^{t_0+h} \Delta\varphi(u(s))\, \mathrm{d}s. \end{aligned}$$

By the strongly continuity property of $E_\alpha(-t^\alpha \mathcal{A})$ over $[0, \infty)$ and the fact that $\Delta\varphi(u) \in C((0, \infty); L^1(\mathbb{R}^3))$, we can see that

$$\lim_{h\to 0^+} I_2 = \Delta\varphi(u(t_0)). \tag{5.27}$$

Combining (5.26) with (5.27), we deduce that w is differential from the right at t_0 and $w'_+(t_0) = -\mathcal{A} u_2(t_0) + \Delta\varphi(u(t_0))$. Analogously, we can also obtain the same limit of $w'_-(t_0)$ as $h \to 0^-$.

Hence, we see that ${}_cD_t^\alpha u \in C((0, \infty); L^1(\mathbb{R}^3))$ and

$${}_cD_t^\alpha u = -\mathcal{A} u + \Delta\varphi(u), \quad t > 0,$$

which completes the proof of Theorem 2.2. □

Acknowledgements

The correspondence author thanks Prof. Keith Promislow for careful reading of the manuscript and insightful suggestions. All authors would like to express sincere thanks to the referee for patient and careful reading and give valuable suggestions and comments to improve the quality of this manuscript greatly.

REFERENCES

[1] B. Andrade, A.N. Carvalho, P.M. Carvalho-Neto, and P. Marín-Rubio, Semilinear fractional differential equations: global solutions, critical nonlinearities and comparison results, Topol. Methods Nonlinear Anal., 45(2) (2015) 439–467.

[2] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer, Berlin 2011.

[3] J. Bricmont, A. Kupiainen, and J. Taskinen, Stability of Cahn–Hilliard fronts, Commun. Pure Appl. Math., 52 (1999), 839–871.

[4] L. A. Caffarelli, and N. E. Muler, An L^∞ bound for solutions of the Cahn–Hilliard equation, Arch. Ration. Mech. Anal., 133 (1995), 129–144.

[5] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, J. Chem. Phys., 28 (1958), 258–267.

[6] R. Caponetto, G. Dongola, L. Fortuna and I. Petrá, Fractional order systems: modeling and control applications, World Scientific, Singapore, 2010.

[7] P. M. Carvalho-Neto, and G. Planas, Mild solutions to the time fractional Navier–Stokes equations in $\mathbb{R}^N$, J. Differ. Equ., 259 (2015), 2948–2980.

[8] P. M. Carvalho-Neto, Fractional differential equations: a novel study of local and global solutions in Banach spaces, PhD thesis, Universidade de Sao Paulo, Sao Carlos, 2013.

[9] S. D. Eidelman, and A. N. Kochubei, Cauchy problem for fractional diffusion equations, J. Differ. Equ., 199 (2004), 211–255.

[10] K. J. Engel, R. Nagel, One-Parameter semigroups for linear evolution equations. Springer, Berlin, 2000.

[11] W. G. Glockle, and T. F. Nonnenmacher, A fractional calculus approach to self-similar protein dynamics, Biophysical Journal, 68 (1995), 46–53.

[12] S. Guo, L. Mei, Z. Zhang, J. Chen, Y. He, and Y. Li, Finite difference/Hermite-Galerkin spectral method for multi-dimensional time-fractional nonlinear reaction-diffusion equation in unbounded domains, Appl. Math. Model., 70 (2019), 246–263.

[13] O. S. Iyiola, and F. D. Zaman, A fractional diffusion equation model for cancer tumor, AIP Advances, 4, 107121 (2014), 1–16.

[14] M. Janev, S. Pilipovic, T. Atanackovic, R. Obradovic, and N. Ralevic, Fully fractional anisotropic diffusion for image denoising, Math. Comput. Modell., 54 (2011), 729–741.

[15] J. Kemppainen, J. Siljander, R. Zacher, Representation of solutions and large-time behavior for fully nonlocal diffusion equations, J. Differ. Equ., 263 (2017), 149–201.

[16] T. A. M. Langlands, B. I. Henry, and S. L. Wearne, Fractional cable equation models for anomalous electrodiffusion in nerve cells: finite domain solutions, SIAM J. Appl. Math., 71 (2011), 1168–1203.

[17] T. A. M. Langlands, B. I. Henry, and S. L. Wearne, Fractional cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions, J. Math. Biol., 59 (2009), 761–808.

[18] L. Li, and J. G. Liu, A generalized definition of caputo derivatives and its application to fractional ODEs, SIAM J. Math. Anal., 50 (2018), 2867–2900.

[19] L. Li, and J. G. Liu, Some compactness criteria for weak solutions of time fractional PDEs, SIAM J. Math. Anal., 50 (2018), 3963–3995.

[20] Y. Li, F. Liu, I. W. Turner, and T. Li, Time-fractional diffusion equation for signal smoothing, Appl. Math. Comput., 326 (2018), 108–116.

[21] L. Li, J. G. Liu, and L. Z. Wang, Cauchy problems for Keller–Segel type time-space fractional diffusion equation, J. Differ. Equ., 265 (2018), 1044–1096.

[22] H. Liu, A. Cheng, H. Wang, and J. Zhao, Time-fractional Allen–Cahn and Cahn–Hilliard phase-field models and their numerical investigation, Comput. Math. Appl., 76 (2018), 1876–1892.

[23] S. Q. Liu, F. Wang, and H. J. Zhao, Global existence and asymptotics of solutions of the Cahn–Hilliard equation, J. Differ. Equ., 238 (2007), 426–469.

[24] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, Ser. Adv. Math. Appl. Sci., 23 (1994) 246–251.

[25] R. Metzler, E. Barkai, and J. Klafter, Anomalous diffusion and relaxation close to thermal equilibrium: A fractional Fokker-Planck equation approach, Physical Review Letters, 82 (1999), 3563–3567.

[26] R. Metzler, and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A Math. Gen., 37 (2004), R161–R208.
[27] C. X. Miao, B. Q. Yuan, B. Zhang, Well-posedness of the Cauchy problem for the fractional power dissipative equations, Nonlinear Analysis, 68 (2008), 461–484.
[28] I. Podlubny, Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers, IEEE Trans. Automat. Control, 44 (1999), 208–214.
[29] J. Prüss, Evolutionary Integral Equations and Applications, Monographs in Mathematics 87, Birkhäuser, Basel, 1993.
[30] J. Prüss, V. Vergara and R. Zacher, Well-posedness and long-time behaviour for the non-isothermal Cahn-Hilliard equation with memory, Discrete Contin. Dyn. Syst., 26 (2010), 625–647.
[31] M. Raberto, E. Scalas, and F. Mainardi, Waiting-times and returns in high-frequency financial data: an empirical study, Phys. A, 314 (2002), 749–755.
[32] M. G. Sakar, O. Saldir, and F. Erdogan, An iterative approximation for time-fractional Cahn-Allen equation with reproducing kernel method, Comput. Appl. Math., 37 (2018), 5951–5964.
[33] S. G. Samko, A.A. Kilbas, and O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.
[34] W. A. Strauss, Decay and asymptotic for $u_{tt} - \Delta u = F(u)$, J. Funct. Anal., 2 (1968), 409–457.
[35] T. Tang, H.J. Yu, and T. Zhou, On Energy Dissipation Theory and Numerical Stability for Time-Fractional Phase-Field Equations, SIAM J. Sci. Comput., 41(6) (2019), A3757–A3778.
[36] N. K. Tripathi, S. Das, S. H. Ong, H. Jafari, and M. M. Al Qurashi, Solution of time-fractional Cahn-Hilliard equation with reaction term using homotopy analysis method, Advances in Mechanical Engineering, 9 (2017).
[37] R. N. Wang, D. H. Chen, T.J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, J. Differ. Equ., 252 (2012), 202–235.
[38] H. C. Yaslan, New analytic solutions of the space-time fractional Cahn–Hilliard equations, Optik, 130 (2017), 990–995.
[39] J. Zhao, L. Chen, and H. Wang, On power law scaling dynamics for time-fractional phase field models during coarsening, Commun. Nonlinear Sci. Numer. Simul., 70 (2019), 257–270.

Hailong Ye, Qiang Liu and Zhi-Min Chen
College of Mathematics and Statistics
Shenzhen University
Shenzhen 518060
China
E-mail: matliu@szu.edu.cn

Hailong Ye
Shenzhen Key Laboratory of Advanced Machine Learing and Applications
Shenzhen University
Shenzhen 518060
China