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On a Timoshenko system with thermal coupling on both the bending moment and the shear force

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Dedicated to Professor Jaime E. Muñoz Rivera on the occasion of his 60th birthday

Abstract. The Timoshenko system is a very well-known model for vibrations of elastic beams, which is given by the coupling of two forces acting on the system: the *shear force* and the *bending moment*. In the non-isothermal case, that is, when the model is subject to the temperature variation, we consider the thermal effect acting on the whole system, that is, we propose a new thermoelastic Timoshenko system by coupling thermal laws on both the shear force and the bending moment under the Fourier's law. Then, we show that such a fully thermoelastic system is exponentially stable without assuming equal wave speeds and also independent of any boundary conditions.

1. Introduction

In the present paper, we are going to address the following thermoelastic Timoshenko model

$$\rho_1\,\varphi_{tt} - k\,(\varphi_x + \psi)_x + m\,\theta_x = 0 \quad \text{in} \quad (0,\, l) \times (0,\, \infty), \tag{1.1}$$

$$\rho_2\,\psi_{tt} - b\,\psi_{xx} + k\,(\varphi_x + \psi) - m\,\theta + \sigma\,\vartheta_x = 0 \quad \text{in} \quad (0,\, l) \times (0,\, \infty), \tag{1.2}$$

$$\rho_3\,\theta_t - c_0\,\theta_{xx} + m\,(\varphi_{xt} + \psi_t) = 0 \quad \text{in} \quad (0,\, l) \times (0,\, \infty), \tag{1.3}$$

$$\rho_4\,\vartheta_t - c_1\,\vartheta_{xx} + \sigma\,\psi_{xt} = 0 \quad \text{in} \quad (0,\, l) \times (0,\, \infty), \tag{1.4}$$

subject to initial conditions

$$\begin{aligned} &\varphi(\cdot,0) = \varphi_0,\ \varphi_t(\cdot,0) = \varphi_1,\ \psi(\cdot,0) = \psi_0,\ \psi_t(\cdot,0) = \psi_1, \\ &\theta(\cdot,0) = \theta_0,\ \vartheta(\cdot,0) = \vartheta_0, \end{aligned} \tag{1.5}$$

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and the following set of different boundary conditions including either full Dirichlet or mixed Dirichlet–Neumann such as

$$\begin{aligned}(a)\varphi(0,t) &= \varphi(l,t) = \psi(0,t) = \psi(l,t)\\ &= \theta(0,t) = \theta(l,t) = \vartheta(0,t) = \vartheta(l,t) = 0,\\ (b)\varphi_x(0,t) &= \varphi_x(l,t) = \psi(0,t) = \psi(l,t)\\ &= \theta(0,t) = \theta(l,t) = \vartheta(0,t) = \vartheta(l,t) = 0,\\ (c)\varphi(0,t) &= \varphi(l,t) = \psi_x(0,t) = \psi_x(l,t)\\ &= \theta(0,t) = \theta(l,t) = \vartheta(0,t) = \vartheta(l,t) = 0,\\ (d)\varphi(0,t) &= \varphi(l,t) = \psi(0,t) = \psi(l,t)\\ &= \theta_x(0,t) = \theta_x(l,t) = \vartheta(0,t) = \vartheta(l,t) = 0,\\ (e)\varphi(0,t) &= \varphi(l,t) = \psi(0,t) = \psi(l,t)\\ &= \theta(0,t) = \theta(l,t) = \vartheta_x(0,t) = \vartheta_x(l,t) = 0,\\ (f)\varphi_x(0,t) &= \varphi_x(l,t) = \psi(0,t) = \psi(l,t)\\ &= \theta(0,t) = \theta(l,t) = \vartheta_x(0,t) = \vartheta_x(l,t) = 0,\\ (g)\varphi(0,t) &= \varphi(l,t) = \psi_x(0,t) = \psi_x(l,t)\\ &= \theta_x(0,t) = \theta_x(l,t) = \vartheta(0,t) = \vartheta(l,t) = 0,\\ (h)\varphi(0,t) &= \varphi(l,t) = \psi(0,t) = \psi(l,t)\\ &= \theta_x(0,t) = \theta_x(l,t) = \vartheta_x(0,t) = \vartheta_x(l,t) = 0.\end{aligned} \tag{1.6}$$

A complete justification from modeling point of view of the particular model (1.1)–(1.4) is presented in Sect. 2 by using theories for elastic and thermoelastic beams/plates as developed, e.g., in [13,21–23,36] in combination with the classical Timoshenko model [48,49]. As clarified in Sect. 2, we say that system (1.1)–(1.4) is a fully thermoelastic Timoshenko system because it has thermal coupling on both the bending moment and the shear force. From stability point of view, it means that we have a fully dissipative system and, therefore, its exponential stability is expected. Indeed, our main result (see Theorem 4.8) states that problem (1.1)–(1.6) is always exponentially stable, independent of any relation among the coefficients and the boundary condition assumed in (1.6). In what follows, we are going to provide a brief existing literature on the subject and then some comparisons.

We start by considering the classical conservative Timoshenko system (see e.g., [48] or [49, Sect. 55]):

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{1.7}$$

$$\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x + \psi) = 0, \tag{1.8}$$

where the positive coefficients are given by $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k = k'GA$, $b = EI$, and whose physical meanings will be clarified in Sect. 2. In this case, a first result in the stabilization scenario is given by Soufyane [45], which asserts that the Timoshenko system (1.7)–(1.8) subject to a weak damping $\beta\psi_t$, $\beta > 0$, is exponentially stable if and only if $\chi = 0$, where from now on χ means the difference of wave speeds

$$\chi := \frac{k}{\rho_1} - \frac{b}{\rho_2} = k'\frac{G}{\rho} - \frac{E}{\rho}. \tag{1.9}$$

Note that $\chi = 0$ is equivalent to $G = E/k'$. Ever since, the condition $\chi = 0$ has been widely used in the stabilization of partially damped Timoshenko systems as we may see in [1,5–7,9,19,20,28,30,32,46,47] and references therein. Moreover, we refer to [8,25,38,42] where it is considered internal or boundary dissipations on both Eqs. (1.7)–(1.8). Therefore, as expected, its exponential stability follows without assuming equal wave speeds $\chi = 0$. We also note that some pioneer results in the stabilization of one-dimensional thermoelastic wave systems can be found in [10,27, 29,44].

Now, we consider some thermoelastic Timoshenko systems that are more related to the subject addressed in this work. Indeed, in Muñoz Rivera and Racke [31] the authors introduced for the fist time the following partially damped thermoelastic Timoshenko system according to Fourier's law

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{1.10}$$

$$\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x + \psi) + \sigma\,\vartheta_x = 0, \tag{1.11}$$

$$\varrho\,\vartheta_t - c\,\vartheta_{xx} + \sigma\,\psi_{xt} = 0, \tag{1.12}$$

where the thermal coupling is considered on the bending moment and the constants c, ϱ, σ are positive, whose physical meaning will be also clarified in Sect. 2. Under the assumption $\chi = 0$, the authors proved that the system with Dirichlet–Neumann boundary conditions is exponentially stable. In addition, it was proved that (1.10)–(1.12) under the boundary condition $\varphi = \psi_x = \theta_x = 0$ is exponentially stable if and only if $\chi = 0$, see e.g., [31, Thms 3.1 and 4.1]. The same result was also obtained by Fernández Sare and Racke [17, Thms 4.6 and 4.7] with $\varphi_x = \psi = \theta_x = 0$ on $x = 0, l$, in the thermoelastic case under Fourier's law and history (memory term) both coupled on the bending moment. This system was also recently addressed by Cardozo et al. [7] with non-constant coefficients and local assumption on equal wave speeds, which complements the results previously considered in [31].

On the other hand, in Almeida Júnior et al. [2] the next thermoelastic Timoshenko system was approached still in accordance with Fourier's law

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + m\,\theta_x \;=\; 0, \tag{1.13}$$

$$\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x + \psi) - m\,\theta \;=\; 0, \tag{1.14}$$

$$\varrho\,\theta_t - c\,\theta_{xx} + m(\varphi_x + \psi)_t \;=\; 0, \tag{1.15}$$

where now the thermal coupling is taken on the shear force. They considered the problem with either Dirichlet–Neumann boundary conditions (D): $\varphi = \psi = \theta = 0$ or (N): $\varphi = \psi_x = \theta_x = 0$. More precisely, in [2, Thms 3.2 and 4.4] the authors have proved that system with boundary condition (N) is exponentially stable iff $\chi = 0$. When $\chi \neq 0$ it is only achieved in [2, Thm 5.1] that system decays polynomially with decay rate depending on the boundary conditions, namely, with rate $t^{-1/4}$ for (D) and optimal rate $t^{-1/2}$ for (N). This latter result was improved by Alves et al. [3, Thm 4.1] where the same polynomial decay rate $t^{-1/2}$ (corresponding to the optimal one) is reached independently of the boundary conditions. System (1.13)–(1.15) with non-constant coefficients and local assumption on equal wave speeds was also considered by Alves et al. [4], where the results complement the previous ones in [2,3].

From the above exposition, one sees that the equal wave speeds assumption ($\chi = 0$) plays a crucial role in the study of uniform (exponential) stability of partially thermoelastic systems like (1.10)–(1.12) and (1.13)–(1.15), among others. Besides, such assumption is also considered in other partially damped thermoelastic Timoshenko systems with different thermal laws, for which, we refer to [12,16,17,39,40] and references therein. Therefore, motivated by this scenario, our main goal is to consider a fully thermoelastic Timoshenko like (1.1)–(1.4) that, besides being an accepted model from mathematical (and physical) point of view under the Fourier's law, it is also exponentially stable independently of both the number χ and the boundary conditions. In Sect. 2 we will see that (1.1)–(1.4) can be physically derived, not only arising by mixing the systems (1.10)–(1.12) and (1.13)–(1.15), and provides a different character in what concerns the stability of solutions when compared to systems (1.10)–(1.12) and (1.13)–(1.15) addressed in [2–4,7,17,31], once the uniform (exponential) stabilization is achieved without assuming the nonphysical condition of equal wave speeds $\chi = 0$. In addition, we also have considered the corresponding system with non-constant coefficients, see Sect. 5. From computation viewpoint, it will be highlighted in Sect. 4 that our stability results follow, in parts, similar strategies as in [3,7].

Another interesting point is that our main result is different from the result on stability for the thermoelastic Bresse system with temperature deviations along the longitudinal and vertical directions, proposed by Liu and Rao [26]. Indeed, in [26] it was considered a thermoelastic Bresse system with temperature coupled on the axial force and the bending moment. Even with such a coupling, the assumption $G = E/k'$ is required in order to obtain exponential stability, see [26, Thm 3.1]. Moreover, in the representative case $G \neq E/k'$ (with real physical meaning), they only proved polynomial stability depending on the boundary conditions and regularity of initial data. Analogous results for thermoelastic Bresse systems (in terms of condition $G = E/k'$) are also provided in [11,15,41,43]. We remember that the equal wave speeds, here translated to $G = E/k'$, is only an assumption from mathematical point of view. Indeed, as remarked in [26,33], we have from the theory of elasticity that these two elastic modulus are related as $G = \dfrac{E}{2(1+\nu)}$, where $\nu \in (0, \frac{1}{2})$ is the Poisson's ratio, which means that the wave speeds are not equal physically since $k' < 1$ and so the identity $2(1+\nu) = k'$ does not happen. For a different approach on thermoelastic plate systems where two temperatures are involved, we refer to Quintanilla and Racke [37].

The remaining paper is organized as follows. In Sect. 2 it is provided a complete justification of the thermoelastic model (1.1)–(1.4). In Sect. 3 it is sketched its well-posedness via semigroup theory, while in Sect. 4 it is proved its exponential stability still using the semigroup theory in combination with an observability result for the resolvent equation associated with (conservative) Timoshenko systems. Last, in Sect. 5 it is addressed a non-homogeneous thermoelastic system and related results.

2. Justification of the model (1.1)–(1.4)

In this section, in order to legitimate the thermoelastic Timoshenko model (1.1)–(1.4) from a mathematical (and physical) viewpoint, we regard some constitutive laws in mathematical–physics that combine the works by Timoshenko [48,49] with elastic/thermoelastic relations provided by Lagnese and Lions [23], Lagnese, Leugering and Schmidt [21,22], Drozdov–Kolmanovskii [13], and Prüss [36], where thin beams/plates are assumed to be homogeneous and elastically/thermally isotropic.

We start by assuming the classical Timoshenko hypotheses for a thin beam/plate as, for instance, in Prüss [36, Chapt. 9] and Drozdov and Kolmanovskii [13, Chapt. 5]. In this way, let us consider a thin 3D beam

$$[0, L] \times \Omega := \{(x_1, x_2, x_3) : \ x_1 \in [0, L] \text{ and } (x_2, x_3) \in \Omega\},$$

of length $L > 0$ and uniform cross section $\Omega \subset \mathbb{R}^2$, which is composed by homogeneous isotropic material under the accepted (and summarized) Timoshenko assumptions:

- **A1.** $(0, 0)$ is the center of Ω so that $\int_\Omega x_3 \mathrm{d}x_2 \mathrm{d}x_3 = \int_\Omega x_2 \mathrm{d}x_2 \mathrm{d}x_3 = 0$;
- **A2.** $\mathrm{diam}\Omega << L$ so that one considers thin beams;
- **A3.** the bending takes place only on the (x_1, x_3)-plane and normal stresses (that is, in the x_2-axis) are negligible in general;
- **A4.** the matrix of stress tensor $\sigma = (\sigma_{ij})_{1 \leq i,j \leq 3}$ is assumed to have only two relevant stresses, namely, σ_{11} and σ_{13}, and the remaining stresses are neglected ($\sigma_{ij} \approx 0$).

Here, a longitudinal section is considered along the (x_1, x_3)-plane, that is, it consists in points of the form $(x_1, 0, x_3)$, hereafter identified with (x_1, x_3) for simplicity, where the bending takes place according to assumption **A3**, see Fig. 1.

Thus, we introduce the displacements and the rotation angle in the (x_1, x_3)-plane by the following notations (see also Fig. 2):

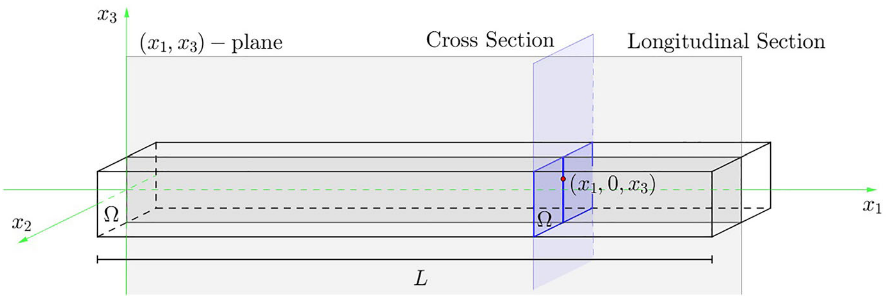

Figure 1. Beam/plate with longitudinal and cross sections

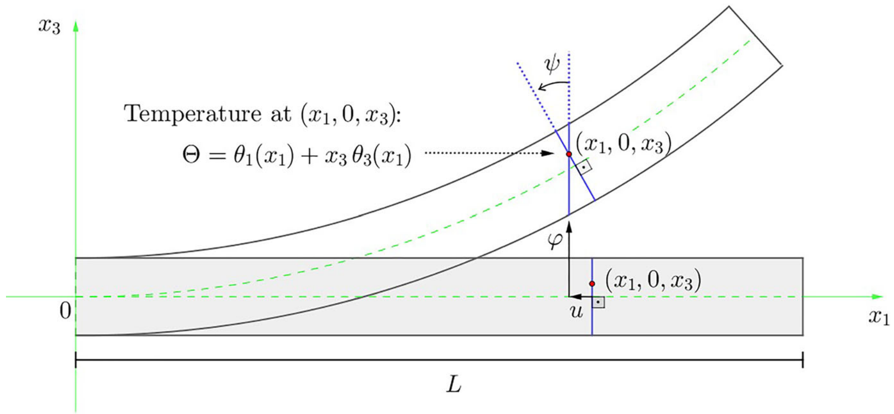

Figure 2. Displacements and temperature distribution in the (x_1, x_3)-plane

- $u = u(x_1, t)$: the longitudinal displacement of points lying on the x_1-axis;
- $\psi = \psi(x_1, t)$: the angle of rotation for the normal to the x_1-axis;
- $w_1(x_1, x_3, t) = u(x_1, t) + x_3\psi(x_1, t)$: longitudinal displacement;
- $w_2(x_1, x_3, t) = \varphi(x_1, t)$: the vertical beam displacement.

In addition to the elastic displacements, we also assume that the beam/plate is subject to an unknown difference of temperature $\Theta(x_1, x_2, x_3, t)$, which clearly contributes to its deformation and whose deviation is measured from a reference state of uniform temperature distribution Θ_0 in the rest position of the beam (no stresses nor strains). In this part we rely on the principles of thermoelasticity (suitable approximations) as developed in [21–23], by restricting ourselves to the reference (x_1, x_3)-plane. Indeed, in such a reference plane we may assume that the temperature distribution takes the form $\Theta(x_1, 0, x_3, t) := \Theta(x_1, x_3, t)$, being constant in each cross section Ω, that is, we assume for simplicity that there is no variation of temperature in the normal x_2-direction. Moreover, because of the thinness of the beam/plate and following the assumptions on page 60 of [22, Chapt. III], see also the identity (6.30) in [23] or else [21, Sect. 2], we introduce the following Taylor's expansion for the temperature distribution in the (x_1, x_3)-plane (with $x_2 = 0$):

- $\Theta(x_1, x_3, t) = \Theta(x_1, 0, x_3, t) = \theta_1(x_1, t) + x_3\theta_3(x_1, t)$,

where θ_1 and θ_2 are temperature components (functions) that may represent the temperature deviations from the reference temperature Θ_0 along the longitudinal and vertical directions. According to [21–23], this is a standard assumption in the theory for very thin beams and, therefore, we hereafter adopt such a "linearization" in the present article for the x_3-variable.

Under these circumstances, we will derive a linear model for thermoelastic Timoshenko beams by taking into account the displacements and the temperature distribution in the plane of reference, see Fig. 2 again. This will be done in some steps as follows.

Stress–Strain relations. Since we are dealing with a homogeneous, elastically and thermally isotropic thin beam/plate, then we consider the following stress–strain relations for the remaining stresses σ_{11} and σ_{13} in **A4**, accordingly to (6.1) on page 26 in [23, Chapt.1]:

$$\sigma_{11} = a_{11}\left(\epsilon_{11} - \epsilon_{11}^T\right) \quad \text{and} \quad \sigma_{13} = a_{13}\left(\epsilon_{13} - \epsilon_{13}^T\right), \tag{2.1}$$

where the coefficients of elasticity a_{11}, a_{13} are independent of Θ. One can consider, for instance, $a_{11} = E$ and $a_{13} = 2k'G$, where E stands for the Young modulus elasticity and G is the shear modulus which is a shear correction coefficient k'. The elastic strains $\epsilon_{11}, \epsilon_{13}$ will be determined as follows according to Timoshenko's laws in elasticity, and ϵ_{11}^T, ϵ_{13}^T denote the thermal strains whose formulation will be theorized in accordance with proper laws for thermoelastic beams/plates.

Elastic strains. Under the above notations, the standard formulas for the components of the infinitesimal elastic strain tensor (see e.g., (2.4) on page 339 in [13]) can be expressed by

$$\epsilon_{11}(x_1, x_3, t) = \frac{\partial w_1}{\partial x_1} = u_{x_1}(x_1, t) + x_3\psi_{x_1}(x_1, t), \tag{2.2}$$

$$\epsilon_{13}(x_1, x_3, t) = \frac{1}{2}\left(\frac{\partial w_1}{\partial x_3} + \frac{\partial w_2}{\partial x_1}\right) = \frac{1}{2}\big[\psi(x_1, t) + \varphi_{x_1}(x_1, t)\big]. \tag{2.3}$$

Thermal strains. According to (6.2) in [23] the thermal strains can be given by

$$\epsilon_{1j}^T = \delta_{1j}\, \epsilon^T, \quad j = 1, 3,$$

where ϵ^T denotes the thermal strain, which depends upon the composition of the beam/plate material under consideration, and $\delta_{1j} > 0$, $j = 1, 3$. In addition, it is assumed that the change of temperature Θ is small when compared to the reference temperature Θ_0 (that is, $|\Theta/\Theta_0| << 1$) and, consequently, one gets the relation

$$\epsilon^T = \alpha\, \Theta,$$

where $\alpha > 0$ is a constant called *coefficient of thermal expansion*. See, for instance, Eqs. (6.17)–(6.18) in [23]. Therefore, combining the last two identities, we obtain the next expressions for the thermal strains

$$\epsilon_{11}^T(x_1, x_3, t) = \alpha\, \delta_{11}\, \Theta(x_1, x_3, t) = \alpha\, \delta_{11}\big[\theta_1(x_1, t) + x_3\, \theta_3(x_1, t)\big], \tag{2.4}$$

$$\epsilon_{13}^T(x_1, x_3, t) = \alpha\, \delta_{13}\, \Theta(x_1, x_3, t) = \alpha\, \delta_{13}\big[\theta_1(x_1, t) + x_3\, \theta_3(x_1, t)\big]. \tag{2.5}$$

Bending and Shear relations. Going back to postulations **A1–A4** and following the identities (9.10)–(9.11) in [36], the conventional formulas to express the bending moment and the shear force are given by

$$\begin{aligned} M(x_1,t) &= \int_\Omega x_3\sigma_{11}(x_1,x_3,t)\,\mathrm{d}x_2\mathrm{d}x_3, \\ S(x_1,t) &= \int_\Omega \sigma_{13}(x_1,x_3,t)\,\mathrm{d}x_2\mathrm{d}x_3, \end{aligned} \tag{2.6}$$

respectively. We note that, for simplicity, we have normalized the equations in (2.6) by the area A and inertial moment I of the cross section Ω, namely,

$$A = \int_\Omega \mathrm{d}x_2\mathrm{d}x_3 \qquad \text{and} \qquad I = \int_\Omega x_3^2\,\mathrm{d}x_2\mathrm{d}x_3.$$

As a first consequence, using identities (2.1), (2.2), (2.4) and (2.6), one can compute the (classical) thermoelastic law for the bending moment

$$\begin{aligned} M(x_1,t) &= E\overbrace{\left(\int_\Omega x_3\,\mathrm{d}x_2\mathrm{d}x_3\right)}^{=0}\big[u_{x_1}(x_1,t)-\alpha\,\delta_{11}\,\theta_1(x_1,t)\big] \\ &\quad + E\overbrace{\left(\int_\Omega x_3^2\,\mathrm{d}x_2\mathrm{d}x_3\right)}^{=I}\big[\psi_{x_1}(x_1,t)-\alpha\,\delta_{11}\,\theta_3(x_1,t)\big], \end{aligned}$$

and then

$$M(x_1,t) = EI\big[\psi_{x_1}(x_1,t)-\alpha\,\delta_{11}\,\theta_3(x_1,t)\big],\quad x_1\in[0,L],\ t\ge 0. \tag{2.7}$$

Remark 2.1. It is worth mentioning that this is the exact moment where the variable $u = u(x_1,t)$, which corresponds to the longitudinal displacement on the x_1-axis, vanishes. This is why such a variable does not appear in the classical elastic (nor in the thermoelastic or in the viscoelastic) Timoshenko systems and could be interpreted as a too small horizontal displacement ($u \approx 0$) when compared to the vertical displacement φ and the rotation angle ψ in the beam deformation.

Moreover, using identities (2.1), (2.3), (2.5) and (2.6), the following (not so classical) thermoelastic law for the shear force comes into the picture

$$\begin{aligned} S(x_1,t) &= 2k'G\overbrace{\left(\int_\Omega \mathrm{d}x_2\mathrm{d}x_3\right)}^{=A}\left[\frac{1}{2}\big(\psi(x_1,t)+\varphi_{x_1}(x_1,t)\big)-\alpha\,\delta_{13}\,\theta_1(x_1,t)\right] \\ &\quad - 2k'G\overbrace{\left(\int_\Omega x_3\,\mathrm{d}x_2\mathrm{d}x_3\right)}^{=0}\big[\alpha\,\delta_{13}\,\theta_3(x_1,t)\big], \end{aligned}$$

that is,

$$S(x_1,t) = 2k'GA\Big[\frac{1}{2}\big(\psi+\varphi_{x_1}\big)(x_1,t) - \alpha\,\delta_{13}\,\theta_1(x_1,t)\Big],\;\; x_1\in[0,L],\;\; t\geq 0. \tag{2.8}$$

The constitutive laws (2.7)–(2.8) provide bending and shear deformations in the context of homogenous thermally isotropic Timoshenko beams. It is worth mentioning that if we neglect thermal effects (e.g., $\epsilon^T = 0$ for $\Theta = 0$), then (2.7)–(2.8) clearly fall on the well-known elastic constitutive relations for the bending moment and shear force as follows

$$M(x_1,t) = EI\psi_{x_1}(x_1,t) \quad \text{and} \quad S(x_1,t) = k'GA\big(\psi+\varphi_{x_1}\big)(x_1,t). \tag{2.9}$$

Heat flux of conduction. For the last thermoelastic relation, we must provide a motion equation of heat conduction for the temperature deviation $\Theta(x_1,x_3,t)$. To do so, we are going to rely on the general Newton's law for the heat flux (see e.g., Eq. (30) in [21] or else (2.12) in [22]) and consider

$$\rho_0 c_v \Theta_t = \big(\Theta_{x_1x_1} + \Theta_{x_3x_3}\big) - \alpha\,\Theta_0\big(\epsilon_{11,t} + \epsilon_{13,t}\big), \tag{2.10}$$

where c_v represents the heat capacity and ρ_0 the density per unit of reference. Thus, taking into account the linearized expansion for Θ and the expressions (2.2)–(2.3) for the strains, also by neglecting the horizontal displacement ($u\approx 0$) in accordance with Remark 2.1, Eq. (2.10) of heat conduction reduces to

$$\rho_0 c_v\Big\{\theta_{1,t} + x_3\theta_{3,t}\Big\} = \Big\{\theta_{1,x_1x_1} + x_3\theta_{3,x_1x_1}\Big\} - \alpha\,\Theta_0\Big\{\frac{1}{2}(\psi+\varphi_{x_1})_t + x_3\psi_{x_1t}\Big\}. \tag{2.11}$$

Moreover, upon taking the average (i.e., integrating (2.11) on Ω) and the inertial moment (i.e., multiplying (2.11) by x_3 and integrating the resulting expression on Ω), we derive the following set of two 1D heat equations for the variables θ_1 and θ_3:

$$\rho_0 c_v\theta_{1,t} = \theta_{1,x_1x_1} - \frac{\alpha}{2}\Theta_0(\varphi_{x_1}+\psi)_t, \tag{2.12}$$

$$\rho_0 c_v\theta_{3,t} = \theta_{3,x_1x_1} - \alpha\Theta_0\psi_{x_1t}. \tag{2.13}$$

We note that Eqs. (2.12)–(2.13) represent the heat flux of conduction under the Fourier's law. However, according to [21,22] we could replace them by more general heat flux laws depending on the material that composes the beam.

Motion equations for Timoshenko beams. We first observe that in (2.7), (2.8), (2.9), (2.12) and (2.13), we only deal with one spatial variable. Thus, for the sake of notation, from now on we will omit the subscript "1" by denoting x_1 simply as x.

In order to deduce the desired thermoelastic systems, including the model (1.1)–(1.4), we additionally consider the classical momentum equations for Timoshenko beams (see [48,49]):

$$\begin{cases} \rho A\varphi_{tt} - S_x = 0, \\ \rho I\psi_{tt} - M_x + S = 0, \end{cases} \tag{2.14}$$

for $(x,t) \in (0,L) \times (0,\infty)$, where ρ represents the mass density per area unit, and the other variables are previously introduced. Keeping this system in mind and regarding the constitutive laws (2.7)–(2.9) along with the heat equations (2.12)–(2.13), we are able to provide the deduction of at least three different models for thermoelastic Timoshenko systems as follows.

Case 1. Fully thermoelastic system: bending moment and shear force with thermal coupling. Replacing M from (2.7) and S from (2.8) in the system (2.14), then it turns into the fully thermoelastic Timoshenko problem:

$$\begin{cases} \rho A\,\varphi_{tt} - k'GA\,(\varphi_x + \psi)_x + 2\alpha\delta_{13}k'GA\,\theta_{1,x} = 0, \\ \rho I\,\psi_{tt} - EI\,\psi_{xx} + k'GA(\phi_x + \psi) + \alpha\delta_{11}EI\,\theta_{3,x} - 2\alpha\delta_{13}k'GA\,\theta_1 = 0. \end{cases} \tag{2.15}$$

This system must be complemented with the governing equations of heat conduction for the variables θ_1 and θ_3. For this purpose, in view of (2.12)–(2.13), we take

$$\begin{cases} \rho_0 c_v\theta_{1,t} - \theta_{1,xx} + \frac{\alpha}{2}\Theta_0(\varphi_x + \psi)_t = 0, \\ \rho_0 c_v\theta_{3,t} - \theta_{3,xx} + \alpha\Theta_0\psi_{xt} = 0. \end{cases} \tag{2.16}$$

We finally observe that system (2.15)–(2.16) is precisely the thermoelastic problem (1.1)–(1.4), with simplified notations

$$\begin{aligned} &\rho_1 = \rho A, \quad \rho_2 = \rho I, \quad k = k'GA, \quad b = EI, \\ &\rho_3 = \frac{4\rho_0 c_v}{\Theta_0}\delta_{13}k'GA, \quad \rho_4 = \frac{\rho_0 c_v}{\Theta_0}\delta_{11}EI, \\ &c_0 = \frac{4}{\Theta_0}\delta_{13}k'GA, \quad c_1 = \frac{1}{\Theta_0}\delta_{11}EI, \quad m = 2\alpha\delta_{13}k'GA, \quad \sigma = \alpha\delta_{11}EI, \\ &\theta = \theta_1, \quad \vartheta = \theta_3, \end{aligned} \tag{2.17}$$

and constitutes the main object of study in the present article. It has to be complemented by appropriate boundary conditions, here considered as those written in (1.6). Last, but not least, we notice that the fully thermoelastic system (2.15)–(2.16) (or (1.1)–(1.4) as well) is enough (and adequate) to determine temperature distribution Θ, in terms of θ_1 and θ_3.

Case 2. Partially thermoelastic systems: either the bending moment or the shear force with thermal coupling. Here the procedure is similar, but also taking (2.9) into account. Now, we replace either in the system (2.14): the couple (M,S) from [(2.7), (2.9)] or else (S,M) from [(2.8), (2.9)]. Then, we consider the heat equation $(2.16)_2$ for $\theta_3 = \vartheta$

or else $(2.16)_1$ for $\theta_1 = \theta$, respectively. This process, under the notations introduced in (2.17), leads precisely to the systems (1.10)–(1.12) or (1.13)–(1.15), respectively, for proper values of ϱ and c.

Therefore, in this case, one obtains two additional partially thermoelastic Timoshenko systems, which have already been studied in the literature as described the introduction. We finally observe that these partially thermoelastic systems are not sufficient to determine the temperature distribution Θ, but only θ_1 to the shear force thermal coupling or else θ_3 in the case of coupling on the bending moment. A similar analysis for thermoelastic plates can be found in [23, Remark 6.3].

3. Semigroup solution

Let us start by defining the phase spaces depending on each boundary condition in (1.6)

$$\mathcal{H} = \begin{cases} H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) \times L^2(0,l) & \text{for } (1.6)_{(a)}, \\ H_*^1(0,l) \times L_*^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) \times L^2(0,l) & \text{for } (1.6)_{(b)}, \\ H_0^1(0,l) \times L^2(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L^2(0,l) \times L^2(0,l) & \text{for } (1.6)_{(c)}, \\ H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L_*^2(0,l) \times L^2(0,l) & \text{for } (1.6)_{(d)}, \\ H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) \times L_*^2(0,l) & \text{for } (1.6)_{(e)}, \\ H_*^1(0,l) \times L_*^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) \times L_*^2(0,l) & \text{for } (1.6)_{(f)}, \\ H_0^1(0,l) \times L^2(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L_*^2(0,l) \times L^2(0,l) & \text{for } (1.6)_{(g)}, \\ H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L_*^2(0,l) \times L_*^2(0,l) & \text{for } (1.6)_{(h)}, \end{cases}$$

where $H_*^1(0,l) = H^1(0,l) \cap L_*^2(0,l)$ and $L_*^2(0,l) = \left\{ u \in L^2(0,l) \mid \int_0^l u(x)\,\mathrm{d}x = 0 \right\}$. It is well-known that $\mathcal{H}$ is a Hilbert space with respect to the norm

$$\|U\|_{\mathcal{H}}^2 = \int_0^l \Big[\rho_1|\Phi|^2 + \rho_2|\Psi|^2 + b|\psi_x|^2 + k|\varphi_x + \psi|^2 + \rho_3|\theta|^2 + \rho_4|\vartheta|^2 \Big] \mathrm{d}x, \tag{3.1}$$

for $U = (\varphi, \Phi, \psi, \Psi, \theta, \vartheta) \in \mathcal{H}$, associated with the inner-product $(\cdot,\cdot)_{\mathcal{H}}$ induced by system on $\mathcal{H}$. Under the above notations, we can rewrite system (1.1)–(1.6) as an abstract first-order Cauchy problem

$$\begin{cases} U_t = \mathcal{A}U, \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{3.2}$$

where $U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \vartheta_1)$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}U := \begin{pmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{m}{\rho_1}\theta_x \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{m}{\rho_2}\theta - \frac{\sigma}{\rho_2}\vartheta_x \\ \frac{c_0}{\rho_3}\theta_{xx} - \frac{m}{\rho_3}(\Phi_x + \Psi) \\ \frac{c_1}{\rho_4}\vartheta_{xx} - \frac{\sigma}{\rho_4}\Psi_x \end{pmatrix} \tag{3.3}$$

for every $U = (\varphi, \Phi, \psi, \Psi, \theta, \vartheta) \in D(\mathcal{A})$, with domain

$$D(\mathcal{A}) = \{U \in \mathcal{H} \mid \varphi, \psi, \theta, \vartheta \in H^2(0,l) \text{ and } (\mathcal{D}) \text{ is satified}\}, \tag{3.4}$$

where

$$(\mathcal{D}) \begin{cases} \theta, \vartheta, \Phi, \Psi \in H_0^1(0,l) & \text{for } (1.6)_{(a)}, \\ \varphi_x, \theta, \vartheta, \Psi \in H_0^1(0,l), \Phi \in H_*^1(0,l) & \text{for } (1.6)_{(b)}, \\ \psi_x, \theta, \vartheta, \Phi, W \in H_0^1(0,l), \Psi \in H_*^1(0,l) & \text{for } (1.6)_{(c)}, \\ \theta_x, \vartheta, \Phi, \Psi \in H_0^1(0,l) & \text{for } (1.6)_{(d)}, \\ \theta, \vartheta_x, \Phi, \Psi \in H_0^1(0,l) & \text{for } (1.6)_{(e)}, \\ \varphi_x, \theta, \vartheta_x, \Psi \in H_0^1(0,l), \Phi \in H_*^1(0,l) & \text{for } (1.6)_{(f)}, \\ \psi_x, \theta_x, \vartheta, \Phi \in H_0^1(0,l), \Psi \in H_*^1(0,l) & \text{for } (1.6)_{(g)}, \\ \theta_x, \vartheta_x, \Phi, \Psi \in H_0^1(0,l) & \text{for } (1.6)_{(h)}. \end{cases} \tag{3.5}$$

The result on existence and uniqueness of solution to problem (3.2), and therefore to the equivalent system (1.1)–(1.6), is stated as follows.

Theorem 3.1. *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be given by (3.3). Then we have:*

1. *If $U_0 \in \mathcal{H}$, then problem (3.2) has a unique mild solution $U \in C^0([0,\infty), \mathcal{H})$.*
2. *If $U_0 \in D(\mathcal{A})$, then the above mild solution is regular one satisfying*

$$U \in C^0([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H}).$$

3. *If $U_0 \in D(\mathcal{A}^n)$, $n \geq 2$ integer, then the above regular solution satisfies*

$$U \in \bigcap_{j=0}^{n} C^{n-j}([0,\infty), D(\mathcal{A}^j)).$$

Proof. According to the general theory in linear semigroups, see e.g., Pazy [34], it is sufficient to prove that operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t) = e^{\mathcal{A}t}$ on $\mathcal{H}$. To do so, it is enough to show that $\mathcal{A}$ is a dissipative operator on $\mathcal{H}$ and $I_d - \mathcal{A}$ maps $D(\mathcal{A})$ onto $\mathcal{H}$. Indeed, given any $U \in D(\mathcal{A})$, standard computations give us

$$\mathrm{Re}(\mathcal{A}U, U)_{\mathcal{H}} = -c_0 \int_0^l |\theta_x|^2 \,\mathrm{d}x - c_1 \int_0^l |\vartheta_x|^2 \,\mathrm{d}x \leq 0, \tag{3.6}$$

independent of the boundary condition assumed in (1.6). Now we are going to prove that $I_d - \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is onto. For commodity, we choose boundary condition $(1.6)_{(g)}$. Under the above notations, the domain in this case is given by

$$D(\mathcal{A}) = \left\{U \in \mathcal{H} \mid \varphi, \psi, \theta, \vartheta \in H^2(0,l),\ \psi_x, \theta_x, \vartheta, \Phi, \in H_0^1(0,l),\ \Psi \in H_*^1(0,l)\right\}.$$

Given $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in H_0^1(0,l) \times L^2(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L_*^2(0,l) \times L^2(0,l)$, we will find a unique function $U \in D(\mathcal{A})$ such that $U - \mathcal{A}U = F$. This last equation is read in terms of its components as follows

$$\varphi - \Phi = f_1, \tag{3.7}$$

$$\rho_1 \Phi - k(\varphi_x + \psi)_x + m\theta_x = \rho_1 f_2, \tag{3.8}$$
$$\psi - \Psi = f_3, \tag{3.9}$$
$$\rho_2 \Psi - b\psi_{xx} + k(\varphi_x + \psi) - m\theta + \sigma \vartheta_x = \rho_2 f_4, \tag{3.10}$$
$$\rho_3 \theta - c_0 \theta_{xx} + m(\Phi_x + \Psi) = \rho_3 f_5, \tag{3.11}$$
$$\rho_4 \vartheta - c_1 \vartheta_{xx} + \sigma \Psi_x = \rho_4 f_6. \tag{3.12}$$

Replacing (3.7) and (3.9) in the remaining Eqs. (3.8), (3.10)–(3.12), we obtain

$$\rho_1 \varphi - k(\varphi_x + \psi)_x + m\theta_x = \rho_1 f_1 + \rho_1 f_2, \tag{3.13}$$
$$\rho_2 \psi - b\psi_{xx} + k(\varphi_x + \psi) - m\theta + \sigma \vartheta_x = \rho_2 f_3 + \rho_2 f_4, \tag{3.14}$$
$$\rho_3 \theta - c_0 \theta_{xx} + m(\varphi_x + \psi) = m f_{1,x} + m f_3 + \rho_3 f_5, \tag{3.15}$$
$$\rho_4 \vartheta - c_1 \vartheta_{xx} + \sigma \psi_x = \sigma f_{3,x} + \rho_4 f_6. \tag{3.16}$$

Using Lax–Milgram theorem, it is easy to conclude that there exists a unique solution $(\varphi, \psi, \theta, \vartheta) \in H_0^1(0,l) \times H_*^1(0,l) \times H_*^1(0,l) \times H_0^1(0,l)$ to the following variational problem related to (3.13)–(3.16)

$$\begin{aligned}
&\rho_1 \int_0^l \varphi\overline{\tilde{\varphi}}\, dx + \rho_2 \int_0^l \psi\overline{\tilde{\psi}}\, dx + \rho_3 \int_0^l \theta\overline{\tilde{\theta}}\, dx + \rho_4 \int_0^l \vartheta\overline{\tilde{\vartheta}}\, dx + k\int_0^l (\varphi_x + \psi)\overline{(\tilde{\varphi}_x + \tilde{\psi})}\, dx \\
&- m\int_0^l \theta\overline{(\tilde{\varphi}_x + \tilde{\psi})}\, dx + b\int_0^l \psi_x \overline{\tilde{\psi}_x}\, dx - \sigma \int_0^l \vartheta \overline{\tilde{\psi}_x}\, dx + c_0 \int_0^l \theta_x \overline{\tilde{\theta}_x}\, dx \\
&+ m\int_0^l (\varphi_x + \psi)\overline{\tilde{\theta}}\, dx + c_1 \int_0^l \vartheta_x \overline{\tilde{\vartheta}_x}\, dx + \sigma \int_0^l \psi_x \overline{\tilde{\vartheta}}\, dx = \int_0^l (\rho_1 f_1 + \rho_1 f_2)\overline{\tilde{\varphi}}\, dx \\
&+ \int_0^l (\rho_2 f_3 + \rho_2 f_4)\overline{\tilde{\psi}}\, dx + \int_0^l (m f_{1,x} + m f_3 + \rho_3 f_5)\overline{\tilde{\theta}}\, dx + \int_0^l (\sigma f_{3,x} + \rho_4 f_6)\overline{\tilde{\vartheta}}\, dx,
\end{aligned}$$

for all $(\tilde{\varphi}, \tilde{\psi}, \tilde{\theta}, \tilde{\vartheta}) \in H_0^1(0,l) \times H_*^1(0,l) \times H_*^1(0,l) \times H_0^1(0,l)$. Then, for standard particular choices of functions $(\tilde{\varphi}, \tilde{\psi}, \tilde{\theta}, \tilde{\vartheta})$ and regularizing properties, we can also conclude that

$$\varphi, \psi, \theta, \vartheta \in H^2(0,l), \quad \psi_x, \theta_x, \vartheta, \Phi \in H_0^1(0,l), \quad \Psi \in H_*^1(0,l), \tag{3.17}$$

with $(\varphi, \psi, \theta, \vartheta)$ satisfying (3.13)–(3.16), which implies that $U = (\varphi, \Phi, \psi, \Psi, \theta, \vartheta) \in D(\mathcal{A})$ solves (3.7)–(3.12) as desired. The proof is analogous to the remaining boundary conditions.

Hence, the proof of Theorem 3.1 is complete. □

4. Exponential stability

Our main result on stability asserts that problem (1.1)–(1.6) is exponentially stable independent of the boundary conditions in (1.6) and the difference of wave speeds χ in (1.9). This achievement will be proved through the semigroup solution $U(t) = e^{\mathcal{A}t}U_0$ by using the following well-known characterization of exponential stability for C_0-semigroups established by Gearhart–Huang–Prüss [18,24,35].

Theorem 4.1. *A C_0-semigroup of contractions $T(t) = e^{At}$ over a Hilbert space H is exponentially stable if and only if*

$$i\mathbb{R} \subseteq \rho(A) \quad \text{and} \quad \limsup_{|\beta|\to\infty} \|(i\beta I_d - A)^{-1}\|_{\mathcal{L}(H)} < \infty, \tag{4.1}$$

where $i\mathbb{R} = \{i\beta \mid \beta \in \mathbb{R}\}$.

We will show property (4.1) for our problem by establishing several lemmas as follows. Our starting point is to consider the resolvent equation

$$i\beta U - \mathcal{A}U = F, \tag{4.2}$$

with $U = (\varphi, \Phi, \psi, \Psi, \theta, \vartheta)^T$, $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ and $\mathcal{A}$ defined in (3.3). Rewriting it in terms of its components we obtain

$$i\beta\varphi - \Phi = f_1, \tag{4.3}$$
$$i\beta\rho_1\Phi - k(\varphi_x + \psi)_x + m\,\theta_x = \rho_1 f_2, \tag{4.4}$$
$$i\beta\psi - \Psi = f_3, \tag{4.5}$$
$$i\beta\rho_2\Psi - b\,\psi_{xx} + k(\varphi_x + \psi) - m\,\theta + \sigma\,\vartheta_x = \rho_2 f_4, \tag{4.6}$$
$$i\beta\rho_3\theta - c_0\,\theta_{xx} + m(\Phi_x + \Psi) = \rho_3 f_5. \tag{4.7}$$
$$i\beta\rho_4\vartheta - c_1\,\vartheta_{xx} + \sigma\Psi_x = \rho_4 f_6. \tag{4.8}$$

Lemma 4.2. *Under the above notations, we have $i\mathbb{R} \subseteq \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ is resolvent set of the linear operator $\mathcal{A}$ given in (3.3).*

Proof. It is not so difficult to prove that $D(\mathcal{A})$ defined in (3.4)–(3.5) is closed and is compactly embedded in $\mathcal{H}$. For example, note that

$$D(\mathcal{A}) = \begin{cases} \left[\left(H^2 \cap H_0^1\right)(0,l) \times H_0^1(0,l)\right]^2 \times \left[\left(H^2 \cap H_0^1\right)(0,l)\right]^2 & \text{for } (1.6)_{(a)}, \\ H_*^2 \times H_*^1(0,l) \times \left(H^2 \cap H_0^1\right)(0,l) \times H_0^1(0,l) \times \left[\left(H^2 \cap H_0^1\right)(0,l)\right]^2 & \text{for } (1.6)_{(b)}, \end{cases}$$

where we denote $H_*^2 := \{\varphi \in H_*^1(0,l) \mid \varphi_x \in H_0^1(0,l)\}$, which means that such a compactness property is standardly verified for $D(\mathcal{A})$. For the remaining conditions one proceeds similarly. Therefore, according to Proposition 5.8 and Corollary 1.15 in [14], the spectrum $\sigma(\mathcal{A}) = \mathbb{C}\backslash\rho(\mathcal{A})$ has only eigenvalues.

Let us suppose that $\mathcal{A}$ possesses an imaginary eigenvalue $\lambda = i\beta \in \sigma(\mathcal{A})$, $\beta \neq 0$, with corresponding eigenvector $U = (\varphi, \Phi, \psi, \Psi, \theta, \vartheta)^T \neq 0$. From (3.6) and (4.2) with $F = 0$, we get

$$c_0 \int_0^l |\theta_x|^2 \,\mathrm{d}x + c_1 \int_0^l |\vartheta_x|^2 \,\mathrm{d}x = 0,$$

from where it follows that $\theta, \vartheta \equiv 0$. Returning to Eqs. (4.3)–(4.8) with $F = 0$, we get $\Phi, \varphi \equiv 0$ and then $\Psi, \psi \equiv 0$, which implies that $U \equiv 0$. But this contradicts the fact that $U \neq 0$ is an eigenvector. Hence, there are no purely imaginary eigenvalues in the spectrum $\sigma(\mathcal{A}) = \mathbb{C}\backslash\rho(\mathcal{A})$, that is, $i\mathbb{R} \subseteq \rho(\mathcal{A})$. □

In what follows, we use the well-known Hölder and Poincaré inequalities several times without mentioning them constantly. Moreover, we denote by $C > 0$ different constants appearing in the estimates and take $|\beta| > 1$ large enough without loss of generality.

Lemma 4.3. *Under the above notations, there exists a constant* $C > 0$ *such that*

$$\|\theta_x\|_{L^2}^2 + \|\vartheta_x\|_{L^2}^2 \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{4.9}$$

Proof. From (3.6) and (4.2) we obtain

$$c_0 \int_0^l |\theta_x|^2 \, \mathrm{d}x + c_1 \int_0^l |\vartheta_x|^2 \, \mathrm{d}x = \mathrm{Re}\,(U, F)_{\mathcal{H}},$$

from where we obtain (4.9). □

In the next results, since we are dealing with several different boundary conditions, we will need to avoid different estimates provided by boundary point-wise terms. In this way, we will first obtain local estimates by using auxiliary cut-off functions. Indeed, let us consider $l_0 \in (0, l)$ and $\delta > 0$ arbitrary numbers such that $(l_0 - \delta, l_0 + \delta) \subset (0, l)$, and a function $s \in C^2(0, l)$ satisfying

$$\mathrm{supp}\, s \subset (l_0 - \delta, l_0 + \delta), \quad 0 \le s(x) \le 1, \quad x \in (0, l), \tag{4.10}$$

and

$$s(x) = 1 \quad \text{for} \quad x \in [l_0 - \delta/2, l_0 + \delta/2]. \tag{4.11}$$

The next lemma can be proved analogously to [3, Proposition 3.3]. For the sake of completeness, we are going to provide the idea of the proof.

Lemma 4.4. *Under the above notations given* $\epsilon > 0$ *there exists a constant* $C_\epsilon > 0$ *such that*

$$\int_{l_0-\delta/2}^{l_0+\delta/2} \left(|\varphi_x + \psi|^2 + |\Phi|^2\right) \mathrm{d}x \le \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2. \tag{4.12}$$

Proof. From expressions (4.3), (4.5) and (4.7), we obtain

$$i\beta\rho_3\theta - c_0\,\theta_{xx} + i\beta m(\varphi_x + \psi) = \rho_3 f_5 + m(f_{1,x} + f_3). \tag{4.13}$$

Taking the multiplier $sk\overline{[\varphi_x + \psi]}$ in (4.13) and performing integration by parts, we have

$$
\begin{aligned}
i\beta k m \int_0^l s|\varphi_x+\psi|^2\,\mathrm{d}x = & \underbrace{-c_0\int_0^l s\,\theta_x\overline{[k(\varphi_x+\psi)_x]}\,\mathrm{d}x}_{:=I_1} \\
& + \underbrace{k\rho_3\int_0^l s\theta\overline{[i\beta(\varphi_x+\psi)]}\,\mathrm{d}x}_{:=I_2} \\
& - kc_0\int_0^l s'\,\theta_x\overline{[\varphi_x+\psi]}\,\mathrm{d}x \\
& + k\int_0^l s\left[\rho_3 f_5+m(f_{1,x}+f_3)\right]\overline{[\varphi_x+\psi]}\,\mathrm{d}x. \qquad (4.14)
\end{aligned}
$$

Using (4.4), one can see that

$$
I_1 = i\beta c_0\rho_1\int_0^l s\,\theta_x\overline{\Phi}\,\mathrm{d}x - c_0 m\int_0^l s\,|\theta_x|^2\,\mathrm{d}x + c_0\rho_1\int_0^l s\theta_x\overline{f_2}\,\mathrm{d}x.
$$

In addition, applying (4.3), (4.5), and integration by parts, we obtain

$$
I_2 = -k\rho_3\int_0^l [s\,\theta]_x\overline{\Phi}\,\mathrm{d}x + k\rho_3\int_0^l s\,\theta\overline{\Psi}\,\mathrm{d}x + k\rho_3\int_0^l s\,\theta\overline{[f_{1,x}+f_3]}\,\mathrm{d}x.
$$

Replacing these two last identities in (4.14) we deduce that

$$
i\beta k m\int_0^l s|\varphi_x+\psi|^2\,\mathrm{d}x \;=\; i\beta c_0\rho_1\int_0^l s\theta_x\overline{\Phi}\,\mathrm{d}x + I_3, \qquad (4.15)
$$

where we set

$$
\begin{aligned}
I_3 = & -c_0 m\int_0^l s\,|\theta_x|^2\,\mathrm{d}x - kc_0\int_0^l s'\,\theta_x\overline{[\varphi_x+\psi]}\,\mathrm{d}x - k\rho_3\int_0^l [s\,\theta]_x\overline{\Phi}\,\mathrm{d}x \\
& + k\rho_3\int_0^l s\,\theta\overline{\Psi}\,\mathrm{d}x \\
& + k\rho_3\int_0^l s\,\theta\overline{[f_{1,x}+f_3]}\,\mathrm{d}x + c_0\rho_1\int_0^l s\theta_x\overline{f_2}\,\mathrm{d}x \\
& + k\int_0^l s\left[\rho_3 f_5+m(f_{1,x}+f_3)\right]\overline{[\varphi_x+\psi]}\,\mathrm{d}x.
\end{aligned}
$$

From estimate (4.9) and keeping in mind the definition of the norm in $\mathcal{H}$, we infer

$$
|I_3| \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}},
$$

for some constant $C>0$. Going back to expression (4.15) and using condition (4.10) on function s, we arrive at

$$
\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x+\psi|^2\,\mathrm{d}x \le\; C\,\|\theta_x\|_{L^2}\|\Phi\|_{L^2} + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}
$$

$$+\ \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\beta|}\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}}. \tag{4.16}$$

Applying Young inequality and estimate (4.9), we conclude

$$\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x+\psi|^2\,\mathrm{d}x\ \leq\ \frac{\epsilon}{2}\|U\|_{\mathcal{H}}^2+C_\epsilon\|F\|_{\mathcal{H}}^2. \tag{4.17}$$

On the other hand, taking the multiplier $-s\overline{\varphi}$ in (4.4), performing integration by parts and applying (4.3), we get

$$\rho_1\int_0^l s|\Phi|^2\,\mathrm{d}x=k\int_0^l s|\varphi_x+\psi|^2\,\mathrm{d}x-k\int_0^l s(\varphi_x+\psi)\overline{\psi}\,\mathrm{d}x+I_4+I_5, \tag{4.18}$$

where

$$I_4=\frac{i}{\beta}m\int_0^l s\,\theta_x\overline{[\Phi+f_1]}\,\mathrm{d}x-\rho_1\int_0^l s[\Phi\overline{f_1}+f_2\overline{\varphi}]\,\mathrm{d}x\quad\text{and}$$

$$I_5=k\int_0^l s'(\varphi_x+\psi)\overline{\varphi}\,\mathrm{d}x.$$

It is easy to see that

$$|I_4|\leq\frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}+\frac{C}{|\beta|}\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},$$

for some constant $C>0$. In addition, from Eqs. (4.3) and (4.5), it follows that

$$|\mathrm{Re}\ I_5|\leq\frac{C}{|\beta|^2}\|U\|_{\mathcal{H}}^2+\frac{C}{|\beta|^2}\|F\|_{\mathcal{H}}^2.$$

Taking the real part in (4.18) and observing that supp $s\subset(l_0-\delta,l_0+\delta)$, we have

$$\begin{aligned}\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2\,\mathrm{d}x\leq\ & C\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x\\
&+\ \psi|^2\,\mathrm{d}x+C\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x+\psi||\psi|\,\mathrm{d}x+\frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}\\
&+\ \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\beta|^2}\|U\|_{\mathcal{H}}^2+\frac{C}{|\beta|^2}\|F\|_{\mathcal{H}}^2\\
\leq\ & C\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x+\psi|^2\,\mathrm{d}x+C\left(\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x+\psi|^2\,\mathrm{d}x\right)^{1/2}\|\psi\|_{L^2}\\
&+\ \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}\\
&+\ \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\beta|^2}\|U\|_{\mathcal{H}}^2+\frac{C}{|\beta|^2}\|F\|_{\mathcal{H}}^2.\end{aligned}$$

From Young's inequality, we get

$$\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2\,\mathrm{d}x\leq\ C\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x+\psi|^2\,\mathrm{d}x+\frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}+\frac{C}{|\beta|}\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}}$$

$$+ \; C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}}.$$

Using estimates (4.16) and (4.9), and Young's inequality once more, we obtain

$$\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2 \, \mathrm{d}x \;\le\; \frac{\epsilon}{2}\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}. \tag{4.19}$$

Therefore, estimates (4.17) and (4.19) lead to

$$\int_{l_0-\delta}^{l_0+\delta} s\left(|\varphi_x+\psi|^2+|\Phi|^2\right)\mathrm{d}x \;\le\; \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}},$$

and observing conditions (4.10)–(4.11) on s, we finally conclude estimate (4.12). □

To the next lemma, we use similar tools as in [7, Lemma 2.10]. We also provide the main idea of the proof for the sake of the reader.

Lemma 4.5. *Under the above notations given $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$\int_{l_0-\delta/2}^{l_0+\delta/2}\left(|\psi_x|^2+|\Psi|^2\right)\mathrm{d}x \le \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}. \tag{4.20}$$

Proof. On the one hand, from expressions (4.5) and (4.8) we obtain

$$\rho_4\,\vartheta - \frac{c_1}{i\beta}\,\vartheta_{xx} + \sigma\,\psi_x = \frac{\sigma}{i\beta}\,f_{3,x} + \frac{\rho_4}{i\beta}\,f_6. \tag{4.21}$$

Taking the multiplier $bs\,\overline{\psi_x}$ in (4.21) and performing integration by parts we get

$$\begin{aligned}\sigma\,b\int_0^l s|\psi_x|^2\,\mathrm{d}x \;=\; & b\,\rho_4\int_0^l s\vartheta_x\overline{\psi}\,\mathrm{d}x + b\,\rho_4\int_0^l s'\vartheta\overline{\psi}\,\mathrm{d}x\\ & - \frac{b\,c_1}{i\beta}\int_0^l s'\vartheta_x\overline{\psi_x}\,\mathrm{d}x \underbrace{-\frac{b\,c_1}{i\beta}\int_0^l s\vartheta_x\overline{\psi_{xx}}\,\mathrm{d}x}_{:=J_1}\\ & + \frac{\sigma\,b}{i\beta}\int_0^l sf_{3,x}\overline{\psi_x}\,\mathrm{d}x + \frac{\rho_4\,b}{i\beta}\int_0^l sf_6\overline{\psi_x}\,\mathrm{d}x.\end{aligned} \tag{4.22}$$

Using (4.6) we rewrite J_1 as

$$\begin{aligned}J_1 \;=\; & c_1\,\rho_2\int_0^l s\vartheta_x\overline{\Psi}\,\mathrm{d}x - \frac{k\,c_1}{i\beta}\int_0^l s\vartheta_x(\overline{\varphi_x+\psi})\,\mathrm{d}x + \frac{m\,c_1}{i\beta}\int_0^l s\vartheta_x\overline{\theta}\,\mathrm{d}x\\ & - \frac{\sigma\,c_1}{i\beta}\int_0^l s|\vartheta_x|^2\,\mathrm{d}x + \frac{c_1\,\rho_2}{i\beta}\int_0^l s\vartheta_x\overline{f_4}\,\mathrm{d}x,\end{aligned}$$

and replacing it in (4.22) we deduce that

$$\sigma\,b\int_0^l s|\psi_x|^2\,\mathrm{d}x \;=\; c_1\,\rho_2\int_0^l s\vartheta_x\overline{\Psi}\,\mathrm{d}x + J_2, \tag{4.23}$$

where

$$J_2 = b\,\rho_4 \int_0^l s\vartheta_x \overline{\psi}\,\mathrm{d}x + b\,\rho_4 \int_0^l s'\vartheta\overline{\psi}\,\mathrm{d}x - \frac{b\,c_1}{i\beta}\int_0^l s'\vartheta_x\overline{\psi_x}\,\mathrm{d}x - \frac{k\,c_1}{i\beta}\int_0^l s\vartheta_x(\overline{\varphi_x+\psi})\,\mathrm{d}x + \frac{m\,c_1}{i\beta}\int_0^l s\vartheta_x\overline{\theta}\,\mathrm{d}x - \frac{\sigma\,c_1}{i\beta}\int_0^l s|\vartheta_x|^2\,\mathrm{d}x + \frac{c_1\,\rho_2}{i\beta}\int_0^l s\vartheta_x\overline{f_4}\,\mathrm{d}x + \frac{\sigma\,b}{i\beta}\int_0^l sf_{3,x}\overline{\psi_x}\,\mathrm{d}x + \frac{b\,\rho_4}{i\beta}\int_0^l sf_6\overline{\psi_x}\,\mathrm{d}x.$$

From estimate (4.9) and Young inequality, one can see that

$$|J_2| \le C\|\vartheta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}},$$

for some constant $C > 0$. Returning to (4.23) and using definition of s we derive

$$\int_{l_0-\delta}^{l_0+\delta} s|\psi_x|^2\,\mathrm{d}x \le C\|\vartheta_x\|_{L^2}\|\Psi\|_{L^2} + C\|\vartheta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}. \tag{4.24}$$

Using Young's inequality and (4.9) once again we arrive at

$$\int_{l_0-\delta}^{l_0+\delta} s|\psi_x|^2\,\mathrm{d}x \le \frac{\epsilon}{2}\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}. \tag{4.25}$$

On the other hand, taking the multiplier $-s\overline{\psi}$ in (4.6), using integration by parts and Eq. (4.5), we have

$$\rho_2\int_0^l s|\Psi|^2\,\mathrm{d}x = b\int_0^l s|\psi_x|^2\,\mathrm{d}x + J_3, \tag{4.26}$$

where

$$J_3 = -\rho_2\int_0^l s\Psi\overline{f_3}\,\mathrm{d}x + b\int_0^l s'\psi_x\overline{\psi}\,\mathrm{d}x + k\int_0^l s\varphi_x\overline{\psi}\,\mathrm{d}x - m\int_0^l s\theta\overline{\psi}\,\mathrm{d}x + \sigma\int_0^l s\vartheta_x\overline{\psi}\,\mathrm{d}x - \rho_2\int_0^l sf_4\overline{\psi}\,\mathrm{d}x + k\int_0^l s|\psi|^2\,\mathrm{d}x.$$

From Eqs. (4.3) and (4.5) it follows that

$$|J_3| \le \frac{C}{|\beta|}\|\psi_x\|_{L^2}\|U\|_{\mathcal{H}} + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + \frac{C}{|\beta|}\|\vartheta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}},$$

for some constant $C > 0$. Using Young's inequality, (4.9) and that $|\beta| > 1$, we obtain

$$|J_3| \le \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Replacing the above estimate in (4.26), using (4.24) and applying Young's inequality once more, results

$$\int_{l_0-\delta}^{l_0+\delta} s|\Psi|^2\,\mathrm{d}x \;\le\; \frac{\epsilon}{2}\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2. \tag{4.27}$$

Hence, adding (4.25) and (4.27), and using assumptions (4.10)–(4.11) on s, we conclude (4.20). □

4.1. Observability inequality and main result

Now we state an observability inequality for Timoshenko systems of conservative type which was first proved by Muñoz Rivera and Ávila [30] and improved by Alves et al. [4]. Let us consider the system

$$i\beta u - v = g_1 \quad \text{in} \quad (0,l), \tag{4.28}$$

$$i\beta\rho_1 v - k(u_x + w)_x = g_2 \quad \text{in} \quad (0,l), \tag{4.29}$$

$$i\beta w - z = g_3 \quad \text{in} \quad (0,l), \tag{4.30}$$

$$i\beta\rho_2 z - b\,w_{xx} + k(u_x + w) = g_4 \quad \text{in} \quad (0,l), \tag{4.31}$$

where $g_1, g_3 \in H_0^1(0,l)$ (or $H_*^1(0,l)$), $g_2, g_4 \in L^2(0,l)$. We denote by V and G the vector-valued functions $V = (u, v, w, z)^T$ and $G = (g_1, g_2, g_3, g_4)^T$, respectively. Besides, given any $a_1, a_2 \in \mathbb{R}$ with $0 \le a_1 < a_2 \le l$, the notations $\|\cdot\|_{a_1,a_2}$ and $\mathcal{I}(\cdot)$ stand for

$$\|V\|_{a_1,a_2}^2 := \int_{a_1}^{a_2} \left(|u_x + w|^2 + |v|^2 + |w_x|^2 + |z|^2\right)\,\mathrm{d}x,$$

$$\mathcal{I}(a_j) := |u_x(a_j) + w(a_j)|^2 + |v(a_j)|^2 + |w_x(a_j)|^2 + |z(a_j)|^2, \quad j = 1, 2.$$

Proposition 4.6. *Under the above notations, let us consider a strong solution* $V = (u, v, w, z)^T$ *of* (4.28)–(4.31) *and any* $0 \le a_1 < a_2 \le l$. *Then there exist constants* $C_0,\ C_1 > 0$ *such that*

$$\mathcal{I}(a_j) \le C_0\|V\|_{a_1,a_2}^2 + C_0\|G\|_{0,l}^2, \quad j = 1, 2, \tag{4.32}$$

$$\|V\|_{a_1,a_2}^2 \le C_1\mathcal{I}(a_j) + C_1\|G\|_{0,l}^2, \quad j = 1, 2. \tag{4.33}$$

Proof. See [30, Lemma 3.2] or [4, Proposition 3.13]. □

Corollary 4.7. *Let* $V = (u, v, w, z)^T$ *be a strong solution of the system* (4.28)–(4.31). *If for some sub-interval* $(a_1, a_2) \subset (0, l)$ *we have*

$$\|V\|_{a_1,a_2}^2 \le \Lambda, \tag{4.34}$$

then there exists a (uniform) constant $C > 0$ *such that*

$$\|V\|_{0,l}^2 \le C\Lambda + C\|G\|_{0,l}^2. \tag{4.35}$$

Proof. See [4, Corollary 3.14] or [3, Corollary 3.8]. □

From the above results, we have finally gathered all tools needed to state and prove our main result on exponential stability. It reads as follows:

Theorem 4.8. *Under the above notations, there exist constants $C, \gamma > 0$ independent of $U_0 \in \mathcal{H}$ such that the semigroup solution $U(t) = e^{\mathcal{A}t}U_0$ satisfies*

$$\|U(t)\|_{\mathcal{H}} \le Ce^{-\gamma t}\|U_0\|_{\mathcal{H}}, \quad t > 0. \tag{4.36}$$

Proof. Let $\epsilon > 0$ be given. From (4.12) and (4.20) we have

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} \left(|\varphi_x+\psi|^2+|\Phi|^2+|\psi_x|^2+|\Psi|^2\right) \mathrm{d}x \ \le\ \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2 := \Lambda,$$

for some constant $C_\epsilon > 0$. In view of (4.3)–(4.6) the function $V := (\varphi, \Phi, \psi, \Psi)^T$ is a solution of (4.28)–(4.31) with

$$g_1 := f_1, \quad g_2 := \rho_1 f_2 - m\,\theta_x, \quad g_3 = f_3, \quad g_4 = \rho_2 f_4 + m\,\theta - \sigma\vartheta_x,$$

and (4.34) is verified with $a_1 = l_0 - \delta/2$ and $a_2 = l_0 + \delta/2$, then Corollary 4.7, Lemma 4.3 and Young's inequality imply that

$$\int_0^l \left(|\varphi_x+\psi|^2+|\Phi|^2+|\psi_x|^2+|\Psi|^2\right) \mathrm{d}x \ \le\ \epsilon C\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2, \tag{4.37}$$

for some constants $C,\ C_\epsilon > 0$. Combining (4.9) and (4.37), we infer

$$\|U\|_{\mathcal{H}}^2 \le \epsilon C\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2, \quad |\beta| > 1 \text{ is large enough.}$$

Taking $\epsilon > 0$ small enough and regarding the resolvent Eq. (4.2), we conclude

$$\|(i\beta I_d - \mathcal{A})^{-1}F\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}, \quad |\beta| \to \infty. \tag{4.38}$$

From (4.38) and Lemma 4.2, we finally obtain property (4.1). Hence, the exponential stability (4.36) follows from Theorem 4.1, which completes the proof of Theorem 4.8. □

5. The non-homogeneous thermoelastic system

In this section, we are going to extend the main result on exponential stability for the thermoelastic system (1.1)–(1.4) (see Theorem 4.8) to the case of non-constant coefficient (here called "non-homogeneous system"). In this way, we consider the following system

$$\rho_1(x)\varphi_{tt} - [k(x)(\varphi_x+\psi)]_x + [m(x)\theta]_x = 0 \quad \text{in} \quad (0,\, l)\times(0,\, \infty), \tag{5.1}$$

$$\rho_2(x)\psi_{tt} - [b(x)\psi_x]_x + k(x)(\varphi_x + \psi) - m(x)\theta + [\sigma(x)\vartheta]_x = 0 \quad \text{in} \quad (0,\, l) \times (0,\, \infty), \tag{5.2}$$

$$\rho_3(x)\theta_t - [c_0(x)\theta_x]_x + m(x)(\varphi_{xt} + \psi_t) = 0 \quad \text{in} \quad (0,\, l) \times (0,\, \infty), \tag{5.3}$$

$$\rho_4(x)\vartheta_t - [c_1(x)\vartheta_x]_x + \sigma(x)\psi_{xt} = 0 \quad \text{in} \quad (0,\, l) \times (0,\, \infty), \tag{5.4}$$

with initial conditions

$$\begin{aligned} &\varphi(\cdot, 0) = \varphi_0,\ \varphi_t(\cdot, 0) = \varphi_1,\ \psi(\cdot, 0) = \psi_0,\ \psi_t(\cdot, 0) = \psi_1, \\ &\theta(\cdot, 0) = \theta_0,\ \vartheta(\cdot, 0) = \vartheta_0, \end{aligned} \tag{5.5}$$

and only Dirichlet boundary conditions—in order to facilitate notations—given by

$$\varphi(0, t) = \varphi(l, t) = \psi(0, t) = \psi(l, t) = \theta(0, t) = \theta(l, t) = \vartheta(0, t) = \vartheta(l, t) = 0. \tag{5.6}$$

Here, we assume that $\rho_1, \rho_2, \rho_3, \rho_4, k, b, c_0, c_1, m, \sigma$ are functions satisfying

$$\begin{aligned} &\rho_1, \rho_2, \rho_3, \rho_4, k, b, c_0, c_1, m, \sigma \in W^{1,\infty}(0, l), \\ &\rho_1, \rho_2, \rho_3, \rho_4, k, b, c_0, c_1, m, \sigma > 0 \text{ in } (0, l). \end{aligned} \tag{5.7}$$

It is worth mentioning that problem (5.1)–(5.6) has the same characteristic as (1.1)–(1.6)$_{(a)}$, being a generalized mathematical case with non-constant coefficients satisfying (5.7). Thus, the essential computations keep unchanged and the result on exponential stability (Theorem 4.8) remains unaltered. Moreover, we also observe that Proposition 4.6 and Corollary 4.7 still hold for systems with non-constant coefficients, see e.g., [4,7].

5.1. Semigroup solution

We consider the standard phase space

$$\mathcal{H} = H_0^1(0, l) \times L^2(0, l) \times H_0^1(0, l) \times L^2(0, l) \times L^2(0, l) \times L^2(0, l)$$

equipped with the same norm as defined in (3.1). In this case, we can also rewrite problem (5.1)–(5.6) as

$$\begin{cases} U_t = \mathcal{A}U, \quad t > 0, \\ U(0) := U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \vartheta_0), \end{cases} \tag{5.8}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is now given by

$$\mathcal{A}U := \begin{pmatrix} \Phi \\ \frac{1}{\rho_1}[k(\varphi_x + \psi)]_x - \frac{1}{\rho_1}[m\theta]_x \\ \Psi \\ \frac{1}{\rho_2}[b\psi_x]_x - \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{m}{\rho_2}\theta - \frac{1}{\rho_2}[\sigma\vartheta]_x \\ \frac{1}{\rho_3}[c_0\theta_x]_x - \frac{m}{\rho_3}(\Phi_x + \Psi) \\ \frac{1}{\rho_4}[c_1\vartheta_x]_x - \frac{\sigma}{\rho_4}\Psi_x \end{pmatrix}, \tag{5.9}$$

for all $U = (\varphi, \Phi, \psi, \Psi, \theta, \vartheta)$ in the domain

$$D(\mathcal{A}) = \left[\left(H^2 \cap H_0^1\right)(0,l) \times H_0^1(0,l)\right]^2 \times \left[\left(H^2 \cap H_0^1\right)(0,l)\right]^2.$$

Under the above notations, one can easily prove that operator $\mathcal{A}$ defined in (5.9) is dissipative in $\mathcal{H}$ with

$$\mathrm{Re}(\mathcal{A}U, U)_{\mathcal{H}} = -\int_0^l c_0(x)|\theta_x(x)|^2 \,\mathrm{d}x - \int_0^l c_1(x)|\vartheta_x(x)|^2 \,\mathrm{d}x \le 0, \quad (5.10)$$

for all $U \in D(\mathcal{A})$. Therefore, the existence and uniqueness result for (5.8) can be stated analogously to Theorem 3.1. In summary, under the assumption (5.7), system (5.1)–(5.6) is well-posed through the semigroup theory.

5.2. Exponential stability

In the present case of non-homogeneous coefficients, our main stability result reads similarly as Theorem 4.8, namely:

Theorem 5.1. *Under the above notations and assumption* (5.7)*, there exist constants* $C, \gamma > 0$, *independent of* $U_0 \in \mathcal{H}$, *such that the semigroup solution* $U(t) = e^{\mathcal{A}t}U_0$ *for* (5.8) *satisfies*

$$\|U(t)\|_{\mathcal{H}} \le Ce^{-\gamma t}\|U_0\|_{\mathcal{H}}, \quad t > 0. \quad (5.11)$$

In other words, the non-homogeneous thermoelastic Timoshenko problem (5.1)–(5.6) *is exponentially stable.*

Proof. The proof of Theorem 5.1 follows the same patterns as provided along the whole Sect. 4 to the proof of Theorem 4.8. Therefore, we omit here for simplicity. □

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