



Attainability of time-periodic flow of a viscous liquid past an oscillating body

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Abstract. A body \mathcal{B} is started from rest by a translational motion in an otherwise quiescent Navier–Stokes liquid filling the whole space. We show, for small data, that if after some time \mathcal{B} reaches a spinless oscillatory motion of period \mathcal{T} , the liquid will eventually execute also a time periodic motion with the same period \mathcal{T} . This result is a suitable generalization of the famous Finn’s starting problem for steady states, to the case of time-periodic motions.

1. Introduction

Consider a rigid body, \mathcal{B} , at rest and completely immersed in a quiescent Navier–Stokes liquid filling the whole three-dimensional space, Ω , outside \mathcal{B} . Next, suppose that at time $t = 0$ (say), \mathcal{B} is smoothly set in translational motion (no spin) and that after the time $t = 1$ (say), its velocity $\eta = \eta(t)$ coincides with a periodic function, $\xi = \xi(t)$, of period \mathcal{T} whose average over the time interval $[0, \mathcal{T}]$ vanishes. In the particular case where both η and ξ are parallel to a given direction, the above means that \mathcal{B} is brought from rest to a regime where it oscillates between two fixed configurations. In the general case, \mathcal{B} is taken from rest to a (spinless) motion where its center of mass moves periodically along a given closed curve.

On physical grounds, it is expected that under the given assumptions, the liquid will eventually reach a time-periodic flow of period \mathcal{T} , at least if the magnitude of η and (possibly) some of its derivatives are not “too large.” This specific circumstance is often referred to as *attainability property* of the flow. In this regard, it is worth mentioning a famous problem of attainability, the so-called Finn’s starting problem [2] where \mathcal{B} accelerates (without spinning) from rest to a given constant translational velocity. In such a case, the terminal flow of the liquid is expected to be at steady state. Finn’s problem was eventually and affirmatively solved by Galdi et al. [5] and, with more general assumptions, very recently by Hishida and Maremonti [10].

In analogy with these results, the main objective of this paper will be to show that under the given hypothesis on the motion of \mathcal{B} , the liquid indeed attains a corresponding time-periodic flow of period \mathcal{T} .

G. P. Galdi: Partially supported by NSF Grant DMS-1614011.

T. Hishida: Partially supported by the Grant-in-Aid for Scientific Research 18K03363 from JSPS.

We shall next give a rigorous mathematical formulation of the problem. Let us begin to observe that the translational velocity $\eta(t)$ can be written as:

$$\eta(t) = h(t)\xi(t),$$

where we assume

$$\begin{aligned} \xi(t + T) &= \xi(t) \quad (t \in \mathbb{R}), & \int_0^T \xi(t) dt &= 0, \\ \xi &\in W^{3,2}(0, T) = W^{3,2}(0, T; \mathbb{R}^3), \end{aligned} \tag{1.1}$$

and

$$h \in C^1(\mathbb{R}; [0, 1]), \quad h(t) = 0 \quad (t \leq 0), \quad h(t) = 1 \quad (t \geq 1). \tag{1.2}$$

The governing equations of the liquid flow, driven by the translational velocity η of the body, are thus given by

$$\begin{cases} \partial_t u + u \cdot \nabla u = \Delta u + \eta(t) \cdot \nabla u - \nabla p_u, \\ \operatorname{div} u = 0 \\ u|_{\partial\Omega} = \eta(t), \end{cases} \quad \text{in } \Omega \times (0, \infty), \tag{1.3}$$

$$\begin{aligned} u &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ u(\cdot, 0) &= 0, \end{aligned}$$

where $u = u(x, t)$ and $p_u = p_u(x, t)$ are, respectively, the velocity vector field and pressure field of the liquid, and Ω (the exterior of the body in \mathbb{R}^3) is assumed to have a sufficiently smooth boundary $\partial\Omega$. Likewise, if the translational velocity of \mathcal{B} is the time-periodic function ξ , it is reasonable to expect that the corresponding velocity field of the liquid $v = v(x, t)$ is time-periodic of period T (T -periodic) as well, and obeys the following equations:

$$\begin{cases} \partial_t v + v \cdot \nabla v = \Delta v + \xi(t) \cdot \nabla v - \nabla p_v, \\ \operatorname{div} v = 0 \\ v|_{\partial\Omega} = \xi(t), \end{cases} \quad \text{in } \Omega \times \mathbb{R}/T\mathbb{Z}, \tag{1.4}$$

$$v \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

where p_v denotes the pressure associated with v .

In [4], the first author showed existence, uniqueness and regularity of a T -periodic solution $(v(t), p_v(t))$ to (1.4) for all “small” $\xi(t)$ satisfying (1.1). Furthermore, he provided a detailed analysis of the asymptotic representation of $v(t)$ at spatial infinity, by showing that the leading term of $v(t)$ is given by a distinctive *steady-state* velocity field $U(x)$ that decays at large spatial distances like $|x|^{-1}$. Therefore, in general, $v(t) \notin L^2(\Omega)$, for all $t \in \mathbb{R}$.

Let

$$u = hv + w, \tag{1.5}$$

where h is the function given in (1.2). Then, from (1.3), we deduce that the “perturbation” $w(t)$ should obey, together with the corresponding pressure $p_w = p_u - hp_v$, the following system of equations:

$$\left. \begin{aligned} \partial_t w + w \cdot \nabla w + h(t)(v \cdot \nabla w + w \cdot \nabla v) \\ = \Delta w + \eta(t) \cdot \nabla w - \nabla p_w + f, \\ \operatorname{div} w = 0 \\ w|_{\partial\Omega} = 0, \\ w \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\ w(\cdot, 0) = 0, \end{aligned} \right\} \text{ in } \Omega \times (0, \infty), \tag{1.6}$$

with the forcing term ($h' := dh/dt$)

$$f := -h'v + (h - h^2)(v - \xi) \cdot \nabla v. \tag{1.7}$$

The desired attainability property consists then in showing that the solution $w(t)$ to (1.6) (exists, is unique and) tends to 0 as $t \rightarrow \infty$ in a suitable norm. In this respect, some comments are in order. Since $u(0) = 0$, one would expect that the solution $u(t)$ to (1.3) has finite energy, namely $u(t) \in L^2(\Omega)$ for all $t \geq 0$. Moreover, as noticed earlier on, $v(t)$ is, in general, *not* in L^2 . Consequently, in view of (1.5), $w(t)$ need not be in $L^2(\Omega)$, as also suggested by the fact that f is *not* in $L^2(\Omega)$. This implies that “energy-based methods” might not be an appropriate tool to analyze the asymptotic behavior of $w(t)$, and one has thus to resort to the more general L^q -theory. This difficulty is analogous to that encountered in Finn’s starting problem, which was in fact solved in [5] thanks to the asymptotic properties of the Oseen semigroup in L^q -spaces, proved for the first time in [13].

However, in comparison with [5], our problem presents the following *two further* complications. (i) The velocity field $v(t)$, $t \in \mathbb{R}$, possesses weaker summability properties at large spatial distances than its steady-state counterpart considered in [5]. This is due to the fact that $\xi(t)$ has zero average, see (1.1), so that, unlike [5], the motion of \mathcal{B} produces no wake structure in the flow. (ii) The non-autonomous character of the principal linear part is the second complication, where the drift term $\eta(t) \cdot \nabla w$ cannot be seen as a perturbation to the main (Stokes) operator, for all sufficiently large times. In order to overcome the difficulty in (i), we adapt to the case at hand the duality method developed by Yamazaki [18] that allows us to handle the additional linear terms $h(t)(v \cdot \nabla w + w \cdot \nabla v)$ in (1.6), in spite of the “poor” summability of v at large distances. As far as the other difficulty, we shall employ the theory recently developed in [7, 8] by the second author, which provides L^q - L^r decay estimates of the evolution operator, $\{T(t, s)\}_{t \geq s \geq 0}$, generated by the *non-autonomous* Oseen operator $-P[\Delta + \eta(t) \cdot \nabla]$ —with P Helmholtz projection on the space of L^q -vector fields—entirely analogous to those available in the autonomous case for Stokes and Oseen semigroups [11, 13, 14].

By suitably combining the above arguments and using the results in [4], in the present paper we are able to show, in particular, the decay to 0 of $w(t)$, as $t \rightarrow \infty$, in

appropriate L^q -spaces; see Theorem 2.1. Moreover, by developing an idea of Koba [12], we shall also show the decay of $w(t)$ in L^∞ -norm (see (2.8)). However, our proof—based on the L^∞ -estimate of the composite operator $T(t, s)P\operatorname{div}$ given in Proposition 3.3—turns out to be simpler and more direct than that given in [12].

The plan of the paper is as follows. In the next section, we shall state the main results, collected in Theorem 2.1. In Sect. 3, we present some results from [4, 7, 8] and deduce some relevant consequences. Sect. 4 is devoted the proof of Theorem 2.1.

Notation $C_{0,\sigma}^\infty(\Omega)$ is the subclass of vector functions u in $C_0^\infty(\Omega)$ with $\operatorname{div} u = 0$. By $L^q(\Omega)$, $1 \leq q \leq \infty$, $W^{m,q}(\Omega)$, $m \geq 0$, ($W^{0,q} \equiv L^q$), we denote usual Lebesgue and Sobolev classes of vector functions, with corresponding norms $\|\cdot\|_q$ and $\|\cdot\|_{m,q}$. Also, $L_\sigma^q(\Omega)$ denotes the completion of $C_{0,\sigma}^\infty(\Omega)$ in $L^q(\Omega)$, and $P : L^q \mapsto L_\sigma^q$ the associated Helmholtz projection [3, 15, 16]. For $1 < p < \infty$ and $1 \leq q \leq \infty$, let $L^{p,q}(\Omega)$ denote the Lorentz space with norm $\|\cdot\|_{p,q}$; see [1] for details about this space. Since P defines a bounded operator on $L^{p,q}(\Omega)$, we set $L_\sigma^{p,q}(\Omega) = P[L^{p,q}(\Omega)]$. Moreover, $D^{m,2}(\Omega)$ stands for the space of (equivalence classes of) functions $u \in L_{loc}^1(\Omega)$ such that $\sum_{|k|=m} \|D^k u\|_2 < \infty$. Obviously, the latter defines a seminorm in $D^{m,2}$. Let B be a function space of spatial variable endowed with seminorm $\|\cdot\|_B$. For $r \in [1, \infty]$, $T > 0$, $L^r(B)$ is the class of functions $u : (0, T) \rightarrow B$ such that

$$\|u\|_{L^r(B)} \equiv \begin{cases} \left(\int_0^T \|u(t)\|_B^r dt \right)^{\frac{1}{r}} < \infty, & \text{if } r \in [1, \infty); \\ \operatorname{ess\,sup}_{t \in [0, T]} \|u(t)\|_B < \infty, & \text{if } r = \infty. \end{cases}$$

Likewise, we put

$$W^{m,r}(B) = \left\{ u \in L^r(B) : \partial_t^k u \in L^r(B), k = 1, \dots, m \right\}.$$

2. Statement of main results

By use of the evolution operator $T(t, s)$ mentioned in the introductory section, problem (1.6) is transformed into the integral equation

$$w(t) = w_0(t) - \int_0^t T(t, s)P \operatorname{div} (Fw)(s) \, ds \tag{2.1}$$

with

$$w_0(t) = \int_0^t T(t, s)Pf(s) \, ds, \tag{2.2}$$

$$Fw = F_v w = w \otimes w + h(w \otimes v + v \otimes w). \tag{2.3}$$

Definition 2.1. We say that

$$w \in C_{w^*}((0, \infty); L_\sigma^{3,\infty}(\Omega)) \tag{2.4}$$

is a solution to (2.1) if the second term on the right-hand side of (2.1) is Bochner integrable in $L^3_\sigma{}^\infty(\Omega)$ and (2.1) is satisfied in $L^3_\sigma{}^\infty(\Omega)$ for every $t > 0$.

Since we see from Lemma 4.1 that $w_0(t) \in L^3_\sigma{}^\infty(\Omega)$, it is reasonable to look for a solution within the class (2.4). We emphasize that Proposition 3.3 plays an important role to verify the well-definedness of the second term on the right-hand side of (2.1) as the Bochner integral. Note, however, that the class (2.4) merely is not enough to ensure its well-definedness. The point is to find a solution being in an auxiliary space $C_{w^*}((0, \infty); L^q_\sigma{}^\infty(\Omega))$ with some $q > 3$ as well as (2.4), which actually implies the Bochner integrability of the second term on the right-hand side of (2.1) even in $L^3_\sigma{}^\infty(\Omega)$ rather than $L^3_\sigma{}^\infty(\Omega)$ thanks to (3.10) for the composite operator $T(t, s)P \operatorname{div}$.

The main result reads

Theorem 2.1. *Suppose (1.1) and (1.2) hold and let $|h'|_0 := \sup_{t \geq 0} |h'(t)|$. For every $\varepsilon \in (0, \frac{1}{4})$, there is a constant $\delta = \delta(\varepsilon)$ such that if*

$$\|\xi\|_{W^{3,2}(0, \mathcal{T})} \leq \frac{\delta}{1 + |h'|_0}, \tag{2.5}$$

then problem (2.1) admits a solution $w(t)$ of class (2.4) with the following properties:

1. The initial condition:

$$\lim_{t \rightarrow 0} \|w(t)\|_{3, \infty} = 0. \tag{2.6}$$

2. There is a constant $C > 0$ such that

$$\|w(t)\|_{3, \infty} \leq C(1 + |h'|_0)\|\xi\|_{W^{3,2}(0, \mathcal{T})} \tag{2.7}$$

for all $t \geq 0$.

3. The solution $w(t)$ is unique among solutions with small $\sup_{t > 0} \|w(t)\|_{3, \infty}$.
4. The solution $w(t)$ belongs to $L^q(\Omega)^3$ for every $q \in (3, \infty]$ and $t > 0$ together with the attainability:

$$\begin{aligned} \|w(t)\|_q &= \begin{cases} O(t^{-1/2+3/2q}), & q \in (3, q_0), \\ O(t^{-1/2+\varepsilon}), & q \in (q_0, \infty), \end{cases} \\ \|w(t)\|_{q_0, \infty} &= O(t^{-1/2+\varepsilon}), \end{aligned} \tag{2.8}$$

as $t \rightarrow \infty$, where $q_0 = 3/2\varepsilon$.

Remark 2.1. The unique existence of the evolution operator $T(t, s)$ or, in other words, the well-posedness of the initial boundary value problem for the linearized system, was successfully proved by Hansel and Rhandi [6] even in the case when the body \mathcal{B} rotates. The key point of their argument is how to overcome difficulties due to the rotational term; in fact, the Tanabe–Sobolevskii theory [17] of parabolic evolution operators does not work in this situation.

Remark 2.2. We apply the theory in [17] to the non-autonomous Oseen operator without rotation. Thus, in such a case, the regularity properties of $T(t, s)$ basically coincide with those of analytic semigroups for the autonomous case. As a consequence, one could show that the solution $w(t)$ in Theorem 2.1 becomes “strong” provided only $h'(t)$, in addition to satisfying (1.2), is Hölder continuous. We will not give details of such a claim, since our main objective is to show the attainability property.

Remark 2.3. We observe that our approach furnishes, in particular, also the stability of the time-periodic solution $v(t)$. In fact, this property can be established by studying an integral equation of the type (2.1) obtained by setting formally $h(t) \equiv 1$ (which implies that the term f in (1.7) vanishes identically) and replacing the function $w_0(t)$ with $\widetilde{w}_0(t) = T(t, 0)w(0)$, where $w(0)$ is the initial perturbation. One can slightly modify the proof of Theorem 2.1 to show that the asymptotic decay property of $w(t)$ stated in (2.8) continues to hold, provided, in addition to (2.5), that $w(0) \in L^3_{\sigma}(\Omega)$ with sufficiently small norm.

3. Preparatory results

Let us begin to recall the following result concerning the existence, uniqueness and asymptotic spatial behavior of solutions to (1.4). A pair of \mathcal{T} -periodic functions (v, p_v) is said to be a solution to (1.4) if every term in (1.4) is continuous in $(x, t) \in \overline{\Omega} \times \mathbb{R}/\mathcal{T}\mathbb{Z}$ and all the equations in (1.4) are pointwisely satisfied.

Proposition 3.1 [4]. *Let ξ satisfy (1.1). Then, there exists a constant $\varepsilon_0 > 0$ such that if*

$$D := \|\xi\|_{W^{3,2}(0,\mathcal{T})} < \varepsilon_0, \tag{3.1}$$

problem (1.4) has one and only one time-periodic solution (v, p_v) of period \mathcal{T} in the class

$$\begin{aligned} v &\in W^{2,2}(D^{2,2}) \cap W^{1,2}(D^{4,2}) \cap W^{2,\infty}(W^{1,2}) \cap L^\infty(D^{3,2}), \\ p_v &\in L^\infty(W^{1,2}) \cap W^{1,2}(D^{3,2}), \end{aligned}$$

with all corresponding norms of (v, p_v) bounded from above by D . Moreover, there exists a constant $C > 0$ such that this solution obeys, in particular, the following estimates:

$$\begin{aligned} (1 + |x|)|v(x, t)| + (1 + |x|^2)\{|\nabla v(x, t)| + |p_v(x, t)|\} \\ + (1 + |x|^3)\{|\nabla^2 v(x, t)| + |\nabla p_v(x, t)|\} \leq C D, \end{aligned} \tag{3.2}$$

for all $(x, t) \in \Omega \times \mathbb{R}/\mathcal{T}\mathbb{Z}$.

Remark 3.1. The constant δ in (2.5) of Theorem 2.1 must be taken smaller than ε_0 in (3.1).

The next result regards the large time behavior of the evolution operator $T(t, s)$ and its adjoint $T(t, s)^*$. These properties, among others, have been established in [7,8].

Proposition 3.2 [7,8]. *Let $m \in (0, \infty)$ and assume*

$$\sup_{t \geq 0} |\eta(t)| + \sup_{t > s \geq 0} \frac{|\eta(t) - \eta(s)|}{t - s} \leq m. \tag{3.3}$$

1. *Let $1 < q < \infty$ and $q \leq r \leq \infty$. Then, there is a constant $C = C(m, q, r, \Omega) > 0$ such that*

$$\|T(t, s)f\|_r \leq C(t - s)^{-(3/q-3/r)/2} \|f\|_q \tag{3.4}$$

for all $t > s \geq 0$, $f \in L^q_\sigma(\Omega)$ and that

$$\|T(t, s)f\|_{r,\infty} \leq C(t - s)^{-(3/q-3/r)/2} \|f\|_{q,\infty} \tag{3.5}$$

for all $t > s \geq 0$ and $f \in L^{q,\infty}_\sigma(\Omega)$.

2. *Let $1 < q \leq r \leq 3$. Then, there is a constant $C = C(m, q, r, \Omega) > 0$ such that*

$$\|\nabla T(t, s)^*g\|_r \leq C(t - s)^{-(3/q-3/r)/2-1/2} \|g\|_q \tag{3.6}$$

for all $t > s \geq 0$, $g \in L^q_\sigma(\Omega)$ and that

$$\|\nabla T(t, s)^*g\|_{r,1} \leq C(t - s)^{-(3/q-3/r)/2-1/2} \|g\|_{q,1} \tag{3.7}$$

for all $t > s \geq 0$ and $g \in L^{q,1}_\sigma(\Omega)$. If in particular $1/q - 1/r = 1/3$ as well as $1 < q < r \leq 3$, then there is a constant $C = C(m, q, \Omega) > 0$ such that

$$\int_0^t \|\nabla T(t, s)^*g\|_{r,1} ds \leq C \|g\|_{q,1} \tag{3.8}$$

for all $t > 0$ and $g \in L^{q,1}_\sigma(\Omega)$

Remark 3.2. In [7,8], the assumption on η is made in terms of the Hölder seminorm that is controlled by the left-hand side of (3.3), which is, in turn, controlled by D ; see (3.1). Estimate (3.5) with $r < \infty$ immediately follows from (3.4) by interpolation. The proof of $L^{q,\infty}$ - L^∞ estimate, that is, (3.5) with $r = \infty$, is not given in [7,8], but it can be easily proved by the use of the semigroup property, following the lines of the proof of (3.9)–(3.10) below with $r = \infty$. The remaining three bounds (3.6)–(3.8) are shown in [8]. However, we emphasize that (3.7) with $r = 3$ does not follow directly from (3.6) by interpolation. The idea of deducing (3.8) from (3.7) is, in fact, due to Yamazaki [18].

We next prove an important consequence of the previous proposition.

Proposition 3.3. *Let $m \in (0, \infty)$ and assume (3.3). The following properties hold.*

1. Let $3/2 \leq q < \infty$ and $q \leq r \leq \infty$. Then, there is a constant $C = C(m, q, r, \Omega) > 0$ such that the composite operator $T(t, s)P \operatorname{div}$ extends to a bounded operator from $L^q(\Omega)^{3 \times 3}$ to $L^r_\sigma(\Omega)$, $r < \infty$, and to $L^\infty(\Omega)^3$ subject to estimate

$$\|T(t, s)P \operatorname{div} F\|_r \leq C(t - s)^{-(3/q-3/r)/2-1/2} \|F\|_q \tag{3.9}$$

for all $t > s \geq 0$ and $F \in L^q(\Omega)^{3 \times 3}$.

2. Let $3/2 < q < r \leq \infty$. Then, there is a constant $C = C(m, q, r, \Omega) > 0$ such that the composite operator $T(t, s)P \operatorname{div}$ extends to a bounded operator from $L^{q,\infty}(\Omega)^{3 \times 3}$ to $L^r_\sigma(\Omega)$, $r < \infty$, and to $L^\infty(\Omega)^3$ subject to estimate

$$\|T(t, s)P \operatorname{div} F\|_r \leq C(t - s)^{-(3/q-3/r)/2-1/2} \|F\|_{q,\infty} \tag{3.10}$$

for all $t > s \geq 0$ and $F \in L^{q,\infty}(\Omega)^{3 \times 3}$.

Proof. By density, it suffices to show (3.9) for $F \in C^\infty_0(\Omega)^{3 \times 3}$. We first consider the case $3/2 \leq q \leq r < \infty$, so that $1 < r' \leq q' \leq 3$. By (3.6), we have

$$\begin{aligned} |\langle T(t, s)P \operatorname{div} F, \varphi \rangle| &= |\langle F, \nabla T(t, s)^* \varphi \rangle| \\ &\leq \|F\|_q \|\nabla T(t, s)^* \varphi\|_{q'} \\ &\leq C(t - s)^{-(3/q-3/r)/2-1/2} \|F\|_q \|\varphi\|_{r'} \end{aligned}$$

for all $t > s \geq 0$ and $\varphi \in L^{r'}_\sigma(\Omega)$, which leads to (3.9) with $r < \infty$. This combined with (3.4) ($r = \infty$) implies that

$$\begin{aligned} \|T(t, s)P \operatorname{div} F\|_\infty &\leq C(t - s)^{-3/4q} \|T((t + s)/2, s)P \operatorname{div} F\|_{2q} \\ &\leq C(t - s)^{-3/2q-1/2} \|F\|_q \end{aligned}$$

yielding (3.9) with $r = \infty$.

Let $3/2 < q \leq r < \infty$, then (3.9) implies

$$\|T(t, s)P \operatorname{div} F\|_{r,\infty} \leq C(t - s)^{-(3/q-3/r)/2-1/2} \|F\|_{q,\infty} \tag{3.11}$$

for all $t > s \geq 0$ and $F \in L^{q,\infty}(\Omega)^{3 \times 3}$. Since

$$\|u\|_r \leq C \|u\|_{r_0,\infty}^{1-\theta} \|u\|_{r_1,\infty}^\theta \tag{3.12}$$

where $1/r = (1 - \theta)/r_0 + \theta/r_1$ as well as $0 < \theta < 1$ and $1 < r_0 < r < r_1 \leq \infty$, we obtain (3.10) from (3.11) as long as $3/2 < q < r < \infty$. This combined with (3.4) ($r = \infty$) leads to (3.10) when $3/2 < q < r = \infty$. The proof is complete. \square

4. Proof of Theorem 2.1

Following Yamazaki [18], we consider the following weak form of (2.1):

$$\langle w(t), \varphi \rangle = \langle w_0(t), \varphi \rangle + \int_0^t \langle (Fw)(s), \nabla T(t, s)^* \varphi \rangle ds \quad \forall \varphi \in C^\infty_{0,\sigma}(\Omega). \tag{4.1}$$

For $q \in [3, \infty)$, let us introduce the space

$$X_q = \{w \in C_{w^*}((0, \infty)); L_{\sigma}^{3,\infty}(\Omega) \cap L_{\sigma}^{q,\infty}(\Omega); [w]_3 + [w]_q < \infty, \lim_{t \rightarrow 0} \|w(t)\|_{3,\infty} = 0\},$$

where

$$[w]_q := \sup_{t>0} t^{1/2-3/2q} \|w(t)\|_{q,\infty}. \tag{4.2}$$

Clearly, X_q becomes a Banach space when endowed with norm $[w]_3 + [w]_q$.

Under the smallness condition (3.1), the solution v obtained in Proposition 3.1 and the force f defined by (1.7) fulfill

$$v(t), f(t) \in L^{3,\infty}(\Omega) \cap L^{\infty}(\Omega)$$

with

$$\begin{aligned} \sup_{t \geq 0} (\|v(t)\|_{3,\infty} + \|v(t)\|_{\infty}) &\leq C\mathbb{D}, \\ \sup_{t \geq 0} (\|f(t)\|_{3,\infty} + \|f(t)\|_{\infty}) &\leq C(|h'|_0 + \mathbb{D})\mathbb{D}, \end{aligned} \tag{4.3}$$

which immediately follows from (3.2). This, combined with (3.5), implies the following lemma.

Lemma 4.1. *Suppose (1.1), (1.2) and (3.1). Then, the function w_0 defined by (2.2) belongs to X_q for every $q \in [3, \infty)$. Moreover, we have $w_0(t) \in L^{\infty}(\Omega)$ for each $t > 0$. Finally, for every $r \in [3, \infty]$, there is a constant $c_r > 0$ such that*

$$\|w_0(t)\|_{r,\infty} \leq c_r (|h'|_0 + \mathbb{D})\mathbb{D} (1+t)^{-1/2+3/2r} \tag{4.4}$$

for all $t > 0$, with \mathbb{D} given in (3.1).

Proof. Let $0 \leq t < t + \tau$, then we have

$$\begin{aligned} &w_0(t + \tau) - w_0(t) \\ &= \int_0^t \{T(t + \tau, s) - T(t, s)\} Pf(s) ds + \int_t^{t+\tau} T(t + \tau, s) Pf(s) ds =: J_1 + J_2. \end{aligned}$$

By (3.5) and (4.3), we know that

$$\|T(t, s) Pf(s)\|_{q,\infty} \leq C(|h'|_0 + \mathbb{D})\mathbb{D} =: C_0$$

with some constant $C = C(q) > 0$ independent of (t, s) for every $q \in [3, \infty)$. From the Lebesgue convergence theorem, we infer $J_1 \rightarrow 0$ as $\tau \rightarrow 0$, whereas it follows at once $J_2 \leq C_0\tau$. For the other case $0 < t/2 < t + \tau < t$, we have

$$w_0(t + \tau) - w_0(t) = \int_0^{t+\tau} \{T(t + \tau, s) - T(t, s)\} Pf(s) ds - \int_{t+\tau}^t T(t, s) Pf(s) ds$$

which goes to zero as $\tau \rightarrow 0$ by the same reasoning as above. Consequently, $w_0(t)$ is even strongly continuous up to $t = 0$ with values in $L^{q,\infty}(\Omega)$ as well as $\|w_0(t)\|_{3,\infty} \rightarrow 0$ ($t \rightarrow 0$). Concerning the estimate in $L^{r,\infty}(\Omega)$ with $r \in [3, \infty]$, we consider only the one involving $\|w_0(t)\|_\infty$, since the other ones are obtained similarly. Since $f(t) = 0$ for $t \geq 1$, we use (3.5) to find

$$\|w_0(t)\|_\infty \leq C \int_0^1 (t-s)^{-1/2} \|Pf(s)\|_{3,\infty} ds \leq Ct^{-1/2}(|h'|_0 + D)D$$

for $t \geq 2$, while we have

$$\|w_0(t)\|_\infty \leq Ct^{1/2}(|h'|_0 + D)D$$

for $t < 2$. We thus obtain the desired estimate. □

Let us begin to prove the uniqueness property. In fact, the solution obtained in Theorem 2.1 is unique in the sense of the following lemma, provided we choose the constant δ in (2.5) smaller than the constant δ_0 defined below.

Lemma 4.2. *There is a constant $\delta_0 > 0$ such that if $D \leq \delta_0$, then the solution to (4.1) is unique in the ball $\{w \in X_3; [w]_3 \leq \delta_0\}$.*

Proof. Let both $w, \tilde{w} \in X_3$ satisfy (4.1). By duality $L^{3,1}(\Omega)^* = L^{\sigma/2,\infty}(\Omega)$ together with the weak-Hölder inequality, we have

$$|\langle w(t) - \tilde{w}(t), \varphi \rangle| \leq C([w]_3 + [\tilde{w}]_3 + [v]_3)[w - \tilde{w}]_3 \int_0^t \|\nabla T(t, s)^* \varphi\|_{3,1} ds$$

for all $\varphi \in C_{0,\sigma}^\infty(\Omega)$. We employ (3.8) and (4.3) to obtain

$$[w - \tilde{w}]_3 \leq c_*([w]_3 + [\tilde{w}]_3 + D)[w - \tilde{w}]_3$$

by duality, which yields the assertion by taking $\delta_0 = 1/4c_*$. □

Given $\varepsilon \in (0, \frac{1}{4})$, we set $q_0 = 3/2\varepsilon \in (6, \infty)$ and intend to find a solution $w \in X_{q_0}$ to (4.1) provided D is small enough. Given $w \in X_{q_0}$ and $t > 0$, we define $(\Psi w)(t)$ by

$$\langle (\Psi w)(t), \varphi \rangle = \int_0^t \langle (Fw)(s), \nabla T(t, s)^* \varphi \rangle ds \quad \forall \varphi \in C_{0,\sigma}^\infty(\Omega).$$

We then find

$$\begin{aligned} [\Psi w]_3 &\leq C([w]_3 + [v]_3)[w]_3, \\ [\Psi w]_{q_0} &\leq C([w]_3 + [v]_3)[w]_{q_0}. \end{aligned} \tag{4.5}$$

The former is deduced along the same lines as in Lemma 4.2, while the latter is verified by splitting the integral as

$$\left(\int_0^{t/2} + \int_{t/2}^t \right) s^{-1/2+3/2q_0} \|\nabla T(t, s)^* \varphi\|_{r,1} ds =: I + II \tag{4.6}$$

where $r \in (3/2, 2)$ is determined by the condition $1/r = 2/3 - 1/q_0$. In fact, in view of (3.7), we get

$$I \leq C \int_0^{t/2} s^{-1/2+3/2q_0} (t-s)^{-1} ds \|\varphi\|_{q'_0,1}$$

that leads to the desired estimate, where $1/q'_0 + 1/q_0 = 1$. Also, employing (3.8), we show

$$II \leq C t^{-1/2+3/2q_0} \int_{t/2}^t \|\nabla T(t, s)^* \varphi\|_{r,1} ds.$$

By the same token, we can show

$$[\Psi w - \Psi \tilde{w}]_r \leq C([w]_3 + [\tilde{w}]_3 + [v]_3)[w - \tilde{w}]_r \quad r \in \{3, q_0\}, \tag{4.7}$$

for all $w, \tilde{w} \in X_{q_0}$.

The above computations are exactly the same as in [9, Section 8]. However, because in our case the equation is non-autonomous, the argument to show the continuity with respect to time is different from the one adopted by Yamazaki [18, Section 3] in which the strong continuity is deduced for $t > 0$. Here, we show merely the weak* continuity for $t > 0$, while we still have strong convergence to 0 at the initial time, namely

$$\|(\Psi w)(t)\|_{3,\infty} \leq C([w]_3 + [v]_3) \sup_{0 < s < t} \|w(s)\|_{3,\infty} \rightarrow 0$$

as $t \rightarrow 0$ (as well as the same property for $w_0(t)$; see Lemma 4.1). Actually, for $0 < t < t + \tau$ and $\varphi \in C_{0,\sigma}^\infty(\Omega)$, let us consider

$$\begin{aligned} \langle (\Psi w)(t + \tau) - (\Psi w)(t), \varphi \rangle &= \int_0^t \langle (Fw)(s), \nabla\{T(t + \tau, s)^* - T(t, s)^*\} \varphi \rangle ds \\ &\quad + \int_t^{t+\tau} \langle (Fw)(s), \nabla T(t + \tau, s)^* \varphi \rangle ds \tag{4.8} \\ &=: III + IV. \end{aligned}$$

Let $r \in (3/2, 2)$ be the same exponent as in (4.6). By using the backward semigroup property, we have

$$\begin{aligned} III &\leq C([w]_3 + [v]_3)[w]_{q_0} \int_0^t s^{-1/2+3/2q_0} \|\nabla T(t, s)^* \{T(t + \tau, t)^* \varphi - \varphi\}\|_{r,1} ds \\ &\leq C([w]_3 + [v]_3)[w]_{q_0} \|T(t + \tau, t)^* \varphi - \varphi\|_{3/2,1} \end{aligned}$$

which goes to zero as $\tau \rightarrow 0$ for all $\varphi \in C_{0,\sigma}^\infty(\Omega)$. Concerning the other part, we have

$$\begin{aligned} IV &\leq C([w]_3 + [v]_3)[w]_{q_0} \int_t^{t+\tau} s^{-1/2+3/2q_0} \|\nabla T(t + \tau, s)^* \varphi\|_{r,1} ds \\ &\leq C([w]_3 + [v]_3)[w]_{q_0} t^{-1/2+3/2q_0} \tau^{1/2-3/2q_0} \|\varphi\|_{3/2,1} \end{aligned}$$

for all $\varphi \in C_{0,\sigma}^\infty(\Omega)$, which implies the strong convergence with values in $L_\sigma^{3,\infty}(\Omega)$ also of this part. Summing up, by density argument, we can state that the left-hand side of (4.8) goes to zero as $\tau \rightarrow 0$ for all $\varphi \in L_\sigma^{3/2,1}(\Omega)$ and also for all $\varphi \in L_\sigma^{q_0',1}(\Omega)$ in view of (4.5). The case $0 < t/2 < t + \tau < t$ is similarly discussed with

$$\begin{aligned} & \langle (\Psi w)(t + \tau) - (\Psi w)(t), \varphi \rangle \\ &= \int_0^{t+\tau} \langle (Fw)(s), \nabla\{T(t + \tau, s)^* - T(t, s)^*\} \varphi \rangle ds \\ & \quad - \int_{t+\tau}^t \langle (Fw)(s), \nabla T(t, s)^* \varphi \rangle ds \end{aligned}$$

to conclude that Ψw is weak* continuous with values in $L_\sigma^{3,\infty}(\Omega)$ and in $L_\sigma^{q_0,\infty}(\Omega)$.

By these results, we can then conclude that $w_0 + \Psi w \in X_{q_0}$, for every $w \in X_{q_0}$. Assume now $D \leq 1$. By taking into account (4.3), (4.4), (4.5) and (4.7), one can easily show the existence of a fixed point $w \in X_{q_0}$ of the map

$$w \mapsto w_0 + \Psi w$$

in a closed ball of X_{q_0} with radius $2(c_3 + c_{q_0})(|h'|_0 + 1)D$, provided $(|h'|_0 + 1)D$ is small enough, where the smallness depends on ε (recall that $q_0 = 3/2\varepsilon > 6$). By Lemma 4.2, it is the only solution to (4.1) in the small within X_3 . From the interpolation inequality (3.12), the solution $w(t)$ satisfies (2.8) for $q \in (3, q_0)$.

For the solution $w(t)$ constructed above, it follows from (3.10) and (4.3) that the second term on the right-hand side of (2.1) is Bochner integrable with values in $L_\sigma^3(\Omega)$; in fact,

$$\int_0^t \|T(t, s)P \operatorname{div} (Fw)(s)\|_3 ds \leq C([w]_3 + [v]_3)[w]_{q_0}$$

for all $t > 0$. The latter, in conjunction with Lemma 4.1, shows that the weak form (4.1) leads, in fact, to the conclusion that the integral equation (2.1) is meaningful in $L_\sigma^{3,\infty}(\Omega)$. Moreover, by means of (3.10) with $r = \infty$, it turns out that the second term on the right-hand side of (2.1) is also Bochner integrable in $L^\infty(\Omega)$ because

$$\int_0^t \|T(t, s)P \operatorname{div} (Fw)(s)\|_\infty ds \leq C \left([w]_{q_0} t^{-1/2} + [v]_{q_0} t^{-3/2q_0} \right) [w]_{q_0} \quad (4.9)$$

for all $t > 0$, where the summability of the integral is ensured since $q_0 > 6$. This together with Lemma 4.1 implies $w(t) \in L^\infty(\Omega)$.

It remains to show (2.8) for the other case $q \in (q_0, \infty]$, $q_0 = 3/2\varepsilon$. To this end, on account of the interpolation inequality (3.12) as well as (4.4) with $r = \infty$, it is enough to prove the decay of the second term on the right-hand side of (2.1) in the L^∞ -norm. The argument that follows is inspired by Koba [12], but, unlike [12], we shall not use a duality procedure; rather, we will directly apply the $L^{q,\infty}$ - L^∞ estimate of the composite operator $T(t, s)P \operatorname{div}$ proved in Proposition 3.3. As a consequence,

the proof is considerably shortened and more direct. For the integral involving the nonlinear term, we have

$$\int_0^t \|T(t, s)P \operatorname{div} (w \otimes w)(s)\|_\infty ds \leq C[w]_{q_0}^2 t^{-1/2}$$

for all $t > 0$ as already observed in (4.9). Let $t > 2$. We split the other part of the integral of (2.1) into two parts

$$\left(\int_0^{t-1} + \int_{t-1}^t \right) \|T(t, s)P \operatorname{div} [h(w \otimes v + v \otimes w)](s)\|_\infty ds =: V + VI.$$

We utilize (3.10) with $r = \infty$ again to find that

$$V \leq C \int_0^{t-1} (t-s)^{-1-\varepsilon} \|v(s)\|_{3,\infty} \|w(s)\|_{q_0,\infty} ds = \int_0^{t/2} + \int_{t/2}^{t-1} =: V_1 + V_2$$

with

$$V_1 \leq CD[w]_{q_0} t^{-1-\varepsilon} \int_0^{t/2} s^{-1/2+\varepsilon} ds = CD[w]_{q_0} t^{-1/2}$$

as well as

$$V_2 \leq CD[w]_{q_0} t^{-1/2+\varepsilon} \int_1^{t/2} s^{-1-\varepsilon} ds \leq CD[w]_{q_0} t^{-1/2+\varepsilon}$$

and that

$$\begin{aligned} VI &\leq C \int_{t-1}^t (t-s)^{-(3/r+3/q_0)/2-1/2} \|v(s)\|_{r,\infty} \|w(s)\|_{q_0,\infty} ds \\ &\leq CD[w]_{q_0} t^{-1/2+\varepsilon} \int_0^1 s^{-(3/r+3/q_0)/2-1/2} ds = CD[w]_{q_0} t^{-1/2+\varepsilon} \end{aligned}$$

where $r \in (3, \infty)$ is chosen in such a way that $1/r + 1/q_0 < 1/3$, see (4.3). The proof is complete.

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Accepted: 23 November 2020