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Refined blow-up criteria for the full compressible Navier–Stokes equations involving temperature

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Abstract. In this paper, inspired by the study of the energy flux in local energy inequality of the 3D incompressible Navier–Stokes equations, we improve almost all the blow-up criteria involving temperature to allow the temperature in its scaling invariant space for the 3D full compressible Navier–Stokes equations. Enlightening regular criteria via pressure $\Pi = \frac{\mathrm{div\,div}}{-\Delta}(u_i u_j)$ of the 3D incompressible Navier–Stokes equations on bounded domain, we generalize Beirao da Veiga's result in (Chin Ann Math Ser B 16:407–412, 1995) from the incompressible Navier–Stokes equations to the isentropic compressible Navier–Stokes system in the case away from vacuum.

1. Introduction

We study the following system of Newton heat-conducting compressible fluid in three-dimensional space

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P(\rho, \theta) - \mu \Delta u - (\mu + \lambda)\nabla \mathrm{div}\, u = 0, \\ c_v[\rho \theta_t + \rho u \cdot \nabla \theta] + P \mathrm{div}\, u - \kappa \Delta \theta = \frac{\mu}{2}\left|\nabla u + (\nabla u)^{\mathrm{tr}}\right|^2 + \lambda(\mathrm{div}\, u)^2, \\ (\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \end{cases} \tag{1.1}$$

where ρ, u, θ stand for the flow density, velocity and the absolute temperature, respectively. The scalar function P represents the pressure, the state equation of which is determined by

$$P = R\rho\theta, \; R > 0, \tag{1.2}$$

and κ is a positive constant. μ and λ are the coefficients of viscosity, which are assumed to be constants, satisfying the following physical restrictions:

$$\mu > 0, \; 2\mu + 3\lambda \geq 0. \tag{1.3}$$

The initial conditions satisfy

$$(\rho_0(x), \; u_0(x), \; \theta_0(x)) \to (0, \; 0, \; 0), \; \text{as } |x| \to \infty. \tag{1.4}$$

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Note that if the triplet $(\rho(x,t), u(x,t), \theta(x,t))$ solves system (1.1), then the triplet $(\rho_\lambda, u_\lambda, \theta_\lambda)$ is also a solution of (1.1) for any $\lambda \in R^+$, where

$$\rho_\lambda = \rho(\lambda^2 t, \lambda x), \ u_\lambda = \lambda u(\lambda^2 t, \lambda x), \ \theta_\lambda = \lambda^2 \theta(\lambda^2 t, \lambda x). \tag{1.5}$$

There have been huge literatures on well-posedness of solutions to compressible Navier–Stokes equations; we only give a brief survey here. For the isentropic case, The first major breakthrough was made by Lions [28], where he first gave the global existence of weak solutions to the compressible Navier–Stokes equations when the constant $\gamma \geq \frac{3N}{N+2}$ for $N = 2$ or 3. Then, Feireisl et al. [12] improved the Lions' work to $\gamma > \frac{3}{2}$ for $N = 3$. In [23], Jiang and Zhang considered the spherical symmetric initial data and relaxed the restriction on γ to the case $\gamma > 1$. Huang et al. [19] obtained the global existence of classical solutions provided the initial energy is sufficiently small, but the oscillation can be large. When the shear viscosity coefficient $\mu = $ constant > 0 and bulk viscosity satisfies $\lambda(\rho) = \rho^\beta$, Vaigant–Kazhikhov [38] showed the two-dimensional system admits a unique strong solution in the periodic domain when $\beta > 3$; it is emphasized that the initial data contain no vacuum and can be arbitrarily large. For the case involving heat conductivity, Feireisl [11] got the existence of variational solutions when the dimension $N \geq 2$. It is noted that this is the very first attempt work in global existence of weak solutions for full compressible Navier–Stokes equations in high dimensions. Matsumura–Nishida [29] obtained the global classical solution for initial data close to a non-vacuum equilibrium in some Sobolev space H^s. Later, Hoff [21] considered the discontinuous case. In [6], the local strong solutions of equations (1.1) with initial data containing vacuum were established by Cho and Kim (for details, see Theorem 2.1 in Sect. 2). On the other hand, when the initial data contain vacuums, finite time blow-up of smooth solutions to the compressible Navier–Stokes system was discussed by Xin [40], Xin and Yan [41] and Jiu et al. [22]. Since then, a number of papers have been devoted to the study of blow-up mechanism of strong solutions mentioned above in (1.1) and many blow-up criteria are established (see for example, [5,7–10,15,18,20,27,31,32,35–37,39] and references therein). In particular, we list some works where vacuum is included as follows:

Suppose that $0 < T^* < \infty$ is the maximal time of existence of a strong solution of system (1.1).

Fan et al. [10]

$$\limsup_{t \to T^*} \left(\|\nabla u\|_{L^1(0,t;L^\infty)} + \|\theta\|_{L^\infty(0,t;L^\infty)} \right) = \infty, \ (\lambda < 7\mu); \tag{1.6}$$

Wen and Zhu [31],

$$\limsup_{t \to T^*} \left(\|\rho\|_{L^\infty(0,t;L^\infty)} + \|\theta\|_{L^\infty(0,t;L^\infty)} \right) = \infty, \ (\lambda < 3\mu); \tag{1.7}$$

Huang et al. [17]

$$\limsup_{t \to T^*} \left(\|\mathrm{div}\, u\|_{L^1(0,t;L^\infty)} + \|u\|_{L^p(0,t;L^q)} \right) = \infty, \ \frac{2}{p} + \frac{3}{q} = 1, \ q > 3; \tag{1.8}$$

Huang and Li [16]

$$\limsup_{t \to T^*} \left(\|\rho\|_{L^\infty(0,t;L^\infty)} + \|u\|_{L^p(0,t;L^q)} \right) = \infty, \quad \frac{2}{p} + \frac{3}{q} = 1, \ q > 3; \qquad (1.9)$$

Li et al. [27]

$$\limsup_{t \to T^*} \left(\|\rho\|_{L^\infty(0,t;L^\infty)} + \|P\|_{L^\infty(0,t;L^\infty)} \right) = \infty, \quad (\lambda < 3\mu, \ \kappa = 0); \qquad (1.10)$$

Wen and Zhu [32],

$$\limsup_{t \to T^*} \left(\|\rho\|_{L^\infty(0,t;L^\infty)} + \|\rho\theta\|_{L^4(0,t;L^{\frac{12}{5}})} \right) = \infty, \quad (\lambda < 3\mu); \qquad (1.11)$$

Wang and Li [39]

$$\limsup_{t \to T^*} \left(\|\mathrm{div}\, u\|_{L^2(0,t;L^\infty)} + \|\theta\|_{L^\alpha(0,t;L^\beta)} \right)$$
$$= \infty, \quad \frac{3}{\alpha} + \frac{2}{\beta} \geq 2, \ \frac{1}{\alpha} + \frac{2}{\beta} \leq 1, \ 1 \leq \alpha \leq 2, \ \beta \geq 4; \qquad (1.12)$$

Choe and Yang [7]

$$\limsup_{t \to T^*} \left(\|\rho\|_{L^\infty(0,t;L^\delta)} + \|\mathrm{div}\, u\|_{L^\infty(0,t;L^3)} \right.$$
$$\left. + \|\Delta\theta\|_{L^\infty(0,t;L^2)} \right) = \infty, \quad \text{for some } \delta \in (1,\infty). \qquad (1.13)$$

The interesting of (1.8) and (1.9) is that they are independent of the temperature and they are in scaling invariant norm in the sense of (1.5). From (1.5), the natural candidate invariant spaces of temperature θ are $L^q(0,T;L^q)$ with $\frac{2}{p} + \frac{3}{q} = 2$. Therefore, a natural question is whether one can show blow-up criteria for the full compressible Navier–Stokes equations involving temperature in its scaling-invariant space. The first objective of this paper is to address this issue, and we obtain

Theorem 1.1. *Suppose (ρ, u, θ) is the unique strong solution in Theorem 2.1 and $\lambda < 3\mu$. If the maximal existence time T^* is finite, then there holds*

$$\limsup_{t \to T^*} \|\rho\|_{L^\infty(0,t;L^\infty)} + \|\theta\|_{L^p(0,t;L^q)} = \infty, \qquad (1.14)$$

where p, q satisfying

$$\frac{2}{p} + \frac{3}{q} = 2, \quad q > \frac{3}{2}.$$

Remark 1.1. Note that (1.14) can be replaced by

$$\limsup_{t \to T^*} \|\mathrm{div}\, u\|_{L^1(0,t;L^\infty)} + \|\theta\|_{L^p(0,t;L^q)} = \infty,$$

or

$$\limsup_{t \to T^*} \|\nabla u\|_{L^1(0,t;L^\infty)} + \|\theta\|_{L^p(0,t;L^q)} = \infty,$$

which improves the known blow-up criteria (1.6).

Remark 1.2. Theorem 1.1 is an extension of corresponding results in (1.7), (1.11), (1.12) and (1.21).

We give some comments on the proof of Theorem 1.1. The proof is motivated by the investigation of regularity of suitable weak solutions to the 3D incompressible Navier–Stokes equations. Suitable weak solutions originated in pioneering work by Scheffer [33] and in the celebrated paper by Caffarelli et al. [3] obey the local energy inequality. Roughly speaking, the energy flux in local energy inequality is $\int_0^T \int |u|^3 \mathrm{d}x\mathrm{d}t$, which can be bounded by (see, e.g., [13,14,30])

$$\int_0^T \int |u|^3 \mathrm{d}x\mathrm{d}t \le C\Big(\|u\|_{L^\infty L^2}^2 + \|\nabla u\|_{L^2 L^2}^2\Big)\|u\|_{L^p L^q}, \quad \frac{2}{p}+\frac{3}{q}=2. \quad (1.15)$$

We would like to mention that the inequality (1.15) plays an important role in the proof of results in [13,14,30].

We turn our attentions back to the 3D compressible Navier–Stokes equations (1.1). Under the hypothesis $\|\rho\|_{L^\infty L^\infty}$ and $\lambda < 3\mu$, we observe that there holds the following energy estimate to system (1.1)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int &\Big[\mu|\nabla u|^2 + (\mu+\lambda)(\mathrm{div}\,u)^2 + \frac{1}{2\mu+\lambda}P^2 \\ &- 2P\mathrm{div}\,u + 2C_3 C_v\rho\theta^2 + 2(C_4+1)\rho|u|^4\Big] \\ &+ \kappa\int|\nabla\theta|^2 + \int\rho|\dot u|^2 + \int|u|^2|\nabla u|^2 \le C\int\rho|\theta|^3 + C\int\rho|u|^2|\theta|^2,\end{aligned} \quad (1.16)$$

where $\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}\,u)^2+\frac{1}{2\mu+\lambda}P^2-2P\mathrm{div}\,u+2C_3 C_v\rho\theta^2+2(C_4+1)\rho|u|^4 \ge \mu|\nabla u|^2+\rho\theta^2+2(C_4+1)\rho|u|^4 > 0$ provided that the positive constant C_3 is suitably large. The key point is that the two terms of right hand side in the preceding inequality are parallel to (1.15). This helps us to prove Theorem 1.1.

Without the restriction $\lambda < 3\mu$, we have

Theorem 1.2. *Suppose (ρ, u, θ) is the unique strong solution in Theorem 2.1. If the maximal existence time T^* is finite, then one of the following results holds, for p, q meeting*

$$\frac{2}{p}+\frac{3}{q}=2,\ q>\frac{3}{2},$$

(1)

$$\limsup_{t\to T^*}\Big(\|\rho\|_{L^\infty(0,t;L^\infty)} + \|div\,u\|_{L^2(0,t;L^3)} + \|\theta\|_{L^p(0,t;L^q)}\Big) = \infty; \quad (1.17)$$

(2)

$$\limsup_{t\to T^*}\Big(\|div\,u\|_{L^1(0,t;L^\infty)} + \|div\,u\|_{L^4(0,t;L^2)} + \|\theta\|_{L^p(0,t;L^q)}\Big) = \infty; \quad (1.18)$$

(3)

$$\limsup_{t \to T^*} \left(\|div\, u\|_{L^2(0,t;L^\infty)} + \|\theta\|_{L^p(0,t;L^q)} \right) = \infty. \tag{1.19}$$

Remark 1.3. One can replace $\|\rho\|_{L^\infty(0,t;L^\infty)}$ in (1.17) by $\|div\, u\|_{L^1(0,t;L^\infty)}$, which improves the known blow-up criteria (1.12).

Though (1.17) and (1.18) involve all the quantities in equations (1.1), they are in scaling-invariant spaces in the sense of (1.5). We explain the motivation of (1.17) and (1.18). It is known that the velocity u (Serrin type), gradient ∇u (Beirao da Veiga type), vorticity curl u or pressure $\Pi = \frac{divdiv}{-\Delta}(u_i u_j)$ in scaling-invariant norms guarantee the regularity of the Leray–Hopf weak solutions to the 3D incompressible Navier–Stokes equations (see, e.g., [1,2,4,13,24–26,34,42,43]). Serrin type criteria for the isentropic compressible fluid were proved by Huang et al. [18]. However, to the knowledge of the authors, even though for the isentropic compressible fluid in the presence of vacuum, the following blow-up criteria are unknown

$$\limsup_{t \to T^*} \left(\|div\, u\|_{L^1(0,t;L^\infty)} + \|\nabla u\|_{L^p(0,t;L^q)} \right) = \infty, \ with\ \frac{2}{p} + \frac{3}{q} = 2, \ q > 3. \tag{1.20}$$

The other case in (1.20) $\frac{3}{2} < q < 3$ can be derived from the result [18] and Sobolev inequality, and $q = 3$ can be derived by a slight variant of the proof of [18]. Hence, it seems that (1.17) and (1.18) without θ are still new results to the isentropic compressible fluid. For the general case (1.20), we can prove it for the strong solutions of the isentropic compressible Navier–Stokes equations in the case away from the vacuum. Before we state the result, we recall the known blow-up criteria for the strong solutions of system (1.1) without vacuum.
Fan and Jiang [9]

$$\lim_{t \nearrow T^\star} \sup \left(\|(\rho, \frac{1}{\rho}, \theta)\|_{L^\infty(0,t;L^\infty)} + \|\rho\|_{L^1(0,t;W^{1,q})} \right.$$
$$\left. + \|\nabla \rho\|_{L^4(0,t;L^2)} \right) = \infty, \ (\lambda < 2\mu) \tag{1.21}$$

Huang and Li [15]

$$\lim_{t \nearrow T^\star} \sup \left(\|\nabla u\|_{L^1(0,t;L^\infty)} + \|\theta\|_{L^2(0,t;L^\infty)} \right) = \infty. \tag{1.22}$$

Sun et al. [35]

$$\lim_{t \nearrow T^\star} \sup \left(\|(\rho, \frac{1}{\rho}, \theta)\|_{L^\infty(0,t;L^\infty)} \right) = \infty, \ (\lambda < 7\mu). \tag{1.23}$$

Then, we consider the case away from vacuum and state the second result as follows:

Theorem 1.3. *Suppose* (ρ, u, θ) *is the unique strong solution in Theorem* 2.2 *in Sect.* 2. *If the maximal existence time* T^* *is finite, then either of the following results holds:*

$$\limsup_{t \to T^*} \left(\|\rho, \ \rho^{-1}\|_{L^\infty(0,t;L^\infty)} + \|div\, u\|_{L^{p_1}(0,t;L^{q_1})} + \|\theta\|_{L^{p_2}(0,t;L^{q_2})} \right) = \infty, \tag{1.24}$$

where the pairs (p_1, q_1) *and* (p_2, q_2) *meet*

$$\frac{2}{p_1} + \frac{3}{q_1} = 2, \ q_1 > \frac{3}{2}; \ \frac{2}{p_2} + \frac{3}{q_2} = 2, \ q_2 > \frac{3}{2}. \tag{1.25}$$

Remark 1.4. This theorem is an improvement in corresponding results in (1.22) and (1.23).

Remark 1.5. Note that we do not need any additional restriction on the viscosity coefficients μ and λ.

The proof of Theorem 1.3 is also enlightened by the study of the 3D incompressible Navier–Stokes equations. Under the natural restriction (1.3), there holds

$$\frac{1}{2}\frac{d}{dt}\int \left[\mu|\nabla u|^2 + (\mu + \lambda)(\text{div } u)^2 + \frac{1}{2\mu + \lambda}P^2 - 2P\text{div } u + 2C_3C_\nu\rho\theta^2\right.$$
$$\left. + \frac{C_4 + 1}{\mu}\rho|u|^4\right] + \kappa\int |\nabla\theta|^2 + \frac{1}{2}\int \rho|\dot{u}|^2 + \int |u|^2|\nabla u|^2$$
$$\leq C\int \rho^2|\theta|^3 + C\int \rho|u|^2|\theta|^2 + C\int |\text{div u}||u|^2|\nabla u|. \tag{1.26}$$

Our observation is that the last term in the right-hand side of (1.26) is similar to the term $\int |\Pi||u|^2|\nabla u|dx$ appearing in the derivation of regular criteria via pressure Π of the 3D incompressible Navier–Stokes equations on bounded domain (see [2,24,25,42]). This criterion was obtained by Kang and Lee [25] until 2010. In the spirit of [25], we can deal with this term to derive the desired estimates.

Theorem 1.3 immediately yields the following result.

Corollary 1.4. *Let* (ρ, u) *be the unique strong solution of the isentropic compressible fluid without initial vacuum. If the maximal existence time* T^* *is finite, then there holds*

$$\limsup_{t\to T^*}\left(\|\rho, \ \rho^{-1}\|_{L^\infty(0,t;L^\infty)} + \|\text{div } u\|_{L^p(0,t;L^q)}\right) = \infty, \tag{1.27}$$

where the pair (p, q) *meets*

$$\frac{2}{p} + \frac{3}{q} = 2, \ q > \frac{3}{2}.$$

Remark 1.6. Although this corollary is valid in the absence of vacuum, it does not require additional assumptions on λ and μ. A special case of (1.27) is that

$$\limsup_{t\to T^*}\|\text{div } u\|_{L^1(0,t;L^\infty)} = \infty. \tag{1.28}$$

In the presence of vacuum, similar blow-up criteria in terms of the divergence (gradient) of the velocity can be found in [20,26,35].

Remark 1.7. For the isentropic compressible fluid in the absence of vacuum, combining the results proved by Huang et al. citeHLX and Corollary 1.4, we obtain the following blow-up criteria in terms of the gradient of the velocity

$$\limsup_{t \to T^*} \left(\|\rho, \ \rho^{-1}\|_{L^\infty(0,t;L^\infty)} + \|\nabla u\|_{L^p(0,t;L^q)} \right) = \infty, \qquad (1.29)$$

or

$$\limsup_{t \to T^*} \left(\|\text{div } u\|_{L^1(0,t;L^\infty)} + \|\nabla u\|_{L^p(0,t;L^q)} \right) = \infty, \qquad (1.30)$$

where the pair $(p, \ q)$ meets

$$\frac{2}{p} + \frac{3}{q} = 2, \ q > \frac{3}{2}.$$

This extends Beirao da Veiga's result in [1] from the incompressible Navier–Stokes equations to the compressible Navier–Stokes system.

The remainder of this paper is structured as follows. In Sect. 2, we first give some notations and recall the local strong solutions of system (1.1) due to Cho and Kim [6]. We establish some auxiliary lemmas under the hypothesis that the upper bound of the density is bounded. Section 3 is devoted to the proof of Theorem 1.1 and Theorem 1.2. Section 4 contains the proof of Theorem 1.3.

2. Notations and some auxiliary lemmas

C is an absolute constant which may be different from line to line unless otherwise stated. For $1 \le p \le \infty$, $L^p(\mathbb{R}^3)$ represents the usual Lebesgue space. The classical Sobolev space $W^{k,p}(\mathbb{R}^3)$ is equipped with the norm $\|f\|_{W^{k,p}(\mathbb{R}^3)} = \sum_{\alpha=0}^{k} \|D^\alpha f\|_{L^p(\mathbb{R}^3)}$. A function f belongs to the homogeneous Sobolev spaces $D^{k,l}$ if $u \in L^1_{\text{loc}}(\mathbb{R}^3)$: $\|\nabla^k u\|_{L^l} < \infty$.

For simplicity, we write

$$L^p = L^p(\mathbb{R}^3), \ H^k = W^{k,2}(\mathbb{R}^3), \ D^k = D^{k,2}(\mathbb{R}^3).$$

We denote the G by the effective viscous flux, that is,

$$G = (2\mu + \lambda)\text{div } u - P.$$

The notation $\dot{v} = v_t + u \cdot \nabla v$ stands for material derivative.

It is well known that

$$\|\nabla G\|_{L^p} \le \|\rho \dot{u}\|_{L^p}, \ \forall p \in (1, +\infty). \qquad (2.1)$$

We recall the local well-posedness of strong solutions to the full compressible Navier–Stokes equations (1.1) due to Cho and Kim [6]. The first result allows initial density contains vacuum and some compatibility conditions are required. The second one is absence of vacuum. Moreover, we refer the reader to [5] the local existence and uniqueness of strong solutions for the isentropic compressible Navier–Stokes system.

Theorem 2.1. *Suppose $u_0, \theta_0 \in D^1(\mathbb{R}^3) \cap D^2(\mathbb{R}^3)$ and*

$$\rho_0 \in W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$$

for some $q \in (3, 6]$. If ρ_0 is nonnegative and the initial data satisfy the compatibility condition

$$Lu_0 + \nabla p(\rho_0) = \sqrt{\rho_0} g_1$$

$$\Delta\theta_0 + \frac{\mu}{2}|\nabla u_0 + (\nabla u_0)^{tr}|^2 + \lambda(div\ u_0)^2 = \sqrt{\rho_0} g_2 \tag{2.2}$$

for vector fields $g_1, g_2 \in L^2(\mathbb{R}^3)$. Then, there exist a time $T \in (0, \infty)$ and unique solution, satisfying

$$(\rho, u, \theta) \in C([0, T]; L^1 \cap H^1 \cap W^{1,q}) \times C([0, T]; D^1 \cap D^2) \times L^2([0, T]; D^{2,q})$$

$$(\rho_t, u_t, \theta_t) \in C([0, T]; L^2 \cap L^q) \times L^2([0, T]; D^1) \times L^2([0, T]; D^1)$$

$$(\rho^{1/2}u_t, \rho^{1/2}\theta_t) \in L^\infty([0, T]; L^2) \times L^\infty([0, T]; L^2). \tag{2.3}$$

Theorem 2.2. *Suppose $u_0, \theta_0 \in D^1(\mathbb{R}^3) \cap D^2(\mathbb{R}^3)$ and*

$$\rho_0 \in W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$$

for some $q \in (3, 6]$. If $\rho_0 > 0$, then there exist a time $T \in (0, \infty)$ and unique solution, satisfying

$$\rho \in C([0, T]; L^1 \cap H^1 \cap W^{1,q}), \quad \inf_{(x,t) \in \mathbb{R}^3 \times [0,T]} \rho > 0,$$

$$u \in C([0, T]; D^1 \cap D^2) \cap L^2([0, T]; W^{2,q}) \tag{2.4}$$

$$\theta \in C([0, T]; D^1 \cap D^2) \cap L^2([0, T]; W^{2,q}).$$

Next, under the hypothesis that the upper bound of the density is bounded, namely

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \le M, \tag{2.5}$$

we derive some useful estimates, which plays an important role in the proof of all our theorems.

Lemma 2.3. *Suppose that (2.5) is valid, then there holds*

$$\frac{1}{2}\frac{d}{dt} \int \left[\mu|\nabla u|^2 + (\mu + \lambda)(div\ u)^2 + \frac{1}{2\mu + \lambda}P^2 - 2P\,div\ u + 2C_3 C_v \rho\theta^2 \right]$$

$$+ \kappa \int |\nabla\theta|^2 + \frac{1}{2}\int \rho|\dot{u}|^2 \tag{2.6}$$

$$\le C_4 \int \rho|\theta|^3 + C_4 \int \rho|u|^2|\theta|^2 + C_4 \int |u|^2|\nabla u|^2.$$

Proof. Taking the L^2 inner product of the temperature equation with θ, by the Cauchy inequality, we infer that

$$
\frac{C_v}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho\theta^2 + \kappa \int |\nabla\theta|^2 \le R \int |\rho\theta^2 \mathrm{div}\, u| + (2\mu + \lambda) \int |\nabla u|^2 \theta
$$
$$
\le C \int \rho^2 |\theta|^3 + C \int |\nabla u|^2 \theta. \tag{2.7}
$$

Multiplying the both sides of the momentum equation by $u\theta$ and using the integration by parts, we get

$$
\mu \int |\nabla u|^2 \theta \le \int |\rho\dot{u}u\theta| + |\int \nabla Pu\theta| + C \int |u||\nabla u||\nabla\theta|
$$
$$
= I + II + III. \tag{2.8}
$$

Thanks to the Cauchy–Schwarz inequality, we find that

$$
I \le \eta \int \rho|\dot{u}|^2 + C(\eta) \int \rho|u|^2 |\theta|^2. \tag{2.9}
$$

According to integration by parts and Young's inequality, we conclude

$$
II = |\int P \mathrm{div}\, u\theta \mathrm{d}x + \int Pu\nabla\theta \mathrm{d}x|
$$
$$
= |R \int \rho \mathrm{div}\, u\theta^2 + R \int \rho\theta u\nabla\theta| \tag{2.10}
$$
$$
\le C \int \rho^2 |\theta|^3 + \frac{\mu}{8} \int |\nabla u|^2 \theta + \varepsilon_1 \int |\nabla\theta|^2 + C \int \rho^2\theta^2|u|^2.
$$

The Cauchy–Schwarz inequality yields that

$$
III \le \varepsilon_1 \int |\nabla\theta|^2 + C \int |u|^2 |\nabla u|^2. \tag{2.11}
$$

Plugging (2.9)–(2.11) into (2.8), we have

$$
\frac{7\mu}{8} \int |\nabla u|^2 \theta \le \eta \int \rho|\dot{u}|^2 + C \int \rho|u|^2 |\theta|^2 + C \int \rho^2 |\theta|^3 + C \int \rho^2\theta^2|u|^2
$$
$$
+ 2\varepsilon_1 \int |\nabla\theta|^2 + C \int |u|^2 |\nabla u|^2. \tag{2.12}
$$

It follows from (2.7) and (2.12) that

$$
\frac{C_v}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho\theta^2 + \frac{\kappa}{2} \int |\nabla\theta|^2
$$
$$
\le C_1 \int \rho^2 |\theta|^3 + C_1\eta \int |\rho\dot{u}|^2 + C_1 \int \rho|u|^2 |\theta|^2 + C_1 \int |u|^2 |\nabla u|^2. \tag{2.13}
$$

Taking the L^2 inner product with u_t in the second equation of (1.1), we get

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}\ u)^2\right] + \int \rho|\dot{u}|^2 \\
= \int \rho\dot{u}(u\cdot\nabla u) + \int P\mathrm{div}\ u_t \mathrm{d}x \\
= J_1 + J_2.
\end{aligned}
\tag{2.14}
$$

The Young inequality ensures that

$$
J_1 \le \frac{1}{4}\int \rho|\dot{u}|^2 + C\int |u|^2|\nabla u|^2.
\tag{2.15}
$$

After a few calculations, by the effective viscous flux $G = (2\mu+\lambda)\mathrm{div}\ u - P$, we arrive at

$$
\begin{aligned}
J_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\int P\mathrm{div}\ u - \int P_t\mathrm{div}\ u \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int P\mathrm{div}\ u - \frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\int P^2 - \frac{1}{2\mu+\lambda}\int P_t G \\
&= J_{21} + J_{22} + J_{23}.
\end{aligned}
\tag{2.16}
$$

Notice that the equation of $\rho E = P + \frac{\rho|u|^2}{2}$ is governed by

$$
(\rho E)_t + \mathrm{div}\ (\rho Eu + Pu) - \kappa\Delta\theta = \mathrm{div}\ \left\{\left[\lambda\mathrm{div}\ uId + \mu(\nabla u + (\nabla u)^{\mathrm{tr}})\right]\cdot u\right\}.
\tag{2.17}
$$

By virtue of (2.17), we see that

$$
\begin{aligned}
J_{23} &= -\frac{1}{2\mu+\lambda}\int (\rho E)_t G + \frac{1}{2\mu+\lambda}\int \left(\frac{\rho|u|^2}{2}\right)_t G \\
&= -\frac{2R}{2\mu+\lambda}\int \rho\theta u\cdot\nabla G - \frac{1}{2\mu+\lambda}\int \rho\frac{|u|^2}{2}u\cdot\nabla G \\
&\quad + \frac{1}{2\mu+\lambda}\int \left\{\left[\lambda\mathrm{div}\ uId + \mu(\nabla u + (\nabla u)')\right]\cdot u\right\}\nabla G \\
&\quad + \frac{\kappa}{2\mu+\lambda}\int \nabla\theta\cdot\nabla G. \\
&\quad + \frac{1}{2\mu+\lambda}\int \left(\frac{\rho|u|^2}{2}\right)_t G.
\end{aligned}
\tag{2.18}
$$

With the help of the Young inequality, (2.1) and (2.5), we get

$$
-\frac{2R}{2\mu+\lambda}\int \rho\theta u\cdot\nabla G \le \frac{\eta}{4}\|\nabla G\|_{L^2}^2 + C\int \rho^2|u|^2|\theta|^2
$$

$$
\le \varepsilon\|\rho\dot{u}\|_{L^2}^2 + C\int \rho|u|^2|\theta|^2
$$

Likewise, there hold

$$\frac{1}{2\mu + \lambda} \int \Big[\lambda \mathrm{div}\, u I d + \mu(\nabla u + (\nabla u)')\Big] u \nabla G \le \frac{\eta}{4} \|\nabla G\|_{L^2}^2 + C \int |u|^2 |\nabla u|^2$$

$$\le \varepsilon \|\rho \dot{u}\|_{L^2}^2 + C \int |u|^2 |\nabla u|^2,$$

$$\frac{\kappa}{2\mu + \lambda} \int \nabla \theta \cdot \nabla G \le \frac{\eta}{4} \|\nabla G\|_{L^2}^2 + C\kappa \int |\nabla \theta|^2 \mathrm{d}x$$

$$\le \varepsilon \|\rho \dot{u}\|_{L^2}^2 + C\kappa \int |\nabla \theta|^2 \mathrm{d}x.$$

Putting together with the above estimates, we have

$$-\frac{1}{2\mu + \lambda} \int (\rho E)_t G \le -\frac{1}{2\mu + \lambda} \int \rho \frac{|u|^2}{2} u \cdot \nabla G + \varepsilon \int \rho |\dot{u}|^2 \mathrm{d}x$$

$$+ C \int \rho |u|^2 |\theta|^2 \mathrm{d}x \qquad (2.19)$$

$$+ C \int |u|^2 |\nabla u|^2 + C\kappa \int |\nabla \theta|^2.$$

We turn our attentions to the last term of (2.18). A straightforward calculation gives

$$\frac{1}{2\mu + \lambda} \int \left(\frac{\rho |u|^2}{2}\right)_t G = \frac{1}{2\mu + \lambda} \int \frac{\rho_t |u|^2}{2} G + \frac{1}{2\mu + \lambda} \int \rho u \cdot u_t G. \qquad (2.20)$$

Taking the advantage of $\rho_t = -\mathrm{div}\, (\rho u)$, the integration by parts, the Young inequality and (3.1), we get

$$\frac{1}{2\mu + \lambda} \int \frac{\rho_t |u|^2}{2} G = -\frac{1}{2\mu + \lambda} \int \frac{\mathrm{div}\, (\rho u)|u|^2}{2} G$$

$$= \frac{1}{2\mu + \lambda} \int \rho u \cdot \nabla u \cdot u G + \frac{1}{2\mu + \lambda} \int \frac{\rho u |u|^2}{2} \cdot \nabla G$$

$$\le C \int \rho |u \cdot \nabla u|^2 + C \int \rho |u|^2 |G|^2$$

$$+ \frac{1}{2\mu + \lambda} \int \frac{\rho u |u|^2}{2} \cdot \nabla G$$

$$\le C \int \rho |u \cdot \nabla u|^2 + C \int \rho |u|^2 (|\nabla u|^2 + R\rho^2 \theta^2)$$

$$+ \frac{1}{2\mu + \lambda} \int \frac{\rho u |u|^2}{2} \cdot \nabla G$$

$$\le C \int |u|^2 |\nabla u|^2 + C \int \rho |u|^2 \theta^2$$

$$+ \frac{1}{2\mu + \lambda} \int \frac{\rho u |u|^2}{2} \cdot \nabla G,$$

$$(2.21)$$

where we have used the fact

$$|G| \le C(|\nabla u| + |\rho\theta|). \tag{2.22}$$

From $\dot{u} = u_t + u \cdot \nabla u$, the Young inequality, (2.22) and (3.1), we find

$$
\begin{aligned}
\frac{1}{2\mu + \lambda} \int \rho u \cdot u_t G &= \frac{1}{2\mu + \lambda} \int \rho u \cdot (\dot{u} - u \cdot \nabla u) G \\
&= \frac{1}{2\mu + \lambda} \int \rho u \cdot \dot{u} G - \int \rho u u \cdot \nabla u G \\
&\le \varepsilon \int \rho |\dot{u}|^2 dx + C \int \rho |u|^2 |G|^2 dx + \int \rho |u|^2 |\nabla u|^2 \\
&\le \varepsilon \int \rho |\dot{u}|^2 dx + C \int |u|^2 |\nabla u|^2 + C \int \rho |u|^2 \theta^2.
\end{aligned}
\tag{2.23}
$$

Inserting (2.21) and (2.23) into (2.20), we obtain

$$
\begin{aligned}
\frac{1}{2\mu + \lambda} \int \left(\frac{\rho |u|^2}{2} \right)_t G &\le \varepsilon \int \rho |\dot{u}|^2 + C \int |u|^2 |\nabla u|^2 \\
&\quad + C \int \rho |u|^2 \theta^2 + \frac{1}{2\mu + \lambda} \int \frac{\rho u |u|^2}{2} \cdot \nabla G.
\end{aligned}
\tag{2.24}
$$

We derive from (2.18), (2.19) and (2.24) that

$$J_{23} \le \varepsilon \int \rho |\dot{u}|^2 + C \int \rho |u|^2 |\theta|^2 + C \int |u|^2 |\nabla u|^2 + C \int |\nabla\theta|^2. \tag{2.25}$$

It follows from (2.14), (2.15), (2.16) and (2.25) that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int \left[\mu |\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2 + \frac{1}{(2\mu + \lambda)} P^2 - 2P \operatorname{div} u \right] + \frac{1}{2} \int \rho |\dot{u}|^2 \\
&\le C_2 \int \rho |u|^2 |\theta|^2 + C_2 \int |u|^2 |\nabla u|^2 + C_2 \kappa \int |\nabla\theta|^2.
\end{aligned}
\tag{2.26}
$$

Since $P = R\rho\theta$, we can choose $C_3 \ge C_2 + 1$ and C_3 sufficiently large to make sure that

$$\mu |\nabla u|^2 + (\mu + \lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu + \lambda} P^2 - 2P \operatorname{div} u + 2C_3 C_v \rho\theta^2 \ge \mu |\nabla u|^2 + \rho\theta^2.$$

Multiplying (2.13) both sides by C_3 and adding it with (2.26), we end up with

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left[\mu|\nabla u|^2+(\mu+\lambda)(\text{div } u)^2+\frac{1}{2\mu+\lambda}P^2-2P\text{div } u+2C_3C_v\rho\theta^2\right]\\
&\quad+C_3\kappa\int|\nabla\theta|^2+\int\rho|\dot{u}|^2\\
&\leq C_1C_3\int\rho^2|\theta|^3+C_1C_3\varepsilon\int|\rho\dot{u}|^2+C_1C_3\int\rho|u|^2|\theta|^2+C_1C_3\int|u|^2|\nabla u|^2\\
&\quad+C_2\int\rho|u|^2|\theta|^2+C_2\int|u|^2|\nabla u|^2+C_2\kappa\int|\nabla\theta|^2\\
&\leq C\int\rho^2\theta^3+C_1C_3\varepsilon\int\rho\dot{u}^2+C\int\rho|u|^2|\theta|^2+C\int|u|^2|\nabla u|^2+C_2\kappa\int|\nabla\theta|^2.
\end{aligned}
$$
(2.27)

Choosing ε sufficiently small to obtain (2.6). This completes the proof of this lemma. □

3. Blow-up criteria with vacuum

3.1. Extra constraint on the coefficients of viscosity

In what follows, we assume that (ρ, u, θ) is a strong solution of (1) in $[0, T) \times \mathbb{R}^3$ with the regularity stated in Theorem 2.1. We will prove Theorem 1.1 by a contradiction argument. Therefore, we assume that

$$
\|\rho\|_{L^\infty(0,T;L^\infty)}+\|\theta\|_{L^p(0,T;L^q)}\leq C, \quad \frac{2}{p}+\frac{3}{q}=2, \text{ where } q>\frac{3}{2}.
$$
(3.1)

First, we follow the arguments of Wen and Zhu [31] and Li et al. [27] to prove the lemma below.

Lemma 3.1. *Suppose that* (3.1) *is valid and* $\lambda < 3\mu$, *then there holds*

$$
\frac{d}{dt}\int\rho|u|^4+\int_{\mathbb{R}^3\cap\{|u|>0\}}|u|^2|\nabla u|^2\leq C_5\int\rho|u|^2|\theta|^2.
$$
(3.2)

Proof. The proof of this lemma is similar as that in [27] and [31]; therefore, we just outline the proof here.

Multiplying the momentum equations by $4|u|^2u$ and integrating on $\mathbb{R}^3$, we find

$$
\begin{aligned}
&\frac{d}{dt}\int\rho|u|^4+\int_{\mathbb{R}^3\cap\{|u|>0\}}4|u|^2\left[\mu|\nabla u|^2+(\lambda+\mu)|\text{div } u|^2+2\mu|\nabla|u||^2\right]\\
&=4\int_{\mathbb{R}^3\cap\{|u|>0\}}\text{div }(|u|^2u)P-8(\mu+\lambda)\int_{\mathbb{R}^3\cap\{|u|>0\}}\text{div } u|u|u\cdot\nabla|u|\\
&=K_1+K_2.
\end{aligned}
$$
(3.3)

Using the Young inequality twice, for $\varepsilon_0 \in (0, \frac{1}{4})$, we get

$$
\begin{aligned}
K_1 &= 4R \int_{\mathbb{R}^3 \cap \{|u|>0\}} (|u|^2 \mathrm{div}\, u + 2u \cdot \nabla u \cdot u)\rho\theta \\
&\leq 2\mu\varepsilon_0 \int |u|^2 |\nabla u|^2 + C \int \rho|u|^2|\theta|^2 + 2\mu\varepsilon_0 \int |u|^2|\nabla u|^2 + C \int \rho|u|^2|\theta|^2 \\
&\leq 4\mu\varepsilon_0 \int |u|^2|\nabla u|^2 + C \int \rho|u|^2|\theta|^2.
\end{aligned}
\tag{3.4}
$$

By the Cauchy inequality, we have

$$
K_2 \leq 4(\lambda + \mu) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2 |\mathrm{div}\, u|^2 + 4(\mu + \lambda) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2 \big|\nabla|u|\big|^2.
\tag{3.5}
$$

Substituting (3.4) and (3.5) into (3.3), we conclude that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int \rho|u|^4 + 4\mu \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2|\nabla u|^2 + 8\mu \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2\big|\nabla|u|\big|^2 \\
&\leq 4\mu\varepsilon_0 \int |u|^2|\nabla u|^2 + C \int \rho|u|^2|\theta|^2 + 4(\mu + \lambda) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2\big|\nabla|u|\big|^2.
\end{aligned}
\tag{3.6}
$$

We have no new ingredient about the bound of the last term of the right-hand side in (3.6); hence, we omit the details here. A slight modified the corresponding proof in [27,31], we derive from (3.6) that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int \rho|u|^4 + \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2\big|\nabla u\big|^2 \leq C \int \rho|u|^2|\theta|^2,
\tag{3.7}
$$

This proves Lemma 3.1. □

Lemma 3.2. *Suppose that* (3.1) *is valid and* $\lambda < 3\mu$, *then there holds*

$$
\sup_{0 \leq t \leq T} \int \Big[\mu|\nabla u|^2 + \rho\theta^2 + \rho|u|^4 \Big] + \kappa \int_0^T \int |\nabla\theta|^2 + \int_0^T \int \rho|\dot{u}|^2 + |u|^2|\nabla u|^2 \leq C.
\tag{3.8}
$$

Proof. Multiplying the inequality (3.7) by $(C_4 + 1)$ and adding the result to the inequality (2.6), we can obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int \Big[\mu|\nabla u|^2 + (\mu + \lambda)(\mathrm{div}\, u)^2 + \frac{1}{2\mu + \lambda}P^2 \\
&\quad - 2P\mathrm{div}\, u + 2C_3 C_\nu \rho\theta^2 + 2(C_4 + 1)\rho|u|^4 \Big] \\
&\quad + \kappa \int |\nabla\theta|^2 + \frac{1}{2} \int \rho|\dot{u}|^2 + \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2\big|\nabla u\big|^2 \\
&\leq C_7 \left(\int \rho|\theta|^3 + \int \rho|u|^2|\theta|^2 \right).
\end{aligned}
\tag{3.9}
$$

At this stage, it suffices to bound the right-hand side of (3.9). Indeed, by the interpolation inequality, (3.1) and the Young inequality imply that

$$
\begin{aligned}
\int \rho |u|^2 |\theta|^2 &= \int \rho^{\frac{1}{2}} |\theta| \rho^{\frac{1}{2}} |u|^2 |\theta| \\
&\le \|\theta\|_{L^q} \|\rho^{\frac{1}{2}}\theta\|_{L^{\frac{2q}{q-1}}} \|\rho^{\frac{1}{2}}|u|^2\|_{L^{\frac{2q}{q-1}}} \\
&\le \|\theta\|_{L^q} \|\rho^{\frac{1}{2}}\theta\|_{L^2}^{1-\frac{3}{2q}} \|\rho^{\frac{1}{2}}\theta\|_{L^6}^{\frac{3}{2q}} \|\rho^{\frac{1}{2}}|u|^2\|_{L^2}^{1-\frac{3}{2q}} \|\rho^{\frac{1}{2}}|u|^2\|_{L^6}^{\frac{3}{2q}} \\
&\le C\|\theta\|_{L^q} \|\rho^{\frac{1}{2}}\theta\|_{L^2}^{1-\frac{3}{2q}} \|\rho^{\frac{1}{2}}|u|^2\|_{L^2}^{1-\frac{3}{2q}} \left(\|\nabla\theta\|_{L^2}^{\frac{3}{q}} + \|\nabla|u|^2\|_{L^2}^{\frac{3}{q}} \right) \\
&\le C\|\theta\|_{L^q}^{\frac{2q}{2q-3}} \|\rho^{\frac{1}{2}}\theta\|_{L^2} \|\rho^{\frac{1}{2}}|u|^2\|_{L^2} + \eta_1 \|\nabla\theta\|_{L^2}^2 + \eta_2 \|\nabla|u|^2\|_{L^2}^2 \\
&\le C\|\theta\|_{L^q}^{\frac{2q}{2q-3}} \left(\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + \|\rho^{\frac{1}{2}}|u|^2\|_{L^2}^2 \right) + \eta_1 \|\nabla\theta\|_{L^2}^2 + \eta_2 \|\nabla|u|^2\|_{L^2}^2
\end{aligned}
\tag{3.10}
$$

By similar above arguments, we can get

$$
\begin{aligned}
\int \rho |\theta|^3 &= \int \rho^{\frac{1}{2}} |\theta| \rho^{\frac{1}{2}} \theta |\theta| \\
&\le \|\theta\|_{L^q} \|\rho^{\frac{1}{2}}\theta\|_{L^{\frac{2q}{q-1}}}^2 \\
&\le \|\theta\|_{L^q} \|\rho^{\frac{1}{2}}\theta\|_{L^2}^{2-\frac{3}{q}} \|\rho^{\frac{1}{2}}\theta\|_{L^6}^{\frac{3}{q}} \\
&\le C\|\theta\|_{L^q} \|\rho^{\frac{1}{2}}\theta\|_{L^2}^{2-\frac{3}{q}} \|\theta\|_{L^6}^{\frac{3}{q}} \\
&\le C\|\theta\|_{L^q} \|\rho^{\frac{1}{2}}\theta\|_{L^2}^{2-\frac{3}{q}} \|\nabla\theta\|_{L^2}^{\frac{3}{q}} \\
&\le C\|\theta\|_{L^q}^{\frac{2q}{2q-3}} \|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + \eta_3 \|\nabla\theta\|_{L^2}^2
\end{aligned}
\tag{3.11}
$$

Substituting (3.10) and (3.11) into (3.9), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int \left[\mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 \right. \\
&\qquad \left. - 2P\operatorname{div} u + 2C_3 C_v \rho\theta^2 + 2(C_4+1)\rho|u|^4 \right] \\
&\quad + \kappa \int |\nabla\theta|^2 + \frac{1}{2}\int \rho|\dot u|^2 + \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2 |\nabla u|^2 \\
&\le C\|\theta\|_{L^q}^{\frac{2q}{2q-3}} \left(\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + \|\rho^{\frac{1}{2}}|u|^2\|_{L^2}^2 \right) \\
&\le C\|\theta\|_{L^q}^{\frac{2q}{2q-3}} \left(\int \left[\mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u \right. \right. \\
&\qquad \left. \left. + 2C_3 C_v \rho\theta^2 + 2(C_4+1)\rho|u|^4 \right] \right),
\end{aligned}
\tag{3.12}
$$

where we used the fact that

$$
\int \left[(\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u + 2C_3 C_v \rho\theta^2 \right] \ge \int \rho\theta^2, \tag{3.13}
$$

provided that the constant C_3 is suitably large enough.

Then, the Gronwall lemma and (3.12) enable us to obtain that

$$\sup_{0 \leq t \leq T} \int (\rho\theta^2 + |\nabla u|^2 + \rho|u|^4) + \int_0^T \int \rho|\dot{u}|^2 + |\nabla\theta|^2 + |u|^2|\nabla u|^2 \leq C. \quad (3.14)$$

$\square$

Proof of Theorem 1.1. With Lemma 3.2 at our disposal, according to (3.1) and (1.9) (alternatively, (1.11)), we completes the proof of this theorem. $\square$

3.2. Without extra constraint on the coefficients of viscosity

As mentioned in the last subsection, it suffices to prove Lemma 3.2 without $\lambda < 3\mu$ to show Theorem 1.2.

Proof of Lemma 3.2. Without $\lambda < 3\mu$ As Lemma 3.1, there holds

$$\frac{d}{dt} \int \rho|u|^4 + \int 4|u|^2 \left[\mu|\nabla u|^2 + (\lambda + \mu)|\text{div } u|^2 + 2\mu|\nabla|u||^2 \right]$$
$$= 4 \int \text{div } (|u|^2 u)P - 8(\mu + \lambda) \int \text{div } u|u||u \cdot \nabla|u| \quad (3.15)$$
$$= L_1 + L_2.$$

Making use of the Young inequality twice, we have

$$L_1 = 4R \int (|u|^2\text{div } u + 2u \cdot \nabla u \cdot u)\rho\theta$$
$$\leq \eta_1 \int |u|^2|\nabla u|^2 + C \int \rho|u|^2|\theta|^2 + \eta_2 \int |u|^2|\nabla u|^2 + C \int \rho|u|^2|\theta|^2 \quad (3.16)$$
$$\leq (\eta_1 + \eta_2) \int |u|^2|\nabla u|^2 + C \int \rho|u|^2|\theta|^2.$$

Similarly,

$$L_2 \leq C(\eta) \int |u|^2|\text{div } u|^2 + \eta \int |u|^2 |\nabla|u||^2. \quad (3.17)$$

Plugging (3.16) and (3.17) into (3.15), we get

$$\frac{d}{dt} \int \rho|u|^4 + 2\mu \int |u|^2|\nabla u|^2 \leq C \int \rho|u|^2|\theta|^2 + C \int |u|^2|\text{div } u|^2. \quad (3.18)$$

Adding (3.18) multiplied by $\frac{C_4+1}{2\mu}$ to (2.6), we have

$$\frac{1}{2}\frac{d}{dt} \int \left[\mu|\nabla u|^2 + (\mu + \lambda)(\text{div } u)^2 + \frac{1}{2\mu + \lambda}P^2 - 2P\text{div } u + 2C_3C_\nu\rho\theta^2 \right.$$
$$\left. + \frac{C_4 + 1}{\mu}\rho|u|^4 \right] + \kappa \int |\nabla\theta|^2 + \frac{1}{2} \int \rho|\dot{u}|^2 + \int |u|^2|\nabla u|^2$$
$$\leq C \int \rho|\theta|^3 + C \int \rho|u|^2|\theta|^2 + C \int |u|^2|\text{div } u|^2.$$
$$(3.19)$$

Case 1:

$$\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\text{div } u\|_{L^2(0,T;L^3)} + \|\theta\|_{L^p(0,T;L^q)} \leq C. \tag{3.20}$$

With the help of Hölder inequality and Sobolev inequality, we get

$$\begin{aligned}
\int |u|^2 |\text{div } u|^2 &\leq C\left(\int |u|^6\right)^{\frac{1}{3}}\left(\int |\text{div } u|^3\right)^{\frac{2}{3}} \\
&\leq C\left(\int |\nabla u|^2\right)\left(\int |\text{div } u|^3\right)^{\frac{2}{3}}.
\end{aligned} \tag{3.21}$$

Inserting (3.10), (3.11) and (3.21) into (3.19), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left[\mu|\nabla u|^2 + (\mu+\lambda)(\text{div } u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\text{div } u + 2C_3 C_\nu \rho\theta^2\right. \\
&\left. + \frac{C_4+1}{\mu}\rho|u|^4\right] + \kappa\int |\nabla\theta|^2 + \frac{1}{2}\int \rho|\dot{u}|^2 + \mu\int |u|^2|\nabla u|^2 \\
&\leq C\|\theta\|_{L^q}^{\frac{2q}{2q-3}}\left(\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + \|\rho^{\frac{1}{2}}|u|^2\|_{L^2}^2\right) + C_8\left(\int |\nabla u|^2\right)\left(\int |\text{div } u|^3\right)^{\frac{2}{3}} \\
&\leq C\|\theta\|_{L^q}^{\frac{2q}{2q-3}}\left(\int \left[\mu|\nabla u|^2 + (\mu+\lambda)(\text{div } u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\text{div } u\right.\right. \\
&\left.\left. + 2C_3 C_\nu \rho\theta^2 + \frac{C_4+1}{\mu}\rho|u|^4\right]\right) + C_8\left(\int |\nabla u|^2\right)\left(\int |\text{div } u|^3\right)^{\frac{2}{3}}.
\end{aligned} \tag{3.22}$$

where we have used (3.13).

Now, (3.20) and Gronwall allow us to obtain Lemma 3.2 without $\lambda < 3\mu$.

Case 2:

$$\|\text{div } u\|_{L^1(0,T;L^\infty)} + \|\text{div } u\|_{L^4(0,T;L^2)} + \|\theta\|_{L^p(0,T;L^q)} \leq C. \tag{3.23}$$

From the above arguments of Case 1, we just need prove the following estimate

$$\begin{aligned}
\int |u|^2 |\text{div } u|^2 &\leq C\left(\int |u|^6\right)^{\frac{1}{3}}\left(\int |\text{div } u|^3\right)^{\frac{2}{3}} \\
&\leq C\left(\int |\nabla u|^2\right)\left(\int |\text{div } u|^3\right)^{\frac{2}{3}} \\
&\leq C\left(\int |\nabla u|^2\right)\left(\int |\text{div } u|^2\right)^{\frac{2}{3}}\|\text{div } u\|_{L^\infty}^{\frac{2}{3}} \\
&\leq C\left(\int |\nabla u|^2\right)\left(\left(\int |\text{div } u|^2\right)^2 + \|\text{div } u\|_{L^\infty}\right),
\end{aligned} \tag{3.24}$$

where we have used the Hölder inequality, Sobolev embedding and interpolation inequality.

Case 3:

$$\|\text{div } u\|_{L^2(0,T;L^\infty)} + \|\theta\|_{L^p(0,T;L^q)} \leq C. \tag{3.25}$$

From (2.17), we have

$$\|\rho\theta\|_{L^\infty(0,T;L^1)} \le C. \tag{3.26}$$

Taking the L^2 inner product of the second equations in (1.1) with u, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \rho|u|^2 + \mu \int |\nabla u|^2 + (\lambda + \mu) \int |\mathrm{div}\, u|^2 \le C\|\mathrm{div}\, u\|_{L^\infty}\|\rho\theta\|_{L^1} \tag{3.27}$$

It follows from (3.26) and (3.27) that

$$\int_0^T \int |\mathrm{div}\, u|^2 \mathrm{d}x \le C. \tag{3.28}$$

From the above arguments of Case 1 and Case 2, we just need prove the following estimate

$$
\begin{aligned}
\int |u|^2 |\mathrm{div}\, u|^2 &\le C \Big(\int |u|^6 \Big)^{\frac{1}{3}} \Big(\int |\mathrm{div}\, u|^3 \Big)^{\frac{2}{3}} \\
&\le C \Big(\int |\nabla u|^2 \Big) \Big(\int |\mathrm{div}\, u|^3 \Big)^{\frac{2}{3}} \\
&\le C \Big(\int |\nabla u|^2 \Big) \Big(\int |\mathrm{div}\, u|^2 \Big)^{\frac{2}{3}} \|\mathrm{div}\, u\|_{L^\infty}^{\frac{2}{3}} \\
&\le C \Big(\int |\nabla u|^2 \Big) \Big(\Big(\int |\mathrm{div}\, u|^2 \Big) + \|\mathrm{div}\, u\|_{L^\infty}^2 \Big).
\end{aligned}
\tag{3.29}
$$

As the same derivation of (3.22), replacing (3.21) by (3.29), we get

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \Big[\mu|\nabla u|^2 + (\mu + \lambda)(\mathrm{div}\, u)^2 + \frac{1}{2\mu + \lambda} P^2 - 2P\mathrm{div}\, u + 2C_3 C_\nu \rho\theta^2 \\
&\quad + \frac{C_4 + 1}{\mu} \rho|u|^4 \Big] + \kappa \int |\nabla\theta|^2 + \frac{1}{2} \int \rho|\dot{u}|^2 + 2 \int |u|^2 |\nabla u|^2 \\
&\le C\|\theta\|_{L^q}^{\frac{2q}{2q-3}} \Big(\|\rho^{\frac{1}{2}}\theta\|_{L^2}^2 + \|\rho^{\frac{1}{2}}|u|^2\|_{L^2}^2 \Big) + C \Big(\int |\nabla u|^2 \Big) \Big(\Big(\int |\mathrm{div}\, u|^2 \Big) + \|\mathrm{div}\, u\|_{L^\infty}^2 \Big) \\
&\le C\|\theta\|_{L^q}^{\frac{2q}{2q-3}} \Big(\int \Big[\mu|\nabla u|^2 + (\mu + \lambda)(\mathrm{div}\, u)^2 + \frac{1}{2\mu + \lambda} P^2 - 2P\mathrm{div}\, u \\
&\quad + 2C_3 C_\nu \rho\theta^2 + \frac{C_4 + 1}{\mu} \rho|u|^4 \Big] \Big) + C \Big(\int |\nabla u|^2 \Big) \Big(\Big(\int |\mathrm{div}\, u|^2 \Big) + \|\mathrm{div}\, u\|_{L^\infty}^2 \Big).
\end{aligned}
\tag{3.30}
$$

Gronwall lemma (3.28), (3.30) and (3.25) yield Lemma 3.2 without $\lambda < 3\mu$. □

4. Blow-up criteria without vacuum

As said in the last of Sect. 3.1, it is enough to show Lemma 3.2 without $\lambda < 3\mu$ to complete the proof of Theorem 1.3 under the following hypothesis

$$\|\rho,\ \rho^{-1}\|_{L^\infty(0,T;L^\infty)} + \|\mathrm{div}\, u\|_{L^{p_1}(0,T;L^{q_1})} + \|\theta\|_{L^{p_2}(0,T;L^{q_2})} \le C, \tag{4.1}$$

for (p_1, q_1) and (p_2, q_2) meeting

$$\frac{2}{p_1} + \frac{3}{q_1} = 2, \ q_1 > \frac{3}{2}, \ \frac{2}{p_2} + \frac{3}{q_2} = 2, \ q_2 > \frac{3}{2}. \tag{4.2}$$

Proof of Lemma 3.2. Without $\lambda < 3\mu$ From (3.15) and (3.16), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \rho |u|^4 + 2\mu \int |u|^2 |\nabla u|^2 \le C \int \rho |u|^2 |\theta|^2 + C \int |\operatorname{div} u| |u|^2 |\nabla u|. \tag{4.3}$$

The interpolation inequality and Sobolev inequality allow us to derive that, for $2 \le \frac{2q_1}{q_1-2} \le 6$,

$$
\begin{aligned}
\|\rho^{\frac{1}{2}} |u|^2\|_{L^{\frac{2q_1}{q_1-2}}} &\le \|\rho^{\frac{1}{2}} |u|^2\|_{L^2}^{1-\frac{3}{q_1}} \|\rho^{\frac{1}{2}} |u|^2\|_{L^6}^{\frac{3}{q_1}} \\
&\le C \|\rho^{\frac{1}{2}} |u|^2\|_{L^2}^{1-\frac{3}{q_1}} \||u|^2\|_{L^6}^{\frac{3}{q_1}} \\
&\le C \|\rho^{\frac{1}{2}} |u|^2\|_{L^2}^{1-\frac{3}{q_1}} \|\nabla |u|^2\|_{L^2}^{\frac{3}{q_1}}.
\end{aligned}
\tag{4.4}
$$

In the light of the Hölder inequality, (4.4), and the Young inequality, we find

$$
\begin{aligned}
\int |\operatorname{div} u| |u|^2 |\nabla u| &\le C \int |\operatorname{div} u| \rho^{\frac{1}{2}} |u|^2 ||\nabla u| \\
&\le C \|\operatorname{div} u\|_{L^{q_1}} \|\rho^{\frac{1}{2}} |u|^2\|_{L^{\frac{2q_1}{q_1-2}}} \|\nabla u\|_{L^2} \\
&\le C \|\operatorname{div} u\|_{L^{q_1}} \|\rho^{\frac{1}{2}} |u|^2\|_{L^2}^{1-\frac{3}{q_1}} \|\rho^{\frac{1}{2}} |u|^2\|_{L^6}^{\frac{3}{q_1}} \|\nabla u\|_{L^2} \\
&\le \eta \|\nabla |u|^2\|_{L^2}^2 + C(\eta) \|\operatorname{div} u\|_{L^{q_1}}^{\frac{2q_1}{2q_1-3}} \|\rho^{\frac{1}{2}} |u|^2\|_{L^2}^{\frac{2(q_1-3)}{2q_1-3}} \|\nabla u\|_{L^2}^{\frac{2q_1}{2q_1-3}} \\
&\le \eta \|\nabla |u|^2\|_{L^2}^2 + C(\eta) \|\operatorname{div} u\|_{L^{q_1}}^{\frac{2q_1}{2q_1-3}} \left(\|\rho^{\frac{1}{2}} |u|^2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right).
\end{aligned}
\tag{4.5}
$$

It follows from (4.3) and (4.5) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int \rho |u|^4 + 2\mu \int |u|^2 |\nabla u|^2 \\
&\le C_8 \int \rho |u|^2 |\theta|^2 + C_8 \left(\int \rho |u|^4 + \int |\nabla u|^2 \right) \|\operatorname{div} u\|_{L^{q_1}}^{\frac{2q_1}{2q_1-3}}.
\end{aligned}
\tag{4.6}
$$

Adding (4.6) multiplied by $\frac{C_4+1}{2\mu}$ to (2.6), we arrive at

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \left[\mu |\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda} P^2 - 2P \operatorname{div} u + 2C_3 C_v \rho \theta^2 \right. \\
&\left. + \frac{C_4+1}{\mu} \rho |u|^4 \right] + \kappa \int |\nabla \theta|^2 + \frac{1}{2} \int \rho |\dot{u}|^2 + 2\mu \int |u|^2 |\nabla u|^2 \\
&\le C \int \rho^2 |\theta|^3 + C \int \rho |u|^2 |\theta|^2 + C \left(\int \rho |u|^4 + \int |\nabla u|^2 \right) \|\operatorname{div} u\|_{L^{q_1}}^{\frac{2q_1}{2q_1-3}}.
\end{aligned}
\tag{4.7}
$$

Then, we use (3.10) and (3.11) to further obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int \Big[&\mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u + 2C_3C_\nu\rho\theta^2 \\
&+ \frac{C_4+1}{\mu}\rho|u|^4 \Big] + \kappa\int|\nabla\theta|^2 + \frac{1}{2}\int\rho|\dot u|^2 + 2\mu\int|u|^2|\nabla u|^2 \\
\leq\ & C\|\theta\|_{L^{q_2}}^{\frac{2q_2}{2q_2-3}}\big(\|\rho^{\frac12}\theta\|_{L^2}^2 + \|\rho^{\frac12}|u|^2\|_{L^2}^2\big) + C\Big(\int\rho|u|^4 + \int|\nabla u|^2\Big)\|\operatorname{div} u\|_{L^{q_1}}^{\frac{2q_1}{2q_1-3}} \\
\leq\ & C\|\theta\|_{L^q}^{\frac{2q}{2q-3}}\Big(\int\Big[\mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}P^2 - 2P\operatorname{div} u \\
&+ 2C_3C_\nu\rho\theta^2 + \frac{C_4+1}{\mu}\rho|u|^4\Big]\Big) + C\Big(\int\rho|u|^4 + \int|\nabla u|^2\Big)\|\operatorname{div} u\|_{L^{q_1}}^{\frac{2q_1}{2q_1-3}}.
\end{aligned}
\tag{4.8}
$$

This proves the whole lemma. □

Acknowledgements

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