J. Evol. Equ. 21 (2021), 151–166
1424-3199/21/010151-16, *published online* April 1, 2020
https://doi.org/10.1007/s00028-020-00573-2

Journal of Evolution Equations

A doubly critical semilinear heat equation in the L^1 space

YASUHITO MIYAMOTO

Abstract. We study the existence and nonexistence for a Cauchy problem of the semilinear heat equation:

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = \phi(x) & \text{in } \mathbb{R}^N \end{cases}$$

in $L^1(\mathbb{R}^N)$. Here, $N \geq 1$, $p = 1 + 2/N$ and $\phi \in L^1(\mathbb{R}^N)$ is a possibly sign-changing initial function. Since $N(p-1)/2 = 1$, the L^1 space is scale critical and this problem is known as a doubly critical case. It is known that a solution does not necessarily exist for every $\phi \in L^1(\mathbb{R}^N)$. Let $X_q := \{\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\phi| \left[\log(e + |\phi|)\right]^q \mathrm{d}x < \infty\} (\subset L^1(\mathbb{R}^N))$. In this paper, we construct a local-in-time mild solution in $L^1(\mathbb{R}^N)$ for $\phi \in X_q$ if $q \geq N/2$. We show that, for each $0 \leq q < N/2$, there is a nonnegative initial function $\phi_0 \in X_q$ such that the problem has no nonnegative solution, using a necessary condition given by Baras–Pierre (Ann Inst Henri Poincaré Anal Non Linéaire 2:185–212, 1985). Since $X_q \subset X_{N/2}$ for $q \geq N/2$, $X_{N/2}$ becomes a sharp integrability condition. We also prove a uniqueness in a certain set of functions which guarantees the uniqueness of the solution constructed by our method.

1. Introduction and main results

We consider the existence and nonexistence for a Cauchy problem of the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = \phi(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 1$, $p = 1 + 2/N$ and ϕ is a possibly sign-changing initial function. When $\phi \in L^\infty(\mathbb{R}^N)$, one can easily construct a solution by using a fixed point argument. When $\phi \notin L^\infty(\mathbb{R}^N)$, the solvability depends on the balance between the strength of the singularity of ϕ and the growth rate of the nonlinearity. Weissler [13] studied the solvability of (1.1), and obtained the following:

Proposition 1.1. *Let $q_c := N(p-1)/2$. Then, the following (i) and (ii) hold:*

(i) (Existence, subcritical and critical cases) Assume either both $q > q_c$ and $q \geq 1$ or $q = q_c > 1$. The problem (1.1) has a local-in-time solution for $\phi \in L^q(\mathbb{R}^N)$.

Mathematics Subject Classification: Primary 35K55; Secondary 35A01, 46E30
Keywords: Local-in-time solution, Fujita exponent, Supersolution, Optimal singularity.

(ii) *(Nonexistence, supercritical case) For each $1 \le q < q_c$, there is $\phi \in L^q(\mathbb{R}^N)$ such that (1.1) has no local-in-time nonnegative solution.*

Let $u(x,t)$ be a function such that u satisfies the equation in (1.1). We consider the scaled function $u_\lambda(x,t) := \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t)$. Then, u_λ also satisfies the same equation. We can easily see that $\|u_\lambda(x,0)\|_q = \|u(x,0)\|_q$ if and only if $q = q_c$. It is well known that q_c is a threshold as Proposition 1.1 shows. However, the case $q = q_c = 1$, i.e., $p = 1 + 2/N$, is not covered by Proposition 1.1, and it is known that there is a nonnegative initial function $\phi \in L^1(\mathbb{R}^N)$ such that (1.1) with $p = 1 + 2/N$ has no local-in-time nonnegative solution. See Brezis–Cazenave [2, Theorem 11], Celik–Zhou [3, Theorem 4.1] or Laister et al. [7, Corollary 4.5] for nonexistence results. See [1,6,11] and references therein for existence and nonexistence results with measures as initial data. In [2, Section 7.5], the case $p = 1 + 2/N$ is referred to as "doubly critical case." Several open problems were given in [2]. It was mentioned in [14, p.32] that (1.1) has a local-in-time solution if $\phi \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $q > 1$. However, a solvability condition was not well studied. See Table 1. For a detailed history about the existence, nonexistence and uniqueness of (1.1), see [3, Section 1].

In this paper, we obtain a sharp integrability condition on $\phi \in L^1(\mathbb{R}^N)$ which determines the existence and nonexistence of a local-in-time solution in the case $p = 1 + 2/N$. We also show that a solution constructed in Theorem 1.3 is unique in a certain set of functions. Throughout the present paper, we define $f(u) := |u|^{p-1}u$. Let $L^q(\mathbb{R}^N)$, $1 \le q \le \infty$, denote the usual Lebesgue space on $\mathbb{R}^N$ equipped with the norm $\|\cdot\|_q$. For $\phi \in L^1(\mathbb{R}^N)$, we define

$$S(t)[\phi](x) := \int_{\mathbb{R}^N} G_t(x-y)\phi(y)\mathrm{d}y,$$

where $G_t(x-y) := (4\pi t)^{-N/2}\exp\left(-\frac{|x-y|^2}{4t}\right)$. The function $S(t)[\phi]$ is a solution of the linear heat equation with initial function ϕ. We give a definition of a solution of (1.1).

Definition 1.2. Let u and $\bar{u}$ be measurable functions on $\mathbb{R}^N \times (0,T)$.

(i) (Integral solution) We call u an integral solution of (1.1) if there is $T > 0$ such that u satisfies the integral equation

$$u(t) = \mathcal{F}[u](t) \ \text{ a.e. } x \in \mathbb{R}^N, \quad 0 < t < T, \quad \text{and} \\ \|u(t)\|_\infty < \infty \text{ for } 0 < t < T, \tag{1.2}$$

where

$$\mathcal{F}[u](t) := S(t)\phi + \int_0^t S(t-s)f(u(s))\mathrm{d}s.$$

(ii) (Mild solution) We call u a mild solution if u is an integral solution and $u(t) \in C([0,T), L^1(\mathbb{R}^N))$.

Table 1. Existence and nonexistence of a local-in-time solution of (1.1) in $L^q(\mathbb{R}^N)$

Ranges of q	$1 \leq q < q_c$ Supercritical	$1 = q = q_c$ Doubly critical	$1 < q = q_c$ Critical	$q > q_c, q \geq 1$ Subcritical
Existence/ nonexistence	Not always Exist Proposition 1.1(ii)	Not always Exist exist: [14, p.32], Theorem 1.3(i) Not exist: [2,3,7], Theorem 1.3(ii)	Exist Proposition 1.1(i)	Exist Proposition 1.1(i)

(iii) We call $\bar{u}$ a supersolution of (1.1) if $\bar{u}$ satisfies the integral inequality $\mathcal{F}[\bar{u}](t) \le \bar{u}(t) < \infty$ for a.e. $x \in \mathbb{R}^N$, $0 < t < T$.

For $0 \le q < \infty$, we define a set of functions by

$$X_q := \left\{ \phi(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} |\phi| \left[\log(e + |\phi|)\right]^q \mathrm{d}x < \infty \right\}.$$

It is clear that $X_q \subset L^1(\mathbb{R}^N)$ and that $X_{q_1} \subset X_{q_2}$ if $q_1 \ge q_2$. The main theorem of the paper is the following:

Theorem 1.3. *Let $N \ge 1$ and $p = 1 + 2/N$. Then, the following (i) and (ii) hold:*

(i) (Existence) If $\phi \in X_q$ for some $q \ge N/2$, then (1.1) has a local-in-time mild solution $u(t)$, and this mild solution satisfies the following:

$$\text{there is } C > 0 \text{ such that } \|u(t)\|_\infty \le C t^{-\frac{N}{2}} (-\log t)^{-q} \text{ for small } t > 0. \tag{1.3}$$

In particular, (1.1) has a local-in-time mild solution for every $\phi \in X_{N/2}$.

(ii) (Nonexistence) For each $0 \le q < N/2$, there is a nonnegative initial function $\phi_0 \in X_q$, which is explicitly given by (4.1), such that (1.1) has no local-in-time nonnegative integral solution, and hence (1.1) has no local-in-time nonnegative mild solution.

Remark 1.4. (i) The function ϕ in Theorem 1.3(i) is not necessarily nonnegative.

(ii) Theorem 1.3 indicates that $X_{N/2} (\subset L^1(\mathbb{R}^N))$ is an optimal set of initial functions for the case $p = 1 + 2/N$ and $X_{N/2}$ is slightly smaller than $L^1(\mathbb{R}^N)$. This situation is different from the case $p > 1 + 2/N$, since (1.1) is always solvable in the scale critical space $L^{N(p-1)/2}$ for $p > 1 + 2/N$ (Proposition 1.1 (i)).

(iii) $L^1(\mathbb{R}^N)$ is larger than the optimal set for $p = 1 + 2/N$. On the other hand, it follows from Proposition 1.1(i) that if $1 < p < 1 + 2/N$, then (1.1) has a solution for all $\phi \in L^1(\mathbb{R}^N)$. Therefore, $L^1(\mathbb{R}^N)$ is small enough for the case $1 < p < 1 + 2/N$.

(iv) The function ϕ_0 given in Theorem 1.3(ii) is modified from $\psi(x)$ given by (1.9). This function comes from Baras–Pierre [1], and Theorem 1.3(ii) is a rather easy consequence of [1, Proposition 3.2]. However, we include Theorem 1.3(ii) for a complete description of the borderline property of $X_{N/2}$.

(v) Laister et al. [7] obtained a necessary and sufficient condition for the existence of a local-in-time nonnegative solution of

$$\begin{cases} \partial_t u = \Delta u + h(u) & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = \phi(x) \ge 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

They showed that when $h(u) = u^{1+2/N}[\log(e + u)]^{-r}$, (1.4) has a local-in-time nonnegative solution for every nonnegative $\phi \in L^1(\mathbb{R}^N)$ if $1 < r < \lambda p$, and (1.4) does not always have if $0 \le r \le 1$. Here, $\lambda > 0$ is a certain constant. Therefore, the optimal growth of $h(u)$ for $L^1(\mathbb{R}^N)$ is slightly smaller than $u^{1+2/N}$.

(vi) The exponent $p = 1 + 2/N$, which is called Fujita exponent, also plays a key role in the study of global-in-time solutions. If $1 < p \le 1 + 2/N$, then every nontrivial nonnegative solution of (1.1) blows up in a finite time. If $p > 1+2/N$, then (1.1) has a global-in-time nonnegative solution. See Fujita [4]. In particular, in the case $p = 1 + 2/N$ we cannot expect a global existence of a classical solution for small initial data.

The next theorem is about the uniqueness of the integral solution in a certain class.

Theorem 1.5. *Let $N \ge 1$, $p = 1 + 2/N$ and $q > N/2$. Then, an integral solution $u(t)$ of (1.1) is unique in the set*

$$\left\{ u(t) \in L^1(\mathbb{R}^N) \;\middle|\; \sup_{0\le t\le T} t^{N/2}(-\log t)^q \, \|u(t)\|_\infty < \infty \right\}. \tag{1.5}$$

Therefore, a solution given by Theorem 1.3 is unique.

Remark 1.6. (i) If there were a solution that does not satisfy (1.5), then the uniqueness fails. However, it seems to be an open problem.

(ii) In the case $q = N/2$, the uniqueness under (1.5) is left open.

(iii) For general p and q, the uniqueness of a solution of (1.1) is known in the set

$$\left\{ u(t) \in L^q(\mathbb{R}^N) \;\middle|\; \sup_{0\le t\le T} t^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{pq}\right)} \|u(t)\|_{pq} < \infty \right\}.$$

See Haraux–Weissler [5] and [13]. For an unconditional uniqueness with a certain range of p and q, see [2, Theorem 4].

(iv) The nonuniqueness in $L^q(\mathbb{R}^N)$ is also known for (1.1). For $p > 1 + 2/N$ and $1 \le q < N(p-1)/2 < p+1$, see [5]. For $p = q = N/(N-2)$, see Ni–Sacks [8] and Terraneo [12].

Let us mention technical details. We assume that $\phi \in X_q$ for some $q \ge N/2$. Using a monotone method, we construct a nonnegative mild solution $w(t)$ of

$$\begin{cases} \partial_t w = \Delta w + f(w) & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = |\phi(x)| & \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

We define $g(u)$ by

$$g(u) := u\left[\log(\rho + |u|)\right]^q, \tag{1.7}$$

where $\rho > 1$ is chosen appropriately. We will see that if $\rho \ge e$, then $g(u)$ is convex for $u \ge 0$ and g plays a crucial role in the construction of the solution of (1.6). In order to construct a nonnegative solution we use a method developed by Robinson–Sierżęga [10] with the convex function g, which was also used in Hisa–Ishige [6]. We define a sequence of functions $(u_n)_{n=0}^\infty$ by

$$\begin{cases} u_n(t) = \mathcal{F}[u_{n-1}](t) \text{ for } 0 \le t < T & \text{if } n \ge 1, \\ u_0(t) = 0. \end{cases} \tag{1.8}$$

Then, we show that $-w(t) \le u_n(t) \le w(t)$ for $0 \le t < T$. Since $|u_n(t)| \le w(t)$, we can extract a convergent subsequence in $C_{\rm loc}(\mathbb{R}^N \times (0,T))$, using a parabolic regularization, the dominated convergence theorem and a diagonal argument. The limit function becomes a mild solution of (1.1).

In the nonexistence part, we use a necessary condition for the existence of a nonnegative solution of (1.1) obtained by Baras–Pierre [1], which is stated in Proposition 2.2 in the present paper. Using their result, one can show that there is $c_0 > 0$ such that if $\phi(x) \ge c_0\psi(x)$ in a neighborhood of the origin, then (1.1) has no nonnegative integral solution. Here,

$$\psi(x) := |x|^{-N}(-\log|x|)^{-\frac{N}{2}-1} \quad \text{for } 0 < |x| < 1/e. \tag{1.9}$$

See also [6]. For each $0 \le q < N/2$, we will see that a modified function ϕ_0, which is given by (4.1), belongs to X_q. We show that ϕ_0 does not satisfy the necessary condition for the existence of an integral solution stated in Proposition 2.2. Hence, (1.1) with ϕ_0 has no nonnegative solution for each $0 \le q < N/2$.

This paper consists of five sections. In Sect. 2, we recall known results including a monotone method, a necessary condition on the existence for (1.1) and L^p-L^q-estimates. In Sect. 3, we prove Theorem 1.3(i). In Sect. 4, we prove Theorem 1.3(ii). In Sect. 5, we prove Theorem 1.5.

2. Preliminaries

First, we recall the monotonicity method.

Lemma 2.1. *Let $0 < T \le \infty$, and let f be a continuous nondecreasing function such that $f(0) \ge 0$. The problem (1.1) has a nonnegative integral solution for $0 < t < T$ if and only if (1.1) has a nonnegative supersolution for $0 < t < T$. Moreover, if a nonnegative supersolution $\bar{u}(t)$ exists, then the solution $u(t)$ obtained in this lemma satisfies $0 \le u(t) \le \bar{u}(t)$.*

Proof. This lemma is well known. See [10, Theorem 2.1] for details. However, we briefly show the proof for readers' convenience.

If (1.1) has an integral solution, then the solution is also a supersolution. Thus, it is enough to show that (1.1) has an integral solution if (1.1) has a supersolution. Let $\bar{u}$ be a supersolution for $0 < t < T$. Let $u_1 = S(t)\phi$. We define u_n, $n = 2, 3, \ldots$, by

$$u_n = \mathcal{F}[u_{n-1}].$$

Then, we can show by induction that

$$0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le \bar{u} < \infty \ \text{ a.e. } x \in \mathbb{R}^N,\ 0 < t < T.$$

This indicates that the limit $\lim_{n\to\infty} u_n(x,t)$ which is denoted by $u(x,t)$ exists for almost all $x \in \mathbb{R}^N$ and $0 < t < T$. By the monotone convergence theorem, we see that

$$\lim_{n\to\infty} \mathcal{F}[u_{n-1}] = \mathcal{F}[u],$$

and hence $u = \mathcal{F}[u]$. Then, u is an integral solution of (1.1). It is clear that $0 \le u(t) \le \bar{u}(t)$. □

Baras–Pierre [1] studied necessary conditions for the existence of an integral solution in the case $p > 1$. See also [6] for details of necessary conditions including Proposition 2.2. The following proposition is a variant of [1, Proposition 3.2].

Proposition 2.2. *Let $N \ge 1$ and $p = 1 + 2/N$. If $u(t)$ is a nonnegative integral solution, i.e., $u(t)$ satisfies (1.2) with a nonnegative initial function ϕ and some $T > 0$, then there exists a constant $\gamma_0 > 0$ depending only on N and p such that*

$$\int_{B(\tau)} \phi(x)\mathrm{d}x \le \gamma_0 |\log \tau|^{-\frac{N}{2}} \text{ for all } 0 < \tau < T, \tag{2.1}$$

where $B(\tau) := \{x \in \mathbb{R}^N \mid |x| < \tau\}$.

Lemma 2.3. *Let $q \ge 0$ be fixed, and let*

$$X_{q,\rho} := \left\{ \phi \in L^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} |\phi| \left[\log(\rho + |\phi|)\right]^q \mathrm{d}x < \infty \right\}. \tag{2.2}$$

Then, $\phi \in X_{q,\rho}$ for all $\rho > 1$ if and only if $\phi \in X_{q,\sigma}$ for some $\sigma > 1$.

Proof. We consider only the case $q > 0$. It is enough to show that $\phi \in X_{q,\rho}$ for all $\rho > 1$ if $\phi \in X_{q,\sigma}$ for some $\sigma > 1$. Let $\rho > 1$ be fixed, and let $\xi(s) := \log(\rho+s)/(\log(\sigma+s))$. By L'Hospital's rule, we see that $\lim_{s\to\infty} \xi(s) = \lim_{s\to\infty}(s+\sigma)/(s+\rho) = 1$. Since $\xi(s)$ is bounded on each compact interval in $[0,\infty)$, we see that $\xi(s)$ is bounded in $[0,\infty)$, and hence there is $C > 0$ such that $\log(\rho + s) \le C \log(\sigma + s)$ for $s \ge 0$. This inequality indicates that $\phi \in X_{q,\rho}$ if $\phi \in X_{q,\sigma}$. □

Because of Lemma 2.1, we do not care about $\rho > 1$ in (2.2). In particular, if $\phi \in X_q$, then $\|g(\phi)\|_1 < \infty$ for every $\rho > 1$.

Proposition 2.4. *(i) Let $N \ge 1$ and $1 \le \alpha \le \beta \le \infty$. There is $C > 0$ such that, for $\phi \in L^\alpha(\mathbb{R}^N)$,*

$$\|S(t)\phi\|_\beta \le Ct^{-\frac{N}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)} \|\phi\|_\alpha \text{ for } t > 0.$$

(ii) Let $N \ge 1$ and $1 \le \alpha < \beta \le \infty$. Then, for each $\phi \in L^\alpha(\mathbb{R}^N)$ and $C_0 > 0$, there is $t_0 = t_0(C_0, \phi)$ such that

$$\|S(t)\phi\|_\beta \le C_0 t^{-\frac{N}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)} \text{ for } 0 < t < t_0.$$

For Proposition 2.4(i) (resp. (ii)), see [9, Proposition 48.4] (resp. [2, Lemma 8]). Note that $C_0 > 0$ in (ii) can be chosen arbitrary small.

We collect various properties of g defined by (1.7).

Lemma 2.5. *Let $q > 0$ and let $g_1(s) := s[\log(\rho + s)]^{-q}$. Then, the following holds:*

(i) If $\rho > 1$, then $g'(s) > 0$ for $s > 0$.
(ii) If $\rho \ge e$, then $g''(s) > 0$ for $s > 0$.
(iii) If $\rho \ge e$, then $g_1(s) \le g^{-1}(s)$ for $s \ge 0$.
(iv) If $\rho > 1$, then there is $C_1 > 0$ such that $g^{-1}(s) \le g_1(C_1 s)$ for $s \ge 0$.
(v) If $\rho > e^{q/(p-1)}$, then $g^{-1}(s)^p/s$ is nondecreasing for $s \ge 0$.
(vi) If $\rho \ge e$, then, for $\phi \in L^1(\mathbb{R}^N)$,

$$S(t)\phi \le g^{-1}(S(t)g(\phi)) \ \ for \ \ t \ge 0.$$

Proof. By direct calculation, we have

$$g'(s) = [\log(\rho + s)]^{q-1}\left\{\log(\rho + s) + \frac{qs}{s+\rho}\right\},$$
$$g''(s) = \frac{q[\log(s+\rho)]^{q-2}}{(s+\rho)^2}\left[s\left\{\log(\rho+s) + q - 1\right\} + 2\rho\log(\rho+s)\right].$$

Thus, (i) and (ii) hold.

(iii) Since $\rho \ge e$, we have

$$\begin{aligned} g(g_1(s)) &= \frac{s}{[\log(\rho+s)]^q}\left[\log\left(\rho + \frac{s}{[\log(\rho+s)]^q}\right)\right]^q \\ &\le \frac{s}{[\log(\rho+s)]^q}[\log(\rho+s)]^q = s \end{aligned} \tag{2.3}$$

for $s \ge 0$. By (i), we see that $g^{-1}(s)$ exists and it is increasing. By (2.3), we see that $g_1(s) \le g^{-1}(s)$ for $s \ge 0$.

(iv) Let $\xi(s) := (g(g_1(s))/s)^{1/q} = \log(\rho + \frac{s}{[\log(\rho+s)]^q})/(\log(\rho+s))$. Then, for each compact interval $I \subset [0, \infty)$, there is $c > 0$ such that $\xi(s) > c$ for $s \in I$. By L'Hospital's rule, we have

$$\lim_{s\to\infty}\xi(s) = \lim_{s\to\infty}\frac{1+\frac{\rho}{s}}{1+\frac{\rho}{s}[\log(\rho+s)]^q}\left\{1 - \frac{1}{1+\frac{\rho}{s}}\frac{q}{\log(\rho+s)}\right\} = 1,$$

and hence there is $c_0 > 0$ such that $\xi(s) \ge c_0$ for $s \ge 0$. Thus, $g^{-1}(c_0^q s) \le g_1(s)$ for $s \ge 0$. Then, the conclusion holds.

(v) By (i), we see that $g(\tau)$ is increasing. Let $s := g(\tau)$. Then, $g^{-1}(s)^p/s = \tau^{p-1}\left[\log(\rho+\tau)\right]^{-q}$. Since $\rho > e^{q/(p-1)}$, we have

$$\frac{d}{d\tau}\frac{\tau^{p-1}}{[\log(\rho+\tau)]^q} = \frac{\tau^{p-2}}{[\log(\rho+\tau)]^{q+1}}\left\{(p-1)\log(\rho+\tau) - \frac{q\tau}{\rho+\tau}\right\} > 0.$$

Thus, $g^{-1}(s)^p/s$ is increasing for $s \ge 0$.

(vi) Because of (ii), g is convex. By Jensen's inequality, we see that $g(S(t)\phi) \le S(t)g(\phi)$. Since g^{-1} exists and g^{-1} is increasing, the conclusion holds. The proof is complete. □

3. Existence

Lemma 3.1. *Let* $N \geq 1$ *and* $p = 1 + 2/N$. *Assume that* $\phi \geq 0$. *If* $\phi \in X_q$ *for some* $q \geq N/2$, *then* (1.1) *has a local-in-time nonnegative mild solution* $u(t)$, *and* $\|u(t)\|_\infty \leq Ct^{-N/2}(-\log t)^{-q}$ *for small* $t > 0$.

Proof. First, we consider the case $q = N/2$. Let $\rho \geq \max\{e^{q/(p-1)}, e\}$ be fixed. Let g be defined by (1.7). Here, $q = N/2$ and g satisfies Lemma 2.5. We define

$$\bar{u}(t) := 2g^{-1}(S(t)g(\phi)).$$

We show that $\bar{u}$ is a supersolution. By Lemma 2.5(vi), we have

$$S(t)\phi \leq g^{-1}(S(t)g(\phi)) = \frac{\bar{u}(t)}{2}. \tag{3.1}$$

Next, we have

$$\begin{aligned}
&\int_0^t S(t-s)f(\bar{u}(s))\mathrm{d}s \\
&\quad = 2^p \int_0^t S(t-s)\left[S(s)g(\phi)\frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)}\right]\mathrm{d}s \\
&\quad \leq 2^p S(t)g(\phi)\int_0^t \left\|\frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)}\right\|_\infty \mathrm{d}s \\
&\quad \leq 2^p g^{-1}(S(t)g(\phi))\left\|\frac{S(t)g(\phi)}{g^{-1}(S(t)g(\phi))}\right\|_\infty \int_0^t \left\|\frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)}\right\|_\infty \mathrm{d}s.
\end{aligned} \tag{3.2}$$

Since $g(\phi) \in L^1(\mathbb{R}^N)$, by Proposition 2.4(ii) we have

$$\|S(t)g(\phi)\|_\infty \leq C_0 t^{-N/2}. \tag{3.3}$$

By Lemma 2.5(v), we see that $g^{-1}(u)^p/u$ is nondecreasing for $u \geq 0$. Using (3.3) and Lemma 2.5(iv), we have

$$\begin{aligned}
&\left\|\frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)}\right\|_\infty \leq \frac{g^{-1}\left(\|S(s)g(\phi)\|_\infty\right)^p}{\|S(s)g(\phi)\|_\infty} \\
&\leq \frac{g^{-1}(C_0 s^{-N/2})^p}{C_0 s^{-N/2}} \leq \frac{C_1^p C_0^{2/N}}{s\left[\log\left(\rho + C_0 C_1 s^{-N/2}\right)\right]^{pq}} \leq \frac{C_0^{2/N} C_1'}{s(-\log s)^{pq}}
\end{aligned} \tag{3.4}$$

for $0 < s < s_0(C_0)$, where C_1' is a constant independent of C_0. Using Lemma 2.5(iii) and (3.3), we have

$$\begin{aligned}
&\left\|\frac{S(t)g(\phi)}{g^{-1}(S(t)g(\phi))}\right\|_\infty \leq \left\|\frac{S(t)g(\phi)}{g_1(S(t)g(\phi))}\right\|_\infty = \left\|\left[\log(\rho + S(t)g(\phi))\right]^q\right\|_\infty \\
&\leq \left[\log(\rho + \|S(t)g(\phi)\|_\infty)\right]^q \leq \left[\log(\rho + C_0 t^{-N/2})\right]^q \leq C_2'(-\log t)^q
\end{aligned} \tag{3.5}$$

for $0 < t < t_0(C_0)$, where g_1 is defined in Lemma 2.5 and C_2' is a constant independent of C_0. By (3.4) and (3.5) we have

$$\begin{aligned}
&\left\|\frac{S(t)g(\phi)}{g^{-1}(S(t)g(\phi))}\right\|_\infty \int_0^t \left\|\frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)}\right\|_\infty \mathrm{d}s \\
&\le C_0^{2/N}C_1'C_2'(-\log t)^q \int_0^t \frac{\mathrm{d}s}{s(-\log s)^{pq}} \\
&= C_0^{2/N}C_1'C_2'(-\log t)^q \frac{2}{N(-\log t)^q} = C_0^{2/N}C_1'C_2'\frac{2}{N} \qquad (3.6)
\end{aligned}$$

for $0 < t < \min\{s_0(C_0), t_0(C_0)\}$. By Proposition 2.4(ii), we can take $C_0 > 0$ such that $2^{p+1}C_0^{2/N}C_1'C_2'/N < 1$. By (3.1), (3.2) and (3.6), we have

$$\mathcal{F}[\bar{u}](t) = S(t)\phi + \int_0^t S(t-s)f(\bar{u}(s))\mathrm{d}s \le \frac{1}{2}\bar{u}(t) + \frac{1}{2}\bar{u}(t) = \bar{u}(t)$$

for small $t > 0$. Thus, there is $T > 0$ such that $\mathcal{F}[\bar{u}] \le \bar{u}$ for $0 < t < T$, and hence $\bar{u}$ is a supersolution. By Lemma 2.1, we see that there is $T > 0$ such that (1.1) has a solution for $0 < t < T$, and $u(t)$ is clearly nonnegative. Moreover,

$$0 \le u(t) \le \bar{u}(t) = 2g^{-1}(S(t)g(\phi)) \le Ct^{-\frac{N}{2}}(-\log t)^{-q}, \qquad (3.7)$$

which is the estimate in the assertion. We show that $u(t) \in C([0,T), L^1(\mathbb{R}^N))$. Since $\left\|g^{-1}(u)\right\|_1 \le C\|u\|_1$, by (3.6) and Proposition 2.4(i) we have

$$\begin{aligned}
\|u(t) - S(t)\phi\|_1 &\le \left\|\int_0^t S(t-s)f(\bar{u}(s))\mathrm{d}s\right\|_1 \le C_0^{2/N}C_1'C_2'\frac{2}{N}\left\|g^{-1}(S(t)g(\phi))\right\|_1 \\
&\le C_0^{2/N}C_1'C_2'\frac{2}{N}C\|S(t)g(\phi)\|_1 \le C_0^{2/N}C_1'C_2'\frac{2}{N}C'\|g(\phi)\|_1 \qquad (3.8)
\end{aligned}$$

for small $t > 0$, where C' is independent of C_0. By Proposition 2.4(ii), we can take $C_0 > 0$ arbitrary small, and hence

$$\|u(t) - S(t)\phi\|_1 \to 0 \text{ as } t \downarrow 0.$$

Since $S(t)$ is a strongly continuous semigroup on $L^1(\mathbb{R}^N)$ (see e.g., [9, Section 48.2]), we have

$$\|u(t) - \phi\|_1 \le \|u(t) - S(t)\phi\|_1 + \|S(t)\phi - \phi\|_1 \to 0 \text{ as } t \downarrow 0. \qquad (3.9)$$

It follows from (3.2) and (3.6) that $\left\|\int_0^t S(t-s)f(\bar{u}(s))\mathrm{d}s\right\|_1 < \infty$ for $0 < t < T$. We see that if $0 < t < T$, then

$$\|u(t+h) - u(t)\|_1 \to 0 \text{ as } h \to 0. \qquad (3.10)$$

By (3.9) and (3.10), we see that $u(t) \in C([0,T), L^1(\mathbb{R}^N))$. The proof of (i) is complete.

Next, we consider the case $q > N/2$. The argument is the same until (3.6). We have

$$\begin{aligned}&\left\|\frac{S(t)g(\phi)}{g^{-1}\left(S(t)g(\phi)\right)}\right\|_{\infty}\int_0^t\left\|\frac{g^{-1}\left(S(s)g(\phi)\right)^p}{S(s)g(\phi)}\right\|_{\infty}\mathrm{d}s\\&\le C_0^{2/N}C_1'C_2'(-\log t)^q\int_0^t\frac{\mathrm{d}s}{s(-\log s)^{pq}}\\&=\frac{C_1^{2/N}C_1'C_2'}{pq-1}(-\log t)^{1-\frac{2q}{N}}\end{aligned}\tag{3.11}$$

instead of (3.6). Since the RHS of (3.11) goes to 0 as $t \downarrow 0$, the rest of the proof is almost the same with obvious modifications. In particular, (3.7) holds even for $q > N/2$. We omit the details. □

We consider (1.6), where ϕ is given in (1.1). By Lemma 3.1, we see that (1.6) has a local-in-time solution which is denoted by $w(t)$. We consider the sequence $(u_n)_{n=0}^{\infty}$ defined by (1.8). Then, the following lemma says that $\|u_n(t)\|_\infty$ can be controlled by $w(t)$.

Lemma 3.2. *Let u_n be as defined by* (1.8)*, and let w be a solution of* (1.6) *on* $(0, T)$*. Then,*

$$-w(t) \le u_n(t) \le w(t) \quad \text{fora.e. } x \in \mathbb{R}^N \quad \text{and } 0 < t < T. \tag{3.12}$$

Proof. It is clear from the definitions of u_0 and $w(t)$ that

$$u_0(t) \le w(t) \ \text{ for } \ 0 < t < T.$$

We assume that $u_{n-1}(t) \le w(t)$ on $(0, T)$. Then, we have

$$\begin{aligned}w(t) = S(t)|\phi| + \int_0^t S(t-s)f(w(s))\mathrm{d}s\\ \ge S(t)\phi + \int_0^t S(t-s)f(u_{n-1}(s))\mathrm{d}s\\ = u_n(t),\end{aligned}$$

and hence $u_n(t) \le w(t)$ for $0 < t < T$. Thus, by induction we see that, for $n \ge 0$,

$$u_n(t) \le w(t) \ \text{ on } \ 0 < t < T. \tag{3.13}$$

It is clear that $u_0(t) \ge -w(t)$ for $0 < t < T$. We assume that $u_{n-1}(t) \ge -w(t)$ on $(0, T)$. Then, we have

$$\begin{aligned}u_n(t) &= S(t)\phi + \int_0^t S(t-s)f(u_{n-1}(s))\mathrm{d}s\\ &\ge -S(t)|\phi| + \int_0^t S(t-s)f(-w(s))\mathrm{d}s = -w(t),\end{aligned}$$

and hence, $u_n(t) \geq -w(t)$ on $(0, T)$. Thus, by induction we see that for $n \geq 0$,

$$-w(t) \leq u_n(t) \text{ on } 0 < t < T. \tag{3.14}$$

By (3.13) and (3.14), we see that (3.12) holds. □

Proof of Theorem 1.3. (i) Let $(u_n)_{n=0}^{\infty}$ be defined by (1.8). Using an induction argument with a parabolic regularity theorem, we can show that, for each $n \geq 1$, $u_n \in C^{2,1}(\mathbb{R}^N \times (0, T))$ and u_n satisfies the equation

$$\partial_t u_n = \Delta u_n + f(u_{n-1}) \text{ in } \mathbb{R}^N \times (0, T)$$

in the classical sense. Let K be an arbitrary compact subset in $\mathbb{R}^N \times (0, T)$, and let K_1, K_2 be two compact sets such that $K \subset K_1 \subset K_2 \subset \mathbb{R}^N \times (0, T)$. Because of Lemma 3.2, $f(u_{n-1})$ is bounded in $C(K_2)$. By a parabolic regularity theorem, we see that u_n is bounded in $C^{\gamma,\gamma/2}(K_1)$. Using a parabolic regularity theorem again, we see that u_{n+1} is bounded in $C^{2+\gamma,1+\gamma/2}(K)$.

In the following, we use a diagonal argument to obtain a convergent subsequence in $\mathbb{R}^N \times (0, T)$. Let $Q_j := \overline{\{x \in \mathbb{R}^N | \, |x| \leq j\}} \times \left[\frac{T}{j+2}, \frac{(j+1)T}{j+2}\right]$. Since $(u_n)_{n=3}^{\infty}$ is bounded in $C^{2,1}(Q_1)$, by Ascoli–Arzerà theorem there is a subsequence $(u_{1,k}) \subset (u_n)$ and $u_1^* \in C(Q_1)$ such that $u_{1,k} \to u_1^*$ in $C(Q_1)$ as $k \to \infty$. Since $(u_{1,k})_{k=1}^{\infty}$ is bounded in $C^{2,1}(Q_2)$, there is a subsequence $(u_{2,k}) \subset (u_{1,n})$ and $u_2^* \in C(Q_2)$ such that $u_{2,k} \to u_2^*$ in $C(Q_2)$ as $k \to \infty$. Repeating this argument, we have a double sequence $(u_{j,k})$ and a sequence (u_j^*) such that, for each $j \geq 1$, $u_{j,k} \to u_j^*$ in $C(Q_j)$ as $k \to \infty$. We still denote $u_{n,n}$ by u_n, i.e., $u_n := u_{n,n}$. It is clear that $u_{j_1}^* \equiv u_{j_2}^*$ in Q_{j_1} if $j_1 \leq j_2$. Since $\mathbb{R}^N \times (0, T) = \bigcup_{j=1}^{\infty} Q_j$, there is $u^* \in C(\mathbb{R}^N \times (0, T))$ such that $u_n \to u^*$ in $C(K)$ as $n \to \infty$ for every compact set $K \subset \mathbb{R}^N \times (0, T)$. In particular,

$$u_n \to u^* \text{ a.e. in } \mathbb{R}^N \times (0, T). \tag{3.15}$$

Let w be a solution of (1.6). It follows from Lemma 3.2 that $|u_n(x,t)| \leq w(x,t)$. Since

$$|G_t(x-y)u_n(y,t)| \leq |G_t(x-y)w(y,t)| \text{ for } y \in \mathbb{R}^N,$$

and

$$G_t(x-y)w(y,t) \in L_y^1(\mathbb{R}^N),$$

by the dominated convergence theorem we see that

$$\begin{aligned} \lim_{n\to\infty} S(t)u_n &= \lim_{n\to\infty} \int_{\mathbb{R}^N} G_t(s-y)u_n(y,t)\mathrm{d}y \\ &= \int_{\mathbb{R}^N} G_t(s-y)u^*(y,t)\mathrm{d}y = S(t)u^*. \end{aligned} \tag{3.16}$$

By (3.2) and (3.6), we see that if $T > 0$ is small, then

$$\int_0^t \int_{\mathbb{R}^N} G_{t-s}(x-y) f(w(y,s)) dyds \le C g^{-1}(S(t)g(\phi)) < \infty$$

for each $(x,t) \in \mathbb{R}^N \times (0,T)$, and hence $G_{t-s}(x-y)f(w(y,s)) \in L^1_{(y,s)}(\mathbb{R}^N \times (0,T))$. Since

$$\begin{aligned} &|G_{t-s}(x-y) f(u_{n-1}(y,s))| \\ &\quad \le |G_{t-s}(x-y) f(w(y,s))| \ \text{ for a.e. } (y,s) \in \mathbb{R}^N \times (0,T) \end{aligned}$$

and

$$G_{t-s}(x-y) f(w(y,s)) \in L^1_{(y,s)}(\mathbb{R}^N \times (0,T)),$$

by the dominated convergence theorem we see that

$$\begin{aligned} \lim_{n\to\infty} \int_0^t S(t-s) f(u_{n-1}(s)) \mathrm{d}s &= \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^N} G_{t-s}(x-y) f(u_{n-1}(y,s)) dyds \\ = \int_0^t \int_{\mathbb{R}^N} G_{t-s}(x-y) f(u^*(y,s)) dyds &= \int_0^t S(t-s) f(u^*(s)) \mathrm{d}s. \qquad (3.17) \end{aligned}$$

Thus, we take a limit of $u_n = \mathcal{F}[u_{n-1}]$. By (3.15), (3.16) and (3.17), we see that $u^*(t) = \mathcal{F}[u^*](t)$ for $0 < t < T$.

Since $|u_n| \le w$, we see that $|u^*| \le w$. Since $|u^*| \le w$ in $\mathbb{R}^N \times (0,T)$, by (3.8) and the arbitrariness of $C_0 > 0$ we have

$$\begin{aligned} \left\| u^*(t) - S(t)\phi \right\|_1 &= \left\| \int_0^t S(t-s) f(u^*(s)) \mathrm{d}s \right\|_1 \\ &\le \left\| \int_0^t S(t-s) f(w(s)) \mathrm{d}s \right\|_1 \to 0 \ \text{ as } \ t \downarrow 0. \end{aligned}$$

Then, $\|u^*(t) - \phi\|_1 \le \|u^*(t) - S(t)\phi\|_1 + \|S(t)\phi - \phi\|_1 \to 0$ as $t \downarrow 0$. Since $\left\| \int_0^t S(t-s) f(w(s)) \right\|_1 < \infty$ for $0 < t < T$, we can show by a similar way to the proof of Lemma 3.1 that $u^*(t) \in C((0,T), L^1(\mathbb{R}^N))$. Thus, $u^*(t) \in C([0,T), L^1(\mathbb{R}^N))$, and hence $u^*(t)$ is a mild solution. Since $|u^*(t)| \le w(t)$, by Lemma 3.1 we have (1.3). The proof of (i) is complete. □

4. Nonexistence

Let $0 \le q < N/2$ be fixed. Then, there is $0 < \varepsilon < N/2 - q$. We define ϕ_0 by

$$\phi_0(x) := \begin{cases} |x|^{-N} (-\log|x|)^{-\frac{N}{2}-1+\varepsilon} & \text{if } |x| < 1/e, \\ 0 & \text{if } |x| \ge 1/e. \end{cases} \qquad (4.1)$$

Lemma 4.1. *Let* $0 \le q < N/2$*, and let* ϕ_0 *be defined by* (4.1)*. Then, the following holds:*

(i) $\phi_0 \in X_q (\subset L^1(\mathbb{R}^N))$.

(ii) The function ϕ_0 *does not satisfy* (2.1) *for any* $T > 0$.

Proof. (i) We write $\phi_0(r) = r^{-N}(-\log r)^{-N/2-1+\varepsilon}$ for $0 < r < 1/e$. Since $\log(e + s) \le 1 + \log s$ for $s \ge 0$, we have

$$\log(e + |\phi_0|) \le 1 - N\log r - \left(\frac{N}{2} + 1 - \varepsilon\right)\log(-\log r) \le -2N\log r \quad (4.2)$$

for $0 < r < 1/e$. Let $B(\tau) := \{x \in \mathbb{R}^N \mid |x| < \tau\}$. Using (4.2), we have

$$\begin{aligned}\int_{B(1/e)} |\phi_0| \left[\log(e + |\phi_0|)\right]^q dx &\le \omega_{N-1}\int_0^{1/e} \frac{(2N)^q(-\log r)^q r^{N-1}\mathrm{d}r}{r^N(-\log r)^{N/2+1-\varepsilon}} \\ &\le (2N)^q\omega_{N-1}\int_0^{1/e}\frac{\mathrm{d}r}{r\,(-\log r)^{N/2+1-q-\varepsilon}} = \frac{(2N)^q\omega_{N-1}}{\frac{N}{2} - q - \varepsilon} < \infty, \quad (4.3)\end{aligned}$$

where ω_{N-1} denotes the area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. By (4.3), we see that $\phi_0 \in X_q$.

(ii) Suppose the contrary, i.e., there exists $\gamma_0 > 0$ such that (2.1) holds. When $0 < \tau < 1/e$, we have

$$\begin{aligned}\int_{B(\tau)} \phi_0(x)\mathrm{d}x &= \omega_{N-1}\int_0^{\tau}\frac{\mathrm{d}r}{r(-\log r)^{N/2+1-\varepsilon}} \\ &= \frac{C}{(-\log\tau)^{N/2-\varepsilon}},\end{aligned}$$

where $C > 0$ is independent of τ. Then,

$$\gamma_0 \ge \frac{\int_{B(\tau)}\phi_0(x)\mathrm{d}x}{(-\log\tau)^{-N/2}} \ge C(-\log\tau)^{\varepsilon} \to \infty \ \text{ as } \ \tau \downarrow 0.$$

which is a contradiction. Thus, the conclusion holds. □

Proof of Theorem 1.3 (ii). Let $0 \le q < N/2$. It follows from Lemma 4.1(i) that $\phi_0 \in X_q$. By Lemma 4.1(ii), we see that there does not exist $\gamma_0 > 0$ such that (2.1) holds. By Proposition 2.2, the problem (1.1) with ϕ_0 has no nonnegative integral solution. □

5. Uniqueness

Proof of Theorem 1.5. Let $q > N/2$. Suppose that (1.1) has two integral solutions $u(t)$ and $v(t)$. Using Young's inequality and the inequality $\|u(t)\|_\infty \le Ct^{-N/2}(-\log t)^{-q}$, we have

$$\begin{aligned}\|u(t)-v(t)\|_1 &\le \int_0^t \left\| G_{t-s} * \left\{ \left(p|u|^{p-1} + p|v|^{p-1} \right)(u-v) \right\} \right\|_1 \mathrm{d}s \\ &\le p \int_0^t \|G_{t-s}\|_1 \left(\|u\|_\infty^{p-1} + \|v\|_\infty^{p-1} \right) \mathrm{d}s \sup_{0\le s\le t} \|u(s)-v(s)\|_1 \\ &\le C \int_0^t \frac{\mathrm{d}s}{\left\{ s^{N/2}(-\log s)^q \right\}^{p-1}} \sup_{0\le s\le t} \|u(s)-v(s)\|_1 .\end{aligned}$$

Since

$$\int_0^t s^{-N(p-1)/2}(-\log s)^{-(p-1)q} \mathrm{d}s = \frac{N(-\log t)^{1-2q/N}}{2q-N}$$

and $1-2q/N < 0$, we can choose $T > 0$ such that $C \int_0^t s^{-N(p-1)/2}(-\log s)^{-(p-1)q} \mathrm{d}s < 1/2$ for every $0 \le t \le T$. Then, we have

$$\sup_{0\le t\le T} \|u(t)-v(t)\|_1 \le \frac{1}{2} \sup_{0\le s\le T} \|u(s)-v(s)\|_1 ,$$

which implies the uniqueness. □

Acknowledgements

The author was supported by JSPS KAKENHI Grant Number 19H01797.

REFERENCES

[1] P. Baras and M. Pierre, Critère d'existence de solutions positives pour des équations semi-linéaires non monotones, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), 185–212.

[2] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math. **68** (1996), 277–304.

[3] C. Celik and Z. Zhou, No local L^1 solution for a nonlinear heat equation, Comm. Partial Differential Equations **28** (2003), 1807–1831.

[4] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), 109–124.

[5] A. Haraux and F. Weissler, Nonuniqueness for a semilinear initial value problem, Indiana Univ. Math. J. **31** (1982), 167–189.

[6] K. Hisa and K. Ishige, Existence of solutions for a fractional semilinear parabolic equation with singular initial data, Nonlinear Anal. **175** (2018), 108–132.

[7] R. Laister, J. Robinson, M. Sierżęga and A. Vidal-López, A complete characterisation of local existence for semilinear heat equations in Lebesgue spaces, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), 1519–1538.

[8] W. Ni and P. Sacks, Singular behavior in nonlinear parabolic equations, Trans. Amer. Math. Soc. **287** (1985), 657–671.

[9] P. Quittner and P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2007. xii+584 pp. ISBN: 978-3-7643-8441-8.

[10] J. Robinson and M. Sierżęga, Supersolutions for a class of semilinear heat equations, Rev. Mat. Complut. **26** (2013), 341–360.

[11] J. Takahashi, Solvability of a semilinear parabolic equation with measures as initial data, Geometric properties for parabolic and elliptic PDE's, Springer Proc. Math. Stat., vol. **176**, Springer, 2016, ISBN: 978-3-319-41538-3, pp. 257–276.

[12] E. Terraneo, Non-uniqueness for a critical non-linear heat equation, Comm. Partial Differential Equations **27** (2002), 185–218.

[13] F. Weissler, Local existence and nonexistence for semilinear parabolic equations in L^p, Indiana Univ. Math. J. **29** (1980), 79–102.

[14] F. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, Israel J. Math. **38** (1981), 29–40.

Yasuhito Miyamoto
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba Meguro-ku
Tokyo 153-8914
Japan
E-mail: miyamoto@ms.u-tokyo.ac.jp