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# Global well-posedness and blow-up solutions of the Cauchy problem for a time-fractional superdiffusion equation

Quanguo Zhang and Yaning Li

*Abstract.* We study the following time-fractional nonlinear superdiffusion equation

$$\begin{cases} {}_0^C D_t^\alpha u - \triangle u = |u|^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^N, \end{cases}$$

where $1 < \alpha < 2$, $p > 1$, $u_0, u_1 \in L^q(\mathbb{R}^N)$ $(q > 1)$ and ${}_0^C D_t^\alpha u$ denotes the Caputo fractional derivative of order $\alpha$. The critical exponents of this problem are determined when $u_1 \equiv 0$ and $u_1 \not\equiv 0$, respectively.

## 1. Introduction

Fractional differential equations are very useful to describe the phenomena of anomalous diffusion, Hamiltonian chaos, dynamical systems with chaotic dynamical behavior, etc. see [16,21,27] and the references therein. In recent years, the time-fractional diffusion equation has received extensive attentions and mathematical treatments have produced many results, see [4,6,7,17–20,23,28,31–33,35–37] and the references therein. For example, in [31], the existence and properties of solutions for a time-fractional equation in a bounded domain were considered by applying the eigenfunction expansions. In [6], the quasilinear abstract time fractional evolution equations were studied in continuous interpolation spaces. Zacher [36] established maximal regularity results of type $L^p$ for abstract parabolic Volterra equations including problems with inhomogeneous boundary data by using the purely operator theories. In [17], the authors gave an $L_q(L_p)$-theory for the semilinear time-fractional equations in $\mathbb{R}^d$ with variable coefficients by classical tools in PDE theories such as the Marcinkiewicz interpolation theorem, the Calderon–Zygmund theorem and perturbation arguments.

This paper is concerned with the blow-up and global existence of solutions to the Cauchy problem for a nonlinear time-fractional superdiffusion equation

$$\begin{cases} {}_0^C D_t^\alpha u - \triangle u = |u|^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $1 < \alpha < 2$, $p > 1$, $u_0, u_1 \in L^q(\mathbb{R}^N)$ $(q > 1)$ and

$$ {}_0^C D_t^\alpha u = \frac{\partial^2}{\partial t^2} {}_0 I_t^{2-\alpha}(u(t,x) - u_t(0,x)t - u(0,x)), $$

${}_0 I_t^{2-\alpha}$ denotes the left Riemann–Liouville fractional integral of order $2-\alpha$.

Our interest in studying problem (1.1) comes from its application as a model for physical systems exhibiting anomalous diffusion. In many complex systems, diffusion processes usually no longer follow Gaussian statistics, and thus, Fick's second law fails to describe the related transport behavior. In classical diffusion, the linear time dependence of the mean squared displacement can be observed, which describes how fast particles diffuse, whereas, in anomalous diffusion, the mean squared displacement of a diffusive particle usually behaves like const$\cdot t^\alpha$ as $t \to \infty$. The diffusion process is called subdiffusion process for $0 < \alpha < 1$, and superdiffusion process for $1 < \alpha < 2$, see, e.g., [12,22].

For the semilinear heat equation

$$ \begin{cases} u_t - \Delta u = |u|^{p-1}u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2} $$

it is well known that the number $p = 1 + \frac{2}{N}$ is the critical exponent of this problem. If $1 < p \le 1 + \frac{2}{N}$ and $u_0 \ge 0$, then any nontrivial solution of (1.2) blows up in a finite time, while if $p > 1 + \frac{2}{N}$ and the initial value $u_0$ is small enough in $L^{q_c}(\mathbb{R}^N)$ where $q_c = \frac{N(p-1)}{2}$, then the solution of (1.2) exists globally. We refer to [30] for details on these results.

For the semilinear wave equation

$$ \begin{cases} u_{tt} - \Delta u = |u|^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.3} $$

the critical exponent is $p_c(N)$, which is the positive root of $(N-1)p^2 - (N+1)p - 2 = 0$. If $1 < p \le p_c(N)$, then global solutions of (1.3) do not exist, provided that $u_0, u_1$ have compact support and satisfy a certain positivity condition, while if $p > p_c(N)$, then solutions with small initial values exist for all time, see Yordanov and the references therein. A slightly less sharp result under much weaker assumptions was obtained by Kato [15]. Kato proved that if $1 < p \le \frac{N+1}{N-1}$, then problem (1.3) admits no global solution.

When $u_1 \equiv 0$, we can rewrite (1.1) as

$$ \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-s)^{\alpha-2}[\Delta u + |u|^p(s,x)]\mathrm{d}s, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.4} $$

Recently, this problem has received the attention of many authors, see, e.g., [1,8,10,11, 13,14,24]. For the linear version of this problem, these articles consider the existence, uniqueness, asymptotic behavior of the solution and the properties of fundamental solution. For the nonlinear version of this problem, Hirato and Miao [14] obtained the global existence of solutions for (1.4) with small initial data in $L^{\frac{(p-1)N}{2}}(\mathbb{R}^N)$ when $p > 1 + \max\{\alpha, \frac{2}{N}\}$. Miao and Yang [24] proved the global existence of self-similar solutions for (1.4) with small initial data in a subset of the critical Besov space. In [1], De Almeida and Ferreira showed the global existence of solutions for (1.4) with small initial data in the critical Morrey space. Let $1 < s < \frac{N(p-1)}{2}$, $\mu = N - \frac{2s}{p-1}$, and let $\|u\|_{s,\mu}$ denote the norm of Morrey space $M_{s,\mu}(\mathbb{R}^N)$. They obtained that if

$$\frac{N-\mu}{s} - \frac{N-\mu}{q} < 2,\ 1 - \frac{1}{\alpha}\frac{p-1}{p} < \frac{s}{q} < \frac{1}{\alpha} \text{ and } \frac{(p-1)s}{s-1} < q < \infty,$$

then there exist $\varepsilon > 0$ and $\delta(\varepsilon)$ such that when $\|u_0\|_{s,\mu} \leq \delta$, problem (1.4) has a mild solution $u \in H_q = \{u \in BC((0,\infty), M_{s,\mu}(\mathbb{R}^N)) \mid t^\beta u \in BC((0,\infty), M_{q,\mu}(\mathbb{R}^N))\}$, where $\beta = \frac{\alpha}{p-1} - \frac{\alpha(N-\mu)}{2q}$.

Recently, there are many papers that considered the existence and nonexistence of the global solutions to semilinear time-fractional subdiffusion equations and semilinear diffusion equations with nonlinear memory.

In [9], Fino and Kirane considered the following heat equation with nonlinear memory

$$\begin{cases} u_t + (-\triangle)^{\frac{\beta}{2}} u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} u \mathrm{d}s, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $p > 1$, $0 < \beta \leq 2$, $0 \leq \gamma < 1$ and $u_0 \in C_0(\mathbb{R}^N)$. They generalized the results of [5] to the case of the fractional differential equation. Using the test function method [25], they obtained the blow-up results of (1.5) and then determined the Fujita critical exponent of this problem.

Zhang and Sun [38] considered the following time-fractional subdiffusion equation

$$\begin{cases} {}_0^C D_t^\alpha u - \triangle u = |u|^{p-1} u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $0 < \alpha < 1$, $p > 1$, and proved that the Fujita critical exponent of (1.6) also is $1 + \frac{2}{N}$. The major difference between problem (1.6) and (1.2) is that the positive solution of (1.6) can exist globally when $p = 1 + \frac{2}{N}$.

Motivated by the aforementioned results, in this paper, we study problem (1.1) and determine the critical exponents of (1.1). In particular, for $u_0, u_1 \in L^q(\mathbb{R}^N)(q > \max\{\frac{Np\alpha}{2}, 1\})$, we will show that the following results.

- (i) If $1 < p \leq 1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha}$ and $u_0 \geq 0$, $u_0 \not\equiv 0$, then all solutions of (1.4) blow up in finite time.
- (ii) If $p > 1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha}$, then the solution of (1.4) exists globally when $\|u_0\|_{L^{q_c}(\mathbb{R}^N)}$ is sufficiently small, where $q_c = \frac{N(p-1)}{2}$.

(iii) If $N > 1$, $1 < p < 1 + \frac{2\alpha}{\alpha N - 2}$ and $u_0, u_1 \geq 0$, $u_1 \not\equiv 0$, then any solution of (1.1) blows up in finite time. If $N = 1$ and $u_0, u_1 \geq 0$, then any nontrivial solution of (1.1) blows up in finite time for every $p > 1$.

(iv) If $N \geq 2$, $p > 1 + \frac{2\alpha}{\alpha N - 2}$ and $\|u_0\|_{L^{q_c}}$, $\|u_1\|_{L^{\tilde{q}_c}}$ are sufficiently small, where $q_c = \frac{N(p-1)}{2}$ and $\tilde{q}_c = \frac{\alpha N(p-1)}{2(\alpha+p-1)}$, then the solution of (1.1) exists globally.

Equation (1.1) interpolates the heat equation and the wave equation. For the case $u_1 \equiv 0$, the critical exponent $1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha} \to 1 + \frac{2}{N}$ as $\alpha \to 1$, which is the Fujita critical exponent of problem (1.2). As $\alpha \to 2$, the critical exponents $1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha}$ and $1 + \frac{2\alpha}{\alpha N - 2}$ tend to $\frac{N+1}{N-1}$, which is an exponent that appears in a paper by Kato [15]. Comparing with the classical results of the heat equation (1.2) and the wave equation (1.3), the conclusions of problem (1.4) are analogous to the results of (1.2), but the difference between the time-fractional equation (1.1) with $u_1 \not\equiv 0$ and the wave equation (1.3) is very apparent.

In [5], Cazenave, Dickstein and Weissler proved that for (1.5) with $\beta = 2$, the Fujita critical exponent is not the one which would be predicted from the scaling properties of the equation. For (1.1) with $u_1 \not\equiv 0$, we can also find the critical exponent by the scaling properties of the equation. But for (1.4), we cannot obtain the critical exponent by the scaling properties of (1.4). In fact, if $u(t, x)$ is a solution of (1.1) with initial values $u_0(x)$ and $u_1(x)$, then, for every $\lambda > 0$, $\lambda^{\frac{2\alpha}{p-1}} u(\lambda^2 t, \lambda^\alpha x)$ is also a solution of (1.1) with initial values $\lambda^{\frac{2\alpha}{p-1}} u_0(\lambda^\alpha x)$ and $\lambda^{\frac{2\alpha}{p-1}+2} u_1(\lambda^\alpha x)$. Since

$$\|\lambda^{\frac{2\alpha}{p-1}} u_0(\lambda^\alpha \cdot)\|_{L^q(\mathbb{R}^N)} = \lambda^{\frac{2\alpha}{p-1} - \frac{\alpha N}{q}} \|u_0\|_{L^q(\mathbb{R}^N)}, \tag{1.7}$$

$$\|\lambda^{\frac{2\alpha}{p-1}+2} u_1(\lambda^\alpha \cdot)\|_{L^q(\mathbb{R}^N)} = \lambda^{\frac{2\alpha+2p-2}{p-1} - \frac{\alpha N}{q}} \|u_1\|_{L^q(\mathbb{R}^N)}, \tag{1.8}$$

it follows that the invariant Lebesgue norms in $L^q(\mathbb{R}^N)$ for (1.7) and (1.8) are given by $q_c = \frac{N(p-1)}{2}$ and $\tilde{q}_c = \frac{\alpha N(p-1)}{2(\alpha+p-1)}$, respectively. Note that $q_c > 1$ if and only if $p > 1 + \frac{2}{N}$, and $\tilde{q}_c > 1$ if and only if $p > 1 + \frac{2\alpha}{\alpha N - 2}$. Thus, one predicts $1 + \frac{2}{N}$ is the critical exponent of (1.4) and $1 + \frac{2\alpha}{\alpha N - 2}$ is the critical exponent of (1.1) with $u_1 \not\equiv 0$. Our results show that $1 + \frac{2}{N}$ is not the critical exponent of (1.1) with $u_1 \equiv 0$ and $1 + \frac{2\alpha}{\alpha N - 2}$ is really the critical exponent of (1.1) with $u_1 \not\equiv 0$.

This paper is organized as follows: In Sect. 2, some preliminaries are presented. In Sect. 3, we give some abstract results that are used to derive our main results in the next sections. Section 4 is devoted to the local existence and uniqueness of mild solutions of problem (1.1). In Sect. 5, we show the blow-up and global existence of the solutions to problem (1.1).

## 2. Preliminaries

In this section, we present some results about the fractional derivatives and the fractional integrals that will be used in the next sections (see [16,27]).

For $T > 0$, $\alpha, \beta \in (0, 2]$, the Riemann–Liouville fractional integrals are defined by

$$ {}_0I_t^\alpha u = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad {}_tI_T^\alpha u = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{u(s)}{(s-t)^{1-\alpha}} \mathrm{d}s. $$

The operators ${}_0I_t^\alpha$ and ${}_tI_T^\alpha$ are bounded on $L^p((0, T))$ for $1 \le p \le +\infty$. Moreover, ${}_0I_t^\alpha {}_0I_t^\beta f = {}_0I_t^{\alpha+\beta} f$ and ${}_tI_T^\alpha {}_tI_T^\beta f = {}_tI_T^{\alpha+\beta} f$ if $f \in L^1((0, T))$. In addition, if $f \in L^p((0, T))$, $g \in L^q((0, T))$ and $p, q \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ \int_0^T {}_0I_t^\alpha f \cdot g \mathrm{d}t = \int_0^T f \cdot {}_tI_T^\alpha g \mathrm{d}t. $$

For $\alpha \in (1, 2]$ and $T > 0$, the Caputo fractional derivatives satisfy that if $g \in AC^2([0, T])$, then ${}_0^C D_t^\alpha g$ and ${}_t^C D_T^\alpha g$ a.e. exist on $[0, T]$ and

$$ {}_0^C D_t^\alpha g = \frac{d^2}{\mathrm{d}t^2} {}_0I_t^{2-\alpha} [g(s) - g'(0)s - g(0)] = {}_0I_t^{2-\alpha} g'', $$
$$ {}_t^C D_T^\alpha g = \frac{d^2}{\mathrm{d}t^2} {}_tI_T^{2-\alpha} [g(t) - g'(T)t - g(T)] = {}_tI_T^{2-\alpha} g''. $$

Assuming that $f \in C^1([0, T])$, ${}_0^C D_t^\alpha f \in L^1(0, T)$, $g \in AC^2([0, T])$ and $g(T) = g'(T) = 0$, we have the following formula of integration by parts

$$ \int_0^T {}_0^C D_t^\alpha f \cdot g \mathrm{d}t = \int_0^T (f(t) - f'(0)t - f(0))\, {}_t^C D_T^\alpha g \mathrm{d}t. \tag{2.1} $$

The Mittag–Leffler function is defined for complex $z \in \mathbb{C}$ as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \beta \in \mathbb{C}, \ \mathrm{Re}(\alpha) > 0, \quad E_\alpha(z) = E_{\alpha,1}(z). $$

It is an entire function and satisfies

$$ {}_0I_t^{2-\alpha} (t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)) = t E_{\alpha,2}(\lambda t^\alpha) \text{ for } \lambda \in \mathbb{C}, \ 1 < \alpha < 2, \tag{2.2} $$
$$ {}_0I_t^{2-\alpha} (t^{\alpha-2} E_{\alpha,\alpha-1}(\lambda t^\alpha)) = E_\alpha(\lambda t^\alpha) \text{ for } \lambda \in \mathbb{C}, \ 1 < \alpha < 2, \tag{2.3} $$
$$ {}_0I_t^{\alpha-1} E_\alpha(\lambda t^\alpha)) = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) \text{ for } \lambda \in \mathbb{C}, \ 1 < \alpha < 2, \tag{2.4} $$
$$ \frac{\mathrm{d}}{\mathrm{d}t} [t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)] = t^{\alpha-2} E_{\alpha,\alpha-1}(\lambda t^\alpha), \ \lambda \in \mathbb{C}, \ t > 0, \ 1 < \alpha < 2, \tag{2.5} $$
$$ {}_0I_t^1 E_\alpha(\lambda t^\alpha) = t E_{\alpha,2}(\lambda t^\alpha). \tag{2.6} $$

Let $0 < \alpha < 2$, $\varepsilon > 0$ and let $\mu$ be a real number such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$. Then the function $E_{\alpha,\beta}(z)$ has the integral representation [27]

$$ E_{\alpha,\beta}(z) = \frac{1}{2\alpha\pi i} \int_{\gamma(\varepsilon,\mu)} \frac{\exp(\zeta^{\frac{1}{\alpha}}) \zeta^{\frac{1-\beta}{\alpha}}}{\zeta - z} d\zeta, \ z \in G^-(\varepsilon, \mu), \tag{2.7} $$

where $\gamma(\varepsilon,\mu)=\{re^{i\mu}|r\geq\varepsilon\}\cup\{re^{-i\mu}|r\geq\varepsilon\}\cup\{\varepsilon e^{i\theta}|-\mu\leq\theta\leq\mu\}$, $\xi^{\frac{1}{\alpha}}$ denotes the principal branch of $\xi^{\frac{1}{\alpha}}$, and $G^{-}(\varepsilon,\mu)$ denotes the domain lying on the left side of the contour $\gamma(\varepsilon,\mu)$. Moreover,

$$E_{\alpha,\beta}(z)=-\sum_{k=1}^{N}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k}+O\left(\frac{1}{z^{N+1}}\right), \tag{2.8}$$

with $|z|\to\infty$, $\mu\leq|\arg(z)|\leq\pi$, and $N\in\mathbb{N}$. In particular,

$$E_{\alpha}(z)=E_{\alpha,1}(z)=-\frac{1}{\Gamma(1-\alpha)}\frac{1}{z}+O\left(\frac{1}{z^2}\right),\ E_{\alpha,\alpha}(z)=-\frac{1}{\Gamma(-\alpha)}\frac{1}{z^2}+O\left(\frac{1}{z^3}\right),$$

with $|z|\to\infty$ and $\mu\leq|\arg(z)|\leq\pi$.

We also need to calculate the Caputo fractional derivative of the following function. For given $T>0$ and $n>1$, let $\varphi(t)=(1-\frac{t}{T})^n$, $t\leq T$. Then

$${}_t^C D_T^{\alpha}\varphi(t)=\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}T^{-\alpha}\left(1-\frac{t}{T}\right)^{n-\alpha},\ t\leq T,$$

(see, e.g., [16]).

For simplicity, in this paper, we use $C$ to denote a positive constant which may vary from line to line, but it is not essential for the analysis of the problem.

## 3. Some abstract results

Let $X$ be a Banach space with norm $\|\cdot\|$. In this section, we suppose that $A$ satisfies the following:

(i) $A: D(A)\subset X\to X$ is a densely defined and closed operator.
(ii) There exist $C>0$ and $\theta\in(0,\pi(1-\frac{\alpha}{2}))$ such that

$$S_\theta=\{\lambda\in\mathbb{C}\mid\lambda\neq 0,\ \theta\leq|\arg\lambda|\leq\pi\}\subset\rho(A)$$

and $\|(\lambda I-A)^{-1}\|\leq\frac{C}{|\lambda|}$, $\lambda\in S_\theta$.

Similar to [20], we define the following two operators.

DEFINITION 3.1. Let $\alpha\in(1,2)$. For every $u_0\in X$, we define the operators $P_\alpha(t)$ and $S_\alpha(t)$ as

$$P_\alpha(t)u_0=\frac{1}{2\pi i}\int_\Gamma E_\alpha(\lambda t^\alpha)(\lambda I+A)^{-1}u_0d\lambda,\ t>0,\ \text{and}\ P_\alpha(0)u_0=u_0, \tag{3.1}$$

$$S_\alpha(t)u_0=\frac{1}{2\pi i}\int_\Gamma E_{\alpha,\alpha}(\lambda t^\alpha)(\lambda I+A)^{-1}u_0d\lambda,\ t>0,\ \text{and}\ S_\alpha(0)u_0=\frac{u_0}{\Gamma(\alpha)}, \tag{3.2}$$

where $\Gamma\in\{\gamma(\varepsilon,\varphi)\subseteq\rho(-A)\mid\varepsilon>0,\ \frac{\pi\alpha}{2}<\varphi<\pi-\theta\}$.

REMARK 3.2. By (2.8) and using Cauchy's integral theorem, we know that $P_\alpha(t)$ and $S_\alpha(t)$ are well defined and independent of $\varphi$ and $\varepsilon$. The operator $P_\alpha(t)$ corresponds to the resolvent family of [29].

Next we give some properties of the operators $P_\alpha(t)$ and $S_\alpha(t)$. Some of these properties have been obtained in [2] and [29]. For the convenience of proving our main results and the completeness of the paper, here we give all the properties.

LEMMA 3.3. *The operators $P_\alpha(t)$ and $S_\alpha(t)$ have the following properties.*

(i) *For every $u_0 \in X$ and $t > 0$, we have*

$$P_\alpha(t)u_0 = \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}u_0 d\lambda,\ \mu \in \left(\frac{\pi\alpha}{2}, \pi - \theta\right),\ \varepsilon > 0,$$

*$P_\alpha(t)u_0 \in C([0,+\infty), X)$ and $P_\alpha(t)u_0 \in D(A)$. Moreover, there exists a constant $C > 0$ such that*

$$\|P_\alpha(t)u_0\| \le C\|u_0\|, \quad \|AP_\alpha(t)u_0\| \le C\frac{\|u_0\|}{t^\alpha}. \tag{3.3}$$

*If $u_0 \notin D(A)$, then $P_\alpha(t)u_0 \notin D^2(A)$.*

(ii) *For every $u_0 \in X$ and $t > 0$, we have*

$$\begin{aligned} S_\alpha(t)u_0 &= t^{1-\alpha}{}_0I_t^{\alpha-1}P_\alpha(t)u_0 \\ &= \frac{t^{1-\alpha}}{2\pi i}\int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{\lambda t}(\lambda^\alpha I + A)^{-1}u_0 d\lambda,\ \mu \in \left(\frac{\pi\alpha}{2}, \pi - \theta\right),\ \varepsilon > 0, \end{aligned}$$

*$S_\alpha(t)u_0 \in C([0,+\infty), X)$ and $S_\alpha(t)u_0 \in D(A^2)$. Moreover, there exists a constant $C > 0$ such that for $t > 0$,*

$$\|S_\alpha(t)u_0\| \le C\|u_0\|,\ \|AS_\alpha(t)u_0\| \le C\frac{\|u_0\|}{t^\alpha},\ \|A^2S_\alpha(t)u_0\| \le C\frac{\|u_0\|}{t^{2\alpha}}. \tag{3.4}$$

*If $u_0 \notin D(A)$, then $S_\alpha(t)u_0 \notin D(A^3)$.*

(iii) *If $u_0 \in X$, then for every $t > 0$, we have $P_\alpha'(t)u_0 = -A[{}_0I_t^{\alpha-1}P_\alpha(t)u_0]$ and*

$$\frac{d^2}{dt^2}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_0 - u_0)] = -AP_\alpha(t)u_0.$$

*Moreover, $\lim_{t\to 0^+} tP_\alpha'(t)u_0 = 0$ for every $u_0 \in X$.*

(iv) *For $u_0 \in X$ and $t > 0$, we know ${}_0^CD_t^\alpha[{}_0I_t^1P_\alpha(t)u_0] = -A[{}_0I_t^1P_\alpha(t)u_0]$ and there exists a constant $C > 0$ such that*

$$\|A[{}_0I_t^1P_\alpha(t)u_0]\| \le \frac{C}{t^{\alpha-1}}\|u_0\|. \tag{3.5}$$

*Proof.* The proof of (i) and (ii) is similar to that of Theorem 3.2–3.5 in [20], so we omit it.

(iii) By the dominated convergence theorem, we obtain that for $u_0 \in X$,

$$P_\alpha'(t)u_0 = \frac{t^{\alpha-1}}{2\pi i}\int_\Gamma \lambda E_{\alpha,\alpha}(t^\alpha\lambda)(\lambda I + A)^{-1}u_0 d\lambda$$

$$= -\frac{t^{\alpha-1}}{2\pi i}\int_\Gamma E_{\alpha,\alpha}(t^\alpha\lambda)A(\lambda I + A)^{-1}u_0 d\lambda,\ t > 0. \tag{3.6}$$

Consequently, $\lim_{t\to 0^+} P'_\alpha(t)u_0 = 0$ for $u_0 \in D(A)$ and there exists a constant $C > 0$ such that

$$\|P'_\alpha(t)u_0\| \le \frac{C}{t}\|u_0\|,\ t > 0. \tag{3.7}$$

From this and a density argument, we see that $\lim_{t\to 0^+} tP'_\alpha(t)u_0 = 0$ for every $u_0 \in X$.

Next we prove $P'_\alpha(t)u_0 = -A[{}_0I_t^{\alpha-1}P_\alpha(t)u_0]$ for $u_0 \in X$ and $t > 0$ by using the approximate method. If $u_0 \in D(A)$, it follows from (2.4) that

$$\begin{aligned} A[{}_0I_t^{\alpha-1}P_\alpha(t)u_0] &= {}_0I_t^{\alpha-1}P_\alpha(t)Au_0 \\ &= \frac{1}{2\pi i}\int_\Gamma t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\lambda)A(\lambda I + A)^{-1}u_0 d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\lambda)u_0 d\lambda \\ &\quad - \frac{1}{2\pi i}\int_\Gamma t^{\alpha-1}\lambda E_{\alpha,\alpha}(t^\alpha\lambda)(\lambda I + A)^{-1}u_0 d\lambda \\ &= -\frac{t^{\alpha-1}}{2\pi i}\int_\Gamma E_{\alpha,\alpha}(t^\alpha\lambda)\lambda(\lambda I + A)^{-1}u_0 d\lambda = -P'_\alpha(t)u_0. \end{aligned} \tag{3.8}$$

For $u_0 \in X$, we choose $u_{0,n} \in D(A)$ such that $u_{0,n} \to u_0$ in $X$. Then (3.7) implies that for every $\delta > 0$, $P'_\alpha(t)u_{0,n} \to P'_\alpha(t)u_0$ in $C([\delta,\infty), X)$ as $n \to \infty$. In addition, in terms of $\|{}_0I_t^{\alpha-1}P_\alpha(t)u_0\| \le CT^{\alpha-1}\|u_0\|$, we deduce that ${}_0I_t^{\alpha-1}P_\alpha(t)u_{0,n} \to {}_0I_t^{\alpha-1}P_\alpha(t)u_0$ in $C([0, T], X)$ as $n \to \infty$. Hence, from the closeness of $A$, we know (3.8) also holds for $u_0 \in X$.

Finally, we prove $\frac{d^2}{dt^2}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_0 - u_0)] = -AP_\alpha(t)u_0$ for every $u_0 \in X$ and $t > 0$ by using the approximate method. Indeed, using (3.6) and (2.5), we know that

$$P''_\alpha(t)u_0 = -\frac{t^{\alpha-2}}{2\pi i}\int_\Gamma E_{\alpha,\alpha-1}(t^\alpha\lambda)A(\lambda I + A)^{-1}u_0 d\lambda,\ t > 0,\ u_0 \in X.$$

So, for $u_0 \in D(A)$, applying (2.3) and Fubini's theorem, we obtain

$${}_0I_t^{2-\alpha}P''_\alpha(t)u_0 = -\frac{1}{2\pi i}\int_\Gamma E_\alpha(\lambda t^\alpha)(\lambda I + A)^{-1}Au_0 d\lambda = -AP_\alpha(t)u_0,\ t > 0.$$

In other words, ${}_0^C D_t^\alpha P_\alpha(t)u_0 = -AP_\alpha(t)u_0$ for $t > 0$ and $u_0 \in D(A)$.

In the general case, we can find $\{u_{0,n}\} \subset D(A)$ such that $u_{0,n} \to u_0$ in $X$. Denote $u_n = P_\alpha(t)u_{0,n}$. Note that

$${}_0^C D_t^\alpha u_n = -Au_n \text{ and } \|u_n\|_X \le C\|u_{0,n}\|_X.$$

Then, for every $T > 0$, $u_n \to P_\alpha(t)u_0$ in $C([0, T], X)$ as $n \to \infty$. Since

$$\|{}_0I_t^{2-\alpha}u_n\| \le \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\|u_n\|_{L^\infty((0,T),X)},\ t \in [0, T],$$

we conclude that ${}_0I_t^{2-\alpha}u_n \to {}_0I_t^{2-\alpha}P_\alpha(t)u_0$ in $C([0,T],X)$. In addition, it follows from (3.3) that

$$\|\frac{d^2}{\mathrm{d}t^2}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_{0,n}-u_{0,n})]\| = \|{}_0^CD_t^\alpha u_n\| \leq \frac{C}{t^\alpha}\|u_{0,n}\|,\ \ t>0.$$

Hence, for every $\delta>0$, there exists $w\in C([\delta,\infty),X)$ such that ${}_0^CD_t^\alpha u_n \to w$ in $C([\delta,\infty),X)$ as $n\to\infty$.

Observing that ${}_0I_t^{2-\alpha}(P_\alpha(t)u_{0,n}-u_{0,n}) \to {}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)$ in $C([0,T],X)$ as $n\to\infty$ and

$${}_0^CD_t^\alpha u_n = \frac{d^2}{\mathrm{d}t^2}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_{0,n}-u_{0,n})] = -Au_n,\ t\in[\delta,\infty),$$

we get

$$w(t) = \frac{d^2}{\mathrm{d}t^2}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)],\ \ t\in[\delta,\infty).$$

It follows from the closeness of $A$ that $w(t)=-AP_\alpha(t)u_0$ for $t\in[\delta,\infty)$. In other words, $\frac{d^2}{\mathrm{d}t^2}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)] = -AP_\alpha(t)u_0$ for $t\in[\delta,\infty)$. Then, by the arbitrariness of $\delta$, we get $\frac{d^2}{\mathrm{d}t^2}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)] = -AP_\alpha(t)u_0$ for $t>0$.

(iv)For $u_0\in X$, $t>0$ and $\mu\in(\frac{\pi\alpha}{2},\pi-\theta)$, we deduce from (2.6) that

$$\begin{aligned}{}_0I_t^1P_\alpha(t)u_0 &= \frac{1}{2\pi i}\int_\Gamma tE_{\alpha,2}(\lambda t^\alpha)(\lambda I+A)^{-1}u_0d\lambda\\ &= \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{\tau t}\tau^{\alpha-2}(\tau^\alpha I+A)^{-1}u_0d\tau.\end{aligned}$$

This implies ${}_0I_t^1P_\alpha(t)u_0\in D(A)$ and

$$A[{}_0I_t^1P_\alpha(t)u_0] = \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{\tau t}\tau^{\alpha-2}d\tau - \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{\tau t}\tau^{2\alpha-2}(\tau^\alpha I+A)^{-1}u_0d\tau.$$

Then there exists a constant $C>0$ such that $\|A[{}_0I_t^1P_\alpha(t)u_0]\|\leq Ct^{1-\alpha}\|u_0\|$ for $t>0$ and $u_0\in X$.

If $u_0\in D(A)$, using (3.8), we have

$$\begin{aligned}{}_0^CD_t^\alpha\,{}_0I_t^1P_\alpha(t)u_0 &= {}_0I_t^{2-\alpha}P'_\alpha(t)u_0 = -{}_0I_t^{2-\alpha}A[{}_0I_t^{\alpha-1}P_\alpha(t)u_0]\\ &= -A[{}_0I_t^1P_\alpha(t)u_0],\ t>0.\end{aligned}$$

An argument similar to the one used in (iii) shows that the above equality also holds for $u_0\in X$. This completes the proof. □

REMARK 3.4. (i) Estimates (3.3) and (3.7) are firstly proved in [29]. Most of the results of (i) and (ii) in Lemma 3.3 are obtained in [2] and [29] by using the properties of the solution operators.

(ii) In [2] and [29], for $u_0 \in D(A)$ and $t \geq 0$, the assertion that ${}_0^C D_t^\alpha P_\alpha(t)u_0 = -AP_\alpha(t)u_0$ was obtained by a generation theorem of the analytic solution operator. In this paper, we prove that this assertion remains true for every $u_0 \in X$ and $t > 0$ by using the approximate method.

The following Lemma further illustrates the regular properties of $S_\alpha(t)$, which are crucial to prove our main results. This Lemma can be obtained by the maximal $L^p$ regularity of time fractional differential equations in [2]. Here, we give a direct proof.

LEMMA 3.5. *Let* $T > 0$ *and* $w = \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)\mathrm{d}s$. *If* $f \in L^q((0,T),X)$, $q \geq 1$, *then* $w \in C([0,T],X)$ *and*

$$ {}_0I_t^{2-\alpha} w = \int_0^t \int_0^{t-\tau} P_\alpha(s) f(\tau)\mathrm{d}s d\tau. $$

*Furthermore, if* $q(\alpha-1) > 1$, *then* $w \in C^{1,\alpha-1-\frac{1}{q}}([0,T],X)$.

*Proof.* Since $\alpha > 1$, we deduce from the dominated convergence theorem that $w \in C([0,T],X)$. By Lemma 3.3(ii), for $u_0 \in X$ and $t > 0$, we know

$$ {}_0I_t^{2-\alpha}(t^{\alpha-1} S_\alpha(t)u_0) = {}_0I_t^{2-\alpha}({}_0I_t^{\alpha-1} P_\alpha(t)u_0) = {}_0I_t^1 P_\alpha(t)u_0 = \int_0^t P_\alpha(s)u_0\mathrm{d}s. $$

Then, Fubini's theorem implies

$$ \begin{aligned} {}_0I_t^{2-\alpha} w &= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) f(\tau) d\tau \mathrm{d}s \\ &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \int_\tau^t (t-s)^{1-\alpha} (s-\tau)^{\alpha-1} S_\alpha(s-\tau) f(\tau)\mathrm{d}s\mathrm{d}\tau \\ &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \int_0^{t-\tau} (t-s-\tau)^{1-\alpha} s^{\alpha-1} S_\alpha(s) f(\tau)\mathrm{d}s\mathrm{d}\tau \\ &= \int_0^t \int_0^{t-\tau} P_\alpha(s) f(\tau)\mathrm{d}s\mathrm{d}\tau. \end{aligned} $$

Next we prove $w \in C^{1,\alpha-1-\frac{1}{q}}([0,T],X)$ if $q(\alpha-1) > 1$. First, we derive $w$ is differentiable on $[0,T]$. Indeed, observing that $w(0) = 0$ and there exists a constant $C > 0$ such that

$$ \left\| \frac{w(t)}{t} \right\| \leq \frac{C}{t} \int_0^t (t-s)^{\alpha-1} \|f(s)\|\mathrm{d}s \leq C t^{\alpha-1-\frac{1}{q}} $$

for $t > 0$, we know $w'(0) = 0$. From Lemma 3.3(ii), we get

$$ \frac{d}{\mathrm{d}t}[t^{\alpha-1} S_\alpha(t)u_0] = \frac{1}{2\pi i} \int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{\lambda t} \lambda(\lambda^\alpha I + A)^{-1} u_0 d\lambda,\ t > 0. $$

This implies that for $t > 0$, $\frac{d}{\mathrm{d}t}[t^{\alpha-1} S_\alpha(t)u_0] \in D(A)$ and

$$ \left\| \frac{d}{\mathrm{d}t}[t^{\alpha-1} S_\alpha(t)u_0] \right\| \leq \frac{C}{t^{2-\alpha}} \|u_0\|, \quad \left\| A\frac{d}{\mathrm{d}t}[t^{\alpha-1} S_\alpha(t)u_0] \right\| \leq \frac{C}{t^2} \|u_0\| \tag{3.9} $$

for some constant $C > 0$.

By (2.7) and using the fact that ${}_0I_t^{2\alpha-2}E_\alpha(\lambda t^\alpha) = t^{2\alpha-2}E_{\alpha,2\alpha-1}(\lambda t^\alpha)$, we have

$$\begin{aligned}{}_0I_t^{2\alpha-2}P_\alpha(t)u_0 &= \frac{1}{2\pi i}\int_\Gamma t^{2\alpha-2}E_{\alpha,2\alpha-1}(\lambda t^\alpha)(\lambda I + A)^{-1}u_0 d\lambda\\ &= \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{t\tau}\tau^{1-\alpha}(\tau^\alpha I + A)^{-1}u_0 d\tau.\end{aligned}$$

Then

$$\begin{aligned}A[{}_0I_t^{2\alpha-2}P_\alpha(t)u_0] &= \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{\lambda t}\lambda^{1-\alpha}u_0 d\lambda - \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\frac{\mu}{\alpha})} e^{\lambda t}\lambda(\lambda^\alpha I + A)^{-1}u_0 d\lambda\\ &= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}u_0 - \frac{\mathrm{d}}{\mathrm{d}t}[t^{\alpha-1}S_\alpha(t)u_0],\ t > 0.\end{aligned} \tag{3.10}$$

Thus, it follows from (3.9) that there exists a constant $C > 0$ such that for $u_0 \in X$ and $t > 0$,

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}[A({}_0I_t^{2\alpha-2}P_\alpha(t)u_0)]\right\| \le \frac{C}{t^{3-\alpha}}\|u_0\|. \tag{3.11}$$

Consequently, the dominated convergence theorem yields that if $f \in L^q((0,T),X)$, $q > \frac{1}{\alpha-1}$, then for $t > 0$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\int_0^t A[{}_0I_{t-s}^{2\alpha-2}P_\alpha(t)f(s)]\mathrm{d}s + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)\mathrm{d}s.$$

This implies that there exists a constant $C > 0$ such that

$$\|\frac{\mathrm{d}w}{\mathrm{d}t}\| \le C\int_0^t (t-s)^{\alpha-2}\|f(s)\|\mathrm{d}s \le C\left(\frac{q-1}{q\alpha-q-1}\right)^{\frac{q-1}{q}} T^{\frac{q\alpha-q-1}{q}}\|f\|_{L^q((0,T),X)}, \tag{3.12}$$

and then $\lim_{t\to 0^+}\frac{\mathrm{d}w}{\mathrm{d}t} = 0 = w'(0)$. Therefore, $w$ is differentiable on $[0,T]$.

Finally, we prove $w'(t) \in C^{\alpha-1-\frac{1}{q}}([0,T],X)$. For $h > 0$ and $t+h \le T$, using (3.9), (3.10) and (3.11), we have

$$\begin{aligned}&\|A[{}_0I_{t+h-s}^{2\alpha-2}P_\alpha(t)f(s) - {}_0I_{t-s}^{2\alpha-2}P_\alpha(t)f(s)]\|\\ &\quad\le C\min\{(t-s)^{\alpha-2},(t-s)^{\alpha-3}h\}\|f(s)\|,\ s\in(0,t)\end{aligned}$$

for some constant $C > 0$. Hence

$$\begin{aligned}&\|\int_0^t [A[{}_0I_{t+h-s}^{2\alpha-2}P_\alpha(t)f(s)] - A[{}_0I_{t-s}^{2\alpha-2}P_\alpha(t)f(s)]\mathrm{d}s\|\\ &\le C\int_0^t \min\{(t-s)^{\alpha-2},(t-s)^{\alpha-3}h\}\|f(s)\|\mathrm{d}s\\ &\le C\left(\int_0^t\left(\min\left\{\frac{1}{(t-s)^{2-\alpha}},\frac{h}{(t-s)^{3-\alpha}}\right\}\right)^{\frac{q}{q-1}}d\tau\right)^{\frac{q-1}{q}}\|f\|_{L^q((0,T),X)}\end{aligned}$$

$$\leq C\left(\int_0^{+\infty}\left(\min\left\{\frac{1}{\tau^{2-\alpha}},\frac{h}{\tau^{3-\alpha}}\right\}\right)^{\frac{q}{q-1}}d\tau\right)^{\frac{q-1}{q}}\|f\|_{L^q((0,T),X)}$$
$$= C\|f\|_{L^q((0,T),X)}h^{\alpha-1-\frac{1}{q}}.$$

On the other hand, by Hölder's inequality, we can easily see that $\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s)\mathrm{d}s \in C^{\alpha-1-\frac{1}{q}}([0,T],X)$ and

$$\begin{aligned}\left\|\int_t^{t+h} A[{}_0I_{t+h-s}^{2\alpha-2}P_\alpha(t)f(s)]\mathrm{d}s\right\| &\leq C\int_t^{t+h}(t-s)^{\alpha-2}\|f(s)\|\mathrm{d}s\\ &= C\|f\|_{L^q((0,T),X)}h^{\alpha-1-\frac{1}{q}}\end{aligned}$$

for some constant $C>0$. Thus, $w\in C^{1,\alpha-1-\frac{1}{q}}([0,T],X)$ if $f\in L^q((0,T),X)$ and $q(\alpha-1)>1$. □

## 4. Local existence

In this section, we give the local existence and uniqueness of the mild solution for problem (1.1).

Let $X=L^q(\mathbb{R}^N)$, $1<q<\infty$, and $A=-\triangle$ with domain $D(A)=\{u\in X \mid \triangle u\in X\}$. Then, $\sigma(A)=[0,+\infty)$ and the operator $A$ satisfies the assumptions of Sect. 3 (see, e.g., [3,26]). Hence, we can define the operators $P_\alpha(t)$ and $S_\alpha(t)$ on $L^q(\mathbb{R}^N)$.

REMARK 4.1. Formally, for $u_0\in X$,

$$P_\alpha(t)u_0=\int_{\mathbb{R}^N}K_\alpha(t,x-y)u_0(y)dy,\ S_\alpha(t)u_0=\int_{\mathbb{R}^N}\tilde{K}_\alpha(t,x-y)u_0(y)dy,$$

where

$$K_\alpha(t,x)=\mathscr{F}^{-1}(E_\alpha(-t^\alpha|\xi|^2)),\quad \tilde{K}_\alpha(t,x)=\mathscr{F}^{-1}(E_{\alpha,\alpha}(-t^\alpha|\xi|^2)),\ t>0,\ x\in\mathbb{R}^N,$$

$\mathscr{F}^{-1}$ denotes the inverse Fourier transform (see [1,14]).

REMARK 4.2. Recently, Kim et al. [17] proved that there exists a function $p(t,x)$ such that $p(t,\cdot)$ is integrable in $\mathbb{R}^N$ and $\mathscr{F}(p(t,\cdot))(\xi)=E_\alpha(-t^\alpha|\xi|^2)$. Then $K_\alpha(t,x)=p(t,x)$ and $\tilde{K}_\alpha(t,x)=t^{1-\alpha}{}_0I_t^{\alpha-1}p(t,x)$.

The following results give the $L^p-L^q$ estimates of the operators $P_\alpha(t)$ and $S_\alpha(t)$, and the regular properties of $S_\alpha(t)$ in Lebesgue space.

LEMMA 4.3. *The operators $P_\alpha(t)$ and $S_\alpha(t)$ have the following properties.*

(i) *If $1<p\leq q\leq+\infty$, $p<+\infty$, and $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}<\frac{2}{N}$, then there exists a constant $C>0$ such that for $t>0$,*

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)}\leq Ct^{-\frac{\alpha N}{2r}}\|u_0\|_{L^p(\mathbb{R}^N)}, \tag{4.1}$$

$$\|\frac{\mathrm{d}}{\mathrm{d}t}[P_\alpha(t)u_0]\|_{L^q(\mathbb{R}^N)} \le Ct^{-1-\frac{\alpha N}{2r}}\|u_0\|_{L^p(\mathbb{R}^N)}, \quad (4.2)$$

$$\|\frac{\mathrm{d}}{\mathrm{d}t}[t^{\alpha-1}S_\alpha(t)u_0]\|_{L^q(\mathbb{R}^N)} \le Ct^{\alpha-2-\frac{\alpha N}{2r}}\|u_0\|_{L^p(\mathbb{R}^N)}, \quad (4.3)$$

$$\|{}_0I_t^1 P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le Ct^{-\frac{\alpha N}{2r}+1}\|u_0\|_{L^p(\mathbb{R}^N)}. \quad (4.4)$$

(ii) *For* $1 < p \le q \le +\infty$ *and* $p < +\infty$, *if* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} < \frac{4}{N}$, *then there exists a constant* $C > 0$ *such that*

$$\|S_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le Ct^{-\frac{\alpha N}{2r}}\|u_0\|_{L^p(\mathbb{R}^N)},\ t > 0. \quad (4.5)$$

(iii) *Let* $T > 0$ *and* $w = \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s)\mathrm{d}s$. *Suppose* $1 < p < +\infty$, $1 < q \le +\infty$ *and* $r \ge p$ *satisfy*

$$\frac{1}{p} - \frac{1}{r} < \frac{2}{N}\left(1 - \frac{1}{\alpha q}\right).$$

*If* $f \in L^q((0,T), L^p(\mathbb{R}^N))$, *then* $w \in C([0,T], L^r(\mathbb{R}^N))$. *In addition, if* $r \in [p, +\infty]$ *satisfies* $1/p - 1/r < 2/N$ *and there is* $\gamma \in [0,1)$ *such that* $\sup_{t\in(0,T)} t^\gamma \|f(t)\|_{L^p(\mathbb{R}^N)} < +\infty$, *then* $w \in C((0,T], L^r(\mathbb{R}^N))$ *and* $w \in C([0,T], L^r(\mathbb{R}^N))$ *provided* $\gamma < \alpha - \frac{\alpha N}{2}(\frac{1}{p} - \frac{1}{r})$.

*Proof.* (i) Using the Gagliardo–Nirenberg inequality, we know that there exists a constant $C > 0$ such that

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le C\|AP_\alpha(t)u_0\|^a_{L^p(\mathbb{R}^N)}\|P_\alpha(t)u_0\|^{1-a}_{L^p(\mathbb{R}^N)},$$

$$\|\frac{\mathrm{d}}{\mathrm{d}t}[P_\alpha(t)u_0]\|_{L^q(\mathbb{R}^N)} \le C\|A\frac{\mathrm{d}}{\mathrm{d}t}[P_\alpha(t)u_0]\|^a_{L^p(\mathbb{R}^N)}\|\frac{\mathrm{d}}{\mathrm{d}t}[P_\alpha(t)u_0]\|^{1-a}_{L^p(\mathbb{R}^N)},$$

$$\|\frac{\mathrm{d}}{\mathrm{d}t}[t^{\alpha-1}S_\alpha(t)u_0]\|_{L^q(\mathbb{R}^N)} \le C\|A\frac{\mathrm{d}}{\mathrm{d}t}[t^{\alpha-1}S_\alpha(t)u_0]\|^a_{L^p(\mathbb{R}^N)}\|\frac{\mathrm{d}}{\mathrm{d}t}[t^{\alpha-1}S_\alpha(t)u_0]\|^{1-a}_{L^p(\mathbb{R}^N)},$$

$$\|{}_0I_t^1 P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le C\|A[{}_0I_t^1 P_\alpha(t)u_0]\|^a_{L^p(\mathbb{R}^N)}\|{}_0I_t^1 P_\alpha(t)u_0\|^{1-a}_{L^p(\mathbb{R}^N)},$$

where $a \in [0,1)$ and $\frac{1}{q} = a(\frac{1}{p} - \frac{2}{N}) + \frac{1-a}{p}$. Therefore, by (3.3), (3.4), (3.9) and (3.5), we know

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le \frac{C}{t^{a\alpha}}\|u_0\|^a_{L^p(\mathbb{R}^N)}\|u_0\|^{1-a}_{L^p(\mathbb{R}^N)} = Ct^{-\frac{N\alpha}{2r}}\|u_0\|_{L^p(\mathbb{R}^N)},$$

and (4.2), (4.3) and (4.4) hold.

(ii) The proof is similar to that of (i); hence, it will be omitted.

(iii) To prove the first part of (iii), without loss of generality, we can assume $1 < q < +\infty$, and we may assume that $f \in L^q((0,T), W^{2,p}(\mathbb{R}^N))$ by using a regularizing sequence. As a result, we obtain $f \in L^q((0,T), L^r(\mathbb{R}^N))$. Hence, the dominated convergence theorem yields $w \in C([0,T], L^r(\mathbb{R}^N))$. It follows from (4.5) that

$$\|w\|_{L^r(\mathbb{R}^N)} \le C\int_0^t (t-s)^{\alpha-1-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|f(s)\|_{L^p(\mathbb{R}^N)}\mathrm{d}s$$

$$\leq C\left(\int_0^t (t-s)^{\left[\alpha-1-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)\right]\frac{q}{q-1}}\mathrm{d}s\right)^{\frac{q-1}{q}}\|f\|_{L^q((0,T),L^p(\mathbb{R}^N))}$$
$$\leq C(T)\|f\|_{L^q((0,T),L^p(\mathbb{R}^N))}.$$

Thus, a approximate argument leads to $w \in C([0,T], L^r(\mathbb{R}^N))$ if $f \in L^q((0,T), L^p(\mathbb{R}^N))$.

Finally, we prove the other assertions of (iii). Assume that $\sup_{t\in(0,T)} t^\gamma \|f(t)\|_{L^p(\mathbb{R}^N)} < +\infty$. Then for $t>0$, $h>0$ and $t+h\leq T$, we deduce from (4.5) that

$$\left\|\int_t^{t+h}(t+h-s)^{\alpha-1}S_\alpha(t+h-s)f(s)\mathrm{d}s\right\|_{L^r(\mathbb{R}^N)} \leq Ct^{-\gamma}h^{\alpha-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}.$$

The rest proof is divided into three cases.

**Case 1.** $1-\alpha+\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})>0$.

Note that

$$\lim_{h\to 0^+} h^{1-\alpha+\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\int_0^{t-h}(t-s)^{\alpha-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)-2}s^{-\gamma}\mathrm{d}s$$
$$=\left(\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)+1-\alpha\right)^{-1}t^{-\gamma}.$$

Then, using (4.3), (4.5) and taking $h$ small, we obtain that there exists a constant $C>0$ such that

$$\left\|\int_0^t[(t+h-s)^{\alpha-1}S_\alpha(t+h-s)f(s)-(t-s)^{\alpha-1}S_\alpha(t-s)f(s)]\mathrm{d}s\right\|_{L^r(\mathbb{R}^N)}$$
$$\leq C\int_{t-h}^t(t-s)^{\alpha-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)-1}s^{-\gamma}\mathrm{d}s + C\int_0^{t-h}(t-s)^{\alpha-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)-2}s^{-\gamma}\mathrm{d}s h$$
$$\leq Ct^{-\gamma}h^{\alpha-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}.$$

**Case 2.** $1-\alpha+\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})=0$.

Since $\lim_{h\to 0^+} h^m\int_0^{t-h}(t-s)^{-1}s^{-\gamma}\mathrm{d}s=0$ for every $m>0$, we get

$$\left\|\int_0^t[(t+h-s)^{\alpha-1}S_\alpha(t+h-s)f(s)-(t-s)^{\alpha-1}S_\alpha(t-s)f(s)]\mathrm{d}s\right\|_{L^r(\mathbb{R}^N)}$$
$$\leq Ch^{1-m}t^{-\gamma}$$

for some constant $C>0$ when $h$ is small enough.

**Case 3.** $1-\alpha+\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})<0$.

Using (4.3), we have

$$\left\|\int_0^t[(t+h-s)^{\alpha-1}S_\alpha(t+h-s)f(s)-(t-s)^{\alpha-1}S_\alpha(t-s)f(s)]\mathrm{d}s\right\|_{L^r(\mathbb{R}^N)}$$
$$\leq Ch\int_0^t(t-s)^{\alpha-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)-2}s^{-\gamma}\mathrm{d}s \leq Ct^{-\gamma}h$$

for some constant $C > 0$.

To summarize what we have proved, we see that $w \in C((0, T], L^r(\mathbb{R}^N))$.

In addition, if $\gamma < \alpha - \frac{\alpha N}{2}(\frac{1}{p} - \frac{1}{r})$, then it is easy to check that there exists a constant $C > 0$ such that

$$\|w(t)\|_{L^r(\mathbb{R}^N)} \leq Ct^{\alpha - \frac{\alpha N}{2}\left(\frac{1}{p} - \frac{1}{r}\right) - \gamma}.$$

This implies $\lim_{t\to 0^+} w(t) = 0$ in $L^r(\mathbb{R}^N)$. Thus, $w \in C([0, T], L^r(\mathbb{R}^N))$. □

REMARK 4.4. Estimates (4.1) and (4.5) are obtained in [1,14] by applying multiplier estimates.

According to Definition 5.1 in [20], we give the definition of the mild solution of (1.1).

DEFINITION 4.5. Let $p > 1$, $1 < \alpha < 2$, $T > 0$ and $u_0, u_1 \in L^q(\mathbb{R}^N)$ for some $q \in (1, +\infty)$. We call that $u$ is a mild solution of problem (1.1) if $u \in C([0, T], L^q(\mathbb{R}^N))$ and satisfies

$$u = P_\alpha(t)u_0 + {}_0I_t^1 P_\alpha(t)u_1 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u|^p \mathrm{d}s.$$

For problem (1.1), we have the following local existence results.

THEOREM 4.6. *Let* $1 < \alpha < 2$ *and* $q_c = \frac{N(p-1)}{2}$. *Let* $u_0, u_1 \in L^q(\mathbb{R}^N)$, $\alpha q_c < q < +\infty$. *Then there exists* $T > 0$ *such that problem* (1.1) *has a mild solution* $u$ *in* $C([0, T], L^q(\mathbb{R}^N)) \cap C((0, T], L^r(\mathbb{R}^N))$ *and* $\sup_{t\in(0,T)} t^{\beta_r}\|u(t)\|_{L^r(\mathbb{R}^N)} < \infty$, *where* $\beta_r = \frac{\alpha N}{2}(\frac{1}{q} - \frac{1}{r})$ *and* $r \in (q, +\infty]$ *satisfies* $\frac{1}{q} - \frac{1}{r} < \frac{2}{N}$. *This solution is unique in the class*

$$\left\{u \in L^\infty_{loc}((0, T), L^{pq}(\mathbb{R}^N)) \mid \sup_{t\in(0,T)} t^{\frac{N\alpha}{2}\left(\frac{1}{q} - \frac{1}{pq}\right)}\|u\|_{L^{pq}(\mathbb{R}^N)} < \infty\right\}.$$

*Furthermore, if* $r$ *satisfies* $pq \leq r \leq +\infty$ *and* $\frac{1}{q} - \frac{1}{r} < \frac{2}{Np\alpha}$, *then* $u$ *can be extended to a maximal interval* $[0, T^*)$ *such that* $u \in C([0, T^*), L^q(\mathbb{R}^N)) \cap C((0, T^*), L^r(\mathbb{R}^N))$ *and either* $T^* = +\infty$ *or* $T^* < +\infty$ *and* $\|u(t)\|_{L^r(\mathbb{R}^N)} \to +\infty$ *as* $t \to T^{*-}$.

*Proof.* For given $T > 0$, let

$$E_{pq,T} = \{u \in L^\infty_{loc}((0, T), L^{pq}(\mathbb{R}^N)) \mid \|u\|_{E_{pq,T}} < \infty\}, \ \|u\|_{E_{pq,T}} = \sup_{t\in(0,T)} t^{\beta_{pq}}\|u(t)\|_{L^{pq}(\mathbb{R}^N)},$$

where $\beta_{pq} = \frac{\alpha N}{2}(\frac{1}{q} - \frac{1}{pq})$. Then, $E_{pq,T}$ is a Banach space. Choose $M > \|u_0\|_{L^q(\mathbb{R}^N)} + T\|u_1\|_{L^q(\mathbb{R}^N)}$ and let $B_K$ denote the closed ball in $E_{pq,T}$ with center 0 and radius $K$. We define the operator $G$ on $E_{pq,T}$ as

$$G(u)(t) = P_\alpha(t)u_0 + \int_0^t P_\alpha(s)u_1 \mathrm{d}s + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u(s)|^p \mathrm{d}s, \ \ u \in E_{pq,T}.$$

It follows from (4.1), (4.4) and (4.5) that there exists a constant $C > 0$ such that for $u \in B_K$ and $t \in (0, T)$,

$$\begin{aligned} t^{\beta_{pq}} \|G(u)(t)\|_{L^{pq}(\mathbb{R}^N)} &\leq C(\|u_0\|_{L^q(\mathbb{R}^N)} + T\|u_1\|_{L^q(\mathbb{R}^N)}) \\ &\quad + Ct^{\beta_{pq}} \int_0^t (t-s)^{\alpha-\beta_{pq}-1} \|u(s)\|^p_{L^{pq}(\mathbb{R}^N)} \mathrm{d}s \\ &\leq C(\|u_0\|_{L^q(\mathbb{R}^N)} + T\|u_1\|_{L^q(\mathbb{R}^N)}) \\ &\quad + CK^p t^{\beta_{pq}} \int_0^t (t-s)^{\alpha-\beta_{pq}-1} s^{-p\beta_{pq}} \mathrm{d}s \\ &\leq C(\|u_0\|_{L^q(\mathbb{R}^N)} + T\|u_1\|_{L^q(\mathbb{R}^N)}) \\ &\quad + CK^p T^{\alpha-p\beta_{pq}} \int_0^1 (1-s)^{\alpha-\beta_{pq}-1} s^{-p\beta_{pq}} \mathrm{d}s \\ &\leq CM + CK^p T^{\alpha-p\beta_{pq}}. \end{aligned} \tag{4.6}$$

The fact that $q > \alpha q_c > q_c$ guarantees that $\alpha - \beta_{pq} > 0$, $p\beta_{pq} < 1$ and $\alpha - p\beta_{pq} > 0$. So, all the integrals above are convergent. Choose $K > 0$ and $T > 0$ so that

$$CM + CK^p T^{\alpha-p\beta_{pq}} \leq K. \tag{4.7}$$

Hence, $G$ maps $B_K$ into itself. Note that

$$\||u|^p - |v|^p\|_{L^q(\mathbb{R}^N)} \leq C(\|u\|^{p-1}_{L^{pq}(\mathbb{R}^N)} + \|v\|^{p-1}_{L^{pq}(\mathbb{R}^N)})\|u-v\|_{L^{pq}(\mathbb{R}^N)}$$

for some constant $C > 0$ independent of $u$ and $v$. Similar calculations show that $G$ is a strict contraction on $B_K$ if $T$ is chosen small enough. Therefore, $G$ possesses a unique fixed point $u$ in $B_K$.

Note that $\sup_{t\in(0,T)} t^{p\beta_{pq}} \||u|^p\|_{L^q(\mathbb{R}^N)} < +\infty$. Then we deduce from Lemma 4.3(iii) and $p\beta_{pq} < \alpha$ that

$$\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u|^p \mathrm{d}s \in C([0, T], L^q(\mathbb{R}^N)).$$

Thus $u \in C([0, T], L^q(\mathbb{R}^N))$.

Since $r > q$ satisfies $1/q - 1/r < 2/N$, using (4.1), (4.4), (4.5) and the fact that $p\beta_{pq} < 1 < \alpha$, we have

$$\begin{aligned} t^{\beta_r} \|u(t)\|_{L^r(\mathbb{R}^N)} &\leq C(\|u_0\|_{L^q(\mathbb{R}^N)} + T\|u_1\|_{L^q(\mathbb{R}^N)}) \\ &\quad + Ct^{\beta_r} \int_0^t (t-s)^{\alpha-\beta_r-1} \|u(s)\|^p_{L^{pq}(\mathbb{R}^N)} \mathrm{d}s \\ &\leq C(\|u_0\|_{L^q(\mathbb{R}^N)} + T\|u_1\|_{L^q(\mathbb{R}^N)}) \\ &\quad + Ct^{\beta_r} \int_0^t (t-s)^{\alpha-\beta_r-1} s^{-p\beta_{pq}} \mathrm{d}s \\ &\leq C(\|u_0\|_{L^q(\mathbb{R}^N)} + T\|u_1\|_{L^q(\mathbb{R}^N)}) \end{aligned}$$

$$+ CT^{\alpha-p\beta_{pq}} \int_0^1 (1-s)^{\alpha-\beta_r-1} s^{-p\beta_{pq}} \mathrm{d}s < +\infty. \tag{4.8}$$

In addition, observe that $u \in E_{pq,T}$ and Lemma 4.3(iii) imply $u \in C((0,T], L^r(\mathbb{R}^N))$. Consequently, $u \in E_{r,T} \cap C((0,T], L^r(\mathbb{R}^N))$.

Next we prove the uniqueness of the solution. Let $u, v \in C([0,T], L^q(\mathbb{R}^N)) \cap E_{pq,T}$ be the mild solutions of (1.1) for some $T > 0$. Suppose $u, v \in B_{K'}$. Then, we can take $T' < T$ small enough such that (4.7) holds with $K$ replaced by $K'$. Thus, $u(t) = v(t)$ for $t \in [0, T']$. When $T' \le t \le T$, we have

$$\|u(t) - v(t)\|_{L^{pq}(\mathbb{R}^N)} \le C \int_{T'}^t (t-s)^{\alpha - \frac{\alpha N(p-1)}{2pq} - 1} \|u(s) - v(s)\|_{L^{pq}(\mathbb{R}^N)} \mathrm{d}s$$

for some constant $C > 0$ independent of $u$ and $v$. Hence, Gronwall's inequality yields $u(t) = v(t)$ for $t \in [T', T]$.

Finally, we prove that the existence of maximal time provided $r$ satisfies $pq \le r \le +\infty$ and $\frac{1}{q} - \frac{1}{r} < \frac{2}{Np\alpha}$. We proceed by considering two cases.

**Case 1.** $pq \le r < +\infty$ and $\frac{1}{q} - \frac{1}{r} < \frac{2}{Np\alpha}$.

Set

$$T^* = \sup\{T > 0 \mid u \in E_{r,T} \cap C((0,T], L^r(\mathbb{R}^N)) \text{ is a mild solution}\}.$$

Assume $T^* < +\infty$ and there exists $M_1 > 0$ such that $\sup_{t \in (0,T^*)} t^{\beta_r} \|u(t)\|_{L^r(\mathbb{R}^N)} \le M_1$. We claim that there exists $\tilde{M}_1 > 0$ such that

$$\sup_{t \in (0,T^*)} t^{\beta_{pq}} \|u(t)\|_{L^{pq}(\mathbb{R}^N)} < \tilde{M}_1 \text{ and } \sup_{t \in (0,T^*)} \|u(t)\|_{L^q(\mathbb{R}^N)} < +\infty. \tag{4.9}$$

In fact, if $r = pq$, we have

$$\begin{aligned}
\|u(t)\|_{L^q(\mathbb{R}^N)} &\le C(\|u_0\|_{L^q(\mathbb{R}^N)} + \|u_1\|_{L^q(\mathbb{R}^N)}) \\
&\quad + C \int_0^t (t-s)^{\alpha-1} \|u(s)\|_{L^{pq}(\mathbb{R}^N)}^p \mathrm{d}s \\
&\le C(\|u_0\|_{L^q(\mathbb{R}^N)} + \|u_1\|_{L^q(\mathbb{R}^N)}) \\
&\quad + C(T^*)^{\alpha - \frac{\alpha N(p-1)}{2q}} \int_0^1 (1-s)^{\alpha-1} s^{-p\beta_{pq}} \mathrm{d}s < +\infty.
\end{aligned}$$

Then the claims are proved.

For the case of $pq < r < +\infty$, since $\frac{p}{r} - \frac{1}{r} < \frac{2}{N}$, $\frac{1}{q} - \frac{1}{r} < \frac{2}{Np\alpha}$ and $\frac{1}{q} - \frac{1}{pq} < \frac{2}{N}$, we can take $n \in \mathbb{N}$ large enough such that

$$\frac{r}{p} < r\left(\frac{pq}{r}\right)^{\frac{1}{n}} < r, \quad \frac{\left(\frac{pq}{r}\right)^{\frac{1}{n}} p - 1}{pq} < \frac{2}{N} \text{ and } \frac{p}{r} - \frac{1}{r\left(\frac{pq}{r}\right)^{\frac{1}{n}}} < \frac{2}{N}.$$

Set $\chi = (\frac{pq}{r})^{\frac{1}{n}}$ and $q_1 = r$, $q_k = q_{k-1}\chi = q_1\chi^{k-1}$, $k = 2, 3, \ldots, n+1$. Observing that $\chi < 1$ and

$$0 < \frac{p}{q_k} - \frac{1}{q_{k+1}} = \frac{1}{\chi^{k-1}}\left(\frac{p}{r} - \frac{1}{r\chi}\right) \le \frac{1}{\chi^{n-1}}\left(\frac{p}{r} - \frac{1}{r\chi}\right)$$

$$= \frac{\chi p}{pq} - \frac{1}{pq} < \frac{2}{N}, \quad k = 1, 2, \ldots, n,$$

$$\frac{1}{q} - \frac{1}{q_k} \le \frac{1}{q} - \frac{1}{q_1} = \frac{1}{q} - \frac{1}{r} < \frac{2}{Np\alpha}, \quad k = 1, 2, \ldots, n+1,$$

we know that if $\sup_{t\in(0,T^*)} t^{\beta_{q_k}} \|u(t)\|_{L^{q_k}(\mathbb{R}^N)} < +\infty$, then there exists a constant $C > 0$ such that

$$\begin{aligned}
&t^{\beta_{q_{k+1}}} \|u(t)\|_{L^{q_{k+1}}(\mathbb{R}^N)} \\
&\quad\le C\|u_0\|_{L^q(\mathbb{R}^N)} + CT^*\|u_1\|_{L^q(\mathbb{R}^N)} \\
&\qquad + Ct^{\beta_{q_{k+1}}} \int_0^t (t-s)^{\alpha - \frac{\alpha N}{2}(\frac{p}{q_k} - \frac{1}{q_{k+1}}) - 1} \|u(s)\|^p_{L^{q_k}(\mathbb{R}^N)} \mathrm{d}s \\
&\quad\le C\|u_0\|_{L^q(\mathbb{R}^N)} + CT^*\|u_1\|_{L^q(\mathbb{R}^N)} \\
&\qquad + Ct^{\beta_{q_{k+1}}} \int_0^t (t-s)^{\alpha - \frac{\alpha N}{2}(\frac{p}{q_k} - \frac{1}{q_{k+1}}) - 1} s^{-p\beta_{q_k}} \mathrm{d}s \\
&\quad\le C(\|u_0\|_{L^q(\mathbb{R}^N)} + T^*\|u_1\|_{L^q(\mathbb{R}^N)}) \\
&\qquad + Ct^{\alpha - \frac{\alpha N(p-1)}{2q}} \int_0^1 (1-s)^{\alpha - \frac{\alpha N}{2}(\frac{p}{q_k} - \frac{1}{q_{k+1}}) - 1} s^{-p\beta_{q_k}} \mathrm{d}s < +\infty.
\end{aligned}$$

Thus, the assumption that $\sup_{t\in(0,T^*)} t^{\beta_r} \|u(t)\|_{L^r(\mathbb{R}^N)} \le M_1$ implies

$$\sup_{t\in(0,T^*)} t^{\beta_{pq}} \|u(t)\|_{L^{pq}(\mathbb{R}^N)} < +\infty,$$

and then $\sup_{t\in(0,T^*)} \|u(t)\|_{L^q(\mathbb{R}^N)} < +\infty$. Therefore, the claims are proved.

Next we verify that $\lim_{t\to T^*-} u(t)$ exists in $L^r(\mathbb{R}^N) \cap L^{pq}(\mathbb{R}^N)$. Indeed, for $\frac{T^*}{2} < t < \tau < T^*$, by the proof of Lemma 4.3(iii) and using (4.1), (4.2), there exists a $\tilde{m} \in (0, 1]$ such that

$$\|u(t) - u(\tau)\|_{L^r(\mathbb{R}^N)} \le C(\tau - t)(\|u_0\|_{L^q(\mathbb{R}^N)} + \|u_1\|_{L^q(\mathbb{R}^N)}) + CM_1^p(\tau - t)^{\tilde{m}}, \tag{4.10}$$

$$\|u(t) - u(\tau)\|_{L^{pq}(\mathbb{R}^N)} \le C(\tau - t)(\|u_0\|_{L^q(\mathbb{R}^N)} + \|u_1\|_{L^q(\mathbb{R}^N)}) + C\tilde{M}_1^p(\tau - t)^{\tilde{m}}. \tag{4.11}$$

Therefore, $\lim_{t\to T^*-} u(t)$ exists in $L^r(\mathbb{R}^N) \cap L^{pq}(\mathbb{R}^N)$. Denote $u_{T^*} = \lim_{t\to T^*-} u(t)$ and define $u(T^*) = u_{T^*}$.

For $h > 0$ and $\delta > 0$, let

$$\tilde{E}_{h,\delta} = \{u \in C([T^*, T^* + h], L^r(\mathbb{R}^N) \cap L^{pq}(\mathbb{R}^N)) \mid u(T^*) = u_{T^*},\ d(u, u_{T^*}) \le \delta\},$$

where

$$d(u, u_{T^*}) = \max_{t\in[T^*,T^*+h]} \|u(t) - u_{T^*}\|_{L^r(\mathbb{R}^N)} + \max_{t\in[T^*,T^*+h]} \|u(t) - u_{T^*}\|_{L^{pq}(\mathbb{R}^N)}.$$

It follows from (4.9) and Lemma 4.3(iii) that $u \in C((0,T^*], L^{pq}(\mathbb{R}^N)\cap L^r(\mathbb{R}^N))$. Then we can define the operator $K$ on $\tilde{E}_{h,\delta}$ as

$$K(v)(t) = P_\alpha(t)u_0 + \int_0^t P_\alpha(s)u_1 \mathrm{d}s + \int_0^{T^*}(t-\tau)^{\alpha-1}S_\alpha(t-\tau)|u(\tau)|^p d\tau + \int_{T^*}^t (t-\tau)^{\alpha-1}S_\alpha(t-\tau)|v(\tau)|^p d\tau,\ v\in \tilde{E}_{h,\delta}.$$

Using Lemmas 4.3, (4.10) and (4.11), we can easily see that $K(v)\in C([T^*,T^*+h], L^r(\mathbb{R}^N)\cap L^{pq}(\mathbb{R}^N))$ and $K(v)(T^*) = u_{T^*}$. For $v\in\tilde{E}_{h,\delta}$ and $t\in[T^*,T^*+h]$, it follows from the same arguments as above that

$$\|K(v)(t)-u_{T^*}\|_{L^r(\mathbb{R}^N)} \le C(t-T^*)(\|u_0\|_{L^q(\mathbb{R}^N)}+\|u_1\|_{L^q(\mathbb{R}^N)}) + CM_1^p(t-T^*)^{\tilde{m}} + C(\|u_{T^*}\|_{L^r(\mathbb{R}^N)}+\delta)^p(t-T^*)^{\alpha-\frac{\alpha N(p-1)}{2r}} \tag{4.12}$$

for some positive constant $C$. Moreover, (4.12) also holds if $r$ is replaced by $pq$. So we can choose $h$ small enough such that $d(u,u_{T^*})\le\delta$.

On the other hand, for every $w, v\in\tilde{E}_{h,\delta}$, there exists a positive constant $C$ such that

$$\begin{aligned}\|Kw-Kv\|_{L^r(\mathbb{R}^N)} &\le C\int_{T^*}^t (t-\tau)^{\alpha-\frac{\alpha N(p-1)}{2r}-1}(\|w\|^{p-1}_{L^r(\mathbb{R}^N)} + \|v\|^{p-1}_{L^r(\mathbb{R}^N)})\|w-v\|_{L^r(\mathbb{R}^N)}d\tau \\ &\le C(\|u_{T^*}\|_{L^r(\mathbb{R}^N)}+\delta)^{p-1}h^{\alpha-\frac{\alpha N(p-1)}{2r}}\max_{t\in[T^*,T^*+h]}\|w-v\|_{L^r(\mathbb{R}^N)},\end{aligned}$$

and

$$\|Kw-Kv\|_{L^{pq}(\mathbb{R}^N)} \le C(\|u_{T^*}\|_{L^{pq}(\mathbb{R}^N)}+\delta)^{p-1}h^{\alpha-\frac{\alpha N(p-1)}{2pq}}\max_{t\in[T^*,T^*+h]}\|w-v\|_{L^{pq}(\mathbb{R}^N)}.$$

Thus, choosing $h$ small enough so that

$$C(\|u_{T^*}\|_{L^r(\mathbb{R}^N)}+\delta)^{p-1}h^{\alpha-\frac{\alpha N(p-1)}{2r}} + C(\|u_{T^*}\|_{L^{pq}(\mathbb{R}^N)}+\delta)^{p-1}h^{\alpha-\frac{\alpha N(p-1)}{2pq}} \le \frac{1}{2}, \tag{4.13}$$

we know $G$ is a strict contraction on $\tilde{E}_{h,\delta}$. So the contraction mapping principle implies $G$ has a fixed point $v\in\tilde{E}_{h,\delta}$.

Define

$$\tilde{u}(t) = \begin{cases} u(t), & t\in[0,T^*],\\ v(t), & t\in[T^*,T^*+h].\end{cases}$$

Since $v(T^*) = G(v(T^*)) = u(T^*)$, one can verify easily that $\tilde{u}\in E_{r,T^*+h}\cap C((0,T^*+h], L^r(\mathbb{R}^N))$ and

$$\tilde{u}(t) = P_\alpha(t)u_0 + \int_0^t P_\alpha(s)u_1\mathrm{d}s + \int_0^t (t-\tau)^{\alpha-1}S_\alpha(t-\tau)|\tilde{u}(\tau)|^p d\tau.$$

Because $u \in E_{pq,T^*+h}$, we know $u \in C([0, T^*+h], L^q(\mathbb{R}^N))$ by Lemma 4.3(iii). Thus, $\tilde{u}(t)$ is a mild solution of (1.1), which contradicts the definition of $T^*$.

**Case 2.** $r = +\infty$ and $\frac{1}{q} < \frac{2}{Np\alpha}$.

The assumption that $\frac{1}{q} < \frac{2}{Np\alpha}$ implies that we can choose $\bar{m} \in (\frac{Np}{2}, +\infty)$ such that $\bar{m} > pq$ and $\frac{1}{q} - \frac{1}{\bar{m}} < \frac{2}{Np\alpha}$. In this case, we also set

$$T^* = \sup\{T > 0 \mid u \in E_{r,T} \cap C((0, T], L^r(\mathbb{R}^N)) \text{ is a mild solution}\}.$$

Assume $T^* < +\infty$, and there exists $M_1 > 0$ such that $\sup_{t\in(0,T^*)} t^{\beta_r}\|u(t)\|_{L^r(\mathbb{R}^N)} \leq M_1$. We can also prove that (4.9) holds. Indeed, for $\tilde{T} < T^*$, $u \in C([0, \tilde{T}], L^q(\mathbb{R}^N))$ because $u$ is the mild solution of (1.1). In addition, for $t \in [\tilde{T}, T^*)$, we conclude that there exists a positive constant $C$ such that

$$\begin{aligned}
\|u(t)\|_{L^q(\mathbb{R}^N)} &\leq C(\|u_0\|_{L^q(\mathbb{R}^N)} + \|u_1\|_{L^q(\mathbb{R}^N)}) + C\int_0^t (t-s)^{\alpha-1}\|u(s)\|^p_{L^{pq}(\mathbb{R}^N)}\mathrm{d}s \\
&\leq C + C\int_0^t (t-s)^{\alpha-1}\|u(s)\|^{p-1}_{L^\infty(\mathbb{R}^N)}\|u(s)\|_{L^q(\mathbb{R}^N)}\mathrm{d}s \\
&\leq C + C\int_0^{\tilde{T}} (t-s)^{\alpha-1}s^{-\frac{\alpha N(p-1)}{2q}}\mathrm{d}s + C\int_{\tilde{T}}^t (t-s)^{\alpha-1}\|u(s)\|_{L^q(\mathbb{R}^N)}\mathrm{d}s \\
&\leq C + C\int_{\tilde{T}}^t \|u(s)\|_{L^q(\mathbb{R}^N)}\mathrm{d}s.
\end{aligned}$$

Then it follows from Gronwall's inequality that $\sup_{t\in(\tilde{T},T^*)}\|u(t)\|_{L^q(\mathbb{R}^N)} < +\infty$. Therefore, $\|u(t)\|_{L^q(\mathbb{R}^N)}$ is bounded on $(0, T^*)$.

Noting that

$$\begin{aligned}
\|u(t)\|_{L^{pq}(\mathbb{R}^N)} &\leq \|u(t)\|^{\frac{p-1}{p}}_{L^\infty(\mathbb{R}^N)}\|u(t)\|^{\frac{1}{p}}_{L^q(\mathbb{R}^N)}, \\
\|u(t)\|_{L^{\bar{m}}(\mathbb{R}^N)} &\leq \|u(t)\|^{1-\frac{q}{\bar{m}}}_{L^\infty(\mathbb{R}^N)}\|u(t)\|^{\frac{q}{\bar{m}}}_{L^q(\mathbb{R}^N)},
\end{aligned}$$

we get $\|u(t)\|_{L^{pq}(\mathbb{R}^N)} \leq \tilde{M}_1 t^{-\frac{\alpha(p-1)N}{2pq}} = \tilde{M}_1 t^{-\beta_{pq}}$ and $\|u(t)\|_{L^{\bar{m}}(\mathbb{R}^N)} \leq Ct^{-\beta_{\bar{m}}}$ for some constants $\tilde{M}$ and $C$. Thus, we get the desired conclusion.

In this case, we can also obtain $\lim_{t\to T^{*-}} u(t)$ exists in $L^\infty(\mathbb{R}^N) \cap L^{pq}(\mathbb{R}^N)$ by an argument similar to the proof used in Case 1. Furthermore, an argument similar to one in Case 1 leads to a contradiction. Then we get the desired conclusion. □

REMARK 4.7. If $p \geq 1 + \frac{2}{\alpha N}$, the assumption $q > \alpha q_c$ in Theorem 4.6 can be weakened to $q > \frac{\alpha Np}{\alpha N+2}$ and $q > q_c$. In fact, the assumption $q > \frac{\alpha Np}{\alpha N+2}$ implies $\frac{\alpha Nq}{(\alpha Np-2q)_+} > 1$, where $(\alpha Np - 2q)_+ = \max\{0, \alpha Np - 2q\}$. In view of $\frac{q}{p} < \frac{\alpha Nq}{(\alpha Np-2q)_+}$, then there exists $\tilde{q} \in (p, pq]$ such that $\frac{\alpha N}{2q} - \frac{1}{p} < \frac{\alpha N}{2\tilde{q}} < \frac{\alpha N}{2q}$, that is, $0 < \frac{\alpha N}{2}(\frac{1}{q} - \frac{1}{\tilde{q}}) < \frac{1}{p}$. By a fixed-point argument in $E_{\tilde{q},T}$, we know if $u_0, u_1 \in L^q(\mathbb{R}^N)$, then there exists $T > 0$ such that $G$ has a unique fixed point

$u \in E_{\tilde{q},T}$. Since $p\beta_{\tilde{q}} < \alpha - \frac{\alpha N}{2}(\frac{p}{\tilde{q}} - \frac{1}{q})$ and $u \in E_{\tilde{q},T}$, it follows from Lemma 4.3(iii) that $u \in C([0,T], L^q(\mathbb{R}^N)) \cap C((0,T], L^{\tilde{q}}(\mathbb{R}^N))$. Thus, problem (1.1) has a unique mild solution $u$ in $E_{\tilde{q},T}$.

## 5. Blow-up and Global existence

In this section, we prove the blow-up and global existence of mild solutions of (1.1). In order to prove the blow-up results by applying the test function method, we firstly give the definition of weak solution of (1.1).

DEFINITION 5.1. Let $1 < \alpha < 2$, $q \geq 1$ and $T > 0$. For $u_0, u_1 \in L^q_{loc}(\mathbb{R}^N)$, we call $u \in L^p((0,T), L^p_{loc}(\mathbb{R}^N))$ is a weak solution of (1.1) if

$$\int_{\mathbb{R}^N}\int_0^T [|u|^p\varphi + (u_0 + tu_1)\,{}_t^C D_T^\alpha \varphi]\mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}^N}\int_0^T u(-\triangle\varphi)\mathrm{d}t\mathrm{d}x + u\,{}_t^C D_T^\alpha \varphi \mathrm{d}t\mathrm{d}x$$

for every $\varphi \in C^{2,2}_{x,t}(\mathbb{R}^N \times [0,T])$ and $\varphi_t \in C^{2,0}_{x,t}(\mathbb{R}^N \times [0,T])$ with $supp_x\varphi \subset\subset \mathbb{R}^N$ and $\varphi(x,T) = 0$, $\varphi_t(x,T) = 0$, where

$$C^{2,2}_{x,t}(\mathbb{R}^N \times [0,T]) = \{f(x,t) \mid f, f_{x_i}, f_{x_i x_i}, f_t, f_{tt} \in C(\mathbb{R}^N \times [0,T]),\ i = 1,2,\ldots,N\},$$
$$C^{2,0}_{x,t}(\mathbb{R}^N \times [0,T]) = \{f(x,t) \mid f, f_{x_i}, f_{x_i x_i} \in C(\mathbb{R}^N \times [0,T]),\ i = 1,2,\ldots,N\}.$$

Moreover, if $T > 0$ can be arbitrarily chosen, then we call $u$ is a global weak solution of (1.1).

The following Lemma gives the relation between weak solutions and mild solutions of (1.1). This Lemma is crucial to prove our blow-up results by using the test function method.

LEMMA 5.2. *Let $T > 0$ and $u_0, u_1 \in L^q(\mathbb{R}^N)$, $q > \max\{\alpha q_c, 1\}$. If $u \in C([0,T], L^q(\mathbb{R}^N))$ is a mild solution obtained by Theorem 4.6, then $u$ is also a weak solution of (1.1).*

*Proof.* Assume that $u \in C([0,T], L^q(\mathbb{R}^N))$ is the mild solution of (1.1). Then

$$\begin{aligned} u - u_0 - u_1 t = {} & P_\alpha(t)u_0 - u_0 - u_1 t + \int_0^t P_\alpha(s)u_1\mathrm{d}s \\ & + \int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|u(\tau)|^p d\tau. \end{aligned}$$

Observing $\sup_{t\in(0,T)} t^{\beta_{pq}}\|u(t)\|_{L^{pq}(\mathbb{R}^N)} < +\infty$ and $p\beta_{pq} < 1$, we know $u \in L^p((0,T), L^{pq}(\mathbb{R}^N))$. Then, it follows from Lemma 3.5 that the following equality holds in $L^q(\mathbb{R}^N)$

$$
\begin{aligned}
{}_0I_t^{2-\alpha}(u-u_0-u_1t) &= {}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)+{}_0I_t^{2-\alpha}[{}_0I_t^1P_\alpha(t)u_1-u_1t]\\
&\quad+\int_0^t\int_0^{t-\tau}P_\alpha(s)|u(\tau)|^p\mathrm{d}s\mathrm{d}\tau.
\end{aligned}
$$

Thus, for every $\varphi\in C_{x,t}^{2,2}(\mathbb{R}^N\times[0,T])$ and $\varphi_t\in C_{x,t}^{2,0}(\mathbb{R}^N\times[0,T])$ with $supp_x\varphi\subset\subset\mathbb{R}^N$, $\varphi(x,T)=0$ and $\varphi_t(x,T)=0$, we have

$$
\int_{\mathbb{R}^N}{}_0I_t^{2-\alpha}(u-u_0-u_1t)\varphi_t\mathrm{d}x=I_1+I_2,
$$

where

$$
\begin{aligned}
I_1&=\int_{\mathbb{R}^N}{}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)\varphi_t\mathrm{d}x+\int_{\mathbb{R}^N}{}_0I_t^{2-\alpha}[{}_0I_t^1P_\alpha(t)u_1-u_1t]\varphi_t\mathrm{d}x,\\
I_2&=\int_{\mathbb{R}^N}\int_0^t\int_0^{t-\tau}P_\alpha(s)|u(\tau)|^p\mathrm{d}s\mathrm{d}\tau\varphi_t\mathrm{d}x.
\end{aligned}
$$

Next we calculate the derivatives of $I_1$ and $I_2$ and find the values of $\int_0^T\frac{\mathrm{d}I_1}{\mathrm{d}t}\mathrm{d}t$ and $\int_0^T\frac{\mathrm{d}I_2}{\mathrm{d}t}\mathrm{d}t$. For $t>0$, it is easy to check that

$$
\begin{aligned}
\frac{\mathrm{d}I_1}{\mathrm{d}t}&=\int_{\mathbb{R}^N}\frac{\mathrm{d}}{\mathrm{d}t}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)]\varphi_t\mathrm{d}x+\int_{\mathbb{R}^N}{}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)\varphi_{tt}\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}({}_0I_t^{2-\alpha}(P_\alpha(t)u_1-u_1))\varphi_t\mathrm{d}x+\int_{\mathbb{R}^N}{}_0I_t^{2-\alpha}({}_0I_t^1P_\alpha(t)u_1-u_1t)\varphi_{tt}\mathrm{d}x,\\
\frac{\mathrm{d}I_2}{\mathrm{d}t}&=\int_{\mathbb{R}^N}\int_0^t\int_0^{t-\tau}P_\alpha(s)|u(\tau)|^p\mathrm{d}s\mathrm{d}\tau\varphi_{tt}\mathrm{d}x+\int_{\mathbb{R}^N}\int_0^tP_\alpha(t-\tau)|u(\tau)|^pd\tau\varphi_t\mathrm{d}x\\
&=I_3+I_4,
\end{aligned}\tag{5.1}
$$

where $I_3$ and $I_4$ denote the first and second integrals of (5.1), respectively.

To evaluate integrals of $\frac{\mathrm{d}I_1}{\mathrm{d}t}$ and $\frac{\mathrm{d}I_2}{\mathrm{d}t}$ on $[0,T]$, we proceed as follows.

First, we calculate the integral $\int_0^T\frac{\mathrm{d}I_1}{\mathrm{d}t}\mathrm{d}t$. Note that if $u_0\in D(A)$, then by Lemma 3.3 (ii), for $t>0$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}[{}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)]=-{}_0I_t^{2-\alpha}(t^{\alpha-1}S_\alpha(t)Au_0)=-A[{}_0I_t^1P_\alpha(t)u_0].\tag{5.2}
$$

So, by an argument similar to the proof of Lemma 3.3 (iii) and using Lemma 3.3 (iv), we know (5.2) remains true for $u_0\in L^q(\mathbb{R}^N)$. Hence, for $t>0$ and $u\in L^q(\mathbb{R}^N)$, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}I_1}{\mathrm{d}t}&=-\int_{\mathbb{R}^N}{}_0I_t^1(P_\alpha(t)u_0)A\varphi_t\mathrm{d}x+\int_{\mathbb{R}^N}{}_0I_t^{2-\alpha}(P_\alpha(t)u_0-u_0)\varphi_{tt}\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}({}_0I_t^{2-\alpha}(P_\alpha(t)u_1-u_1))\varphi_t\mathrm{d}x+\int_{\mathbb{R}^N}{}_0I_t^{2-\alpha}({}_0I_t^1P_\alpha(t)u_1-u_1t)\varphi_{tt}dx.
\end{aligned}\tag{5.3}
$$

Integrating (5.3) from 0 to $T$ and using (5.2), one get

$$\int_0^T \frac{dI_1}{dt}dt = \int_0^T \int_{\mathbb{R}^N} P_\alpha(t)u_0 A\varphi dxdt + \int_0^T \int_{\mathbb{R}^N} (P_\alpha(t)u_0 - u_0)_t^C D_T^\alpha \varphi dxdt + \int_0^T \int_{\mathbb{R}^N} {}_0I_t^1 P_\alpha(t)u_1 A\varphi dxdt + \int_0^T \int_{\mathbb{R}^N} ({}_0I_t^1 P_\alpha(t)u_1 - u_1 t)_t^C D_T^\alpha \varphi dxdt. \tag{5.4}$$

Next we evaluate the integral $\int_0^T \frac{\mathrm{d}I_2}{\mathrm{d}t}\mathrm{d}t$. It follows from (5.1) and Lemma 3.5 that

$$\int_0^T I_3 dt = \int_0^T \int_{\mathbb{R}^N} ({}_0I_t^{2-\alpha} w)\varphi_{tt} dxdt = \int_0^T \int_{\mathbb{R}^N} w(t)_0^C D_t^\alpha \varphi dxdt, \tag{5.5}$$

where $w(t) = \int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|u(\tau)|^p d\tau$.

For convenience, denote $f(t) = |u(t)|^p$. Then $f \in L^1([0,T], L^q(\mathbb{R}^N))$. We use a approximate argument to get the value of $\int_0^T I_4 dt$. In fact, if $f \in L^1((0,T), D(A))$, then

$$\int_0^T I_4 dt = -\int_0^T \int_{\mathbb{R}^N} f(t)\varphi dxdt + \int_0^T \int_{\mathbb{R}^N} \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s) f(s)\mathrm{d}s \varphi dxdt = -\int_0^T \int_{\mathbb{R}^N} f(t)\varphi dxdt + \int_0^T \int_{\mathbb{R}^N} \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)\mathrm{d}s A\varphi dxdt. \tag{5.6}$$

In the general case, we can choose $f_n \in L^1((0,T), D(A))$ such that $f_n \to f$ in $L^1((0,T), L^q(\mathbb{R}^N))$ as $n \to \infty$. Taking $n \to \infty$, we know equality (5.6) also holds for $f \in L^1([0,T], L^q(\mathbb{R}^N))$.

Combining (5.5) and (5.6), we get

$$\int_0^T \frac{dI_2}{dt}dt = \int_0^T \int_{\mathbb{R}^N} w(t)\, {}_0^C D_t^\alpha \varphi dxdt - \int_0^T \int_{\mathbb{R}^N} |u|^p \varphi dxdt + \int_0^T \int_{\mathbb{R}^N} \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u(s)|^p \mathrm{d}s A\varphi \mathrm{d}x\mathrm{d}t. \tag{5.7}$$

As a result, we deduce from (5.4) and (5.7) that

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \int_{\mathbb{R}^N} {}_0I_t^{2-\alpha}(u - u_0 - u_1 t)\varphi_t dxdt = \int_0^T \frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{\mathrm{d}I_2}{\mathrm{d}t}\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^N} (P_\alpha(t)u_0 + {}_0I_t^1 P_\alpha(t)u_1)A\varphi dxdt + \int_0^T \int_{\mathbb{R}^N} (u - u_0 - u_1 t)\, {}_t^C D_T^\alpha \varphi dxdt \\ &\quad - \int_0^T \int_{\mathbb{R}^N} |u|^p \varphi dxdt + \int_0^T \int_{\mathbb{R}^N} \int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|u|^p d\tau A\varphi dxdt \\ &= -\int_0^T \int_{\mathbb{R}^N} u\Delta\varphi dxdt + \int_0^T \int_{\mathbb{R}^N} (u - u_0 - u_1 t)\, {}_t^C D_T^\alpha \varphi dxdt - \int_0^T \int_{\mathbb{R}^N} |u|^p \varphi dxdt. \end{aligned}$$

In other words, $u$ is a weak solution of (1.1). This completes the proof. □

We say the solution $u$ of problem (1.1) blows up in a finite time $T$ if

$$\lim_{t\to T^-} \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

First, we give a blow-up result of problem (1.1).

THEOREM 5.3. *Let* $u_0, u_1 \in L^q(\mathbb{R}^N)$, $q > \max\{\frac{Np\alpha}{2}, 1\}$. *If*

$$\int_{\mathbb{R}^N} u_0(x)\chi(x)dx > 3^{\frac{1}{p-1}} \text{ and } \int_{\mathbb{R}^N} u_1(x)\chi(x)dx \ge 0,$$

*where* $\chi(x) = (\int_{\mathbb{R}^N} e^{-\sqrt{N^2+|x|^2}}dx)^{-1}e^{-\sqrt{N^2+|x|^2}}$, *then any mild solution of* (1.1) *blows up in a finite time.*

*Proof.* Take $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \psi(x) \le 1$ and

$$\psi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2. \end{cases}$$

Let $\psi_n(x) = \psi(\frac{x}{n})$, $n = 1, 2, \dots$ and $\varphi_T \in C^2([0,T])$ with $\varphi_T(T) = 0$, $\varphi_T'(T) = 0$ and $\varphi_T \ge 0$. Assume $u \in C([0,T], L^q(\mathbb{R}^N))$ is a mild solution of (1.1). Since $q > \frac{Np\alpha}{2}$, we get from Theorem 4.6 that $u \in C((0,T], L^\infty(\mathbb{R}^N))$ and $\sup_{t\in(0,T)} t^{\frac{N\alpha}{2q}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} < +\infty$. This implies $u \in L^p((0,T), L^\infty(\mathbb{R}^N))$. It follows from Lemma 5.2 that $u$ is also a weak solution of (1.1). Then, taking $\varphi(x,t) = \chi(x)\psi_n(x)\varphi_T(t)$ as a test function in the definition of weak solution, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^T u^p\chi\psi_n\varphi_T dtdx + \int_{\mathbb{R}^N}\int_0^T (u_0+tu_1)\chi\psi_n\, {}_t^C D_T^\alpha \varphi_T dtdx \\ = \int_{\mathbb{R}^N}\int_0^T u[-\Delta(\chi\psi_n)]\varphi_T dtdx + u\chi\psi_n\, {}_t^C D_T^\alpha\varphi_T dtdx. \qquad (5.8)\end{aligned}$$

A simple calculation shows that

$$\triangle\chi = \Big(-\frac{N}{\sqrt{N^2+|x|^2}} + \frac{|x|^2}{N^2+|x|^2} + \frac{|x|^2}{(N^2+|x|^2)^{\frac{3}{2}}}\Big)\chi.$$

Hence, $|\triangle\chi| \le 3\chi$. Note that $\triangle(\chi\psi_n) = (\triangle\chi)\psi_n + 2\nabla\chi\cdot\nabla\psi_n + (\triangle\psi_n)\chi$. Then by (5.8), letting $n\to\infty$ and using the dominated convergence theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^T [|u|^p\chi\varphi_T + (u_0+u_1t)\chi\, {}_t^C D_T^\alpha\varphi_T]dxdt \\ \le \int_{\mathbb{R}^N}\int_0^T (3|u|\chi\varphi_T + |u\chi\, {}_t^C D_T^\alpha\varphi_T|)dxdt. \qquad (5.9)\end{aligned}$$

Denote $f(t) = \int_{\mathbb{R}^N} |u|\chi dx$. Since $u \in C([0,T], L^q(\mathbb{R}^N))$, we know $f \in C([0,T])$. It follows from Jensen's inequality and (5.9) that

$$\int_0^T (f^p - 3f)\varphi_T dt + \int_0^T (B_0 + B_1 t){}_t^C D_T^\alpha\varphi_T dt \le \int_0^T f|{}_t^C D_T^\alpha\varphi_T|dt, \qquad (5.10)$$

where $B_0 = \int_{\mathbb{R}^N} u_0\chi dx$ and $B_1 = \int_{\mathbb{R}^N} u_1\chi dx \geq 0$. Take $\varphi_T = {}_tI_T^\alpha\tilde{\psi}(t)$ where $\tilde{\psi} \in C_0^\infty((0,T))$ and $\tilde{\psi} \geq 0$. Then we deduce from (5.10) that

$$\int_0^T {}_0I_t^\alpha(f^p - 3f)\tilde{\psi}dt = \int_0^T (f^p - 3f)_tI_T^\alpha\tilde{\psi}(t)dt \leq \int_0^T (f - B_0 - B_1t)\tilde{\psi}dt,$$

where we have used the fact that ${}_t^CD_T^\alpha {}_tI_T^\alpha\tilde{\psi}(t) = \tilde{\psi}(t)$. This implies

$$ {}_0I_t^\alpha(f^p - 3f) + B_0 + B_1t \leq f(t),\ t \in [0,T]. \tag{5.11}$$

Noting that

$$3^{\frac{1}{p-1}} < B_0 = \int_{\mathbb{R}^N} u_0\chi dx \leq \int_{\mathbb{R}^N} |u_0|\chi dx = f(0)$$

and using the continuity of $f$, we know that there exists $t_1 > 0$ such that $f(t) > 3^{\frac{1}{p-1}}$ for $t \in [0,t_1]$. Set

$$t^* = \sup\{s \in [0,T] \mid f(t) > 3^{\frac{1}{p-1}},\ t \in [0,s]\}.$$

Then $0 < t^* \leq T$. Suppose that $t^* < T$. By the definition of $t^*$, one gets $f(t) \geq 3^{\frac{1}{p-1}}$ for $t \in [0,t^*]$. Thus, $f^p(t) \geq 3f(t)$ for $t \in [0,t^*]$. Consequently, it follows from (5.11) that

$$f(t) \geq B_0 + B_1t \geq B_0 > 3^{\frac{1}{p-1}},\ t \in [0,t^*].$$

In particular, $f(t^*) > 3^{\frac{1}{p-1}}$. By the continuity of $f$, we obtain a contradiction. Thus $t^* = T$ and then $f(t) \geq B_0$ for $t \in [0,T]$ by (5.11).

Taking $\varphi_T(t) = (1 - \frac{t}{T})^k$ ($k \geq \max\{2, \frac{p\alpha}{p-1}\}$) in (5.10), we know there exists a constant $C > 0$ such that for every $\varepsilon > 0$,

$$\int_0^T (f^p - 3f)\varphi_T dt + CB_0T^{1-\alpha} + CB_1T^{2-\alpha} \leq \varepsilon\int_0^T f^p\varphi_T dt + C(\varepsilon)T^{1-\frac{p\alpha}{p-1}},$$

where we have used $\int_0^T {}_t^CD_T^\alpha\varphi_1(t) = \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)}T^{1-\alpha}$. Choose $\varepsilon$ small enough such that $B_0 > (\frac{3}{1-\varepsilon})^{\frac{1}{p-1}}$. Then $f(t) \geq B_0 > (\frac{3}{1-\varepsilon})^{\frac{1}{p-1}}$ for $t \in [0,T]$. Thus $B_0 \leq CT^{\alpha-\frac{p\alpha}{p-1}}$ for some constant $C > 0$. Assuming that $u$ exists globally, we get $B_0 = 0$ by taking $T \to \infty$, which contradicts $B_0 > 3^{\frac{1}{p-1}}$. Consequently, Theorem 4.6 guarantees that $u$ blows up in a finite time. □

Next we give the main results of this paper. For $u_1 \equiv 0$, we have the following results.

THEOREM 5.4. *Let $u_0 \in L^q(\mathbb{R}^N)$, $q > \max\{\frac{Np\alpha}{2}, 1\}$ and $u_0 \not\equiv 0$, $u_1 \equiv 0$.*

(i) *If $1 < p \leq 1 + \frac{2\alpha}{\alpha N+2-2\alpha}$, and $u_0 \geq 0$ or $u_0 \in L^m(\mathbb{R}^N)$ for some $m \in [1, \frac{\alpha N}{\alpha N+2-2\alpha})$, then any mild solution of (1.1) blows up in a finite time.*

(ii) *If* $p > 1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha}$ *and* $\|u_0\|_{L^{q_c}(\mathbb{R}^N)}$ *is sufficiently small, where* $q_c = \frac{N(p-1)}{2}$, *then the mild solution of* (1.1) *exists globally.*

*Proof.* (i) Let $\Phi \in C_0^\infty(\mathbb{R}^N)$ such that $\Phi(x) = 1$ for $|x| \le 1$, $\Phi(x) = 0$ for $|x| > 2$ and $0 \le \Phi(x) \le 1$.

In the case of $p < 1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha}$, for $T > 0$, we define $\phi_T(x) = (\Phi(T^{-\frac{\alpha}{2}}x))^{\frac{2p}{p-1}}$ and take $\varphi_T(t) = (1 - \frac{t}{T})^k$, $k \ge \max\{2, \frac{p\alpha}{p-1}\}$. Assume that $u$ is a mild solution of (1.1). Then, by Lemma 5.2, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^T & |u|^p \phi_T \varphi_T + u_0 \phi_T {}_t^C D_T^\alpha \varphi_T dxdt \\ &= \int_{\mathbb{R}^N}\int_0^T u(-\Delta\phi_T)\varphi_T + u\phi_T({}_t^C D_T^\alpha \varphi_T)dxdt. \qquad (5.12)\end{aligned}$$

By a simple calculation, we get

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2}\phi_T(x) =& \frac{2p(p+1)}{(p-1)^2} T^{-\alpha} (\Phi(T^{-\frac{\alpha}{2}}x))^{\frac{2}{p-1}} \Phi_{x_i}^2 (T^{-\frac{\alpha}{2}}x) \\ &+ \frac{2p}{p-1} T^{-\alpha} (\Phi(T^{-\frac{\alpha}{2}}x))^{\frac{p+1}{p-1}} \Phi_{x_i x_i}(T^{-\frac{\alpha}{2}}x),\ i = 1, 2, \ldots, N.\end{aligned}$$

Observing that $|\Phi| \le 1$ and $\Phi \in C_0^\infty(\mathbb{R}^N)$, one see that

$$|\Delta\phi_T| \le CT^{-\alpha}(\Phi(T^{-\frac{\alpha}{2}}x))^{\frac{2}{p-1}} = CT^{-\alpha}\phi_T^{\frac{1}{p}}$$

for some positive constant $C$ independent of $T$. Since $k \ge \frac{p\alpha}{p-1}$ and

$${}_t^C D_T^\alpha \varphi_T(t) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} T^{-\alpha}(1 - \frac{t}{T})^{k-\alpha}, \quad t \le T,$$

it follows that there exists a constant $C > 0$ such that $|{}_t^C D_T^\alpha \varphi_T| \le CT^{-\alpha}\varphi_T^{\frac{1}{p}}$. Combining the above estimates and observing $0 \le \phi_T \le 1$ and $0 \le \varphi_T \le 1$, we derive

$$|(-\Delta\phi_T)\varphi_T + \phi_T({}_t^C D_T^\alpha \varphi_T)| \le CT^{-\alpha}\phi_T^{\frac{1}{p}}\varphi_T^{\frac{1}{p}} \qquad (5.13)$$

for some positive constant $C$ independent of $T$. Thus, using Hölder's inequality, we deduce from (5.12) and (5.13) that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^T [|u|^p \phi_T \varphi_T + u_0\phi_T({}_t^C D_T^\alpha \varphi_T)]dxdt &\le CT^{-\alpha}\int_{\mathbb{R}^N}\int_0^T |u|\phi_T^{\frac{1}{p}}\varphi_T^{\frac{1}{p}}dxdt \\ \le CT^{-\alpha+(1+\frac{\alpha N}{2})\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}\int_0^T |u|^p\phi_T\varphi_T dxdt\right)^{\frac{1}{p}}. \qquad (5.14)\end{aligned}$$

In other words,

$$\int_{\mathbb{R}^N}\int_0^T |u|^p\phi_T\varphi_T dxdt + CT^{1-\alpha}\int_{\mathbb{R}^N} u_0\phi_T dx \le CT^{1+\frac{\alpha N}{2}-\frac{p\alpha}{p-1}}. \qquad (5.15)$$

Note that if $u_0 \in L^m(\mathbb{R}^N)$ then $T^{1-\alpha}|\int_{\mathbb{R}^N} u_0\phi_T dx| \le CT^{1-\alpha+\frac{\alpha N(m-1)}{2m}}$ by Hölder's inequality, and if $u_0 \ge 0$ then $\int_{\mathbb{R}^N} u_0\phi_T dx \ge 0$. Thus, the second term of the left hand of (5.15) is either nonnegative or convergent to 0 as $T \to +\infty$. Suppose that $u$ exists globally. Then observing $1 + \frac{\alpha N}{2} - \frac{p\alpha}{p-1} < 0$ and taking $T \to \infty$, we deduce from (5.15) that

$$\int_0^{+\infty}\int_{\mathbb{R}^N} |u(t,x)|^p dxdt = 0.$$

This implies $u \equiv 0$ which contradicts $u_0 \not\equiv 0$.

For the case of $p = 1 + \frac{2\alpha}{\alpha N+2-2\alpha}$, suppose that $u$ is a global weak solution of (1.1). Then it follows from (5.15) that $\int_0^{+\infty}\int_{\mathbb{R}^N} |u|^p dxdt < +\infty$. In this case, we define $\phi_{T,L}(x) = (\Phi(LT^{-\frac{\alpha}{2}}x))^{\frac{2p}{p-1}}$, $0 < L < T$, and take $\varphi(t,x) = \phi_{T,L}(x)\varphi_T(t)$ as a test function. Note that there exists a constant $C > 0$ independent of $T$ and $L$ such that

$$\begin{aligned}|(-\triangle\phi_{T,L})\varphi_T + \phi_{T,L}({}_t^C D_T^\alpha\varphi_T)| \le{}& CL^2T^{-\alpha}\phi_{T,L}^{\frac{1}{p}}\varphi_T^{\frac{1}{p}}\chi_{\{T^{\frac{\alpha}{2}}L^{-1}\le|x|\le 2T^{\frac{\alpha}{2}}L^{-1}\}}\\ &+CT^{-\alpha}\phi_{T,L}\varphi_T^{\frac{1}{p}},\end{aligned}$$

where $\chi_{\{T^{\frac{\alpha}{2}}L^{-1}\le|x|\le 2T^{\frac{\alpha}{2}}L^{-1}\}}$ denotes the characteristic function of the set $\{x \mid T^{\frac{\alpha}{2}}L^{-1} \le |x| \le 2T^{\frac{\alpha}{2}}L^{-1}\}$. Then

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_0^T |u|^p\phi_{T,L}\varphi_T dxdt + CT^{1-\alpha}\int_{\mathbb{R}^N} u_0\phi_{T,L}dx\\ &\le CL^2\Big(\int_0^T\int_{T^{\frac{\alpha}{2}}\le L|x|\le 2T^{\frac{\alpha}{2}}} |u|^p dxdt\Big)^{\frac{1}{p}} + CT^{-\frac{p\alpha}{p-1}}\int_{\mathbb{R}^N}\int_0^T \phi_{T,L}dxdt\\ &= CL^2\Big(\int_0^T\int_{T^{\frac{\alpha}{2}}\le L|x|\le 2T^{\frac{\alpha}{2}}} |u|^p dxdt\Big)^{\frac{1}{p}} + CL^{-N}\int_{|x|\le 2}[\Phi(x)]^{\frac{2p}{p-1}}dx. \qquad (5.16)\end{aligned}$$

Thus, letting $T \to +\infty$, we deduce from (5.16) that

$$\int_0^{+\infty}\int_{\mathbb{R}^N} |u|^p dxdt \le CL^{-N}\int_{|x|\le 2}[\Phi(x)]^{\frac{2p}{p-1}}dx,$$

which implies $u \equiv 0$ by taking $L \to +\infty$. This contradicts again the assumption that $u_0 \not\equiv 0$.

Hence, by Theorem 4.6, we know $u$ blows up in a finite time.

(ii) We construct the global solution of (1.1) by the contraction mapping principle. Since $p > 1 + \frac{2\alpha}{\alpha N+2-2\alpha}$, we know

$$\frac{\alpha N(p-1)}{2(p\alpha - p + 1)} > 1. \qquad (5.17)$$

In terms of (5.17) and $\frac{(p-1)N}{2p} < \frac{\alpha N(p-1)}{2(p\alpha-p+1)}$, we can choose $r > p > 1 + \frac{2\alpha}{\alpha N+2-2\alpha}$ and $r < pq_c$ such that

$$\frac{\alpha}{p-1} - \frac{1}{p} < \frac{\alpha N}{2r} < \frac{\alpha}{p-1}. \qquad (5.18)$$

Let

$$\beta = \frac{\alpha N}{2}\left(\frac{1}{q_c} - \frac{1}{r}\right) = \frac{\alpha}{p-1} - \frac{\alpha N}{2r}. \tag{5.19}$$

Using (5.18) and (5.19), one verifies that

$$0 < p\beta < 1, \quad \alpha = \frac{\alpha N(p-1)}{2r} + (p-1)\beta. \tag{5.20}$$

Note that (5.19) and (5.20) imply $0 < \frac{1}{q_c} - \frac{1}{r} < \frac{2}{N}$. Then, it follows from (4.1) that for $u_0 \in L^{q_c}(\mathbb{R}^N)$,

$$\sup_{t>0} t^{\beta} \|P_\alpha(t)u_0\|_{L^r(\mathbb{R}^N)} = \eta < +\infty. \tag{5.21}$$

Let

$$Y = \{u \in L^\infty_{loc}((0,\infty), L^r(\mathbb{R}^N)) \mid \|u\|_Y < \infty\},$$

where $\|u\|_Y = \sup_{t>0} t^\beta \|u(t)\|_{L^r(\mathbb{R}^N)}$. For $u \in Y$, we define

$$\Psi(u)(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u(s)|^p \mathrm{d}s.$$

Set $B_M = \{u \in Y \mid \|u\|_Y \le M\}$, where $M > 0$ is to be chosen sufficiently small. By Hölder's inequality, Lemmas 4.3(ii) and (5.20), there exists a constant $C > 0$ such that for any $u, v \in B_M$ and $t \ge 0$,

$$\begin{aligned}
t^\beta \|\Psi(u) - \Psi(v)\|_{L^r(\mathbb{R}^N)} &\le Ct^\beta \int_0^t (t-s)^{\alpha-1-\frac{\alpha N}{2}\left(\frac{p}{r}-\frac{1}{r}\right)} \||u|^p - |v|^p\|_{L^{\frac{r}{p}}(\mathbb{R}^N)} \mathrm{d}s \\
&\le Ct^\beta \int_0^t (t-s)^{\alpha-1-\frac{\alpha N(p-1)}{2r}} (\|u\|^{p-1}_{L^r(\mathbb{R}^N)} + \|v\|^{p-1}_{L^r(\mathbb{R}^N)})\|u-v\|_{L^r(\mathbb{R}^N)} \mathrm{d}s \\
&\le CM^{p-1} t^{\beta - p\beta - \frac{\alpha N(p-1)}{2r} + \alpha} \int_0^1 (1-\tau)^{-\frac{\alpha N(p-1)}{2r} + \alpha - 1} \tau^{-p\beta} d\tau \|u-v\|_Y \\
&\le CM^{p-1}\|u-v\|_Y.
\end{aligned}$$

Thus, if we choose $M$ small enough such that $CM^{p-1} < \frac{1}{2}$, then $\|\Psi(u) - \Psi(v)\|_Y \le \frac{1}{2}\|u-v\|_Y$.

On the other hand, since

$$t^\beta \|\Psi(u)(t)\|_{L^r(\mathbb{R}^N)} \le \eta + CM^p t^\beta \int_0^t (t-s)^{-\frac{\alpha N}{2}\left(\frac{p}{r}-\frac{1}{r}\right)-1+\alpha} s^{-p\beta} \mathrm{d}s \le \eta + CM^p, \; t \in [0, +\infty),$$

$\Psi$ maps $B_m$ into itself if $\eta$ and $M$ are chosen small enough. Therefore, $\Psi$ is a strict contraction. Then the contraction mapping principle implies $\Psi$ has a fixed point $u \in B_M$, that is, (1.1) has a mild solution $u \in B_M$.

We now have to show $u \in C([0,\infty), L^q(\mathbb{R}^N)) \cap C((0,\infty), L^\infty(\mathbb{R}^N))$, where $u$ is the solution just constructed.

First, we prove that for $T > 0$ small enough, the solution $u \in C([0, T], L^q(\mathbb{R}^N)) \cap C((0, T], L^\infty(\mathbb{R}^N))$. In fact, the above proof shows that $u$ is the unique solution in

$$B_{M,T} = \left\{ u \in L^\infty_{loc}((0, T), L^r(\mathbb{R}^N)) \mid \sup_{0<t<T} t^\beta \|u(t)\|_{L^r(\mathbb{R}^N)} \leq M \right\}.$$

Since $u_0 \in L^q(\mathbb{R}^N) \cap L^{q_c}(\mathbb{R}^N)$ and $r > q_c$, we know $u_0 \in L^{\tilde{q}}(\mathbb{R}^N)$ for every $\tilde{q} \in (q_c, q)$ and $\tilde{q} < r$. Observe that the assumption $p > 1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha}$ implies $p > 1 + \frac{2}{\alpha N}$ and $q_c > \frac{\alpha N p}{\alpha N + 2}$. Then, using Theorem 4.6 and Remark 4.7, we know that (1.1) has a unique solution $\tilde{u} \in C([0, T], L^q(\mathbb{R}^N) \cap L^{\tilde{q}}(\mathbb{R}^N)) \cap C((0, T], L^\infty(\mathbb{R}^N))$ if $T$ is small enough, and $\sup_{0<t<T} t^{\frac{\alpha N}{2q}} \|\tilde{u}(t)\|_{L^\infty(\mathbb{R}^N)} < +\infty$. Note that $\tilde{q} > q_c$ and there exists a constant $C > 0$ such that

$$t^\beta \|\tilde{u}(t)\|_{L^r(\mathbb{R}^N)} \leq t^\beta \|\tilde{u}(t)\|^{1-\frac{\tilde{q}}{r}}_{L^\infty(\mathbb{R}^N)} \|\tilde{u}(t)\|^{\frac{\tilde{q}}{r}}_{L^{\tilde{q}}(\mathbb{R}^N)} \leq C t^{\frac{\alpha N}{2}\left(\frac{1}{q_c} - \frac{1}{\tilde{q}}\right)} \|\tilde{u}(t)\|^{\frac{\tilde{q}}{r}}_{L^{\tilde{q}}(\mathbb{R}^N)}$$

for $t \in (0, T)$. It follows that we can take $T$ small enough such that $\sup_{0<t<T} t^\beta \|\tilde{u}(t)\|_{L^r(\mathbb{R}^N)} \leq M$. Thus, by uniqueness, $u \equiv \tilde{u}$ for $t \in [0, T]$. Consequently, $u \in C((0, T], L^\infty(\mathbb{R}^N)) \cap C([0, T], L^q(\mathbb{R}^N) \cap L^{\tilde{q}}(\mathbb{R}^N))$.

Finally, we prove $u \in C([T, \infty), L^q(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$. Indeed, for $t > T$, we have

$$\begin{aligned} u - P_\alpha(t)u_0 &= \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u|^p \mathrm{d}s \\ &= \int_0^T (t-s)^{\alpha-1} S_\alpha(t-s)|u|^p \mathrm{d}s + \int_T^t (t-s)^{\alpha-1} S_\alpha(t-s)|u|^p \mathrm{d}s \\ &= I_5 + I_6. \end{aligned}$$

Since $u \in C([0, T], L^{\tilde{q}}(\mathbb{R}^N)) \cap C((0, T], L^\infty(\mathbb{R}^N))$ and $\sup_{0<t<T} t^{\frac{\alpha N}{2q}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} < \infty$, we obtain $I_5 \in C([T, \infty), L^\infty(\mathbb{R}^N)) \cap C([T, \infty), L^{\tilde{q}}(\mathbb{R}^N))$ by an argument similar to the proof of Lemma 4.3(iii).

For given $T_1 > T$, $|u|^p \in L^\infty((T, T_1), L^{\frac{r}{p}}(\mathbb{R}^N))$. Because $r > \frac{N(p-1)}{2}$, we can choose $\tilde{m} > r$ such that $\frac{N}{2}(\frac{p}{r} - \frac{1}{\tilde{m}}) < 1$. Observing $0 < \frac{p}{r} - \frac{1}{\tilde{q}} < \frac{p}{r} - \frac{1}{\tilde{m}} < \frac{2}{N}$, an argument similar to the one used in Lemma 4.3(iii) shows that $I_6 \in C([T, T_1], L^{\tilde{m}}(\mathbb{R}^N) \cap L^{\tilde{q}}(\mathbb{R}^N))$. By the arbitrariness of $T_1$, we know $I_6 \in C([T, \infty), L^{\tilde{m}}(\mathbb{R}^N) \cap L^{\tilde{q}}(\mathbb{R}^N))$. Note that the term $P_\alpha(\cdot)u_0 \in C([T, \infty), L^{\tilde{m}}(\mathbb{R}^N) \cap L^{\tilde{q}}(\mathbb{R}^N))$. Consequently, $u \in C([T, \infty), L^{\tilde{m}}(\mathbb{R}^N)) \cap C([0, \infty), L^{\tilde{q}}(\mathbb{R}^N))$.

Let $\chi = \frac{\tilde{m}}{r}$. Observe that $\chi > 1$ and

$$\frac{N}{2}\left(\frac{p}{r\chi^{i-1}} - \frac{1}{r\chi^i}\right) < 1, \; i = 1, 2, \ldots.$$

Repeating the above arguments, we deduce that if $u \in C([T, \infty), L^{r\chi^{i-1}}(\mathbb{R}^N))$ then $u \in C([T, \infty), L^{r\chi^i}(\mathbb{R}^N))$. After finite steps, we get

$$\frac{p}{r\chi^i} < \frac{2}{N}.$$

Then $u \in C((0,\infty), L^{\infty}(\mathbb{R}^N))$. Therefore, $u \in C([0,+\infty), L^q(\mathbb{R}^N)) \cap C((0,\infty), L^{\infty}(\mathbb{R}^N))$. This completes the proof. □

For $u_1 \not\equiv 0$, we have the following results.

THEOREM 5.5. *Let $N \geq 2$, $q_c = \frac{N(p-1)}{2}$ and $\tilde{q}_c = \frac{\alpha N(p-1)}{2(\alpha+p-1)}$. Assume that $u_0, u_1 \in L^q(\mathbb{R}^N)$ for some $q > \max\{\frac{Np\alpha}{2}, 1\}$ and $u_1 \geq 0$, $u_1 \not\equiv 0$.*

(i) *If $1 < p < 1 + \frac{2\alpha}{\alpha N-2}$, and $u_0 \geq 0$ or $u_0 \in L^m(\mathbb{R}^N)$ for some $m \in [1, \frac{\alpha N}{\alpha N-2})$, then any mild solution of (1.1) blows up in a finite time.*

(ii) *If $p > 1 + \frac{2\alpha}{\alpha N-2}$, and $\|u_0\|_{L^{q_c}(\mathbb{R}^N)}$ and $\|u_1\|_{L^{\tilde{q}_c}(\mathbb{R}^N)}$ are sufficiently small, then the mild solution of (1.1) exists globally.*

*Proof.* (i) By an argument similar to the proof of Theorem 5.4(i), we get

$$\int_{\mathbb{R}^N}\int_0^T |u|^p\phi_T\varphi_T + (u_0+u_1t)\phi_T({}_t^C D_T^{\alpha}\varphi_T)dxdt \leq CT^{-\alpha}\int_{\mathbb{R}^N}\int_0^T |u|\phi_T^{\frac{1}{p}}\varphi_T^{\frac{1}{p}}dxdt$$
$$\leq CT^{-\alpha+\left(1+\frac{\alpha N}{2}\right)\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}\int_0^T |u|^p\phi_T\varphi_T dxdt\right)^{\frac{1}{p}}. \tag{5.22}$$

Note that if $u_0 \geq 0$, then it follows from (5.22) that

$$T^{2-\alpha}\int_{\mathbb{R}^N} u_1\phi_T dx \leq CT^{1+\frac{\alpha N}{2}-\frac{p\alpha}{p-1}}, \tag{5.23}$$

and if $u_0 \in L^m(\mathbb{R}^N)$ then

$$\begin{aligned} T^{2-\alpha}\int_{\mathbb{R}^N} u_1\phi_T dx &\leq CT^{1+\frac{\alpha N}{2}+\frac{p\alpha}{p-1}} + CT^{1-\alpha}\int_{\mathbb{R}^N}|u_0|\phi_T dx \\ &\leq CT^{1+\frac{\alpha N}{2}+\frac{p\alpha}{p-1}} + CT^{1-\alpha+\frac{\alpha N(m-1)}{2m}} \end{aligned} \tag{5.24}$$

by Hölder's inequality. In addition, in terms of $1 < p < 1 + \frac{2\alpha}{\alpha N-2}$ and $m < \frac{\alpha N}{\alpha N-2}$, we know $\frac{\alpha N}{2} - 1 - \frac{\alpha}{p-1} < 0$ and $\frac{\alpha N(m-1)}{2m} < 1$. Thus, if the solution of (1.1) exists globally, then taking $T \to \infty$, we obtain $u_1 \equiv 0$ by (5.23) and (5.24), which contradicts $u_1 \not\equiv 0$.

(ii) We also construct the global solution of (1.1) by the contraction mapping principle.

In this case, since $p \geq 1 + \frac{2\alpha}{\alpha N-2} > 1 + \frac{2\alpha}{\alpha N+2-2\alpha}$, we know (5.17) also holds. In view of

$$p \geq 1 + \frac{2\alpha}{\alpha N-2} > 1 + \frac{2-\alpha N+\sqrt{(\alpha N-2)^2+16\alpha(\alpha-1)}}{4(\alpha-1)},$$

we have

$$\frac{N(p-1)}{2p(2-p)_+} \geq \frac{\alpha N(p-1)}{2p(2\alpha-p\alpha+p-1)_+} > 1.$$

Note that $\frac{(p-1)N}{2p} < \frac{\alpha N(p-1)}{2(p\alpha-p+1)}$ and $\frac{(p-1)N}{2p} < \frac{\alpha N(p-1)}{2p(2\alpha-p\alpha+p-1)_+}$. So we can choose $r > p$ and $r < pq_c$ such that (5.18) and

$$\frac{\alpha}{p-1} - \alpha < \frac{\alpha}{p-1} + 1 - \alpha < \frac{\alpha N}{2r} \tag{5.25}$$

hold. It follows from (5.25) and (5.18) that $0 < \frac{1}{q_c} - \frac{1}{r} < \frac{2}{N}$ and $0 < \frac{1}{\tilde{q}_c} - \frac{1}{r} < \frac{2}{N}$. Hence, if $u_0 \in L^{q_c}(\mathbb{R}^N)$ and $u_1 \in L^{\tilde{q}_c}(\mathbb{R}^N)$, then (4.1) and (4.4) imply

$$\sup_{t>0} t^{\beta}(\|P_\alpha(t)u_0\|_{L^r(\mathbb{R}^N)} + \|{}_0I_t^1 P_\alpha(t)u_1\|_{L^r(\mathbb{R}^N)}) = \eta < +\infty.$$

The rest proof is similar to that of Theorem 5.4(ii), so we omit it. □

REMARK 5.6. The condition $p > 1 + \frac{2\alpha}{\alpha N-2}$ in Theorem 5.5(ii) is required just for guaranteeing $\tilde{q}_c > 1$ and $\sup_{t>0} t^{\beta}\|{}_0I_t^1 P_\alpha(t)u_1\|_{L^r(\mathbb{R}^N)} \leq C\|u_1\|_{L^{\tilde{q}_c}(\mathbb{R}^N)}$. Hence, if one can prove that estimate (4.4) remains true for $u_1 \in L^1(\mathbb{R}^N)$, then the conclusion of Theorem 5.5(ii) will also be true for $p = 1 + \frac{2\alpha}{\alpha N-2}$. Note that $\triangle$ is the infinitesimal generator of the heat semigroup on $L^1(\mathbb{R}^N)$. Thus, the spectral angle of $-\triangle$ on $L^1(\mathbb{R}^N)$ is less than or equal to some $\theta_0 \in [0, \frac{\pi}{2})$. So the assumption (ii) in Sect. 3 holds when $\alpha$ is close to 1. Consequently, for $u_1 \in L^1(\mathbb{R}^N)$, (4.4) holds at least when $\alpha$ is close to 1. Therefore, the conclusion of Theorem 5.5(ii) is true for $p = 1 + \frac{2\alpha}{\alpha N-2}$ at least when $\alpha$ is close to 1.

REMARK 5.7. From (5.23), we know any nontrivial mild solution of (1.1) blows up in a finite time if $N = 1$ and $u_0, u_1 \geq 0$.

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*Quanguo Zhang*
*Department of Mathematics*
*Luoyang Normal University*
*Luoyang 471022 Henan*
*People's Republic of China*
*E-mail: zhangqg07@163.com*

*Yaning Li*
*College of Mathematics & Statistics*
*Nanjing University of Information*
*Science & Technology*
*Nanjing Jiangsu 210044*
*People's Republic of China*
*E-mail: liyn11@126.com*