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Remark on the global regularity of 2D MHD equations with almost Laplacian magnetic diffusion

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Abstract. Whether or not the classical solutions of the two-dimensional (2D) incompressible magnetohydrodynamics (MHD) equations with only Laplacian magnetic diffusion (without velocity dissipation) are globally well posed is a difficult problem and remains completely open. In this paper, we establish the global regularity of solutions to the 2D incompressible MHD equations with almost Laplacian magnetic diffusion in the whole space. This result can be regarded as a further improvement and generalization of the previous works. Consequently, our result is more closer to the resolution of the global regularity issue on the 2D MHD equations with standard Laplacian magnetic diffusion.

1. Introduction

In this article, we focus on the global regularity problem concerning the 2D generalized incompressible magnetohydrodynamic (GMHD) equations of the form in the whole space

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = (b \cdot \nabla)b, & x \in \mathbb{R}^2, \ t > 0, \\ \partial_t b + (u \cdot \nabla)b + \mathcal{L}b = (b \cdot \nabla)u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \ b(x,0) = b_0(x), \end{cases} \quad (1.1)$$

where $u = (u_1(x,t), u_2(x,t))$ denotes the velocity, $p = p(x,t)$ the scalar pressure, and $b = (b_1(x,t), b_2(x,t))$ the magnetic field of the fluid. $u_0(x)$ and $b_0(x)$ are the given initial data satisfying $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. The operator $\mathcal{L}$ is a Fourier multiplier with symbol $|\xi|^2 g(\xi)$, namely

$$\widehat{\mathcal{L}b}(\xi) = |\xi|^2 g(\xi)\widehat{b}(\xi),$$

with $g(\xi) = g(|\xi|)$ a radial non-decreasing smooth function satisfying the following two conditions

(a) g obeys that

$$g(\xi) > 0 \quad \text{for all} \ \xi \neq 0;$$

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(b) g is of the Mikhlin–Hömander type, namely, a constant $\widetilde{C} > 0$ such that

$$|\partial_\xi^k g(\xi)| \leq \widetilde{C}|\xi|^{-k}|g(\xi)|, \quad k \in \{1, 2, 3\}, \ \forall \xi \neq 0.$$

The MHD equations model the complex interaction between the fluid dynamic phenomena, such as the magnetic reconnection in astrophysics and geomagnetic dynamo in geophysics, plasmas, liquid metals, and salt water (see, e.g., [7,15]). The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn creates forces on the fluid and also changes the magnetic field itself. Besides their important physical applications, the MHD equations and GMHD equations are also mathematically significant. Due to the physical background and mathematical relevance, the MHD equations attracted quite a lot of attention lately from various authors. One of the fundamental problems concerning the MHD equations is whether physically relevant regular solutions remain smooth for all time or they develop finite time singularities. Actually, there is a considerable body of literature on the global regularity of the MHD equations with different form of the dissipation (see, e.g., [4–6,8,10,11,19–27] and the references cited therein), and here for our purpose, we only recall the notable works about the 2D fractional MHD equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + (-\Delta)^\alpha u + \nabla p = (b \cdot \nabla)b, & x \in \mathbb{R}^2, \ t > 0, \\ \partial_t b + (u \cdot \nabla)b + (-\Delta)^\beta b = (b \cdot \nabla)u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \ b(x,0) = b_0(x), \end{cases} \tag{1.2}$$

where $(-\Delta)^\gamma$ is defined by the Fourier transform, namely

$$\widehat{(-\Delta)^\gamma f}(\xi) = |\xi|^{2\gamma} \hat{f}(\xi).$$

We remark the convention that by $\alpha = 0$ we mean that there is no dissipation in $(1.2)_1$, and similarly $\beta = 0$ represents that there is no diffusion in $(1.2)_2$. It is well known that the 2D MHD equations with both Laplacian dissipation and magnetic diffusion ($\alpha = \beta = 1$) have the global smooth solution (e.g., [18]). In the completely inviscid case ($\alpha = \beta = 0$), the question of whether smooth solution of the 2D MHD equations develops singularity in finite time appears to be out of reach. Therefore, it is natural to examine the MHD equations with fractional dissipation (see, e.g., [6,8,10,11,14,17,20,24–26,28] and the references cited therein). Let us review several works about the system (1.2), which are very closely related to our study. To the best of our knowledge, the issue of the global regularity for 2D resistive MHD ($\alpha = 0, \beta = 1$) is still a challenging open problem (see [5,13]) as we are unable to derive the key *a priori* estimate of $\|\nabla j\|_{L_t^1 L^\infty}$. But if more dissipation is added, then the corresponding system does admit a unique global regular solution. On the one hand, when $\alpha > 0, \beta = 1$, the global regularity issue was solved by Fan et al. [8]. Very recently, Yuan and Zhao improved this work as they obtained the global regularity of

solutions requiring the dissipative operators weaker than any power of the fractional Laplacian. On the other hand, when $\alpha = 0, \beta > 1$, the global regularity of smooth solutions for the corresponding system was established in works [6,11] with quite different approach, which was also improved by Agelas [1] with the $(-\Delta)^\beta b$ $(\beta > 1)$ replaced by $-\Delta\big(\log(e - \Delta)\big)^\kappa b$ $(\kappa > 1)$.

Inspired by the previous works, the aim of this paper is to weaken the operator $\mathcal{L}$ as possible as one can, without losing the global regularity of the system (1.1). More precisely, our main result reads as follows.

THEOREM 1.1. *Let* $(u_0, b_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *for any s > 2 and satisfy* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Assume that* $g(\xi) = g(|\xi|)$ *is a radial non-decreasing smooth function satisfying the conditions* (a)–(b) *and the following growth condition: for any given finite* $T \in (0, \infty)$ *such that*

$$\int_{A_T}^\infty \frac{\mathrm{d}r}{rg(r)} = C_T < \infty, \tag{1.3}$$

where $A_T > 0$ *is the unique solution of the following equation*

$$x^2 g(x) = \frac{1}{T},$$

then the system (1.1) *admits a unique global solution such that*

$$u, \ b \in L^\infty\big([0, T]; H^s(\mathbb{R}^2)\big), \quad b \in L^2\big([0, T]; H^{s+1}(\mathbb{R}^2)\big).$$

REMARK 1.2. On the one hand, A_T is a decreasing function of T which obviously satisfies

$$\lim_{T\to\infty} A_T = 0.$$

On the other hand, C_T is a non-decreasing function of T which can satisfy the condition

$$\lim_{T\to\infty} C_T = \infty.$$

REMARK 1.3. The typical examples of g satisfying the conditions (a)–(b) and (1.3) are

$$g(r) = r^{\mu_1} \ \text{with} \ \mu_1 > 0;$$
$$g(r) = \big(\ln(1 + r)\big)^{\mu_2} \ \text{with} \ \mu_2 > 1;$$
$$g(r) = r^{\mu_3}\big(\ln(1 + r)\big)^{\mu_4} \ \text{with} \ \mu_3 > 0, \ \mu_4 \geq 0;$$
$$g(r) = \ln(1 + r)\Big(\ln\big(1 + \ln(1 + r)\big)\Big)^{\mu_5} \ \text{with} \ \mu_5 > 1.$$

REMARK 1.4. In [1], Agelas proved the global regularity of the smooth solution of the system (1.1) with the operator given by $\mathcal{L} = -\Delta\big(\log(e - \Delta)\big)^\kappa$ for $\kappa > 1$. Obviously, one can easily check that $\big(\log(e + |\xi|^2)\big)^\kappa$ for $\kappa > 1$ satisfies our conditions (a)–(b) and (1.3). Clearly, our result generalizes the work [1] and improves the previous works [6,11] which require the dissipative $(-\Delta)^\beta b$ with $\beta > 1$.

REMARK 1.5. For the system (1.1), it remains an open problem whether there exists a global smooth solution when the function $g(\xi)$ is a positive constant.

The plan of this paper is as follows: We collect some useful properties of the operator $\mathcal{L}$ in Sect. 2, and then we prove Theorem 1.1 in Sect. 3. Throughout this paper, the letter C denotes various positive and finite constants whose exact values are unimportant and may vary from line to line.

2. The properties of the operator $\mathcal{L}$

This section is devoted to establishing some properties of the operator $\mathcal{L}$, which are the key components in proving our main theorem. To begin with, we consider the following linear inhomogeneous equation

$$\begin{cases} \partial_t W + \mathcal{L}W = f, \\ W(x, 0) = W_0(x). \end{cases} \tag{2.1}$$

Thanks to the Fourier transform method, the solution of the linear inhomogeneous equation (2.1) can be explicitly given by

$$W(t) = K(t) * W_0 + \int_0^t K(t - \tau) * f(\tau) \, d\tau, \tag{2.2}$$

where the kernel function K satisfies

$$K(x, t) = \mathcal{F}^{-1}\left(e^{-t|\xi|^2 g(\xi)}\right)(x). \tag{2.3}$$

Now we will establish the following estimate which plays an essential role in proving our main theorem.

LEMMA 2.1. *For any $k \geq 0$ and $s > k - 1$, there exists a constant C depending only on s and k such that for any $t > 0$*

$$\int_{\mathbb{R}^2} |\xi|^{2s} g(\xi)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi \leq C t^{-(s+1)} g(A_t)^{-(s-k+1)}, \tag{2.4}$$

where $A_t > 0$ is the unique solution of the following equation

$$x^2 g(x) = \frac{1}{t}. \tag{2.5}$$

Proof of Lemma 2.1. We split the integral into the following two parts

$$\int_{\mathbb{R}^2} |\xi|^{2s} g(\xi)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi = \int_{|\xi| \leq R} |\xi|^{2s} g(\xi)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi$$
$$+ \int_{|\xi| \geq R} |\xi|^{2s} g(\xi)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi,$$

where $R > 0$ will be fixed hereafter. Recalling $s > k - 1$, direct computations yield that the first part admits the following bound

$$
\int_{|\xi| \leq R} |\xi|^{2s} g(\xi)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi = \int_{|\xi| \leq R} |\xi|^{2s} \big(2t|\xi|^2\big)^{-k} \big(2t|\xi|^2 g(\xi)\big)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi
$$
$$
\leq C t^{-k} \int_{|\xi| \leq R} |\xi|^{2s-2k} \, d\xi
$$
$$
\leq C t^{-k} \int_0^R r^{2s-2k+1} \, dr
$$
$$
\leq C t^{-k} R^{2(s-k+1)}, \tag{2.6}
$$

where in the second line, we have used the simple fact

$$
\max_{\lambda \geq 0} (\lambda^k e^{-\lambda}) \leq C(k). \tag{2.7}
$$

Thanks to the above fact (2.7), by some simple computations, we can estimate the remainder term as follows:

$$
\int_{|\xi| \geq R} |\xi|^{2s} g(\xi)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi
$$
$$
= \int_{|\xi| \geq R} |\xi|^{2s} \big(t|\xi|^2\big)^{-k} \big(t|\xi|^2 g(\xi)\big)^k e^{-t|\xi|^2 g(\xi)} e^{-t|\xi|^2 g(\xi)} \, d\xi
$$
$$
\leq C t^{-k} \int_{|\xi| \geq R} |\xi|^{2s-2k} e^{-t|\xi|^2 g(\xi)} \, d\xi
$$
$$
\leq C t^{-k} \int_{|\xi| \geq R} |\xi|^{2s-2k} e^{-t|\xi|^2 g(R)} \, d\xi
$$
$$
\leq C t^{-k} \int_R^\infty r^{2s-2k+1} e^{-tg(R)r^2} \, dr
$$
$$
\leq C t^{-(s+1)} g(R)^{-(s-k+1)}. \tag{2.8}
$$

Combining (2.6) and (2.8), we get

$$
\int_{\mathbb{R}^2} |\xi|^{2s} g(\xi)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi \leq C t^{-k} R^{2(s-k+1)} + C t^{-(s+1)} g(R)^{-(s-k+1)}
$$
$$
\leq C t^{-(s+1)} \Big((R^2 t)^{(s-k+1)} + g(R)^{-(s-k+1)} \Big).
$$

Now taking

$$
R^2 t = \frac{1}{g(R)} \quad \text{or} \quad R^2 g(R) = \frac{1}{t} \quad \text{or} \quad R = A_t,
$$

we eventually deduce

$$
\int_{\mathbb{R}^2} |\xi|^{2s} g(\xi)^k e^{-2t|\xi|^2 g(\xi)} \, d\xi \leq 2C t^{-(s+1)} g(R)^{-(s-k+1)} = 2C t^{-(s+1)} g(A_t)^{-(s-k+1)}.
$$

Thus, we get the desired result. This concludes the proof of Lemma 2.1. $\qquad \square$

With the help of Lemma 2.1, one can conclude the following results.

LEMMA 2.2. *For any $s \geq 0$, there exists a constant C depending only on s such that for any $t > 0$*

$$\|K(t)\|_{\dot{H}^s} \leq C t^{-\frac{s+1}{2}} g(A_t)^{-\frac{s+1}{2}}. \tag{2.9}$$

In particular, it holds

$$\|K(t)\|_{L^\infty} \leq C t^{-1} g(A_t)^{-1}, \tag{2.10}$$

where A_t is given by (2.5).

Proof of Lemma 2.2. According to (2.4), one immediately obtains

$$
\begin{aligned}
\|K(t)\|_{\dot{H}^s}^2 &= \int_{\mathbb{R}^2} |\xi|^{2s} |\widehat{K}(\xi, t)|^2 \, d\xi \\
&= \int_{\mathbb{R}^2} |\xi|^{2s} e^{-2t|\xi|^2 g(\xi)} \, d\xi \\
&\leq C t^{-(s+1)} g(A_t)^{-(s+1)}.
\end{aligned}
$$

By the simple interpolation inequality and (2.9), we have

$$
\begin{aligned}
\|K(t)\|_{L^\infty} &\leq C \|K(t)\|_{L^2}^{\frac{1}{2}} \|K(t)\|_{\dot{H}^2}^{\frac{1}{2}} \\
&\leq C t^{-\frac{1}{4}} g(A_t)^{-\frac{1}{4}} t^{-\frac{3}{4}} g(A_t)^{-\frac{3}{4}} \\
&= C t^{-1} g(A_t)^{-1}.
\end{aligned}
$$

Therefore, we finally conclude the proof of Lemma 2.2.   □

Next, we would like to show the following lemma.

LEMMA 2.3. *There exists a constant C such that for any $t > 0$*

$$\|\nabla^2 K(t)\|_{L^1} \leq C t^{-1} g(A_t)^{-1}, \tag{2.11}$$

where A_t is given by (2.5).

Proof of Lemma 2.3. Since for any radial function g in $\mathbb{R}^n$, one may check that

$$\widehat{g}(x) = \check{g}(x),$$

where $\check{g}$ denotes the inverse Fourier transform of g. Since $\widehat{K}(., t)$ is a radial function, then we infer that $K(., t)$ is also a radial function. Based on this observation and the following interpolation inequality (see the end of this lemma)

$$\|\widehat{h}\|_{L^1} \leq C \|h\|_{L^2}^{\frac{1}{2}} \|h\|_{\dot{H}^2}^{\frac{1}{2}}, \tag{2.12}$$

it yields that

$$
\begin{aligned}
\|\nabla^2 K(t)\|_{L^1} &= \|\mathcal{F}(\xi^2 \check{K}(\xi, t))\|_{L^1} \\
&= \|\mathcal{F}(\xi^2 \widehat{K}(\xi, t))\|_{L^1} \\
&\leq C \|\xi^2 \widehat{K}(\xi, t)\|_{L^2}^{\frac{1}{2}} \|\nabla_\xi^2(\xi^2 \widehat{K}(\xi, t))\|_{L^2}^{\frac{1}{2}},
\end{aligned}
\tag{2.13}
$$

where we have denoted by ξ^2 the matrix $\xi \otimes \xi$. By (2.9), one gets

$$
\|\xi^2 \widehat{K}(\xi, t)\|_{L^2} \leq C t^{-\frac{3}{2}} g(A_t)^{-\frac{3}{2}}.
\tag{2.14}
$$

It follows from some direct computations that

$$
|\nabla_\xi^2(\xi^2 \widehat{K}(\xi, t))| \leq C(|\widehat{K}(\xi, t)| + |\xi| \, |\nabla_\xi \widehat{K}(\xi, t)| + |\xi|^2 |\nabla_\xi^2 \widehat{K}(\xi, t)|).
$$

Recalling $\widehat{K}(\xi, t) = \mathrm{e}^{-t|\xi|^2 g(\xi)}$ and the property $|\partial_\xi^k g(\xi)| \leq \widetilde{C} |\xi|^{-k} |g(\xi)|$, $k \in \{1, 2, 3\}$, it is not hard to check

$$
\begin{aligned}
|\nabla_\xi \widehat{K}(\xi, t)| &\leq C(t|\xi| \, |g(\xi)| + t|\xi|^2 |\partial_\xi g(\xi)|)|\widehat{K}(\xi, t)| \\
&\leq C t|\xi| \, |g(\xi)| \, |\widehat{K}(\xi, t)|
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla_\xi^2 \widehat{K}(\xi, t)| &\leq C \left(t^2 |\xi|^2 |g(\xi)|^2 + t^2 |\xi|^4 |\partial_\xi g(\xi)|^2 + t|g(\xi)| + t|\xi| |\partial_\xi g(\xi)| + t|\xi|^2 |\partial_\xi^2 g(\xi)| \right) \\
&\quad \times |\widehat{K}(\xi, t)| \\
&\leq C(t^2 |\xi|^2 |g(\xi)|^2 + t|g(\xi)|)|\widehat{K}(\xi, t)|.
\end{aligned}
\tag{2.15}
$$

Therefore, we obtain

$$
|\nabla_\xi^2(\xi^2 \widehat{K}(\xi, t))| \leq C(1 + t|\xi|^2 |g(\xi)| + t^2 |\xi|^4 |g(\xi)|^2)|\widehat{K}(\xi, t)|.
$$

We appeal to (2.4) to get

$$
\begin{aligned}
\|\nabla_\xi^2(\xi^2 \widehat{K}(\xi, t))\|_{L^2}^2 &\leq C \int_{\mathbb{R}^2} \mathrm{e}^{-2t|\xi|^2 g(\xi)} \, \mathrm{d}\xi + C t^2 \int_{\mathbb{R}^2} |\xi|^4 g(\xi)^2 \mathrm{e}^{-2t|\xi|^2 g(\xi)} \, \mathrm{d}\xi \\
&\quad + C t^4 \int_{\mathbb{R}^2} |\xi|^8 g(\xi)^4 \mathrm{e}^{-2t|\xi|^2 g(\xi)} \, \mathrm{d}\xi \\
&\leq C t^{-1} g(A_t)^{-1} + C t^2 t^{-3} g(A_t)^{-1} + C t^4 t^{-5} g(A_t)^{-1} \\
&\leq C t^{-1} g(A_t)^{-1}.
\end{aligned}
\tag{2.16}
$$

Inserting (2.14) and (2.16) into (2.13) yields

$$
\|\nabla^2 K(t)\|_{L^1} \leq C t^{-1} g(A_t)^{-1},
$$

which is nothing but the desired estimate (2.11). Finally, let us show (2.12)

$$
\begin{aligned}
\|\widehat{h}\|_{L^1} &= \int_{\mathbb{R}^2} |\widehat{h}(\xi)| \, d\xi \\
&= \int_{|\xi| \le N} |\widehat{h}(\xi)| \, d\xi + \int_{|\xi| \ge N} |\widehat{h}(\xi)| \, d\xi \\
&= \int_{|\xi| \le N} |\widehat{h}(\xi)| \, d\xi + \int_{|\xi| \ge N} |\xi|^{-2} |\xi|^2 |\widehat{h}(\xi)| \, d\xi \\
&\le CN \Big(\int_{|\xi| \le N} |\widehat{h}(\xi)|^2 \, d\xi \Big)^{\frac{1}{2}} + CN^{-1} \Big(\int_{|\xi| \ge N} |\xi|^4 |\widehat{h}(\xi)|^2 \, d\xi \Big)^{\frac{1}{2}} \\
&\le CN \|h\|_{L^2} + CN^{-1} \|h\|_{\dot{H}^2} \\
&\le C \|h\|_{L^2}^{\frac{1}{2}} \|h\|_{\dot{H}^2}^{\frac{1}{2}},
\end{aligned}
$$

where in the last line, we have chosen N as

$$
N = \Big(\frac{\|h\|_{\dot{H}^2}}{\|h\|_{L^2}} \Big)^{\frac{1}{2}}.
$$

We thus complete the proof of the lemma.   □

Now we are ready to show the following key lemma.

LEMMA 2.4. *Under the assumptions of Theorem* 1.1, *there holds*

$$
\int_0^T \|\nabla K(t)\|_{L^2} \, dt + \int_0^T \|K(t)\|_{L^2}^2 \, dt + \int_0^T \|K(t)\|_{L^\infty} \, dt
$$
$$
+ \int_0^T \|\nabla^2 K(t)\|_{L^1} \, dt \le C(T).
$$

Proof of Lemma 2.4. Making use of Lemmas 2.2 and 3.3, it follows that

$$
\|\nabla K(t)\|_{L^2} + \|K(t)\|_{L^2}^2 + \|K(t)\|_{L^\infty} + \|\nabla^2 K(t)\|_{L^1} \le C t^{-1} g(R)^{-1},
$$

where R satisfies

$$
R^2 g(R) = \frac{1}{t}.
$$

Taking R-derivative of the both sides of the above equation, it is easy to check

$$
2R g(R) + R^2 g'(R) = -\frac{1}{t^2} \frac{dt}{dR}
$$

or equivalently

$$
dt = -t^2 \big(2R g(R) + R^2 g'(R) \big) dR = -R^{-4} g(R)^{-2} \big(2R g(R) + R^2 g'(R) \big) dR.
$$

By direct computations, we have that

$$
\begin{aligned}
\int_0^T t^{-1} g(R)^{-1}\, dt &= -\int_{+\infty}^{A_T} R^{-2} g(R)^{-2} \big(2Rg(R) + R^2 g'(R)\big)\, dR \\
&= \int_{A_T}^{+\infty} R^{-2} g(R)^{-2} \big(2Rg(R) + R^2 g'(R)\big)\, dR \\
&= 2\int_{A_T}^{+\infty} R^{-1} g(R)^{-1}\, dR + \int_{A_T}^{+\infty} g(R)^{-2} g'(R)\, dR \\
&= 2\int_{A_T}^{+\infty} R^{-1} g(R)^{-1}\, dR + g(A_T)^{-1} - g(+\infty)^{-1} \\
&\le 2\int_{A_T}^{+\infty} R^{-1} g(R)^{-1}\, dR + T A_T^2 \\
&\le C(T),
\end{aligned}
$$

where we have applied (1.3) and the fact $A_T^2 g(A_T) = \frac{1}{T}$ or $g(A_T)^{-1} = T A_T^2$. Consequently, it is obvious to check that

$$
\int_0^T \|\nabla K(t)\|_{L^2}\, dt + \int_0^T \|K(t)\|_{L^2}^2\, dt + \int_0^T \|K(t)\|_{L^\infty}\, dt
$$
$$
+ \int_0^T \|\nabla^2 K(t)\|_{L^1}\, dt \le C(T).
$$

This concludes the proof of the lemma.   □

In order to show the nonnegativity of $K(x, t)$, let us recall the definition of positive definite function.

DEFINITION 2.5. The real-valued function g defined on the 2D Euclidean space $\mathbb{R}^2$ is positive definite if the matrix

$$
\big(g(x_i - x_j)\big)_{1 \le i, j \le 2}
$$

is nonnegative definite for every set of points $x_1,\ x_2 \in \mathbb{R}^2$.

The following is the celebrated Bochner's theorem (see [2, 16]).

LEMMA 2.6. *Let g be a bounded continuous function on $\mathbb{R}^2$. Then the matrix*

$$
\big(g(\eta_i - \eta_j)\big)_{1 \le i, j \le 2} = \begin{pmatrix} g(0) & g(\eta_1 - \eta_2) \\ g(\eta_1 - \eta_2) & g(0) \end{pmatrix}
$$

is nonnegative definite for every set of points $\eta_1,\ \eta_2 \in \mathbb{R}^2$ if, and only if, there is a positive finite Borel measure φ on $\mathbb{R}^2$ such that

$$
g(x) = (\mathcal{F}\varphi)(x).
$$

Next, we go into the proof of the nonnegativity of $K(x, t)$.

LEMMA 2.7. *Let $K(x, t)$ be given by* (2.3)*, then it holds for any $t > 0$,*

$$K(x, t) \geq 0. \tag{2.17}$$

Proof of Lemma 2.7. One may check that for any $t > 0$,

$$h(\xi) = e^{-t|\xi|^2 g(\xi)}$$

satisfies

$$h(0) = 1, \qquad 0 \leq h(\xi) \leq 1.$$

One can also show that the matrix

$$\Big(h(\eta_i - \eta_j)\Big)_{1 \leq i, j \leq 2} = \begin{pmatrix} 1 & h(\eta_1 - \eta_2) \\ h(\eta_1 - \eta_2) & 1 \end{pmatrix}$$

is nonnegative definite for every set of points η_1, $\eta_2 \in \mathbb{R}^2$. This yields that h is a positive definite function. By Bochner's theorem, it is clear that $K(x, t) \geq 0$. This concludes the proof of Lemma 2.7. $\qquad \square$

In order to get the estimate $\|b\|_{L_t^\infty L^\infty}$, we need to show the following lemma, whose proof is inspired by [9, Lemma 3.8].

LEMMA 2.8. *There exists a constant C depending on t such that for any $t > 0$*

$$\|K(t)\|_{L^1} = 1. \tag{2.18}$$

Proof of Lemma 2.8. According to the definition of $K(x, t)$, we have

$$K(x, t) = \int_{\mathbb{R}^2} e^{ix \cdot \xi} e^{-t|\xi|^2 g(\xi)} \, d\xi.$$

For $\mu > 0$, we obtain by integrating by parts

$$|x|^\mu x_j^2 K(x, t) = \int_{\mathbb{R}^2} |x|^\mu e^{ix \cdot \xi} \partial_{\xi_j}^2 \Big(e^{-t|\xi|^2 g(\xi)}\Big) \, d\xi.$$

Keep in mind the fact

$$|x|^\mu e^{ix \cdot \xi} = \Lambda_\xi^\mu e^{ix \cdot \xi},$$

where Λ^μ with $0 < \mu < 2$ is defined by

$$\Lambda^\mu g(x) = C_1 \int_{\mathbb{R}^2} \frac{g(x) - g(x - y)}{|y|^{2+\mu}} \, dy$$

with an absolute constant $C_1 > 0$. As a consequence, we deduce

$$|x|^\mu x_j^2 K(x, t) = \int_{\mathbb{R}^2} e^{ix \cdot \xi} \Lambda^\mu \partial_{\xi_j}^2 \Big(e^{-t|\xi|^2 g(\xi)}\Big) \, d\xi.$$

We further get

$$|x|^\mu x_j^2 |K(x,t)| \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|h(\xi) - h(\xi - y)|}{|y|^{2+\mu}} \, dy \, d\xi,$$

where

$$h(\xi) = \partial_{\xi_j}^2 \left(e^{-t|\xi|^2 g(\xi)} \right).$$

We decompose the integral into two parts to obtain

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|h(\xi) - h(\xi - y)|}{|y|^{2+\mu}} \, dy \, d\xi = \overline{A} + \underline{A},$$

where

$$\overline{A} = \int_{\mathbb{R}^2} \int_{|y| \geq \frac{|\xi|}{2}} \frac{|h(\xi) - h(\xi - y)|}{|y|^{2+\mu}} \, dy \, d\xi, \quad \underline{A} = \int_{\mathbb{R}^2} \int_{|y| \leq \frac{|\xi|}{2}} \frac{|h(\xi) - h(\xi - y)|}{|y|^{2+\mu}} \, dy \, d\xi.$$

Thanks to (2.15), it ensures

$$|h(\xi)| \leq C(t^2 |\xi|^2 |g(\xi)|^2 + t|g(\xi)|) e^{-t|\xi|^2 g(\xi)}.$$

It thus gives

$$\overline{A} \leq C \int_{\mathbb{R}^2} \int_{|y| \geq \frac{|\xi|}{2}} \frac{|h(\xi)|}{|y|^{2+\mu}} \, dy \, d\xi + C \int_{\mathbb{R}^2} \int_{|y| \geq \frac{|\xi|}{2}} \frac{|h(\xi - y)|}{|y|^{2+\mu}} \, dy \, d\xi$$

$$\leq C \int_{\mathbb{R}^2} \int_{|y| \geq \frac{|\xi|}{3}} \frac{|h(\xi)|}{|y|^{2+\mu}} \, dy \, d\xi$$

$$\leq C \int_{\mathbb{R}^2} \frac{|h(\xi)|}{|\xi|^\mu} \, d\xi$$

$$\leq C \int_{\mathbb{R}^2} \frac{t^2 |\xi|^2 |g(\xi)|^2 e^{-t|\xi|^2 g(\xi)}}{|\xi|^\mu} \, d\xi + C \int_{\mathbb{R}^2} \frac{t|g(\xi)| e^{-t|\xi|^2 g(\xi)}}{|\xi|^\mu} \, d\xi.$$

By (2.7), we can easily check that for any $\mu \in (0, 2)$

$$\int_{\mathbb{R}^2} \frac{t^2 |\xi|^2 |g(\xi)|^2 e^{-t|\xi|^2 g(\xi)}}{|\xi|^\mu} \, d\xi$$

$$= C \int_0^\infty \frac{r t^2 r^2 |g(r)|^2 e^{-tr^2 g(r)}}{r^\mu} \, dr$$

$$= C \int_0^1 \frac{r t^2 r^2 |g(r)|^2 e^{-tr^2 g(r)}}{r^\mu} \, dr + C \int_1^\infty \frac{r t^2 r^2 |g(r)|^2 e^{-tr^2 g(r)}}{r^\mu} \, dr$$

$$\leq C t^2 \int_0^1 r^{3-\mu} \, dr + \int_1^\infty \frac{t^2 r^4 |g(r)|^2 e^{-tr^2 g(r)}}{r^{1+\mu}} \, dr$$

$$\leq Ct^2 + C \int_1^\infty \frac{1}{r^{1+\mu}} \, dr$$
$$\leq C + Ct^2.$$

Similarly, we also have

$$\int_{\mathbb{R}^2} \frac{t|g(\xi)|e^{-t|\xi|^2 g(\xi)}}{|\xi|^\mu} \, d\xi \leq C + Ct.$$

Consequently, we get

$$\overline{A} \leq C + Ct^2.$$

To estimate the term $\underline{A}$, we use the mean value theorem

$$|h(\xi) - h(\xi - y)| \leq |y| \sup_{\eta \text{ is between } \xi \text{ and } \xi - y} |\nabla h(\eta)|.$$

Recalling the property $|\partial_\xi^k g(\xi)| \leq \widetilde{C}|\xi|^{-k}|g(\xi)|$, $k \in \{1, 2, 3\}$, we can show

$$|\nabla h(\eta)| \leq |\nabla_\eta^3 \widehat{K}(\eta, t)|$$
$$\leq C(t^3|\eta|^3|g(\eta)|^3 + t^3|\eta|^6|\partial_\eta g(\eta)|^3 + t^2|\eta||g(\eta)|^2 + t^2|\eta|^2|g(\eta)| \, |\partial_\eta g(\eta)|$$
$$+ t^2|\eta|^3|\partial_\eta g(\eta)|^2 + t^2|\eta|^3|g(\eta)| \, |\partial_\eta^2 g(\eta)| + t^2|\eta|^4|\partial_\eta g(\eta)| \, |\partial_\eta^2 g(\eta)|$$
$$+ t|\partial_\eta g(\eta)| + t|\eta| \, |\partial_\eta^2 g(\eta)| + t|\eta|^2|\partial_\eta^3 g(\eta)|)|\widehat{K}(\eta, t)|$$
$$\leq C(t^3|\eta|^3|g(\eta)|^3 + t^2|\eta| \, |g(\eta)|^2 + t|\eta|^{-1}|g(\eta)|)e^{-t|\eta|^2 g(\eta)}.$$

Noticing the fact $|y| \leq \frac{|\xi|}{2}$ and η is between $\xi - y$ and ξ, we thus have

$$\frac{1}{2}|\xi| \leq |\eta| \leq \frac{5}{2}|\xi|,$$

which implies

$$|h(\xi) - h(\xi - y)| \leq C|y|(t^3|\xi|^3|g(\kappa\xi)|^3 + t^2|\xi| \, |g(\kappa\xi)|^2 + t|\xi|^{-1}|g(\kappa\xi)|)$$
$$e^{-Ct|\xi|^2 g(\kappa\xi)},$$

where κ is a fixed constant satisfying $\kappa \in [\frac{1}{2}, \frac{5}{2}]$. Now it is not difficult to deduce for any $\mu \in (0, 1)$

$$\underline{A} \leq C \int_{\mathbb{R}^2} \int_{|y| \leq \frac{|\xi|}{2}} \frac{(t^3|\xi|^3|g(\kappa\xi)|^3 + t^2|\xi| \, |g(\kappa\xi)|^2 + t|\xi|^{-1}|g(\kappa\xi)|)e^{-Ct|\xi|^2 g(\kappa\xi)}}{|y|^{1+\mu}} \, dy \, d\xi$$
$$\leq C \int_{\mathbb{R}^2} |\xi|^{1-\mu}(t^3|\xi|^3|g(\kappa\xi)|^3 + t^2|\xi| \, |g(\kappa\xi)|^2 + t|\xi|^{-1}|g(\kappa\xi)|)e^{-Ct|\xi|^2 g(\kappa\xi)} \, d\xi$$
$$\leq C \int_0^\infty (t^3 r^{5-\mu}|g(\kappa r)|^3 + t^2 r^{3-\mu} \, |g(\kappa r)|^2 + t r^{1-\mu}|g(\kappa r)|)e^{-Ctr^2 g(\kappa r)} \, dr$$
$$:= \underline{A}_1 + \underline{A}_2,$$

where

$$A_1 = C \int_0^1 \left(t^3 r^{5-\mu} |g(\kappa r)|^3 + t^2 r^{3-\mu} |g(\kappa r)|^2 + t r^{1-\mu} |g(\kappa r)| \right) e^{-Ctr^2 g(\kappa r)} \, dr,$$

$$A_2 = C \int_1^\infty \left(t^3 r^{5-\mu} |g(\kappa r)|^3 + t^2 r^{3-\mu} |g(\kappa r)|^2 + t r^{1-\mu} |g(\kappa r)| \right) e^{-Ctr^2 g(\kappa r)} \, dr.$$

It is obvious that

$$A_1 \leq C + C t^3.$$

For the remainder term A_2, it follows from (2.7) that

$$A_2 = C \int_1^\infty r^{-1-\mu} t^3 r^6 |g(\kappa r)|^3 e^{-tr^2 g(\kappa r)} \, dr + C \int_1^\infty r^{-1-\mu} t^2 r^4 |g(\kappa r)|^2 e^{-tr^2 g(\kappa r)} \, dr$$

$$+ C \int_1^\infty r^{-1-\mu} t r^2 |g(\kappa r)| e^{-tr^2 g(\kappa r)} \, dr$$

$$\leq C \int_1^\infty r^{-1-\mu} \, dr$$

$$\leq C.$$

We therefore get

$$\overline{A} \leq C + C t^3.$$

Combining the above estimates, we finally obtain

$$|x|^\mu x_j^2 |K(x, t)| \leq C + C t^3, \quad j = 1, 2.$$

It should be noted that

$$K(x, t) = \int_{\mathbb{R}^2} e^{ix \cdot \xi} e^{-t|\xi|^2 g(\xi)} \, d\xi$$

$$= \int_{|\xi| \leq 1} e^{ix \cdot \xi} e^{-t|\xi|^2 g(\xi)} \, d\xi + \int_{|\xi| \geq 1} e^{ix \cdot \xi} e^{-t|\xi|^2 g(\xi)} \, d\xi$$

$$\leq \int_{|\xi| \leq 1} d\xi + \int_{|\xi| \geq 1} e^{-t|\xi|^2 g(1)} \, d\xi$$

$$\leq C + C t^{-1},$$

which gives

$$|K(x, t)| \leq C(t).$$

This further implies

$$(1 + |x|^{2+\mu}) |K(x, t)| \leq C(t).$$

Combining the above facts yields that

$$\|K(t)\|_{L^1} \le C(t).$$

Thanks to the nonnegativity of $K(x, t)$ obtained in Lemma 2.7, we further have

$$\|K(x, t)\|_{L^1} = \int_{\mathbb{R}^2} |K(x, t)| \, dx = \int_{\mathbb{R}^2} K(x, t) \, dx = \widehat{K}(0, t) = 1.$$

The lemma is proved.   □

3. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, the global existence and unique-ness of smooth solution to the system (1.1). Before the proof, we will state a notation. For a quasi-Banach space X and for any $0 < T \le \infty$, we use standard notation $L^p(0, T; X)$ or $L_T^p(X)$ for the quasi-Banach space of Bochner measurable functions f from $(0, T)$ to X endowed with the norm

$$\|f\|_{L_T^p(X)} := \begin{cases} \left(\int_0^T \|f(., t)\|_X^p \, dt \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup_{0 \le t \le T} \|f(., t)\|_X, & p = \infty. \end{cases}$$

By the classical hyperbolic method, there exists a finite time T_0 such that the system (1.1) is local well-posedness in the interval $[0, T_0]$ in H^s with $s > 2$. Thus, it is sufficient to establish *a priori* estimates in the interval $[0, T]$ for the given $T > T_0$.

We first state the basic L^2-estimate of the system (1.1).

LEMMA 3.1. *Let (u_0, b_0) satisfy the conditions stated in Theorem 1.1, then it holds*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2 \int_0^t \|\mathcal{L}^{\frac{1}{2}} b(\tau)\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2, \qquad (3.1)$$

where $\mathcal{L}^{\frac{1}{2}}$ is defined by

$$\widehat{\mathcal{L}^{\frac{1}{2}} b}(\xi) = |\xi| \sqrt{g(\xi)} \, \widehat{b}(\xi).$$

Proof of Lemma 3.1. Taking the inner products of (1.1)$_1$ with u and (1.1)$_2$ with b, adding the results and invoking the following cancelation identity

$$\int_{\mathbb{R}^2} (b \cdot \nabla) b \cdot u \, dx + \int_{\mathbb{R}^2} (b \cdot \nabla) u \cdot b \, dx = 0,$$

we obtain that

$$\frac{1}{2} \frac{d}{dt} \left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \right) + \|\mathcal{L}^{\frac{1}{2}} b\|_{L^2}^2 = 0,$$

where we have used the following estimate due to the Plancherel theorem

$$\int_{\mathbb{R}^2} \mathcal{L}b\, b\, \mathrm{d}x = \int_{\mathbb{R}^2} \widehat{\mathcal{L}b}(\xi)\, \widehat{b}(\xi)\, \mathrm{d}\xi = \int_{\mathbb{R}^2} |\xi|^2 g(\xi)|\widehat{b}(\xi)|^2\, \mathrm{d}\xi$$
$$= \int_{\mathbb{R}^2} |\widehat{\mathcal{L}^{\frac{1}{2}}b}(\xi)|^2\, \mathrm{d}\xi = \|\mathcal{L}^{\frac{1}{2}}b\|_{L^2}^2.$$

Integrating the above inequality implies (3.1). This ends the proof of Lemma 3.1. $\square$

In order to get the H^1 estimate on (u, b), we apply $\nabla\times$ to the MHD equations (1.1) to obtain the governing equations for the vorticity $\omega := \nabla \times u = \partial_{x_1} u_2 - \partial_{x_2} u_1$ and the current $j := \nabla \times b = \partial_{x_1} b_2 - \partial_{x_2} b_1$ as follows:

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = (b \cdot \nabla)j, \\ \partial_t j + (u \cdot \nabla)j + \mathcal{L}j = (b \cdot \nabla)\omega + T(\nabla u, \nabla b), \end{cases} \tag{3.2}$$

where $T(\nabla u, \nabla b) = 2\partial_{x_1} b_1 (\partial_{x_2} u_1 + \partial_{x_1} u_2) - 2\partial_{x_1} u_1 (\partial_{x_2} b_1 + \partial_{x_1} b_2)$.

We now prove that any classical solution of (1.1) admits a global H^1-bound, as stated in the following lemma.

LEMMA 3.2. *Let (u_0, b_0) satisfy the conditions stated in Theorem 1.1, then it holds for any $t \in [0, T]$*

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^t \|\mathcal{L}^{\frac{1}{2}}j(\tau)\|_{L^2}^2\, \mathrm{d}\tau \le C(T, u_0, b_0). \tag{3.3}$$

Proof of Lemma 3.2. Taking the inner products of $(3.2)_1$ with ω, $(3.2)_2$ with j, adding them up and using the incompressible condition as well as the following fact

$$\int_{\mathbb{R}^2} (b \cdot \nabla j)\omega\, \mathrm{d}x + \int_{\mathbb{R}^2} (b \cdot \nabla \omega)j\, \mathrm{d}x = 0,$$

it yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2\right) + \|\mathcal{L}^{\frac{1}{2}}j\|_{L^2}^2 = \int_{\mathbb{R}^2} T(\nabla u, \nabla b)j\, \mathrm{d}x.$$

We obtain by direct computations that

$$\|\nabla\phi\|_{L^2}^2 = C\|\xi\widehat{\phi}(\xi)\|_{L^2}^2$$
$$= C\int_{\mathbb{R}^2} |\xi|^2|\widehat{\phi}(\xi)|^2\, \mathrm{d}\xi$$
$$= C\int_{|\xi|\le 1} |\xi|^2|\widehat{\phi}(\xi)|^2\, \mathrm{d}\xi + C\int_{|\xi|\ge 1} |\xi|^2|\widehat{\phi}(\xi)|^2\, \mathrm{d}\xi$$
$$= C\int_{|\xi|\le 1} |\widehat{\phi}(\xi)|^2\, \mathrm{d}\xi + C\int_{|\xi|\ge 1} \frac{1}{g(\xi)}|\xi|^2 g(\xi)|\widehat{\phi}(\xi)|^2\, \mathrm{d}\xi$$

$$\leq C \int_{\mathbb{R}^2} |\widehat{\phi}(\xi)|^2 \, d\xi + C \int_{|\xi| \geq 1} \frac{1}{g(1)} |\xi|^2 g(\xi) |\widehat{\phi}(\xi)|^2 \, d\xi$$

$$\leq C_1 \|\phi\|_{L^2}^2 + C_2 \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2}^2,$$

which implies

$$\|\nabla \phi\|_{L^2} \leq C_1 \|\phi\|_{L^2} + C_2 \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2}. \tag{3.4}$$

Therefore, we get by virtue of the easy interpolation inequality that

$$\int_{\mathbb{R}^2} T(\nabla u, \nabla b) j \, dx \leq C \|\nabla u\|_{L^2} \|\nabla b\|_{L^4} \|j\|_{L^4}$$

$$\leq C \|\omega\|_{L^2} \|j\|_{L^4}^2$$

$$\leq C \|\omega\|_{L^2} \|j\|_{L^2} \|\nabla j\|_{L^2}$$

$$\leq C \|\omega\|_{L^2} \|\nabla b\|_{L^2} \|\nabla j\|_{L^2}$$

$$\leq C \|\omega\|_{L^2} \left(\|b\|_{L^2} + \|\mathcal{L}^{\frac{1}{2}} b\|_{L^2} \right) \left(\|j\|_{L^2} + \|\mathcal{L}^{\frac{1}{2}} j\|_{L^2} \right)$$

$$\leq \frac{1}{2} \|\mathcal{L}^{\frac{1}{2}} j\|_{L^2}^2 + C \left(1 + \|b\|_{L^2}^2 + \|\mathcal{L}^{\frac{1}{2}} b\|_{L^2}^2 \right) \left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 \right).$$

As a result, one has

$$\frac{d}{dt} \left(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \right) + \|\mathcal{L}^{\frac{1}{2}} j\|_{L^2}^2$$

$$\leq C \left(1 + \|b\|_{L^2}^2 + \|\mathcal{L}^{\frac{1}{2}} b\|_{L^2}^2 \right) \left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 \right).$$

From the classical Gronwall's lemma and (3.1), we easily get (3.3). The proof of the lemma is now achieved. □

Making use of the global bounds obtained in Lemma 3.2, we obtain a global bound for the $L_t^\infty L^\infty$-norm of b as follows.

LEMMA 3.3. *Let (u_0, b_0) satisfy the conditions stated in Theorem 1.1, then it holds for any $t \in [0, T]$*

$$\|b\|_{L_t^\infty L^\infty} \leq C(T, u_0, b_0). \tag{3.5}$$

Proof of Lemma 3.3. We rewrite the second equation of (1.1) as

$$\partial_t b + \mathcal{L} b = \nabla \cdot (b \otimes u) - \nabla \cdot (u \otimes b).$$

By (2.2), we can check that

$$b(t) = K(t) * b_0 + \int_0^t K(t - \tau) * [\nabla \cdot (b \otimes u) - \nabla \cdot (u \otimes b)](\tau) \, d\tau.$$

Taking L^∞-norm in terms of space variable and using the Young's inequality, it implies

$$\|b(t)\|_{L^\infty} \leq \|K(t) * b_0\|_{L^\infty} + \int_0^t \|\nabla K(t-\tau) * [(b \otimes u) - (u \otimes b)](\tau)\|_{L^\infty} \, d\tau$$

$$\leq C\|K(t)\|_{L^1}\|b_0\|_{L^\infty} + C \int_0^t \|\nabla K(t-\tau)\|_{L^2}\|[(b \otimes u) - (u \otimes b)](\tau)\|_{L^2} \, d\tau$$

$$\leq C\|K(t)\|_{L^1}\|b_0\|_{L^\infty} + C \int_0^t \|\nabla K(t-\tau)\|_{L^2}\|u(\tau)\|_{L^4}\|b(\tau)\|_{L^4} \, d\tau$$

$$\leq C\|K(t)\|_{L_t^\infty L^1}\|b_0\|_{L^\infty} + C\|\nabla K(\tau)\|_{L_t^1 L^2}\|u(\tau)\|_{L_t^\infty L^4}\|b(\tau)\|_{L_t^\infty L^4}$$

$$\leq C(T, u_0, b_0),$$

where in the last line, we have used the following fact

$$\int_0^t \|\nabla K(\tau)\|_{L^2} \, d\tau < \infty. \tag{3.6}$$

Consequently, the proof of Lemma 3.3 is completed. $\qquad\square$

With the help of the global bounds obtained in the above lemmas, we obtain a global bound for the $L_t^2 L^\infty$-norm of j.

LEMMA 3.4. *Let (u_0, b_0) satisfy the conditions stated in Theorem* 1.1, *then it holds for any $t \in [0, T]$*

$$\int_0^t \|j(\tau)\|_{L^\infty}^2 \, d\tau \leq C(T, u_0, b_0). \tag{3.7}$$

Proof of Lemma 3.4. We rewrite the second equation of (3.2) as

$$\partial_t j + \mathcal{L}j = \nabla \cdot (b \otimes \omega) - \nabla \cdot (u \otimes j) + T(\nabla u, \nabla b).$$

We again make use of (2.2) to deduce

$$j(t) = K(t) * j_0$$
$$+ \int_0^t K(t-\tau) * [\nabla \cdot (b \otimes \omega) - \nabla \cdot (u \otimes j) + T(\nabla u, \nabla b)](\tau) \, d\tau. \tag{3.8}$$

Taking L^∞-norm in terms of space variable and using the Young's inequality, we conclude

$$\|j(t)\|_{L^\infty} \leq \|K(t) * j_0\|_{L^\infty} + \int_0^t \|\nabla K(t-\tau) * [(b \otimes \omega) - (u \otimes j)](\tau)\|_{L^\infty} \, d\tau$$

$$+ \int_0^t \|K(t-\tau) * T(\nabla u, \nabla b)(\tau)\|_{L^\infty} \, d\tau$$

$$\leq C\|K(t)\|_{L^2}\|j_0\|_{L^2} + C \int_0^t \|K(t-\tau)\|_{L^\infty}\|T(\nabla u, \nabla b)(\tau)\|_{L^1} \, d\tau$$

$$+ C \int_0^t \|\nabla K(t-\tau)\|_{L^2}(\|(b \otimes \omega)(\tau)\|_{L^2} + \|(u \otimes j)(\tau)\|_{L^2}) \, d\tau.$$

Take L^2-norm in terms of time variable and use the convolution Young's inequality as well as the estimates (3.5)–(3.7) to show

$$\begin{aligned}
\|j(t)\|_{L_t^2 L^\infty} &\leq C\|K(t)\|_{L_t^2 L^2}\|j_0\|_{L^2} + C\|K(t)\|_{L_t^1 L^\infty}\|T(\nabla u, \nabla b)\|_{L_t^2 L^1} \\
&\quad + C\|\nabla K(t)\|_{L_t^1 L^2}\left(\|b \otimes \omega\|_{L_t^2 L^2} + \|u \otimes j\|_{L_t^2 L^2}\right) \\
&\leq C\|K(t)\|_{L_t^2 L^2}\|j_0\|_{L^2} + C\|K(t)\|_{L_t^1 L^\infty}\|\omega\|_{L_t^2 L^2}\|j\|_{L_t^2 L^2} \\
&\quad + C\|\nabla K(t)\|_{L_t^1 L^2}\left(\|b\|_{L_t^2 L^\infty}\|\omega\|_{L_t^\infty L^2} + \|u\|_{L_t^4 L^4}\|j\|_{L_t^4 L^4}\right) \\
&\leq C(T, u_0, b_0),
\end{aligned}$$

where we have applied the following facts:

$$\int_0^t \|K(\tau)\|_{L^2}^2 \, d\tau \leq C(T, u_0, b_0), \qquad \int_0^t \|K(\tau)\|_{L^\infty} \, d\tau \leq C(T, u_0, b_0), \quad (3.9)$$

$$\|u\|_{L_t^4 L^4} \leq C\|u\|_{L_t^\infty L^2}^{\frac{1}{2}}\|\omega\|_{L_t^2 L^2}^{\frac{1}{2}} \leq C(T, u_0, b_0)$$

and by (3.4)

$$\begin{aligned}
\|j\|_{L_t^4 L^4} &\leq C\|j\|_{L_t^\infty L^2}^{\frac{1}{2}}\|\nabla j\|_{L_t^2 L^2}^{\frac{1}{2}} \\
&\leq C\|j\|_{L_t^\infty L^2}^{\frac{1}{2}}\left(\|j\|_{L_t^2 L^2}^{\frac{1}{2}} + \|\mathcal{L}^{\frac{1}{2}} j\|_{L_t^2 L^2}^{\frac{1}{2}}\right) \\
&\leq C(T, u_0, b_0).
\end{aligned} \tag{3.10}$$

This completes the proof of Lemma 3.4. $\square$

The next lemma plays a significant role in obtaining the global bound for (u, b) in H^s with $s > 2$ for the system (1.1).

LEMMA 3.5. *Let (u_0, b_0) satisfy the conditions stated in Theorem 1.1, then it holds for any $t \in [0, T]$*

$$\int_0^t \|\nabla j(\tau)\|_{L^\infty} \, d\tau \leq C(T, u_0, b_0), \tag{3.11}$$

$$\|\omega(t)\|_{L^\infty} \leq C(T, u_0, b_0), \tag{3.12}$$

$$\int_0^t \|\nabla b(\tau)\|_{L^\infty}^2 \, d\tau \leq C(T, u_0, b_0). \tag{3.13}$$

Proof of Lemma 3.5. Coming back to (3.8), we get

$$j(t) = K(t) * j_0 + \int_0^t K(t - \tau) * [\nabla \cdot (b \otimes \omega) - \nabla \cdot (u \otimes j) + T(\nabla u, \nabla b)](\tau) \, d\tau.$$

Applying the gradient ∇ to the both sides of the above equality, taking L^∞-norm in terms of space variable, and using the Young's inequality, it follows that

$$\|\nabla j(t)\|_{L^\infty} \le \|\nabla K(t) * j_0\|_{L^\infty} + \int_0^t \|\nabla^2 K(t-\tau) * [(b \otimes \omega) - (u \otimes j)](\tau)\|_{L^\infty} \, d\tau$$

$$+ \int_0^t \|\nabla K(t-\tau) * T(\nabla u, \nabla b)(\tau)\|_{L^\infty} \, d\tau$$

$$\le C\|\nabla K(t)\|_{L^2}\|j_0\|_{L^2} + C \int_0^t \|\nabla K(t-\tau)\|_{L^2}\|T(\nabla u, \nabla b)(\tau)\|_{L^2} \, d\tau$$

$$+ C \int_0^t \|\nabla^2 K(t-\tau)\|_{L^1}(\|(b \otimes \omega)(\tau)\|_{L^\infty} + \|(u \otimes j)(\tau)\|_{L^\infty}) \, d\tau.$$

Now we obtain by taking L^1-norm in terms of time variable and appealing to the convolution Young's inequality

$$\|\nabla j(t)\|_{L_t^1 L^\infty}$$
$$\le C\|\nabla K(t)\|_{L_t^1 L^2}\|j_0\|_{L^2} + C\|\nabla K(t)\|_{L_t^1 L^2}\|T(\nabla u, \nabla b)\|_{L_t^1 L^2}$$
$$+ C\|\nabla^2 K(t)\|_{L_t^1 L^1}\left(\|b \otimes \omega\|_{L_t^1 L^\infty} + \|u \otimes j\|_{L_t^1 L^\infty}\right)$$
$$\le C\|\nabla K(t)\|_{L_t^1 L^2}\|j_0\|_{L^2} + C\|\nabla K(t)\|_{L_t^1 L^2}\|\omega\|_{L_t^2 L^4}\|j\|_{L_t^2 L^4}$$
$$+ C\|\nabla^2 K(t)\|_{L_t^1 L^1}\left(\|b\omega\|_{L_t^1 L^\infty} + \|u\|_{L_t^2 L^\infty}\|j\|_{L_t^2 L^\infty}\right)$$
$$\le C\|\nabla K(t)\|_{L_t^1 L^2}\|j_0\|_{L^2} + C\|\nabla K(t)\|_{L_t^1 L^2}\|\omega\|_{L_t^\infty L^2}^{\frac{1}{2}}\|\omega\|_{L_t^1 L^\infty}^{\frac{1}{2}}\|j\|_{L_t^\infty L^2}^{\frac{1}{2}}\|j\|_{L_t^1 L^\infty}^{\frac{1}{2}}$$
$$+ C\|\nabla^2 K(t)\|_{L_t^1 L^1}\left(\|b\|_{L_t^\infty L^\infty}\|\omega\|_{L_t^1 L^\infty} + \|u\|_{L_t^\infty L^2}^{\frac{1}{2}}\|\omega\|_{L_t^1 L^\infty}^{\frac{1}{2}}\|j\|_{L_t^2 L^\infty}\right)$$
$$= H_1(t) + H_2(t)\|\omega\|_{L_t^1 L^\infty}^{\frac{1}{2}} + H_3(t)\|\omega\|_{L_t^1 L^\infty}, \tag{3.14}$$

where $H_l(t)$ ($l = 1, 2, 3$) are given by

$$H_1(t) \equiv C\|\nabla K(t)\|_{L_t^1 L^2}\|j_0\|_{L^2}; \qquad H_3(t) \equiv C\|b\|_{L_t^\infty L^\infty}\|\nabla^2 K(t)\|_{L_t^1 L^1};$$

$$H_2(t) \equiv C\left(\|\nabla K(t)\|_{L_t^1 L^2}\|\omega\|_{L_t^\infty L^2}^{\frac{1}{2}}\|j\|_{L_t^\infty L^2}^{\frac{1}{2}}\|j\|_{L_t^1 L^\infty}^{\frac{1}{2}} + \|\nabla^2 K(t)\|_{L_t^1 L^1}\|u\|_{L_t^\infty L^2}^{\frac{1}{2}}\|j\|_{L_t^2 L^\infty}\right).$$

Thanks to (2.11), (3.3), (3.5), (3.6), and (3.7), it is easy to show that $H_l(t)$ ($l = 1, 2, 3$) are non-decreasing functions satisfying

$$H_l(t) \le C(T, u_0, b_0), \quad \forall t \in [0, T].$$

Consequently, it thus follows from (3.14) that

$$\int_0^t \|\nabla j(\tau)\|_{L^\infty} \, d\tau \le H_1(t) + \frac{1}{4}H_2(t) + H_2(t)\|\omega\|_{L_t^1 L^\infty} + H_3(t)\|\omega\|_{L_t^1 L^\infty}$$

$$= H_1(t) + \frac{1}{4}H_2(t) + (H_2(t) + H_3(t))\int_0^t \|\omega(\tau)\|_{L^\infty} \, d\tau$$

$$\le \zeta_1(T) + \zeta_2(T)\int_0^t \|\omega(\tau)\|_{L^\infty} \, d\tau, \tag{3.15}$$

where

$$\zeta_1(T) := H_1(T) + \frac{1}{4}H_2(T) < \infty, \qquad \zeta_2(T) := H_2(T) + H_3(T) < \infty.$$

Multiplying the vorticity ω equation of (3.2) by $|\omega|^{p-2}\omega$ and integrating over $\mathbb{R}^2$ with respect to variable x, it holds

$$\frac{1}{p}\frac{d}{dt}\|\omega(t)\|_{L^p}^p = \int_{\mathbb{R}^2}(b \cdot \nabla j)\omega|\omega|^{p-2}\,dx$$
$$\leq \|b\|_{L^\infty}\|\nabla j\|_{L^p}\|\omega\|_{L^p}^{p-1}.$$

We thus have

$$\frac{d}{dt}\|\omega(t)\|_{L^p} \leq \|b\|_{L^\infty}\|\nabla j\|_{L^p}.$$

Integrating in time, the outcome is

$$\|\omega(t)\|_{L^p} \leq \|\omega(0)\|_{L^p} + \int_0^t \|b(\tau)\|_{L^\infty}\|\nabla j(\tau)\|_{L^p}\,d\tau.$$

Letting $p \to \infty$, it has

$$\|\omega(t)\|_{L^\infty} \leq \|\omega(0)\|_{L^\infty} + \int_0^t \|b(\tau)\|_{L^\infty}\|\nabla j(\tau)\|_{L^\infty}\,d\tau.$$

By (3.5), we conclude

$$\|\omega(t)\|_{L^\infty} \leq \|\omega(0)\|_{L^\infty} + C(T)\int_0^t \|\nabla j(\tau)\|_{L^\infty}\,d\tau. \tag{3.16}$$

Now letting

$$G(t) := \zeta_1(T) + \zeta_2(T)\int_0^t \|\omega(\tau)\|_{L^\infty}\,d\tau,$$

it follows from (3.15) that

$$\int_0^t \|\nabla j(\tau)\|_{L^\infty}\,d\tau \leq G(t).$$

Thanks to (3.16), we easily get

$$\frac{d}{dt}G(t) = \zeta_2(T)\|\omega(t)\|_{L^\infty}$$
$$\leq \zeta_2(T)\big(\|\omega(0)\|_{L^\infty} + C(T)\int_0^t \|\nabla j(\tau)\|_{L^\infty}\,d\tau\big)$$
$$\leq \zeta_2(T)\big(\|\omega(0)\|_{L^\infty} + C(T)G(t)\big). \tag{3.17}$$

The following key estimate is an easy consequence of the Gronwall's lemma

$$G(t) \leq C(T, u_0, b_0),$$

which further implies that

$$\int_0^t \|\nabla j(\tau)\|_{L^\infty} \, d\tau \leq C(T, u_0, b_0).$$

From the above estimate combined with (3.16), we get

$$\|\omega(t)\|_{L^\infty} \leq C(T, u_0, b_0).$$

The following interpolation inequality

$$\|\nabla b\|_{L^\infty} \leq C \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\nabla j\|_{L^\infty}^{\frac{1}{2}},$$

as well as (3.3) and (3.11) yields that

$$\int_0^t \|\nabla b(\tau)\|_{L^\infty}^2 \, d\tau \leq C(T, u_0, b_0).$$

This achieves the proof of Lemma 3.5. □

The global H^s estimate To show the global bound for (u, b) in H^s with $s > 2$, we apply Υ^s with $\Upsilon := (I - \Delta)^{\frac{1}{2}}$ to the equations u and b and take the L^2 inner product of the resulting equations with $(\Upsilon^s u, \Upsilon^s b)$ to obtain the energy inequality

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2) + \|\mathcal{L}^{\frac{1}{2}} b\|_{H^s}^2$$

$$= - \int_{\mathbb{R}^2} [\Upsilon^s, u \cdot \nabla] u \cdot \Upsilon^s u \, dx + \int_{\mathbb{R}^2} [\Upsilon^s, b \cdot \nabla] b \cdot \Upsilon^s u \, dx$$

$$- \int_{\mathbb{R}^2} [\Upsilon^s, u \cdot \nabla] b \cdot \Upsilon^s b \, dx + \int_{\mathbb{R}^2} [\Upsilon^s, b \cdot \nabla] u \cdot \Upsilon^s b \, dx$$

$$:= J_1 + J_2 + J_3 + J_4,$$

where $[a, b]$ is the standard commutator notation, namely $[a, b] = ab - ba$. By means of the following Kato-Ponce inequality (see [12])

$$\|[\Upsilon^s, f]g\|_{L^p} \leq C(\|\nabla f\|_{L^\infty} \|\Upsilon^{s-1} g\|_{L^p} + \|g\|_{L^\infty} \|\Upsilon^s f\|_{L^p}), \quad 1 < p < \infty,$$

one can deduce that

$$J_1 \leq C \|\nabla u\|_{L^\infty} \|u\|_{H^s}^2, \qquad J_2 \leq C \|\nabla b\|_{L^\infty} \left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2 \right),$$

$$J_3 + J_4 \leq C \left(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} \right) \left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2 \right).$$

Consequently, it implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2\right) + \left\|\mathcal{L}^{\frac{1}{2}}b\right\|_{H^s}^2 \tag{3.18}$$
$$\leq C\left(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right)\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right).$$

To bound the term $\|\nabla u\|_{L^\infty}$ with $\|\omega\|_{L^\infty}$, we need the following Sobolev extrapolation inequality with logarithmic correction (see e.g., [3])

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq C\left(1 + \|u\|_{L^2(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)}\ln(e + \|u\|_{H^s(\mathbb{R}^2)})\right), \quad s > 2.$$

Consequently, it enables us to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2\right) + \left\|\mathcal{L}^{\frac{1}{2}}b\right\|_{H^s}^2$$
$$\leq C\left(1 + \|\omega\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right)\ln\left(e + \|u\|_{H^s}^2 + \|b\|_{H^s}^2\right)$$
$$\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right). \tag{3.19}$$

Applying the log-Gronwall type inequality as well as the estimates (3.12) and (3.13), we eventually obtain

$$\|u(t)\|_{H^s} + \|b(t)\|_{H^s} + \int_0^t \|\mathcal{L}^{\frac{1}{2}}b(\tau)\|_{H^s}^2 \,\mathrm{d}\tau \leq C(t),$$

which together with (3.4) implies

$$\int_0^t \|b(\tau)\|_{H^{s+1}}^2 \,\mathrm{d}\tau \leq C(t).$$

This is nothing but the desired global H^s estimates.

With the global bounds in the previous lemmas at our disposal, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. With a priori estimates achieved in the previous lemmas, it is a standard procedure to complete the proof of Theorem 1.1. The proof is achieved by using a standard procedure. First, we seek the solution of a regularized system. In order to do this, we recall the mollification of $\varrho_N f$ given by

$$(\varrho_N f)(x) = N^2 \int_{\mathbb{R}^2} \eta\big(N(x-y)\big)f(y)\,\mathrm{d}y,$$

where $0 \leq \eta(|x|) \in C_0^\infty(\mathbb{R}^2)$ satisfies $\int_{\mathbb{R}^2} \eta(y)\,\mathrm{d}y = 1$. Now we regularize our system (1.1) as follows:

$$\begin{cases} \partial_t u^N + \mathbb{P}\varrho_N((\varrho_N u^N \cdot \nabla)\varrho_N u^N) = \mathbb{P}\varrho_N((\varrho_N b^N \cdot \nabla)\varrho_N b^N), \\ \partial_t b^N + \varrho_N((\varrho_N u^N \cdot \nabla)\varrho_N b^N) + \mathcal{J}\varrho_N b^N = \varrho_N((\varrho_N b^N \cdot \nabla)\varrho_N u^N), \\ \nabla \cdot u^N = \nabla \cdot b^N = 0, \\ u^N(x, 0) = \varrho_N u_0(x), \quad b^N(x, 0) = \varrho_N b_0(x), \end{cases}$$

where $\mathbb{P}$ denotes the Leray projection operator (onto divergence-free vector fields). For any fixed $N > 0$, using properties of mollifiers and following the same argument used in proving the previous lemmas, it is not difficult to establish the global bound, for any $t \in (0, \infty)$,

$$\|u^N(t)\|_{H^s} + \|b^N(t)\|_{H^s} + \int_0^t \|\mathcal{L}^{\frac{1}{2}}b^N(\tau)\|_{H^s}^2 \, d\tau \le C(t)$$

$$\int_0^t \|b^N(\tau)\|_{H^{s+1}}^2 \, d\tau \le C(t).$$

Now the standard Alaoglu's theorem allows us to obtain the global existence of the classical solution (u, b) to (1.1). The uniqueness can also be easily established. This completes the proof of Theorem 1.1. □

Acknowledgements

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