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Time periodic traveling wave solutions for a Kermack–McKendrick epidemic model with diffusion and seasonality

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Abstract. In this paper, we study the time periodic traveling wave solutions for a Kermack–McKendrick SIR epidemic model with individuals diffusion and environment heterogeneity. In terms of the basic reproduction number R_0 of the corresponding periodic ordinary differential model and the minimal wave speed c^*, we establish the existence of periodic traveling wave solutions by the method of super- and sub-solutions, the fixed-point theorem, as applied to a truncated problem on a large but finite interval, and the limiting arguments. We further obtain the nonexistence of periodic traveling wave solutions for two cases involved with R_0 and c^*.

1. Introduction

In this paper, we are interested in the following time periodic reaction–diffusion epidemic system

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = d_1 \Delta S(t,x) - \beta(t) S(t,x) I(t,x), & t>0,\ x\in\mathbb{R}, \\ \frac{\partial I(t,x)}{\partial t} = d_2 \Delta I(t,x) + \beta(t) S(t,x) I(t,x) - \gamma(t) I(t,x), & t>0,\ x\in\mathbb{R}, \\ \frac{\partial R(t,x)}{\partial t} = d_3 \Delta R(t,x) + \gamma(t) I(t,x), & t>0,\ x\in\mathbb{R}, \end{cases} \tag{1.1}$$

which describes the evolution of an epidemic within a spatially distributed population of individuals in a seasonal forcing environment. Here, $S(t,x)$, $I(t,x)$ and $R(t,x)$ denote the densities of the susceptible, infected and recovered/removed individuals at time t and located at the spatial position $x \in \mathbb{R}$, respectively. The positive constants d_1, d_2 and d_3 are the diffusion rates for the susceptible, infected and recovered/removed individuals, respectively. The disease transmission rate $\beta(t)$ and the recovery rate $\gamma(t)$ are all positive T-periodic continuous functions in t.

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The kinetic system of (1.1) is

$$\begin{cases} \frac{dS(t)}{dt} = -\beta(t)S(t)I(t), & t > 0, \\ \frac{dI(t)}{dt} = \beta(t)S(t)I(t) - \gamma(t)I(t), & t > 0, \\ \frac{dR(t)}{dt} = \gamma(t)I(t), & t > 0, \end{cases} \tag{1.2}$$

which has been deeply studied by Bacaër and Gomes [2], where they observed somewhat counterintuitive conclusions quite different from what is in a constant environment, that is, the classic Kermack–McKendrick SIR epidemic model [26] (see also [1,5]):

$$\begin{cases} \frac{dS(t)}{dt} = -\beta S(t)I(t), & t > 0, \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t), & t > 0, \\ \frac{dR(t)}{dt} = \gamma I(t), & t > 0, \end{cases} \tag{1.3}$$

but almost compatible with occurrence. The consequence in [2] implies that the behavior of epidemics under the seasonal forcing is not a straightforward generalization of the known results in a constant environment. In fact, it was reported that the transmission rates and the recovery rates of many epidemics can be significantly impacted by seasonality, see London and Yorke [30] for the yearly outbreaks of measles, chickenpox and mumps, and Hethcote and Yorke [21] for the seasonal oscillation of gonorrhea. In particular, London and Yorke [30] pointed out that there are two significant factors influencing the dynamics of epidemics and contributing to the one-year periodicity of the contact rate: (i) weather/climatic factors such as temperature and relative humidity; (ii) social behavior (contact patterns) influenced by public holidays (children due to school terms), vacations. Figure 1 in [8] also states that most human respiratory pathogens exhibit and annual increase in incidence each winter, although there are variations in the timing of onset and magnitude of the increase. For more on the impact of the seasonality in epidemic models, we refer to [19,20,32] and a review paper [6]. Here, we would like to emphasize that models (1.2) and (1.3) are usually used to describe the transmission of disease whose time scale is rather fast with respect to the vital dynamic of the population. Therefore, the vital dynamics is not incorporated into (1.2) and (1.3) and the total population number remains invariant in the transmission process of the epidemic. As mentioned above, the transmission dynamics of many epidemics such as measles, chickenpox, mumps and gonorrhea [21,30] are significantly influenced by seasonality. On the other hand, the total number of the population usually remains (almost) invariant within several years. Thus, if we neglect (or do not consider) the effect of the vital dynamics of the population, then the system such as (1.2) (and (1.1)) is rather reasonable and should do duty for an admonition to interpret the epidemics influenced by seasonality.

To consider the propagation dynamics of (1.1), in which the random walk of individuals and the seasonality are incorporated, traveling wave solution is a key topic. For the autonomous version of system (1.1), namely

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = d_1 \Delta S(t,x) - \beta S(t,x) I(t,x), & t > 0,\ x \in \mathbb{R}, \\ \frac{\partial I(t,x)}{\partial t} = d_2 \Delta I(t,x) + \beta S(t,x) I(t,x) - \gamma I(t,x), & t > 0,\ x \in \mathbb{R}, \\ \frac{\partial R(t,x)}{\partial t} = d_3 \Delta R(t,x) + \gamma I(t,x), & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.4}$$

there has been extensive research on the traveling waves for the first two equations of (1.4) (the R equation can be decoupled). Kallen [24] and Kallen et al. [25] have studied the existence of traveling wave solutions when $d_1 = 0$. Particularly, Hosnono and Ilyas [22] proved that there admits a pair of traveling wave solution $(S(x+ct), I(x+ct))$ satisfying $S(-\infty) = S_0 > 0, S(+\infty) = S^{\infty} < S_0, I(\pm\infty) = 0$ for each $c \geq c^* = 2\sqrt{\beta S_0 d_2 (1 - \gamma/\beta S_0)}$ when the basic reproduction number $R_0 := \frac{\beta S_0}{\gamma}$ of system (1.3) is larger than unit, which represents the transition from the initial disease-free equilibrium $(S_0, 0, 0)$ to another disease- free state $(S^{\infty}, 0, 0)$ with S^{∞} being determined by the model coefficients. Since then, there have been extensive investigations on traveling wave solutions of system (1.4) (see, e.g., [18,23,40] and references therein), and its variants such as age-infection structure [10,11], delays or non-local delays [34], spatially discrete structure [17] and non-local dispersal case [37]. We also refer to [9] for the long-term behavior of (1.4) with spatial heterogeneity ($d_1 = 0$).

In the current work, we are concerned with time periodic traveling wave solutions (see the definition in the next section) for problem (1.1). Since system (1.1) involved with the same non-monotone structure as system (1.4), which implies that (1.1) does not have comparison principle, the theory and methods for monotone periodic systems (see, e.g., [13,28,41,42]) are no longer effective. In addition, differently from system (1.4), problem (1.1) gives rise to a periodic parabolic system of wave profile, which leads to failure for the approaches in the aforementioned literatures to system (1.4). Recently, Wang et al. [35] studied time periodic traveling wave solutions for the following periodic and diffusive SIR model with standard incidence:

$$\begin{cases} \frac{\partial}{\partial t} S(t,x) = d_1 \Delta S(t,x) - \frac{\beta(t) S(t,x) I(t,x)}{S(t,x) + I(t,x)}, \\ \frac{\partial}{\partial t} I(t,x) = d_2 \Delta I(t,x) + \frac{\beta(t) S(t,x) I(t,x)}{S(t,x) + I(t,x)} - \gamma(t) I(t,x), \\ \frac{\partial}{\partial t} R(t,x) = d_3 \Delta R(t,x) + \gamma(t) I(t,x). \end{cases} \tag{1.5}$$

Here, $S(t,x)$, $I(t,x)$ and $R(t,x)$ denote the densities of the susceptible, infected and recovered individuals at time t and in location x, respectively. The coefficients in (1.5) represent the same meaning as in system (1.1). It should be pointed out that the incidence in (1.5) reflects the recovered individuals is removed from the population and not involved in the contact and disease transmission (see [33]). They proved that if the basic reproduction number $R_0 := \frac{\int_0^T \beta(t) dt}{\int_0^T \gamma(t) dt}$ of kinetic system of (1.5) is larger than unit, there exists a critical value $c^* = 2\sqrt{\frac{1}{d_2 T} \int_0^T [\beta(t) - \gamma(T)] dt}$ such that for any wave speed $c > c^*$, system (1.5) admits a time periodic traveling wave solution. Furthermore, they obtained the nonexistence of periodic traveling wave solutions for two cases:(i) $R_0 \le 1$; (ii) $R_0 > 1$ and $c < c^*$. The literature [35] makes an elementary attempt and provides a novel train of thought to solve the existence of time periodic traveling wave solutions for periodic and non-monotone systems.

Note that mass action in (1.1) and standard incidence infection mechanism in (1.5) are widely adopted in modeling infectious diseases transmission. From the epidemiological perspective, the mass action is appropriate for modeling contact between infectious individuals and susceptible individuals in small population size, while utilizing the standard incidence frequently depends on population size, that is, it is suitable for larger population size. Another observation is that the basic reproduction number of the kinetic system associated with (1.1) is dependent on population size (see Sect. 2), while the basic reproduction number of kinetic system of (1.5) is independent of population size. The aforementioned difference on two incidence functions leads to some distinction on mathematical analysis in the corresponding models. In addition, in view of the bilinear incidence (or mass action infection mechanism) in system (1.1), the derivation of existence of periodic traveling wave solutions to (1.1) becomes much more challenging. Precisely speaking, it is difficult to verify the boundedness of I. On the other hand, the method on the nonexistence of periodic traveling wave solutions of (1.5) when $R_0 := \frac{\int_0^T \beta(t)dt}{\int_0^T \gamma(t)dt} > 1$ and $c < c^*$, can be hardly applied to system (1.1). Motivated by the ideas in [10,35,39], we shall consider the truncated problem on a finite interval and apply the limiting arguments to deal with the periodic traveling wave problem associated with (1.1). This will extend the research strategy on periodic traveling wave solutions for periodic and non-monotone systems. Here, we emphasize that in [39] a similar argument was used to establish the existence of periodic traveling wave solution for a time periodic and delayed reaction–diffusion equation without quasi-monotonicity, which describes the growth of mature population of a single species living in a fluctuating environment.

The rest of this paper is organized as follows. In the next section, by constructing a suitable pair of super- and sub-solutions and applying the Schauder's fixed-point theorem to a similar problem on a bounded domain, we then use some *a priori* estimations and a limiting procedure to establish the existence of the periodic traveling wave solutions. Section 3 is devoted to the study of the nonexistence of periodic traveling wave solutions for two cases. A brief discussion completes the paper.

2. Existence of periodic traveling waves

In this section, we focus on the existence of the non-trivial and time periodic traveling waves $(\phi(t, z), \psi(t, z))$ of system (1.1). Since the R equation of system (1.1) can be decoupled, it is sufficient to consider the following system

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = d_1 \Delta S(t, x) - \beta(t) S(t, x) I(t, x), & t > 0,\ x \in \mathbb{R}, \\ \frac{\partial I(t,x)}{\partial t} = d_2 \Delta I(t, x) + \beta(t) S(t, x) I(t, x) - \gamma(t) I(t, x) & t > 0,\ x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Time periodic traveling waves to system (2.1) are defined to be solutions of the form

$$\begin{pmatrix} S(t,x) \\ I(t,x) \end{pmatrix} = \begin{pmatrix} \phi(t,x+ct) \\ \psi(t,x+ct) \end{pmatrix}, \quad \begin{pmatrix} \phi(t+T,z) \\ \psi(t+T,z) \end{pmatrix} = \begin{pmatrix} \phi(t,z) \\ \psi(t,z) \end{pmatrix} \tag{2.2}$$

satisfying

$$\begin{pmatrix} \phi(t,\pm\infty) \\ \psi(t,\pm\infty) \end{pmatrix} = \begin{pmatrix} \phi_\pm(t) \\ \psi_\pm(t) \end{pmatrix},$$

where c is called the wave speed, $z = x + ct$ is the moving coordinate and $\begin{pmatrix} \phi_+(t) \\ \psi_+(t) \end{pmatrix}$ and $\begin{pmatrix} \phi_-(t) \\ \psi_-(t) \end{pmatrix}$ are two periodic solutions of the corresponding kinetic system:

$$\begin{cases} \frac{dS}{dt} = -\beta(t)S(t)I(t), \\ \frac{dI}{dt} = \beta(t)S(t)I(t) - \gamma(t)I(t). \end{cases} \tag{2.3}$$

Such solutions (ϕ, ψ) must satisfy the following system:

$$\begin{cases} \phi_t(t,z) = d_1\phi_{zz}(t,z) - c\phi_z(t,z) - \beta(t)\phi(t,z)\psi(t,z), \\ \psi_t(t,z) = d_2\psi_{zz}(t,z) - c\psi_z(t,z) + \beta(t)\phi(t,z)\psi(t,z) - \gamma(t)\psi(t,z). \end{cases} \tag{2.4}$$

This system is posed on $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and is supplemented with the following asymptotic boundary conditions

$$\phi(t,-\infty) = S_0,\ \phi(t,\infty) = S^\infty,\ \psi(t,\pm\infty) = 0 \text{ uniformly in } t \in \mathbb{R}. \tag{2.5}$$

Here, $S_0 > 0$ is a constant, and $(S_0, 0)$ is the initial disease-free steady state. The parameter $c > 0$ is the wave speed, while constant $S^\infty \geq 0$ describes the density of susceptible individuals after the epidemic.

Our basic procedure to prove the existence of periodic traveling wave solutions is as follows. Firstly, by constructing some suitable super- and sub-solutions for (2.4), we obtain a closed and convex set Γ_N of initial functions lying between the sub- and super-solutions. Secondly, we consider the truncated problem posed on the bounded domain and define a nonlinear solution operator $\mathcal{F}$ on Γ_N, and then, we apply the Schauder's fixed- point theorem to $\mathcal{F}$ after verifying the complete continuity of it. Finally, on the basis of some proposed *a priori* estimations of the obtained fixed point of $\mathcal{F}$, a limiting procedure can be used to extend the bounded interval to $\mathbb{R}$, and then, the existence of periodic traveling wave solutions is established. By similar arguments to [35], we further verify the asymptotic boundary conditions for periodic traveling wave solutions.

2.1. Construction of sub- and super-solutions

Linearizing system (2.4) at the disease-free steady state $(S_0, 0)$, we have the following equation:

$$\tilde{I}_t(t,x) = d_2\tilde{I}_{zz} - c\tilde{I}_z(t,z) + (S_0\beta(t) - \gamma(t))\tilde{I}(t,x).$$

Define

$$\Theta_c(\lambda) = d_2\lambda^2 - c\lambda + \varrho,\ c \in \mathbb{R}, \lambda \in \mathbb{R} \tag{2.6}$$

where $\varrho := \frac{1}{T}\int_0^T (S_0\beta(t) - \gamma(t))\,\mathrm{d}t$. Clearly, $\varrho > 0$ if the basic reproduction number $R_0 := \frac{S_0\int_0^T \beta(t)\mathrm{d}t}{\int_0^T \gamma(t)\mathrm{d}t} > 1$. In what follows, we always assume that $R_0 > 1$. Let

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4d_2\varrho}}{2d_2},\ \lambda_2 = \frac{c + \sqrt{c^2 - 4d_2\varrho}}{2d_2}$$

if $c > c^* := 2\sqrt{d_2\varrho}$. Then, we have $\Theta_c(\lambda_1) = \Theta_c(\lambda_2) = 0$ and $\Theta_c(\lambda) < 0, \forall\lambda \in (\lambda_1, \lambda_2)$.

Fixing $c > c^*$, we set

$$K(t) := \exp\left(\int_0^t [d_2\lambda_1^2 - c\lambda_1 + (S_0\beta(s) - \gamma(s)]\mathrm{d}s\right).$$

It is easy to see that $K(t)$ is T-periodic. We further define the following functions

$$\phi^+(t,z) := S_0,\ \phi^-(t,z) := \max\{S_0(1 - M_1e^{\epsilon_1 z}), 0)\},$$
$$\psi^+(t,z) := K(t)e^{\lambda_1 z},\ \psi^-(t,z) := \max\{K(t)e^{\lambda_1 z}(1 - M_2e^{\epsilon_2 z}), 0\},$$

where M_i and ϵ_i, $i = 1, 2$ are all positive constants and will be determined below. Then, we can inductively establish the following results.

Lemma 2.1. *The function $\psi^+(t,z) = K(t)e^{\lambda_1 z}$ satisfies the following linear equation:*

$$\psi_t = d_2\psi_{zz} + c\psi_z - (\beta(t)S_0 - \gamma(t))\,\psi. \tag{2.7}$$

Lemma 2.2. *For sufficiently small ϵ_1 such that $0 < \epsilon_1 < \lambda_1$ and sufficiently large $M_1 > 1$, the function ϕ^- satisfies*

$$\phi_t - d_1\phi_{zz} + c\phi_z \le -\beta(t)\phi\psi^+ \tag{2.8}$$

for any $z \ne z_1 := -\epsilon_1^{-1}\ln M_1$.

Proof. In case where $z > -\epsilon_1^{-1}\ln M_1$, we have $\phi^-(t,z) = 0$, which implies (2.8) holds.

In case where $z < -\epsilon_1^{-1} \ln M_1$, then $\phi^-(t, z) = S_0 \left(1 - M_1 e^{\epsilon_1 z}\right)$. Thus, we need only to prove that

$$d_1 S_0 M_1 \epsilon_1^2 e^{\epsilon_1 z} - c S_0 M_1 \epsilon_1 e^{\epsilon_1 z} \leq -\beta(t) S_0 \left(1 - M_1 e^{\epsilon_1 z}\right) K(t) e^{\lambda_1 z}.$$

That is,

$$M_1 \epsilon_1 \left(c - d_1 \epsilon_1\right) \geq \beta(t) \left(1 - M_1 e^{\epsilon_1 z}\right) K(t) e^{(\lambda_1 - \epsilon_1) z}.$$

So for $z < z_1 := -\epsilon_1^{-1} \ln M_1$, it is sufficient to verify

$$M_1 \epsilon_1 \left(c - d_1 \epsilon_1\right) \geq \beta(t) K(t) e^{-\epsilon_1^{-1} (\lambda_c - \epsilon_1) \ln M} = \beta(t) K(t) M_1^{-\epsilon_1^{-1} (\lambda_1 - \epsilon_1)}, \quad \forall t \in \mathbb{R}.$$

Since both $\beta(t)$ and $K(t)$ are positive T-periodic functions, the above inequality is valid as long as we choose $M_1 = 1/\epsilon_1$ with $\epsilon_1 > 0$ sufficiently small and $0 < \epsilon_1 < \lambda_1$. □

Lemma 2.3. *Suppose $\epsilon_2 > 0$ is sufficiently small such that $\epsilon_2 < \min\{\epsilon_1, \lambda_2 - \lambda_1\}$, and M_2 is sufficiently large such that $-\epsilon_2^{-1} \ln M_2 < -\epsilon_1^{-1} \ln M_1$. Then, the function ψ^- satisfies*

$$\psi_t - d_2 \psi_{zz} + c \psi_z \leq \beta(t) \phi^- \psi - \gamma(t) \psi \tag{2.9}$$

for any $z \neq z_2 := -\epsilon_2^{-1} \ln M_2$.

Proof. Choose M_2 large enough to ensure that $-\epsilon_2^{-1} \ln M_2 < -\epsilon_1^{-1} \ln M_1$. For $z > z_2 := -\epsilon_2^{-1} \ln M_2$, one has $\psi^-(t, z) = 0$, and hence, the inequality (2.9) holds.

When $z < z_2 := -\epsilon_2^{-1} \ln M_2$, $\psi^-(t, z) = K(t) e^{\lambda_1 z} \left(1 - M_2 e^{\epsilon_2 z}\right)$ and $\phi^-(t, z) = S_0 (1 - M_1 e^{\epsilon_1 z})$. In order to obtain (2.9), we only need to verify the following inequality

$$\psi_t^- - d_2 \psi_{zz}^- + c \psi_z^- - \left(\beta(t) S_0 - \gamma(t)\right) \psi^- \leq \beta(t) \left(\phi^- - S_0\right) \psi^-. \tag{2.10}$$

By the expression of $K(t)$ and ψ^-, it follows that

$$\begin{aligned}
&\psi_t^- - d_2 \psi_{zz}^- + c \psi_z^- - \left(\beta(t) S_0 - \gamma(t)\right) \psi^- \\
&\quad = K'(t) e^{\lambda_1 z} \left(1 - M_2 e^{\epsilon_2 z}\right) - d_2 \Big[\lambda_1^2 K(t) e^{\lambda_1 z} \left(1 - M_2 e^{\epsilon_2 z}\right) - \lambda_1 \epsilon_2 M_2 K(t) e^{(\lambda_1 + \epsilon_2) z} \\
&\qquad - (\lambda_1 + \epsilon_2) \epsilon_2 M_2 K(t) e^{(\lambda_1 + \epsilon_2) z}\Big] + c \Big[\lambda_1 K(t) e^{\lambda_1 z} \left(1 - M_2 e^{\epsilon_2 z}\right) - \epsilon_2 M_2 K(t) e^{(\lambda_1 + \epsilon_2) z}\Big] \\
&\qquad - [\beta(t) S_0 - \gamma(t)] K(t) e^{\lambda_1 z} \left(1 - M_2 e^{\epsilon_2 z}\right) \\
&\quad = e^{\lambda_1 z} \Big\{K'(t) - d_2 \lambda_1^2 K(t) + c \lambda_1 K(t) - [\beta(t) S_0 - \gamma(t)] K(t)\Big\} \\
&\qquad - M_2 e^{(\lambda_1 + \epsilon_2) z} \Big\{K'(t) - d_2 (\lambda_1 + \epsilon_2)^2 K(t) + c(\lambda_1 + \epsilon_2) K(t) - [\beta(t) S_0 - \gamma(t)] K(t)\Big\} \\
&\quad = -M_2 e^{(\lambda_1 + \epsilon_2) z} K(t) \Big\{\Big[d_2 \lambda_1^2 - c \lambda_1\Big] - \Big[d_2 (\lambda_1 + \epsilon_2)^2 - c(\lambda_1 + \epsilon_2)\Big]\Big\} \\
&\quad = M_2 e^{(\lambda_1 + \epsilon_2) z} K(t) \Theta_c (\lambda_1 + \epsilon_2).
\end{aligned}$$

Then, the inequality (2.10) is equivalent to

$$M_2 e^{\epsilon_2 z}\Theta_c(\lambda_1+\epsilon_2) \leq -\beta(t)S_0 M_1 e^{\epsilon_1 z}(1-M_2 e^{\epsilon_2 z}) \tag{2.11}$$

Owing to $\epsilon_1 < \lambda_2 - \lambda_1$, we have $\lambda_1+\epsilon_2 \in (\lambda_1, \lambda_2)$, and hence,

$$\Theta_c(\lambda_1+\epsilon_2) = d_2(\lambda_1+\epsilon_2)^2 - c(\lambda_1+\epsilon_2) + \varrho < 0.$$

Since $\beta(t)$ is positive and T-periodic in $\mathbb{R}$, the inequality (2.11) is true if and only if

$$-M_2\Theta_c(\lambda_1+\epsilon_2) \geq \beta(t)S_0 M_1 e^{(\epsilon_1-\epsilon_2)z}.$$

Thus, when $z < -\epsilon_2^{-1}\ln M_2$, we need to show

$$-M_2\Theta_c(\lambda_1+\epsilon_2) \geq \beta(t)S_0 M_1 M_2^{-(\epsilon_1-\epsilon_2)/\epsilon_2}$$

for all $t \in [0,T]$. The last inequality holds true when we choose sufficiently small $\epsilon_2 < \epsilon_1$ and M_2 large enough. □

2.2. Reduction to a fixed-point problem

Take $N > -z_2$. Define

$$\Gamma_N := \left\{ (\tilde{\phi},\tilde{\psi}) \in C(\mathbb{R}\times[-N,N],\mathbb{R}^2): \begin{array}{l} \tilde{\phi}(t,z) = \tilde{\phi}(t+T,z),\ \forall\, t\in\mathbb{R},\ z\in[-N,N], \\ \tilde{\psi}(t,z) = \tilde{\psi}(t+T,z),\ \forall\, t\in\mathbb{R},\ z\in[-N,N]; \\ \phi^-(t,z) \leq \tilde{\phi}(t,z) \leq \phi^+(t,z),\ \forall t\in\mathbb{R},\ z\in[-N,N], \\ \psi^-(t,z) \leq \tilde{\psi}(t,z) \leq \psi^+(t,z),\ \forall t\in\mathbb{R},\ z\in[-N,N]; \\ \tilde{\phi}(t,\pm N) = \phi^-(t,\pm N),\ \forall t\in\mathbb{R}, \\ \tilde{\psi}(t,\pm N) = \psi^-(t,\pm N),\ \forall t\in\mathbb{R} \end{array} \right\}.$$

For any given $(\tilde{\phi},\tilde{\psi}) \in \Gamma_N$, define maps

$$f_1[\tilde{\phi},\tilde{\psi}](t,z) = \alpha_1\tilde{\phi}(t,z) - \beta(t)\tilde{\phi}(t,z)\tilde{\psi}(t,z)$$

and

$$f_2[\tilde{\phi},\tilde{\psi}](t,z) = \alpha_2\tilde{\psi}(t,z) + \beta(t)\tilde{\phi}(t,z)\tilde{\psi}(t,z) - \gamma(t)\tilde{\psi}(t,z),$$

where α_1 and α_2 are positive constants and satisfy $\alpha_1 > \max_{t\in[0,T]}\beta(t)K(t)e^{\lambda_1 N}$ and $\alpha_2 > \max_{t\in[0,T]}\gamma(t)$, respectively. Let $\mathcal{A}_i u = d_i\partial_{zz}u - c\partial_z u - \alpha_i u$, $i=1,2$. Fix a $(\tilde{\phi},\tilde{\psi}) \in \Gamma_N$. Consider the following linear parabolic initial boundary value problem:

$$\begin{cases} \partial_t\phi(t,z) - \mathcal{A}_1\phi(t,z) = f_1[\tilde{\phi},\tilde{\psi}](t,z), & t>0, z\in[-N,N], \\ \partial_t\psi(t,z) - \mathcal{A}_2\psi(t,z) = f_2[\tilde{\phi},\tilde{\psi}](t,z), & t>0, z\in[-N,N], \\ \phi(0,z) = \phi_0(z),\quad \psi(0,z)=\psi_0(z), & z\in[-N,N],\ \phi_0,\psi_0\in C([-N,N]), \\ \phi(t,\pm N) = G_1(t,\pm N),\quad \psi(t,\pm N) = G_2(t,\pm N), & t\geq 0, \end{cases} \tag{2.12}$$

where $G_1(t,z) = \frac{1}{2}\phi^-(t,-N) - \frac{z}{2N}\phi^-(t,-N)$ and $G_2(t,z) = \frac{1}{2}\psi^-(t,-N) - \frac{z}{2N}\psi^-(t,-N)$ for all $t \in [0,T]$ and $z \in [-N,N]$. It is easy to see that $G_1(t,\pm N) = \phi^-(t,\pm N), G_2(t,\pm N) = \psi^-(t,\pm N)$ for $t \in \mathbb{R}$, and the function G_i is T-periodic and belongs to $C^{1,2}(\mathbb{R}\times[-N,N])$ for $i = 1,2$. Let $V_1(t,z) = \phi(t,z) - G_1(t,z)$, $V_2(t,z) = \psi(t,z) - G_2(t,z)$ and $\tilde{G}_i = \mathcal{A}_i G_i(t,z) - \partial_t G_i(t,z)$. Then, the problem (2.12) reduces to the following system on (V_1, V_2) :

$$\begin{cases} \partial_t V_i(t,z) - \mathcal{A}_i V_i(t,z) = f_i[\tilde{\phi},\tilde{\psi}](t,z) + \tilde{G}_i(t,z), \ i=1,2, t>0, z \in [-N,N], \\ V_1(0,z) = \phi_0(z) - G_1(0,z), \quad V_2(0,z) = \phi_0(z) - G_2(0,z), \quad z \in [-N,N], \\ V_i(t,\pm N) = 0, \qquad i = 1,2,\ t \geq 0. \end{cases} \tag{2.13}$$

Define the realization of $\mathcal{A}_i$ in $C([-N,N])$ with homogeneous Dirichlet boundary condition,

$$D(A_i^0) = \left\{ u \in \bigcap_{p\geq 1} W^{2,p}_{loc}((-N,N)) : u, \mathcal{A}_i u \in C([-N,N]), u|_{\pm N} = 0 \right\}, \ A_i^0 u = \mathcal{A}_i u,\ i=1,2.$$

Let $T_i(t)_{t\geq 0}$ be the strongly continuous analytic semigroup generated by $A_i^0 : D(A_i^0) \subset C([-N,N]) \to C([-N,N])$ (see, e.g., [7,31]). It is easy to see that

$$T_i(t)w(x) = e^{-\alpha_i t}\int_{-N}^{N} \Gamma_i(t,x,y)w(y)dy, \quad i = 1,2,\ w(\cdot) \in C([-N,N]), \tag{2.14}$$

for $t > 0, x \in [-N,N]$, where $\Gamma_i, i = 1,2$ is the Green function associated with $d_i\partial_{xx} - c\partial_x, i = 1,2$ and Dirichlet boundary condition. Then, system (2.13) can be rewritten as the following integral system

$$\begin{cases} V_1(t,z) = T_1(t)\left(\phi_0 - G_1(0)\right)(z) + \int_0^t T_1(t-s)\left(f_1[\tilde{\phi},\tilde{\psi}](s) + \tilde{G}_1(s)\right)(z)ds, \\ V_2(t,z) = T_2(t)\left(\psi_0 - G_2(0)\right)(z) + \int_0^t T_2(t-s)\left(f_2[\tilde{\phi},\tilde{\psi}](s) + \tilde{G}_2(s)\right)(z)ds \end{cases} \tag{2.15}$$

for all $t \geq 0$ and $z \in [-N,N]$. Then, $(\phi(t,z),\psi(t,z))$ satisfies that

$$\begin{cases} \phi(t,z) = T_1(t)\left(\phi_0 - G_1(0)\right)(z) + \int_0^t T_1(t-s)\left(f_1[\tilde{\phi},\tilde{\psi}](s) + \tilde{G}_1(s)\right)(z)ds + G_1(t,z), \\ \psi(t,z) = T_2(t)\left(\psi_0 - G_2(0)\right)(z) + \int_0^t T_2(t-s)\left(f_2[\tilde{\phi},\tilde{\psi}](s) + \tilde{G}_2(s)\right)(z)ds + G_2(t,z) \end{cases} \tag{2.16}$$

for all $t \geq 0$ and $z \in [-N,N]$. We call a solution of (2.16) as a mild solution of (2.12). Since $f_i[\tilde{\phi},\tilde{\psi}] \in C(\mathbb{R}\times[-N,N])$ and $f_i[\tilde{\phi},\tilde{\psi}](t,\cdot) \in C([-N,N])$, it follows from [31, Theorem 5.1.17] that the functions ϕ and ψ defined by (2.16) belong to $C([0,2T]\times[-N,N])\cap C^{\theta,2\theta}([\epsilon,2T]\times[-N,N])$ for every $\epsilon\in(0,2T)$ and $\theta\in(0,1)$.

Define a set

$$\Gamma'_N := \left\{ (\phi_0,\psi_0)\in C([-N,N],\mathbb{R}^2) : \begin{array}{r} \phi^-(0,z)\le\phi_0(z)\le\phi^+(0,z),\ z\in[-N,N], \\ \psi^-(0,z)\le\psi_0(z)\le\psi^+(0,z),\ z\in[-N,N], \\ \phi_0(\pm N)=\phi^-(0,\pm N),\ \psi_0(\pm N)=\psi^-(0,\pm N) \end{array} \right\}$$

with the usual supreme norm. Obviously, Γ'_N is a closed and convex set.

Lemma 2.4. *For any $(\phi_0,\psi_0)\in\Gamma'_N$, let $(\phi_N(t,z;\phi_0,\psi_0),\psi_N(t,z;\phi_0,\psi_0))$ be the solutions of the system (2.16) with the initial value (ϕ_0,ψ_0). Then,*

$$\phi^-(t,z)\le\phi_N(t,z;\phi_0,\psi_0)\le\phi^+(t,z),\quad \psi^-(t,z)\le\psi_N(t,z;\phi_0,\psi_0)\le\psi^+(t,z)$$

for $(t,z)\in[0,+\infty)\times[-N,N]$.

Proof. Let us first recall that for the given $(\tilde\phi,\tilde\psi)\in\Gamma_N$, there hold

$$\begin{aligned}&\phi^-(t,z)\le\tilde\phi(t,z)\le\phi^+(t,z),\quad \psi^-(t,z)\le\tilde\psi(t,z)\le\psi^+(t,z),\\ &\forall(t,z)\in\mathbb{R}\times[-N,N],\end{aligned}$$

while every $(\phi_0,\psi_0)\in\Gamma'_N$ satisfies

$$\phi^-(0,z)\le\phi_0(z)\le\phi^+(0,z),\quad \psi^-(0,z)\le\psi_0(z)\le\psi^+(0,z),\quad \forall z\in[-N,N].$$

We are ready to prove that $\phi_N(t,z;\phi_0,\psi_0)\le\phi^+(t,z)$ for all $t\ge0$ and $z\in[-N,N]$. Let $\overline{\phi}$ be the solution of the following equation

$$\begin{aligned}\overline{\phi}(t) &= T_1(t)\,(\phi_0-G_1(0))+\int_0^t T_1(t-s)\left(f_1[\phi^+,\psi^-](s)+\tilde G_1(s)\right)ds\\ &\quad +G_1(t),\quad t\ge0.\end{aligned}$$

Since $f_1[\tilde\phi,\tilde\psi]\le f_1[\phi^+,\psi^-]$, we have

$$\phi_N(t,\cdot;\phi_0,\psi_0)\le\overline{\phi}(t,\cdot;\phi_0,\psi_0),\quad \forall t\ge0. \tag{2.17}$$

In addition, since $f_1[\phi^+,\psi^-]\in C^{\theta/2,\theta}(\mathbb{R}\times[-N,N])$ for some $\theta\in(0,1)$, it follows from [31, Theorems 5.1.18 and 5.1.19] that $\overline{\phi}\in C([0,+\infty)\times[-N,N])$ is differentiable with respect to t in $(0,+\infty)\times[-N,N]$, $\overline{\phi}(t,\cdot)$ belongs to $W^{2,p}_{\mathrm{loc}}((-N,N))$ for every $p\ge1$, and $\partial_t\overline{\phi},\mathcal{A}_1\overline{\phi}\in C^{\theta/2,\theta}([\delta,+\infty)\times[-N,N])$ for any $\delta>0$. As a consequence, we see that $\overline{\phi}\in C([0,+\infty)\times[-N,N])\cap C^{1,2}((0,+\infty)\times[-N,N])$ and satisfies that

$$\begin{cases}\partial_t\overline{\phi}(t,z)-\mathcal{A}_1\overline{\phi}(t,z)=f_1[\phi^+,\psi^-](t,z), & t>0, z\in[-N,N],\\ \overline{\phi}(0,z)=\phi_0(z), & z\in[-N,N],\\ \overline{\phi}(t,\pm N)=G_1(t,\pm N)=\phi^-(t,\pm N), & t\ge0.\end{cases}$$

On the other hand, it is easy to see that ϕ^+ satisfies

$$\begin{cases} \partial_t \phi^+(t,z) - \mathcal{A}_1 \phi^+(t,z) = \alpha_1 \phi^+(t,z) \geq f_1[\phi^+, \psi^-](t,z), & t > 0, z \in [-N,N], \\ \phi^+(0,z) = S_0 \geq \phi_0(z), & z \in [-N,N], \\ \phi^+(t,\pm N) = S_0 \geq G_1(t,\pm N) = \phi^-(t,\pm N), & t \geq 0. \end{cases}$$

Thus, the parabolic comparison principle indicates that

$$\overline{\phi}(t,z) \leq \phi^+(t,z), \quad \forall (t,z) \in [0,+\infty) \times [-N,N]. \tag{2.18}$$

In view of (2.17) and (2.18), we have that

$$\phi_N(t,z;\phi_0,\psi_0) \leq \overline{\phi}(t,z;\phi_0,\psi_0) \leq \phi^+(t,z), \quad \forall (t,z) \in [0,+\infty) \times [-N,N].$$

Let $\underline{\phi}$ be the solution of the following equation

$$\underline{\phi}(t) = T_1(t)\left(\phi_0 - G_1(0)\right) + \int_0^t T_1(t-s)\left(f_1[\phi^-,\psi^+](s) + \tilde{G}_1(s)\right) ds + G_1(t), \quad t \geq 0.$$

Thus, we have

$$\phi_N(t,\cdot;\phi_0,\psi_0) \geq \underline{\phi}(t,\cdot;\phi_0,\psi_0), \quad \forall t \geq 0, \tag{2.19}$$

because of $f_1[\tilde{\phi},\tilde{\psi}] \geq f_1[\phi^-,\psi^+]$. Additionally, since $f_1[\phi^-,\psi^+] \in C^{\theta/2,\theta}(\mathbb{R} \times [-N,N])$ for some $\theta \in (0,1)$, we conclude from [31, Theorems 5.1.18 and 5.1.19] that $\underline{\phi} \in C([0,+\infty) \times [-N,N]) \cap C^{1,2}((0,+\infty) \times [-N,N])$ satisfies that

$$\begin{cases} \partial_t \underline{\phi}(t,z) - \mathcal{A}_1 \underline{\phi}(t,z) = f_1[\phi^-,\psi^+](t,z), & t > 0, z \in [-N,N], \\ \underline{\phi}(0,z) = \phi_0(z), & z \in [-N,N], \\ \underline{\phi}(t,\pm N) = G_1(t,\pm N) = \phi^-(t,\pm N), & t \geq 0. \end{cases}$$

Let $\underline{\phi}^* \equiv 0$. Then, $\underline{\phi}^*$ satisfies

$$\partial_t \underline{\phi}^*(t,z) - \mathcal{A}_1 \underline{\phi}^*(t,z) \leq f_1[\phi^-,\psi^+], \quad t \in [0,+\infty), z \in [-N,N],$$

and hence, the parabolic comparison principle implies that $\underline{\phi}(t,z) \geq 0$ for all $t \in [0,+\infty)$ and $z \in [-N,N]$. When $(t,z) \in \mathbb{R} \times (-\infty, z_1)$, it follows from Lemma 2.2 that $\phi^-(t,z) = S_0(1 - M_1 e^{\epsilon_1 z})$ satisfies (2.8). Thus,

$$\partial_t \left(\underline{\phi}(t,z) - \phi^-(t,z)\right) - \mathcal{A}_1 \left(\underline{\phi}(t,z) - \phi^-(t,z)\right) \geq 0,$$
$$(t,z) \in (0,+\infty) \times [-N, z_1).$$

Hence, it follows from the maximum principle [15, Chapter 2, Theorem 1] that

$$\underline{\phi}(t,z) \geq \phi^-(t,z), \quad (t,z) \in (0,+\infty) \times [-N, z_1).$$

Note that $\phi^-(t,z) = \max\{S_0(1 - M_1 e^{\epsilon_1 z}), 0\}$. Therefore, we further have that

$$\phi_N(t,z;\phi_0,\psi_0) \geq \underline{\phi}(t,z;\phi_0,\psi_0) \geq \phi^-(t,z), \quad \forall (t,z) \in [0,+\infty) \times [-N,N].$$

In the following, we consider $\psi_N(t,z;\phi_0,\psi_0)$ for $t \in [0,+\infty)$ and $z \in [-N,N]$. Let $\overline{\psi}$ be the solution of the following equation

$$\overline{\psi}(t) = T_2(t)\left(\psi_0 - G_2(0)\right) + \int_0^t T_2(t-s)\left(f_2[\phi^+,\psi^+](s) + \tilde{G}_2(s)\right) ds + G_2(t), \quad t \geq 0.$$

Clearly,

$$\overline{\psi}(t,\cdot;\phi_0,\psi_0) \geq \psi_N(t,\cdot;\phi_0,\psi_0), \quad \forall t \in [0,+\infty).$$

On the other hand, since $f_2[\phi^-,\psi^+] \in C^{\theta/2,\theta}([0,T]\times[-N,N])$ for some $\theta \in (0,1)$, it follows from [31, Theorems 5.1.18 and 5.1.19] that $\overline{\psi} \in C([0,+\infty) \times [-N,N]) \cap C^{1,2}((0,+\infty) \times [-N,N])$ satisfies that

$$\begin{cases} \partial_t \overline{\psi}(t,z) - \mathcal{A}_2 \overline{\psi}(t,z) = f_2[\phi^+,\psi^+](t,z), & t > 0, z \in [-N,N], \\ \overline{\psi}(0,z) = \psi_0(z), & z \in [-N,N], \\ \overline{\psi}(t,\pm N) = G_2(t,\pm N) = \psi^-(t,\pm N), & t \geq 0. \end{cases}$$

In view of Lemma 2.1, (2.7) can be rewritten as

$$\begin{cases} \partial_t \psi^+(t,z) - \mathcal{A}_2 \psi^+(t,z) = P[\psi^+](t,z), \quad t \in (0,+\infty), z \in [-N,N], \\ \psi^+(0,z) = e^{\lambda_1 z}, \quad z \in [-N,N], \\ \psi^+(t,\pm N) = K(t) e^{\pm \lambda_1 N}, \quad t \in [0,+\infty), \end{cases}$$

where $P[\psi^+](t,z) = \alpha_2 \psi^+ + \beta(t) S_0 \psi^+ - \gamma(t)\psi^+$, $(t,z) \in \mathbb{R} \times [-N,N]$. Since $P[\psi^+](t,z) \geq f_2[\phi^+,\psi^+](t,z)$ for $t \in (0,+\infty)$ and $z \in [-N,N]$, $\psi^+(0,\cdot) \geq \psi_0(\cdot)$ and $\psi^+(\cdot,\pm N) \geq G_2(\cdot,\pm N)$, we can conclude from the parabolic comparison principle that

$$\overline{\psi}(t,z) \leq \psi^+(t,z), \quad \forall (t,z) \in [0,+\infty) \times [-N,N].$$

Thus, we further have that

$$\psi_N(t,z;\phi_0,\psi_0) \leq \overline{\psi}(t,z;\phi_0,\psi_0) \leq \psi^+(t,z), \quad \forall (t,z) \in [0,+\infty) \times [-N,N].$$

Finally, we show that $\psi_N(t,z;\phi_0,\psi_0) \geq \psi^-(t,z)$ for all $t \in [0,+\infty)$ and $z \in [-N,N]$. Let $\underline{\psi}$ be the solution of the following equation

$$\underline{\psi}(t) = T_2(t)\left(\psi_0 - G_2(0)\right) + \int_0^t T_2(t-s)\left(f_2[\phi^-,\psi^-](s) + \tilde{G}_2(s)\right) ds + G_2(t), \quad t \geq 0.$$

It is obvious that $\underline{\psi}(t,\cdot;\phi_0,\psi_0) \leq \psi_N(t,\cdot;\phi_0,\psi_0)$ for all $t \geq 0$. In addition, since $f_2[\phi^+,\psi^-] \in C^{\theta/2,\theta}([0,T] \times [-N,N])$ for some $\theta \in (0,1)$, it follows from [31, Theorems 5.1.18 and 5.1.19] that $\underline{\psi} \in C([0,+\infty)\times[-N,N]) \cap C^{1,2}((0,+\infty)\times[-N,N])$ satisfies that

$$\begin{cases} \partial_t \underline{\psi}(t,z) - \mathcal{A}_2 \underline{\psi}(t,z) = f_2[\phi^-,\psi^-](t,z), & t\in(0,T], z\in[-N,N],\\ \underline{\psi}(0,z) = \psi_0(z) \geq 0, & z\in[-N,N],\\ \underline{\psi}(t,\pm N) = G_2(t,\pm N) = \psi^-(t,\pm N) \geq 0, & t\in[0,T]. \end{cases}$$

Let $\underline{\psi}^*(t,z)\equiv 0$. Then, $\underline{\psi}^*(t,z)$ satisfies

$$\partial_t \underline{\psi}^*(t,z) - \mathcal{A}_2 \underline{\psi}^*(t,z) \leq f_2[\phi^-,\psi^-], \quad t\in[0,+\infty), z\in[-N,N],$$

and hence, the parabolic comparison principle implies that $\underline{\psi}(t,z)\geq 0$ for all $t\in[0,+\infty)$ and $z\in[-N,N]$. When $(t,z)\in\mathbb{R}\times(-\infty,z_2)$, we see that $\psi^-(t,z) = K(t)e^{\lambda_1 z}(1-M_2 e^{\epsilon_2 z})$. Thus, by Lemma 2.3, we have

$$\partial_t\left(\underline{\psi}(t,z)-\psi^-(t,z)\right) - \mathcal{A}_2\left(\underline{\psi}(t,z)-\psi^-(t,z)\right) \geq 0,$$
$$\forall (t,z)\in(0,+\infty)\times[-N,z_2).$$

Consequently, the maximum principle [15, Chapter 2, Theorem 1] yields that

$$\underline{\psi}(t,z)\geq\psi^-(t,z), \quad \forall (t,z)\in[0,+\infty)\times[-N,z_2).$$

Therefore, we further have that

$$\psi_N(t,z;\phi_0,\psi_0)\geq\underline{\psi}(t,z;\phi_0,\psi_0)\geq\psi^-(t,z), \quad \forall (t,z)\in[0,+\infty)\times[-N,N].$$

This completes the proof. □

For a given $(\tilde{\phi},\tilde{\psi})\in\Gamma_N$, we define a map $F_{(\tilde{\phi},\tilde{\psi})}:\Gamma_N'\to C([-N,N],\mathbb{R}^2)$ by

$$F_{(\tilde{\phi},\tilde{\psi})}[\phi_0,\psi_0](\cdot) = \left(\phi_N(T,\cdot;\phi_0,\psi_0),\psi_N(T,\cdot;\phi_0,\psi_0)\right),$$

where $(\phi_N(t,z;\phi_0,\psi_0),\psi_N(t,z;\phi_0,\psi_0))$ is the solution of (2.12). With the aid of Lemma 2.4 and the periodicity of ϕ^-, ψ^-, ϕ^+ and ψ^+, we have $F_{(\tilde{\phi},\tilde{\psi})}\left(\Gamma_N'\right)\subset\Gamma_N'$. Clearly, Γ_N' is a complete metric space with a distance induced by the supreme norm. For any $\left(\phi_0^1,\psi_0^1\right), \left(\phi_0^2,\psi_0^2\right)\in\Gamma_N'$, it follows from (2.14) and (2.16) that

$$\begin{aligned} &\left\|\phi_N(T,\cdot;\phi_0^1,\psi_0^1)-\phi_N(T,\cdot;\phi_0^2,\psi_0^2)\right\|_{C([-N,N])}\\ &= \sup_{z\in[-N,N]}\left|e^{-\alpha T}\int_{-N}^{N}\Gamma_1(T,z,y)\left(\phi_0^1(y)-\phi_0^2(y)\right)dy\right|\\ &\leq e^{-\alpha_1 T}\left\|\phi_0^1-\phi_0^2\right\|_{C([-N,N])}. \end{aligned}$$

Similarly, we have

$$\left\|\psi_N(T,\cdot;\phi_0^1,\psi_0^1)-\psi_N(T,\cdot;\phi_0^2,\psi_0^2)\right\|_{C([-N,N])}\leq e^{-\alpha_2 T}\left\|\psi_0^1-\psi_0^2\right\|_{C([-N,N])}.$$

Since $e^{-\alpha T} < 1$, we see that $F_{(\tilde{\phi},\tilde{\psi})} : \Gamma'_N \to \Gamma'_N$ is a contraction map. It then follows from the Banach fixed-point theorem that $F_{(\tilde{\phi},\tilde{\psi})}$ admits a unique fixed point $(\phi_0^*, \psi_0^*) \in \Gamma'_N$. Let $(\hat{\phi}_N^*(t,z), \hat{\psi}_N^*(t,z)) = \big(\phi_N(t,z;\phi_0^*,\psi_0^*), \psi_N(t,z;\phi_0^*,\psi_0^*)\big)$ for all $t \in [0,+\infty)$ and $z \in [-N,N]$, where $(\phi_N(t,z;\phi_0^*,\psi_0^*), \psi_N(t,z;\phi_0^*,\psi_0^*))$ is the solution of (2.16) with initial value (ϕ_0^*, ψ_0^*). In view of $(\phi_0^*(z), \psi_0^*(z)) = (\phi_N(T,z;\phi_0^*,\psi_0^*), \psi_N(T,z;\phi_0^*,\psi_0^*))$, we get $(\hat{\phi}_N^*(t+T,z), \hat{\psi}_N^*(t+T,z)) = (\hat{\phi}_N^*(t,z), \hat{\psi}_N^*(t,z))$ for all $t \in [0,+\infty)$ and $z \in [-N,N]$. Define $(\phi_N^*(t,z), \psi_N^*(t,z)) = (\hat{\phi}_N^*(t-kT,z), \hat{\psi}_N^*(t-kT,z))$ for $t \in \mathbb{R}$ and $z \in [-N,N]$, where $k \in \mathbb{Z}$ satisfies $kT \le t \le (k+1)T$. Then, $(\phi_N^*(t+T,z), \psi_N^*(t+T,z)) = (\phi_N^*(t,z), \psi_N^*(t,z))$ for all $t \in \mathbb{R}$ and $z \in [-N,N]$. According to Lemma 2.4, we see that $(\phi_N^*, \psi_N^*) \in \Gamma_N$. Moreover, (ϕ_N^*, ψ_N^*) satisfies

$$\begin{cases} \phi_N^*(t) = T_1(t-s)\left(\phi_N^*(s) - G_1(s)\right) + \int_s^t T_1(t-\theta)\left(f_1[\tilde{\phi},\tilde{\psi}](\theta) + \tilde{G}_1(\theta)\right)d\theta + G_1(t), \\ \psi_N^*(t) = T_2(t-s)\left(\psi_N^*(s) - G_2(s)\right) + \int_s^t T_2(t-\theta)\left(f_2[\tilde{\phi},\tilde{\psi}](\theta) + \tilde{G}_2(\theta)\right)d\theta + G_2(t) \end{cases} \tag{2.20}$$

for all $t \ge s$. On the basis of the above discussion, we obtain the following theorem.

Theorem 2.5. *For any given $(\tilde{\phi},\tilde{\psi}) \in \Gamma_N$, there exists a unique $(\phi_N^*, \psi_N^*) \in \Gamma_N$ such that (2.20) holds.*

Following Theorem 2.5, we can define an operator $\mathcal{F} : \Gamma_N \to \Gamma_N$ by $\mathcal{F}(\tilde{\phi},\tilde{\psi}) = (\phi_N^*, \psi_N^*)$. We further show the properties of the operator $\mathcal{F}$.

Lemma 2.6. *The operator $\mathcal{F} : \Gamma_N \to \Gamma_N$ is completely continuous.*

Proof. For any $(\tilde{\phi},\tilde{\psi}) \in \Gamma_N$, there holds $f_i[\tilde{\phi},\tilde{\psi}](\cdot,\cdot) \in C(\mathbb{R}\times[-N,N])$ and $f_i[\tilde{\phi},\tilde{\psi}](t+T,z) = f_i[\tilde{\phi},\tilde{\psi}](t,z)$ for $i = 1,2$, $(t,z) \in \mathbb{R}\times[-N,N]$. Note that $f_i[\tilde{\phi},\tilde{\psi}]$, $i = 1,2$ are uniformly bounded with respect to $(\tilde{\phi},\tilde{\psi}) \in \Gamma_N$. For any given $(\tilde{\phi},\tilde{\psi}) \in \Gamma_N$, let $(\phi_N^*, \psi_N^*) = \mathcal{F}(\tilde{\phi},\tilde{\psi}) \in \Gamma_N$. By [31, Theorem 5.1.17], it follows from (2.20) with $s = 0$ that $\phi_N^*, \psi_N^* \in C^{\theta/2,\theta}([T,2T]\times[-N,N])$ for every $\theta \in (0,1)$ and there exists $C_i(\theta) > 0$, $i = 1,2$ such that

$$\begin{aligned} &\|\phi_N^*\|_{C^{\theta/2,\theta}([T,2T]\times[-N,N])} \\ &\quad \le C_1(\theta)\left(T^{-\theta/2}\|\phi_N^*(0)-G_1(0)\|_\infty + \|f_1[\tilde{\phi},\tilde{\psi}]\|_\infty + \|G_1\|_{C^{0,1}}\right) \end{aligned}$$

and

$$\begin{aligned} &\|\psi_N^*\|_{C^{\theta/2,\theta}([T,2T]\times[-N,N])} \\ &\quad \le C_2(\theta)\left(T^{-\theta/2}\|\psi_N^*(0)-G_2(0)\|_\infty + \|f_2[\tilde{\phi},\tilde{\psi}]\|_\infty + \|G_2\|_{C^{0,1}}\right). \end{aligned}$$

Since ϕ_N^*, ψ_N^* are T-periodic, we have that $\phi_N^*, \psi_N^* \in C^{\theta/2,\theta}(\mathbb{R}\times[-N,N])$, and there exists $K_0^i(\theta) > 0$, $i = 1,2$ such that

$$\left\|\phi_N^*\right\|_{C^{\theta/2,\theta}(\mathbb{R}\times[-N,N])} \leq K_0^1(\theta), \quad \left\|\psi_N^*\right\|_{C^{\theta/2,\theta}(\mathbb{R}\times[-N,N])} \leq K_0^2(\theta),$$

which implies that $\mathcal{F}$ is compact on Γ_N.

We further prove the continuity of $\mathcal{F}$. For any $(\tilde{\phi}_i, \tilde{\psi}_i) \in \Gamma_N$, $i = 1, 2$, there exists a positive constant M such that $|\tilde{\phi}_i(t, z)| \leq M$ and $|\tilde{\psi}_i(t, z)| \leq M$ for $i = 1, 2$, $t \in \mathbb{R}$ and $z \in [-N, N]$, and let $(\phi_{i,N}^*(t, z; \tilde{\phi}_i, \tilde{\psi}_i),\ \psi_{i,N}^*(t, z; \tilde{\phi}_i, \tilde{\psi}_i)) = \mathcal{F}(\tilde{\phi}_i, \tilde{\psi}_i)$, $i = 1, 2$. By virtue of (2.14) and (2.20), we have

$$\begin{aligned}
&\phi_{i,N}^*(T, z; \tilde{\phi}_i, \tilde{\psi}_i)\\
&\quad = e^{-\alpha_1 T}\int_{-N}^{N}\Gamma_1(T, z, y)\left[\phi_{i,N}^*(0, y) - G_1(0, y)\right]dy + G_1(T, z)\\
&\qquad + \int_0^T e^{-\alpha_1 s}\int_{-N}^{N}\Gamma_1(s, z, y)\left(f_1[\tilde{\phi}_i, \tilde{\psi}_i](T-s, y) + \tilde{G}_1(T-s, y)\right)dyds
\end{aligned}$$

and

$$\begin{aligned}
&\psi_{i,N}^*(T, z; \tilde{\phi}_i, \tilde{\psi}_i)\\
&\quad = e^{-\alpha_2 T}\int_{-N}^{N}\Gamma_2(T, z, y)\left[\phi_{i,N}^*(0, y) - G_2(0, y)\right]dy + G_2(T, z)\\
&\qquad + \int_0^T e^{-\alpha_2 s}\int_{-N}^{N}\Gamma_2(s, z, y)\left(f_2[\tilde{\phi}_i, \tilde{\psi}_i](T-s, y) + \tilde{G}_2(T-s, y)\right)dyds.
\end{aligned}$$

Then, there holds

$$\begin{aligned}
&\left|\phi_{1,N}^*(T, z; \tilde{\phi}_1, \tilde{\psi}_1) - \phi_{2,N}^*(T, z; \tilde{\phi}_2, \tilde{\psi}_2)\right|\\
&\quad \leq e^{-\alpha_1 T}\int_{-N}^{N}\Gamma_1(T, z, y)\left|\phi_{1,N}^*(0, y) - \phi_{2,N}^*(0, y)\right|dy\\
&\qquad + \int_0^T e^{-\alpha_1 s}\int_{-N}^{N}\Gamma_1(s, z, y)\Big[\beta(T-s)\tilde{\phi}_1(T-s, y)\left(\tilde{\psi}_1(T-s, y) - \tilde{\psi}_2(T-s, y)\right)\\
&\qquad + \beta(T-s)\tilde{\psi}_2(T-s, y)\left(\tilde{\phi}_1(T-s, y) - \tilde{\phi}_2(T-s, y)\right)\Big]dyds\\
&\quad \leq e^{-\alpha_1 T}\left\|\phi_{1,N}^*(0) - \phi_{2,N}^*(0)\right\|_{C([-N,N])} + \tilde{\beta}M(1 - e^{-\alpha_1 T})\left\|\tilde{\psi}_1 - \tilde{\psi}_2\right\|\\
&\qquad + \tilde{\beta}M(1 - e^{-\alpha_1 T})\left\|\tilde{\phi}_1 - \tilde{\phi}_2\right\|,
\end{aligned}$$

where $\tilde{\beta} := \max_{t\in[0,T]}\beta(t)$. Since $\phi_{i,N}^*(t+T, z; \tilde{\phi}_i, \tilde{\psi}_i) = \phi_{i,N}^*(t, z; \tilde{\phi}_i, \tilde{\psi}_i)$ for all $t \in \mathbb{R}$ and $z \in [-N, N]$, we can get from the above inequality that

$$\left\|\phi_{1,N}^*(0) - \phi_{2,N}^*(0)\right\|_{C([-N,N])} \leq \tilde{\beta}M\left\|\tilde{\psi}_1 - \tilde{\psi}_2\right\| + \tilde{\beta}M\left\|\tilde{\phi}_1 - \tilde{\phi}_2\right\|.$$

Additionally, $\phi^*_{i,N}(t,z;\tilde{\phi}_i,\tilde{\psi}_i)$ satisfies

$$\begin{aligned}&\phi^*_{i,N}(t,z;\tilde{\phi}_i,\tilde{\psi}_i)\\&\quad= e^{-\alpha_1 t}\int_{-N}^{N}\Gamma_1(t,z,y)\left[\phi^*_{i,N}(0,y)-G_1(0,y)\right]dy+G_1(t,z)\\&\qquad+\int_0^t e^{-\alpha_1 s}\int_{-N}^{N}\Gamma_1(s,z,y)\Big(f_1[\tilde{\phi}_i,\tilde{\psi}_i](t-s,y)+\tilde{G}_1(t-s,y)\Big)\,dyds.\end{aligned}$$

Thus, by similar arguments to above, it is not difficult to show that $\phi^*_N(t,z;\tilde{\phi},\tilde{\psi})$ is continuous in $(\tilde{\phi},\tilde{\psi})$. Similarly, we can prove that $\psi^*_N(t,z;\tilde{\phi},\tilde{\psi})$ is continuous in $(\tilde{\phi},\tilde{\psi})$. The proof is complete. □

With the aid of Lemma 2.6, we can conclude from the Shauder's fixed-point theorem that $\mathcal{F}$ admits a fixed point $(\phi^*_N,\psi^*_N)\in\Gamma_N$. In particular, $(\phi^*_N(t+T,\cdot),\psi^*_N(t+T,\cdot))=(\phi^*_N(t,\cdot),\psi^*_N(t,\cdot))$ for all $t\in\mathbb{R}$. Note that $\phi^*_N,\psi^*_N\in C^{\theta/2,\theta}(\mathbb{R}\times[-N,N])$ for some $\theta\in(0,1)$. By [31, Theorem 5.1.18 and 5.1.19], we have that $\phi^*_N,\psi^*_N\in C^{1,2}(\mathbb{R}\times[-N,N])$ satisfy

$$\begin{cases}\partial_t\phi^*_N(t,z)=d_1\partial_{zz}\phi^*_N(t,z)-c\partial_z\phi^*_N(t,z)-\beta(t)\phi^*_N(t,z)\psi^*_N(t,z),\\ \qquad\qquad (t,z)\in\mathbb{R}\times[-N,N],\\ \partial_t\psi^*_N(t,z)=d_2\partial_{zz}\psi^*_N(t,z)-c\partial_z\psi^*_N(t,z)+\beta(t)\phi^*_N(t,z)\psi^*_N(t,z)-\gamma(t)\psi^*_N(t,z),\\ \phi^*_N(t,\pm N)=\phi^-(t,\pm N),\quad \psi^*_N(t,\pm N)=\psi^-(t,\pm N),\quad t\in\mathbb{R}.\end{cases}\tag{2.21}$$

The following theorem lists some local uniform estimates on ϕ^*_N and ψ^*_N.

Theorem 2.7. *Let $p\geq 2$. For any given $Z>0$, there exists a constant $C(p,Z)>0$ such that for sufficiently large $N>\max\{Z,-z_2\}$, there hold*

$$\|\phi^*_N\|_{W^{1,2}_p([0,T]\times[-Z,Z])},\ \|\phi^*_N\|_{W^{1,2}_p([0,T]\times[-Z,Z])}\leq C.$$

Furthermore, there exists a constant $C'(Z)>0$ such that for any $z_0\in\mathbb{R}$, there hold

$$\|\phi^*_N\|_{C^{(1+\theta)/2,1+\theta}([0,T]\times[z_0-Z,z_0+Z])},\ \|\phi^*_N\|_{C^{(1+\theta)/2,1+\theta}([0,T]\times[z_0-Z,z_0+Z])}\leq C'\tag{2.22}$$

for sufficiently large $N>\max\{Z+|z_0|,-z_2\}$, where $\theta\in(0,1)$.

Proof. Fix $Z>0$ and $z_0\in\mathbb{R}$. Let $N>\max\{Z+|z_0|,-z_2\}$. In view of the above discussion, we see that

$$\begin{cases}\partial_t\phi^*_N(t,z)=d_1\partial_{zz}\phi^*_N(t,z)-c\partial_z\phi^*_N(t,z)-\beta(t)\phi^*_N(t,z)\psi^*_N(t,z),\\ \partial_t\psi^*_N(t,z)=d_2\partial_{zz}\psi^*_N(t,z)-c\partial_z\psi^*_N(t,z)+\beta(t)\phi^*_N(t,z)\psi^*_N(t,z)-\gamma(t)\psi^*_N(t,z)\end{cases}$$

for all $(t, z) \in \mathbb{R} \times (-N, N)$. Since $(\phi_N^*, \psi_N^*) \in \Gamma_N$, there exists a $M > 0$ independent of N such that

$$\sup_{(t,z)\in\mathbb{R}\times[-N,N]} \phi_N^*(t, z) < M, \qquad \sup_{(t,z)\in\mathbb{R}\times[-N,N]} \psi_N^*(t, z) < M.$$

Let $W_N^1(t, z) := e^{-\frac{c(z-z_0)}{2d_1}} \phi_N^*(t, z)$, $W_N^2(t, z) := e^{-\frac{c(z-z_0)}{2d_2}} \psi_N^*(t, z)$ for any $t \in \mathbb{R}$ and $z \in [-N, N]$. It then follows that

$$\begin{aligned}
\partial_t W_N^1(t, z) &= d_1 \partial_{zz} W_N^1(t, z) - \frac{c^2}{4d_1} e^{-\frac{c(z-z_0)}{2d_1}} \phi_N^*(t, z) - \beta(t)\phi_N^*(t, z)\psi_N^*(t, z),\\
\partial_t W_N^2(t, z) &= d_2 \partial_{zz} W_N^2(t, z) - \frac{c^2}{4d_2} e^{-\frac{c(z-z_0)}{2d_2}} \psi_N^*(t, z) + \beta(t)\phi_N^*(t, z)\psi_N^*(t, z)\\
&\quad -\gamma(t)\psi_N^*(t, z)
\end{aligned}$$

for any $(t, z) \in \mathbb{R} \times (-N, N)$. For $(t', z') \in \mathbb{R}^2$ and $r > 0$, define

$$Q((t', z'), r) := \left\{(t, z) \in \mathbb{R}^2 \,\middle|\, |z - z'| < r, |t - t'| < r, t < t'\right\}.$$

For the given $Z > 0$, take $R = \max\{2Z, \sqrt{3T}\}$. Define

$$\begin{aligned}
h_N^1(t, z) &= -\frac{c^2}{4d_1} e^{-\frac{c(z-z_0)}{2d_1}} \phi_N^*(t, z) - \beta(t)\phi_N^*(t, z)\psi_N^*(t, z),\\
h_N^2(t, z) &= -\frac{c^2}{4d_2} e^{-\frac{c(z-z_0)}{2d_2}} \psi_N^*(t, z) + \beta(t)\phi_N^*(t, z)\psi_N^*(t, z) - \gamma(t)\psi_N^*(t, z).
\end{aligned}$$

According to [29, Proposition 7.14], for $N > 72R + |z_0|$, there exists a constant $C_1(p, R)$ independent of N, such that

$$\begin{aligned}
&\left\|\partial_z W_N^i\right\|_{L^p(Q((2T,z_0),2R))}\\
&\quad \le C_1\left(\left\|W_N^i\right\|_{L^p(Q((2T,z_0),72R))} + \left\|h_N^i\right\|_{L^p(Q((2T,z_0),72R))}\right), \; i = 1, 2.
\end{aligned}$$

This implies that there exists a constant $C_2(p, R)$, which is independent of N, such that

$$\left\|\partial_z \phi_N^*\right\|_{L^p(Q((2T,z_0),2R))}, \left\|\partial_z \psi_N^*\right\|_{L^p(Q((2T,z_0),2R))} \le C_2.$$

In view of the equations for ϕ_N^* and ψ_N^*, we further conclude from [29, Proposition 7.18] that there exists a constant $C_3(p, R)$ independent of N, such that

$$\begin{aligned}
\left\|\partial_{zz}\phi_N^*\right\|_{L^p(Q((2T,z_0),R))} + \left\|\partial_t\phi_N^*\right\|_{L^p(Q((2T,z_0),R))} &\le C_3,\\
\left\|\partial_{zz}\psi_N^*\right\|_{L^p(Q((2T,z_0),R))} + \left\|\partial_t\psi_N^*\right\|_{L^p(Q((2T,z_0),R))} &\le C_3,
\end{aligned}$$

As a consequence, there exists a constant $C(p, R)$, which is independent of N, such that

$$\left\|\phi_N^*\right\|_{W_p^{1,2}(Q((2T,z_0),R))}, \left\|\psi_N^*\right\|_{W_p^{1,2}(Q((2T,z_0),R))} \le C.$$

On account of $[0, T] \times [-Z, Z] \subset Q((2T, 0), R)$, we have

$$\|\phi_N^*\|_{W_p^{1,2}([0,T]\times[-Z,Z])}, \|\psi_N^*\|_{W_p^{1,2}([0,T]\times[-Z,Z])} \le C.$$

Here, R merely depends on Z, and then, C only relies on Z and p.

Take $p > 3$. Then, the embedding theorem indicates that

$$\phi_N^*, \psi_N^* \in C^{(1+\theta)/2,1+\theta}([0, T] \times [z_0 - Z, z_0 + Z]) \quad \text{for some } \theta \in (0, 1)$$

and

$$\|\phi_N^*\|_{C^{(1+\theta)/2,1+\theta}([0,T]\times[-Z,Z])}, \|\psi_N^*\|_{C^{(1+\theta)/2,1+\theta}([0,T]\times[-Z,Z])} \le C',$$

where $C' > 0$ is a constant depending upon p and Z. □

Let (ϕ_N^*, ψ_N^*) be the solution of the system (2.21), and we further have the following estimations.

Proposition 2.8. *There exists a constant C_0 such that*

$$\frac{1}{T}\int_{-N}^{N}\int_0^T \beta(t)\phi_N^*(t,z)\psi_N^*(t,z)dtdz < C_0, \quad \frac{1}{T}\int_{-N}^{N}\int_0^T \psi_N^*(t,z)dtdz < C_0$$

for any $N > -z_2$. In particular, $\int_0^T \partial_z\phi_N^*(t,z)dt \le 0$ for $z \in [-N, N]$ and $N > -z_2$.

Proof. For $z \in [-N, N]$, define

$$\Phi^*(z) = \frac{1}{T}\int_0^T \phi_N^*(t,z)dt, \quad \Psi^*(z) = \frac{1}{T}\int_0^T \psi_N^*(t,z)dt,$$
$$\Phi^\pm(z) = \frac{1}{T}\int_0^T \phi^\pm(t,z)dt, \quad \Psi^\pm(z) = \frac{1}{T}\int_0^T \psi^\pm(t,z)dt.$$

Clearly,

$$\Phi^-(z) \le \Phi^*(z) \le \Phi^+(z), \quad \Psi^-(z) \le \Psi^*(z) \le \Psi^+(z), \ \forall z \in [-N, N].$$

In view of (2.21), we have

$$c\Phi_z^* = d_1\Phi_{zz}^* - \frac{1}{T}\int_0^T \beta(t)\psi_N^*(t,z)\psi_N^*(t,z)dt, \quad \forall z \in [-N, N], \tag{2.23}$$

where the subscripts $_z$ and $_{zz}$ represent the first derivative and the second derivative for one function on z, respectively. It follows from (2.23) that

$$\begin{aligned}\left(e^{-cz/d_1}\Phi_z^*\right)_z &= e^{-cz/d_1}\left(\Phi_{zz}^* - c\Phi_z^*/d_1\right)\\ &= \frac{e^{-cz/d_1}}{d_1T}\int_0^T \beta(t)\phi_N^*(t,z)\psi_N^*(t,z)dt, \quad \forall z \in [-N, N].\end{aligned}$$

Integrating two sides of the last equality from $z \in [-N, N)$ to N yields

$$\Phi_z^*(z) = e^{-c(N-z)/d_1}\Phi_z^*(N) - \frac{1}{d_1 T}\int_z^N e^{-c(\xi-z)/d_1}\int_0^T \beta(t)\phi_N^*(t,z)\psi_N^*(t,z)dtd\xi.$$

Since $\Phi^*(z) \geq 0 = \Phi^*(N) = \Phi^-(N)$ for $z \in [-N, N]$, we have that $\Phi_z^*(N) \leq 0$, and hence, $\Phi_z^*(z) \leq 0$ for $z \in [-N, N]$. In particular, $\Phi_z^*(z) \not\equiv 0$. Making an integration from $-N$ to N for Eq. (2.23), we obtain

$$\begin{aligned}
&\frac{1}{T}\int_{-N}^N\int_0^T \beta(t)\phi_N^*(t,z)\psi_N^*(t,z)dtdz \\
&\quad = c\left(\Phi^*(-N) - \Phi^*(N)\right) + d_1\left(\Phi_z^*(N) - \Phi_z^*(-N)\right), \\
&\quad \leq (c + d_1)S_0
\end{aligned} \tag{2.24}$$

due to $\Phi^*(-N) \leq S_0$ and

$$\Phi_z^*(-N) \geq \Phi_z^-(-N) = \frac{d}{dz}\left(\frac{1}{T}\int_0^T \phi^-(t,z)dt\right)\bigg|_{z=-N} = -S_0 M_1\epsilon_1 e^{-\epsilon_1 N} \geq -S_0.$$

Let $\hat{\gamma} := \min_{t\in[0,T]}\gamma(t)$ and $\tilde{\gamma} := \max_{t\in[0,T]}\gamma(t)$. Then, Ψ^* satisfies

$$\begin{aligned}
-d_2\Psi_{zz}^* + c\Psi_z^* + \hat{\gamma}\Psi^* = &\frac{1}{T}\int_0^T \beta(t)\phi_N^*(t,z)\psi_N^*(t,z)dt \\
&- \frac{1}{T}\int_0^T \left(\gamma(t) - \hat{\gamma}\right)\psi_N^*(t,z)dt.
\end{aligned}$$

Integrating the two sides of the last equality on $[-N, N]$, we have

$$\begin{aligned}
\int_{-N}^N \Psi^*(z)dz \leq &\frac{d_2}{\hat{\gamma}}\left(\Psi_z^*(N) - \Psi_z^*(-N)\right) + \frac{c}{\hat{\gamma}}\left(\Psi^*(-N) - \Psi^*(N)\right) \\
&+ \frac{1}{\hat{\gamma}T}\int_{-N}^N\int_0^T \beta(t)\phi_N^*(t,z)\psi_N^*(t,z)dtdz.
\end{aligned}$$

Since $\Psi_z^*(N) \leq 0, \Psi_z^*(-N) \geq \Psi_z^-(-N) > 0, \Psi^*(-N) = \Psi^-(-N)$ and the inequality (2.24) holds, we can conclude from the last equality that

$$\int_{-N}^N \Psi^*(z)dz \leq \frac{1}{\hat{\gamma}}\left(c\Psi^-(-N) + cS_0 + d_1 S_0\right).$$

Thus, there exists a constant $C_0 > 0$ independent of $N > -z_2$ such that

$$\frac{1}{T}\int_{-N}^N\int_0^T \beta(t)\phi_N^*(t,z)\psi_N^*(t,z)dtdy < C_0, \quad \frac{1}{T}\int_{-N}^N\int_0^T \psi_N^*(t,z)dtdz < C_0.$$

This completes the proof. □

2.3. Existence of periodic traveling waves

This subsection is concerned with the existence of periodic traveling waves.

Theorem 2.9. *Assume that* $R_0 > 1$. *For any* $c > c^*$, *the system* (2.1) *admits a time periodic traveling wave solution* (ϕ^*, ψ^*) *satisfying* (2.4) *and* (2.5). *Furthermore, there hold* $0 < \frac{1}{T}\int_0^T \psi^*(t,z)dt \le S_0 - S^\infty$ *for any* $z \in \mathbb{R}$, *and*

$$\frac{1}{T}\int_{-\infty}^{\infty}\int_0^T \gamma(t)\psi^*(t,z)dtdz$$
$$= \frac{1}{T}\int_{-\infty}^{\infty}\int_0^T \beta(t)\phi^*(t,z)\psi^*(t,z)dtdz = c[S_0 - S^\infty].$$

Proof. The proof is divided into four parts.

I. Existence of periodic traveling waves Let $\{N_m\}$ be an increasing sequence such that $N_m \ge -z_2$ and $\lim_{m\to+\infty} N_m = +\infty$. It then follows that the solutions $(\phi^*_{N_m}, \psi^*_{N_m}) \in \Gamma_{N_m}$ satisfy Theorem 2.7 and (2.21). In light of the periodicity of $(\phi^*_{N_m}, \psi^*_{N_m})$ in $t \in \mathbb{R}$, we can extract a subsequence of $(\phi^*_{N_m}, \psi^*_{N_m})$, still denoted by $(\phi^*_{N_m}, \psi^*_{N_m})$, tending toward functions $(\phi^*, \psi^*) \in C(\mathbb{R}^2)$ in the following topologies

$$(\phi^*_{N_m}, \psi^*_{N_m}) \to (\phi^*, \psi^*) \text{ in } C^{\frac{1+\beta}{2},1+\beta}_{\mathrm{loc}}(\mathbb{R}^2),\ \text{ in } H^1_{\mathrm{loc}}(\mathbb{R}^2) \text{ weakly and in } L^2_{\mathrm{loc}}(\mathbb{R}, H^2_{\mathrm{loc}}(\mathbb{R}))\ weakly, \tag{2.25}$$

where $\beta \in (0,\theta)$ and $\theta \in (0,1)$ is given in (2.22). It is obvious that $(\phi^*, \psi^*) \in C^{\frac{1+\beta}{2},1+\beta}(\mathbb{R}^2) \cap H^1_{\mathrm{loc}}(\mathbb{R}^2) \cap L^2_{\mathrm{loc}}(\mathbb{R}, H^2_{\mathrm{loc}}(\mathbb{R}))$. Since $(\phi^*_{N_m}, \psi^*_{N_m})$ is T-periodic in t, we have $(\phi^*(t+T,z), \psi^*(t+T,z)) = (\phi^*(t,z), \psi^*(t,z))$ for all $t \in \mathbb{R}$ and $z \in \mathbb{R}$, and hence, the estimation (2.22) implies that for any $N > 0$, there exists a constant $C_3 > 0$ such that

$$\|\phi^*\|_{C^{\frac{1+\beta}{2},1+\beta}_{[0,T]\times[-N,N]}(\mathbb{R}^2)} + \|\psi^*\|_{C^{\frac{1+\beta}{2},1+\beta}_{[0,T]\times[-N,N]}(\mathbb{R}^2)} \le C_3. \tag{2.26}$$

Let $u, v \in C_0^\infty(\mathbb{R}^2)$ be given. Then, for sufficiently large $m \in \mathbb{N}$ satisfying $\mathrm{supp}(u)\times \mathrm{supp}(v) \subset \mathbb{R}\times(-N_m, N_m)$, we have that $(\phi^*_{N_m}, \psi^*_{N_m})$ satisfy the equalities

$$\int\int_{\mathbb{R}^2} \partial_t u(t,z)\phi^*_{N_m}(t,z)dtdz - d_1\int\int_{\mathbb{R}^2} \partial_z u(t,z)\partial_z\phi^*_{N_m}(t,z)dtdz$$
$$= c\int\int_{\mathbb{R}^2} u(t,z)\partial_z\phi^*_{N_m}(t,z)dtdz + \int\int_{\mathbb{R}^2} \beta(t)u(t,z)\phi^*_{N_m}(t,z)\psi^*_{N_m}dtdz$$

and

$$\int\int_{\mathbb{R}^2} \partial_t v(t,z)\psi^*_{N_m}(t,z)dtdz - d_2\int\int_{\mathbb{R}^2} \partial_z v(t,z)\partial_z\psi^*_{N_m}(t,z)dtdz$$
$$= c\int\int_{\mathbb{R}^2} v(t,z)\partial_z\psi^*_{N_m}(t,z)dtdz - \int\int_{\mathbb{R}^2} \beta(t)v(t,z)\phi^*_{N_m}(t,z)\psi^*_{N_m}dtdz$$

$$+\int\int_{\mathbb{R}^2} \gamma(t)v(t,z)\psi^*_{N_m} dtdz.$$

On the basis of (2.25), we have that (ϕ^*, ψ^*) satisfy

$$\int\int_{\mathbb{R}^2} \partial_t u(t,z)\phi^*(t,z)dtdz - d_1 \int\int_{\mathbb{R}^2} \partial_z u(t,z)\partial_z\phi^*(t,z)dtdz$$
$$= c\int\int_{\mathbb{R}^2} u(t,z)\partial_z\phi^*(t,z)dtdz + \int\int_{\mathbb{R}^2} \beta(t)u(t,z)\phi^*(t,z)\psi^* dtdz$$

and

$$\int\int_{\mathbb{R}^2} \partial_t v(t,z)\psi^*(t,z)dtdz - d_2 \int\int_{\mathbb{R}^2} \partial_z v(t,z)\partial_z\psi^*(t,z)dtdz$$
$$= c\int\int_{\mathbb{R}^2} v(t,z)\partial_z\psi^*(t,z)dtdz - \int\int_{\mathbb{R}^2} \beta(t)v(t,z)\phi^*(t,z)\psi^* dtdz$$
$$+\int\int_{\mathbb{R}^2} \gamma(t)v(t,z)\psi^* dtdz$$

for any $u, v \in C_0^\infty(\mathbb{R}^2)$. Then, we conclude that (ϕ^*, ψ^*) satisfy

$$\begin{cases} \partial_t\phi^*(t,z) = d_1\partial_{zz}\phi^*(t,z) - c\partial_z\phi^*(t,z) - \beta(t)\phi^*(t,z)\psi^*(t,z), \\ \partial_t\psi^*(t,z) = d_2\partial_{zz}\psi^*(t,z) - c\partial_z\psi^*(t,z) + \beta(t)\phi^*(t,z)\psi^*(t,z) - \gamma(t)\psi^*(t,z) \end{cases}$$

almost everywhere in $(t,z) \in \mathbb{R}^2$. Consider the following Cauchy problem

$$\begin{cases} \partial_t w_1(t,z) = d_1\partial_{zz}w_1(t,z) - c\partial_z w_1(t,z) - \beta(t)\phi^*(t,z)\psi^*(t,z), \ t > 0, z \in \mathbb{R}, \\ \partial_t w_2(t,z) = d_2\partial_{zz}w_2(t,z) - c\partial_z w_2(t,z) + \beta(t)\phi^*(t,z)\psi^*(t,z) - \gamma(t)\psi^*(t,z), \\ w_1(0,z) = \phi^*(0,z), \ w_2(0,z) = \psi^*(0,z), \quad z \in \mathbb{R}. \end{cases} \tag{2.27}$$

Clearly, $(\phi^*(t,z), \psi^*(t,z))$ is a strong solution of (2.27). Moreover, [31, Theorem 5.1.3 and 5.1.4] imply that (ϕ^*, ψ^*) is the unique strong solution of (2.27), and hence, $\phi^*, \psi^* \in C^{1+\frac{\nu}{2},2+\nu}(\mathbb{R}^2)$ for some $\nu \in (0,1)$ and satisfy (2.4), that is,

$$\begin{cases} \partial_t\phi^*(t,z) = d_1\partial_{zz}\phi^*(t,z) - c\partial_z\phi^*(t,z) - \beta(t)\phi^*(t,z)\psi^*(t,z), \\ \partial_t\psi^*(t,z) = d_2\partial_{zz}\psi^*(t,z) - c\partial_z\psi^*(t,z) + \beta(t)\phi^*(t,z)\psi^*(t,z) - \gamma(t)\psi^*(t,z) \end{cases} \tag{2.28}$$

for $(t,z) \in \mathbb{R}^2$. Furthermore, it follows from Proposition 2.8 that there exists a constant $C_0 > 0$ such that

$$\frac{1}{T}\int_{-\infty}^{\infty}\int_{0}^{T}\beta(t)\phi^*(t,z)\psi^*(t,z)dtdz < C_0, \quad \frac{1}{T}\int_{-\infty}^{\infty}\int_{0}^{T}\psi^*(t,z)dtdz < C_0. \tag{2.29}$$

Note that (ϕ^*, ψ^*) satisfies that

$$\phi^-(t,z) \le \phi^*(t,z) \le S_0, \quad \psi^-(t,z) \le \psi^*(t,z) \le \psi^+(t,z), \quad (t,z) \in \mathbb{R}^2,$$

and hence, there hold $\phi^*(t,z) \to S_0$ and $\psi^*(t,z) \to 0$ uniformly for $t \in \mathbb{R}$, as $z \to -\infty$.

II. The asymptotic behavior of ψ^* as $z \to +\infty$ Define $\Psi(z) = \frac{1}{T}\int_0^T \psi^*(t,z)dt$. Then, $\Psi(z)$ satisfies

$$\begin{aligned} -d_2\Psi_{zz} + c\Psi_z + \hat{\gamma}\Psi = &\frac{1}{T}\int_0^T \beta(t)\phi^*(t,z)\psi^*(t,z)dt \\ &-\frac{1}{T}\int_0^T \left(\gamma(t) - \hat{\gamma}\right)\psi^*(t,z)dt, \end{aligned} \tag{2.30}$$

where $\hat{\gamma}$ is defined as in the proof of Proposition 2.8. Denote by

$$\hat{\lambda}^{\pm} := \frac{c \pm \sqrt{c^2 + 4d_2\hat{\gamma}}}{2d_2}$$

the two roots of the characteristic equation

$$-d_2\lambda^2 + c\lambda + \hat{\gamma} = 0.$$

In addition, denote

$$\hat{\rho} := d_2\left(\hat{\lambda}^+ - \hat{\lambda}^-\right) = \sqrt{c^2 + 4d_2\hat{\gamma}}.$$

Clearly, $\hat{\lambda}^- < 0 < \hat{\lambda}^+$. It follows from (2.30) and (2.29) that

$$\begin{aligned} \Psi(z) = &\frac{1}{\hat{\rho}T}\int_{-\infty}^{z} e^{\hat{\lambda}^-(z-y)}\left[\int_0^T \beta(t)\phi^*(t,y)\psi^*(t,y) - \int_0^T \left(\gamma(t)-\hat{\gamma}\right)\psi^*(t,y)\right]dtdy \\ &+\frac{1}{\hat{\rho}T}\int_{z}^{\infty} e^{\hat{\lambda}^+(z-y)}\left[\int_0^T \beta(t)\phi^*(t,y)\psi^*(t,y) - \int_0^T \left(\gamma(t)-\hat{\gamma}\right)\psi^*(t,y)\right]dtdy \end{aligned}$$

and

$$\begin{aligned} \Psi_z(z) = &\frac{\hat{\lambda}^-}{\hat{\rho}T}\int_{-\infty}^{z} e^{\hat{\lambda}^-(z-y)}\left[\int_0^T \beta(t)\phi^*(t,y)\psi^*(t,y) - \int_0^T \left(\gamma(t)-\hat{\gamma}\right)\psi^*(t,y)\right]dtdy \\ &+\frac{\hat{\lambda}^+}{\hat{\rho}T}\int_{z}^{\infty} e^{\hat{\lambda}^+(z-y)}\left[\int_0^T \beta(t)\phi^*(t,y)\psi^*(t,y) - \int_0^T \left(\gamma(t)-\hat{\gamma}\right)\psi^*(t,y)\right]dtdy \\ \le &\frac{\hat{\lambda}^-}{\hat{\rho}T}\int_{-\infty}^{z} e^{\hat{\lambda}^-(z-y)}\int_0^T \beta(t)\phi^*(t,y)\psi^*(t,y)dtdy \end{aligned}$$

$$
\begin{aligned}
&+\frac{\hat{\lambda}^{+}}{\hat{\rho}T}\int_{z}^{\infty}e^{\hat{\lambda}^{+}(z-y)}\int_{0}^{T}\beta(t)\phi^{*}(t,y)\psi^{*}(t,y)dtdy\\
=&\frac{\hat{\lambda}^{-}}{\hat{\rho}T}\int_{0}^{\infty}e^{\hat{\lambda}^{-}y}\int_{0}^{T}\beta(t)\phi^{*}(t,z-y)\psi^{*}(t,z-y)dtdy\\
&+\frac{\hat{\lambda}^{+}}{\hat{\rho}T}\int_{-\infty}^{0}e^{\hat{\lambda}^{+}y}\int_{0}^{T}\beta(t)\phi^{*}(t,z-y)\psi^{*}(t,z-y)dtdy.
\end{aligned}
$$

Since $\hat{\lambda}^{-}<0<\hat{\lambda}^{+}$ and $\hat{\rho}:=d_2\left(\hat{\lambda}^{+}-\hat{\lambda}^{-}\right)$, we have

$$
|\Psi_z(z)|\le\frac{1}{d_2T}\int_{-\infty}^{\infty}\int_{0}^{T}\beta(t)\phi^{*}(t,z)\psi^{*}(t,z)dtdz.
$$

It then follows from the integrability of $\int_0^T\beta(t)\phi^*(t,\cdot)\psi^*(t,\cdot)dt$ on $\mathbb{R}$ that Ψ_z is uniformly bounded. Consequently, following $\int_{-\infty}^{\infty}\Psi(z)dz<C_0$, we must have $\Psi(z)\to 0$ as $z\to\infty$. We further apply Harnack inequalities ([35, Lemma 2.9] (see also [14]) with $\tau=-T,\theta=T$ and $D:=D_z=(z-\frac{1}{4},z+\frac{1}{4}),U=(z-\frac{1}{2},z+\frac{1}{2}),\Omega=(z-1,z+1)$ with $z\in\mathbb{R}$) for the second equation of system (2.28), we have

$$
\begin{aligned}
\sup_{(0,T)\times D}\psi^{*}(t,y)&\le C_0'\inf_{(2T,3T)\times D}\psi^{*}(t,z)\\
&=C_0'\min_{[2T,3T]\times\overline{D}}\psi^{*}(t,y)\\
&\le C_0'\min_{\overline{D}}\psi^{*}(0,y),
\end{aligned}
$$

where C_0' is a positive constant independent of D. Since ψ^* is periodic in time t, $\psi^*(t,z)\to 0$ uniformly for $t\in\mathbb{R}$, as $z\to\infty$. As a consequence, there holds $\psi^*(t,z)\le C_0$ for $(t,z)\in\mathbb{R}^2$.

III. The asymptotic behavior of ϕ^* as $z\to\infty$ By virtue of the estimate (2.26) and Laudau type inequalities (see, e.g., [4,27]), we have

$$
\left|\phi_z^*\right|_{L^\infty([0,T]\times(-\infty,M])}\le 2\left|\phi^*-S_0\right|^{\frac{1}{2}}_{L^\infty([0,T]\times(-\infty,M])}\left|\phi_{zz}^*\right|^{\frac{1}{2}}_{L^\infty([0,T]\times(-\infty,M])}.
$$

Consequently,

$$
\lim_{z\to-\infty}\phi_z^*(t,z)=0\ \text{ uniformly for } t\in\mathbb{R}.
$$

Define $\Phi(z)=\frac{1}{T}\int_0^T\phi^*(t,z)dt$. It is obvious that $\Phi_z(z)\to 0$ as $z\to-\infty$. It then follows from the first equation of system (2.28) that

$$
c\Phi_z=d_1\Phi_{zz}-\frac{1}{T}\int_0^T\beta(t)\phi^*(t,z)\psi^*(t,z)dt. \tag{2.31}
$$

It is easy to see from the last equation

$$
\left(e^{-cz/d_1}\Phi_z\right)_z=e^{-cz/d_1}\left(\Phi_{zz}-c\Phi_z/d_1\right)=\frac{e^{-cz/d_1}}{d_1T}\int_0^T\beta(t)\phi^*(t,z)\psi^*(t,z)dt.
$$

Since $\frac{1}{T}\int_0^T \beta(t)\phi^*(t,z)\psi^*(t,z)dt$ is integrable on $\mathbb{R}$, an integration from z to ∞ for the last equality yields

$$e^{-cz/d_1}\Phi_z(z) = -\frac{1}{d_1 T}\int_z^\infty e^{-cy/d_1}\int_0^T \beta(t)\phi^*(t,y)\psi^*(t,y)dtdy,$$

which implies that $\Phi_z(z) < 0$ for $z \in \mathbb{R}$, and hence, $\Phi(\infty)$ exists and $\Phi(\infty) < \Phi(-\infty) = S_0$. It follows from the Barbălat's lemma (see, e.g., [3,12]) that $\Phi_z(z) \to 0$ as $z \to \infty$. Integrating two sides of (2.31) from $-\infty$ to ∞ on z leads to

$$\frac{1}{T}\int_{-\infty}^\infty \int_0^T \beta(t)\phi^*(t,z)\psi^*(t,z)dtdz = c[S_0 - \Phi(\infty)] = c[S_0 - S^\infty],$$

where $S^\infty := \Phi(\infty) < S_0$.

By similar arguments to [35, Theorem 2.10], we prove that $\phi^*(t,z) \to S^\infty$ uniformly for $t \in \mathbb{R}$, as $z \to \infty$. In the light of T-periodicity of ϕ^*, it is sufficient to show

$$\limsup_{z\to\infty} \max_{t\in[0,T]} \phi^*(t,z) =: S_+^\infty = S^\infty = S_-^\infty := \liminf_{z\to\infty} \min_{t\in[0,T]} \phi^*(t,z).$$

Clearly, there exist $\{t_n\}$ and $\{z_n\}$ satisfying $\{t_n\} \subset [0,T]$ and $z_n \to \infty$ (as $n \to \infty$), respectively, such that

$$\lim_{n\to\infty} \phi^*(t_n, z_n) = S_+^\infty.$$

Let $\phi_n(t,z) = \phi^*(t+t_n, z+z_n)$, $\psi_n(t,z) = \psi^*(t+t_n, z+z_n)$, $\forall n \in \mathbb{N}, t \in \mathbb{R}, z \in \mathbb{R}$. Based on the estimation (2.26) and the uniform boundedness of Φ, Φ_z, Ψ and Ψ_z, there exists a subsequence of $(\phi_n(t,z), \psi_n(t,z))$, still denoted by $(\phi_n(t,z), \psi_n(t,z))$, converging to $(\phi_*(t,z), 0)$ in $C_{\text{loc}}^{\nu/2,\nu}(\mathbb{R}\times\mathbb{R})$ for some $\nu \in (0,1)$, as $n \to \infty$. Particularly, we have $\phi_*(0,0) = S_+^\infty$ and

$$\phi_*(t+T,z) = \phi_*(t,z), \quad \phi_*(t,z) \le S_+^\infty, \quad \forall (t,z) \in \mathbb{R}\times\mathbb{R}.$$

Since $\{t_n\} \subset [0,T]$, without loss of generality, let $t_n \to t^* \in [0,T]$. Then, $\phi_*^+(t,z) = \phi_*(t - t^*, z)$ satisfies

$$\begin{aligned}\phi_*^+(t) &= T_1(t)\phi_*^+(0) + \int_0^t T_1(t-s) f_1[\phi_*^+, 0](s)ds \\ &= T_1(t)\phi_*^+(0) + \int_0^t T_1(t-s)\alpha_1\phi_*^+(s)ds.\end{aligned}$$

Accordingly, $\phi_*^+(t,z)$ satisfies

$$\partial_t\phi_*^+(t,z) = d_1\partial_{zz}\phi_*^+(t,z) - c\partial_z\phi_*^+(t,z), \quad (t,z) \in \mathbb{R}\times\mathbb{R}.$$

As a result of $\phi_*^+(t^*,0) = S_+^\infty$ and $\phi_*^+(t,z) \le S_+^\infty$, the maximum principle indicates that $\phi_*^+(t,z) \equiv S_+^\infty$ for $t < t^*$. Since ϕ_*^+ is T-periodic in t, we have $\phi_*^+(t,z) \equiv S_+^\infty$, $\forall t \in \mathbb{R}$, and hence $\Phi_*^+(z) := \frac{1}{T}\int_0^T \phi_*^+(t,z)dt \equiv S_+^\infty$. On the other hand,

$$\begin{aligned}\Phi_*^+(z) &= \frac{1}{T}\int_0^T \phi_*^+(t,z)dt = \frac{1}{T}\int_0^T \phi_*(t-t^*,z)dt\\ &= \lim_{n\to\infty}\frac{1}{T}\int_0^T \phi_n(t-t^*,z)dt\\ &= \lim_{n\to\infty}\frac{1}{T}\int_0^T \phi^*(t-t^*+t_n,z+z_n)dt\\ &= S^\infty,\end{aligned}$$

which implies $S_+^\infty = S^\infty$. Thus, $\limsup_{z\to\infty}\max_{t\in[0,T]}\phi^*(t,z) = S^\infty$. Similarly, we can prove $\liminf_{z\to\infty}\min_{t\in[0,T]}\phi^*(t,z) = S^\infty$. This implies that $\phi_*^+(t,z)$ converges to S^∞ uniformly in $t\in\mathbb{R}$ as $z\to\infty$.

IV. The properties for ψ^* We use the similar arguments to [35, Theorem 2.10](see also [33]) check on the properties for ψ^*. Since $\Psi(z)$ satisfies

$$-d_2\Psi_{zz} + c\Psi_z = \frac{1}{T}\int_0^T \beta(t)\phi^*(t,z)\psi^*(t,z)dt - \frac{1}{T}\int_0^T \gamma(t)\psi^*(t,z)dt, \tag{2.32}$$

an integration of (2.32) on $\mathbb{R}$ yields

$$\begin{aligned}&\frac{1}{T}\int_{-\infty}^\infty\int_0^T \gamma(t)\psi^*(t,z)dtdz\\ &\quad= \frac{1}{T}\int_{-\infty}^\infty\int_0^T \beta(t)\phi^*(t,z)\psi^*(t,z)dtdz = c[S_0 - S^\infty].\end{aligned}$$

Similar to the aforementioned proof on the asymptotic behavior of $\phi_z^*(t,z)$ as $z\to-\infty$, we can show that

$$\lim_{z\to\pm\infty}\psi_z^*(t,z) = 0 \tag{2.33}$$

uniformly for $t\in\mathbb{R}$. For any $z\in\mathbb{R}$, define a function

$$\begin{aligned}\Psi^{**}(z) &= \frac{1}{cT}\int_{-\infty}^z\int_0^T \gamma(t)\psi^*(t,y)dtdy\\ &\quad+\frac{1}{cT}\int_z^\infty e^{c/d_2(z-y)}\int_0^T \gamma(t)\psi^*(t,y)dtdy.\end{aligned} \tag{2.34}$$

It is not difficult to see that $\Psi^{**}(z)$ satisfies the following equation:

$$c\Psi_z^{**}(z) = d_2\Psi_{zz}^{**}(z) + \frac{1}{T}\int_0^T \gamma(t)\psi^*(t,y)dt,\quad \forall z\in\mathbb{R}.$$

By means of (2.33) and L'Hôpital's rule, it follows that

$$\lim_{z\to-\infty}\Psi^{**}(z) = 0,\ \lim_{z\to\infty}\Psi^{**}(z) = \frac{1}{cT}\int_{-\infty}^\infty\int_0^T \gamma(t)\psi^*(t,y)dy = S_0 - S^\infty$$

and

$$\lim_{z\to\pm\infty} \Psi_z^{**}(z) = 0.$$

Define a new function

$$\hat{\Psi}(z) := \Psi(z) + \Psi^{**}(z), \quad \forall z \in \mathbb{R},$$

where $\Psi(z) = \frac{1}{T}\int_0^T \psi^*(t,z)dt$. On the basis of (2.33) and (2.34) that

$$c\hat{\Psi}_z(z) = d_2\hat{\Psi}_{zz}(z) + \frac{1}{T}\int_0^T \beta(t)\phi^*(t,z)\psi^*(t,z)dt, \quad \forall z \in \mathbb{R}.$$

Multiplying two sides of the above equation by $e^{-c/d_2 z}$ and integrating from z to ∞, we have

$$\hat{\Psi}_z(z) = \frac{1}{d_2T}\int_z^\infty e^{c/d_2(z-y)}\int_0^T \beta(t)\phi^*(t,y)\psi^*(t,y)dtdy.$$

Then, it is obvious that $\hat{\Psi}(z)$ is non-decreasing in $\mathbb{R}$. Note that $\lim_{z\to\infty}\hat{\Psi}(z) = S_0 - S^\infty$. Hence, $\hat{\Psi}(z) \le S_0 - S^\infty$ for all $z \in \mathbb{R}$. In view of the definition of $\hat{\Psi}(z)$ and $\Psi^*(z)$, we conclude that $\Psi(z) \le \hat{\Psi}(z) \le S_0 - S^\infty$ for all $z \in \mathbb{R}$, that is, $0 \le \frac{1}{T}\int_0^T \psi^*(t,z)dt \le S_0 - S^\infty$ for any $z \in \mathbb{R}$. The proof is complete. $\square$

3. Nonexistence of periodic traveling waves

In this section, our task is to investigate the nonexistence of time periodic traveling waves for two cases. Firstly, we prove that there is no time periodic traveling wave in the case where $R_0 \le 1$.

Theorem 3.1. *Assume that* $R_0 = \frac{S_0\int_0^T \beta(t)\mathrm{d}t}{\int_0^T \gamma(t)\mathrm{d}t} \le 1$. *Then, for any* $c \ge 0$, *there is no time periodic traveling wave solutions* (ϕ, ψ) *satisfying the asymptotic boundary conditions* (2.5) *uniformly for* $t \in \mathbb{R}$.

Proof. By contradiction, we assume that there exists a time periodic, non-trivial and nonnegative solution $(\phi(t,z), \psi(t,z))$ of (2.4) satisfying (2.5) uniformly for $t \in \mathbb{R}$, that is,

$$\begin{cases} \phi_t(t,z) = d_1\phi_{zz}(t,z) - c\phi_z(t,z) - \beta(t)\phi(t,z)\psi(t,z), \\ \psi_t(t,z) = d_2\psi_{zz}(t,z) - c\psi_z(t,z) + \beta(t)\phi(t,z)\psi(t,z) - \gamma(t)\psi(t,z) \end{cases}$$

and

$$\phi(t,-\infty) = S_0,\ \phi(t,\infty) = S^\infty,\ \psi(t,\pm\infty) = 0 \text{ uniformly in } t \in \mathbb{R}.$$

Due to the T-periodicity of $\psi(t,z)$ and the parabolic maximum principle, it follows that $\psi(t,z) > 0$ for $t \in \mathbb{R}$, $z \in \mathbb{R}$. In addition, it is not difficult to show that $\phi(t,z) \leq S_0$ for $t \in \mathbb{R}$, $z \in \mathbb{R}$. In fact, suppose that there exists (t_0, x_0) such that $S(t_0, x_0) > S_0$. Thus,

$$0 = \frac{\partial S(t,x)}{\partial t}\bigg|_{(t_0,x_0)} = d_1 \Delta S(t,x)\Big|_{(t_0,x_0)} - \beta(t_0) S(t_0,x_0) I(t_0,x_0) < 0,$$

which is a contradiction. Let $\overline{\psi}(t) = \int_{-\infty}^{\infty} \psi(t,z)dz$. Then, by the asymptotical boundary conditions (2.5) and (2.33), we have

$$\frac{d}{dt}\overline{\psi}(t) = (\beta(t)S_0 - \gamma(t))\,\overline{\psi}(t) + f(t), \qquad \forall t \in \mathbb{R},$$

where

$$f(t) = \beta(t) \int_{-\infty}^{\infty} (\phi(t,z) - S_0)\,\psi(t,z)dz < 0, \quad \forall t \in \mathbb{R}.$$

It is easy to see that $\overline{\psi}(t+T) = \overline{\psi}(t)$, $f(t+T) = f(t)$, $\forall t \in \mathbb{R}$. According to the positivity of $\overline{\psi}(t)$, we see that

$$\frac{\frac{d}{dt}\overline{\psi}(t)}{\overline{\psi}(t)} = (\beta(t)S_0 - \gamma(t)) + \frac{f(t)}{\overline{\psi}(t)}, \qquad \forall t \in \mathbb{R}.$$

Integrating both two sides of the above equality from 0 to T , we obtain

$$0 = \int_0^T (\beta(t)S_0 - \gamma(t))\,dt + \int_0^T \frac{f(t)}{\overline{\psi}(t)}dt < 0$$

due to the periodicity and positivity of $\overline{\psi}(t)$ and $R_0 = \frac{S_0 \int_0^T \beta(t)\mathrm{d}t}{\int_0^T \gamma(t)\mathrm{d}t} \leq 1$. This is a contradiction. □

Next, we prove the nonexistence of periodic traveling waves for the case where $R_0 > 1$ and $c < c^*$.

Theorem 3.2. *Assume that* $R_0 > 1$ *and* $0 < c < c^* = 2\sqrt{d_2 \varrho} = 2\sqrt{\frac{d_2 \int_0^T (S_0\beta(t) - \gamma(t))}{T}}$. *System* (2.4) *does not have a time periodic traveling waves* (ϕ, ψ) *satisfying* (2.5) *uniformly for* $t \in \mathbb{R}$.

Proof. Suppose, by contradiction, that there exists such a traveling wave solution $(\phi(t, x+ct), \psi(t, x+ct))$ satisfying (2.5) for some $c < c^* = 2\sqrt{\frac{d_2 \int_0^T (S_0\beta(t)-\gamma(t))}{T}}$. Since $R_0 = \frac{S_0 \int_0^T \beta(t)dt}{\int_0^T \gamma(t)dt}$, we have $\int_0^T [\beta(t)S_0 - \gamma(t)]dt > 0$, and hence, there exists a sufficiently small $\delta_0 > 0$ such that $\int_0^T [\beta(t)(S_0 - \delta_0) - \gamma(t)]\mathrm{d}t > 0$. For each $\delta \in (0, \delta_0)$, define ϱ^δ by

$$\varrho^\delta = \frac{1}{T} \int_0^T [\beta(t)(S_0 - \delta) - \gamma(t)]dt.$$

We fix a $\delta \in (0, 1)$ such that $c < 2\sqrt{d_2 \varrho^\delta}$. Since $\lim_{z\to-\infty} \phi(t, z) = S_0, \forall t \in \mathbb{R}$, we can choose a $M_\delta > 0$ such that $S_0 - \delta \le \phi(t, z) \le S_0 + \delta, \forall z < -M_\delta$ uniformly for $t \in \mathbb{R}$. Fix a $c_0 \in (c, 2\sqrt{d_2 \varrho^\delta})$ and let $M_{c_0} = \frac{\sqrt{4d_2\varrho^\delta - c_0^2}}{2d_2}$. Define

$$Q^\delta(t) = \exp\left(\int_0^t [\beta(s)(S_0 - \delta) - \gamma(s)]\mathrm{d}s - \varrho^\delta t\right).$$

Clearly,

$$\frac{\mathrm{d}Q^\delta(t)}{\mathrm{d}t} = [\beta(t)(S_0 - \delta) - \gamma(t)]Q^\delta(t) - \varrho^\delta Q^\delta(t).$$

We consider a function $w_{c_0}(t, z) := e^{\frac{c_0 z}{2d_2}} \sin(M_{c_0} z)Q^\delta(t)$. It is easy to verify that $w_{c_0}(t, z)$ satisfy $w_{c_0}(t+T, z) = w_{c_0}(t, z)$ for $z \in \mathbb{R}$. Further, some direct manipulation yields

$$\begin{aligned} \partial_t w_{c_0}(t, z) = d_2\partial_{zz} w_{c_0}(t, z) - c_0\partial_z w_{c_0}(t, z) + [\beta(t)(S_0 - \delta) \\ - \gamma(t)]w_{c_0}(t, z),\ t > 0, z \in \mathbb{R}. \end{aligned}$$

Let $k_0 \in \mathbb{N}^+$ such that $\frac{(2k_0-1)\pi}{M_{c_0}} > M_\delta$. Then, let $y_1 = -\frac{2k_0\pi}{M_{c_0}}$, $y_2 = -\frac{(2k_0-1)\pi}{M_{c_0}}$. Clearly, $\sin(M_{c_0} y_1) = \sin(M_{c_0} y_2) = 0$, $\sin(M_{c_0} z) > 0, \forall z \in (y_1, y_2)$. Since $\psi(0, z)$ is strictly positive on $[y_1, y_2]$, then there exists an $\epsilon > 0$ such that $\epsilon w_0(0, z) \le \psi(0, z),\ \forall z \in [y_1, y_2]$. Consider the function $\phi(t, x+(c-c_0)t)$ and $\psi(t, x+(c-c_0)t)$, $\forall t > 0, x \in [y_1, y_2]$. Denote $\hat{\psi}(t, x) := \psi(t, x + (c - c_0)t)$. Since $(\phi(t, z), \psi(t, z))$ is a solution of system (2.4), we have

$$\begin{aligned} \partial_t\hat{\psi}(t, x) = d_2\partial_{xx}\hat{\psi}(t, x) - c_0\partial_x\hat{\psi}(t, x) + \beta(t)\phi(t, x + (c - c_0)t)\hat{\psi}(t, x) \\ - \gamma(t)\hat{\psi}(t, x) \end{aligned}$$

Since $\phi(t, z) \ge S_0-\delta, \forall z < -M_\delta$ uniformly for $t \in \mathbb{R}$, it follows from above equality that $\hat{\psi}$ satisfies

$$\partial_t\hat{\psi}(t, x) \ge d_2\partial_{xx}\hat{\psi}(t, x) - c_0\partial_x\hat{\psi}(t, x) + \beta(t)(S_0 - \delta)\hat{\psi}(t, x) - \gamma(t)\hat{\psi}(t, x)$$

for all $t > 0$ and $x \in [y_1, y_2]$. In view of $c - c_0 < 0$ and $y_1 < y_2 < -M_\delta$, we have $x + (c - c_0)t \le -M_\delta, \forall t \ge 0, x \in [y_1, y_2]$. Let $\check{\psi}(t, x) := \psi(t, x + (c - c_0)t) - \epsilon w_{c_0}(t, x) = \hat{\psi}(t, x) - \epsilon w_{c_0}(t, x)$ for all $t \ge 0$ and $x \in [y_1, y_2]$. Then, we can derive that

$$\begin{cases} \partial_t\check{\psi}(t, x) \ge d_2\partial_{xx}\check{\psi}(t, x) - c_0\partial_x\check{\psi}(t, x) \\ +\beta(t)(S_0 - \delta)\check{\psi}(t, x) - \gamma(t)\check{\psi}(t, x),\ t \ge 0, x \in [y_1, y_2], \\ \check{\psi}(0, x) \ge 0,\ x \in [y_1, y_2], \\ \check{\psi}(t, y_j) \ge 0,\ j = 1, 2. \end{cases}$$

In view of the maximum principle of the parabolic equations, we are led to the conclusion that $\check{\psi} \ge 0$ for all $t > 0$ and $x \in [y_1, y_2]$, which implies that $\psi(t, x + (c - c_0)t) \ge \epsilon w_{c_0}(t, x)$ for all $t > 0$ and $x \in [y_1, y_2]$. Since $c - c_0 < 0$, there is a contradiction that $\psi(t, x + (c - c_0)t) \to 0$ as $t \to +\infty$. $\square$

4. Discussion

In this paper, we investigated time periodic traveling waves for system (1.1) with bilinear incidence in a seasonal forcing environment. To overcome the unboundedness of mass action (bilinear incidence) function, we considered a truncated problem on a large but finite interval and applied the limiting arguments to obtain the existence of periodic traveling waves for each $c > c^*$ when $R_0 > 1$. In addition, we also proved the nonexistence of periodic traveling waves for either $R_0 \leq 1$ or $c < c^*$ and $R_0 > 1$. The idea and method of this paper also apply to other periodic and non-monotone evolution systems provided that some new techniques are developed for the verification of the asymptotic boundary condition. Unfortunately, we cannot prove the existence of time periodic traveling waves with critical wave speed $c = c^*$, which remains an open problem for future investigation. The substantial difficulty is again due to the unboundedness of bilinear incidence, which makes the construction of proper sub- and super-solutions much more challenging (if not impossible). At the same time, since system (1.1) is non-autonomous and non-monotone, and the I-component of the periodic traveling wave with wave speed $c > c^*$ is a time periodic pulse wave, it is also difficult to get the existence of critical periodic traveling wave by taking the limit of a sequence of periodic traveling wave with wave speeds c_n, where $c_n > c^*$ and $c_n \to c^*$, see [36,39]. Nevertheless, when the bilinear incidence is replaced by the standard incidence in (1.1) [i.e., system (1.5)], Zhang and Wang [38] recently proved the existence of time periodic traveling wave with the minimal wave speed c^* by constructing sub- and super-solutions similar to those for some autonomous systems, see [16,43] and the references therein.

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