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Asymptotic behavior of solutions to a class of diffusive predator–prey systems

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Abstract. We study the large time behavior of a class of diffusive predator–prey systems posed on the whole Euclidean space. By studying a family of similar problems with all possible spatial translations, we first prove the asymptotic persistence of the prey for the spatially heterogeneous case under certain assumptions on the coefficients. Then, applying this persistence theorem, we prove the convergence of the solution to the unique positive equilibrium for the spatially homogeneous case, under certain restrictions on the space dimension and the predation coefficient.

1. Introduction

In this paper, we are concerned with the large time behavior of a class of diffusive predator–prey systems posed on the whole space $\mathbb{R}^N$ that takes the following form

$$\begin{cases} \partial_t u - D\Delta u = u\left[g(x,u) - \Pi(x,u)v\right], \quad t > 0, \quad x \in \mathbb{R}^N, \\ \partial_t v - \Delta v = r(x)v\left[1 - \delta(x)v/u\right], \quad t > 0, \quad x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

Here $u = u(t, x)$ and $v = v(t, x)$ denote, respectively, the densities of the prey and the predator at time $t > 0$ and spatial location $x \in \mathbb{R}^N$. In the first equation of system (1.1), the function g denotes the intrinsic growth rate of the prey while the function $u\Pi(x, u)$ denotes the functional response of the predation. In the second equation of system (1.1), r and δ are given nonnegative, continuous and bounded functions such that r denotes the growth rate and u/δ denotes the carrying capacity of the predator. Finally, $D > 0$ denotes the normalized diffusion coefficient of the prey.

The above class of systems contains as special cases the so-called Leslie–Gower and Holling–Tanner diffusive models that correspond to the following specific functions:

$$g(x,u) = \lambda(x) - \alpha(x)u, \quad \Pi(x,u) = \beta(x),$$

$$g(x,u) = \lambda(x) - \alpha(x)u, \quad \Pi(x,u) = \frac{\beta(x)}{1+\gamma(x)u},$$

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where the functions λ, α, β and γ are nonnegative continuous and bounded on $\mathbb{R}^N$.

The aim of this work is to prove that the prey component of solutions of (1.1) does not asymptotically quench as time becomes large. This will allow us to provide a sufficient condition to ensure that the solution converges to some equilibrium solution for some specific homogeneous functions.

In this paper, we set $\mathbb{R}_+ := [0, \infty)$. We also introduce for any uniformly continuous and bounded function $f : \mathbb{R}^N \to \mathbb{R}$ the hull of f, denoted by $\mathcal{H}(f)$, defined by

$$\mathcal{H}(f) = \mathrm{cl}\left(\left\{\sigma_y f,\ y \in \mathbb{R}^N\right\}\right) \ \text{ with } \ \sigma_y f(\cdot) = f(\cdot + y),$$

where the closure (denoted by cl) is taken with respect to the topology of $C_{\mathrm{loc}}\left(\mathbb{R}^N\right)$. In other words, one has

$$\widehat{f} \in \mathcal{H}(f) \Leftrightarrow\ \exists \{y_n\} \subset \mathbb{R}^N \text{ such that } \widehat{f}(x) = \lim_{n\to\infty} f(x + y_n) \ \text{ in } \ C_{\mathrm{loc}}(\mathbb{R}^N).$$

Hereafter, for each $p \geq 1$, the set $W^{1,2;p}_{\mathrm{loc}}\left((0,\infty) \times \mathbb{R}^N\right)$ is consisted of functions $\varphi = \varphi(t, x)$ defined on $(0, \infty) \times \mathbb{R}^N$ such that

$$\varphi,\ \partial_t\varphi,\ \partial_{x_i}\varphi \text{ and } \partial^2_{x_i,x_j}\varphi \ \text{ belong to } \ L^p_{\mathrm{loc}}\left((0,\infty) \times \mathbb{R}^N\right)$$

for all indices $i = 1, \ldots, N$ and $j = 1, \ldots, N$. Also, the set $W^{2;p}_{\mathrm{loc}}\left(\mathbb{R}^N\right)$ consists of functions $\varphi = \varphi(x)$ defined on $\mathbb{R}^N$ such that

$$\varphi,\ \partial_{x_i}\varphi \ \text{ and } \ \partial^2_{x_i,x_j}\varphi \ \text{ belong to } L^p_{\mathrm{loc}}\left(\mathbb{R}^N\right)$$

for all indices $i = 1, \ldots, N$ and $j = 1, \ldots, N$.

In order to state our main results, let us firstly introduce the general assumptions which we shall use in this work. Our main set of assumptions reads as follows.

ASSUMPTION 1.1. *The function* $g : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ *is continuous and satisfies*

(i) *there exists some constant* $M > 0$ *such that* $g(x, u) \leq 0$ *for all* $x \in \mathbb{R}^N$ *and* $u \geq M$;
(ii) *the set* $g\left(\mathbb{R}^N \times [0, M]\right)$ *is bounded and* g *is uniformly continuous on* $\mathbb{R}^N \times [0, M]$;
(iii) *there exists a constant* $\gamma > 0$ *such that*

$$\lambda_1\left(-D\Delta - \widehat{g}(x); \mathbb{R}^N\right) \leq -\gamma, \quad \forall \widehat{g} \in \mathcal{H}\left(g(\cdot, 0)\right).$$

Here, if we set $L = D\Delta + \widehat{g}(x)$, $\lambda_1\left(-L; \mathbb{R}^N\right)$ *denotes the generalized principal eigenvalue of the elliptic operator* L, *that is defined by*

$$\lambda_1(-L; \mathbb{R}^N) = \sup\left\{\lambda \in \mathbb{R} : \exists \phi \in W^{2;N}_{\mathrm{loc}}(\mathbb{R}^N) \text{ such that } \phi > 0,\ (L + \lambda)\phi \leq 0 \text{ in } \mathbb{R}^N\right\}.$$

REMARK 1.1. Here we have used the notion of generalized principal eigenvalue for which we refer the reader to [4] (see also [5] and the references therein). Since the operator L arising in the above assumption is self-adjoint, this principal eigenvalue can also be formulated using the usual Rayleigh quotient as follows:

$$\lambda_1(-L;\mathbb{R}^N) = \inf_{\phi\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\left(D|\nabla\phi|^2+\widehat{g}(x)\phi^2\right)\mathrm{d}x}{\int_{\mathbb{R}^N}\phi^2\mathrm{d}x}.$$

ASSUMPTION 1.2. *The function* $\Pi:\mathbb{R}^N\times\mathbb{R}_+\to\mathbb{R}$ *is continuous and satisfies*

(i) *the set* $\Pi\left(\mathbb{R}^N\times\mathbb{R}_+\right)\subset[0,\infty)$*;*
(ii) *the set* $\Pi\left(\mathbb{R}^N\times[0,M]\right)$ *is bounded and* Π *is uniformly continuous on* $\mathbb{R}^N\times[0,M]$.

ASSUMPTION 1.3. *The functions* $r,\delta:\mathbb{R}^N\to\mathbb{R}_+$ *are uniformly continuous and bounded and they satisfy*

(i) *there exists* $\delta_0>0$ *such that* $\delta(x)\geq\delta_0$ *for all* $x\in\mathbb{R}^N$*;*
(ii) *any function* $\widehat{r}\in\mathcal{H}(r)$ *satisfies* $\widehat{r}(x)>0$ *almost everywhere for* $x\in\mathbb{R}^N$.

Our first result is concerned with the asymptotic positivity of the u-component. To state our result, we consider the set $\mathcal{S}$ of all solutions defined by

$$\mathcal{S}=\left\{(u,v)\in\mathcal{X}\times\mathcal{X}\ :\ 0<u\leq M, 0\leq v\leq M/\delta_0, (u,v)\text{ satisfies (1.1)}\right\},$$

wherein we have set

$$\mathcal{X}:=C\left(\mathbb{R}_+\times\mathbb{R}^N\right)\cap\bigcap_{p\geq1}W^{1,2;p}_{\mathrm{loc}}\left((0,\infty)\times\mathbb{R}^N\right).$$

Before going to our first result, note that when system (1.1) is supplemented with some initial data $(u_0,v_0)\in C(\mathbb{R}^N)\times C(\mathbb{R}^N)$ such that

$$0<\inf_{\mathbb{R}^N}u_0\leq u_0(x)\leq M,\quad 0\leq v_0(x)\leq M/\delta_0,\quad \forall x\in\mathbb{R}^N, \tag{1.2}$$

then it has (at least) a globally defined solution $(u,v)\in\mathcal{X}\times\mathcal{X}$ such that $u>0$ and $v\geq0$. Moreover, although the parabolic comparison principle does not apply to this system of equations, it separately applies to each component of that problem and ensures that the following properties hold true

$$0<u(t,x)\leq M,\quad 0\leq v(t,x)<M/\delta_0,\quad \forall t>0,\quad x\in\mathbb{R}^N.$$

The first inequality is ensured by the positivity of Π (see Assumption 1.2(i)) and Assumption 1.1(i). The upper bound for v follows from the upper bound for u and the uniform lower bound for the function δ as stated in Assumption 1.3. As a consequence, $\mathcal{S}$ is non-empty.

We are now able to state our first main result as follows.

THEOREM 1.4. (Asymptotic persistence of the prey) *Let assumptions 1.1–1.3 be satisfied. Then there exists a positive constant ε small enough such that for all $x \in \mathbb{R}^N$ and all $(u, v) \in \mathcal{S}$ one has*

$$\liminf_{t\to\infty} u(t, x) \geq \varepsilon.$$

The proof of this result is based on uniform persistence theory, for which we refer the reader to the paper of Hale and Waltman [9], the paper of Magal and Zhao [11] and the monograph of Smith and Thieme [16]. However, the usual theory does not directly apply to the problem we consider in this work. On the one hand, we are not able to fully characterize the set of initial data ensuring the existence of a solution of (1.1). On the other hand, the weak regularity we assume in this work does not ensure the uniqueness of the solutions. As a consequence, it is difficult to associate a strongly continuous semigroup on a complete metric space to system (1.1). Moreover, since the value of $\varepsilon > 0$ appearing in Theorem 1.4 holds for any $x \in \mathbb{R}^N$, the natural solution set $\mathcal{S}$ for problem (1.1) defined above is not sufficiently large to prove the above persistence result. To overcome this difficulty, we shall also act by space translation on the solutions. However, our system is not invariant under space translations, since the problem we consider is spatially heterogeneous. To resolve this issue, we consider the set of solutions for a family of problems (see ($\mathcal{P}_\Psi$) below) in the form of (1.1) taking into account all possible spatial translations.

Based on Theorem 1.4, we shall investigate the large time behavior of the problem (1.1) with the specific homogeneous functions

$$g(x, u) = 1 - u, \quad \Pi(x, u) \equiv k > 0, \quad r(x) \equiv r > 0, \quad \delta(x) \equiv 1. \tag{1.3}$$

Our next aim is to show that when $v_0 \not\equiv 0$ the solution (u, v) of (1.1) with initial data (u_0, v_0) satisfying (1.2) converges locally uniformly as $t \to \infty$ to the unique positive equilibrium point (u^*, v^*) defined by

$$\left(u^*, v^*\right) = \frac{1}{k+1}(1, 1).$$

Observe also that $(u, v)(t, x) \to (1, 0)$ as $t \to \infty$ locally uniformly with respect to $x \in \mathbb{R}^N$, when $v_0 \equiv 0$. Such a property is usually referred to the so-called "air trigger effect." We refer to Aronson and Weinberger [3] for the derivation of such a property for scalar reaction–diffusion equation of monostable type.

The question of converging to the positive equilibrium point has already been solved in [7] in the case where the parameter k satisfies $k \in (0, 1)$. Indeed, in that case it is easy to obtain that the u-component stays uniformly far away from the singular boundary $u = 0$. This crucial property allows the author to use a sandwiching technique by constructing a suitable decreasing sequence of invariant rectangles. In that setting, namely $0 < k < 1$, since $v \leq 1$ the function u satisfies

$$\partial_t u - D\Delta u \geq u(1 - k - u), \quad t > 0, \quad x \in \mathbb{R}^N,$$

and, since $1 - k > 0$, this prevents the u-component to be too close to $u = 0$. Of course, when $k \geq 1$ such a simple argument can no longer be used.

In this paper, we first prove that the u-component cannot quench for all time for arbitrary parameter $k > 0$. Then we develop some refined arguments in order to prove that (u, v) is asymptotically constant. More precisely, we obtain the following result.

THEOREM 1.5. *Let* $(u, v) \in \mathcal{S}$ *be any solution of* (1.1) *with* (1.3) *such that* $v > 0$ *on* $\mathbb{R}_+ \times \mathbb{R}^N$. *Then one has*

$$\lim_{t\to\infty} (u, v)(t, x) = \left(u^*, v^*\right)$$

locally uniformly for $x \in \mathbb{R}^N$, *if one of the following conditions is satisfied*

(i) *The dimension* N *is arbitrary and* $k \in (0, 1]$;

(ii) *The dimension* $N = 2$ *and* $k \in (0, 1/s_0)$ *where* $s_0 \in (1/5, 1/4)$ *is the unique solution of the polynomial equation* $32s^3 + 16s^2 - s - 1 = 0$.

The proof of this theorem is given in Sect. 4 below. Note that the sandwiching technique in [7] for $k \in (0, 1)$ can be extended to the case $k = 1$, but cannot be further extended to $k > 1$. Here, for $k > 1$, we develop another technique based on energy arguments using the Lyapunov function proposed by Du and Hsu in [6] in which the global stability of (u^*, v^*) for a similar system posed on a bounded spatial domain was studied.

The rest of this paper is organized as follows. In the next section, we shall provide some preliminary estimates which will be used later. Then Sect. 3 is devoted to the proof of Theorem 1.4. Finally, the proof of Theorem 1.5 is given in Sect. 4.

2. Preliminary estimates

In this section, we shall recall and derive some important estimates for the solutions of system (1.1). For this reason, instead of only dealing with the solution of the initial value problem (1.1), we will state our estimates for a larger class of solutions.

2.1. Harnack inequalities and maximum principle

Let $\Omega \subset \mathbb{R} \times \mathbb{R}^N$ be a bounded open set such that $(0, 0) \in \Omega$. For $R > 0$, we denote by Q_R the parabolic cylinder defined by

$$Q_R = (-R^2, 0) \times B_R, \quad \text{with} \quad B_R = \{x \in \mathbb{R}^N : \|x\| < R\}.$$

A function $w : \overline{\Omega} \to [0, \infty)$ is said to be a Lipschitz sub- (resp. super-) solution of the heat equation on Ω if w is a Lipschitz continuous function on $\overline{\Omega}$ satisfying the inequality

$$\int_\Omega \partial_t w \varphi \mathrm{d}t\mathrm{d}x + \int_\Omega \nabla w \cdot \nabla \varphi \mathrm{d}t\mathrm{d}x \leq 0 \text{ (resp. } \geq 0)$$

for all Lipschitz continuous test functions $\varphi : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ with $\operatorname{supp} \varphi \subset \Omega$.

We first recall from [8,12] the following two Harnack inequalities for sub- and super-solutions of the heat equation.

LEMMA 2.1. (Harnack inequality for Lipschitz sub-solution) *For each* $p > 1$, $0 < \theta < \tau < 1$, *there exists a constant* $C = C(p, \theta, \tau, \Omega) > 1$ *such that for any* $R > 0$ *with* $Q_R \subset \Omega$ *one has*

$$\sup_{Q_{\theta R}} w \leq C R^{-\frac{N+2}{p}} \|w\|_{L^p(Q_{\tau R})}$$

for any positive Lipschitz sub-solution w *of the heat equation on* Ω.

LEMMA 2.2. (Harnack inequality for Lipschitz super-solution) *For each* $R > 0$ *such that* $(-4R^2, 0) \times B_R \subset \Omega$, *there exist* $p = p(R) > 1$ *and* $M = M(R) > 1$ *such that*

$$\|w\|_{L^p((-4R^2, -3R^2) \times B_R)} \leq M \inf_{Q_R} w$$

for any positive Lipschitz super-solution w *of the heat equation on* Ω.

We continue this section by recalling a strong maximum principle for Sobolev super-solution of a parabolic equation (cf. [8]).

LEMMA 2.3. *Let* $T > 0$ *be given and* $p > N + 2$. *Let* $u \in W^{1,2;p}_{\text{loc}}\left((0, T) \times \mathbb{R}^N\right)$ *be a function such that*

$$\partial_t u - \Delta u \geq 0 \text{ a.e. in } (0, T) \times \mathbb{R}^N.$$

If $u(0, \cdot) \geq 0$ *then either* $u(t, \cdot) \equiv 0$ *or* $u(t, x) > 0$ *for all* $t > 0$ *and* $x \in \mathbb{R}^N$.

2.2. A family of problems and basic estimates

As explained in the introduction, the proof of Theorem 1.4 requires the introduction of a suitable family of problems related to (1.1) in order to take into account spatial translations. To that aim, let us recall that the functions $g(x, u)$ and $\Pi(x, u)$ are both uniformly continuous and bounded from $\mathbb{R}^N \times [0, M]$ into $\mathbb{R}$ (see Assumptions 1.1 and 1.2). Next we set

$$\mathcal{K} = \operatorname{cl}\left(\left\{\left(g(\cdot + y, \cdot), \Pi(\cdot + y, \cdot), \sigma_y r, \sigma_y \delta\right),\ y \in \mathbb{R}^N\right\}\right), \tag{2.1}$$

where the closure is taken for the topology on $C_{\text{loc}}\left(\mathbb{R}^N \times [0, M]\right)^2 \times C_{\text{loc}}\left(\mathbb{R}^N\right)^2$. Because of the uniform continuity assumption, let us further observe that the set $\mathcal{K}$ is uniformly equi-continuous. Moreover, from the definition of the set $\mathcal{K}$, one also has the following boundedness property:

$$|g^*(x, u)| \leq \sup_{y \in \mathbb{R}^N,\ v \in [0, M]} |g(y, v)|, \quad \forall (x, u) \in \mathbb{R}^N \times [0, M],$$

$$0 \leq \Pi^*(y, v) \leq \sup_{y \in \mathbb{R}^N,\ v \in [0, M]} \Pi(y, v), \quad \forall (x, u) \in \mathbb{R}^N \times [0, M],$$

$$r^*(x) \leq \overline{r} := \sup_{x \in \mathbb{R}^N} r(x), \quad 0 < \delta_0 \leq \delta^*(x) \leq \overline{\delta} := \sup_{x \in \mathbb{R}^N} \delta(x), \quad \forall x \in \mathbb{R}^N,$$

for any $(g^*, \Pi^*, r^*, \delta^*) \in \mathcal{K}$. Therefore, $\mathcal{K}$ is compact with respect to the topology on $C_{\text{loc}}\left(\mathbb{R}^N \times [0, M]\right)^2 \times C_{\text{loc}}\left(\mathbb{R}^N\right)^2$.

Next, for each $\Psi = (g^*, \Pi^*, r^*, \delta^*) \in \mathcal{K}$ we consider the problem

$$\begin{cases} \partial_t u - D\Delta u = u\left[g^*(x, u) - \Pi^*(x, u)v\right], & t > 0, \quad x \in \mathbb{R}^N, \\ \partial_t v - \Delta v = r^*(x)v\left(1 - \delta^*(x)v/u\right), & t > 0, \quad x \in \mathbb{R}^N. \end{cases} \tag{$\mathcal{P}_\Psi$}$$

Similarly as in the introduction, for each $\Psi \in \mathcal{K}$, we consider the set of solutions of $(\mathcal{P}_\Psi)$ defined by

$$\mathcal{S}_\Psi = \left\{(u, v) \in \mathcal{X} \times \mathcal{X} \ : \ 0 < u \leq M, 0 \leq v \leq M/\delta_0, (u, v) \text{ satisfying } (\mathcal{P}_\Psi)\right\},$$

and we consider the set $\overline{\mathcal{S}}$ defined by

$$\overline{\mathcal{S}} = \bigcup_{\Psi \in \mathcal{K}} \mathcal{S}_\Psi. \tag{2.2}$$

Here let us observe that if $(u, v) \in \overline{\mathcal{S}}$ then, for each $x_0 \in \mathbb{R}^N$, the function $(\tilde{u}, \tilde{v})(t, x) := (u, v)(t, x + x_0)$ belongs to $\overline{\mathcal{S}}$.

In the following, we shall derive some basic uniform estimates for the solution set $\overline{\mathcal{S}}$. Our first estimate reads as follows.

LEMMA 2.4. *Let* $0 < \theta < \tau < 1$ *be given. Set* $\overline{r} = \sup\{r(x),\ x \in \mathbb{R}^N\}$. *For each* $R > 0$, *there exists a constant* C_R *such that for all* $(u, v) \in \overline{\mathcal{S}}$ *one has*

$$\sup_{x \in Q_{\theta R}} \mathrm{e}^{-\overline{r}t} v(t + h, x) \leq C_R \left(\iint_{(-\tau R^2, 0) \times B_{\tau R}} \mathrm{e}^{-2\overline{r}t} v^2(s + h, y)\mathrm{d}s\mathrm{d}y\right)^{1/2} \tag{2.3}$$

for any $h > R^2$.

Proof. Since $u > 0$, the function v is a classical solution of the following differential inequality

$$\partial_t v - \Delta v - \overline{r}v \leq 0, \quad t > 0, \quad x \in \mathbb{R}^N.$$

Let $R > 0$ be given. For a fixed value $h > R^2$, we consider the function $w^h(t, x) := \mathrm{e}^{-\overline{r}t} v(t + h, x)$ that satisfies the inequality

$$\partial_t w^h - \Delta w^h \leq 0, \quad t > -h, \quad x \in \mathbb{R}^N.$$

Then, applying Lemma 2.1 on $\Omega = Q_R$ with $p = 2$, there exists some constant $C = C(R, \theta, \tau)$ such that (2.3) is satisfied. □

Our second result deals with estimates for the u-component as follows.

LEMMA 2.5. *For each* $R > 0$, *there exist some constants* $M > 1$ *and* $p > 1$ *such that for all* $(u, v) \in \overline{\mathcal{S}}$ *one has*

$$\iint_{(-4R^2, -3R^2) \times B_R} u^p(t + h, x)\mathrm{d}t\mathrm{d}x \leq M\left[\inf_{(t,x) \in Q_R} u(t + h, x)\right]^p.$$

for any $h > 4R^2$.

Proof. Since $(u, v) \in \overline{\mathcal{S}}$, the function u satisfies

$$\partial_t u - D\Delta u \geq -\widehat{k}u,$$

for some constant $\widehat{k} > 0$ such that for any $(g^*, \Pi^*, r^*, \delta^*) \in \mathcal{K}$

$$g^*(x, u) - \Pi^*(x, u)v \geq -\widehat{k}, \ \forall x \in \mathbb{R}^N, \quad (u, v) \in \overline{\mathcal{S}}.$$

One may apply Lemma 2.2 to the smooth function $w^h(t, x) := e^{\widehat{k}t}u(t + h, x)$ and the lemma follows. □

Before going to the proof of our main result, we need to derive some further local estimates.

LEMMA 2.6. *For each $R > 0$ and $T > 0$, there exists a constant $K = K(R, T) > 0$ such that for all $\Psi = (g^*, \Pi^*, r^*, \delta^*) \in \mathcal{K}$ and $(u, v) \in \mathcal{S}_\Psi$ one has*

$$\int_0^T \int_{B_R} |\nabla v(t, x)|^2 \, dx dt \leq K, \tag{2.4}$$

$$\int_0^T \int_{B_R} r^*(x)\delta^*(x)\frac{v^2}{u}(t, x) dx dt \leq K. \tag{2.5}$$

Proof. Let $R > 0$ and $T > 0$ be given. Consider the function $\varphi \in C^\infty\left(\mathbb{R}^N\right)$ defined by

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{(R+1)^2 - |x|^2}\right), & \text{if } x \in B_{R+1}, \\ 0, & \text{elsewhere,} \end{cases}$$

and let us observe that $\nabla\varphi/\sqrt{\varphi} \in L^2(B_{R+1})$. Now, fix $(u, v) \in \mathcal{S}_\Psi$. Then multiplying the v-equation by φv and integrating over B_{R+1} yields

$$\frac{d}{dt}\int_{B_{R+1}} \varphi\frac{v^2}{2} dx + \int_{B_{R+1}} \varphi\,|\nabla v|^2 \, dx = -\int_{B_{R+1}} \sqrt{\varphi}\nabla v \cdot \left(\frac{v\nabla\varphi}{\sqrt{\varphi}}\right) dx$$
$$+ \int_{B_{R+1}} r^*(x)\varphi v^2 dx - \int_{B_{R+1}} r^*(x)\delta^*(x)\varphi\frac{v^3}{u} dx.$$

However, one has

$$\int_{B_{R+1}} \sqrt{\varphi}\nabla v \cdot \left(-\frac{v\nabla\varphi}{\sqrt{\varphi}}\right) dx \leq \frac{1}{2}\int_{B_{R+1}} \varphi|\nabla v|^2 dx + \frac{\|v\|_\infty^2}{2}\int_{B_{R+1}} \frac{|\nabla\varphi|^2}{\varphi} dx;$$

and this ensures that

$$\frac{d}{dt}\int_{B_{R+1}} \varphi\frac{v^2}{2} dx + \frac{1}{2}\int_{B_{R+1}} \varphi|\nabla v|^2 dx \leq \|v\|_\infty^2 \int_{B_{R+1}} \left[\frac{1}{2}\frac{|\nabla\varphi|^2}{\varphi} + r^*(x)\varphi\right] dx.$$

As a consequence, since $v \leq M/\delta_0$ and $r^*(x) \leq \overline{r}$, there exists a positive constant $K_1 = K_1(R, T)$ such that

$$\int_0^T \int_{B_{R+1}} \varphi |\nabla v|^2 \mathrm{d}x \leq K_1(R, T),$$

and (2.4) follows.

To prove (2.5), let us multiply the v-equation by φ and integrate over B_{R+1}. This yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{B_{R+1}} \varphi v \mathrm{d}x + \int_{B_{R+1}} \varphi r^*(x)\delta^*(x) \frac{v^2}{u} \mathrm{d}x \\ &\quad = - \int_{B_{R+1}} \nabla\varphi \cdot \nabla v \mathrm{d}x + \int_{B_{R+1}} r^*(x)\varphi v \mathrm{d}x. \end{aligned} \tag{2.6}$$

However, one gets

$$- \int_{B_{R+1}} \nabla\varphi \cdot \nabla v \mathrm{d}x \leq \frac{1}{2} \int_{B_{R+1}} \left| \frac{\nabla\varphi}{\sqrt{\varphi}} \right|^2 \mathrm{d}x + \frac{1}{2} \int_{B_{R+1}} \varphi |\nabla v|^2 \mathrm{d}x;$$

and we infer from the above computations that

$$- \int_0^T \int_{B_{R+1}} \nabla\varphi \cdot \nabla v \mathrm{d}x\mathrm{d}t \leq 2K_1(R, T).$$

Finally, integrating (2.6) over $(0, T)$, there exists a constant $K_2 = K_2(R, T) > 0$ such that

$$\int_0^T \int_{B_{R+1}} r^*(x)\delta^*(x)\varphi \frac{v^2}{u} \mathrm{d}x\mathrm{d}t \leq K_2,$$

and estimates (2.5) follows. □

As a consequence of the above lemma, one obtains that the set $\overline{\mathcal{S}}$ is relatively compact for some suitable local topology. To that aim, let us recall that for each $p \geq 1$ and $\alpha > 0$ such that $\alpha < 2 - \frac{N+2}{p}$ then, for each $\in [-\infty, \infty)$, the embedding $W^{1,2;p}_{\mathrm{loc}}((a, \infty) \times B_R) \hookrightarrow C^{\frac{\alpha}{2},\alpha}_{\mathrm{loc}}\left((a, \infty) \times \mathbb{R}^N\right)$ is compact.

We are able to state the compactness with respect to time translation action of the set $\overline{\mathcal{S}}$. To that aim let us introduce the time translation operator of $\overline{\mathcal{S}}$. For each $h \geq 0$ and $(u, v) \in \mathcal{S}$ we denote by $\tau_h \cdot (u, v)$ the function defined by

$$\tau_h \cdot (u, v)(t, x) = (u, v)\,(t + h, x), \quad \forall t \geq -h, \quad x \in \mathbb{R}^N.$$

Then our next result reads as follows.

PROPOSITION 2.7. *For each $p \geq 1$, the set $\overline{\mathcal{S}}$ is bounded in $W^{1,2;p}_{\mathrm{loc}}\left((0, \infty) \times \mathbb{R}^N\right) \times L^\infty\left((0, \infty) \times \mathbb{R}^N\right)$ and relatively compact with respect to the topology of*

$$C^{\frac{\alpha}{2},\alpha}_{\mathrm{loc}}\left((0, \infty) \times \mathbb{R}^N\right) \times L^2_{\mathrm{loc}}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$$

for any $\alpha \in (0, 2)$. Moreover, let $\{h_n\}_{n\geq 0} \subset \mathbb{R}_+$ be a given sequence such that $h_n \to \infty$ and let $\{U_n\}_{n\geq 0}$ be a sequence of functions with $U_n \in \tau_{h_n}\overline{\mathcal{S}}$ for all $n \geq 0$. Then $\{U_n\}_{n\geq 0}$ is relatively compact with respect to the topology of $C^{\frac{\alpha}{2},\alpha}_{\mathrm{loc}}\left(\mathbb{R} \times \mathbb{R}^N\right) \times L^2_{\mathrm{loc}}\left(\mathbb{R} \times \mathbb{R}^N\right)$ for any $\alpha \in (0, 2)$.

Proof. The proof of this compactness result follows from usual parabolic regularity for the u-component, while the compactness for the v-component follows from Lemma 2.6 and Simon compactness theorem (see [15]). Indeed, fix a smooth and compactly supported function $\varphi \in \mathcal{D}\left([0,\infty)\times\mathbb{R}^N\right)$. One has, for any $\Psi = (g^*, \Pi^*, r^*, \delta^*) \in \mathcal{K}$ and any $(u,v)\in\mathcal{S}_\Psi$,

$$\partial_t(\varphi v) = \varphi\partial_t v + v\partial_t\varphi = (\varphi\Delta v) + \left(\varphi r^*(x)v - \varphi r^*(x)\delta^*(x)\frac{v^2}{u} + v\partial_t\varphi\right). \quad (2.7)$$

However, because of Lemma 2.6, the first term in the right-hand side of (2.7) is bounded in the dual space $\left(H^1\left((0,\infty)\times\mathbb{R}^N\right)\right)'$, while the second term is bounded in $L^1\left((0,\infty)\times\mathbb{R}^N\right)$. Hence, for each $\varphi\in\mathcal{D}([0,\infty)\times\mathbb{R}^N)$, φv is bounded in $L^\infty((0,\infty)\times\mathbb{R}^N)$ and $\partial_t(\varphi v)$ is bounded in $L^1((0,\infty)\times\mathbb{R}^N)+\left(H^1\left((0,\infty)\times\mathbb{R}^N\right)\right)'$. Therefore, Simon compactness theorem applies and ensures that $\left\{\varphi v, (u,v)\in\overline{\mathcal{S}}\right\}$ is relatively compact in $L^1_{\text{loc}}\left(\mathbb{R}_+\times\mathbb{R}^N\right)$ and thus in $L^2_{\text{loc}}\left(\mathbb{R}_+\times\mathbb{R}^N\right)$. This completes the proof of the first statement of the proposition. The second statement follows from the same arguments and the proposition is proved. □

With the above estimates and the local compactness property, we are now in position to prove Theorem 1.4 in the next section.

3. Proof of Theorem 1.4

In this section, we aim at proving Theorem 1.4. To reach this goal, we will make use of the following crucial lemma.

Lemma 3.1 (Weak persistence of $\mathcal{S}$). *There exists* $\varepsilon>0$ *such that for all* $(u,v)\in\overline{\mathcal{S}}$ *one has*

$$\limsup_{t\to\infty} u(t,x)\geq\varepsilon$$

for all $x\in\mathbb{R}^N$.

Proof. In order to prove this lemma, we shall argue by contradiction by assuming that for each $n\geq 1$ there exist $x_n\in\mathbb{R}^N$, $\Psi_n\in\mathcal{K}$ and $U_n=(u_n,v_n)\in\mathcal{S}_{\Psi_n}$ and $t_n\geq 0$ such that

$$\lim_{n\to\infty} t_n=\infty \ \text{ and } \ u_n(t_n+t,x_n)\leq\frac{1}{n},\ \ \forall n\geq 1,\ \ \forall t\geq 0. \quad (3.1)$$

Now, consider the sequence of maps (u^n,v^n) defined on $[-t_n,\infty)\times\mathbb{R}^N$ by

$$\left(u^n,v^n\right)(t,x) := (u_n,v_n)(t+t_n,x+x_n),\ \ \forall n\geq 1.$$

Note also that (3.1) re-writes as

$$u^n(t,0)\leq\frac{1}{n},\ \ \forall t\geq 0,\ \ n\geq 1. \quad (3.2)$$

We now divide the proof of Lemma 3.1 into three steps. We will first investigate the behavior of the sequence $\{u^n\}_{n\geq 1}$ as $n \to \infty$. Then we will derive some uniform convergence for the sequence $\{v^n\}_{n\geq 1}$ to finally reach a contradiction and thus complete the proof of the lemma.

First step In this first step, we will prove that for each $R > 0$ and $h > 0$ one has

$$\lim_{n\to\infty} \sup_{(t,x)\in[-h,\infty)\times\overline{B_R}} u^n(t,x) = 0. \tag{3.3}$$

To that aim let us first observe that due to Proposition 2.7 up to a subsequence there exists a pair of functions $(u^\infty, v^\infty) : \mathbb{R} \times \mathbb{R}^N \to [0, M] \times [0, M/\delta_0]$ such that

$$u^n \to u^\infty \text{ in } C_{\text{loc}}\left(\mathbb{R} \times \mathbb{R}^N\right) \text{ and weakly in } W^{1,2;p}_{\text{loc}}\left(\mathbb{R} \times \mathbb{R}^N\right) \text{ for all } p \in [1,\infty);$$
$$v^n \to v^\infty \text{ strongly in } L^2_{\text{loc}}\left(\mathbb{R} \times \mathbb{R}^N\right) \text{ and weakly* in } L^\infty_{\text{loc}}\left(\mathbb{R} \times \mathbb{R}^N\right).$$

Let us write $\Psi_n = \left(g_n^*, \Pi_n^*, r_n^*, \delta_n^*\right)$. In addition, possibly along a subsequence, one assume that the uniformly bounded sequence of function $g_n^*(x, u^n) - \Pi_n^*(x, u^n)\, v^n$ converges weakly* in $L^\infty_{\text{loc}}\left(\mathbb{R} \times \mathbb{R}^N\right)$ toward some bounded function $K(t, x)$.

Now note that the function u^n satisfies the following equation on $(-t_n, \infty) \times \mathbb{R}^N$

$$\partial_t u^n - D\Delta u^n = u^n\left(g_n^*\left(x, u^n\right) - \Pi_n^*\left(x, u^n\right) v^n\right),$$

so that passing to the limit $n \to \infty$ and using the above convergence property ensure that the function pair (u^∞, v^∞) satisfies the following equation:

$$\partial_t u^\infty - D\Delta u^\infty = u^\infty K(t, x), \text{ a.e. } (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Moreover, (3.2) ensures that $u^\infty(t, 0) = 0$ for all $t \geq 0$. Since $u^\infty \leq M$ and $v^\infty \leq M/\delta_0$ and K is bounded, the function u^∞ satisfies the differential inequality

$$\partial_t u^\infty - D\Delta u^\infty \geq -\widehat{k} u^\infty, \text{ a.e. } (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

for some constant $\widehat{k} > 0$ such that $K \geq -\widehat{k}$. Finally, since u^∞ is nonnegative and belongs to $W^{1,2;p}_{\text{loc}}\left(\mathbb{R} \times \mathbb{R}^N\right)$ for any large p and $u^\infty(t, 0) = 0$ for all $t \geq 0$, Lemma 2.3 ensures that $u^\infty(t, x) = 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

Due to the above arguments, in order to prove (3.3) we shall argue by contradiction by assuming that there exists a sequence $\tau_n \to \infty$ such that

$$\limsup_{n\to\infty} \sup_{x\in B_R} u^n(\tau_n, x) > 0. \tag{3.4}$$

Now we replace the sequence of function (u^n, v^n) by $(\tilde{u}^n, \tilde{v}^n) := (u^n, v^n)(\cdot + \tau_n, \cdot)$. Note that (3.2) re-writes as

$$\tilde{u}^n(t, 0) = u^n(t + \tau_n, 0) \leq \frac{1}{n}, \ \forall t \geq -\tau_n, \ n \geq 1.$$

Hence using the same arguments as above with $(\tilde{u}^n, \tilde{v}^n)$ instead of (u^n, v^n) leads us to a contradiction of (3.4) and this proves (3.3).

Second step We shall now investigate the behavior of the sequence $\{v^n\}_{n\geq 1}$ and we shall prove that for each $R > 0$ one has:

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,\infty)\times\overline{B_R}} v^n(t,x) = 0. \tag{3.5}$$

We first show that $v^\infty \equiv 0$. To that aim observe that v^n satisfies the equation

$$u^n \partial_t v^n - u^n \Delta v^n = r_n^*(x) v^n u^n - r_n^*(x)\delta_n^*(x)\left(v^n\right)^2.$$

Up to a subsequence one may assume that

$$\left(g_n^*, \Pi_n^*, r_n^*, \delta_n^*\right)(x) \to (g^*, \Pi^*, r^*, \delta^*)(x) \text{ locally uniformly,}$$

for some function $r^* \in \mathcal{H}(r)$ and $\delta^* \in \mathcal{H}(\delta)$. Recall that due to Assumption 1.3(ii), $r^* > 0$ a.e.

Let $T > 0$ be given and $\varphi \in \mathcal{D}((-T,T)\times B_T)$. Then multiplying the above equation by φ and integrating over $\mathbb{R}\times\mathbb{R}^N$ yields for any large enough n:

$$\begin{aligned}
&-\iint_{\mathbb{R}\times\mathbb{R}^N} \partial_t\varphi u^n v^n - \iint_{\mathbb{R}\times\mathbb{R}^N} \varphi\partial_t u^n v^n \mathrm{d}x\mathrm{d}t \\
&\quad + \iint_{\mathbb{R}\times\mathbb{R}^N} \nabla\varphi\cdot u^n\nabla v^n + \iint_{\mathbb{R}\times\mathbb{R}^N} \varphi\nabla u^n\cdot\nabla v^n \mathrm{d}x\mathrm{d}t \\
&= \iint_{\mathbb{R}\times\mathbb{R}^N} \varphi r_n^*(x) v^n u^n \mathrm{d}x\mathrm{d}t - \iint_{\mathbb{R}\times\mathbb{R}^N} r_n^*(x)\delta_n^*(x)\varphi\left(v^n\right)^2 \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.6}$$

However, since ∇v^n is bounded in $L^2((-T,T)\times B_T)$, one may assume, possibly along a subsequence, that

$$\nabla v^n \to \nabla v^\infty \text{ weakly in } L^2((-T,T)\times B_T).$$

Furthermore, due to the first step, since u^n is bounded in $W^{1,2;p}((-T,T)\times B_T)$ one may assume, possibly along a subsequence, that

$$\begin{aligned}
&\nabla u^n \to \nabla u^\infty = 0 \text{ uniformly on } [-T,T]\times\overline{B_T}, \\
&\partial_t u^n \to \partial_t u^\infty = 0 \text{ weakly in } L^2((-T,T)\times B_T).
\end{aligned}$$

Hence passing to the limit $n\to\infty$ in (3.6) it yields:

$$\iint_{\mathbb{R}\times\mathbb{R}^N} r^*(x)\delta^*(x)\varphi\left(v^\infty\right)^2 = 0.$$

Since this true for arbitrary test function $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^N)$ and since $r^* > 0$ $a.e.$ and $\delta^* \geq \delta_0 > 0$, we obtain that $v^\infty = 0$.

Now, using the uniform convergence of u^n as stated in (3.3) and the above arguments one can obtain the stronger convergence property:

$$\lim_{n\to\infty} \sup_{h\geq 0} \|v^n(h+.,.)\|_{L^2(Q_R)} = 0$$

for each $R > 0$. As a consequence of the above uniform converge with respect to the local L^2-norm, one can apply the Harnack inequality as stated in Lemma 2.4 to conclude that for each $R > 0$ one has

$$\lim_{n\to\infty} \sup_{h\geq 0} \sup_{(t,x)\in Q_R} v^n(t+h,x) = 0,$$

from which (3.5) follows.

Third step From (3.3) and (3.5), we shall reach a contradiction. For that purpose, we consider for each $R > 0$ and each continuous and bounded function ρ defined in $\mathbb{R}^N$ the principal Laplace Dirichlet eigenvalue $\lambda_R[\rho]$ on the ball B_R defined by

$$\begin{cases} \varphi \in C\left(\overline{B_R}\right) \cap C^2(B_R), \\ -\left(D\Delta + \rho(x)\right)\varphi = \lambda_R \varphi \text{ in } B_R, \\ \varphi = 0 \text{ on } \partial B_R \text{ and } \varphi > 0 \text{ in } B_R. \end{cases} \tag{3.7}$$

Recalling from the result of Agmon in [1] (see also [4,5]) that one has

$$\lim_{R\to\infty} \lambda_R[\rho] = \lambda_1(-D\Delta - \rho(x); \mathbb{R}^N).$$

Due to Assumption 1.1(iii), choose $R > 0$ large enough so that $\lambda_R[g^*(\cdot,0)] < 0$ (see Assumption 1.1(ii)). Now observe that

$$\lim_{n\to\infty} \lambda_R\left[g_n^*(\cdot,0)\right] = \lambda_R\left[g^*(\cdot,0)\right] < -\gamma/2.$$

Then fix $n_0 \geq 0$ large enough such that

$$\lambda_R\left[g_n^*(\cdot,0)\right] < -\gamma/4, \ \forall n \geq n_0.$$

Now let $\eta > 0$ small enough be given such that

$$\gamma/4 > 2\eta, \tag{3.8}$$

and choose $\varepsilon > 0$ small enough such that

$$g_n^*(x,u) \geq g_n^*(x,0) - \eta, \quad \forall u \in [0,\varepsilon], \quad \forall n \geq n_0.$$

(Here recall that the set $\mathcal{K}$ is uniformly equi-continuous). Next because of (3.3) and (3.5), let us fix $n \geq n_0$ large enough such that

$$g_n^*\left(x,u^n\right) - \Pi_n^*(x,u^n)v^n \geq g_n^*(x,0) - 2\eta \text{ for all } (t,x) \in [0,\infty) \times \overline{B_R}.$$

Hence, recalling that n is fixed, the function u^n satisfies

$$\partial_t u^n - D\Delta u^n \geq \left[g_n^*(x,0) - 2\eta\right] u^n, \quad t>0, \quad x \in B_R.$$

For notational simplicity, we write $\lambda_R = \lambda_R\left[g_n^*(\cdot,0)\right] < 0$. Let φ_R be the corresponding principal eigenfunction normalized by $\|\varphi_R\|_\infty = 1$ of the eigenvalue problem (3.7).

Consider now the function $\underline{u}(t,x) := \mathrm{e}^{(-2\eta-\lambda_R)t}\varphi_R(x)$ on $[0,\infty)\times B_R$ that satisfies

$$\begin{cases} \partial_t \underline{u} - D\Delta \underline{u} = \left[g_n^*(x,0) - 2\eta\right]\underline{u}, \text{ for } t>0, \ x \in B_R; \\ \underline{u} = 0 \ \text{ on } \ [0,\infty)\times \partial B_R. \end{cases}$$

Then, choosing $\kappa > 0$ small enough such that $\kappa \leq \min_{\overline{B_R}} u^n(0,\cdot)$, one obtains from the comparison principle that

$$\kappa \underline{u}(t,x) \leq u^n(t,x) \ \ \forall t>0, \ x \in B_R.$$

However, due to the choice of η in (3.8), one has $-2\eta - \lambda_R > 0$. Thus one obtains that $\lim_{t\to\infty} u^n(t,0) = \infty$, a contradiction with $u^n \leq M$. This completes the proof of Lemma 3.1. □

We are now able to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Let us argue by contradiction by assuming that there exist a sequence $(u_n, v_n) \in \mathcal{S}$ and $x_n \in \mathbb{R}^N$ such that for all $n \geq 1$ one has

$$\liminf_{t\to\infty} u_n(t,x_n) \leq \frac{1}{n}.$$

Using Lemma 3.1, there exist a sequence $t_n \to \infty$ and a sequence $l_n \geq 0$ such that for all n large one has

$$u_n(t_n,x_n) = \frac{\varepsilon}{2}, \quad u_n(t,x_n) \leq \frac{\varepsilon}{2}, \ \forall t \in [t_n, t_n + l_n], \quad u_n(t_n + l_n, x_n) = \frac{1}{n}.$$

Consider the sequence of functions (u^n, v^n) defined by

$$\left(u^n, v^n\right)(t,x) = \tau_{t_n}\cdot\sigma_{x_n}(u_n,v_n)(t,x) = (u_n,v_n)(t+t_n, x_n + x),$$

that is a solution on $(-t_n,\infty)$ of the problem ($\mathcal{P}_\Psi$) with $\Psi = \Psi_n$ defined by

$$\Psi_n = \left(g(\cdot + x_n,\cdot), \Pi(\cdot + x_n,\cdot), \sigma_{x_n} r, \sigma_{x_n}\delta\right).$$

Since $t_n \to \infty$, possibly along a subsequence, one has

$$\begin{aligned} &\left(u^n, v^n\right) \to \left(u^\infty, v^\infty\right) \text{ in } C_{\mathrm{loc}}\left(\mathbb{R}\times\mathbb{R}^N\right) \times L^2_{\mathrm{loc}}\left(\mathbb{R}\times\mathbb{R}^N\right), \\ &\Psi_n \to \Psi_\infty := \left(g^*, \Pi^*, r^*, \delta^*\right) \in \mathcal{K} \text{ locally uniformly}, \\ &l_n \to l := \liminf_{n\to\infty} l_n, \end{aligned}$$

and wherein the function $(u^\infty, v^\infty) \in \mathcal{X} \times L^\infty\left(\mathbb{R}\times\mathbb{R}^N\right)$ satisfies

$$(\partial_t - D\Delta)\, u^\infty = u^\infty\left(g^*\left(x, u^\infty\right) - \Pi^*\left(x, u^\infty\right) v^\infty\right),$$
$$u^\infty(0,0) = \frac{\varepsilon}{2} \ \text{ and } \ u^\infty(t,0) \le \frac{\varepsilon}{2}\ \forall t \in [0, l).$$

Let us further observe that $l = \infty$. Indeed, if $l < \infty$, then the function u^∞ satisfies in addition $u^\infty(l, 0) = 0$. Note that latter condition contradicts $u^\infty(0, 0) > 0$ and the comparison principle as stated in Lemma 2.3.

As a consequence of the above construction, $u^\infty > 0$ on $\mathbb{R}\times\mathbb{R}^N$ and satisfies

$$u^\infty(t,0) \le \frac{\varepsilon}{2}, \quad \forall t \ge 0,$$

while the function pair (u^∞, v^∞) (for positive time) belongs to $\mathcal{S}_{\Psi_\infty}$. This contradicts Lemma 3.1 and completes the proof of Theorem 1.4. □

4. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. To that aim, consider $(u, v) \in \mathcal{S}$, a solution of (1.1) with (1.3), with $v > 0$. Note that $M = \delta_0 = 1$ so that we have

$$0 < u < 1 \ \text{ and } \ 0 < v < 1.$$

Take a sequence $t_n \to \infty$ and assume, up to the extraction of a subsequence, that

$$\tau_{t_n}.(u, v)(t, x) = (u, v)(t + t_n, x) \to (u_\infty, v_\infty)(t, x)$$

locally uniformly for $(t, x) \in \mathbb{R}\times\mathbb{R}^N$.

To prove Theorem 1.5, it is sufficient to prove that

$$(u_\infty, v_\infty)\,(t,x) \equiv \left(u^*, v^*\right) := \left(\frac{1}{1+k}, \frac{1}{1+k}\right). \tag{4.1}$$

Let us firstly observe that, due to Theorem 1.4, one has, for all $(t, x) \in \mathbb{R}\times\mathbb{R}^N$, $u_\infty(t, x) \ge \varepsilon$ for some given $\varepsilon > 0$. Then we claim that the following holds true.

CLAIM 4.1. *One has*

$$v_\infty(t,x) \ge \varepsilon, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Proof. Let $\theta \in (0, 1/2)$ be given. Fix $R > 0$ large enough such that, for each continuous positive function w_0, the solution $w = w(t, x)$ of the problem

$$\begin{cases} \partial_t w - \Delta w = w\left(1 - \frac{w}{(1-\theta)\varepsilon}\right), \ t > 0, \ x \in B_R, \\ w(0,\cdot) = w_0, \ \text{ and } \ w(t,x) = 0, \ t > 0, \ x \in \partial B_R, \end{cases}$$

satisfies

$$\lim_{t\to\infty} w(t,0) \geq (1-2\theta)\,\varepsilon.$$

Now fix $h \in \mathbb{R}^N$ and observe that, due to Theorem 1.4, there exists $t_0 > 0$ large enough such that

$$u(t,h+x) \geq (1-\theta)\varepsilon,\ \forall t \geq t_0,\ x \in B_R.$$

Hence the function $v^h(t,x) := v(t,h+x)$ satisfies

$$\partial_t v^h - \Delta v^h \geq r v^h \left(1 - \frac{v^h}{(1-\theta)\varepsilon}\right),\ t > t_0,\ x \in B_R.$$

Since $v^h > 0$, one obtains from the comparison principal that

$$\liminf_{t\to\infty} v^h(t,0) = \liminf_{t\to\infty} v(t,h) \geq (1-2\theta)\varepsilon.$$

Hence we have proved that for each $\theta > 0$ small enough, and each $h \in \mathbb{R}^N$, one has

$$\liminf_{t\to\infty} v(t,h) \geq (1-2\theta)\varepsilon.$$

Letting $\theta \to 0$, Claim 4.1 follows. □

From this claim, we obtain that there exists $\eta > 0$ small enough such that

$$(u_\infty, v_\infty)(t,x) \in P := [\varepsilon, 1-\eta]^2,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{4.2}$$

while (u_∞, v_∞) is a bounded entire solution of (1.1) with (1.3). In addition, because of parabolic estimates, this function pair is C^∞ on $\mathbb{R}\times\mathbb{R}^N$ and has bounded derivatives of any order.

From now on, we split our arguments into two parts. We first complete the proof of Theorem 1.5(i) using a sandwiching argument and then we go to the proof of Theorem 1.5(ii) by developing Lyapunov like arguments.

Proof of Theorem 1.5(i). Here we prove Theorem 1.5(i). Since the result is already known for $k \in (0,1)$ we shall only focus on the proof of the case $k = 1$. To that aim we fix $k = 1$ and we set

$$u_{\min} = \inf_{\mathbb{R}\times\mathbb{R}^N} u_\infty,\ u_{\max} = \sup_{\mathbb{R}\times\mathbb{R}^N} u_\infty,\ v_{\min} = \inf_{\mathbb{R}\times\mathbb{R}^N} v_\infty \text{ and } v_{\max} = \sup_{\mathbb{R}\times\mathbb{R}^N} v_\infty$$

Here recall that $(u_\infty, v_\infty)(t,x) \in P$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ so that

$$\varepsilon \leq u_{\min},\ v_{\min} \text{ and } u_{\max},\ v_{\max} < 1.$$

Next observe that

$$(\partial_t - \Delta)v_\infty = r v_\infty \left(1 - \frac{v_\infty}{u_\infty}\right) \geq r v_\infty \left(1 - \frac{v_\infty}{u_{\min}}\right) \text{ on } \mathbb{R}\times\mathbb{R}^N,$$

so that one deduces that $v_\infty \geq u_{\min}$ on $\mathbb{R} \times \mathbb{R}^N$. A similar argument also provides $v_\infty \leq u_{\max}$ on $\mathbb{R} \times \mathbb{R}^N$. This re-writes as

$$u_{\min} \leq v_{\min} \quad \text{and} \quad v_{\max} \leq u_{\max} \tag{4.3}$$

Now from the u-equation one gets

$$(\partial_t - D\Delta)u_\infty \leq u_\infty(1 - u_\infty - v_{\min}) \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^N.$$

Hence one gets

$$u_\infty \leq 1 - v_{\min} \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^N \quad \text{and} \quad u_{\max} \leq 1 - v_{\min}. \tag{4.4}$$

Now also observe that

$$(\partial_t - D\Delta)u_\infty \geq u_\infty(1 - u_\infty - v_{\max}) \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^N.$$

This yields $u_\infty \geq 1 - v_{\max}$ on $\mathbb{R} \times \mathbb{R}^N$ and $u_{\min} + v_{\max} \geq 1$. Hence we infer from (4.3) and (4.4) that

$$1 \geq u_{\min} + u_{\max} \geq 1 \;\Rightarrow\; u_{\min} + u_{\max} = 1.$$

Next, due to (4.4), one has $v_{\min} \leq 1 - u_{\max} = u_{\min}$, and since $u_{\min} \leq v_{\min}$ (see (4.3)), one concludes that $v_{\min} = u_{\min}$. On the other hand, coupling $u_{\min} \geq 1 - v_{\max}$ and $v_{\max} \leq u_{\max}$, yields $u_{\max} = v_{\max}$.

As a consequence, take a sequence $(t_n, x_n) \in \mathbb{R} \to \mathbb{R}^N$ such that $v(t_n, x_n) \to v_{\min}$ and let us show that $(u_n, v_n)(t, x) := (u, v)(t + t_n, x + x_n)$ converge locally uniformly up to a subsequence to $(u_{\min}, v_{\min})$. To that aim observe that the limit function (up to a subsequence) denoted by $(U, V)(t, x)$ satisfies $V(0, 0) = \inf_{\mathbb{R}\times\mathbb{R}^N} V = v_{\min}$ and the function $W := V - v_{\min} \geq 0$ is a entire solution of

$$(\partial_t - \Delta)W - r\,(W + v_{\min})\left[1 - \frac{W + v_{\min}}{U}\right] = 0 \text{ with } W \geq 0 \text{ and } W(0, 0) = 0$$

Hence, since $u_{\min} = v_{\min}$, we obtain

$$(\partial_t - \Delta)W - r\,(W + v_{\min})\,W/v_{\min} \geq 0 \quad \text{with} \quad W \geq 0 \quad \text{and} \quad W(0, 0) = 0.$$

This implies that $W \equiv 0$, namely $V \equiv v_{\min}$. Hence, we deduce from the above W-equation that $U \equiv v_{\min}$. Finally, form the U-equation, we get

$$1 - 2v_{\min} = 0, \quad \text{i.e.,} \quad v_{\min} = \frac{1}{2}.$$

Therefore, since $v_{\max} = 1 - v_{\min}$ and $u_{\max} = 1 - v_{\min}$ one obtains that $v_{\min} = v_{\max} = 1/2$ and $u_{\max} = u_{\min} = 1/2$. □

Now, we assume that $N = 2$. In order to prove (4.1), let $\rho:\mathbb{R} \to \mathbb{R}_+$ be a C^∞ compactly supported function such that

$$\rho(-z) = \rho(z),\ \forall z \in \mathbb{R},$$
$$\rho(z) = 1 \text{ if } z \in [0, 1],\quad \rho(z) = 0 \ \text{ if } \ |z| \geq 2.$$

Then set the function $\pi:\mathbb{R}^2 \to \mathbb{R}_+$ defined by $\pi = \rho^{\otimes 2}$, that is

$$\pi(x) = \rho(x_1)\rho(x_2) \ \text{ for } \ x = (x_1, x_2) \in \mathbb{R}^2.$$

Consider also the function $V : (0, \infty)^2 \to \mathbb{R}_+$ defined by

$$V(u, v) = \int_{u^*}^{u} \frac{\xi^2 - (u^*)^2}{\xi^2} \mathrm{d}\xi + c \int_{v^*}^{v} \frac{\eta - v^*}{\eta} \mathrm{d}\eta,$$

where $c > 0$ is a parameter that will be specified later. This function has been proposed by Du and Hsu in [6], as a suitable Lyapunov function, to prove the global stability of the unique positive equilibrium for the reaction–diffusion system (1.1) with (1.3) posed on a bounded domain with Neumann boundary conditions. Here we make use of the function V to prove (4.1) for the case $N = 2$.

Before going to this argument, let us first give some preparation computations. First, note that the function V defined above can be re-written as follows:

$$\begin{cases} V(u, v) = V_1(u) + cV_2(v), \\ \text{with } V_1(u) = (u - u^*)^2/u,\ \ V_2(v) = v^* G\left(v/v^*\right) \text{ and } G(x) = x - \ln x - 1. \end{cases}$$

Hence $V \geq 0$ on the set P, if we choose $\eta > 0$ in (4.2) sufficiently small. Moreover, there exists $\kappa \in (0, 1)$ small enough such that

$$\kappa \left(|u - u^*|^2 + |v - v^*|^2\right) \leq V(u, v) \leq \kappa^{-1} \left(|u - u^*|^2 + |v - v^*|^2\right),\ \ \forall (u, v) \in P.$$

Now define the vector field $F : P \to \mathbb{R}^2$ by

$$F(u, v) := (u(1 - u - kv), rv(1 - v/u)).$$

Then we have

CLAIM 4.2. *If $k \leq s_0^{-1}$ (see Theorem 1.5(ii) for the definition of s_0), then there exist two constants $c > 0$ and $\alpha > 0$ such that*

$$\nabla V(u, v) \cdot F(u, v) \leq -\alpha V(u, v)\ \forall (u, v) \in P.$$

Moreover, $V_1'' \geq 0$ and $V_2'' \geq 0$ on $[\varepsilon, 1 - \eta]$.

The proof of this computational claim follows from the arguments and computations developed by Du and Hsu in [6] in proving their global stability result (see [6, Theorem 2.2]).

We are now able to complete the proof of Theorem 1.5(ii) as follows.

Proof of Theorem 1.5(ii). To that aim let us consider for each $R > 0$ the function $t \mapsto \mathcal{E}_R(t)$ defined by

$$\mathcal{E}_R(t) = \int_{\mathbb{R}^2} \pi\left(\frac{x}{R}\right) V\left(u_\infty, v_\infty\right)(t, x)\mathrm{d}x.$$

We compute the time derivative of $\mathcal{E}_R$ as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_R(t) &= \int_{\mathbb{R}^2} \pi\left(\frac{x}{R}\right) \nabla V\left(u_\infty, v_\infty\right) \cdot (\partial_t u_\infty, \partial_t v_\infty)\mathrm{d}x \\ &= \int_{\mathbb{R}^2} \pi\left(\frac{x}{R}\right) \left[V_1'\left(u_\infty\right) D\Delta u_\infty + cV_2'\left(v_\infty\right) \Delta v_\infty\right] \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2} \pi\left(\frac{x}{R}\right) \nabla V\left(u_\infty, v_\infty\right) \cdot F\left(u_\infty, v_\infty\right) \mathrm{d}x.\end{aligned}$$

Hence, using Claim 4.2, one obtains that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_R(t) \leq -\alpha\mathcal{E}_R(t) + I_R^1(t) + I_R^2(t), \quad \forall t \in \mathbb{R},$$

where

$$\begin{aligned}I_R^1(t) &:= D\int_{\mathbb{R}^2} \pi\left(\frac{x}{R}\right) V_1'\left(u_\infty\right) \Delta u_\infty \mathrm{d}x, \\ I_R^2(t) &:= c\int_{\mathbb{R}^2} \pi\left(\frac{x}{R}\right) V_2'\left(v_\infty\right) \Delta v_\infty \mathrm{d}x.\end{aligned}$$

Now observe that for all $t \in \mathbb{R}$ one gets

$$\begin{aligned}I_R^1(t) &= -\frac{D}{R}\int_{\mathbb{R}^2} \nabla\pi\left(\frac{x}{R}\right) \cdot \nabla\left(V_1\left(u_\infty\right)\right) \mathrm{d}x - D\int_{\mathbb{R}^2} \pi\left(\frac{x}{R}\right) V_1''\left(u_\infty\right) |\nabla u_\infty|^2 \mathrm{d}x \\ &= \frac{D}{R^2}\int_{\mathbb{R}^2} \Delta\pi\left(\frac{x}{R}\right) V_1\left(u_\infty\right) \mathrm{d}x - \int_{\mathbb{R}^2} \pi\left(\frac{x}{R}\right) V_1''\left(u_\infty\right) |\nabla u_\infty|^2 \mathrm{d}x.\end{aligned}$$

As a consequence of the boundedness of u_∞ and recalling that $V_1'' \geq 0$, one obtains

$$I_R^1(t) \leq K_1 \text{ for all } t \in \mathbb{R}$$

for some positive constant K_1 independent of $R > 0$. Using the same computation arguments, one also obtains that there exists some constant $K_2 > 0$ independent of $R > 0$ such that

$$I_R^2(t) \leq K_2 \text{ for all } t \in \mathbb{R}.$$

Thus, setting $K = K_1 + K_2$, one deduces that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_R(t) \leq -\alpha\mathcal{E}_R(t) + K,\ \forall t \in \mathbb{R}.$$

The above differential inequality implies that

$$\mathcal{E}_R(t) \leq \frac{K}{\alpha},\ \forall\ t \in \mathbb{R},\ R > 0,$$

and, by letting $R \to \infty$, it ensures that there exists some positive constant, still denoted by K, such that

$$\int_{\mathbb{R}^2} \left|u_\infty(t,x) - u^*\right|^2 \mathrm{d}x + \int_{\mathbb{R}^2} \left|v_\infty(t,x) - v^*\right|^2 \mathrm{d}x \le K, \ \forall t \in \mathbb{R}.$$

Finally, using the above integrability property the function $\mathcal{E}(t) := \lim_{R\to\infty} \mathcal{E}_R(t)$ is well defined and turns out to be a bounded and decreasing energy functional along the entire solution (u_∞, v_∞). More precisely, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + D\int_{\mathbb{R}^2} V_1''(u_\infty)\,|\nabla u_\infty|^2\,\mathrm{d}x + c\int_{\mathbb{R}^2} V_2''(v_\infty)\,|\nabla v_\infty|^2\,\mathrm{d}x \le -\alpha\mathcal{E}(t).$$

Standard argument allows us to conclude that $(u_\infty, v_\infty)(t,x) \equiv (u^*, v^*)$. This concludes the proof of (4.1) and thus the proof of Theorem 1.5(ii). □

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