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# Energy scattering for a class of the defocusing inhomogeneous nonlinear Schrödinger equation

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*Abstract.* In this paper, we consider a class of the defocusing inhomogeneous nonlinear Schrödinger equation

$$i\partial_t u + \Delta u - |x|^{-b}|u|^\alpha u = 0, \quad u(0) = u_0 \in H^1,$$

with $b, \alpha > 0$. We first study the decaying property of global solutions for the equation when $0 < \alpha < \alpha^\star$ where $\alpha^\star = \frac{4-2b}{d-2}$ for $d \geq 3$. The proof makes use of an argument of Visciglia (Math Res Lett 16(5):919–926, 2009). We next use this decay to show the energy scattering for the equation in the case $\alpha_\star < \alpha < \alpha^\star$, where $\alpha_\star = \frac{4-2b}{d}$.

## 1. Introduction

Consider the Cauchy problem for the inhomogeneous nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u + \mu|x|^{-b}|u|^\alpha u = 0, \\ \qquad\qquad u(0) = u_0, \end{cases} \tag{INLS}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $u_0 : \mathbb{R}^d \to \mathbb{C}$, $\mu = \pm 1$ and $\alpha, b > 0$. The parameters $\mu = 1$ and $\mu = -1$ correspond to the focusing and defocusing cases, respectively. The case $b = 0$ is the well-known nonlinear Schrödinger equation which has been studied extensively over the last three decades. In the end of the last century, it was suggested that stable high power propagation can be achieved in a plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel (see [14] and [17]). In this situation, the beam propagation can be modeled by the inhomogeneous nonlinear Schrödinger equation of the form

$$i\partial_t u + \Delta u + K(x)|u|^\alpha u = 0. \tag{1.1}$$

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The (INLS) is a particular case of (1.1) with $K(x) = |x|^{-b}$. Equation (1.1) has been attracted a lot of interest in a past several years. Bergé in [1] studied formally the stability condition for soliton solutions of (1.1). Towers–Malomed in [24] observed by means of variational approximation and direct simulations that a certain type of time-dependent nonlinear medium gives rise to completely stable beams. Merle in [19] and Raphaël–Szeftel in [21] studied (1.1) for $k_1 < K(x) < k_2$ with $k_1, k_2 > 0$. Fibich–Wang in [11] investigated (1.1) with $K(x) := K(\epsilon|x|)$ where $\epsilon > 0$ is small and $K \in C^4(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The case $K(x) = |x|^b$ with $b > 0$ is studied by many authors (see e.g. [3,18] and [26] and references therein).

Before reviewing known results for the (INLS), we recall some facts for this equation. We first note that the (INLS) is invariant under the scaling

$$u_\lambda(t,x) := \lambda^{\frac{2-b}{\alpha}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

An easy computation shows

$$\|u_\lambda(0)\|_{\dot{H}^\gamma(\mathbb{R}^d)} = \lambda^{\gamma + \frac{2-b}{\alpha} - \frac{d}{2}} \|u_0\|_{\dot{H}^\gamma(\mathbb{R}^d)}.$$

The critical Sobolev exponent is thus defined by

$$\gamma_{\mathrm{c}} := \frac{d}{2} - \frac{2-b}{\alpha}. \tag{1.2}$$

Moreover, the (INLS) has the following conserved quantities:

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 \mathrm{d}x = M(u_0), \tag{1.3}$$

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t,x)|^2 - \frac{\mu}{\alpha+2}|x|^{-b}|u(t,x)|^{\alpha+2}\mathrm{d}x = E(u_0). \tag{1.4}$$

The well-posedness for the (INLS) was first studied by Genoud–Stuart in [12, Appendix] by using an argument of Cazenave [2, Chapter 3] which does not use Strichartz estimates. More precisely, they showed that the focusing (INLS) with $0 < b < \min\{2, d\}$ is well-posed in $H^1(\mathbb{R}^d)$:

- locally if $0 < \alpha < \alpha^\star$,
- globally for any initial data if $0 < \alpha < \alpha_\star$,
- globally for small initial data if $\alpha_\star \leq \alpha < \alpha^\star$,

where $\alpha_\star$ and $\alpha^\star$ are defined by

$$\alpha_\star := \frac{4-2b}{d}, \quad \alpha^\star := \begin{cases} \frac{4-2b}{d-2} & \text{if } d \geq 3, \\ \infty & \text{if } d = 1, 2. \end{cases} \tag{1.5}$$

In the case $\alpha = \alpha_\star$ ($L^2$-critical), Genoud in [13] showed that the focusing (INLS) with $0 < b < \min\{2, d\}$ is globally well-posed in $H^1(\mathbb{R}^d)$ assuming $u_0 \in H^1(\mathbb{R}^d)$ and

$$\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)},$$

where $Q$ is the unique nonnegative, radially symmetric, decreasing solution of the ground state equation

$$\Delta Q - Q + |x|^{-b}|Q|^{\frac{4-2b}{d}} Q = 0.$$

Also, Combet–Genoud in [5] established the classification of minimal mass blow-up solutions for the focusing $L^2$-critical (INLS).

In the case $\alpha_\star < \alpha < \alpha^\star$, Farah in [8] showed that the focusing (INLS) with $0 < b < \min\{2, d\}$ is globally well-posedness in $H^1(\mathbb{R}^d)$ by assuming $u_0 \in H^1(\mathbb{R}^d)$ and

$$\begin{aligned} E(u_0)^{\gamma_c} M(u_0)^{1-\gamma_c} &< E(Q)^{\gamma_c} M(Q)^{1-\gamma_c}, \\ \|\nabla u_0\|^{\gamma_c}_{L^2(\mathbb{R}^d)} \|u_0\|^{1-\gamma_c}_{L^2(\mathbb{R}^d)} &< \|\nabla Q\|^{\gamma_c}_{L^2(\mathbb{R}^d)} \|Q\|^{1-\gamma_c}_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{1.6}$$

where $Q$ is the unique nonnegative, radially symmetric, decreasing solution of the ground state equation

$$\Delta Q - Q + |x|^{-b}|Q|^{\alpha} Q = 0.$$

He also proved that if $u_0 \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |x|^2 \mathrm{d}x) =: \Sigma$ satisfies (1.6) and

$$\|\nabla u_0\|^{\gamma_c}_{L^2(\mathbb{R}^d)} \|u_0\|^{1-\gamma_c}_{L^2(\mathbb{R}^d)} > \|\nabla Q\|^{\gamma_c}_{L^2(\mathbb{R}^d)} \|Q\|^{1-\gamma_c}_{L^2(\mathbb{R}^d)}, \tag{1.7}$$

then the corresponding solution blows up in finite time. Afterward, Farah–Guzman in [9] and [10] proved that the above global solution scatters in $H^1(\mathbb{R}^d)$ under the radial condition of the initial data.

In [7], the author showed the existence of finite time blow-up $H^1$-solutions for the focusing $L^2$-critical and $L^2$-supercritical (INLS).

Guzman in [16] used Strichartz estimates and the contraction mapping argument to establish the well-posedness for the (INLS) in Sobolev spaces. Precisely, he showed that:

- if $0 < \alpha < \alpha_\star$ and $0 < b < \min\{2, d\}$, then the (INLS) is locally well-posed in $L^2(\mathbb{R}^d)$. It is then globally well-posed in $L^2(\mathbb{R}^d)$ by the mass conservation.
- if $0 < \alpha < \widetilde{\alpha}$, $0 < b < \widetilde{b}$ and $\max\{0, \gamma_c\} < \gamma \leq \min\left\{\frac{d}{2}, 1\right\}$ where

$$\widetilde{\alpha} := \begin{cases} \frac{4-2b}{d-2\gamma} & \text{if } \gamma < \frac{d}{2}, \\ \infty & \text{if } \gamma = \frac{d}{2} \end{cases} \quad \text{and} \quad \widetilde{b} := \begin{cases} \frac{d}{3} & \text{if } d = 1, 2, 3, \\ 2 & \text{if } d \geq 4, \end{cases} \tag{1.8}$$

  then the (INLS) is locally well-posed in $H^\gamma(\mathbb{R}^d)$.
- if $\alpha_\star < \alpha < \widetilde{\alpha}$, $0 < b < \widetilde{b}$ and $\gamma_c < \gamma \leq \min\left\{\frac{d}{2}, 1\right\}$, then the (INLS) is globally well-posed in $H^\gamma(\mathbb{R}^d)$ for small initial data.

In particular, Guzman proved the following local well-posedness in the energy space for the (INLS).

THEOREM 1. [16] *Let* $d \geq 2, 0 < b < \widetilde{b}$ *and* $0 < \alpha < \alpha^\star$. *Then the* (INLS) *is locally well-posed in* $H^1(\mathbb{R}^d)$. *Moreover, the global solutions to the defocusing* (INLS) *satisfy* $u \in L^p_{\mathrm{loc}}(\mathbb{R}, W^{1,q}(\mathbb{R}^d))$ *for any Schrödinger admissible pair* $(p, q)$.

Recently, the author in [6] improved the range of $b$ in Theorem 1 in the two and three-dimensional spatial spaces. More precisely, he proved the following:

THEOREM 2. [6] *Let*

$$d \geq 4, \quad 0 < b < 2, \quad 0 < \alpha < \alpha^\star,$$

*or*

$$d = 3, \quad 0 < b < 1, \quad 0 < \alpha < \alpha^\star,$$

*or*

$$d = 3, \quad 1 \leq b < \frac{3}{2}, \quad 0 < \alpha < \frac{6-4b}{2b-1},$$

*or*

$$d = 2, \quad 0 < b < 1, \quad 0 < \alpha < \alpha^\star.$$

*Then the* (INLS) *is locally well-posed in* $H^1(\mathbb{R}^d)$. *Moreover, the global solutions to the defocusing* (INLS) *satisfy* $u \in L^p_{\mathrm{loc}}(\mathbb{R}, W^{1,q}(\mathbb{R}^d))$ *for any Schrödinger admissible pair* $(p, q)$.

The results of Guzman [16] and Dinh [6] about the local well-posedness of (INLS) in $H^1(\mathbb{R}^d)$ are a bit weaker than the one of Genoud–Stuart [12]. Precisely, they do not treat the case $d = 1$, and there is a restriction on the validity of $b$ when $d = 2$ or 3. Note also that the author in [6] pointed out that one cannot expect a similar result as Theorems 1 or 2 holds in the one-dimensional case by using Strichartz estimates. Although the result showed by Genoud–Stuart is strong, but one does not know whether the global solutions to the defocusing (INLS) belong to $L^p_{\mathrm{loc}}(\mathbb{R}, W^{1,q}(\mathbb{R}^d))$ for any Schrödinger admissible pair $(p, q)$. This property plays an important role in proving the energy scattering for the defocusing (INLS).

Note that the local well-posedness (which is also available for the defocusing case) of Genoud–Stuart in [12] and the conservations of mass and energy immediately give the global well-posedness in $H^1(\mathbb{R}^d)$ for the defocusing (INLS). In [6], the author used the pseudo-conformal conservation law to show the decaying property of global solutions by assuming the initial data in $\Sigma$ [see the definition before (1.7)]. In particular, he showed that in the case $\alpha \in [\alpha_\star, \alpha^\star)$, global solutions have the same decay as solutions of the linear Schrödinger equation, that is for $2 \leq q \leq \frac{2d}{d-2}$ when $d \geq 3$ or $2 \leq q < \infty$ when $d = 2$ or $2 \leq q \leq \infty$ when $d = 1$,

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{q}\right)}, \quad \forall t \neq 0.$$

This allows the author to prove the scattering in $\Sigma$ for a certain class of the defocusing (INLS). We refer the reader to [6] for more details.

The main purpose of this paper is to show the energy scattering for the defocusing (INLS). Before stating our results, let us recall the two known methods to prove the energy scattering for the nonlinear Schrödinger equation (NLS). The first one is to use the classical Morawetz inequality to derive the decay of global solutions, and then use it to prove the global Strichartz bound of solutions (see e.g. [15] and [20] or [2]). The second one is to use the interaction Morawetz inequality to derive directly the global Strichartz bound for solutions (see e.g. [4,23] and references therein). With the global Strichartz bound at hand, the energy scattering follows easily. Note also that Visciglia in [25] used the interaction Morawetz inequality to show the decaying property of global solutions for the (NLS) in any dimensions. This approach is a complement to [15] where the classical Morawetz inequality only allowed to prove the decaying property in spatial dimensions greater than or equal to three. It is worth noticing that the (INLS) does not enjoy the conservation of momentum which is crucial to prove the interaction Morawetz-type inequality (see e.g. [4]). We thus do not attempt to show the interaction Morawetz-type inequality for the defocusing (INLS). It is also not clear to us that the techniques of [15] and [20] can be applied for the defocusing (INLS). Fortunately, we are able to use the classical Morawetz-type inequality and an argument of [25] to show the decaying property of global solutions for the defocusing (INLS). More precisely, we have the following decaying property of global solutions to the defocusing (INLS).

THEOREM 3. *Let* $d \geq 3, 0 < b < 2$ *and* $0 < \alpha < \alpha^\star$. *Let* $u_0 \in H^1(\mathbb{R}^d)$ *and* $u \in C(\mathbb{R}, H^1(\mathbb{R}^d))$ *be the unique global solution to the defocusing* (INLS). *Then,*

$$\lim_{t\to\pm\infty} \|u(t)\|_{L^q(\mathbb{R}^d)} = 0, \tag{1.9}$$

*for every* $q \in (2, 2^\star)$, *where* $2^\star := \frac{2d}{d-2}$.

The proof of this result is based on the classical Morawetz-type inequality and an argument of Visciglia in [25]. The classical Morawetz-type inequality related to the defocusing (INLS) is derived by using the same argument of that for the classical (NLS). This inequality is enough to prove the decaying property for global solutions of the defocusing (INLS) by following the technique of [25]. Note that in [25], the author used the interaction Morawetz inequality to show the decay of solutions for the defocusing (NLS) in any dimensions. We expect that the decay (1.9) still holds in dimensions 1 and 2. But it is not clear to us how to prove it at the moment.

Using the decaying property given in Theorem 3, we are able to show the energy scattering for the defocusing (INLS). Due to the singularity of $|x|^{-b}$, the scattering result does not cover the full range of exponents as in Theorem 2. Our main result is the following:

THEOREM 4. *Let*

$$d \geq 4, \quad 0 < b < 2, \quad \alpha_\star < \alpha < \alpha^\star,$$

*or*

$$d = 3, \quad 0 < b < \frac{5}{4}, \quad \alpha_\star < \alpha < 3 - 2b.$$

*Let $u_0 \in H^1(\mathbb{R}^d)$ and $u$ be the unique global solution to the defocusing* (INLS). *Then there exist $u_0^\pm \in H^1(\mathbb{R}^d)$ such that*

$$\lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} u_0^\pm\|_{H^1(\mathbb{R}^d)} = 0.$$

The proof of this result is based on a standard argument as for the nonlinear Schrödinger equation (see e.g. [2, Chapter 7]). Because of the singularity $|x|^{-b}$, one needs to be careful in order to control the nonlinearity in terms of decaying norms and Strichartz norms. The singularity also leads to a restriction on the ranges of $b$ and $\alpha$ compared to those in Theorem 3. We expect that the same result still holds true in the two-dimensional case. This expectation will be possible if one can show the same decay as in Theorem 3 in 2D.

The plan of this paper is as follows. In Sect. 2, we introduce some notations and give some preliminary results related to our problem. We also derive classical Morawetz-type inequalities for the defocusing (INLS) in Sect. 2. The proof of the decaying property of Theorem 3 is given in Sect. 3. Section 4 is devoted to the proof of the scattering result of Theorem 4.

## 2. Preliminaries

In the sequel, the notation $A \lesssim B$ denotes an estimate of the form $A \leq CB$ for some constant $C > 0$. The constant $C$ may change from line to line.

### 2.1. Nonlinearity

Let $F(x,z) := |x|^{-b} f(z)$ with $b > 0$ and $f(z) := |z|^\alpha z$. The complex derivatives of $f$ are

$$\partial_z f(z) = \frac{\alpha+2}{2}|z|^\alpha, \quad \partial_{\bar{z}} f(z) = \frac{\alpha}{2}|z|^{\alpha-2} z^2.$$

We have for $z, w \in \mathbb{C}$,

$$f(z) - f(w) = \int_0^1 \Big( \partial_z f(w + \theta(z-w))(z-w) + \partial_{\bar{z}} f(w + \theta(z-w))\overline{z-w} \Big) d\theta.$$

Thus,

$$|F(x,z) - F(x,w)| \lesssim |x|^{-b}(|z|^\alpha + |w|^\alpha)|z-w|. \tag{2.1}$$

To deal with the singularity $|x|^{-b}$, we have the following remark.

REMARK 1. [16] Let $B := B(0,1) = \{x \in \mathbb{R}^d : |x| < 1\}$ and $B^c := \mathbb{R}^d \backslash B$. Then

$$\||x|^{-b}\|_{L^\gamma(B)} < \infty \quad \text{if} \quad \frac{d}{\gamma} > b,$$

and

$$\||x|^{-b}\|_{L^\gamma(B^c)} < \infty \quad \text{if} \quad \frac{d}{\gamma} < b.$$

2.2. Strichartz estimates

Let $J \subset \mathbb{R}$ and $p, q \in [1, \infty]$. We define the mixed norm

$$\|u\|_{L_t^p(J, L_x^q)} := \Big( \int_J \Big( \int_{\mathbb{R}^d} |u(t,x)|^q \mathrm{d}x \Big)^{\frac{p}{q}} \Big)^{\frac{1}{p}}$$

with a usual modification when either $p$ or $q$ is infinity. When there is no risk of confusion, we may write $L_t^p L_x^q$ instead of $L_t^p(J, L_x^q)$. We also use $L_{t,x}^p$ when $p = q$.

DEFINITION 1. A pair $(p, q)$ is said to be **Schrödinger admissible**, for short $(p, q) \in S$, if

$$(p,q) \in [2,\infty]^2, \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

We denote for any spacetime slab $J \times \mathbb{R}^d$,

$$\|u\|_{S(L^2,J)} := \sup_{(p,q)\in S} \|u\|_{L_t^p(J,L_x^q)}, \quad \|v\|_{S'(L^2,J)} := \inf_{(p,q)\in S} \|v\|_{L_t^{p'}(J,L_x^{q'})}. \tag{2.2}$$

We next recall well-known Strichartz estimates for the linear Schrödinger equation. We refer the reader to [2] and [22] for more details.

PROPOSITION 1. *Let $u$ be a solution to the linear Schrödinger equation, namely*

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta} F(s)\mathrm{d}s,$$

*for some data $u_0$, $F$. Then,*

$$\|u\|_{S(L^2,\mathbb{R})} \lesssim \|u_0\|_{L_x^2} + \|F\|_{S'(L^2,\mathbb{R})}. \tag{2.3}$$

2.3. Classical Morawetz-type inequality

In this section, we derive classical Morawetz inequalities for the defocusing (INLS) by following an argument of [23]. Given a smooth real-valued function $a$, we define the Morawetz action by

$$M_a(t) := 2 \int_{\mathbb{R}^d} \nabla a(x) \cdot \mathrm{Im}\,(\overline{u}(t,x) \nabla u(t,x))\mathrm{d}x. \tag{2.4}$$

By a direct computation, we have the following result.

LEMMA 1. [23] *If $u$ is a smooth-in-time and Schwartz-in-space solution to*

$$i\partial_t u + \Delta u = N(u),$$

*with $N(u)$ satisfying* $\mathrm{Im}\,(N(u)\overline{u}) = 0$*, then we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M_a(t) = &-\int \Delta^2 a(x)|u(t,x)|^2\mathrm{d}x + 4\sum_{j,k=1}^{d}\int \partial^2_{jk}a(x)\mathrm{Re}\,(\partial_k u(t,x)\partial_j\overline{u}(t,x))\mathrm{d}x\\ &+ 2\int \nabla a(x)\cdot\{N(u),u\}_p(t,x)\mathrm{d}x,\end{aligned} \tag{2.5}$$

*where* $\{f,g\}_p := \mathrm{Re}\,(f\nabla\overline{g} - g\nabla\overline{f})$ *is the momentum bracket.*

We refer the reader to [23, Lemma 5.3] for the proof of this result. Note that if $N(u) = F(x,u) = |x|^{-b}|u|^\alpha u$, then we have[1]

$$\{N(u),u\}_p = -\frac{\alpha}{\alpha+2}\nabla(|x|^{-b}|u|^{\alpha+2}) - \frac{2}{\alpha+2}\nabla(|x|^{-b})|u|^{\alpha+2}. \tag{2.6}$$

In particular, we have the following result.

COROLLARY 1. *If $u$ is a smooth-in-time and Schwartz-in-space solution to the defocusing* (INLS)*, then we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M_a(t) = &-\int \Delta^2 a(x)|u(t,x)|^2\mathrm{d}x + 4\sum_{j,k=1}^{d}\int \partial^2_{jk}a(x)\mathrm{Re}\,(\partial_k u(t,x)\partial_j\overline{u}(t,x))\mathrm{d}x\\ &+\frac{2\alpha}{\alpha+2}\int \Delta a(x)|x|^{-b}|u(t,x)|^{\alpha+2}\mathrm{d}x\\ &-\frac{4}{\alpha+2}\int \nabla a(x)\cdot\nabla(|x|^{-b})|u(t,x)|^{\alpha+2}\mathrm{d}x.\end{aligned} \tag{2.7}$$

With the help of Corollary 1, we obtain the following classical Morawetz-type inequalities for the defocusing (INLS).

PROPOSITION 2. *Let $d \geq 3$, $0 < b < 2$ and $u$ be a solution to the defocusing* (INLS) *on the spacetime slab $J\times\mathbb{R}^d$. Then*

$$\int_J\int_{\mathbb{R}^d}|x|^{-b-1}|u(t,x)|^{\alpha+2}\mathrm{d}x\mathrm{d}t < \infty. \tag{2.8}$$

*Proof.* We consider $a(x) = |x|$. An easy computation shows

$$\partial_j a(x) = \frac{x_j}{|x|}, \quad \partial^2_{jk}a(x) = \frac{1}{|x|}\Big(\delta_{jk} - \frac{x_jx_k}{|x|^2}\Big),$$

[1] See "Appendix" for the proof.

for $j,k=1,\ldots,d$. This implies

$$\nabla a(x)=\frac{x}{|x|},\quad \Delta a(x)=\frac{d-1}{|x|},$$

and

$$-\Delta^2 a(x)=-(d-1)\Delta\Big(\frac{1}{|x|}\Big)=\begin{cases}4\pi(d-1)\delta_0 & \text{if } d=3,\\ \frac{(d-1)(d-3)}{|x|^3} & \text{if } d\geq 4,\end{cases}$$

where $\delta_0$ is the Dirac delta function. Since $a$ is a convex function, it is well-known that

$$\sum_{j,k=1}^{d}\partial^2_{jk}a re(\partial_k u\partial_j\overline{u})\geq 0.$$

Therefore, applying (2.7) with $a(x)=|x|$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{|x|}(t)\geq\frac{2\alpha(d-1)+4b}{\alpha+2}\int|x|^{-b-1}|u(t,x)|^{\alpha+2}\mathrm{d}x.$$

Thus,

$$\begin{aligned}\int_J\int_{\mathbb{R}^d}|x|^{-b-1}|u(t,x)|^{\alpha+2}\mathrm{d}x\mathrm{d}t&\lesssim\sup_{t\in J}|M_{|x|}(t)|\\ &\lesssim\|u(t)\|_{L^\infty_t(J,L^2_x)}\|\nabla u(t)\|_{L^\infty_t(J,L^2_x)}<\infty.\end{aligned}$$

The last estimate follows from the conservations of mass and energy. □

REMARK 2. The above method breaks down for $d\leq 2$ since the distribution $-\Delta^2(|x|)$ is not positive anymore. In this case, one can adapt an argument of Nakanishi in [20] to show

$$\int_J\int_{\mathbb{R}^d}\frac{t^2}{(t^2+|x|^2)^{\frac{3}{2}}}|x|^{-b}|u(t,x)|^{\alpha+2}\mathrm{d}x\mathrm{d}t<\infty. \tag{2.9}$$

However, we do not know whether the estimate (2.9) is sufficient to prove the decay of global solutions to the defocusing (INLS).

## 3. Decay of global solutions

In this section, we will give the proof of Theorem 3. To do so, we follow the argument of Visciglia in [25]. Let us start with the following result.

LEMMA 2. *Let $d\geq 3$, $0<b<2$ and $0<\alpha<\alpha^\star$. Let $\chi\in C^\infty_0$ be a cutoff function and $\psi_n\in H^1_x$ be a sequence such that*

$$\sup_{n\in\mathbb{N}}\|\psi_n\|_{H^1_x}<\infty,\quad\textit{and}\quad\psi_n\rightharpoonup\psi\textit{ weakly in } H^1_x.$$

*Let $v_n$ and $v \in C(\mathbb{R}, H^1_x)$ be the corresponding solutions to the defocusing* (INLS) *with initial data $\psi_n$ and $\psi$, respectively. Then for every $\epsilon > 0$, there exists $T(\epsilon) > 0$ and $n(\epsilon) \in \mathbb{N}$ such that*

$$\sup_{t\in(0,T(\epsilon))} \|\chi(v_n(t) - v(t))\|_{L^2_x} \leq \epsilon, \quad \forall n > n(\epsilon). \tag{3.1}$$

*Proof.* By the conservations of mass and energy,

$$\sup_{t\in\mathbb{R}, n\in\mathbb{N}} \left\{ \|v_n(t)\|_{H^1_x}, \|v(t)\|_{H^1_x} \right\} < \infty. \tag{3.2}$$

By Rellich's compactness lemma, up to a subsequence,

$$\lim_{n\to\infty} \|\chi(\psi_n - \psi)\|_{L^2_x} = 0. \tag{3.3}$$

Now let $w_n(t, x) := \chi(x)v_n(t, x)$ and $w(t, x) := \chi(x)v(t, x)$. It is easy to see that

$$i\partial_t w_n = -\Delta w_n + 2\nabla\chi \cdot \nabla v_n + v_n\Delta\chi + \chi|x|^{-b}|v_n|^\alpha v_n, \quad w_n(0) = \chi\psi_n,$$

and

$$i\partial_t w = -\Delta w + 2\nabla\chi \cdot \nabla v + v\Delta\chi + \chi|x|^{-b}|v|^\alpha v, \quad w(0) = \chi\psi.$$

Thus, by Duhamel formula,

$$\begin{aligned} w_n(t) - w(t) = e^{it\Delta}(\chi(\psi_n - \psi)) &- i\int_0^t e^{i(t-s)\Delta}\Big(2\nabla\chi \cdot \nabla(v_n(s) - v(s)) \\ &+ (v_n(s) - v(s))\Delta\chi\Big)\mathrm{d}s \\ &- i\int_0^t e^{i(t-s)\Delta}\Big(\chi|x|^{-b}(|v_n(s)|^\alpha v_n(s) - |v(s)|^\alpha v(s))\Big)\mathrm{d}s. \end{aligned} \tag{3.4}$$

Due to the singularity of $|x|^{-b}$, we need to consider two cases:

**Case 1** The support of $\chi$ does not contain the origin. In this case, the proof follows as in [25, Lemma 1.1]. For reader's convenience, we recall some details. Denote $J = (0, T)$. Let us introduce the following Schrödinger admissible pair $(p, q)$:

$$p = \frac{8}{(d-2)\alpha}, \quad q = \frac{4d}{2d - (d-2)\alpha}.$$

Using Strichartz estimates, we get

$$\begin{aligned} \|w_n - w\|_{L^p_t(J, L^q_x)} \lesssim \|\chi(\psi_n - \psi)\|_{L^2_x} &+ \|\nabla\chi \cdot \nabla(v_n - v)\|_{L^1_t(J, L^2_x)} \\ &+ \|(v_n - v)\Delta\chi\|_{L^1_t(J, L^2_x)} + \|\chi|x|^{-b}(|v_n|^\alpha v_n - |v|^\alpha v)\|_{L^{p'}_t(J, L^{q'}_x)}. \end{aligned} \tag{3.5}$$

We use (3.2), Hölder's inequality and the Sobolev embedding $H^1_x \subset L_x^{\frac{2d}{d-2}}$ to get

$$\begin{aligned}\|w_n - w\|_{L^p_t(J,L^q_x)} &\lesssim \|\chi(\psi_n - \psi)\|_{L^2_x} + |J| \\ &\quad + \|\chi(v_n - v)\|_{L^{p'}_t(J,L^q_x)} \sup_{t\in J}\left(\|v_n(t)\|^\alpha_{L_x^{\frac{2d}{d-2}}} + \|v(t)\|^\alpha_{L_x^{\frac{2d}{d-2}}}\right) \\ &\lesssim \|\chi(\psi_n - \psi)\|_{L^2_x} + |J| + |J|^{1-\frac{2}{p}}\|v_n - v\|_{L^p_t(J,L^q_x)}.\end{aligned}$$

We learn from the above estimate and (3.3) that for every $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ and $T(\epsilon) > 0$ such that

$$\|w_n - w\|_{L^p_t(I(\epsilon),L^q_x)} \leq \epsilon, \tag{3.6}$$

for all $n > n(\epsilon)$, where $I(\epsilon) = (0, T(\epsilon))$. By applying again Strichartz estimate and arguing as above, we obtain

$$\|w_n - w\|_{L^\infty_t(I(\epsilon),L^2_x)} \lesssim \|\chi(\psi_n - \psi)\|_{L^2_x} + C|I(\epsilon)| + |I(\epsilon)|^{1-\frac{2}{p}}\|w_n - w\|_{L^p_t(I(\epsilon),L^q_x)}.$$

Combining this estimate with (3.2) and (3.6), we prove (3.1).

**Case 2** The support of $\chi$ contains the origin. Without loss of generality, we assume that $\mathrm{supp}(\chi) \subset B$, where $B$ is the ball centered at the origin and of radius 1. Since we are considering $0 < \alpha < \frac{4-2b}{d-2}$, there exists $\delta \in \left(0, \frac{2-b}{2(d-2)}\right)$ such that $\alpha = \frac{4-2b}{d-2} - 4\delta$. Let us choose a Schrödinger admissible pair $(p, q)$ with

$$p = \frac{4}{2-(d-2)\delta}, \quad q = \frac{2d}{(d-2)(1+\delta)}.$$

In the view of (3.5), it suffices to bound $\|\chi|x|^{-b}(|v_n|^\alpha v_n - |v|^\alpha v)\|_{L^{p'}_t(J,L^{q'}_x)}$. To do this, we use Hölder's inequality, Sobolev embedding and (3.2) to get

$$\begin{aligned}&\|\chi|x|^{-b}(|v_n|^\alpha v_n - |v|^\alpha v)\|_{L^{p'}_t(J,L^{q'}_x)} \\ &\quad \leq \||x|^{-b}(|v_n|^\alpha v_n - |v|^\alpha v)\|_{L^{p'}_t(J,L^{q'}_x(B))} \\ &\quad \lesssim \||x|^{-b}\|_{L^\gamma_x(B)}\||v_n|^\alpha v_n - |v|^\alpha v\|_{L^{p'}_t(J,L^r_x)} \\ &\quad \lesssim \|v_n - v\|_{L^{p'}_t(J,L^q_x)} \sup_{t\in I}\left(\|v_n(t)\|^\alpha_{L_x^{\frac{2d}{d-2}}} + \|v(t)\|^\alpha_{L_x^{\frac{2d}{d-2}}}\right) \\ &\quad \lesssim |J|^{\frac{(d-2)\delta}{2}}\|v_n - v\|_{L^p_t(J,L^q_x)},\end{aligned} \tag{3.7}$$

where

$$\gamma = \frac{d}{(d-2)\delta + b}, \quad r = \frac{2d}{4-2b+(d-2)(1-3\delta)}.$$

By Remark 1, $\||x|^{-b}\|_{L^\gamma_x(B)} < \infty$ provided $\frac{d}{\gamma} > b$, and it is easy to check that

$$\frac{d}{\gamma} = (d-2)\delta + b > b.$$

With (3.7) at hand, we argue as in Case 1 to have (3.1). □

REMARK 3. It is not hard to check that Lemma 2 still holds true for any $d \geq 1, 0 < b < \min\{2, d\}$ and $0 < \alpha < \alpha^\star$.

We are now able to prove the decaying property of global solutions to the defocusing (INLS).

*Proof of Theorem 3.* We only consider the case $t \to +\infty$, the case $t \to -\infty$ is treated similarly. We first note that by interpolating between $L^2_x$-norm, $L^{2^\star}_x$-norm and $L^q_x$ with $2 < q < 2^\star$, it suffices to prove (1.9) for $q = 2 + \frac{4}{d}$. We next recall the following localized Gagliardo–Nirenberg inequality

$$\|\varphi\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}_x} \leq C\Big(\sup_{x\in\mathbb{R}^d} \|\varphi\|_{L^2(Q_1(x))}\Big)^{\frac{4}{d}} \|\varphi\|^2_{H^1_x}, \tag{3.8}$$

where $Q_r(x)$ is the cubic in $\mathbb{R}^d$ centered at $x$ whose edge has length $r$. To see (3.8), we take a recovering of $\mathbb{R}^d$ with disjoint cubes $Q_1(x_j)$ and associate with a positive partition of unity $\sum_j \chi_j = 1$. Using the usual Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}
\|\varphi\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}_x} &= \sum_j \int \chi_j(x)|\varphi(x)|^{2+\frac{4}{d}} \mathrm{d}x = \sum_j \|\tilde{\chi}_j\varphi\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}_x} \\
&\lesssim \sum_j \|\tilde{\chi}_j\varphi\|^2_{\dot{H}^1_x} \|\tilde{\chi}_j\varphi\|^{\frac{4}{d}}_{L^2_x} \\
&\lesssim \|\varphi\|_{H^1_x} \sup_{x\in\mathbb{R}^d} \left(\|\varphi\|^{\frac{4}{d}}_{L^2(Q_1(x))}\right),
\end{aligned}$$

where the functions $\tilde{\chi}_j = \chi_j^{\frac{d}{2d+4}}$ satisfy $\tilde{\chi}_j \in C^\infty_0(\mathbb{R}^d)$, $0 \leq \tilde{\chi}_j \leq 1$ and $\operatorname{supp}(\tilde{\chi}_j) \subset Q_1(x_j)$.

Let $u$ be the global solution to the defocusing (INLS) with initial data $u_0 \in H^1_x$. The conservations of mass and energy show that

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1_x} < \infty.$$

Assume by the absurd that there is a sequence $t_n \to \infty$ such that

$$\|u(t_n)\|_{L^{2+\frac{4}{d}}_x} \geq \epsilon_0 > 0, \tag{3.9}$$

for all $n \in \mathbb{N}$. By applying (3.8) with $\varphi \equiv u(t_n, x)$, we see from (3.9) that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ such that

$$\|u(t_n)\|_{L^2(Q_1(x_n))} \geq \epsilon_1 > 0, \tag{3.10}$$

for all $n \in \mathbb{N}$. We now set $\psi_n(t, x) := u(t_n, x + x_n)$. By the conservations of mass and energy,

$$\sup_{n\in\mathbb{N}} \|\psi_n\|_{H^1_x} < \infty.$$

Thus, up to a subsequence, there exists $\psi \in H^1$ such that $\psi_n \rightharpoonup \psi$ weakly in $H^1_x$. By Rellich's compactness lemma, up to a subsequence, we have

$$\lim_{n\to\infty} \|\psi_n - \psi\|_{L^2(Q_1(0))} = 0. \tag{3.11}$$

We also have from (3.10) that $\|\psi_n\|_{L^2(Q_1(0))} \geq \epsilon_1$. Thus, (3.11) ensures that there exists a positive real number still denoted by $\epsilon_1$ such that

$$\|\psi\|_{L^2(Q_1(0))} \geq \epsilon_1. \tag{3.12}$$

Let us now introduce $v_n(t,x)$ and $v(t,x)$ as the solutions to

$$\begin{cases} i\partial_t v_n + \Delta v_n - |x-x_n|^{-b}|v_n|^\alpha v_n = 0, \\ v_n(0) = \psi_n, \end{cases}$$

and

$$\begin{cases} i\partial_t v + \Delta v - |x-x_n|^{-b}|v|^\alpha v = 0, \\ v(0) = \psi, \end{cases}$$

Let $\chi$ be any cutoff function supported in $Q_2(0)$ such that $\chi \equiv 1$ on $Q_1(0)$. We have from (3.12) and a continuity argument that there exists $T_1 > 0$ such that

$$\inf_{t\in(0,T_1)} \|\chi v(t)\|_{L^2_x} \geq \frac{\epsilon_1}{2}.$$

Next, applying Lemma 2, there exists $T_2 > 0$ and $N \in \mathbb{N}$ such that

$$\sup_{t\in(0,T_2)} \|\chi(v_n(t) - v(t)\|_{L^2_x} \leq \frac{\epsilon_1}{4},$$

for all $n > N$. Thus, we get for all $t \in (0, T_0)$ with $T_0 = \min\{T_1, T_2\}$ and all $n > N$,

$$\|\chi v_n(t)\|_{L^2_x} \geq \|\chi v(t)\|_{L^2_x} - \|\chi(v_n(t) - v(t)\|_{L^2_x} \geq \frac{\epsilon_1}{4}.$$

By the choice of $\chi$, we have for all $t \in (0, T_0)$ and all $n > N$,

$$\|v_n(t)\|_{L^2(Q_2(0))} \geq \frac{\epsilon_1}{4}. \tag{3.13}$$

By the uniqueness of local solution to the (INLS),

$$v_n(t,x) = u(t+t_n, x+x_n).$$

Thus, for all $t \in (t_n, t_n + T_0)$ and all $n > N$,

$$\|u(t)\|_{L^2(Q_2(x_n))} \geq \frac{\epsilon_1}{4}. \tag{3.14}$$

Moreover, as $\lim_{n\to\infty} t_n = +\infty$, we can suppose[2] that $t_{n+1} - t_n > T_0$ for $n > N$. By Hölder's inequality,

$$\|u(t)\|_{L^{\alpha+2}(Q_2(x_n))} \gtrsim \|u(t)\|_{L^2(Q_2(x_n))} \geq \frac{\epsilon_1}{4}, \tag{3.15}$$

for all $t \in (t_n, t_n + T_0)$ and all $n > N$.

The classical Morawetz inequality (2.8) combined with (3.15) imply

$$\begin{aligned} \infty &> \int_0^{+\infty} \int_{\mathbb{R}^d} |x|^{-b-1}|u(t,x)|^{\alpha+2} \mathrm{d}x\mathrm{d}t \\ &\gtrsim \sum_{n>N} \int_{t_n}^{t_n+T_0} \int_{Q_2(x_n)} |u(t,x)|^{\alpha+2} \mathrm{d}x\mathrm{d}t \\ &\gtrsim \sum_{n>N} \left(\frac{\epsilon_1}{4}\right)^{\alpha+2} T_0 = \infty. \end{aligned}$$

This is impossible, and the proof is complete. □

## 4. Scattering property

In this section, we give the proof of the scattering property given in Theorem 4. To do this, we use Strichartz estimates and the decaying property given in Theorem 3 to obtain a bound on global solutions. The scattering property follows easily from the standard argument.

LEMMA 3. *Let $d$, $b$ and $\alpha$ be as in Theorem 4. Let $u$ be a solution to the defocusing* (INLS) *on a spacetime slab $J \times \mathbb{R}^d$ and $t_0 \in J$. Then there exists $\theta_1, \theta_2 \in (0, \alpha)$ and $q_1, q_2 \in (2, 2^\star)$ such that*

$$\|u - e^{it\Delta}u(t_0)\|_{S(J)} \lesssim \|u\|^{\alpha-\theta_1}_{L^\infty_t(J, L^{q_1}_x)} \|u\|^{1+\theta_1}_{S(J)} + \|u\|^{\alpha-\theta_2}_{L^\infty_t(J, L^{q_2}_x)} \|u\|^{1+\theta_2}_{S(J)},$$

*where $\|u\|_{S(J)} := \| \langle\nabla\rangle u\|_{S(L^2,J)}$.*

*Proof.* By Duhamel's formula, the solution to the defocusing (INLS) can be written as

$$u(t) = e^{it\Delta}u(t_0) - i\int_{t_0}^t e^{i(t-s)\Delta}|x|^{-b}|u(s)|^\alpha u(s)\mathrm{d}s.$$

The Strichartz estimate (2.3) implies

$$\|u - e^{it\Delta}u(t_0)\|_{S(J)} \lesssim \||x|^{-b}|u|^\alpha u\|_{S'(L^2,J)} + \|\nabla(|x|^{-b}|u|^\alpha u)\|_{S'(L^2,J)}.$$

[2] One can reduce the value of $T_0$ and increase the value of $N$ if necessary.

We next bound

$$\begin{aligned}\||x|^{-b}|u|^\alpha u\|_{S'(L^2,J)} &\leq \||x|^{-b}|u|^\alpha u\|_{S'(L^2(B),J)} \\ &\quad + \||x|^{-b}|u|^\alpha u\|_{S'(L^2(B^c),J)} =: A_1 + A_2, \\ \|\nabla(|x|^{-b}|u|^\alpha u)\|_{S'(L^2,J)} &\leq \|\nabla(|x|^{-b}|u|^\alpha u)\|_{S'(L^2(B),J)} \\ &\quad + \|\nabla(|x|^{-b}|u|^\alpha u)\|_{S'(L^2(B^c),J)} =: B_1 + B_2.\end{aligned}$$

<u>On $B$</u> By Hölder's inequality and Remark 1,

$$\begin{aligned}A_1 \leq \||x|^{-b}|u|^\alpha u\|_{L_t^{p_1'}(J,L_x^{q_1'}(B))} &\lesssim \||x|^{-b}\|_{L_x^{\gamma_1}(B)}\||u|^\alpha u\|_{L_t^{p_1'}(J,L_x^{\upsilon_1})} \\ &\lesssim \|\|u\|^{\alpha+1}_{L_x^{q_1}}\|_{L_t^{p_1'}(J)} \\ &\lesssim \|u\|^{\alpha-\theta_1}_{L_t^\infty(J,L_x^{q_1})}\|u\|^{1+\theta_1}_{L_t^{p_1}(J,L_x^{q_1})},\end{aligned}$$

provided that $(p_1, q_1) \in S$ and

$$\frac{1}{q_1'} = \frac{1}{\gamma_1} + \frac{1}{\upsilon_1}, \quad \frac{d}{\gamma_1} > b, \quad \frac{1}{\upsilon_1} = \frac{\alpha+1}{q_1}, \quad \theta \in (0,\alpha), \quad \frac{1}{p_1'} = \frac{1+\theta_1}{p_1}.$$

This implies

$$\frac{d}{\gamma_1} = d - \frac{d(\alpha+2)}{q_1} > b, \quad p_1 = \theta_1 + 2 \in (2, \alpha+2). \tag{4.1}$$

The first condition in (4.1) is equivalent to $q_1 > \frac{d(\alpha+2)}{d-b}$. Let us choose

$$q_1 = \frac{d(\alpha+2)}{d-b} + \epsilon, \tag{4.2}$$

for some $0 < \epsilon \ll 1$ to be chosen later. Since $\alpha_\star < \alpha < \alpha^\star$, by taking $\epsilon > 0$ sufficiently small, it is easy to see that $q_1 \in (2, 2^\star)$. It remains to check $p_1 < \alpha + 2$. Since $(p_1, q_1) \in S$, we need to show

$$\frac{2}{p_1} = \frac{d}{2} - \frac{d}{q_1} > \frac{2}{\alpha+2} \quad \text{or} \quad \frac{d}{q_1} < \frac{d(\alpha+2)-4}{2(\alpha+2)}.$$

It is in turn equivalent to

$$d(\alpha+2)(d\alpha - 4 + 2b) + \epsilon(d-b)[d(\alpha+2)-4] > 0.$$

Since $\alpha > \alpha_\star = \frac{4-2b}{d}$, the above inequality holds true by taking $\epsilon > 0$ small enough. We thus obtain

$$A_1 \lesssim \|u\|^{\alpha-\theta_1}_{L_t^\infty(J,L_x^{q_1})}\|u\|^{1+\theta_1}_{L_t^{p_1}(J,L_x^{q_1})} \lesssim \|u\|^{\alpha-\theta_1}_{L_t^\infty(J,L_x^{q_1})}\|u\|^{1+\theta_1}_{S(J)}. \tag{4.3}$$

We next bound

$$B_1 \leq \||x|^{-b}\nabla(|u|^\alpha u)\|_{S'(L^2(B),J)} + \||x|^{-b-1}|u|^\alpha u\|_{S'(L^2(B),J)} =: B_{11} + B_{12}.$$

By the fractional chain rule, we estimate

$$\begin{aligned} B_{11} \leq \||x|^{-b}\nabla(|u|^\alpha u)\|_{L_t^{p_1'}(J,L_x^{q_1'}(B))} &\lesssim \||x|^{-b}\|_{L_x^{\gamma_1}(B)} \|\nabla(|u|^\alpha u)\|_{L_t^{p_1'}(J,L_x^{\upsilon_1})} \\ &\lesssim \| \|u\|^\alpha_{L_x^{q_1}} \|\nabla u\|_{L_x^{q_1}} \|_{L_t^{p_1'}(J)} \\ &\lesssim \|u\|^{\alpha-\theta_1}_{L_t^\infty(J,L_x^{q_1})} \|u\|^{\theta_1}_{L_t^{p_1}(J,L_x^{q_1})} \|\nabla u\|_{L_t^{p_1}(J,L_x^{q_1})}, \end{aligned}$$

provided that $(p_1, q_1) \in S$ and

$$\frac{1}{q_1'} = \frac{1}{\gamma_1} + \frac{1}{\upsilon_1}, \quad \frac{d}{\gamma_1} > b, \quad \frac{1}{\upsilon_1} = \frac{\alpha+1}{q_1}, \quad \frac{1}{p_1'} = \frac{1+\theta_1}{q_1}, \quad \theta_1 \in (0, \alpha).$$

This implies

$$\frac{d}{\gamma_1} = d - \frac{d(\alpha+2)}{q_1} > b, \quad p_1 = \theta_1 + 2 \in (2, \alpha+2).$$

This condition is exactly (4.1). Therefore, we choose $q_1$ as in (4.2) and get

$$B_{11} \lesssim \|u\|^{\alpha-\theta_1}_{L_t^\infty(J,L_x^{q_1})} \|u\|^{\theta_1}_{L_t^{p_1}(J,L_x^{q_1})} \|\nabla u\|_{L_t^{p_1}(J,L_x^{q_1})} \lesssim \|u\|^{\alpha-\theta_1}_{L_t^\infty(J,L_x^{q_1})} \|u\|^{1+\theta_1}_{S(J)}. \tag{4.4}$$

We next bound

$$\begin{aligned} B_{12} \leq \||x|^{-b-1}|u|^\alpha u\|_{L_t^{p_1'}(J,L_x^{q_1'}(B))} &\lesssim \||x|^{-b-1}\|_{L_x^{\gamma_1}(B)} \||u|^\alpha u\|_{L_t^{p_1'}(J,L_x^{\upsilon_1})} \\ &\lesssim \| \|u\|^\alpha_{L_x^{q_1}} \|u\|_{L_x^{n_1}} \|_{L_t^{p_1'}(J)} \\ &\lesssim \|u\|^{\alpha-\theta_1}_{L_t^\infty(J,L_x^{q_1})} \|u\|^{\theta_1}_{L_t^{p_1}(J,L_x^{q_1})} \|u\|_{L_t^{p_1}(J,L_x^{n_1})}. \end{aligned}$$

<u>When $d \geq 4$</u> We use the homogeneous Sobolev embedding $\|u\|_{L_x^{n_1}} \lesssim \|\nabla u\|_{L_x^{q_1}}$ to have

$$B_{12} \lesssim \|u\|^{\alpha-\theta_1}_{L_t^\infty(J,L_x^{q_1})} \|u\|^{\theta_1}_{L_t^{p_1}(J,L_x^{q_1})} \|\nabla u\|_{L_t^{p_1}(J,L_x^{q_1})}.$$

The above estimates hold true provided that $(p_1, q_1) \in S$ and

$$\frac{1}{q_1'} = \frac{1}{\gamma_1} + \frac{1}{\upsilon_1}, \quad \frac{d}{\gamma_1} > b+1, \quad \frac{1}{\upsilon_1} = \frac{\alpha}{q_1} + \frac{1}{n_1}, \quad \frac{1}{p_1'} = \frac{1+\theta_1}{p_1}, \quad \theta_1 \in (0, \alpha),$$

and

$$q_1 < d, \quad \frac{1}{n_1} = \frac{1}{q_1} - \frac{1}{d}.$$

Note that the last condition allows us to use the homogeneous Sobolev embedding. The above requirements imply

$$\frac{d}{\gamma_1} = d - \frac{d(\alpha+2)}{q_1} + 1 > b+1, \quad p_1 = \theta_1 + 2 \in (2, \alpha+2).$$

This is exactly (4.1). We thus choose $q_1$ as in (4.2). Note that by taking $\epsilon > 0$ small enough, the requirement $q_1 < d$ is satisfied if

$$\frac{d(\alpha+2)}{d-b} < d \quad \text{or} \quad \alpha < d-b-2. \tag{4.5}$$

Since $d \geq 4$, it is easy to check that $\alpha^\star = \frac{4-2b}{d-2} \leq d-b-2$. We thus get for $d \geq 4$, $0 < b < 2$ and $\alpha_\star < \alpha < \alpha^\star$,

$$B_{12} \lesssim \|u\|^{\alpha-\theta_1}_{L^\infty_t(J,L^{q_1}_x)} \|u\|^{1+\theta_1}_{S(J)}. \tag{4.6}$$

When $d=3$ We first note that (4.5) does not hold true. We use instead the inhomogeneous Sobolev embedding $\|u\|_{L^{n_1}_x} \lesssim \| \langle \nabla \rangle u\|_{L^{q_1}_x}$ to have

$$B_{12} \lesssim \|u\|^{\alpha-\theta_1}_{L^\infty_t(J,L^{q_1}_x)} \|u\|^{\theta_1}_{L^{p_1}_t(J,L^{q_1}_x)} \| \langle \nabla \rangle u\|_{L^{p_1}_t(J,L^{q_1}_x)}.$$

The above estimate holds true provided that $(p_1, q_1) \in S$ and

$$\frac{1}{q_1'} = \frac{1}{\gamma_1} + \frac{1}{\upsilon_1}, \quad \frac{3}{\gamma_1} > b+1, \quad \frac{1}{\upsilon_1} = \frac{\alpha}{q_1} + \frac{1}{n_1}, \quad \frac{1}{p_1'} = \frac{1+\theta_1}{p_1}, \quad \theta_1 \in (0,\alpha),$$

and

$$3 < q_1, \quad n_1 \in (q_1, \infty) \quad \text{or} \quad \frac{1}{n_1} = \frac{\tau}{q_1}, \quad \tau \in (0,1).$$

Here the last condition ensures the inhomogeneous Sobolev embedding. The above requirements imply

$$\frac{3}{\gamma_1} = 3 - \frac{3(\alpha+1+\tau)}{q_1} > b-1 \quad \text{or} \quad \frac{3(\alpha+1+\tau)}{q_1} < 2-b.$$

Let us choose

$$q_1 = \frac{3(\alpha+1+\tau)}{2-b} + \epsilon,$$

for some $0 < \epsilon \ll 1$ to be chosen later. It remains to check

$$q_1 \in (3,6), \quad p_1 \in (2, \alpha+2).$$

By taking $\epsilon > 0$ small enough, the condition $q_1 \in (3,6)$ implies

$$1-b-\tau < \alpha < 3-2b-\tau. \tag{4.7}$$

Since $(p_1, q_1) \in S$, the condition[3] $p_1 < \alpha+2$ is equivalent to

$$\frac{3}{2} - \frac{3}{q_1} = \frac{2}{p_1} > \frac{2}{\alpha+2}.$$

[3] Note that $q_1 < 6$ implies $p_1 > 2$.

The above condition is then equivalent to

$$3[3\alpha^2 + (1+2b)\alpha + 4b - 6 + \tau(3\alpha+2)] + \epsilon(2-b)(3\alpha+2) > 0.$$

By taking $\epsilon > 0$ sufficiently small, the above inequality holds true provided that

$$3\alpha^2 + (1+2b)\alpha + 4b - 6 + \tau(3\alpha+2) > 0. \tag{4.8}$$

Now, if we take $\tau$ closed to 0, (4.7) and (4.8) imply

$$1-b < \alpha < 3-2b, \quad \alpha > \frac{-1-2b+\sqrt{4b^2-44b+73}}{6}.$$

Combining this with the assumption $\frac{4-2b}{3} = \alpha_\star < \alpha < \alpha^\star = 4-2b$, we have

$$\frac{4-2b}{3} < \alpha < 3-2b, \quad 0 < b < \frac{5}{4}. \tag{4.9}$$

We thus obtain for $d=3$ and $\alpha, b$ as in (4.9),

$$B_{12} \lesssim \|u\|^{\alpha-\theta_1}_{L^\infty_t(J,L^{q_1}_x)} \|u\|^{1+\theta_1}_{S(J)}. \tag{4.10}$$

<u>On $B^c$</u> By Hölder's inequality and Remark 1,

$$\begin{aligned} A_2 \leq \||x|^{-b}|u|^\alpha u\|_{L^{p_2'}_t(J,L^{q_2'}_x(B^c))} &\lesssim \||x|^{-b}\|_{L^{\gamma_2}_x(B^c)} \||u|^\alpha u\|_{L^{p_2'}_t(J,L^{\upsilon_2}_x)} \\ &\lesssim \|\|u\|^{\alpha+1}_{L^{q_2}_x}\|_{L^{p_2'}_t(J)} \\ &\lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)} \|u\|^{1+\theta_2}_{L^{p_2}_t(J,L^{q_2}_x)}, \end{aligned}$$

provided that $(p_2, q_2) \in S$ and

$$\frac{1}{q_2'} = \frac{1}{\gamma_2} + \frac{1}{\upsilon_2}, \quad \frac{d}{\gamma_2} < b, \quad \frac{1}{\upsilon_2} = \frac{\alpha+1}{q_2}, \quad \frac{1}{p_2'} = \frac{1+\theta_2}{p_2}, \quad \theta_2 \in (0,\alpha).$$

This implies

$$\frac{d}{\gamma_2} = d - \frac{d(\alpha+2)}{q_2} < b, \quad p_2 = \theta_2 + 2 \in (2, \alpha+2). \tag{4.11}$$

The first condition in (4.11) implies $q_2 < \frac{d(\alpha+2)}{d-b}$. Let us choose

$$q_2 = \frac{d(\alpha+2)}{d-b} - \epsilon, \tag{4.12}$$

for some $0 < \epsilon \ll 1$ to be chosen later. By taking $\epsilon > 0$ small enough, the assumption $\alpha_\star < \alpha < \alpha^\star$ ensures $q_2 \in (2, 2^\star)$. It remains to check $p_2 < \alpha+2$. Since $(p_2, q_2) \in S$, it is equivalent to

$$\frac{d}{2} - \frac{d}{q_2} = \frac{2}{p_2} > \frac{2}{\alpha+2} \quad \text{or} \quad \frac{d}{q_2} < \frac{d(\alpha+2)-4}{2(\alpha+2)}.$$

A direct computation shows that the above condition is equivalent to

$$d(\alpha+2)(d\alpha-4+2b)-\epsilon(d-b)[d(\alpha+2)-4]>0.$$

Since $\alpha > \frac{4-2b}{d}$, the above inequality holds true by taking $\epsilon > 0$ small enough. We thus get

$$A_2 \lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)}\|u\|^{1+\theta_2}_{L^{p_2}_t(J,L^{q_2}_x)} \lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)}\|u\|^{1+\theta_2}_{S(J)}. \tag{4.13}$$

We next bound

$$B_2 \leq \||x|^{-b}\nabla(|u|^\alpha u)\|_{S'(L^2(B^c),J)} + \||x|^{-b-1}|u|^\alpha u\|_{S'(L^2(B^c),J)} =: B_{21}+B_{22}.$$

By the fractional chain rule, Hölder's inequality and Remark 1,

$$\begin{aligned} B_{21} \leq \||x|^{-b}\nabla(|u|^\alpha u)\|_{L^{p_2'}_t(J,L^{q_2'}_x(B^c))} &\lesssim \||x|^{-b}\|_{L^{\gamma_2}_x(B^c)}\|\nabla(|u|^\alpha u)\|_{L^{p_2'}_t(J,L^{\upsilon_2}_x)} \\ &\lesssim \|\|u\|^\alpha_{L^{q_2}_x}\|\nabla u\|_{L^{q_2}_x}\|_{L^{p_2'}_t(J)} \\ &\lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)}\|u\|^{\theta_2}_{L^{p_2}_t(J,L^{q_2}_x)}\|\nabla u\|_{L^{p_2}_t(J,L^{q_2}_x)}, \end{aligned}$$

provided that $(p_2,q_2)\in S$ and

$$\frac{1}{q_2'}=\frac{1}{\gamma_2}+\frac{1}{\upsilon_2},\quad \frac{d}{\gamma_2}<b,\quad \frac{1}{\upsilon_2}=\frac{\alpha+1}{q_2},\quad \frac{1}{p_2'}=\frac{1+\theta_2}{p_2},\quad \theta_2\in(0,\alpha).$$

These conditions are exactly those for $A_2$. We thus choose $q_2$ as in (4.12) and obtain

$$B_{21} \lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)}\|u\|^{\theta_2}_{L^{p_2}_t(J,L^{q_2}_x)}\|\nabla u\|_{L^{p_2}_t(J,L^{q_2}_x)} \lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)}\|u\|^{1+\theta_2}_{S(J)}. \tag{4.14}$$

It remains to treat $B_{22}$. By Hölder's inequality and Remark 1,

$$\begin{aligned} B_{22} \leq \||x|^{-b-1}|u|^\alpha u\|_{L^{p_2'}_t(J,L^{q_2'}_x(B^c))} &\lesssim \||x|^{-b-1}\|_{L^{\gamma_2}_x(B^c)}\||u|^\alpha u\|_{L^{p_2'}_t(J,L^{\upsilon_2}_x)} \\ &\lesssim \|\|u\|^\alpha_{L^{q_2}_x}\|u\|_{L^{n_2}_x}\|_{L^{p_2'}_t(J)} \\ &\lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)}\|u\|^{\theta_2}_{L^{p_2}_t(J,L^{q_2}_x)}\|u\|_{L^{p_2}_t(J,L^{n_2}_x)}, \end{aligned}$$

provided that

$$\frac{1}{q_2'}=\frac{1}{\gamma_2}+\frac{1}{\upsilon_2},\quad \frac{d}{\gamma_2}<b+1,\quad \frac{1}{\upsilon_2}=\frac{\alpha}{q_2}+\frac{1}{n_2},\quad \frac{1}{p_2'}=\frac{1+\theta_2}{p_2},\quad \theta_2\in(0,\alpha). \tag{4.15}$$

As for $B_{12}$, we separate two cases: $d\geq 4$ and $d=3$.

<u>When $d\geq 4$</u> We use the homogeneous Sobolev embedding $\|u\|_{L^{n_2}_x} \lesssim \|\nabla u\|_{L^{q_1}_x}$ provided that

$$q_2<d,\quad \frac{1}{n_2}=\frac{1}{q_2}-\frac{1}{d}.$$

Thus, (4.15) implies

$$\frac{d}{\gamma_2} = d - \frac{d(\alpha+2)}{q_2} + 1 < b + 1 \text{ or } d - \frac{d(\alpha+2)}{q_2} < b, \quad p_2 = \theta_2 + 2 \in (2, \alpha+2).$$

This condition is exactly (4.11). We thus choose $q_2$ as in (4.12). Note that by taking $\epsilon > 0$ small enough, this condition holds true if we have

$$\frac{d(\alpha+2)}{d-b} < d \quad \text{or} \quad \alpha < d - b - 2.$$

Since $d \geq 4$, we always have $\frac{4-2b}{d-2} < d - b - 2$. Therefore, the last estimate holds true for $\alpha_\star < \alpha < \alpha^\star$. We obtain for $d \geq 4$, $0 < b < 2$ and $\alpha_\star < \alpha < \alpha^\star$,

$$B_{22} \lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)} \|u\|^{\theta_2}_{L^{p_2}_t(J,L^{q_2}_x)} \|\nabla u\|_{L^{p_2}_t(J,L^{q_2}_x)} \lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)} \|u\|^{1+\theta_2}_{S(J)}. \tag{4.16}$$

<u>When $d = 3$</u> We use the inhomogeneous Sobolev embedding $\|u\|_{L^{n_2}_x} \lesssim \|\langle\nabla\rangle u\|_{L^{q_2}_x}$ provided that

$$q_2 > 3, \quad n_2 \in (q_2, \infty) \quad \text{or} \quad \frac{1}{n_2} = \frac{\tau}{q_2}, \quad \tau \in (0,1).$$

Thus, (4.15) implies

$$\frac{3}{\gamma_2} = 3 - \frac{3(\alpha+1+\tau)}{q_2} < b+1 \quad \text{or} \quad \frac{3(\alpha+1+\tau)}{q_2} > 2 - b.$$

Let us choose

$$q_2 = \frac{3(\alpha+1+\tau)}{2-b} - \epsilon,$$

for some $0 < \epsilon \ll 1$ to be chosen later. We need to check $q_2 \in (3,6)$ and $p_2 \in (2, \alpha+2)$. By taking $\epsilon > 0$ sufficiently small, these conditions hold true if we have

$$1 - b - \tau < \alpha < 3 - 2b - \tau, \quad 3\alpha^2 + (1+2b)\alpha + 4b - 6 + \tau(3\alpha+2) > 0.$$

Taking $\tau$ closed to 0, we have

$$1 - b < \alpha < 3 - 2b, \quad \alpha > \frac{-1 - 2b + \sqrt{4b^2 - 44b + 73}}{6}.$$

By the assumption $\frac{4-2b}{3} < \alpha < 4 - 2b$, we see that $b$ and $\alpha$ satisfy (4.9). Therefore, we get for $d = 3$ and $b, \alpha$ as in (4.9),

$$B_{22} \lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)} \|u\|^{\theta_2}_{L^{p_2}_t(J,L^{q_2}_x)} \|\langle\nabla\rangle u\|_{L^{p_2}_t(J,L^{q_2}_x)} \lesssim \|u\|^{\alpha-\theta_2}_{L^\infty_t(J,L^{q_2}_x)} \|u\|^{1+\theta_2}_{S(J)}. \tag{4.17}$$

Collecting (4.3), (4.4), (4.6), (4.10), (4.13), (4.14), (4.16) and (4.17), we complete the proof. □

COROLLARY 2. *Let $d, b$ and $\alpha$ be as in Theorem 4. Let $u_0 \in H^1(\mathbb{R}^d)$ and $u$ be the unique global solution to the defocusing* (INLS). *Then*

$$u \in L^p_t(\mathbb{R}, W^{1,q}_x),$$

*for any Schrödinger admissible pair $(p, q)$.*

*Proof.* By applying Lemma 3 with $J = (T, t)$ and using the decaying property given in Theorem 3, we see that there exist $\theta_1, \theta_2 \in (0, \alpha)$ such that

$$\|u\|_{S((T,t))} \lesssim \|u(T)\|_{H^1_x} + \epsilon_1(T)\|u\|^{1+\theta_1}_{S((T,t))} + \epsilon_2(T)\|u\|^{1+\theta_2}_{S((T,t))},$$

where $\epsilon_1(T), \epsilon_2(T) \to 0$ as $T \to +\infty$. By the conservations law and the continuity argument (see e.g. [2, Lemma 7.7.4] or [22, Section 1.3]), we learn that for $T$ large enough,

$$\|u\|_{S((T,t))} \leq C,$$

for some $C > 0$ independent of $t$. We thus get $u \in L^p_t((T, +\infty), W^{1,q}_x)$ for any $(p, q) \in S$. Similarly, we prove as well that $u \in L^p_t((-\infty, -T), W^{1,q}_x)$ for any $(p, q) \in S$. Combining these facts and the local well-posedness given in Theorem 2, we obtain $u \in L^p_t(\mathbb{R}, W^{1,q}_x)$ for any Schrödinger admissible pair $(p, q)$. □

We are now able to prove Theorem 4. The proof is based on a standard argument (see e.g. [2, Section 8.3] or [22, Section 3.6]).

*Proof of Theorem 4.* Let $u$ be the global solution to the defocusing (INLS) with initial data $u_0 \in H^1_x$. By Duhamel's formula,

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}|x|^{-b}|u(s)|^\alpha u(s)\mathrm{d}s. \tag{4.18}$$

As in the proof of Lemma 3, we see that there exists $\theta_1, \theta_2 \in (0, \alpha)$ and $q_1, q_2 \in (2, 2^\star)$ such that

$$\|\langle\nabla\rangle(|x|^{-b}|u|^\alpha u)\|_{S'(L^2,\mathbb{R})} \lesssim \|u\|^{\alpha-\theta_1}_{L^\infty_t(\mathbb{R},L^{q_1}_x)}\|u\|^{1+\theta_1}_{S(\mathbb{R})} + \|u\|^{\alpha-\theta_2}_{L^\infty_t(\mathbb{R},L^{q_2}_x)}\|u\|^{1+\theta_2}_{S(\mathbb{R})}.$$

Thus, Theorem 3 and Corollary 2 imply

$$\|\langle\nabla\rangle(|x|^{-b}|u|^\alpha u)\|_{S'(L^2,\mathbb{R})} < \infty. \tag{4.19}$$

Let $0 < t_1 < t_2 < +\infty$. By Strichartz estimates and (4.19), we have

$$\|e^{-it_2\Delta}u(t_2) - e^{-it_1\Delta}u(t_1)\|_{H^1_x} \lesssim \|\langle\nabla\rangle(|x|^{-b}|u|^\alpha u)\|_{S'(L^2,(t_1,t_2))} \to 0,$$

as $t_1, t_2 \to +\infty$. This implies that the limit

$$u_0^+ := \lim_{t\to+\infty} e^{-it\Delta}u(t)$$

exists in $H^1_x$. Moreover,

$$u(t) - e^{it\Delta}u_0^+ = -i \int_t^{+\infty} e^{i(t-s)\Delta}|x|^{-b}|u(s)|^\alpha u(s)\mathrm{d}s.$$

Applying again Strichartz estimates, we get

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}u_0^+\|_{H^1_x} = 0.$$

This shows the energy scattering for positive time, the one for negative time is treated similarly. The proof is complete. □

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## Appendix

In this "Appendix", we will give the proof of (2.6). Let $N(u) = F(x,u) = |x|^{-b}|u|^\alpha u$. We compute

$$\begin{aligned}\nabla\left(|x|^{-b}|u|^{\alpha+2}\right) &= \nabla\left(|x|^{-b}\right)|u|^{\alpha+2} + |x|^{-b}\nabla\left(|u|^{\alpha+2}\right)\\ &= \nabla\left(|x|^{-b}\right)|u|^{\alpha+2} + (\alpha+2)|x|^{-b}|u|^\alpha \mathrm{Re}\ (\nabla u\overline{u})\\ &= \nabla\left(|x|^{-b}\right)|u|^{\alpha+2} + (\alpha+2)|x|^{-b}|u|^\alpha \mathrm{Re}\ (u\nabla\overline{u}).\end{aligned}$$

Similarly,

$$\nabla\left(|x|^{-b}|u|^{\alpha+2}\right) = \nabla\left(|x|^{-b}|u|^\alpha\overline{u}u\right) = \nabla\left(|x|^{-b}|u|^\alpha\overline{u}\right)u + |x|^{-b}|u|^\alpha\overline{u}\nabla u,$$

or

$$\nabla\left(|x|^{-b}|u|^\alpha\overline{u}\right)u = \nabla\left(|x|^{-b}|u|^{\alpha+2}\right) - |x|^{-b}|u|^\alpha\overline{u}\nabla u.$$

Therefore,

$$
\begin{aligned}
\{N(u), u\}_p &= \mathrm{Re}\left(|x|^{-b}|u|^\alpha u\nabla\overline{u} - u\nabla(|x|^{-b}|u|^\alpha\overline{u})\right) \\
&= 2\mathrm{Re}\left(|x|^{-b}|u|^\alpha u\nabla\overline{u}) - \nabla(|x|^{-b}|u|^{\alpha+2}\right) \\
&= \frac{2}{\alpha+2}\left(\nabla(|x|^{-b}|u|^{\alpha+2}) - \nabla(|x|^{-b})|u|^{\alpha+2}\right) - \nabla\left(|x|^{-b}|u|^{\alpha+2}\right) \\
&= -\frac{\alpha}{\alpha+2}\nabla\left(|x|^{-b}|u|^{\alpha+2}\right) - \frac{2}{\alpha+2}\nabla\left(|x|^{-b}\right)|u|^{\alpha+2}.
\end{aligned}
$$

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