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Topologies and measures on the space of functions of bounded variation taking values in a Banach or metric space

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Abstract. We study functions of bounded variation with values in a Banach or in a metric space. In finite dimensions, there are three well-known topologies; we argue that in infinite dimensions there is a natural fourth topology. We provide some insight into the structure of these four topologies. In particular, we study the meaning of convergence, duality and regularity for these topologies and provide some useful compactness criteria, also related to the classical Aubin–Lions theorem. After this we study the Borel σ-algebras induced by these topologies, and we provide some results about probability measures on the space of functions of bounded variation, which can be used to study stochastic processes of bounded variation.

1. Introduction

Functions of bounded variation have a broad range of applications, including materials science, chemistry, image processing, or more generally, models that involve jumps and intervals of differentiability or even quiescence. There are even some applications to random processes, e.g. [19,25], which was in fact one of the motivations behind the current work, see [29]. Many properties of functions of bounded variation and their corresponding topologies can be found in the standard works [17,34] and [3]. In this paper, we study functions of bounded variation $\mathrm{BV}(0,T;Z)$ mapping an interval $(0,T)$ to an infinite-dimensional codomain,[1] see, for example, [1,10,22,30,31]. In the general setting, we take Z to be a metric space; some stronger results will require the case where $Z = X^*$, the Banach dual of a Banach space X.

The aim is to provide a systematic study of topological and measurability properties of the space of functions of bounded variation mapping to an infinite-dimensional codomain. For completeness, we also provide an overview of properties of functions of bounded variation themselves; most of those results are either straightforward generalisations of the finite-dimensional case, as, for example, outlined in [3], or are otherwise known from the literature, see, for example, [1]. However, we are not aware of a comparable overview in the literature that would fit our purpose.

In finite dimensions, three topologies are commonly used: the *norm*, *strict* and *weak-** topologies. What is commonly called the weak-* topology is actually slightly stronger than what should be the called the weak-* topology from a functional-analytic

[1] For functions of bounded variations on an infinite-dimensional domain, see, e.g. [2,4,5,13].

perspective. Since the bounded norm balls are known to be compact in the stronger topology, this distinction is rarely made. However, we shall see that this is no longer true for functions mapping to an infinite-dimensional codomain. This is a subtle difference between the finite- and the infinite-dimensional settings. Therefore, in infinite dimensions one needs to distinguish between the (functional analytic) weak-* topology and what we coin the *hybrid* topology, which coincides with what is usually called the weak-* topology in finite dimensions.

Since the norm topology is too strong for many practical purposes, we focus on the weak-*, hybrid and strict topologies. In particular, we:

- Prove criteria for convergence;
- Prove compactness criteria, from which we also derive a generalised Aubin–Lions result;
- Characterise the dual space of the space $\mathrm{BV}(0, T; Z)$ equipped with the different topologies;
- Study several topological properties like separability, metrisability on subsets, Sousliness, perfect normality and complete regularity.

The study of Borel σ-algebras and measures on the space of functions of bounded variation seems to be relatively untrodden ground in the literature, despite its relevance. This relevance lies mostly in the application to Banach or metric-space-valued stochastic jump processes. Most standard tools in the theory of stochastic processes require a Polish space; therefore, one commonly works with the space of càdlàg functions equipped with the Skorokhod (J1-) topology. A natural question is then whether such tools are still available when working on the functions of bounded variation. Unfortunately, none of the mentioned topologies are Polish: the strong topology is metrisable but not separable, the strict topology is metrisable but not complete, and the hybrid and weak-$*$ topologies are not metrisable. Therefore, the standard tools mentioned above may no longer work in those topologies. Although the weak-$*$ topology is probably too weak for many practical purposes, the hybrid topology turns out to be strong enough to be useful but weak and regular enough to be tractable. In particular, it turns out that the hybrid topology is perfectly normal and completely regular, which is sufficient to establish:

- A generalised Portmanteau Theorem;
- A generalised forward Prokhorov Theorem;
- That tightness plus convergence of finite-dimensional distributions imply convergence of the path measures.

This implies that the space of functions of bounded variation equipped with the hybrid topology can be a suitable alternative to the càdlàg functions with the Skorokhod topology, which is commonly used in the study of stochastic processes.

1.1. Overview

The structure of the article is as follows: In Sect. 2, we outline the setting of this paper. We provide the definitions of variation and prove a number of fundamental results such as the existence of càdlàg-representatives, equivalence of the various notions of variation and the existence of time derivatives. In Sect. 3, we study the properties of the three topologies that we have introduced above and provide a number of convergence, duality, regularity and compactness theorems for each of them. In Sect. 4, we derive a number of measure-theoretic results. Most notably, we first show a number of facts about the σ-algebras corresponding to the strong, strict and hybrid topologies and then provide generalised versions of the Portmanteau theorem, Prokhorov's theorem, and a criteria for convergence of measures. For completeness, in "Appendix" we recall the notions of Banach-valued measures, integrals against Banach-valued measures, regularisation and topologies on Banach-valued measures.

2. Set-up

We first define various notions of variation and the space of functions of bounded variation. Next, we show a number of continuity properties of BV-functions that follow from purely metric considerations; in particular, we recall [18, p.109] that every BV-function is continuous up to countably many points and that the càdlàg representative can be used to calculate the variation. After this, we show that all concepts of variation coincide. Finally, we introduce the time derivative of a function of bounded variation and present a number of its properties.

2.1. Notions of variation

Let (Z, d) be a metric space. There are many different notions of variation; we shall see in Sect. 2.3 that these generally coincide. Most of these notions can easily be defined in a metric space, but some notions require a Banach predual space. To define those notions, we will use the embedding of the metric space Z in the Banach space $\mathrm{Lip}_0(Z)^*$ as we will explain below; the resulting notions are consistent with an alternative generalisation of Ambrosio [1].

2.1.1. Pointwise variation

The first, classical notion of variation is the *pointwise variation* (see, for example, [9,16]), defined for a (pointwise defined) function $f : (0, T) \to Z$ as

$$\mathrm{pvar}(f) := \sup_{0<t_0<t_1<\cdots<t_n<T} \sum_{i=1}^{n} \mathrm{d}\big(f(t_{i-1}), f(t_i)\big), \tag{2.1}$$

where the supremum runs over all finite partitions of the interval $(0, T)$. The pointwise variation is often used to define functions of bounded variation in fields where one is interested in the values at every single timepoint, like the field of energetic solutions for rate-independent systems and in non-smooth mechanics (see [26] and [28, Ch. 1]), which is closely connected to our theory for the Banach-valued case.

2.1.2. Essential pointwise variation

The pointwise defined functions of bounded pointwise variation have some mathematical drawbacks, for example: one cannot define weak derivatives. Therefore, one usually works on functions in $L^1\big(0, T; Z\big)$. If Z is a Banach space, the meaning of this space is clear; otherwise, the $L^1(0, T; Z)$ is a metric space defined as follows over equivalence classes of measurable functions $f \in \mathcal{M}(0, T; Z)$, where $\mathcal{M}(0, T; Z)$ are the measurable functions from $(0, T)$ to Z. More precisely, define the metric $\rho_{L^1}(f, g) := \int_0^T \mathrm{d}\big(f(t), g(t)\big)\mathrm{d}t$ for two functions $f, g \in \mathcal{M}(0, T; Z)$. We write $f \sim g$ if $f = g$ a.e., that is, if $\rho_{L^1}(f, g) = 0$. Fix a point $z_0 \in Z$. This point will play the role of the zero element in a Banach space; all results that we present are trivially invariant under the choice of this point. One then defines the metric space $L^1(0, T; Z) := \{f \in \mathcal{M}(0, T; Z) \ : \ \rho_{L^1}(z_0, f) < \infty\}/\sim$, which is complete whenever Z is complete. For a function $f \in L^1\big(0, T; Z\big)$, we can now define the *essential pointwise variation*,

$$\mathrm{epvar}(f) := \inf_{g\sim f} \mathrm{pvar}(g). \tag{2.2}$$

We recall in this context that the definitions of $\mathrm{pvar}(g)$ and $\mathrm{epvar}(f)$ do not rely on $L^1(0, T; Z)$-regularity [11]. However, every function g with bounded pointwise variation is almost everywhere continuous, i.e. measurable, and bounded and hence lies in $L^1(0, T; Z)$. The space of functions of bounded variation is defined accordingly:

$$\mathrm{BV}(0, T; Z) := \Big\{f \in L^1(0, T; Z) : \mathrm{epvar}(f) < \infty\Big\}.$$

If Z is a Banach space, then the natural norm on this space is simply $\|f\|_{\mathrm{BV}} := \|f\|_{L^1} + \mathrm{epvar}(f)$. If Z is a pure metric space, then defining a metric ρ_{BV} is a nontrivial task since $\mathrm{epvar}(f - g)$ is not well defined. This will require the embedding into the space of Lipschitz functions, as explained below.

2.1.3. Variation

The third and fourth notions can only be defined whenever $Z = X^*$ is a dual Banach space, since it involves integrating against test functions. In what follows, we consider the space $C_0(0, T; X)$ of continuous functions $(0, T) \to X$, vanishing in 0 and T, with its corresponding dual space $\mathrm{rca}(0, T; X^*)$ of regular X^*-valued measures, see Definition A.1 in "Appendix A.1"). To shorten notation we write, for $f \in L^1(0, T; X^*)$ or $\mu \in \mathrm{rca}(0, T; X^*)$ and $\phi \in C_0(0, T; X)$,

$$\langle\!\langle \phi, f \rangle\!\rangle := \int_0^T \langle \phi(t), f(t) \rangle \,\mathrm{d}t \qquad \text{or} \qquad \langle\!\langle \phi, \mu \rangle\!\rangle := \int_0^T \langle \phi(t), \mu(\mathrm{d}t) \rangle \,, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between the Banach space X and its Banach dual X^* (see "Appendix A.2"). It is now possible to introduce two integral-based variations for $f \in L^1(0, T; X^*)$:

$$\operatorname{var}(f) := \sup_{\substack{\phi \in C_0^1(0,T;X): \\ \|\phi\|_\infty \le 1}} -\langle\!\langle \dot{\phi}, f \rangle\!\rangle, \tag{2.4}$$

and

$$\operatorname{varw}(f) := \sup_{\substack{\phi \in C_b^1(0,T;X): \\ \|\phi\|_\infty \le 1}} \langle \phi(T), f(T-) \rangle - \langle \phi(0), f(0+) \rangle - \langle\!\langle \dot{\phi}, f \rangle\!\rangle, \tag{2.5}$$

provided the right limit $f(0+)$ at 0 and the left limit $f(T-)$ at T exist; otherwise, we set the value to $+\infty$. Naturally, (2.5) is just an extension of (2.4) which includes the boundary; we shall see in Sect. 2.3 that this extension does not change the value.

2.1.4. Embedding into Lipschitz functions

Following, for example, [1], one can note that in the Banach-valued case, every predual element $x \in X$ induces a Lipschitz functional $x^* \mapsto \langle x, x^* \rangle$ for which $\langle x, 0 \rangle = 0$. Motivated by this observation, the predual space is replaced by the (potentially much larger class of) Lipschitz functions,

$$\operatorname{Lip}_0(Z) := \Big\{ \xi : Z \to \mathbb{R} \text{ for which } \|\xi\|_{\operatorname{Lip}(Z)} < \infty \text{ and } \xi(z_0) = 0 \Big\},$$

equipped with the Lipschitz constant as norm:

$$\|\xi\|_{\operatorname{Lip}(Z)} := \sup_{\substack{z_1, z_2 \in Z: \\ z_1 \neq z_2}} \frac{|\xi(z_2) - \xi(z_1)|}{\mathrm{d}(z_2, z_1)}.$$

This Lipschitz norm is basically the global metric slope of ξ, see [6, Defn. 1.2.4], and is sometimes also known as the Cheeger derivative. To mimic the notation for the Banach case while emphasising the one-sided linearity, we write

$$\langle \xi, z] := \xi(z) \qquad \text{for } \xi \in \operatorname{Lip}_0(Z) \text{ and } z \in Z, \qquad \text{and}$$

$$\langle\!\langle \phi, f]\!] := \int_0^T \langle \phi(t), f(t)] \,\mathrm{d}t \qquad \text{for } \phi \in C_0\big(0, T; \operatorname{Lip}_0(Z)\big) \text{ and } f \in L^1(0, T; Z).$$

We now introduce the canonical embedding

$$\delta_\cdot : Z \to \mathrm{Lip}_0(Z)^*, \qquad z \mapsto \delta_z \quad \text{with}$$
$$_{\mathrm{Lip}_0(Z)}\langle \xi, \delta_z \rangle_{\mathrm{Lip}_0(Z)^*} := \langle \xi, z] = \xi(z) \quad \forall \xi \in \mathrm{Lip}_0(Z). \tag{2.6}$$

Any function $f \in \mathrm{BV}(0, T; Z)$ can then be associated with a Banach space-valued function $\delta_f(t) = \delta_{f(t)}$. However, the space $\mathrm{Lip}_0(Z)$ should be interpreted as a generalisation of the dual of Z rather than the predual; this leads to some differences between the Banach and the metric cases. For example, we can only define a measure-valued derivative $\dot{\delta}_f$ that takes values in $\mathrm{Lip}_0(Z)^*$ rather than in Z. Moreover, if $Z = X^*$ happens to be dual Banach space and we use the embedding, it is not immediately clear whether the notions of variation in X^* coincide with the notions in $\mathrm{Lip}_0(X^*)^*$. In Theorem 2.10, we prove that in the metric setting $\mathrm{epvar}(f) = \mathrm{epvar}\big(\delta_f\big)$; we then find in Corollary 2.11 that for $Z = X^*$ all notions of variations coincide. This shows that the following metric on the space $\mathrm{BV}(0, T; Z)$,

$$\rho_{\mathrm{BV}}(f, g) := \rho_{L^1}(f, g) + \mathrm{epvar}\big(\delta_f - \delta_g\big), \tag{2.7}$$

generalises the distance induced by the norm

$$\|f\|_{\mathrm{BV}} := \|f\|_{L^1(0,T;X^*)} + \mathrm{epvar}(f)$$

on $\mathrm{BV}(0, T; X^*)$.

2.2. Continuity-related properties of BV-functions

In this section, we work with pointwise variation and so we can work with functions taking values in a metric space. A crucial fact will be the existence of right and left limits, which will lead to a proof of Theorem 2.10. The first step is to show the existence of such limits for pointwise defined functions.

PROPOSITION 2.1. (Existence of right and left limits, [18, Sect. 2.5.16]) *Let Z be a complete metric space and $g : (0, T) \to Z$ satisfy* $\mathrm{pvar}(g) < \infty$. *Then g is continuous up to a countable subset of* $(0, T)$ *and g has left- and right-sided limits:*

$$g(t-) := \lim_{\substack{s\to t\\ s<t}} g(s) \equiv \lim_{s \nearrow t} g(s), \qquad \textit{and} \qquad g(t+) := \lim_{\substack{s\to t\\ s>t}} g(s) \equiv \lim_{s \searrow t} g(s)$$

for all $t \in (0, T)$, and one-sided limits at the end points.

Proof. Let us write

$$\mathrm{pvar}(g; (0, t]) := \sup_{0<t_0<t_1<\cdots<t_n\le t} \sum_{i=1}^{n} \mathrm{d}\big(g(t_{i-1}), g(t_i)\big). \tag{2.8}$$

This is a monotonely increasing function of t and bounded above by $\mathrm{pvar}(g)$ so it has at most countably many jumps. Let t be a continuity point of $\mathrm{pvar}(g; (0, t])$. Then we find

$$\lim_{\tau\to t} \mathrm{d}\,(g(t), g(\tau)) \le \lim_{\tau\to t} \big|\mathrm{pvar}(g; (0, t]) - \mathrm{pvar}(g; (0, \tau])\big| = 0\,.$$

Let $t \in [0, T)$. If there were a monotone sequence $t_n \searrow t$ such that $g(t_n)$ did not converge, then the sequence cannot be Cauchy, i.e. for some $\epsilon > 0$ one can pass to a subsequence such that $\mathrm{d}\big(g(t_n), g(t_{n+1})\big) \geq \epsilon$ for all n. This would imply that $\mathrm{pvar}(g) = \infty$, which is a contradiction. Similarly, we prove existence of left limits on $(0, T]$. $\square$

Using this fact, we can now prove that the right and left limits of a BV-equivalence class are well defined.

PROPOSITION 2.2. (Uniqueness of right and left limits) *Let Z be a complete metric space and $f \in \mathrm{BV}(0, T; Z)$. Then the left and right limits of f are independent from the chosen representative and thus uniquely defined.*

Proof. Take two representatives g_1, g_2 of f, i.e. $g_1(t) = g_2(t)$ almost everywhere, and $\mathrm{pvar}(g_1), \mathrm{pvar}(g_2) < \infty$. By Proposition 2.1, there exists a countable set $I \subset \mathbb{N}$ such that $g_1|_{[0,T]\setminus I}$ and $g_2|_{[0,T]\setminus I}$ are continuous and thus $g_1(t) = g_2(t)$ for all $t \notin I$. Using the triangle inequality, we infer that

$$\begin{aligned}\mathrm{pvar}(\mathrm{d}(g_1, g_2)) &= \mathrm{pvar}\big(\|\delta_{g_1} - \delta_{g_2}\|\big)\\ &\leq \mathrm{pvar}\big(\delta_{g_1}\big) + \mathrm{pvar}\big(\delta_{g_2}\big) = \mathrm{pvar}(g_1) + \mathrm{pvar}(g_2)\,.\end{aligned}$$

Therefore, the left and right limits of $\mathrm{d}(g_1, g_2) = 0$ exist in all $t \in (0, T)$ with one-sided limits at the end points. Hence, $g_1(t-) = g_2(t-)$ for all $t \in (0, T]$ and $g_1(t+) = g_2(t+)$ for all $t \in [0, T)$. $\square$

Note that the last proposition can be proved without the δ-formalism, but applying the δ-duality shortens the proof due to the fact that pvar is a norm in the dual Banach space setting. By Proposition 2.2, we can construct a càdlàg version of a BV-function. We prove here that this version is in fact a minimiser for essential pointwise variation. Later on in Corollary 2.20 we prove that the càdlàg version can be related to the derivative.

PROPOSITION 2.3. *Let Z be a complete metric space and $f \in \mathrm{BV}(0, T; Z)$. Define $f_{\mathrm{cadlag}}(t) := f(t+)$. Then $f_{\mathrm{cadlag}} = f$ a.e. and* $\mathrm{epvar}(f) = \mathrm{pvar}\big(f_{\mathrm{cadlag}}\big)$.

Proof. First note that $f_{\mathrm{cadlag}}(t) = f(t)$ wherever f is continuous and by Proposition 2.1 the discontinuity points are a countable set and thus of measure 0, which proves the first statement. Because of this, one has $\mathrm{epvar}(f) \leq \mathrm{pvar}\big(f_{\mathrm{cadlag}}\big)$. Suppose the inequality to be strict, then there exists an a $g = f$ a.e. such that $\mathrm{pvar}(g) < \mathrm{pvar}\big(f_{\mathrm{cadlag}}\big)$. By Proposition 2.2 $f_{\mathrm{cadlag}} \equiv g_{\mathrm{cadlag}}$ and so $\mathrm{pvar}(g) < \mathrm{pvar}\big(g_{\mathrm{cadlag}}\big)$, thus there exists an $\epsilon > 0$ and a finite partition $0 < t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ such that

$$\mathrm{pvar}(g) + \epsilon \leq \sum_{i=1}^{n} \mathrm{d}\left(g_{\mathrm{cadlag}}(t_{i-1}), g_{\mathrm{cadlag}}(t_i)\right).$$

However, we may also find $s_i \in (t_i, t_{i+1})$ such that

$$\max_{i=0,\dots,n} \mathrm{d}\left(g(s_i), g_{\mathrm{cadlag}}(t_i)\right) < \frac{\epsilon}{3n}$$

and thus $\mathrm{pvar}(g) + \epsilon \le \mathrm{pvar}(g) + \frac{2\epsilon}{3}$, which is a contradiction as $\mathrm{pvar}(g) < \infty$. □

We end this subsection with two results that are instrumental in proving our Compactness theorem 3.21. For an open subinterval $I \subset (0, T)$ let

$$\mathrm{pvar}(f; I) := \sup_{\substack{0<t_0<t_1<\cdots<t_n<T:\\ t_0,t_n\in I}} \sum_{i=1}^{n} \mathrm{d}\big(f(t_{i-1}), f(t_i)\big),$$

then the following rule for combining variation holds:

PROPOSITION 2.4. *Let Z be a complete metric space, $0 < T_1 < T_2$ and $f \in \mathrm{BV}(0, T_2; Z)$, then*

$$\begin{aligned}&\mathrm{pvar}(f; (0, T_2))\\ &\quad = \mathrm{pvar}(f; (0, T_1)) + \mathrm{pvar}(f; (T_1, T_2)) + \mathrm{d}(f(T_1-), f(T_1)) + \mathrm{d}(f(T_1), f(T_1+)).\end{aligned}$$

LEMMA 2.5. *Let Z be a complete metric space, $f \in \mathrm{BV}(0, T; Z)$, $\epsilon > 0$ and $\sigma: (0, T) \to (0, T)$ be measurable and satisfy $|\sigma(t) - t| \le \epsilon$ for all $t \in (0, T)$, then*

$$\int_0^T \mathrm{d}(f(t), f(\sigma(t)+))\,\mathrm{d}t \le 3\epsilon\,\mathrm{epvar}(f).$$

Proof. By Proposition 2.3, one may replace $f(t)$ with $f_{\mathrm{cadlag}}(t)$. Let n be the largest integer no greater than T/ϵ; then, implicitly intersecting domains of integration with $(0, T)$ the integral can be broken into smaller integrals:

$$\int_0^T \mathrm{d}(f(t), f(\sigma(t)+))\,\mathrm{d}t = \sum_{i=1}^{n+1} \int_{(i-1)\epsilon}^{i\epsilon} \mathrm{d}\big(f_{\mathrm{cadlag}}(t), f_{\mathrm{cadlag}}(\sigma(t))\big)\mathrm{d}t$$

However, for $t \in ((i-1)\epsilon, i\epsilon) \cap (0, T)$ it follows that $\sigma(t) \in ((i-2)\epsilon, (i+1)\epsilon) \cap (0, T)$ and so

$$\mathrm{d}(f(t), f(\sigma(t)+)) \le \mathrm{pvar}\big(f_{\mathrm{cadlag}}; ((i-2)\epsilon, (i+1)\epsilon) \cap (0, T)\big).$$

Thus

$$\int_0^T \mathrm{d}(f(t), f(\sigma(t)+))\,\mathrm{d}t \le \epsilon \sum_{i=1}^{n+1} \mathrm{pvar}\big(f_{\mathrm{cadlag}}; ((i-2)\epsilon, (i+1)\epsilon) \cap (0, T)\big)$$

and the result follows after allowing for some triple counting since pvar is subadditive by Proposition 2.4 . □

2.3. Equivalence of notions of variation

In this section, we compare the various notions of variations, first for the Banach space setting, and then for the metric space setting.

2.3.1. Equivalence of notions of variation for Banach-valued functions

We now prove in two parts that all notions of variations coincide.

PROPOSITION 2.6. *Let X^* be a dual Banach space and let $f \in \mathrm{BV}(0, T; X^*)$, then*

$$\mathrm{var}(f) = \mathrm{epvar}(f).$$

Proof. By Proposition 2.2, we can canonically identify any $f \in \mathrm{BV}(0, T; X^*)$ with its càdlàg version, and then by Proposition 2.3 one sees that $\mathrm{epvar}(f) = \mathrm{pvar}(f)$. We can choose $0 < t_0 < t_1 < \cdots < t_n < T$ and $\xi_i, \zeta_i \in X$ with $\|\xi_i\|_X, \|\zeta_i\|_X = 1$ such that

$$\begin{aligned} \mathrm{pvar}(f) &\leq \sum_{i=1}^{n} \|f(t_{i-1}) - f(t_i)\| + \epsilon \\ &\leq \sum_{i=1}^{n} \Big\{ \|f(t_{i-1}) - f(t_i-)\| + \|f(t_i-) - f(t_i)\| \Big\} + \epsilon \\ &\leq \sum_{i=1}^{n} \Big\{ \langle \xi_i, f(t_{i-1}) - f(t_i-)\rangle \Big\} + \sum_{i=1}^{n} \Big\{ \langle \zeta_i, f(t_i-) - f(t_i)\rangle \Big\} + 2\epsilon. \end{aligned} \tag{2.9}$$

We now estimate both sums separately. Since $\xi_i \circ f \in \mathrm{BV}(0, T; \mathbb{R})$, each term in the first sum is bounded by the variation $\mathrm{var}(\xi_i \circ f; (t_{i-1}, t_i))$ of $\xi_i \circ f$, restricted to the interval (t_{i-1}, t_i). Due to this BV-regularity, we can take $\phi_i \in C_c^1(0, T; \mathbb{R})$ with $\operatorname{supp} \phi_i \subset (t_{i-1}, t_i)$, $0 \leq \phi_i \leq 1$ and

$$\langle \xi_i, f(t_{i-1}) - f(t_i-)\rangle \leq \mathrm{var}(\xi_i \circ f; (t_{i-1}, t_i)) \leq \int_{t_{i-1}}^{t_i} \dot{\phi}_i(t)\, \langle \xi_i, f(t)\rangle \,\mathrm{d}t + \frac{\epsilon}{n}.$$

Now define $\Phi\colon (0, T) \to X$ by $\Phi(t) := \sum_{i=1}^{n} \phi_i(t)\xi_i$. Then $\Phi \in C_c^1\big(0, T; X\big)$ with $\|\Phi\|_\infty \leq 1$ and,

$$\sum_{i=1}^{n} \Big\{ \langle \xi_i, f(t_{i-1}) - f(t_i-)\rangle \Big\} \leq \langle\langle \dot{\Phi}, f\rangle\rangle + \epsilon\,.$$

For some $\delta > 0$, one has $\Phi\,|_{\cup_{i=1}^{n}(t_i-\delta, t_i+\delta)} \equiv 0$. We now exploit this flexibility to deal with possible jumps at the ends of the intervals in the second sum of (2.9). Define $\psi \in C_c^1(\mathbb{R}; \mathbb{R})$ through

$$\dot{\psi}(s) = \begin{cases} -4s & \text{if} \quad 0 \le s < \frac{1}{2} \\ 4(s - \frac{1}{2}) - 2 & \text{if} \quad \frac{1}{2} \le s < 1 \\ 0 & \text{if} \quad s \ge 1 \end{cases}, \qquad \dot{\psi}(s) = -\dot{\psi}(-s)\,, \qquad \psi(-1) = 0\,.$$

Since $\zeta_i \circ f$ is right continuous and has left limits, we obtain that

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{t_i - \delta}^{t_i + \delta} \dot{\psi}\Big(\frac{t - t_i}{\delta}\Big) \langle \zeta_i, f(t) \rangle \, \mathrm{d}t = \langle \zeta_i, f(t_i-) - f(t_i) \rangle\,.$$

Thus we can pick δ sufficiently small so that $\Psi(t) := \sum_{i=1}^n \psi\left(\frac{t-t_i}{\delta}\right)\zeta_i$ lies in $C_c^1\big(0, T; X\big)$ with $\|\Psi\|_\infty \le 1$ and $\mathrm{supp}\Phi \cap \mathrm{supp}\Psi = \emptyset$, and

$$\sum_{i=1}^n \Big\{ \langle \zeta_i, f(t_i-) - f(t_i) \rangle \Big\} \le \langle\!\langle \dot{\Psi}, f \rangle\!\rangle + \epsilon.$$

Continuing with (2.9), we find that

$$\mathrm{pvar}(f) \le \langle\!\langle \dot{\Phi} + \dot{\Psi}, f \rangle\!\rangle + 4\epsilon \tag{2.10}$$

where even $\Phi + \Psi \in C_c^1\big(0, T; X\big)$ with $\|\Phi + \Psi\|_\infty \le 1$. Since for all $\varepsilon > 0$ we can construct Φ, Ψ such that (2.10) holds, we obtain

$$\mathrm{pvar}(f) \le \sup_{\substack{\Phi \in C_c^1(0,T;X): \\ \|\Phi\|_\infty \le 1}} \langle\!\langle \dot{\Phi}, f \rangle\!\rangle = \mathrm{var}(f).$$

For the converse, it is sufficient to establish $\mathrm{var}(f) \le \mathrm{pvar}(f) = \mathrm{epvar}(f)$ since we still identify f with its càdlàg representative. For $n \in \mathbb{N}$ define $f_n \in \mathrm{BV}(0, T; X^*)$ by the piecewise constant approximation

$$f_n(t) := \sum_{i=1}^n f\Big(\frac{(i-1)T}{n}\Big) \mathbb{1}_{\left[\frac{(i-1)T}{n}, \frac{iT}{n}\right)}(t)$$

and note that these are càdlàg by construction and satisfy$\rho_{L^1}(f_n, f) \to 0$. Further

$$\begin{aligned} \mathrm{var}(f_n) &= \sup_{\substack{\Phi \in C_0^1(0,T;X): \\ \|\Phi\|_\infty = 1}} \sum_{i=1}^n \int_{\frac{(i-1)T}{n}}^{\frac{iT}{n}} \Big\langle \dot{\Phi}(t), f\Big(\frac{(i-1)T}{n}\Big) \Big\rangle \mathrm{d}t \\ &= \sup_{\substack{\Phi \in C_0^1(0,T;X): \\ \|\Phi\|_\infty = 1}} \sum_{i=1}^n \Big\langle \Phi\Big(\frac{iT}{n}\Big) - \Phi\Big(\frac{(i-1)T}{n}\Big), f\Big(\frac{(i-1)T}{n}\Big) \Big\rangle \\ &\le \sum_{i=1}^{n-1} \Big\| f\big(\tfrac{(i-1)T}{n}\big) - f\big(\tfrac{iT}{n}\big) \Big\| \le \mathrm{pvar}(f). \end{aligned}$$

Since $\operatorname{var}(f)$ is the supremum over functionals continuous in the L^1 topology, it is lower semicontinuous, so that:

$$\operatorname{var}(f) \leq \liminf_n \operatorname{var}(f_n) \leq \operatorname{pvar}(f).$$

□

The next result can also be proved by explicit estimates in the style of the previous proof, the proof given uses material presented in the following sections and is therefore somewhat shorter.

PROPOSITION 2.7. *Let X^* be a dual Banach space and let $f \in \mathrm{BV}(0, T; X^*)$, then*

$$\operatorname{var}(f) = \operatorname{varw}(f).$$

Proof. By definition, we find $\operatorname{var}(f) \leq \operatorname{varw}(f)$. For the other direction of the equality $\operatorname{var}(f) = \operatorname{varw}(f)$, choose a sequence of functions $\psi_\eta \in C_b^1([0, 1); \mathbb{R})$ such that $\psi_\eta(0) = 1$, ψ_η is non-increasing and $\psi_\eta(\eta) = 0$. For an arbitrary $\phi \in C_b^1(0, T; X)$, we write $\phi_\eta(t) := \phi(t)\left(1 - \psi_\eta(t) - \psi_\eta(T - t)\right)$ which is now in $C_0^1(0, T; X)$. Then

$$\begin{aligned}
&\langle \phi(T), f(T-)\rangle - \langle \phi(0), f(0+)\rangle - \langle\!\langle \dot{\phi}, f\rangle\!\rangle \\
&\quad = \langle \phi(T), f(T-)\rangle - \langle \phi(0), f(0+)\rangle - \langle\!\langle \dot{\phi}_\eta, f\rangle\!\rangle + \langle\!\langle \dot{\phi}_\eta - \dot{\phi}, f\rangle\!\rangle \\
&\quad \leq \langle \phi(T), f(T-)\rangle - \langle \phi(0), f(0+)\rangle + \operatorname{var}(f) - \langle\!\langle \left(\psi_\eta + \psi_\eta(T - \cdot)\right) \dot{\phi}, f\rangle\!\rangle \\
&\qquad - \langle\!\langle \phi \left(\dot{\psi}_\eta - \dot{\psi}_\eta(T - \cdot)\right), f\rangle\!\rangle \\
&\quad \to \langle \phi(T), f(T-)\rangle - \langle \phi(0), f(0+)\rangle + \operatorname{var}(f) - \langle \phi(T), f(T-)\rangle + \langle \phi(0), f(0+)\rangle \\
&\quad = \operatorname{var}(f)
\end{aligned}$$

where we used the continuity of ϕ and the existence of left and right limits of f together with $\int_0^T \dot{\psi}_\eta(t)\,\mathrm{d}t = 1$. Taking the supremum over $\phi \in C_b^1(0, T; X)$ proves the claim. □

To summarise the results of Propositions 2.3, 2.6 and 2.7, we now have the equivalence of all notions of variations:

COROLLARY 2.8. *Let X^* be a dual Banach space, then for all $f \in \mathrm{BV}(0, T; X^*)$*

$$\operatorname{epvar}(f) = \operatorname{pvar}\left(f_{\text{cadlag}}\right) = \operatorname{var}(f) = \operatorname{varw}(f).$$

2.3.2. Equivalence of notions of variation for metric-space-valued functions

We now investigate the relation between the variations in the metric and the Banach setting. As explained in Sect. 2.1, the canonical embedding $\delta \colon Z \to \operatorname{Lip}_0(Z)^*$ plays a crucial role. We note that the space $\operatorname{Lip}_0(Z)$ is never empty—just consider $\xi(z) := \mathrm{d}(z, z_0)$, and that the embedding is continuous and injective, due to the following result, which is trivial to obtain but can be considered the heart of the concept of δ_f.

LEMMA 2.9. *For all* $z_1, z_2 \in Z$

$$\left\|\delta_{z_1} - \delta_{z_2}\right\|_{\mathrm{Lip}_0(Z)^*} = \sup_{\substack{\xi \in \mathrm{Lip}_0(Z): \\ \|\xi\|_{\mathrm{Lip}(Z)}=1}} \langle \xi, z_1] - \langle \xi, z_2] = \mathrm{d}(z_1, z_2). \tag{2.11}$$

Proof. The inequality $\sup_{\|\xi\|_{\mathrm{Lip}(Z)}=1} \langle \xi, z_1] - \langle \xi, z_2] \le \mathrm{d}(z_1, z_2)$ holds by definition. Equality follows for the choice $\xi(z) := \mathrm{d}(z, z_2) - \mathrm{d}(z_0, z_2)$. □

This lemma guarantees that $\delta_f \in \mathrm{BV}\big(0, T; \mathrm{Lip}_0(Z)^*\big)$ if $f \in \mathrm{BV}(0, T; Z)$. More precisely, we have

THEOREM 2.10. *Let Z be a complete metric space. For every $g : (0, T) \to Z$,*

$$\mathrm{pvar}(g) = \mathrm{pvar}\big(\delta_g\big),$$

and for every $f \in \mathrm{BV}(0, T; Z)$,

$$\mathrm{epvar}(f) = \mathrm{epvar}\big(\delta_f\big).$$

Proof. The first statement is a simple consequence of equation (2.11). For the second statement, use $\delta_f(t) = \delta_{f(t)}$ and continuity of the mapping δ to see that $(\delta_f)_{\mathrm{cadlag}} \equiv \delta_{(f_{\mathrm{cadlag}})}$. Together with Proposition 2.3 and the first statement, this implies that

$$\mathrm{epvar}(f) = \mathrm{pvar}\big(f_{\mathrm{cadlag}}\big) = \mathrm{pvar}\big(\delta_{(f_{\mathrm{cadlag}})}\big) = \mathrm{pvar}\big((\delta_f)_{\mathrm{cadlag}}\big) = \mathrm{epvar}\big(\delta_f\big).$$

□

Combining this result with Theorem 2.8 yields.

COROLLARY 2.11. *Let X^* be a dual Banach space. For every $f \in \mathrm{BV}(0, T; X^*)$:*

$$\mathrm{var}(f) = \mathrm{varw}(f) = \mathrm{epvar}(f) = \mathrm{epvar}\big(\delta_f\big) = \mathrm{varw}\big(\delta_f\big) = \mathrm{var}\big(\delta_f\big).$$

REMARK 2.12. (i) As a by-product of Corollary 2.11, it also follows that

$$\begin{aligned}\mathrm{var}(f) &= \sup_{\phi \in C_0^1(0,T;X)} \int_0^T \big\langle \dot\phi(t), f(t) \big\rangle \,\mathrm{d}t \\ &= \sup_{\psi \in C_0^1(0,T;\mathrm{Lip}_0(X^*))} \int_0^T \big\langle \dot\psi(t), f(t)\big] \,\mathrm{d}t = \mathrm{var}\big(\delta_f\big),\end{aligned}$$

and so the space of test functions used in defining $\mathrm{var}(\cdot)$ for $\mathrm{BV}(0, T; X^*)$ can be extended from $C_0\big(0, T; X\big)$ to $C_0\big(0, T; \mathrm{Lip}_0(X^*)\big)$ [which includes $C_0\big(0, T; X^{**}\big)$] without changing what is meant by variation.

(ii) There is another well-established concept for the variation of a metric-space-valued function, which was studied by Ambrosio [1] to define the space $\mathrm{BV}(\Omega; Z)$, $\Omega \subset \mathbb{R}^n$. For the case $n = 1$, his results read as follows: given

$f\colon (0,T) \to Z$, the variation of f is defined to be the smallest measure $\sigma_f \in \mathrm{rca}(0,T;\mathbb{R})$ such that

$$\forall \varphi \in \mathrm{Lip}_0(Z),\ \|\varphi\|_{\mathrm{Lip}_0(Z)} = 1,\ B \subset (0,T) \quad \sigma_f(B) \geq |\partial\varphi \circ f|(B)\,,$$

where $|\partial\varphi \circ f|(B)$ is the variation/Stieltjes measure of $\varphi(f)$ over B, see Theorem A.3. This definition by Ambrosio coincides with $\mathrm{epvar}(f)$. Our approach allows us to identify σ_f with $|\dot{\delta}_f|$.

2.4. Time derivatives of BV-functions

In this section, we introduce the measure-valued time derivative of a function of bounded variation and prove a number of properties related to this derivative. The concept of Banach-measure-valued derivatives for BV-functions was already outlined by Dinculeanu [15, Sect. 17], and recently used by Recupero [30] in the Hilbert space setting to study rate-independent systems. We briefly note that there are also other related notions of time derivatives, such as the reduced derivative [27, App. A] and Darboux-sums [26, App. B.5]. Moreover, the Kurzweil integral $\int_0^T \phi \,\mathrm{d} f$ can be used to give sense to the right-hand side of (2.12) without introducing a time derivative, see, e.g. [20,21].

First we show the existence of a measure-valued time derivative:

THEOREM 2.13. *(Existence of measure-valued derivatives) Let* $f \in \mathrm{BV}(0,T;Z)$.

(i) If $Z = X^*$ is a dual Banach space, then there exists a unique finite measure $\dot{f} \in \mathrm{rca}\big(0,T;X^*\big)$ with $\|\dot{f}\|_{\mathrm{TV}} = \mathrm{var}(f)$ and such that

$$-\langle\!\langle \dot{\phi}, f \rangle\!\rangle = \langle\!\langle \phi, \dot{f} \rangle\!\rangle \qquad \text{for all } \phi \in C_0^1(0,T;X)\,. \tag{2.12}$$

(ii) If Z is a complete metric space, then there exists a unique finite measure $\dot{\delta}_f \in \mathrm{rca}\big(0,T;\mathrm{Lip}_0(Z)^*\big)$ with $\|\dot{\delta}_f\|_{\mathrm{TV}} = \mathrm{var}(f)$ and such that

$$-\langle\!\langle \dot{\phi}, f]\!] = \langle\!\langle \phi, \dot{\delta}_f \rangle\!\rangle \qquad \text{for all } \phi \in C_0^1\big(0,T;\mathrm{Lip}_0(Z)\big)\,. \tag{2.13}$$

REMARK 2.14. The existence of a time derivative as a Stieltjes measure was already formulated in [15, III.17.2 Theorem 1] for general Banach spaces. However, this result does not provide an interpretation in the sense of a dual pairing with C_0^1-functions, which makes it necessary to restrict to dual spaces.

Proof of Theorem 2.13. The map $\phi \mapsto -\langle\!\langle \dot{\phi}, f \rangle\!\rangle$ is clearly linear and bounded:

$$\big|-\langle\!\langle \dot{\phi}, f \rangle\!\rangle\big| \leq \|\phi\|_{C_0(0,T;X)} \mathrm{var}(f) \qquad \forall \phi \in C_0^1(0,T;X)\,.$$

By denseness and by the Banach-valued Riesz–Markov–Kakutani theorem A.7, the claim follows. The proof for the metric case is the same if we replace X by $\mathrm{Lip}_0(Z)$. $\square$

REMARK 2.15. A word of warning is appropriate here: even in the case $Z = X^*$ and for differentiable f one does not in general have $\delta_{\dot f} = \dot\delta_f$ viewed as elements of rca$\big(0, T; \mathrm{Lip}_0(X^*)^*\big)$. To see this, take an $f \in W^{1,1}(0, T; X^*)$ and a $\phi \in C_0^1\big(0, T; C_b^1(X^*)\big)$ and apply two partial integrations to get ($D_{x^*}\langle\phi(t), x^*]$ denotes the Gâteaux derivative of $\phi(t)$ in x^*):

$$\langle\!\langle \phi, \dot\delta_f \rangle\!\rangle = \int_0^T {}_{X^{**}}\big\langle D_{f(t)}\langle\phi(t), f(t)], \dot f(t)\big\rangle_{X^*}\,\mathrm{d}t, \qquad \text{and}$$

$$\langle\!\langle \phi, \delta_{\dot f} \rangle\!\rangle = \int_0^T \langle\phi(t), \dot f(t)]\,\mathrm{d}t.$$

Hence in general the two only agree when integrated against test functions that are linear in f, that is $\phi \in C_0(0, T; X)$.

In many cases, the measure-valued derivative can itself be identified with a function. This is captured by the following definition and result.

DEFINITION 2.16. Let Z be a metric space and let $f \in \mathrm{BV}(0, T; Z)$. We say that f is p-absolutely continuous, $1 \le p \le \infty$, if there exists $v \in L^p(0, T)$ such that

$$\mathrm{d}\left(f_{\mathrm{cadlag}}(t), f_{\mathrm{cadlag}}(\tau)\right) \le \int_t^\tau v(s)\,\mathrm{d}s \qquad \forall 0 \le t \le \tau \le T\,. \tag{2.14}$$

In the next lemma, we show that *p-absolutely continuity* is equivalent to L^p-regularity of the derivative . This result is known in the literature for reflexive Banach spaces (see, for instance, [6, Rem. 1.1.3]).

LEMMA 2.17. (Absolute continuity) *Let Z be a complete separable metric space. A function $f \in \mathrm{BV}(0, T; Z)$ is p-absolutely continuous if and only if $\dot\delta_f \in L^p(0, T; \mathrm{Lip}_0(Z)^*)$. Let $Z = X^*$ be a dual Banach space. Then a function $f \in \mathrm{BV}(0, T; X^*)$ is p-absolutely continuous if and only if $\dot f \in L^p(0, T; X^*)$. Furthermore, $v = |\dot\delta_f|$ is optimal in (2.14).*

Proof. We first prove the Banach case. Let $\dot f \in L^p(0, T; X^*)$. Then (2.14) holds for $v(t) := \|\dot f(t)\|_{X^*}$. On the other hand, let (2.14) hold. As explained in "Appendix A.1", we may take the supremum over finite sub-intervals in the definition of the $\mathbb{R}$-valued measure $|\dot f|$. Therefore, we get that for every interval $(a, b] \subset (0, T)$,

$$\begin{aligned}
|\dot f|(a, b] &= \sup_{a<t_0<\ldots\le t_n\le b} \sum_{i=1}^n \big\|\dot f\big((t_{i-1}, t_i]\big)\big\|_{X^*} \\
&\overset{(2.15)}{=} \sup_{a<t_0<\ldots\le t_n\le b} \sum_{i=1}^n \big\|f(t_i+) - f(t_{i-1}+)\big\|_{X^*} \\
&\overset{(2.14)}{\le} \sup_{a<t_0<\ldots\le t_n\le b} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} v(s)\,\mathrm{d}s \le \int_{(a,b]} v(s)\,\mathrm{d}s.
\end{aligned}$$

By the classical Nikodym theorem, there exists a measurable function $w : (0, T) \to [0, 1]$ such that $|\dot{f}|(\mathrm{d}s) = v(s)w(s)\,\mathrm{d}s$. By the generalised Lebesgue–Nikodym theorem A.8, there exists $u : (0, T) \to X^*$ with $\|u(t)\|_{X^*} \le 1$ for every $t \in (0, T)$ such that for all $\phi \in C_c(0, T; X)$

$$\langle\!\langle \phi, \dot{f} \rangle\!\rangle = \int_0^T \langle \phi(t), u(t) \rangle \, |\dot{f}|(\mathrm{d}t) = \int_0^T \langle \phi(t), u(t) \rangle \, v(t)w(t)\,\mathrm{d}t\,.$$

This implies $\dot{f} = uvw$ and hence $\|\dot{f}\|_{L^p(0,T;X^*)} \le \|v\|_{L^p(0,T)}$ and $v = |\dot{f}|$ is optimal in (2.14).

The general metric case now follows immediately. □

We now show how a function of bounded variation can be reconstructed from its derivative. For this, we first need two lemmas which can be proved the same way as in [3, Example 1.75, Proposition 3.2 and Theorem 3.27].

LEMMA 2.18. *Let $Z = X^*$ be a dual Banach space, $\mu \in \mathrm{rca}(0, T; X^*)$, and define $g(t) := \mu\big((0, t]\big)$ for all $t \in (0, T)$. Then $g \in \mathrm{BV}(0, T; X^*)$ and $\mu = \dot{g} = \partial g$, where ∂g is the Stieltjes measure from Theorem A.3.*

LEMMA 2.19. *Let $Z = X^*$ be a dual Banach space. If $u, v \in \mathrm{BV}(0, T; X^*)$ such that $\dot{u} = \dot{v}$, then $u \equiv v + c$ for some constant $c \in X^*$.*

As a corollary of Lemmas 2.18, 2.19 and Theorem A.3, we obtain the following.

COROLLARY 2.20. *Let X be a Banach space with dual space X^* and let $f \in \mathrm{BV}(0, T; X^*)$. Then the càdlàg version of Proposition 2.3 can be written as*

$$f_{\mathrm{cadlag}}(t) = \dot{f}\big((0, t]\big) + f(0+). \tag{2.15}$$

Similarly, if Z is a metric space and $f \in \mathrm{BV}(0, T; Z)$ then

$$\delta_{f_{\mathrm{cadlag}}(t)} = \big(\delta_f\big)_{\mathrm{cadlag}}(t) = \dot{\delta}_f\big((0, t]\big) + \delta_{f(0+)}.$$

We now prove a number of useful results related to the derivative: a product rule, a mollification, an approximation result, and an integration-by-parts formula. The following two statements (Proposition 2.21 and Theorem 2.22) can be proved along the same lines as [3, Prop. 3.2, Thm. 3.9], using Lemma's A.10 and A.11 from "Appendix A.3".

PROPOSITION 2.21. *Let X be a separable Banach space and $f \in \mathrm{BV}(0, T; X^*)$.*

1. *For any Lipschitz function $\psi : (0, T) \to \mathbb{R}$ the product $f\psi \in \mathrm{BV}(0, T; X^*)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f\psi) = f\dot{\psi} + \dot{f}\psi\,.$$

2. *If $\psi_\eta \in C_c^\infty(\mathbb{R})$ is a mollifier with $\operatorname{supp}\psi_\eta \subset [-\eta, \eta]$ then*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_\eta * f)(t) = \big(\psi_\eta * \dot{f}\big)(t) \qquad \forall t \in (\eta, T - \eta)\,.$$

THEOREM 2.22. *(Approximation of* BV$(0,T;X^*)$*-functions by smooth functions.) Let X be a separable Banach space and let $f \in L^1(0,T;X^*)$. Then $f \in BV(0,T;X^*)$ if and only if there exists a sequence of functions $(f_\epsilon)_{\epsilon>0}$ with $f_\epsilon \in C^\infty(0,T;X^*)$ such that for all ϵ it holds*

$$\|f - f_\epsilon\|_{L^1(0,T;X^*)} < \epsilon\,, \qquad \left\|\dot{f}_\epsilon\right\|_{L^1(0,T;X^*)} \leq \operatorname{epvar}(f) + \epsilon. \tag{2.16}$$

Finally we give an integration-by-parts formula for the general metric case:

THEOREM 2.23. *(Integration by parts) Let Z be a complete metric space, and $f \in$ BV$(0,T;Z)$ and $\phi \in W^{1,\infty}\big(0,T;\mathrm{Lip}_0(Z)\big)$ (ϕ is identified with its Lipschitz continuous representative). Then*

$$\langle\!\langle \dot{\phi}, f]\!] + \langle\!\langle \phi, \dot{\delta}_f \rangle\!\rangle = \langle \phi(T-), f(T-)] - \langle \phi(0+), f(0+)].$$

Proof. Let $k \in \mathbb{N}$ satisfy $1/k < T/4$. Choose functions $\chi, \rho \in C_c^2(\mathbb{R};\mathbb{R})$ such that $\mathbb{1}_{[1/k,T-1/k]} \geq \chi \geq \mathbb{1}_{[2/k,T-2/k]}$ and $\rho \geq 0$ with support contained in $[-1/2k, 1/2k]$ and $\int \rho(t)\mathrm{d}t = 1$. Define $\phi_k \in C_c^2\left(0,T;\mathrm{Lip}_0(Z)\right)$ by $\phi_k := (\chi\phi) * \rho$. Then, since $k > 4/T$,

$$\begin{aligned}
&\left|\langle\!\langle \dot{\phi}, f]\!] + \langle\!\langle \phi, \dot{\delta}_f \rangle\!\rangle - \phi(T-)\left(f(T-)\right) + \phi(0+)\left(f(0+)\right)\right| \\
&\quad \leq \left|\langle\!\langle \dot{\phi}, f]\!] - \langle\!\langle \dot{\phi}_k, f]\!] - \phi(T-)\left(f(T-)\right) + \phi(0+)\left(f(0+)\right)\right| \\
&\qquad + \left|\langle\!\langle \dot{\phi}_k, f]\!] + \langle\!\langle \phi_k, \dot{\delta}_f \rangle\!\rangle\right| + \left|\langle\!\langle \phi, \dot{\delta}_f \rangle\!\rangle - \langle\!\langle \phi_k, \dot{\delta}_f \rangle\!\rangle\right|.
\end{aligned}$$

Now $\left|\langle\!\langle \dot{\phi}_k, f]\!] + \langle\!\langle \phi_k, \dot{\delta}_f \rangle\!\rangle\right|$ is 0 by the definition of $\dot{f}$ since $\phi_k \in C_c^2\left(0,T;\mathrm{Lip}_0(Z)\right)$. Also $\|\phi_k\|_{L^\infty(0,T;\mathrm{Lip}_0(Z))} \leq \|\phi\|_{L^\infty(0,T;\mathrm{Lip}_0(Z))}$, and $\lim_{k\to\infty} \phi_k(t) = \phi(t)$ for every $t \in (0,T)$, so by dominated convergence

$$\left|\langle\!\langle \phi, \dot{\delta}_f \rangle\!\rangle - \langle\!\langle \phi_k, \dot{\delta}_f \rangle\!\rangle\right| \leq \int_0^T \|\phi(t) - \phi_k(t)\|_{\mathrm{Lip}_0(Z)} \left|\dot{\delta}_f\right|(\mathrm{d}t) \xrightarrow{k\to\infty} 0.$$

The remaining term can be estimated by noting $\frac{\mathrm{d}}{\mathrm{d}t}\phi_k \equiv (\chi\dot{\phi}) * \rho + (\dot{\chi}\phi) * \rho$, using Lemma 2.5 and noting that if $s, t \searrow 0$ then $\phi(s)\left(f(t)\right) \to \phi(0+)\left(f(0+)\right)$ along with the analogous result for $T-$. □

REMARK 2.24. For the case $Z = X^*$ we obtain a stronger integration-by-parts result with $\dot{\delta}_f$ replaced by $\dot{f}$.

3. Topologies on the space of functions of bounded variation

Although the space of functions of bounded variation is a Banach (or metric) space, the norm topology is too fine for many purposes and in order to achieve convergence and compactness results one introduces coarser topologies: the weak-* and the strict topologies, see [3, Defn. 3.11 and 3.14]. As mentioned in introduction, in infinite dimensions one needs to distinguish between the (functional analytic) weak-* and what we call the hybrid topology.

In Sect. 3.1, we define these four topologies and explain why this distinction is relevant in infinite-dimensional spaces. We then investigate a number of important properties of the topologies. Although the norm topology is clearly metrisable, it is not separable and it is rarely possible to establish precompactness results; therefore, we restrict our analysis to the weak-*, hybrid and strict topologies, subsequently in Sects. 3.2, 3.3 and 3.4. For each of these three topologies, we will characterise convergence, the dual space, discuss regularity properties, and give sufficient conditions for compactness. Section 3.4 on the strict topology includes generalisations of [30].

Some of the results in this section hold when Z is a general metric space, but many others require the dual Banach space structure $Z = X^*$. These results then also hold in the metric setting after embedding $\mathrm{BV}(0, T; Z)$ into $\mathrm{BV}\big(0, T; \mathrm{Lip}_0(Z)^*\big)$.

3.1. Definition of the topologies

We now present four distinct topologies for the space $\mathrm{BV}(0, T; X^*)$, in decreasing order of fineness. All four topologies have equivalent formulations in terms of open (semi-)balls. To simplify presentation, we define these topologies by their corresponding notions of convergence. We emphasise that the definition of vague convergence given in Definition A.12 uses dual pairings with functions in $C_0(0, T; X)$ and not just $C_c(0, T; X)$. This is important because we are not simply dealing with probability measures.

It should be noted that the hybrid and weak-* topologies are not necessarily first-countable, so the topologies are defined through their convergent *nets* rather than through convergent sequences, see, for example, [12, Sect. A.2].

DEFINITION 3.1. *(Topologies on $BV(0, T; X^*)$)* Let X^* be a dual Banach space, and let $(f_n)_n$ be a net and f an element in $\mathrm{BV}(0, T; X^*)$. We say that

f_n converges to f in the *norm* or *strong* topology whenever:

$$f_n \Rrightarrow f \qquad :\Longleftrightarrow \qquad f_n \xrightarrow{L^1} f \quad \text{and} \quad \|\dot{f}_n - \dot{f}\|_{\mathrm{TV}} \to 0, \tag{3.1}$$

f_n converges to f in the *strict* topology whenever:

$$f_n \overset{\text{strict}}{\Longrightarrow} f \qquad :\Longleftrightarrow \qquad f_n \xrightarrow{L^1} f \quad \text{and} \quad \|\dot{f}_n\|_{\mathrm{TV}} \to \|\dot{f}\|_{\mathrm{TV}}, \tag{3.2}$$

f_n converges to f in the *hybrid* topology whenever:

$$f_n \rightrightarrows f \qquad :\Longleftrightarrow \qquad f_n \xrightarrow{L^1} f \quad \text{and} \quad \dot{f}_n \xrightarrow{\text{vague}} \dot{f}, \tag{3.3}$$

f_n converges to f in the *weak-** topology whenever:

$$f_n \rightleftharpoons f \qquad :\Longleftrightarrow \qquad f_n \xrightarrow{\text{vague}} f \quad \text{and} \quad \dot{f}_n \xrightarrow{\text{vague}} \dot{f}. \tag{3.4}$$

Observe that the strong topology is induced by the norm $\|\cdot\|_{\mathrm{BV}}$. The term *strict* convergence is used in [3, Def. 3.14]. It is slightly stronger than the hybrid convergence, see Proposition A.14, and it is clearly metrisable, see (3.11). The term *weak-** convergence is appropriate since $\mathrm{BV}(0, T; X^*)$ is isometrically isomorphic to a dual space, see [3, Rem. 3.12] and Proposition 3.6. We named the convergence (3.3) *hybrid* since it is a combination of the strong convergence for the functions and weak-* convergence for the distributional time derivatives. We have not (yet) been able to determine whether the hybrid topology is a *mixed* topology in the sense of Wiweger [35]; it certainly topologies the two-norm convergence of sequences, which was one of Wiweger's motivations.

For finite-dimensional X, weak-* and hybrid convergence coincide whenever the net is uniformly bounded in the BV-norm. Therefore, the distinction between the two is rarely made explicit. However, this is no longer true in the infinite-dimensional setting, as the following example shows.

EXAMPLE 3.2. Suppose $X = X^*$ is a separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, and define the sequence of constant functions $f_n(t) \equiv e_n$. This sequence has uniformly bounded norm $\|f_n\|_{\mathrm{BV}} = \|f_n\|_{L^1(0,T;X^*)} = T$, and $f_n \rightharpoonup 0$ but certainly not $f_n \rightrightarrows 0$. In particular, since $\mathrm{BV}(0, T; X^*)$ can be identified with a dual space, Banach–Alaoglu gives compactness of bounded BV-balls in the weak-* topology, but not in the hybrid topology.

Using the canonical embedding $\delta : Z \to \mathrm{Lip}_0(Z)^*$ and Theorem 2.10, one can easily generalise Definition 3.1 to the metric case:

DEFINITION 3.3. *(Topologies on $BV(0, T; Z)$)* Let Z be a metric space and let $(f_n)_n$ be a net and f an element in $\mathrm{BV}(0, T; Z)$. We say that

f_n converges to f in the *strong* topology whenever:

$$f_n \Rrightarrow_m f \quad :\Longleftrightarrow \quad f_n \xrightarrow{L^1} f \quad \text{and} \quad \left\|\dot{\delta}_{f_n} - \dot{\delta}_f\right\|_{\mathrm{TV}} \to 0, \tag{3.5}$$

f_n converges to f in the *strict* topology whenever:

$$f_n \overset{\text{strict}}{\rightrightarrows}_m f \quad :\Longleftrightarrow \quad f_n \xrightarrow{L^1} f \quad \text{and} \quad \|\dot{\delta}_{f_n}\|_{\mathrm{TV}} \to \|\dot{\delta}_f\|_{\mathrm{TV}}, \tag{3.6}$$

f_n converges to f in the *hybrid* topology whenever:

$$f_n \rightrightarrows_m f \quad :\Longleftrightarrow \quad f_n \xrightarrow{L^1} f \quad \text{and} \quad \dot{\delta}_{f_n} \xrightarrow{\text{vague}}_m \dot{\delta}_f, \tag{3.7}$$

f_n converges to f in the *weak-** topology whenever:

$$f_n \rightharpoonup_m f \quad :\Longleftrightarrow \quad \delta_{f_n} \xrightarrow{\text{vague}} \delta_f \quad \text{and} \quad \dot{\delta}_{f_n} \xrightarrow{\text{vague}}_m \dot{\delta}_f. \tag{3.8}$$

Observe that the strong topology is indeed induced by the metric (2.7). In our notation, we have included the subscript "$_m$", since it is not a priori clear whether these notions coincide with the notions of Definition 3.1 if $Z = X^*$ is a dual Banach space. However, we will show in Proposition 3.25 that the notions of strict convergence indeed coincide, and similarly for the strong convergence, see Remark 3.26. In Propositions 3.5 and 3.14, we show that the metric and Banach versions of the weak-$*$ and hybrid topologies agree at least on sequences; it is still an open question whether the topologies are the same, i.e. whether they agree on all nets.

3.2. The weak-$*$ topology

Recall from (3.4) and (3.8) that $f_n \overset{*}{\rightharpoonup} f$ whenever $f_n \xrightarrow{\text{vague}} f$ and $\dot{f}_n \xrightarrow{\text{vague}} \dot{f}$, and for the metric case $f_n \overset{*}{\rightharpoonup}_m f$ whenever $\delta_{f_n} \xrightarrow{\text{vague}} \delta_f$ and $\dot{\delta}_{f_n} \xrightarrow{\text{vague}}_m \dot{\delta}_f$.

3.2.1. Characterisation of convergence

PROPOSITION 3.4. *Let $(f_n)_n$ be a net and f an element in* $\mathrm{BV}(0, T; Z)$. *If $Z = X^*$ is a dual Banach space, then*

$$f_n \overset{*}{\rightharpoonup} f \quad \Longleftarrow \quad f_n \xrightarrow{\text{vague}} f \quad \text{and} \quad \sup_n \operatorname{var}(f_n) < \infty,$$

and equivalence holds when $(f_n)_n$ is a sequence. If Z is a complete metric space, then the same result holds if we replace $f_n \xrightarrow{\text{vague}} f$ by $\delta_{f_n} \xrightarrow{\text{vague}} \delta_f$.

Proof. Let X^* be a dual Banach space and $f_n \xrightarrow{\text{vague}} f$ a convergent net in $\mathrm{BV}(0, T; Z)$ with $\sup_n \operatorname{var}(f_n) < \infty$. Now approximate an arbitrary test function $\phi \in C_0(0, T; X)$ by a sequence $(\phi_k)_k \subset C_0^1(0, T; X)$ such that $\|\phi - \phi_k\|_\infty \to 0$. Since the variation is lower semicontinuous in the vague topology, we automatically get $\operatorname{var}(f) \le \sup_n \operatorname{var}(f_n)$. It then follows that

$$\begin{aligned}
\left|\langle\langle \phi, \dot{f}_n \rangle\rangle - \langle\langle \phi, \dot{f} \rangle\rangle\right| &\le \left|\langle\langle \phi_k, \dot{f}_n - \dot{f} \rangle\rangle\right| + \left|\langle\langle \phi - \phi_k, \dot{f}_n - \dot{f} \rangle\rangle\right| \\
&\le \left|\langle\langle \dot{\phi}_k, f_n - f \rangle\rangle\right| + 2\|\phi - \phi_k\|_\infty \sup_{\hat{n}} \operatorname{var}(f_{\hat{n}}) \\
&\underset{n}{\to} 2\|\phi - \phi_k\|_\infty \sup_{\hat{n}} \operatorname{var}(f_{\hat{n}}) \underset{k}{\to} 0,
\end{aligned}$$

which together with $f_n \xrightarrow{\text{vague}} f$ shows that $f_n \overset{*}{\rightharpoonup} f$.

On the other hand, if $(f_n)_n$ is a weak-* convergent *sequence*, then $\sup_n \langle\langle \phi, f_n \rangle\rangle < \infty$ and by Banach–Steinhaus it follows that $\sup_n \operatorname{var}(f_n) < \infty$.

The proof in the metric case is analogous once we replace f and $\dot{f}$ by δ_f and $\dot{\delta}_f$. □

We now compare the Banach-case Definition 3.1 of convergence with the metric-case Definition 3.3.

PROPOSITION 3.5. *Let $Z = X^*$ be a dual Banach space. Let $(f_n)_n$ be a net and f an element in* $\mathrm{BV}(0, T; X^*)$. *Then*

$$f_n \rightrightarrows f \qquad \Longleftarrow \qquad f_n \rightrightarrows_m f.$$

Moreover, if $\sup_n \operatorname{epvar}(f_n) < \infty$, *for example because $(f_n)_n$ is a sequence, then the implication is in fact an equivalence.*

Proof. If $f_n \rightrightarrows_m f$, then $f_n \rightrightarrows f$ since $C_0(0, T; X) \hookrightarrow C_0(0, T; \mathrm{Lip}_0(X^*))$. The converse is Proposition 3.4. □

3.2.2. Duality

We first show that the space of functions of bounded variation can itself be regarded as a dual space. This theorem works as in the finite-dimensional case (see [3, Rem. 3.12]), and we include it here to provide the full details. To shorten notation, we introduce the spaces:

$$\begin{aligned}\Phi &:= C_0(0, T; X) \times C_0(0, T; X) \quad \text{and} \\ \Phi_{\partial t} &:= \left\{(\dot{\phi}_2, \phi_2) : \phi_2 \in C_c^\infty(0, T; X)\right\} \subset \Phi,\end{aligned}$$

both equipped with the norm $\|\phi\|_\Phi := \sup_{t\in(0,T)} \|\phi_1(t)\|_X + \sup_{t\in(0,T)} \|\phi_2(t)\|_X$.

THEOREM 3.6. *Let $Z = X^*$ be a dual Banach space. Then the Banach space $\big(\mathrm{BV}(0, T; X^*), \|\cdot\|_{\mathrm{BV}}\big)$ is isometrically isomorphic to $\big(\Phi/\overline{\Phi_{\partial t}}\big)^*$, and the weak-* convergence corresponds to the convergence defined in* (3.4).

Proof. Observe that for any $f \in \mathrm{BV}(0, T; X^*)$, by Theorem 2.13 the derivative $\dot{f}$ is well defined as an object in $C_0(0, T; X)^* \cong \mathrm{rca}(0, T; X^*)$. Define the map $T : \mathrm{BV}(0, T; X^*) \to \Phi^*$ by $Tf := (\hat{f}, \dot{f})$, where $\hat{f}(\mathrm{d}t) := f(t)\,\mathrm{d}t$. We can then characterise the annihilator of the closure $\overline{\Phi_{\partial t}}$ as (see [32, Sect. 4.6])

$$\begin{aligned}\overline{\Phi_{\partial t}}^{\perp} &:= \left\{\mu \in \Phi^* : \langle \mu, \phi\rangle = 0 \text{ for all } \phi \in \overline{\Phi_{\partial t}}\right\} \\ &= \left\{\mu \in \Phi^* : \langle \mu, \phi\rangle = 0 \text{ for all } \phi \in \Phi_{\partial t}\right\} \\ &= \operatorname{Ran} T. \end{aligned} \tag{3.9}$$

The first equality follows immediately from the fact that $\Phi_{\partial t}$ is strongly dense in its own closure. For the second equality, the direction $\supseteq$ follows immediately from the definitions of $\dot{f}$ and $\Phi_{\partial t}$. For the direction $\subseteq$, pick a $\mu \in \Phi^*$ for which $\langle \mu, \phi\rangle = \langle \mu_1, \dot{\phi}_2\rangle + \langle \mu_2, \phi_2\rangle = 0$ for all $\phi_2 \in C_c^\infty(0, T; X)$. If we define $f(t) := \mu_2\big((0, t]\big)$, then Lemma 2.18 yields that $f \in \mathrm{BV}(0, T; X^*)$ and $\langle f, \dot{\phi}_2\rangle = -\langle \mu_2, \phi_2\rangle = \langle \mu_1, \dot{\phi}_2\rangle$ for all $\phi_2 \in C_c^\infty(0, T; X)$. We therefore find that $\mu_1(\mathrm{d}t) = f(t)\,\mathrm{d}t$, and hence indeed $\mu \in \operatorname{Ran} T$, which proves equality (3.9).

Exploiting (3.9), by [32, Th. 4.9(b)] there exists a isometric isomorphism $\tau : \big(\Phi/\overline{\Phi_{\partial t}}\big)^* \to \overline{\Phi_{\partial t}}^{\perp} = \operatorname{Ran} T$. It is easily verified that $\|f\|_{\mathrm{BV}} = \|Tf\|_{\Phi^*}$. Therefore, the map $\tau^{-1} \circ T$ is an isometric isomorphism between $\mathrm{BV}(0, T; X^*)$ and $\big(\Phi/\overline{\Phi_{\partial t}}\big)^*$.

Finally, the desired weak-* convergence is characterised by convergence against $\Phi/\overline{\Phi_{\partial t}}$. In fact, it again suffices to test against functions in $\Phi/\Phi_{\partial t}$ because of the (strong) density. Then by definition,

$$\begin{aligned} f_n \overset{*}{\rightharpoonup} f :&\iff (\tau^{-1} \circ T)(f_n) \overset{*}{\rightharpoonup} (\tau^{-1} \circ T)(f) \\ :&\iff {}_{\operatorname{Ran} T}\big\langle (f_n \,\mathrm{d}x, \dot{f}_n), \overline{(\psi_1, \psi_2)} \big\rangle_{\Phi/\Phi_{\partial_t}} \to {}_{\operatorname{Ran} T}\big\langle (f \,\mathrm{d}x, \dot{f}), \overline{(\psi_1, \psi_2)} \big\rangle_{\Phi/\Phi_{\partial_t}} \\ &\quad \text{for all } \overline{(\psi_1, \psi_2)} = (\psi_1, \psi_2) + (\dot{\phi}_2, \phi_2) \in \Phi/\Phi_{\partial t} \\ &\iff \langle f_n, \psi_1 \rangle + \langle \dot{f}_n, \psi_2 \rangle \to \langle f, \psi_1 \rangle + \langle \dot{f}, \psi_2 \rangle \quad \text{for all } \psi_1, \psi_2 \in C_0(0, T; X) \\ &\iff f_n \rightrightarrows f. \end{aligned}$$

□

Now that we have a predual at hand, it is easy to see what the dual space for the weak-* topology is.

COROLLARY 3.7. *Let $Z = X^*$ be a dual Banach space. Then the dual space $\big(\mathrm{BV}(0, T; X^*), \text{weak-}*\big)^*$ is isomorphic to $\Phi/\overline{\Phi_{\partial t}}$.*

Proof. This is a general property of weak-* topologies, see, for example, [12, Th. V.1.3]. □

3.2.3. Regularity

Using Theorem 3.6, we can deduce many topological properties of the weak-* topology. In general, weak-$*$ topologies are not metrisable. Nevertheless, the compact sets are metrisable under a separability assumption. For this we first state the following simple lemma.

LEMMA 3.8. *Let X be a Banach space. Then $C_0(0, T; X)$ is separable if and only if X is separable.*

Proof. Let X be separable, with countable dense subset $Q \subset X$. Take a countable dense subset $\Psi \subset C_0(0, T)$. Then the countable set $\big\{ \sum_{i=1}^{\infty} \psi_i(t) q_i : (\psi_i)_i \subset \Psi, (q_i)_i \subset Q \big\}$ lies dense in $C_0(0, T; X)$. On the other hand, assume that $C_0(0, T; X)$ has a countable dense subset Λ. Take a function $\psi \in C_0(0, T)$ with $\psi(T/2) = 1$. Then for any arbitrary $x \in X$ there exists a sequence $(\lambda_n)_n \subset \Lambda$ such that $\lambda_n \to \psi x$. Let $\pi_{T/2} : C_0(0, T; X) \to X$ with $\pi_{T/2}[\phi] := \phi(T/2)$. By continuity of this evaluation map, we get that $\pi_{T/2}[\lambda_n] \to \pi_{T/2}[\psi x] = x$. Hence, the countable set $\pi_{T/2}[\Lambda]$ lies dense in X. □

From this we deduce that:

PROPOSITION 3.9. *Let $Z = X^*$ be a dual Banach space. All weak-$*$ compact sets in $\mathrm{BV}(0, T; X^*)$ are metrisable if and only if X is separable.*

Proof. By Lemma 3.8, the predual $\Phi/\overline{\Phi_{\partial t}} \subset C_0(0, T; X) \times C_0(0, T; X)$ from Theorem 3.6 is separable if and only if X is separable. The claim then follows from [9, Th. III.25]. □

PROPOSITION 3.10. *Let* $Z = X^*$ *where* X *is a separable Banach space. Then the topological space* $\big(\mathrm{BV}(0, T; X^*), \text{weak-}*\big)$ *is separable.*

Proof. Again by Lemma 3.8 the predual $\Phi/\overline{\Phi}_{\partial t} \subset C_0(0, T; X) \times C_0(0, T; X)$ of $\mathrm{BV}(0, T; X^*)$ is separable. By Corollary 3.7, this space $\Phi/\overline{\Phi}_{\partial t}$ is also the dual of $\big(\mathrm{BV}(0, T; X^*), \text{weak-}*\big)$. It then follows [9, Th. III.23] that the space $\big(\mathrm{BV}(0, T; X^*), \text{weak-}*\big)$ is also separable. □

3.2.4. Compactness criteria

Again by Theorem 3.6 it is easy to get compactness:

COROLLARY 3.11. *Let* X *be a Banach space, then any set of bounded* BV*-norm is relatively compact in* $\big(\mathrm{BV}(0, T; X^*), \text{weak-}*\big)$.

Proof. By Banach–Alaoglu. □

REMARK 3.12. Again after using the embedding $\delta_z : Z \to \mathrm{Lip}_0(Z)^*$, the same argument applies to the case where Z is a metric space. However, the limit of a relatively compact sequence/net in $\mathrm{BV}(0, T; Z)$ might end up in the bigger space $\mathrm{BV}\big(0, T; \mathrm{Lip}_0(Z)^*\big)$. In an abstract sense, such limit can be interpreted as a Young measure.

3.3. The hybrid topology

Recall from (3.3) and (3.7) that $f_n \Longrightarrow f$ whenever $f_n \xrightarrow{L^1} f$ and $\dot{f}_n \xrightarrow{\text{vague}} \dot{f}$, and for the metric case $f_n \Longrightarrow_m f$ whenever $f_n \xrightarrow{L^1} f$ and $\dot{\delta}_{f_n} \xrightarrow{\text{vague}}_m \dot{\delta}_f$.

3.3.1. Characterisation of convergence

PROPOSITION 3.13. *Let* Z *be a complete metric space,* $(f_n)_n$ *be a net and* f *an element in* $\mathrm{BV}(0, T; Z)$*, then*

$$f_n \Longrightarrow_m f \quad \Longleftarrow \quad f_n \xrightarrow{L^1} f \quad \text{and} \quad \sup_n \mathrm{var}(f_n) < \infty,$$

and equivalence holds when $(f_n)_n$ *is a sequence.*

Proof. The convergence $f_n \xrightarrow{L^1} f$ implies $\langle\!\langle \phi, \dot{f}_n \rangle\!\rangle \to \langle\!\langle \phi, \dot{f} \rangle\!\rangle$ for all test functions $\phi \in C_0^1(0, T; X)$ in the case $Z = X^*$ with X Banach, and $\langle\!\langle \phi, \dot{\delta}_{f_n} \rangle\!\rangle \to \langle\!\langle \phi, \dot{\delta}_f \rangle\!\rangle$ for all $\phi \in C_0^1(0, T; \mathrm{Lip}_0(Z)^*)$ in the general metric case. The proof is then the same as for Proposition 3.4. □

The previous result even holds if one weakens the condition on the variations by only requiring that there is some n_0 in the index set of the net such that $\sup_{n \geq n_0} \mathrm{var}(f_n) < \infty$.

We again compare the Banach-case Definition 3.1 of convergence with the metric-case Definition 3.3.

PROPOSITION 3.14. *Let $Z = X^*$ be a dual Banach space. Let $(f_n)_n$ be a net and f an element in* $\mathrm{BV}(0, T; X^*)$. *Then*

$$f_n \Rrightarrow f \qquad\qquad \Longleftarrow \qquad\qquad f_n \Rrightarrow_m f.$$

Moreover, if $\sup_n \operatorname{epvar}(f_n) < \infty$, *for example because $(f_n)_n$ is a sequence, then the implication is in fact an equivalence.*

Proof. The proof is the same as the proof of Proposition 3.5 if we replace $f_n \xrightarrow{\text{vague}} f$ by $f_n \xrightarrow{L^1} f$. □

This result extends to the case where there is an n_0 in the index set of the net such that $\sup_{n \geq n_0} \operatorname{epvar}(f_n) < \infty$, but we are not able to determine whether $f_n \Rrightarrow f$ implies $f_n \Rrightarrow_m f$ for arbitrary nets and thus whether the metric version of the hybrid topology is strictly finer than the Banach version when $Z = X^*$ a dual Banach space. We write τ_{hybrid} for the topology corresponding to $\Rrightarrow$ and $\tau_{\text{hybrid},m}$ for the topology corresponding to $\Rrightarrow_m$.

3.3.2. Duality

PROPOSITION 3.15. *Let $Z = X^*$ be a dual Banach space. Suppose a functional* $l : \mathrm{BV}(0, T; X^*) \to \mathbb{R}$ *is linear and τ_{hybrid}-continuous, then*

$$lf = \Psi f + \langle\!\langle \phi, \dot{f} \rangle\!\rangle$$

for some $\Psi \in L^1(0, T; X^*)^*$ and $\phi \in C_0(0, T; X)$. In particular, if X^{**} has the Radon–Nikodym property with respect to the Lebesgue measure on $(0, T)$, then $\Psi f = \langle\!\langle \psi, f \rangle\!\rangle$ for some $\psi \in L^\infty(0, T; X^{**})$.

Proof. By linearity, we only need to consider $|lf| < 1$. Since l is hybrid continuous, the inverse image $l^{-1}\big((-1, 1)\big)$ contains a hybrid open set from the subbase, centred around 0. Hence, one can find an $a \geq 0$, $n \in \mathbb{N}$ and $(\phi_i)_{i=1}^n \subset C_0\,(0, T; X)$ such that

$$\Big\{ f \in \mathrm{BV}\big(0, T; X^*\big) \colon a\, \|f\|_{L^1} < 1, \big|\langle\!\langle \phi_i, \dot{f} \rangle\!\rangle\big| < 1, i = 1, \ldots, n \Big\} \subset l^{-1}\big((-1, 1)\big).$$

Here, we rescaled a and ϕ_i such that the (semi-) balls have radii 1. Observe that all ϕ_i may be zero; this reflects the fact that l is also L^1-continuous. Similarly a could be 0 as well. Now define $\Phi : \mathrm{BV}(0, T; X^*) \to \mathbb{R}^n$ by

$$\Phi f = \big(\langle\!\langle \phi_1, \dot{f} \rangle\!\rangle, \ldots, \langle\!\langle \phi_n, \dot{f} \rangle\!\rangle\big)$$

and set

$$\ker(\Phi) = \bigcap_{i=1}^n \big\{ f \in \mathrm{BV}\big(0, T; X^*\big) \colon \langle\!\langle \phi_i, \dot{f} \rangle\!\rangle = 0 \big\}.$$

If $a = 0$, then $\ker(\Phi) \subset \ker(l)$ and from [32, Lemma 3.9] it follows that $lf = \langle\langle \phi, \dot{f} \rangle\rangle$ where $\phi = \sum_{i=1}^{n} \alpha_i \phi_i \in C_0(0, T; X)$ for some numbers $\alpha_i \in \mathbb{R}$.

If $a > 0$, then for $f \in \ker(\Phi)$ with $a \|f\|_{L^1} < 1$ it holds $|lf| < 1$ so $l|_{\ker(\Phi)}$ is L^1-continuous and can be extended to an L^1-continuous linear functional Ψ on all of $\mathrm{BV}(0, T; X^*)$ with norm at most a. We then have $\ker(\Phi) \subset \ker(l - \Psi)$ and so one can now find ϕ as in the case $a = 0$.

Finally, if X^{**} has the Radon–Nikodym property, then $L^1(0, T; X^*)^*$ can be identified with $L^\infty(0, T; X^{**})$ [14, Ch. IV, Th. 1]. □

It is easy to see that $\Psi_1 f + \langle\langle \phi_1, \dot{f} \rangle\rangle = \Psi_2 f + \langle\langle \phi_2, \dot{f} \rangle\rangle$ for all $f \in \mathrm{BV}(0, T; X^*)$ if and only if $(\Psi_1 - \Psi_2) f = -\langle\langle \phi_1 - \phi_2, \dot{f} \rangle\rangle$. Thus if X^{**} has the Radon–Nikodym property, the hybrid dual space may be identified with

$$L^\infty\left(0, T; X^{**}\right) \times C_0\left(0, T; X\right) / \left\{\left(\phi, \dot{\phi}\right) : \phi \in W_0^{1,\infty}\left(0, T; X\right)\right\}$$

3.3.3. Regularity

PROPOSITION 3.16. *Let Z be a complete, separable metric space. Then the space $\left(\mathrm{BV}(0, T; Z), \tau_{\mathrm{hybrid},m}\right)$ is also separable and if X^* is a dual Banach space then the result is also true for $\left(\mathrm{BV}(0, T; X^*), \tau_{\mathrm{hybrid}}\right)$.*

Proof. The proof of Proposition 3.32 applies in both cases. □

Recall that the hybrid topology is not metrisable, and not even sequential. Nonetheless, we shall see in Sect. 4 that it has many ‘good’ properties due to the fact that it is perfectly normal. In order to prove this, we first show that the space is completely regular and Souslin.

PROPOSITION 3.17. *Let Z be a complete, separable metric space and X^* a dual Banach space, then $\left(\mathrm{BV}(0, T; Z), \tau_{\mathrm{hybrid},m}\right)$ and $\left(\mathrm{BV}(0, T; X^*), \tau_{\mathrm{hybrid}}\right)$ are both Souslin, i.e. the continuous image of a (in this case unspecified) Polish space.*

Proof. For any $n \in \mathbb{N}$ define the balls:

$$A_n := \{f \in \mathrm{BV}(0, T; Z) \colon \mathrm{var}(f) \leq n\}.$$

By Proposition 3.13 $\tau_{\mathrm{hybrid},m}$ restricted to A_n is the same as the L^1 topology also restricted to A_n. Moreover, the L^1 subspace topology on each A_n is Polish. Thus $\mathrm{BV}(0, T; Z) = \bigcup_n A_n$, which is Souslin by [8, Th. 6.6.6]. The proof for τ_{hybrid} is the same. □

THEOREM 3.18. *Let Z be a complete, separable metric space and X^* a dual Banach space. Then $\left(\mathrm{BV}(0, T; Z), \tau_{\mathrm{hybrid},m}\right)$ and $\left(\mathrm{BV}(0, T; X^*), \tau_{\mathrm{hybrid}}\right)$ are perfectly normal topological spaces.*

Proof. The spaces are Souslin by Proposition 3.17 and completely regular, because the hybrid topologies are locally convex topologies. The statement then follows from [8, Theorem 6.7.7]. □

Although the hybrid topology is not metrisable, we do have the following simple result.

PROPOSITION 3.19. *Let Z be a complete separable metric space and X^* a dual Banach space. Then $\tau_{\text{hybrid},m}$ is metrisable on its own compact subsets of* $\mathrm{BV}(0, T; Z)$ *and τ_{hybrid} is metrisable on its own compact subsets of* $\mathrm{BV}(0, T; X^*)$.

Proof. We mainly follow the idea of [32, Th. 3.16] □

3.3.4. Compactness criteria

The next two results also hold in the case that $Z = X^*$ a dual Banach space provided one.

THEOREM 3.20. *Let Z be a complete metric space. If $\mathcal{F} \subset \mathrm{BV}(0, T; Z)$ is relatively compact in $\big(\mathrm{BV}(0, T; Z), \rho_{L^1}\big)$ and $\sup_{f\in\mathcal{F}} \mathrm{epvar}(f) < \infty$, then $\mathcal{F}$ is (both topologically and sequentially) relatively compact in $\big(\mathrm{BV}(0, T; Z), \tau_{\text{hybrid},m}\big)$ and if $Z = X^*$ a dual Banach space the compactness results also hold with respect to τ_{hybrid}.*

Proof. For any net or sequence in $\mathcal{F}$, there exists a ρ_{L^1}-convergent subnet or subsequence, respectively. Recall from Corollary 2.11 that $\mathrm{var}\big(\delta_f\big) = \mathrm{epvar}(f)$. Hence by Proposition 3.13, the subnet or subsequence is also hybrid convergent. □

THEOREM 3.21. *Let Z be a complete metric space, and let $\mathcal{F} \subset \mathrm{BV}(0, T; Z)$ satisfy*

1. $\sup_{f\in\mathcal{F}} \int_0^T \mathrm{d}(z_0, f(t))\mathrm{d}t + \mathrm{epvar}(f) < \infty$,
2. *For some countable and dense $Q \subset (0, T)$, there exist compact sets $K_q \subset Z$ with $\bigcup_{f\in\mathcal{F}} \{f(q+)\} \subset K_q$ for all $q \in Q$,*

then $\mathcal{F}$ is (both topologically and sequentially) relatively $\tau_{\text{hybrid},m}$-compact and if $Z = X^*$ is a dual Banach space, the result also holds in τ_{hybrid}.

We note that a closely related result was obtained by Mainik and Mielke [24] within the classical pvar-setting of BV-Functions.

Proof. We will establish the relative compactness of $\mathcal{F}$ in the ρ_{L^1} topology and then use Theorem 3.20. The L^1-compactness proof is an adaptation of the standard proof of the Arzela–Ascoli compactness result [16, IV.6, Th. 7] for sets of continuous functions. Since the ρ_{L^1}-topology is clearly metric, it is sufficient to show that every sequence in $\mathcal{F}$ has a converging subsequence.

Take a sequence $(f_n)_n$ in $\mathcal{F}$, identify each function with its càdlàg representative. We divide the remaining proof into four steps.

1. Since the K_q is compact for every $q \in Q$, by a diagonal argument we can construct a subsequence, which we will also denote $(f_n)_n$, such that $f_n(q+)$ converges for each $q \in Q$, and we denote these limits $\tilde{f}(q)$.

2. For any $t \in (0, T)$, one can now define

$$\tilde{f}(t) = \lim_{q \in Q, q > t, q \to t} \tilde{f}(q).$$

To see that this limit is well defined, pick an arbitrary $t \in (0, T)$ and suppose the converse. Then as in the proof of Proposition 2.1 there would be an $\epsilon > 0$ and a sequence $(q_i)_i \subset Q$ converging monotonely to t from above such that $\mathrm{d}\big(\tilde{f}(q_i), \tilde{f}(q_{i+1})\big) \geq \epsilon$. Then for any $N \in \mathbb{N}$ one could find an n_0 such that $\mathrm{d}\big(f_n(q_i+), \tilde{f}(q_i)\big) < \epsilon/N$ for all $n \geq n_0$ and for all $i = 1, \ldots, N$. This would imply that $\mathrm{pvar}(f_n) = \mathrm{epvar}(f_n) \geq (N-2)\epsilon$, which is a contradiction for large N.

3. Let $\epsilon > 0$, $k \in \mathbb{N}$ and $0 < t_0 < t_1 < \cdots < t_k < T$ then by the construction of $\tilde{f}$ one can find $n \in \mathbb{N}$ and $q_i \in Q$ satisfying $0 < q_0 < q_1 < \cdots < q_k < T$ and $q_i > t_i$ such that $\max_i \mathrm{d}\left(\tilde{f}(t_i), f_n(q_i)\right) < \epsilon/k$. Thus

$$\mathrm{epvar}\left(\tilde{f}\right) = \mathrm{pvar}\left(\tilde{f}\right) \leq \sup_{g \in \mathcal{F}} \mathrm{epvar}(g).$$

4. Now take an arbitrary $\epsilon > 0$. Since Q is dense one can find $0 < \widetilde{q}_1 < \cdots < \widetilde{q}_N < T$, all in Q such that $\max_{i=2,\ldots,N} (\widetilde{q}_i - \widetilde{q}_{i-1}) < \epsilon$ and $\widetilde{q}_1, T - \widetilde{q}_N < \epsilon$. Let

$$\sigma(t) := \widetilde{q}_1 \mathbb{1}_{(0,\widetilde{q}_1]}(t) + \sum_{i=2}^{N-1} \widetilde{q}_i \mathbb{1}_{(\widetilde{q}_{i-1},\widetilde{q}_i]}(t) + \widetilde{q}_N \mathbb{1}_{(\widetilde{q}_N,T)}(t).$$

Then, using Lemma 2.5 (the f_n and $\tilde{f}$ are càdlàg) and the pointwise convergence of the f_n on Q

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \mathrm{d}\left(f_n(t), \tilde{f}(t)\right) \mathrm{d}t \\
&\quad\leq \lim_{n\to\infty} \int_0^T \mathrm{d}\left(f_n(t+), f_n\left(\sigma(t)+\right)\right) \mathrm{d}t + \lim_{n\to\infty} \int_0^T \mathrm{d}\left(f_n\left(\sigma(t)+\right), \tilde{f}\left(\sigma(t)\right)\right) \mathrm{d}t \\
&\qquad + \int_0^T \mathrm{d}\left(\tilde{f}\left(\sigma(t)\right), \tilde{f}(t)\right) \mathrm{d}t \\
&\quad\leq 3\epsilon \sup_n \mathrm{epvar}(f_n) + \lim_{n\to\infty} \sum_{i=1}^{N} (\widetilde{q}_i - \widetilde{q}_{i-1})\, \mathrm{d}\left(f_n\left(\widetilde{q}_i+\right), g\left(\widetilde{q}_i\right)\right) + 3\epsilon\, \mathrm{epvar}(g) \\
&\quad\leq 6\epsilon \sup_{g \in \mathcal{F}} \mathrm{epvar}(g).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary $\lim_{n\to\infty} \rho_{L^1}(f_n, g) = 0$ and hence $\mathcal{F}$ is indeed L^1 compact.

□

We stress that the above compactness results are stated for a metric space Z without the embedding to the larger space $\mathrm{Lip}_0(Z)^*$. In particular, relative compactness of a set $\mathcal{F}$ in $\mathrm{BV}(0, T; Z)$ implies that the closure of $\mathcal{F}$ remains in $\mathrm{BV}(0, T; Z)$.

As an important application of Theorem 3.21, we provide the following generalisation of the classical Lemma by Aubin and Lions:

THEOREM 3.22. *Let $1 \leq p < \infty$. Let X^* be the dual of a Banach space and let Y, Z be Banach spaces such that $Y \hookrightarrow Z$ compactly and $Z \hookrightarrow X^*$ continuously. Denote*

$$\mathrm{BV}^p(0, T; Y, X^*) := \left\{ f \in L^p(0, T; Y) \;:\; \dot{f} \in \mathrm{rca}(0, T; X^*) \right\}.$$

Then, the embedding $\mathrm{BV}^p(0, T; Y, X^*) \hookrightarrow L^p(0, T; Z)$ *is compact.*

Proof. By a contradiction argument, we easily verify that any bounded set $\mathcal{F} \subset \mathrm{BV}(0, T; X^*)$ is bounded in $L^\infty(0, T; X^*)$. Hence, due to Simon [33, Sect. 8, Theorem 5] we only have to show that every bounded set $\mathcal{F} \subset \mathrm{BV}^p(0, T; Y, X^*)$ satisfies

$$\lim_{h \to 0} \sup_{f \in \mathcal{F}} \| f(h + \cdot) - f(\cdot) \|_{L^1(0, T-h; X^*)} = 0 .$$

This can be verified as follows:

$$\int_0^{T-h} |f(t+h) - f(t)| \mathrm{d}t \leq \int_0^{T-h} \int_t^{t+h} \mathrm{d}|\dot{f}|(s) \leq 2h \int_0^T \mathrm{d}|\dot{f}|(s) .$$

□

Let us also mention that the classic result [3, Th. 3.23] coincides with Theorem 3.21 in case $Z = \mathbb{R}$; it is proven via the Arzela–Ascoli theorem for compactness in the space of continuous functions, by substantially the same compact containment condition and diagonal argument as the present theorem. More generally, if Z is locally compact, then Condition 2 of Theorem 3.21 is redundant, since the functions are almost everywhere in bounded Z-balls, which are automatically compact. On the other hand, Condition 2 is not necessary as the following example shows.

EXAMPLE 3.23. Suppose $Z = X = X^*$ is a separable Hilbert space with orthonormal basis $(e_n)_{n \in \mathbb{N}_0}$, and define the sequence of functions

$$f_n(t) := \begin{cases} \left(n(t - \frac{1}{2}T) + 1\right) e_n & t \in \left[\frac{1}{2}T - \frac{1}{n}, \frac{1}{2}T\right] \\ \left(n(\frac{1}{2}T - t) + 1\right) e_n & t \in \left[\frac{1}{2}T, \frac{1}{2}T + \frac{1}{n}\right] \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Then clearly $\lim_{n\to\infty}\int_0^T \mathrm{d}(0, f_n(t))\,\mathrm{d}t = 0$ and the derivative also converge vaguely to the 0-measure. For any $t \in (0, T/2) \cup (T/2, T)$, the value $f_n(t)$ lies in the compact set $\{0\}$ for n large enough, but how large n should be depends on t. Therefore, Condition 2 cannot be satisfied even though the sequence is hybrid convergent.

3.4. The strict topology

Recall from (3.2) and (3.6) that $f_n \overset{\text{strict}}{\Longrightarrow} f$ whenever both $f_n \xrightarrow{L^1} f$ and $\|\dot{f}_n\|_{\mathrm{TV}} \to \|\dot{f}\|_{\mathrm{TV}}$, and for the metric case $f_n \overset{\text{strict}}{\Longrightarrow}_m f$ whenever both $f_n \xrightarrow{L^1} f$ and $\|\dot{\delta}_{f_n}\|_{\mathrm{TV}} \to \|\dot{\delta}_f\|_{\mathrm{TV}}$. The strict topology was recently also studied in [30] in the context of rate-independent systems, where the functions take values in a Hilbert space. Below, we will transfer some of the results from [30], to the dual Banach setting, in case the original proofs apply also in this case.

3.4.1. Characterisation of convergence

Let Z be a metric space. By its definition, the strict topology on $\mathrm{BV}(0, T; Z)$ is metrisable by

$$\rho_{\text{strict}}(f, g) := \rho_{L^1}(f, g) + \big|\|\dot{\delta}_g\|_{\mathrm{TV}} - \|\dot{\delta}_f\|_{\mathrm{TV}}\big|. \tag{3.11}$$

The following Proposition is stated for convergent sequences and may actually fail for convergent nets, see the discussion in "Appendix A.4". However, since the strict topology is metrisable, it is fully characterised by its convergent sequences.

PROPOSITION 3.24. *Let Z be a metric space, and let $(f_n)_n$ be a sequence and f an element in* $\mathrm{BV}(0, T; Z)$. *If $Z = X^*$ is a dual Banach space, then*

$$f_n \overset{\text{strict}}{\Longrightarrow} f \quad \Longrightarrow \quad f_n \xrightarrow{L^1} f \quad \text{and} \quad \dot{f}_n \xrightarrow{\text{narrow}} \dot{f}.$$

If Z is a complete metric space, then

$$f_n \overset{\text{strict}}{\Longrightarrow}_m f \quad \Longrightarrow \quad f_n \xrightarrow{L^1} f \quad \text{and} \quad \dot{\delta}_{f_n} \xrightarrow{\text{narrow}} \dot{\delta}_f.$$

Proof. For a strictly convergent sequence $f_n \overset{\text{strict}}{\Longrightarrow} f$, we have $\sup_{n\ge N} \mathrm{var}(f_n) < \infty$. By Proposition 3.13, we thus have in particular $\dot{f}_n \xrightarrow{\text{vague}} \dot{f}$. Using Proposition A.14, this implies together with $\|\dot{f}_n\|_{\mathrm{TV}} \to \|\dot{f}\|_{\mathrm{TV}}$ that $\dot{f}_n \xrightarrow{\text{narrow}} \dot{f}$. □

It turns out that if $Z = X^*$, then the topologies induced by $\overset{\text{strict}}{\Longrightarrow}$ and $\overset{\text{strict}}{\Longrightarrow}_m$ coincide:

PROPOSITION 3.25. *Let $Z = X^*$ be a dual Banach space. Let $(f_n)_n$ be a net and f an element in* $\mathrm{BV}(0, T; X^*)$. *Then*

$$f_n \overset{\text{strict}}{\Longrightarrow} f \qquad \Longleftrightarrow \qquad f_n \overset{\text{strict}}{\Longrightarrow}_m f.$$

Proof. Observe that the L^1 norm and metric are equivalent, and that Corollary 2.11 and Theorem 2.13 imply for $Z = X^*$ that $\|\dot{f}\|_{\mathrm{TV}} = \mathrm{epvar}(f) = \mathrm{epvar}\big(\delta_f\big) = \|\dot{\delta}_f\|_{\mathrm{TV}}$ and similarly $\|\dot{f}_n\|_{\mathrm{TV}} = \|\dot{\delta}_{f_n}\|_{\mathrm{TV}}$. □

REMARK 3.26. The same argument shows that if $Z = X^*$, then strong topologies induced by $\Rrightarrow$ and $\Rrightarrow_m$ also coincide.

REMARK 3.27. It is still an open question whether the converse of Proposition 3.24 is also true; since the converse of Proposition A.14 does not hold (see Remark A.15), the proof of this statement seems to be more involved than in the case $X = \mathbb{R}$.

We conclude with a number of properties from [30], which can be proven completely analogue to the original statement:[2]

LEMMA 3.28. ([30, Lemma 4.5]) *Let $Z = X^*$ be a dual Banach space, and let $(f_n)_n$ be a sequence and f an element in* $\mathrm{BV}(0, T; X^*)$ *such that $f_n \overset{\text{strict}}{\Longrightarrow} f$. Then $|\dot{f}_n|(a, b) \to |\dot{f}|(a, b)$ for every $a, b \in (0, T)$ such that $\dot{f}(\{a\}) = 0 = \dot{f}(\{b\})$.*

From this, similar results can trivially be obtained for the metric case, using the isometry of the embedding $\delta : Z \to \mathrm{Lip}_0(Z)^*$:

LEMMA 3.29. (Generalisation of [30, Lemma 4.6]) *Let Z be a complete separable metric space, and let $(f_n)_n$ be a sequence and f an element in* $\mathrm{BV}(0, T; Z)$ *such that $f_n \overset{\text{strict}}{\Longrightarrow}_m f$. Then, assuming càdlàg representatives, $f_n(t) \to f(t)$ for every $t \in (0, T)$ with $\dot{f}(\{t\}) = 0$. Moreover $f_n(0+) \to f(0+)$ and $f_n(T-) \to f(T-)$.*

LEMMA 3.30. (Generalisation of [30, Corollary 4.8]) *Let Z be a complete separable metric space, and let $(f_n)_n$ be a sequence and f an element in* $\mathrm{BV}(0, T; Z) \cap C(0, T; Z)$ *such that $f_n \overset{\text{strict}}{\Longrightarrow}_m f$. Then $f_n \to f$ uniformly.*

3.4.2. Duality

PROPOSITION 3.31. *Let $Z = X^*$ be a dual Banach space. Suppose a functional $l\colon \mathrm{BV}(0, T; X^*) \to \mathbb{R}$ is linear and strictly continuous, then*

$$lf = \Psi f + \langle\!\langle \phi, \dot{f} \rangle\!\rangle$$

for some $\Psi \in L^1(0, T; X^*)^*$ and $\phi \in C_b(0, T; X)$. In particular, if X^{**} has the Radon–Nikodym property with respect to the Lebesgue measure on $(0, T)$, then $\Psi f = \langle\!\langle \psi, f \rangle\!\rangle$ for some $\psi \in L^\infty(0, T; X^{**})$.

Proof. This exactly follows the proof of Proposition 3.15. □

[2] We mention that that paper uses left-continuous functions, whereas we use the càdlàg functions. However, in the proofs this distinction is not relevant.

3.4.3. Regularity

PROPOSITION 3.32. *Let Z be complete and separable (in its metric topology), then* $\big(\mathrm{BV}(0,T;Z),\text{strict}\big)$ *is separable.*

Proof. Let $f \in \mathrm{BV}(0,T;Z)$. For $n \in \mathbb{N}$ define the piecewise constant function $g_n \in \mathrm{BV}(0,T;Z)$ by

$$g_n(t) \equiv f\left(\frac{(i-1)T}{n}+\right) \qquad \text{for} \quad t \in \left[\frac{(i-1)T}{n}, \frac{iT}{n}\right), \quad i = 1,\dots,n.$$

First note that $\rho_{L^1}(g_n, f) \leq \mathrm{var}(f)T/n \to 0$ and further that

$$\mathrm{pvar}(f_n) = \sum_{i=1}^{n} \mathrm{d}\Big(f\big(\tfrac{(i-1)T}{n}+\big), f\big(\tfrac{iT}{n}+\big)\Big) \leq \mathrm{pvar}\big(f_{\mathrm{cadlag}}\big) = \mathrm{var}(f).$$

Since Z is separable one can take z_n^i from a countable subset such that

$$\mathrm{d}\Big(z_n^i, f\big(\tfrac{iT}{n}+\big)\Big) \leq \frac{1}{2n^2} \quad \forall n, i \leq n.$$

Define the piecewise constant function $f_n \in \mathrm{BV}(0,T;Z)$ by

$$f_n(t) \equiv z_n^i \qquad \text{for } t \in \left[\frac{(i-1)T}{n}, \frac{iT}{n}\right), \quad i = 1,\dots,n.$$

Then $\rho_{L^1}(f_n, f) \leq \rho_{L^1}(f_n, g_n) + \rho_{L^1}(g_n, f) \to 0$ and $\mathrm{var}(f_n) \leq \mathrm{var}(f) + 1/n$ so by the L^1 lower semi-continuity of the variation $\mathrm{var}(f_n) \to \mathrm{var}(f)$. □

The strict metric is not a complete metric: consider $f_n \in \mathrm{BV}(0,1;\mathbb{R})$, $f_n(t) = \mathbb{1}_{(0,1/n)}(t)$. Then $\mathrm{var}(f_n) = 1$ for all n so $\rho_{\mathrm{strict}}(f_n, f_m) = |n-m|/nm$. The sequence f_n is thus Cauchy for ρ_{strict}, but $\rho_{L^1}(f_n, 0) \to 0$ so it cannot converge in the strict metric.

3.4.4. Compactness criteria

As mentioned in Remark 3.27, it is still unclear whether the strict topology is the same as the topology characterised by the convergence

$$f_n \xrightarrow{L^1} f \quad \text{and} \quad \dot{f}_n \xrightarrow{\text{narrow}} \dot{f}, \qquad \text{or} \qquad f_n \xrightarrow{L^1} f \quad \text{and} \quad \dot{\delta}_{f_n} \xrightarrow{\text{narrow}} \dot{\delta}_f, \tag{3.12}$$

the latter being in the metric case. It is the latter topology for which we state a compactness result:

PROPOSITION 3.33. *Let Z be a metric space or a dual Banach space, let $\mathcal{F} \subset \mathrm{BV}(0,T;Z)$ be compact in the hybrid topology and suppose additionally that the set $\big\{|\dot{\delta}_f| : f \in \mathcal{F}\big\} \subset \mathrm{rca}(0,T)$ is tight. Then $\mathcal{F}$ is compact in the topology characterised by* (3.12).

Proof. By Lemma A.13, every hybridly convergent subnet of an arbitrary net is also convergent in the sense of (3.12). □

4. Measures on the space of functions of bounded variation

In this section, we study probability measures $\mathcal{P}\big(\mathrm{BV}(0,T;Z)\big)$ on the space of functions of bounded variation. The results of this section can be used to study of stochastic processes with bounded variation paths. Recall that white-noise-driven processes are never of bounded variation, but jump processes usually are.

Naturally, the definition of the space of probability measures depends on the σ-algebra. In Sect. 4.1 we study the Borel σ-algebra generated by the topologies introduced in the previous section. In Sect. 4.2 we study convergence and compactness for sequences of probability measures, where again the space of functions of bounded variation may be equipped with several topologies. The challenge is here that none of these topologies is Polish. In this respect, the theories that we present are generalisations of standard tools in probability. In particular, we show a Portmanteau theorem, a forward Prokhorov theorem, and a theorem that deduces convergence of the path measure from convergence of the finite-dimensional distributions.

It turns out that the weak-* topology is too coarse for these results, but all three results hold in the hybrid topology. Related topologies are used in the stochastics literature in [7,19].

REMARK 4.1. The Skorokhod J1 topology, commonly used in the study of stochastic processes, can be applied to $\mathrm{BV}(0,T;Z)$ since every function can be identified with its càdlàg representative. We cannot completely characterise the relationship between the two topologies. However, there is a sequence that converges hybridly but not J1, namely: $f_n(t) := \mathbb{1}_{[\frac{1}{2}-\frac{1}{2n},\frac{1}{2}+\frac{1}{2n}]}(t)e_n$, where e_n is the countable orthonormal basis of a Hilbert space, as in Example 3.2.

Because measure theory is based on countable numbers of operations, many results will require the metric topology on Z to be separable.

4.1. Measurability

First we consider possible σ-algebras. As usual, the Borel σ-algebra generated by a topology is the smallest σ-algebra containing all open sets, the Baire σ-algebra is the smallest σ-algebra making all continuous functions measurable. Let $\pi_t : \mathrm{BV}(0,T;Z) \to Z$ with

$$\pi_t(f) = f(t+) \tag{4.1}$$

for $t \in [0,T)$ and note that by Proposition 2.1 this is well defined on L^1-equivalence classes. The product σ-algebra is defined to be the smallest σ-algebra making all the π_t measurable.

Most results in this section hold true for $\mathrm{BV}(0,T;Z)$ with the strong, strict and hybrid topology. The key fact that makes this work is the following:

PROPOSITION 4.2. *Let Z be a complete metric space, and let* $\mathrm{BV}(0, T; Z)$ *be equipped with the strong or the strict topology; alternatively, let Z also be separable, and let* $\mathrm{BV}(0, T; Z)$ *be equipped with the hybrid topology. Then in each of these three cases the Baire and the Borel* σ*-algebras coincide.*

Proof. The norm and the strict topologies are metrisable and therefore perfectly normal. If Z is separable, then by Proposition 3.18 the hybrid topology is perfectly normal. By [8, Prop. 6.3.4] the Baire and Borel σ-algebras of a perfectly normal topological spaces coincide. □

For brevity we will write σ_{L^1} for the Borel σ-algebra generated by the ρ_{L^1} topology on $\mathrm{BV}(0, T; Z)$.

PROPOSITION 4.3. *Let Z be a complete, separable metric space, then the time-evaluation functions* π *from* (4.1) *are measurable with respect to* σ_{L^1}.

Proof. The metric topology on Z is separable so every open set can be written as a countable union of open balls $U_{z,\delta} := \{y \in Z : \mathrm{d}(y, z) < \delta\}$ for $z \in Z$ and $\delta > 0$. Hence it suffices to prove $\pi_t^{-1}\left(U_{z,\delta}\right) \in \sigma_{L^1}$ for arbitrary z and δ.

Now note that

$$f(t+) \in U_{z,\delta} \iff \lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathrm{d}\left(f(s), z\right) \mathrm{d}s < \delta.$$

Further, for fixed ϵ, t, z the function $f \mapsto \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathrm{d}\left(f(s), z\right) \mathrm{d}s$ is L^1-continuous and thus σ_{L^1} measurable, so that the limit as $\epsilon \searrow 0$ is also σ_{L^1} measurable. □

PROPOSITION 4.4. *Let Z be a complete, separable metric space. Then the function* $f \mapsto \mathrm{var}(f)$ *is measurable with respect to* σ_{L^1} *on* $\mathrm{BV}(0, T; Z)$.

Proof. Use Proposition 4.3 and note that the variation can be written as the supremum over rational partitions. □

From Proposition 4.4, one sees that $A_n := \{f \in \mathrm{BV}(0, T; Z) : \mathrm{var}(f) \le n\} \in \sigma_{L^1}$. By Proposition 3.13, the hybrid and L^1-topologies coincide on each A_n and thus the associated Borel σ-algebras are equal on each A_n. In particular, any hybrid open set, U, can be written as

$$U = \bigcup_{n \in \mathbb{N}} U \cap A_n = \bigcup_{n \in \mathbb{N}} V_n \cap A_n \tag{4.2}$$

for some $V_n \in \sigma_{L^1}$. It then follows that:

COROLLARY 4.5. *The Borel σ-algebras generated by the* L^1 *and hybrid topologies on* $\mathrm{BV}(0, T; Z)$ *are identical.*

THEOREM 4.6. *Let Z be a complete, separable metric space. Then the Borel σ-algebras of the topological spaces* $(\mathrm{BV}(0, T; Z), \tau_{\mathrm{hybrid},m})$ *and* $(\mathrm{BV}(0, T; Z), \rho_{L^1})$ *are both equal to the product σ-algebra.*

Proof. Using Propositions 4.3 and 4.5, it is sufficient to prove that all L^1-measurable functions are product measurable. To this end, note that

$$\begin{aligned}\rho_{L^1}(f,g) &= \int_0^T \mathrm{d}\big(f(t),g(t)\big)\,\mathrm{d}t = \int_0^T \mathrm{d}\big(f(t+),g(t+)\big)\,\mathrm{d}t\\ &= \lim_{n\to\infty} \tfrac{T}{n}\sum_{i=1}^{n} \mathrm{d}\big(f(\tfrac{iT}{n}+),g(\tfrac{iT}{n}+)\big),\end{aligned}$$

and that for fixed $n\in\mathbb{N}$, $f\in \mathrm{BV}(0,T;Z)$ the map $g\mapsto \sum_{i=1}^n \mathrm{d}\big(f(\frac{iT}{n}+),g(\frac{iT}{n}+)\big)$ is measurable in the product σ-algebra (defined above as making the evaluations π_t measurable). Since countable limits of measurable functions are measurable, the balls $\{g\in \mathrm{BV}(0,T;Z) : \rho_{L^1}(f,g)<\epsilon\}$ are therefore also measurable in the product σ-algebra. Finally, Z and $\mathrm{BV}(0,T;Z)$ are separable so that every open set can be written as a countable union of such balls. □

4.2. Convergence and compactness theorems

We now study *narrow* (often also called weak or weak-*) convergence and compactness in the space of Borel probability measures $\mathcal{P}\big(\mathrm{BV}(0,T;Z)\big)$ on the space $\mathrm{BV}(0,T;Z)$ equipped with the strong, strict or hybrid topology. Narrow convergence of probability measures in this space is characterised by the following Portmanteau theorem:

PROPOSITION 4.7. (Portmanteau Theorem, [8, Th. 8.2.10]) *Let Z be a metric space, and let $\mathrm{BV}(0,T;Z)$ be equipped with the strong or the strict topology; alternatively, let Z be a separable metric space, and let $\mathrm{BV}(0,T;Z)$ be equipped with the hybrid topology. Let $(\nu_n)_n$ be a net and ν be an element of the Borel probability measures $\mathcal{P}\big(\mathrm{BV}(0,T;Z)\big)$. Then the following statements are equivalent:*

- *(i)* $\nu_n \rightharpoonup \nu$, *that is,* $\int\Phi\,\mathrm{d}\nu_n\to\int\Phi\,\mathrm{d}\nu$ *for all* $\Phi\in C_b\big(\mathrm{BV}(0,T;Z)\big)$,
- *(ii)* $\limsup_n \nu_n(\mathcal{F})\le\nu(\mathcal{F})$ *for all closed sets* $\mathcal{F}\subset \mathrm{BV}(0,T;Z)$,
- *(iii)* $\liminf_n \nu_n(\mathcal{U})\le\nu(\mathcal{U})$ *for all open sets* $\mathcal{U}\subset \mathrm{BV}(0,T;Z)$,
- *(iv)* $\lim_n \nu_n(\mathcal{C})\le\nu(\mathcal{C})$ *for all continuity sets* $\mathcal{C}\subset \mathrm{BV}(0,T;Z)$, *i.e.* $\nu\big(\overline{\mathcal{C}}\backslash\mathring{\mathcal{C}}\big)=0$.

Proof. This is an immediate consequence of the fact that the topologies are perfectly normal [8, Th. 8.2.10]. The strong and strict topologies are clearly metrisable and therefore perfectly normal. By Theorem 3.18, the hybrid topology is also perfectly normal. □

The forward part of Prokhorov's theorem also holds:

PROPOSITION 4.8. (Generalised Prokhorov) *Let Z be a complete separable metric space. Then a tight collection of probability measures defined on the space $\big(\mathrm{BV}(0,T;Z),\tau_{\mathrm{hybrid},m}\big)$ is topologically and sequentially compact. If $Z = X^*$ a dual Banach space, then the result also holds for probability measures on $\big(\mathrm{BV}(0,T;X^*),\tau_{\mathrm{hybrid}}\big)$.*

Proof. Both BV spaces are Souslin by Proposition 3.17 so [8, Theorem 7.4.3] shows that the probability measures are Radon measures. (By Theorem 4.6, it is no restriction to assume that they are Borel.) Hybrid compact sets are metrisable by Proposition 3.19 and so the result follows from [8, Theorem 8.6.7]. □

PROPOSITION 4.9. *Let Z be a complete separable metric space, and let the space $\mathrm{BV}(0,T;Z)$ be equipped with $\tau_{\mathrm{hybrid},m}$, or let $Z = X^*$ be a dual Banach space, and let $\mathrm{BV}(0,T;X^*)$ be equipped with τ_{hybrid}. Let $(\nu_n)_n$ be a sequence or net and ν be an element of the Borel probability measures $\mathcal{P}\big(\mathrm{BV}(0,T;Z)\big)$. For each $t \in (0,T)$ define the finite-dimensional distributions by $\pi_t\#\nu_n(\mathcal{Z}) := \nu_n\big(\pi_t^{-1}(\mathcal{Z})\big)$ and similarly for ν. Assume:*

(i) The sequence ν_n is tight;
(ii) The finite-dimensional distributions $\pi_t\#\nu_n \rightharpoonup \pi_t\#\nu$ converge narrowly for each $t \in (0,T)$.

Then the sequence converges narrowly $\nu_n \rightharpoonup \nu$.

Proof. By Proposition 4.8, the net (or sequence) $(\nu_n)_n$ has a narrowly convergent subnet (or subsequence). Since the finite-dimensional distributions converge, any cluster point agrees with ν on the finite-dimensional distributions. Because of Theorem 4.6, a measure is uniquely characterised by its finite-dimensional distributions and hence ν is the unique limit. □

Appendix A: Preliminaries on Banach-valued measures and integration

We summarise the main concepts of the theory of Banach-valued measures. For a deeper insight into this subject, we refer the reader to the classical books [14,15] and [16, § IV.10] and the recent monograph by Ma [23]. In what follows, we mostly stick to the presentation in [23]. While the Bochner theory of Banach-valued functions has become very popular among analysts, this seems not to be the case for Banach-valued measures. Banach-valued measures are defined similarly to classical measures and one often can prove the intuitive analogues of classical results from ($\mathbb{R}$- or $\mathbb{C}$-) measure theory. The theory of Banach-valued measures is useful to understand the time derivative of a X^*-valued function of bounded variation.

A.1. Banach-valued measures

Let $\mathcal{B}$ denote the set of all Borel-sets of the interval $(0,T)$ and let X be a Banach space. For a set function $\mu : \mathcal{B} \to X$, we define the set function $|\mu| : \mathcal{B} \to \mathbb{R}$ through ([23, Paragraph 17-4.1])

$$|\mu|(A) := \sup_{P\in\mathcal{Q}(0,T)} \sum_{D\in P} \|\mu(A\cap D)\|_X \qquad \forall A \in \mathcal{B}\,. \tag{A.1}$$

where the supremum is taken over all finite families P of disjoint subsets of $(0, T)$. Observe that it is not a priori clear whether it suffices to take the supremum over intervals $0 < t_1, \ldots < t_n < T$, like in Definition 2.1 of the pointwise variation. However, in the case where A is the full interval $(0, T)$ and $\mu \in \mathrm{rca}(0, T; X^*)$, we get by Theorem A.7 that $|\mu|(0, T) = \|\mu\|_{\mathrm{TV}}$, the total variation norm defined in (A.4).

DEFINITION A.1. [15,16,23] Let X be a Banach space and $\mathcal{B}$ be the Borel σ-algebra on $(0, T)$.

- A set function $\mu : \mathcal{B} \to X$ is called an X-valued measure on $(0, T)$ if
  1. It is of finite total variation, i.e. $|\mu|(0, T) < \infty$ and
  2. It is countably additive, i.e. for every disjoint union $A = \bigcup_{j\in\mathbb{N}} B_j$, where $A \in \mathcal{B}$ and $B_j \in \mathcal{B}$ for all $j \in \mathbb{N}$

$$\mu(A) = \sum_{j\in\mathbb{N}} \mu(B_j)\,.$$

- An X-valued measure μ is called *regular* if for every $A \in \mathcal{B}$ and every $\varepsilon > 0$, there exists a compact (some authors only require closed, but here closed and compact are equivalent) set $K \subset A$ and an open set $G \supset A$ such that for all $A' \in \mathcal{B}$ with $K \subseteq A' \subseteq G$ it holds $\|\mu(A') - \mu(A)\|_X \leq \frac{\varepsilon}{2}$.
- We denote

$$\mathrm{rca}(0, T; X) := \big\{\mu : \mathcal{B} \to X \quad \text{regular } X\text{-valued measure}\big\},$$

which is a Banach space with norm $\mu \mapsto |\mu|\,(0, T) = \|\mu\|_{\mathrm{TV}}$ by Theorem A.7 or [16, Chapter III §7].

We make a few observations about this definition. First of all, this definition only allows for finite measures, that is, measures of finite total variation. This is related to the fact that the measures take values in a Banach space, where each element $x \in X$ is of finite norm (see, e.g. [16, IV.10, Cor. 2]). Secondly, we note that the countable additivity is equivalent to $\mu(A_j) \to \mu(A)$ whenever $A_j \supset A_{j+1}$ for all j and $\bigcap A_j = A$ [23, 17-5.4]. Thirdly, we point out that every X-valued regular Borel measure of finite total variation has the direct sum property of [15, Def. 10-7.1].

THEOREM A.2. *Suppose* $\mu \in \mathrm{rca}\,(0, T; X)$ *then* $|\mu| \in \mathrm{rca}\,(0, T; \mathbb{R})$.

Proof. That $|\mu|$ is a measure (a countably additive set function) is the content of [16, III.4.7], however all (real-valued) Borel measures on a metric space are regular by [8, Theorem 7.1.7]. An extensive discussion leads up to the statement of this result as [15, Chapter 3 §15 Proposition 21]. □

For a map $g : (0, T) \to X$, we define the set function ∂g through the sets:

$$\partial g\big((a, b]\big) := g(b) - g(a).$$

The well definedness of the extension of ∂g to arbitrary measurable subsets of $(0, T)$ is not a priori clear. However, recalling the function $t \mapsto \mathrm{pvar}(g; (0, t])$ from (2.8), we find the following important theorem.

THEOREM A.3. *([23, Ths. 17-7.4 and 17-7.9]) Let X be a Banach space and let $g : (0,T) \to X$ be of finite pointwise variation. Then ∂g defines a Banach-valued measure on $\mathcal{B}$ if and only if g is right continuous. In this case, $|\partial g| = \partial_t \mathrm{pvar}_{(0,t]}(g)$ and ∂g is called the Stieltjes' measure induced by g.*

A.2. Integration theory for Banach-valued measures

Let X, Y, Z be Banach spaces with a bilinear continuous mapping $p_{XY} : X \times Y \to Z$, $\mathcal{B}$ the Borel algebra on $(0,T)$ and let $\mu : \mathcal{B} \to Y$ be a Y-valued measure. In this section, we introduce the Z-valued integral $\int_0^T p_{XY}\big(\phi(t), \mu(\mathrm{d}t)\big)$. Recall from "Appendix A.1" the definition of the $\mathbb{R}$-valued measure $|\mu|$. With this we define the space

$$L^p_\mu(0,T;X,Y) := \left\{\phi : (0,T) \to X \ : \|\phi\|^p_{L^p_\mu(0,T;X,Y)} := \int_0^T \|\phi(t)\|^p_X \, |\mu|(\mathrm{d}t) < \infty\right\}.$$

A $\mathcal{B}$-step X-map $\phi : (0,T) \to X$ (or simple function) is of the form $\sum_{j=1}^N \alpha_j \chi_{A_j}$, where $\alpha_j \in X$ and χ_{A_j} is the indicator function for the set $A_j \in \mathcal{B}$. We define the step-integral

$$I_{p_{XY}}(\phi) := \sum_{j=1}^N p_{XY}\left(\alpha_j, \mu(A_j)\right) \quad \text{for } \phi = \sum_{j=1}^N \alpha_j \chi_{A_j}. \tag{A.2}$$

This map can be extended to an integral in the following way (see [23, Sect. 21]). First, one can show that for every $\phi \in L^p_\mu(0,T;X,Y)$ there exists a sequence of $\mathcal{B}$-step X-maps ϕ_n, such that $\|\phi_n(t)\|_X \uparrow \|\phi(t)\|_X$ and $\phi_n(t) \to \phi(t)$ for $|\mu|$-almost every $t \in (0,T)$. Then, for such approximating sequences $(\phi_n)_n$, one can show that the limit $\lim_{n\to\infty} I_{p_{XY}}(\phi_n)$ is independent of the choice of the sequence $(\phi_n)_n$. This defines the integral $\int_0^T p_{XY}\big(\phi(t), \mu(\mathrm{d}t)\big)$.

For $1 \le p < \infty$, one can show that the set of $\mathcal{B}$-step X-maps is dense in $L^p_\mu(0,T;X,Y)$. This in turn implies that the functions $C_c(0,T;X)$ are dense in $L^p_\mu(0,T;X,Y)$. From [23, Th. 21-2.11], we obtain that for every integrable map $\phi \in L^1_\mu(0,T;X,Y)$, we have

$$\left\| \int_0^T p_{XY}(\phi, \mathrm{d}\mu) \right\|_Z \le \|p_{XY}\| \int_0^T \|\phi\|_X \, \mathrm{d}|\mu| \, . \tag{A.3}$$

REMARK A.4. The above definition of the integral is very general. Let us mention four possible settings here:

(A) Let $Y = \mathbb{R}$, $Z = X$ is a Banach space, and $p_{XY}(x,y) := xy$. Then the theory in [23] is the commonly used Bochner theory.
(B) The case $Y = Z$ and $X = \mathbb{R}$ is a further connection to Bochner theory.
(C) Let X and Z be Banach spaces, and let $Y = L(X;Z)$ with $p_{XY}(x,y) := y(x)$.
(D) Let X be a Banach space, $Y = X^*$ and $Z = \mathbb{R}$ with $p_{XY}(x,y) := \langle x, y\rangle$. This is the setting of the main content of this paper.

The following generalisations of the classical Riesz–Markov–Kakutani result on the duality between $C_c(0,T;\mathbb{R})^*$ and $\operatorname{rca}(0,T;\mathbb{R})$ will turn out to be very useful in the proof of Theorem 2.13. For this we first define:

DEFINITION A.5. Let X and Z be Banach spaces. A linear $U : C_c(0,T;X) \to Z$ is called dominated if there exists a regular positive Borel measure ν such that

$$\|U(\phi)\|_Z \le \int_0^T \|\phi(t)\|_X \ \nu(\mathrm{d}t) \qquad \forall \phi \in C_c(0,T;X)\,.$$

PROPOSITION A.6. *([15, §19, Prop. 2 and Th. 3]) Let $U : C_c(0,T;X) \to \mathbb{R}$ be linear. Then U is dominated if and only if $\|U\|_{C_c(0,T;X)^*} < \infty$.*

The previous result will enable us to apply the generalised Riesz–Markov–Kakutani result to the settings (C) and (D) from Remark A.4.

THEOREM A.7. *([15, §19, Th. 2])*

(i) Assume X and Z are Banach spaces, $Y = L(X;Z)$ with $p_{XY}(x,y) := y(x)$. Then there exists an isomorphism between the dominated linear operators $U : C_c(0,T;X) \to Z$ and $\operatorname{rca}\big(0,T;L(X;Z)\big)$, given by the equality

$$U(\phi) = \int_0^T p_{XY}(\phi, \mathrm{d}\mu)\,.$$

(ii) Assume X is a Banach space, $Y = X^*$ and $Z = \mathbb{R}$ with $p_{XY}(x,y) := \langle x, y\rangle$. Then there exists an isomorphism between the dual $C_c(0,T;X)^* = C_0(0,T;X)^*$ and $\operatorname{rca}(0,T;X^*)$, with $|\mu|(0,T) = \|U\|_{C_c(0,T;X)^*} = \|\mu\|_{\mathrm{TV}}$, where

$$\|\mu\|_{\mathrm{TV}} := \sup_{\substack{\phi \in C_0(0,T;X):\\ \|\phi\|_\infty \le 1}} \langle\!\langle \phi, \mu \rangle\!\rangle. \tag{A.4}$$

We finally cite the following Lebesgue–Nikodym theorem. Recall in this context, every X^*-valued Borel measure μ of finite variation has the direct sum property of [15].

THEOREM A.8. *(General Lebesgue–Nikodym theorem, [15, §13, Th. 4]) Let $\mu : \mathcal{B} \to X^*$ be a measure of finite total variation. There exists a function $u : (0,T) \to X^*$ such that $\|u(t)\|_{X^*} = 1$ for $|\mu|$-almost every $t \in (0,T)$ and*

$$\int_0^T \langle \phi, \mathrm{d}\mu \rangle = \int_0^T \langle \phi, u \rangle \,\mathrm{d}|\mu| \qquad \forall \phi \in C_c(0,T;X)\,.$$

Theorem A.8 is less general than the classical Radon–Nikodym theorem as it only postulates the existence of a density for μ with respect to $|\mu|$ instead of a density with respect to a general real measure ν. If in the statement of Theorem A.8, we would want to replace $|\mu|$ by a general real measure ν, the space X^* would need to satisfy the Radon–Nikodym property. This holds, for example, if X is reflexive or if X^* is separable, see [14].

REMARK A.9. Using the Lebesgue–Nikodym theorem, we can decompose any $\mu \in \mathrm{rca}(0, T; X^*)$ into an absolute continuous part μ^c, an atomic part μ^a and a diffuse singular part μ^d (without atoms). This can be seen as follows: The measure $|\mu|$ can be decomposed into absolute continuous, atomic and diffuse singular parts $|\mu| = |\mu|^c + |\mu|^a + |\mu|^d$, see [3] (3.26). By Theorem A.8, there exists an $u : (0, T) \to X^*$ such that $\mu = u|\mu|$, and thus, we can set $\mu^c = u|\mu|^c$, $\mu^a = u|\mu|^a$ and $\mu^d = u|\mu|^d$.

A.3. Regularisation

In this section, we recall some standard regularisation results that are needed to prove Proposition 2.21 and Theorem 2.22.

Given $\mu \in \mathrm{rca}(0, T; X^*)$ and $\psi \in C_c(\mathbb{R})$, we define the convolution through

$$\psi * \mu(t) := \int_{\mathbb{R}} \varphi(t-s)\, \mu(\mathrm{d}s)\,.$$

Of particular interest are measures of the form $\mu(\mathrm{d}t) = f(t)\,\mathrm{d}t$ for some $f \in L^p(0, T; X^*)$. Recall the definition of $L^p_\mu(0, T; X, Y)$ from "Appendix A.2". We now have the following lemma:

LEMMA A.10. *Let $\left(\psi_\eta\right)_{\eta>0} \subset C_c(\mathbb{R})$ be a Dirac-sequence and let $1 \le p < \infty$. For all $\phi \in L^p(0, T; X, Y, \mu)$, we have $\psi_\eta * \phi \to \phi$ in $L^p_\mu(0, T; X, Y)$ as $\eta \to 0$.*

Proof. If $\left(\psi_\eta\right)_{\eta>0} \subset C_c^\infty(\mathbb{R})$ is a Dirac-sequence, one can use the denseness of $C_c(0, T; X)$ in $L^p(0, T; X, Y, \mu)$ and the uniform convergence of $\psi_\eta * \phi \to \phi$ for $\phi \in C_c(0, T; X)$. □

LEMMA A.11. *Let $\psi \in C_c^\infty(\mathbb{R})$ be non-negative, symmetric, with support in $(-1, 1)$ and with total mass $\int_{\mathbb{R}} \psi(t)\,\mathrm{d}t = 1$, and define the family of mollifiers $\psi_\eta(t) := \eta^{-1}\psi(t/\eta)$. For any $\mu \in \mathrm{rca}(0, T; X^*)$ and $\eta > 0$, the functions $\psi_\eta * \mu$ belong to $C^\infty(0, T; X^*)$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\psi_\eta * \mu)(t) = (\dot{\psi}_\eta * \mu)(t)$. Moreover, the measures $\psi_\eta * \mu$ converge weakly-$*$ to μ as $\eta \to 0$ and the following estimate holds for all Borel-sets $I \subset (0, T)$:*

$$\int_I |\psi_\eta * \mu|(t)\mathrm{d}t \le |\mu|\left(\bigcup_{t\in I}(t-\eta, t+\eta)\right),$$

Proof. The proof follows the lines of [3, Theorem 2.2]. □

A.4. Topologies on Banach-valued measures

In this section, we recall the most relevant topologies on the space $\mathrm{rca}(0, T; X^*)$, where X is a Banach space. Although $\mathrm{rca}(0, T; X^*)$ is a Banach space with norm $\|\cdot\|_{\mathrm{TV}}$, its norm topology is too strong for many practical purposes. Instead, we mostly work with the weak-* topology, which is sometimes called the *vague* topology. On the other hand, motivated by the duality between $C_b(0, T; X)$ and the space of finite, finitely additive regular signed Borel set functions (see [16, Th. IV.6.2] for the finite-dimensional version), one often also works with the topology induced by duality with $C_b(0, T; X)$, which is sometimes called the *narrow* or *weak* topology. To avoid confusion, we will avoid calling these topologies weak or weak-*, and stick to vague and narrow instead. To be more precise, we define:

DEFINITION A.12. Let $(\mu_n)_n$ be a net and μ an element in $\mathrm{rca}(0, T; X^*)$. We say that

μ_n converges to μ in the *vague* topology whenever:

$$\mu_n \xrightarrow{\text{vague}} \mu \quad :\Longleftrightarrow \quad \langle\!\langle \phi, \mu_n \rangle\!\rangle \to \langle\!\langle \phi, \mu \rangle\!\rangle \qquad \text{for all } \phi \in C_0(0, T; X), \tag{A.5}$$

μ_n converges to μ in the *narrow* topology whenever:

$$\mu_n \xrightarrow{\text{narrow}} \mu \quad :\Longleftrightarrow \quad \langle\!\langle \phi, \mu_n \rangle\!\rangle \to \langle\!\langle \phi, \mu \rangle\!\rangle \qquad \text{for all } \phi \in C_b(0, T; X). \tag{A.6}$$

Moreover, we will say that a net $(f_n)_n \subset L^1(0, T; X^*)$ converges to an element $f \in L^1(0, T; X^*)$ in the vague or narrow topology whenever the measures $(f_n(t)\,\mathrm{d}t)_n$ converge to $f(t)\,\mathrm{d}t$ in the vague or narrow topology, respectively.

The vague topology is not metrisable, since it is really a weak-* topology; in particular, it cannot be fully characterised through its convergent sequences. It should be noted that vague convergence is often also defined as convergence against compactly supported test functions $\phi \in C_c(0, T; X)$, in which case it would be metrisable. For measures of uniformly bounded finite total variation (e.g. probability measures), the two notions coincide; this is not the case in the present work and we work with test functions in $C_0(0, T; X)$.

Clearly, the narrow topology is stronger than the vague topology. As in the case of real-valued measures, vague convergence can be strengthened by a tightness argument.

LEMMA A.13. *Let $(\mu_n)_n$ be a net and μ an element in* $\mathrm{rca}(0, T; X^*)$. *If $\mu_n \xrightarrow{\text{vague}} \mu$ and the net $(|\mu_n|)_n \subset \mathrm{rca}(0, T)$ is tight, then $\mu_n \xrightarrow{\text{narrow}} \mu$.*

Proof. Take an arbitrary test function $\phi \in C_b(0, T; X)$ and an arbitrary $\epsilon > 0$. By the tightness, there exists a compact set $K_\epsilon \subset (0, T)$ for which $|\mu_n|(K_\epsilon^c) \le \epsilon$ for all n and without loss of generality we can assume that $|\mu|(K_\epsilon^c) < \epsilon$ since $|\mu|$ is regular. Take a test function $\psi \in C_0(0, T; X)$ such that $\phi|_{K_\epsilon} \equiv \psi|_{K_\epsilon}$. Then

$$\begin{aligned}\big|\langle\!\langle \phi, \mu_n - \mu \rangle\!\rangle\big| &\le \big|\langle\!\langle \psi, \mu_n - \mu \rangle\!\rangle\big| + \big(\|\psi\|_\infty + \|\phi\|_\infty\big)\big(|\mu_n|(K_\epsilon^c) + |\mu|(K_\epsilon^c)\big)\\ &< \big|\langle\!\langle \psi, \mu_n - \mu \rangle\!\rangle\big| + 2\epsilon\big(\|\psi\|_\infty + \|\phi\|_\infty\big) \to 2\epsilon\big(\|\psi\|_\infty + \|\phi\|_\infty\big),\end{aligned}$$

which proves the statement as ϵ was arbitrary. □

We can also strengthen vague convergence if one knows that the total variations converge. The proof requires sequences; the argument breaks down for general nets since a convergent net does not necessarily form a compact set.

PROPOSITION A.14. *Let* $(\mu_n)_n$ *be a sequence and* μ *an element in* $\mathrm{rca}(0, T; X^*)$. *If* $\mu_n \xrightarrow{\text{vague}} \mu$ *and* $\|\mu_n\|_{\mathrm{TV}} \to \|\mu\|_{\mathrm{TV}}$, *then* $\mu_n \xrightarrow{\text{narrow}} \mu$.

Proof. First we show by standard approximation arguments that the $|\mu_n|$ converge narrowly, which implies the tightness of the variation measures. Then we exploit the tightness to strengthen the vague convergence to the narrow convergence.

Recall from Theorem A.7 that

$$\big\||\mu_n|\big\|_{\mathrm{TV}} = \|\mu_n\|_{\mathrm{TV}} \to \|\mu\|_{\mathrm{TV}} = \big\||\mu|\big\|_{\mathrm{TV}}. \tag{A.7}$$

Therefore, the variation measures $|\mu_n|$ are bounded and a subsequence converges vaguely to some finite, positive measure $\nu \in \mathrm{rca}(0, T)$. Because of Theorem A.8, for any $\phi \in C_0(0, T; X)$ and along the convergent subsequence,

$$\langle\!\langle \phi, \mu \rangle\!\rangle \leftarrow \langle\!\langle \phi, \mu_n \rangle\!\rangle \le \int_0^T \|\phi(t)\|_X \; |\mu_n|(\mathrm{d}t) \to \int_0^T \|\phi(t)\|_X \; \nu(\mathrm{d}t).$$

Taking the supremum over test function yields the inequality,

$$\begin{aligned}\|\mu\|_{\mathrm{TV}} = \sup_{\phi \in C_0(0,T;X)} \langle\!\langle \phi, \mu \rangle\!\rangle &\le \sup_{\phi \in C_0(0,T;X)} \int_0^T \|\phi(t)\|_X \; \nu(\mathrm{d}t) \\ &\le \sup_{\psi \in C_0(0,T)} \int_0^T \psi(t)\, \nu(\mathrm{d}t) = \|\nu\|_{\mathrm{TV}}.\end{aligned}$$

However, the other direction is immediately from the vague lower semi-continuity of the total variation and so $\|\nu\|_{\mathrm{TV}} = \|\mu\|_{\mathrm{TV}}$. Together with (A.7) and $|\mu_n| \xrightarrow{\text{vague}} \nu$, this implies that $|\mu_n| \xrightarrow{\text{narrow}} \nu$. Hence by the (real-valued) Prokhorov Theorem the measures $|\mu_n|$ are tight, and by Lemma A.13 the measures μ_n convergence narrowly to μ. □

REMARK A.15. The converse statement is, in general, wrong. To see this let X be a separable Hilbert space with basis $(e_n)_{n\in\mathbb{N}}$ and set $\mu_n = \mathrm{e}_n \mathcal{L}$. Then

$$\forall f \in C_b(0, T; X) \qquad \int_0^T \langle f, \mathrm{d}\mu_n \rangle = \langle \int_0^T f \,\mathrm{d}t, \mathrm{e}_n \rangle \to 0$$

but $\|\mu_n\|_{TV} = 1$ for every n.

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