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Solving a nonlinear variation of the heat equation: self-similar solutions of the second kind and other results

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Abstract. This paper studies regular self-similar solutions of the following diffusion equation

$$u_t + \gamma |u_t| = \Delta u \quad \text{in } \mathbb{R}^N \times]0, \infty[,$$

where $-1 < \gamma < 1$. The analysis is focused on radial symmetric solutions $u(x,t) = t^{-\alpha/2} f(\eta)$ with $\alpha > 0$ and $\eta = \|x\|/\sqrt{t}$. Closed representation is obtained in terms of confluent hypergeometric functions. Employing specific properties of these special functions, oscillatory and symptotic aspects of f are obtained. It is demonstrated that such features are governed by increasing and unbounded sequences of exponents $\alpha_0 < \alpha_1 < \cdots$, as in other diffusion equations. These exponents are determined by solving a system of transcendental equations related to specific roots of Kummer and Tricomi functions. As these cannot be determined using dimensional analysis, it is concluded that they are anomalous. For each exponent α_k, linear approximation when γ is close to zero is also presented. Finally, relationships with previous results as well as an extension to other fully nonlinear parabolic equations are discussed.

1. Introduction

Several physical and mathematical aspects associated with nonlinear partial differential equations (PDEs) often are captured from particular solutions [10,18,25]. An example of this fact is the class of self-similar solutions [3,8,23]. For parabolic equations, the structure of such functions is closely related to several general features such as intermediate asymptotic, large-time behavior, spatial rate decay among others [3,11]. Since these results also are applied on more complex models by means comparison arguments, having closed representations of these solutions for a specific equation can be useful in the development of qualitative analysis from different approaches [10,22].

This work studies self-similar solutions of the following diffusion equation

$$u_t + \gamma |u_t| = \Delta u, \quad -1 < \gamma < 1, \tag{1}$$

posed in $\mathbb{R}^N \times \mathbb{R}^+$ with $N \geq 1$. This equation was formulated in [4], to model an elasto-plastic filtration processes throughout an irreversible medium (case $0 <$

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$\gamma < 1$). Previous results for solving this framework, commonly known as "Barenblatt equation of elasto-plastic filtration," has been developed principally on nonnegative solutions (see [2,3,5,20]). Our attention in (1) arises from their mathematical structure which is related to other more complex nonlinear models. Specifically, considering $\kappa = (1+\gamma)/(1-\gamma)$ and $t \mapsto \kappa t$, Eq. (1) is written as

$$u_t = \begin{cases} \max\{\kappa \Delta u, \Delta u\}, & \text{when } \kappa < 1, \\ \min\{\kappa \Delta u, \Delta u\}, & \text{when } \kappa > 1. \end{cases} \tag{2}$$

Hence, the theory of viscosity solutions is an applicable framework for the study, and therefore, results for (1) could be used in the understanding of elemental features of uniformly parabolic equations [6,9,13,17].

We analyze solutions of the form:

$$u(x,t) = t^{-\alpha/2} f(\eta) \quad \text{with} \quad \eta = \|x\| t^{-1/2}, \quad \alpha > 0, \tag{3}$$

and f regular, i.e., radial symmetric self-similar solutions with negative homogeneity. The first proposal here is to construct as well as to show oscillatory and asymptotic features of f. We describe each profile as solution of nonlinear Cauchy problem in the classic sense. Following some ideas used in [5,20], the solution is obtained solving a collection linear Cauchy problems. Closed expressions of f are described in terms of confluent hypergeometric functions (caloric functions). Employed properties of these special functions and oscillatory and asymptotic behaviors of f are presented. The $\mathcal{C}^{2,1}$ regularity of the profile is direct consequence of the nonlinearity of the ODE.

Our first main results reads as follows.

THEOREM 1. *There exists an increasing and unbounded sequence of similarity exponents* $0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots$ *such that the associated profile satisfies*

$$f(\eta) \sim \eta^{\alpha_k - N} e^{-(1+(-1)^k\gamma)\eta^2/4} \quad \textit{for } \eta \textit{ large,} \tag{4}$$

whereas for the case $\alpha \neq \alpha_k$*, the asymptotic behavior is given by* $f(\eta) \sim \eta^{-\alpha}$*. Moreover, for each* α *such that* $\alpha_k < \alpha \leq \alpha_{k+1}$*,* $f(\eta)$ *has exactly* $k+1$ *zeros in* $[0,\infty[$ *and if* $0 < \alpha \leq \alpha_0$*, it holds that* $f(\eta) > 0$ *on* $[0,\infty[$.

There exist several descriptions for nonnegative solutions. In particular, the existence of one-dimensional solution such that $u(x,0) = 0$ as $x \neq 0$ (source-like solution) was early shown in [19] and constructed in [5] using parabolic functions, indicating that α_0 should be anomalous. For arbitrary $N \geq 1$, in [20] were presented the existence and asymptotic spatial behaviors of classical nonnegative solutions (i.e., cases $0 < \alpha \leq \alpha_0$). Additionally, in [20] it is shown that the self-similar solutions attract all the solutions with nonnegative and continuous initial data that decay sufficiently at infinity.

Although physical approaches have focused on nonnegative solutions, our interest on sign change solutions is based on the similarity with other parabolic models. Particularly, for the heat equation (case $\gamma = 0$) the sequence indicated in Theorem 1 is explicitly described by $\alpha_k = N + 2k$ ($k = 0, 1, \ldots$), with f representing Kummer functions (below). For the porous medium equation, similar results also exist. In such case, the class conformed by compactly supported self-similar (with sign changes) was presented in [14] where the sequence of anomalous exponents is increasing but bounded (see also [7,15]). These results can be rewritten conveniently in an equivalent form for the evolution p-Laplacian equation [16] (see [24] for further results and details).

Results similar to Theorem 1 were developed in [6], where the main goal was to prove the existence of solutions for (2), which are not $\mathcal{C}^{2,1}$. For v satisfying the equation in the classic sense, the authors analyzed $w = v_t$ with positive homogeneity and developed a complete description of the class of locally bounded self-similar functions w. Additionally, a technique allows to relate w with $\tilde{w} = u_t$ where $\tilde{w}$ has negatively self-similar structure. We use this technique to prove that both u and u_t have self-similar structure, and therefore, there exists an explicit relationship between the families of regular self-similar solutions of (1) with negative and positive homogeneity. Moreover, we prove a linear correspondence between the anomalous exponents. Our second proposal includes descriptions of specific features of the anomalous exponents, as formulated in the following theorem:

THEOREM 2. *Consider $\nu = (N/2) - 1$ and L_m^ν the (n, ν)-Laguerre polynomial. Let α_k as Theorem 1 and v_{k+1} the $(k+1)$-th root of L_{k+1}^ν. For each $k = 0, 1, 2 \ldots$, it follows that $\alpha_k = \alpha_k(\gamma)$ is regular function of γ satisfying*

$$\frac{\partial \alpha_k(0)}{\partial \gamma} = (-1)^k \frac{v_{k+1}^\nu [L_k^\nu(v_{k+1})]^2}{e^{v_{k+1}}} \frac{\Gamma(k+2)((N/2)+k)}{\Gamma((N/2)+k)} \tag{5}$$

with $\alpha_k(0) = N + 2k$ the exponent in the sequence for the heat equation.

Note that the case $k = 0$, previously analyzed in [2], is directly recovered from (5) by taking $L_0(z) = 1$ and $v_1 = N/2$. Herein, we use a similar procedure which includes the application of the implicit function theorem (IFT), alongside a self-adjoint representation of the linear ODEs used in the description of the Cauchy problem related to f.

The present work is organized as follows. The Cauchy problem for f is constructed in Sect. 2.1, and the procedure to solve it in Sect. 2.2. Auxiliary results related to oscillatory features are studied in Sect. 2.3. The construction of each f and the description of asymptotic behaviors are developed in Sect. 3. Section 4 presents the proofs of the main results of the investigation. Section 5 discusses some relationships with other fully nonlinear parabolic equations, as well as previous results for (1) obtained in [6]. Finally, Sect. 6 presents the conclusions of this research.

2. Preliminaries

2.1. Nonlinear Cauchy problem for the profiles $f(\eta)$

Consider $u(x,t) = t^{-\alpha/2} f(\|x\|/\sqrt{t})$ solution of (1),with f regular function. Directly

$$\frac{\partial u}{\partial t} = -t^{-(1+\alpha/2)}(\alpha f + \eta f')/2 \quad \text{and} \quad \Delta u = t^{-(1+\alpha/2)}(f'' + (N-1)f'/\eta), \quad (6)$$

where the prime denotes differentiation with respect to η. The nonlinearity of the PDE shall be synthesized using

$$\sigma(s) = \begin{cases} -(1-\gamma)/2, & s > 0, \\ -(1+\gamma)/2, & s \leq 0. \end{cases} \quad (7)$$

As $f(0) > 0$ it is conveniently normalized by $f(0) = 1$, whereas $f'(0) = 0$ it is imposed for regularity of $u(x,t)$ at $x = 0$, the profile f satisfies the following nonlinear Cauchy problems

$$\begin{cases} f'' + \frac{N-1}{\eta} f' = \sigma(\alpha f + \eta f') \cdot [\alpha f + \eta f'], & \eta > 0, \\ f(0) = 1, \\ f'(0) = 0. \end{cases} \quad (8)$$

The existence and uniqueness results follow from the Lipschitz regularity of the expression in the right side in the ODE (8). The solution f shall be described by means a collection of solutions of linear second-order ODEs that each of them can be expressed as Kummer equation [see (12) and (14)].

On the other hand, from (6) we notice that u_t also has similarity structure. In this case the profile is given by

$$F(\eta) = (\eta f'(\eta) + \alpha f(\eta))/2. \quad (9)$$

Using $\eta f'' = -(N-1)f' + \eta\sigma(F)F$ can be proved that $F' = (\alpha + 2 - N)f' + \eta\sigma(F)F$. Differentiating the later expression follows

$$F'' + \frac{N-1}{\eta} F' = \sigma(F)[(\alpha+1)F + \eta F']. \quad (10)$$

Representation for F is derived from recurrence relations of caloric functions. The initial conditions for F are considered as $F(0) = \alpha/2$ and $F'(0) = 0$.

2.2. Outline of the construction of the self-similar solutions

By means Sturm's oscillatory results can be proved that for each nonnegative integer n there exists α^* such that if $\alpha \geq \alpha^*$ then F has at least n zeros.

Considering that F has exactly n zeros, namely $0 < \eta_1 < \eta_2 < \cdots < \eta_n$, the profile f is defined by

$$f(\eta) = \begin{cases} f_1(\eta), & 0 \le \eta \le \eta_1, \\ \vdots & \\ f_{n+1}(\eta), & \eta_n \le \eta, \end{cases} \tag{11}$$

with $f_m(\eta)$ satisfying $(m = 1, 2, \ldots, n+1)$

$$\begin{cases} f_m'' + \frac{N-1}{\eta} f_m' = -\frac{(1+(-1)^m\gamma)}{2}(\alpha f_m + \eta f_m'), & \eta_{m-1} < \eta < \eta_m, \\ f_m(\eta_{m-1}) = f_{m-1}(\eta_{m-1}), & \\ f_m'(\eta_{m-1}) = f_{m-1}'(\eta_{m-1}). & \end{cases} \tag{12}$$

Here, $\eta_0 = 0$ and $\eta_{n+1} = \infty$. For each $m = 1, 2, \ldots, n+1$, f_m is constructed considering

$$s = -\frac{(1+(-1)^m\gamma)}{4}\eta^2, \quad g_m(s) = f_m(\eta). \tag{13}$$

From these substitutions, the ODE in (12) becomes

$$s g_m'' + ((N/2) - s) g_m' - (\alpha/2) g_m = 0, \tag{14}$$

which corresponds to Kummer-type second-order ODE with parameters $a = \alpha/2$ and $b = N/2$. Thus, each $f_m(\eta)$ is linear combination of confluent hypergeometric functions, specifically Kummer and Tricomi functions (see [1] Section 13).

Using the initial condition in (8), it follows that $f_1(\eta)$ is described through the Kummer function with the following series representation

$$M(\alpha/2, N/2; s) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha/2)_k}{(N/2)_k} \frac{s^k}{k!}. \tag{15}$$

Here, $(\lambda)_k$ denotes the Pochhammer symbol

$$(\lambda)_0 = 1, \quad (\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(k)} = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1). \tag{16}$$

From recurrence relations of this special function [see (30)], for the existence of η_1 it is necessary that

$$\alpha > (N-2)^+ = \max\{N-2, 0\}. \tag{17}$$

This relation is related to necessary and sufficiently condition for the existence of real (and positive) zeros of the Kummer function (below). This condition is related to the asymptotic feature shown in [20] which establishes $\alpha_0 \to (N-2)^+$ when $\gamma \to -1$ (see Theorem 3.2 in such article).

Finally, considering (17) we have that $f(\eta) = f_1(\eta)$ for $\alpha \le (N-2)^+$ and therefore their description is obtained using (15).

2.3. Auxiliary results on oscillatory behaviors

Let $\eta_1 < \eta_2 < \cdots < \eta_n$ be the n positive roots of the equation $\eta f' + \alpha f = 0$.

LEMMA 1. *The function $f(\eta)$ changes sign exactly once in $]\eta_{m-1}, \eta_m[$.*

Proof. Assume that $f(\eta)$ has two zeros, namely z_{m-1} and z_m. As $f'(z_{m-1}) \cdot f'(z_m) < 0$, we obtain a contradiction with the assumption that $\eta_{m-1} < \eta_m$ are consecutive. Now, assume that $f(\eta)$ does not change sign in $]\eta_1, \eta_2[$. Knowing that $f(\eta_1) > 0$, we continue working under the assumption $f(\eta) > 0$ in $]\eta_1, \eta_2[$ and therefore $f'(\eta) < 0$ in such interval (from $F(\eta; \alpha) < 0$ in $]\eta_1, \eta_2[$).

Directly from the definition of F and the form of the ODE for f in the branch $]\eta_1, \eta_2[$, we get $\left(e^{(1+\gamma)\eta^2/4} F\right)' = (\alpha - (N-2)) e^{(1+\gamma)\eta^2/4} f'$. Integrating the later relation over $[\eta_1, \eta_2]$ and using $F(\eta_1) = F(\eta_2) = 0$, we get

$$\int_{\eta_1}^{\eta_2} (\alpha - (N-2)) e^{(1+\gamma)\eta^2/4} f' d\eta = 0$$

Under the assumption that $f'(\eta) < 0$ in $]\eta_1, \eta_2[$ and knowing $\alpha > (N-2)^+$, a contradiction is obtained. Thus, the function $f(\eta)$ changes sign exactly once in the interval $]\eta_1, \eta_2[$. Finally, as

$$\mathrm{Sign}(\eta f_m' + \alpha f_m) = (-1)^{m+1} \qquad \text{when } \eta_{m-1} < \eta < \eta_m, \quad m = 1, 2, \ldots, n, \tag{18}$$

following the previous arguments, we obtain the result in each branch $]\eta_{m-1}, \eta_m[$. □

LEMMA 2. *Let $z_{m-1} < z_m$ be two consecutive zeros of $f(\eta)$. The function $F(\eta)$ changes sign exactly once in $]z_{m-1}, z_m[$.*

Proof. As $F(z_{m-1}) \cdot F(z_m) < 0$, $F(\eta)$ has at least one zero over $]z_{m-1}, z_m[$. If we assume that it has two zeros, we obtain a contradiction with Lemma 1. □

From the explicit representation of each f_m [see (28)] and using the implicit function theorem (IFT), it follows that η_m depends on α. Thus, it is possible to define a continuous function $h_m(\alpha) = f_m(\eta_m)$. The following result is a direct consequence of Lemma 1.

LEMMA 3. *The functions $h_m(\alpha)$ do not change signs.*

Proof. From Lemma 1 $f(\eta)$ follows $f(\eta_m) \cdot f(\eta_{m+1}) < 0$. As F is more oscillatory than f, the result is obtained from $f(\eta_1) > 0$. □

Through the above result, we have

$$\mathrm{Sign}(f(\eta_m)) = (-1)^{m+1} \qquad \text{for each } m = 1, 2, \ldots, n \tag{19}$$

Let us complete this section with the following result which will be used later to characterize the decay rate of $f(\eta)$ for large η.

LEMMA 4. *The function* $f(\eta)$ *does not change sign in* $]\eta_n, \infty[$.

Proof. From (18), the sign of $F(\eta)$ is determined by the parity of n. Let us consider the case n is odd, i.e., $\alpha f + \eta f' < 0$ in $]\eta_n, \infty[$ and from (19) we have $f(\eta_n) > 0$. On the other hand, by means Lemma 2 we know that f cannot change sign more than once in $]\eta_n, \infty[$. Hence, the proof is obtained by contradiction arguments assuming that there exists $\eta_* \in [\eta_n, \infty[$ such that $f(\eta_*) = 0$. From Lemma 1, we have $f(\eta) < 0$ in $]\eta_*, \infty[$. As $f(\eta) \to 0$ when $\eta \to \infty$, there exists $\eta^* > \eta_*$ such that $f'(\eta^*) = 0$ and $f'(\eta) > 0$ when $\eta > \eta^*$.

Using $f''(\eta; \alpha) \to 0$ when $\eta \to \infty$ and the ODE in (8) for $\eta > \eta^*$, we get $F \to 0$ as $\eta \to \infty$. Similar to the proof of Lemma 1, if $f'(\eta) > 0$ in $]\eta_n, \infty[$ and knowing that $\alpha > N - 2$, we have $(\alpha + 1)f' > (N - 1)f'$, i.e., $F' > -(1 + \gamma)\eta F/2$. Integrating $[\eta^*, \eta[$ and considering $F(\eta^*) = \alpha f(\eta^*) < 0$ follow

$$-F(\eta) > -\alpha f(\eta^*)e^{-(1+\gamma)(\eta^*)^2/4}e^{(1+\gamma)\eta^2/4},$$

and therefore, F is unbounded, obtaining a contradiction. Finally, if we assume $\alpha f + \eta f' > 0$ over $]\eta_n, \infty[$, taking $\tilde{F}(\eta) = -F(\eta)$, we obtain similar contradictions. □

To finish this subsection, we comment that through Lemmas 1 and 4, we get the following separation relation

$$\eta_1 < z_1 < \eta_2 < z_2 < \eta_3 < \cdots < \eta_{m-1} < z_{m-1} < \eta_m, \tag{20}$$

where $z_1 < z_2 < \cdots < z_{m-1}$ are the first $m - 1$ roots of f. This feature shall be used to obtain the oscillatory results of f.

3. Construction of f: closed representations and asymptotic behaviors

Given $\alpha > (N - 2)^+$ fixed, throughout this section we assume that $F(\eta)$ has exactly n positive zeros, namely $\eta_1 < \eta_2 < \cdots < \eta_n$. We also consider f satisfying the Cauchy problem (8) for this selected α. In the next presentation, we consider the following notations

$$a = \alpha/2,\ b = N/2,\ \lambda_m = -(1 + (-1)^m\gamma),\ s_m = \lambda_m(\eta_{m-1})^2/4. \tag{21}$$

Here, $M(a, b; s)$, $U(a, b; s)$ are used to denote the Kummer and Tricomi confluent hypergeometric functions and $\mathcal{W}_m$ to denote the Wronskian

$$\mathcal{W}\{M(a, b; s), U(a, b; s)\} = -\Gamma(b)s^{-b}e^s/\Gamma(a), \tag{22}$$

at s_m. We also studied the behavior of $f(\eta)$ for η large using the asymptotic representations of the Kummer and Tricomi functions given by

$$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}\left[\sum_{k=0}^{N}\frac{(a)_k(a-b+1)_k}{k!(-z)^k} + \mathcal{O}(|s|^{-N})\right] \quad (z<0), \tag{23a}$$

$$U(a,b;z) = z^{-a}\left[\sum_{k=0}^{N}\frac{(a)_k(a-b+1)_k}{k!(-z)^{-k}} + \mathcal{O}(|s|^{-N})\right]. \tag{23b}$$

The representation in (23a) is not valid for $b-a=-l$ with $l=0,1,2,\ldots$. In such case, the behavior is obtained by applying the Kummer transformation $M(a,b;z) = e^{-z}M(b-a,b;-z)$ leaving

$$M(a,b;z) = e^{-z}\sum_{k=0}^{l}\frac{(b-a)_k}{(b)_k}\frac{(-z)^k}{k!}. \tag{24}$$

In the study of Gaussian-type decay, it used the following confluent hypergeometric functions [see formula (10.09) in [21]]

$$V(a,b;z) = \frac{\Gamma(a)}{\Gamma(b)}e^{(b-a)i\pi}M(a,b;z) - \frac{\Gamma(a)}{\Gamma(b-a)}e^{bi\pi}U(a,b;z), \tag{25}$$

where $V(a,b;z)$ is also introduced as follows [see (10.03) in [21]]

$$V(a,b;z) = e^z U(b-a,b;-z). \tag{26}$$

Thus, the asymptotic representation of $V(a,b;z)$ is given by [see formula (10.02) in [21]]

$$V(a,b;z) = e^z(-z)^{a-b}\left[\sum_{k=0}^{N}\frac{(b-a)_k(1-a)_k}{k!z^k} + O(z^{-N})\right] \quad (z<0). \tag{27}$$

See Chapter 13 in [1] and Chapter 9 in [21] for further details for these special functions.

3.1. Closed representations

Let $2 \le m \le n$. From (14) we have[1]

$$f_m(\eta) = A_m(\alpha)M(a,b;\lambda_m\eta^2/4) + B_m(\alpha)U(a,b;\lambda_m\eta^2/4). \tag{28}$$

Using $f_m(\eta_{m-1}) = f_{m-1}(\eta_{m-1})$, $f'_m(\eta_{m-1}) = f'_{m-1}(\eta_{m-1})$, and $F_k(\eta_k)=0$, follows

$$\begin{aligned} A_m(\alpha) &= \frac{f_{m-1}(\eta_{m-1})}{s_m\mathcal{W}_m}\big(s_mU'(a,b;s_m) + aU(a,b;s_m)\big),\\ B_m(\alpha) &= -\frac{f_{m-1}(\eta_{m-1})}{s_m\mathcal{W}_m}\big(s_mM'(a,b;s_m) + aM(a,b;s_m)\big), \end{aligned} \tag{29}$$

[1] For $m=1$ we get $A_1(\alpha)=1$ and $B_1(\alpha)=0$.

From the recurrence relations (see formulae 13.4.11 and 13.4.23 in [1])

$$\begin{aligned} aM(a,b;s)+sM'(a,b;s) &= aM(a+1,b;s),\\ aU(a,b;s)+sU'(a,b;s) &= a(1+a-b)U(a+1,b;s), \end{aligned} \tag{30}$$

the constants in (29) are expressed as

$$A_m(\alpha) = a\frac{f_{m-1}(\eta_{m-1})}{s_m\mathcal{W}_m}(1+a-b)U(a+1,b;s_m), \tag{31a}$$

$$B_m(\alpha) = -a\frac{f_{m-1}(\eta_{m-1})}{s_m\mathcal{W}_m}M(a+1,b;s_m). \tag{31b}$$

On the other hand, each η_m is determined by means

$$A_m(\alpha)M(a+1,b;\lambda_m\eta^2/4) = -B_m(\alpha)(1+a-b)U(a+1,b;\lambda_m\eta^2/4).$$

From (31), the later relation becomes

$$U(a+1,b;s_m)M(a+1,b;\lambda_m\eta^2/4) = U(a+1,b;\lambda_m\eta^2/4)M(a+1,b;s_m). \tag{32}$$

3.2. Asymptotic representations

From (11), the asymptotic behavior of f is studied from

$$f_{n+1}(\eta) = A_{n+1}(\alpha)M(a,b;s)+B_{n+1}(\alpha)U(a,b;s), \tag{33}$$

with $s=\lambda_{n+1}\eta^2/4$ and $A_{n+1}(\alpha)$, $B_{n+1}(\alpha)$ described in (31).
Firstly, for η large and $\alpha = N+2l$ (with $l=0,1,2,\ldots$), if $B_2(\alpha)\neq 0$, $A_{n+1}(\alpha)$ $M(a,b;s)$ is recessive, whereas $B_{n+1}(\alpha)U(a,b;s)$ is dominant. Thus, in such case we have that

$$\begin{aligned} f_{n+1}(\eta) &\sim B(\alpha)U(a,b;s)\\ &\sim -\frac{\Gamma(a)}{\Gamma(b)}(-s_{n+1})^{b-1}f_n(\eta_n)M(a+1,b;s_{n+1})(-s)^{-a}+\mathcal{O}(|s|^{-(1+a)}) \end{aligned} \tag{34}$$

For the case $\alpha\neq N+2l$, from (25) we write

$$f_{n+1}(\eta) = C_{n+1}(\alpha)M(a,b;s)-B_{n+1}(\alpha)\frac{\Gamma(b-a)}{\Gamma(a)}e^{-b\pi i}V(a,b;s), \tag{35}$$

with

$$C_{n+1}(\alpha) = A_{n+1}(\alpha)+\frac{\Gamma(b-a)}{\Gamma(b)}e^{-ai\pi}B_{n+1}(\alpha). \tag{36}$$

From (31), the later relation becomes

$$C_{n+1}(\alpha) = a\frac{\Gamma(b-a)}{\Gamma(a)}\frac{f_n(\eta_n)}{W_{n+1}s_{n+1}}e^{-bi\pi}V(a+1,b,s_{n+1}). \tag{37}$$

We notice that $V(a,b;s)$ is recessive and $M(a,b;s)$ is dominant for η large. Hence, the change in the asymptotic behavior of $f_{n+1}(\eta)$ is related to the condition $C_{n+1}(\alpha)=0$.

Using Lemma 3, we get $(f_n(\eta_n))^2 > 0$ and therefore the condition $C_{n+1}(\alpha) = 0$ is reduced as

$$U(b-a-1, b; (1-(-1)^n\gamma)\eta_n^2/4) = 0, \tag{38}$$

where $V(a, b; s) = e^s U(b-a, b; -s)$. On the other hand, when $C_{n+1}(\alpha) = 0$, $f_{n+1}(\eta)$ takes the form

$$\begin{aligned} f_{n+1}(\eta) &= -B_{n+1}(\alpha)\frac{\Gamma(b-a)}{\Gamma(a)}e^{-b\pi i}V(a, b; s) \\ &= \frac{a\Gamma(b-a)}{\Gamma(b)}(-s_{n+1})^{b-1} f_n(\eta_n)M(b-a-1, b; -s_{n+1})V(a, b; s). \end{aligned} \tag{39}$$

Knowing that $U(b-a, b; -s) > 0$ for $-s$ large [see (23b)] and $(-1)^{n+1} f_n(\eta_n) > 0$ (see Lemma 3), $f_{n+1}(\eta)$ has the sign of

$$(-1)^{n+1}\Gamma(b-a) \cdot M(b-a-1, b; -s_{n+1}). \tag{40}$$

Finally, from (23) and (27) the asymptotic behaviors for η large are described by

$$f_{n+1}(\eta) \sim [\tilde{C}_{n+1}(\alpha) + \mathcal{O}(\eta^{-2})]\eta^{-\alpha}, \tag{41a}$$

$$f_{n+1}(\eta) \sim [D_{n+1}(\alpha) + \mathcal{O}(\eta^{-2})]\eta^{\alpha-N}e^{-(1-(-1)^n\gamma)\eta^2/4}, \tag{41b}$$

where the first representation is obtained when $C_{n+1}(\alpha) \neq 0$, while the second one for $C_{n+1}(\alpha) = 0$. Here, $\tilde{C}_{n+1}(\alpha) = ((1-(-1)^n\gamma)/4)^{-\alpha/2}C_{n+1}(\alpha)$ and

$$D_{n+1}(\alpha) = \frac{\alpha\Gamma(b-a)}{2\Gamma(b)}(-\lambda_{n+1})^{b-a}(-s_{n+1})^{b-1} f_n(\eta_n)M(b-a-1, b; -s_{n+1}). \tag{42}$$

We notice that the sign of $D_{n+1}(\alpha)$ is determined from (40).

4. Proofs of Theorems

Proof Theorem 1. Fixing $k \geq 1$, we assume the existence of exponents $\alpha_0 < \cdots < \alpha_{k-1}$. The profile associated with α_{k-1} is defined through (11) with $n = k$, and its asymptotic behavior is given in (41b) from $C_k(\alpha_{k-1}) = 0$. From the behavior indicated in (23) and applying Lemmas 3, 4, we get the existence of $\alpha^* > \alpha_{k-1}$, the largest value for which f has exactly one zero on $]\eta_k, \infty[$, namely z_k. The existence of such exponent follows from Sturm's comparison arguments. Using Lemma 1, let $\eta_{k+1} > z_k$ be the largest zero of F defined in (9) [see relation (20)]. For each $\alpha_{k-1} < \alpha < \alpha^*$, the profile f is defined in (11) with $n = k+1$ where f_{k+1} is described in (35). From Lemma 4, we get $(C_{k+1}(\alpha))^2 > 0$ and therefore the asymptotic representation is given in (41a). Using continuity arguments follows $C_{k+1}(\alpha^*) = 0$, obtaining the asymptotic Gaussian-type representation (41b). Taking $\alpha_k = \alpha^*$ and knowing the existence of α_0, the proof is obtained from induction procedure. □

Proof Theorem 2. Let $\alpha_0 < \cdots < \alpha_{k-1}$ be the anomalous exponents indicated in Theorem 1. Consider $\alpha > \alpha_{k-1}$ such that F has exactly $k+1$ roots, namely $\eta_1 < \cdots < \eta_{k+1}$. For $i = 1, \ldots, k+1$, let $G_i(\eta_1, \ldots, \eta_{k+1}, \alpha, \gamma) = \eta_i f_i' + \alpha f_i$ and $G_{k+2}(\eta_1, \ldots, \eta_{k+1}, \alpha, \gamma) = \eta_{k+1}(f_{k+1}'/f_{k+1} - f_{k+2}'/f_{k+2})$. Consider $G : \mathbb{R}^{k+3} \to \mathbb{R}^{k+2}$ given by $G = (G_1, \ldots, G_{k+2})$. The regularity of each f_i implies that $G \in \mathcal{C}^1$. We study features of α_k employing arguments of implicit function from the system $G_i = 0$. Using the equations $G_1 = \cdots = G_{k+1} = 0$, we get $\eta_1, \ldots, \eta_{k+1}$ as functions of γ. On the other hand, from the last equation in the system, we get

$$\frac{\partial \alpha_k}{\partial \gamma} = -\frac{(\partial/\partial\gamma)G_{k+2}}{(\partial/\partial\alpha)G_{k+2}}. \tag{43}$$

Each term in the right side is obtained from (12). Let $P_m(\eta) = \eta^{N-1}e^{\lambda_m \eta^2/4}$ and $\varphi_m = P_m f_m'/f_m$ defined in $]\eta_{m-1}, \eta_m[\backslash\{z_{m-1}\}$. We use $\lambda_m = (1 + (-1)^m\gamma)$ and z_{m-1} to denote the unique root of f_m in $]\eta_{m-1}, \eta_m[$ (see Lemma 1). Using the self-adjoint representation of each (12), we get

$$\varphi_m' = -(\alpha/2)\lambda_m P_m - \varphi_m^2/P_m.$$

Considering $\phi_m = (\partial/\partial\alpha)\varphi_m$ and taking the derivative with respect to α in the later equation, we get

$$(f_m^2\phi_m)' = -\lambda_m P_m f_m^2/2. \tag{44}$$

As the limit $f_m^2\phi_m$ at $\eta \to z_{m-1}$ is well defined, Eq. (44) can be integrated over $[\eta_{m-1}, \eta_m]$ obtaining[2]

$$[f_m^2\phi_m](\eta_m) - [f_m^2\phi_m](\eta_{m-1}) = -(\lambda_m/2)\int_{\eta_{m-1}}^{\eta_m} P_m(w) f_m^2(w)\mathrm{d}w. \tag{45}$$

Taking $\gamma \to 0$ and considering linear combination of each (45) ($m = k+1, \ldots, 1$), we have

$$f^2(\eta_{k+1})\phi_{k+1}(\eta_{k+1}) = f_1^2(0)\phi_1(0) - (1/2)\int_0^{\eta_{k+1}} P(w) f^2(w)\mathrm{d}w$$

with $P(\eta) = \eta^{N-1}e^{\eta^2/4}$ and $f(w) = M(\alpha_k(0)/2, N/2, -\eta^2/4)$. From the initial condition in (8), it follows that $\phi_m(0) = 0$, and using the Kummer transformation, we obtain

$$\phi_{k+1}(\eta_{k+1}) = -\frac{1}{2f^2(\eta_{k+1})}\int_0^{\eta_{k+1}} w^{N-1}e^{-w^2/4}[M(-k, N/2, w^2/4)]^2\mathrm{d}w \tag{46}$$

where $(N - \alpha_k(0))/2 = -k$. On the other hand, knowing that $f_{k+2}^2\phi_{k+2} \to 0$ as $\eta \to \infty$, the integration of (45) over $[\eta_{k+1}, \infty[$ becomes

$$-[f_{k+2}^2\phi_{k+2}](\eta_{k+1}) = -(\lambda_{k+2}/2)\int_{\eta_{k+1}}^{\infty} P_{k+2}(w) f_{k+2}^2(w)\mathrm{d}w.$$

[2] Here, $[f_m^2\phi_m](\eta_m)$ denotes $f_m^2(\eta_m)\phi_m(\eta_m)$, similar as $[f_m^2\phi_m](\eta_{m-1})$.

Considering $\gamma \to 0$ in the later equation and combined with (46), we obtain the following relation

$$\phi_{k+1}(\eta_{k+1}) - \phi_{k+2}(\eta_{k+1}) = -\frac{1}{2f^2(\eta_{k+1})}\int_0^\infty w^{N-1}e^{w^2/4}f^2(w)\mathrm{d}w. \tag{47}$$

Let $\nu = (N/2) - 1$ and L_m^ν the (n, ν)-Laguerre polynomial. Using the Kummer transformation and the relation $M(-k, N/2, x) = k!L_k^\nu(x)/(N/2)_k$, from (47), it follows

$$\frac{\partial G_{k+2}}{\partial \alpha} = -2^{N-1}\frac{\eta_{k+1}^{2-N}e^{\eta_{k+1}^2/4}}{2(L_k^\nu(\eta_{k+1}^2/4))^2}\int_0^\infty z^\nu e^{-z}(L_k^\nu(z))^2\mathrm{d}z$$

Denoting by $v_{k+1} = \eta_{k+1}^2/4$ the $(k+1)$-th root of L_{k+1}^ν [see recurrence relation (30)] and using the formula 7.414. 3 in [12], the first relation can be obtained

$$\frac{\partial G_{k+2}}{\partial \alpha} = -\frac{v_{k+1}^{-\nu}e^{v_{k+1}}}{[L_k^\nu(v_{k+1})]^2}\frac{\Gamma((N/2)+k))}{k!}. \tag{48}$$

For the calculation of the numerator in (43), the auxiliary variables $z^i = -\lambda_i\eta^2/4$ and $\varphi_i(z^i) = f_i(\eta)$ were used, where $\eta f_i'/f_i = 2z^i\varphi_i'/\varphi_i$.
Directly $(\partial/\partial\gamma)(z^i\varphi_i'/\varphi_i) = (z^i\varphi_i'/\varphi_i)'(\partial/\partial\gamma)z^i$. As $(\partial/\partial)z^i = (-1)^i(-\eta^2/4)$, from the ODE (14) and taking $\gamma \to 0$, we have

$$\frac{\partial G_{k+2}}{\partial \gamma} = 4(-1)^k(k+1)((N/2)+k). \tag{49}$$

Considering (48) and (49) into (43) follows the relation given in (5). □

We comment that the integral relation used in (48) is valid for $\nu > 0$, i.e., for $N \geq 3$. The cases $N = 1, 2$ are derived directly from the representations of $L_0(z)$, $L_1(z)$ and $\Gamma(z)$.

5. Applications and brief extensions

Close results from Theorem 1 were developed in [6] for positively homogeneous functions

$$w(x,t) = (-t)^{\beta/2}G(\|x\|/\sqrt{-t}), \quad t < 0,\ \beta > 0, \tag{50}$$

with $w = v_t$, and v solution of (2). The authors show the existence of a unbounded sequence of similarity exponents β for which (50) is locally bounded. Employing the procedure presented in Sect. 3, we notice that representation for each profile G is obtained in terms of confluent hypergeometric. Moreover, from recurrence formulae indicated in (30), it follows that v also has self-similar structure with profile g for which $G(\eta) = (1/2)(\tilde{\eta}g' - (\beta+2)g)$, with $\eta = \|x\|/\sqrt{-t}$ and $g(0) = 1$.

Considering Appell transform arguments presented in Section 4 in [6], the later feature is synthesized as follows

$$\alpha_k = \beta_{k+1} + N - 2. \tag{51}$$

Thus, profiles for counterexample to $\mathcal{C}^{2,1}$ regularity are described by

$$g(\eta) = \begin{cases} M(-1-\beta/2, N/2, (1-\gamma)\eta^2/4), & \text{when } \eta \leq \eta_1, \\ \tilde{\lambda} U(-1-\beta/2, N/2, (1+\gamma)\eta^2/4), & \text{when } \eta \geq \eta_1, \end{cases} \tag{52}$$

where $\tilde{\lambda} = \lambda/(a(1+a-b))$ with $a = -1-(\beta/2)$ and $b = N/2$. The values of β_1 and η_1 are obtained from

$$M(b-(a+1), b, z) = 0, \tag{53a}$$

$$U(b-(a+1), b, \lambda z) = 0, \tag{53b}$$

with $a = \alpha_0/2$, $b = N/2$ and $\lambda = (1+\gamma)/(1-\gamma)$. From the results presented in [2] and using (51), it follows

$$\frac{\beta_1}{2} = 1 + \frac{2(N/2)^{N/2}}{\Gamma(N/2)e^{N/2}}\gamma + o(\gamma), \tag{54}$$

for $\gamma < 0$ sufficiently close to zero. Moreover, it is possible to show that α_0 has a change of concavity at $\gamma = 0$, and therefore, from (54) a upper estimation of the Hölder exponents for the parabolic regularity of v can be obtained.

A remarkable fact of the previous analysis is the relationship between similarity solutions and some general aspects for solutions of parabolic equations. In this sense, we comment that our attention in (1) is related to their simple mathematical structure associated with HJB-type equation of the form

$$u_t = F(D^2u), \quad \text{in } \mathbb{R}^N \times]0, \infty[, \tag{55}$$

with $F(\cdot)$ elliptic operator defined on matrix space and D^2u the Hessian matrix of u respect to x (see [9] for further details in general models described in terms of HJB equations).

A special class of Eq. (55) is defined considering the following matrix operators

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM), \quad \mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM), \tag{56}$$

where $\mathcal{A}_{\lambda,\Lambda}$ denotes the set of symmetric matrix with eigenvalues belong in $[\lambda, \Lambda]$. These extremal operators are useful for the qualitative analysis of equations (55) with F satisfying

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \leq F(M) \leq \mathcal{M}^+_{\lambda,\Lambda}(M). \tag{57}$$

Through comparison arguments, preliminaries features for equations (55) can be obtained from results on PDE with diffusion term defined by means (56).

Some features treated in the current work can be extended for these extremal diffusion equations. In such cases, radial self-similar solutions are described by f satisfying similar Cauchy problem as (8). Such solutions can be useful to construct sub- and superviscosity solutions.

6. Conclusions

The Barenblatt equation of elasto-plastic filtration was analytically investigated. From the nonlinearity of this equation, several results treated here can be applied on other equations with more complex structure.

The proposed procedure is based on the connection between linear Cauchy problems, which allows constructing radial symmetric self-similar solutions in terms of confluent hypergeometric functions. The arguments in the proofs were mostly developed using the properties of these special functions.

The analysis was developed for $-1 < \gamma < 1$, knowing that the results for the linear heat equation are obtained when $\gamma \to 0$. Oscillatory and asymptotic features of f are qualitatively similar in the full parametric range. Nevertheless, differences in the behaviors of the anomalous exponents relative to the sign exist, as can be observed in the sign of the derivative calculated in Theorem 2.

Although physical approaches for this equations had been derived in the past, such as descriptions of the features of nonnegative solutions, this article studied the complete family of self-similar solutions with the form (3), obtaining a full description for the self-similar solutions of the second kind, as well as elemental properties for the anomalous exponents.

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