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Higher-order nonlinear Schrödinger equations with singular data

NAKAO HAYASHI, PAVEL I. NAUMKIN AND TAKAYOSHI OGAWA

Abstract. We consider the Cauchy problem for the higher-order nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \frac{1}{2k}\left(-\partial_x^2\right)^k u = \lambda \left|u\right|^{2p} u, \ \ (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}, \end{cases}$$

where $k, p \in \mathbb{N}$, $k \geq 2$, $\lambda \in \mathbb{C}$. We prove local existence of solutions for the case of singular initial data $u_0(x)$ including the Dirac delta function.

1. Introduction

We consider the Cauchy problem for the higher-order nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u - \frac{1}{2k}\left(-\partial_x^2\right)^k u = \lambda \left|u\right|^{2p} u, \ \ (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $k, p \in \mathbb{N}$, $k \geq 2$, $\lambda \in \mathbb{C}$. We are interested in the case of singular initial data $u_0(x)$ including the Dirac delta function. When $k = 1$, then (1.1) converts the well-known nonlinear Schrödinger equation. Equation (1.1) with $k = 2$ appears in the description of deep water waves [3], in the study of the influence of the higher-order dispersion on the propagation of intense laser beams in a bulk medium with Kerr nonlinearity [8,9] and also for the motion of a vortex filament in an incompressible fluid [5]. We consider the problem with the data which are not in $\mathbf{L}^2$. In the case of $k = 1$, namely for

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda \left|u\right|^{2p} u, \ \ (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

there are some works in which (1.2) was considered with the data which are not in $\mathbf{L}^2$. In [15], local well-posedness was studied of (1.2) with cubic nonlinearity $p = 1, n = 1$ and initial data $u_0 \in \mathbf{L}^q$, $1 < q < 2$. Local well-posedness with a critical

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nonlinearity $p = \frac{1}{n}$ for $n = 1, 2$ was shown in [13] in the homogeneous Sobolev spaces $u_0 \in \overset{\circ}{\mathbf{H}}{}^{0,\alpha} \cap \overset{\circ}{\mathbf{H}}{}^{0,\beta}$, $0 \le \alpha < \frac{n}{2} < \beta \le n$, where

$$\overset{\circ}{\mathbf{H}}{}^{0,\alpha} = \left\{ \varphi \in \mathbf{S}'; \|\varphi\|_{\overset{\circ}{\mathbf{H}}{}^{0,\alpha}} = \left\| |x|^{\alpha} \varphi \right\|_{\mathbf{L}^2} < \infty \right\}.$$

Global existence of small solutions of (1.2) with $\operatorname{Im}\lambda \le 0$ was also obtained in [13]. We note that $\mathbf{L}^1$ and $\overset{\circ}{\mathbf{H}}{}^{0,\frac{1}{2}}$ are scaling invariant for (1.2) when $n = 1$ and $\overset{\circ}{\mathbf{H}}{}^{0,1}$ is scaling invariant when $n = 2$. Therefore function spaces used in [15], [13] are related to the scaling invariant spaces. The initial value problems for systems of nonlinear Schrödinger equations including the system

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \overline{u}v, \\ i\partial_t u + \frac{1}{2}\Delta v = u^2, \end{cases}$$

were considered in [12,14] for the space dimension $n = 2$ in function spaces related to invariant spaces by the similar ideas as those in [13,15], respectively. Furthermore in [14], the case of $\mathbf{L}^1$ data is treated for the above system for $n = 1$ by using the method of [15]. In the previous works [12–15], proofs depend on the gauge-invariant nonlinearities and the property of Schrödinger evolution group which was represented by the formula

$$\mathcal{F}\mathcal{U}_s(-t)|u|^{2p}u = t^{-\frac{p}{n}}\mathcal{F}M^{-1}\mathcal{F}^{-1}|\mathcal{F}M\mathcal{U}_s(-t)u|^{2p}\mathcal{F}M\mathcal{U}_s(-t)u,$$

where $\mathcal{U}_s(t) = e^{i\frac{1}{2}t\Delta}$ and $M = e^{\frac{1}{2}it|x|^2}$. Higher-order dispersive equations do not have this property. On the other hand, the kernel $e^{-it\frac{1}{2k}\left(-\partial_x^2\right)^k}1$ for $k > 1$ has a space decay property, which comes from the dispersivity. The order of space decay of the kernel becomes large when k becomes large, and it will be used to obtain the desired result below. This property differs from the usual Schrödinger evolution group.

There are a lot of works concerning quadratic nonlinear Schrödinger equations (1.2) with $|u|\,u$ replaced by u^2 for $n = 1, 2$ in papers [2,11] for $n = 1$, and [1,11] for $n = 2$ on time local well-posedness and ill-posedness for $\mathbf{H}^s$ with negative s. Time local well-posedness for $\mathbf{H}^s$ for $s \ge -1$ if $n = 1$ and $s > -1$ if $n = 2$, and ill-posedness for $s < -1$ if $n = 1$ and $s \le -1$ if $n = 2$ were shown in [1,2], respectively. In paper [7], time local ill-posedness was improved in the scale invariant Besov space $\mathbf{B}^{-1}_{2,\sigma} \supset \mathbf{H}^{-1}$ for $2 < \sigma \le \infty$ if $n = 1$ and $\mathbf{H}^{-1}$ for $n = 2$.

If we take the scaled function $u_\mu(t) = \mu^{\frac{k}{p}} u\left(\mu^{2k}t, \mu x\right)$ with $\mu > 0$, then $u_\mu(t)$ satisfies equations (1.1) with the scaled initial data $u_{0,\mu}(x) = \mu^{\frac{k}{p}} u_0(\mu x)$. Therefore equation (1.1) with $p = 2k$ is called $\mathbf{L}^2$ critical, since $\left\|u_{0,\mu}\right\|_{\mathbf{L}^2} = \|u_0\|_{\mathbf{L}^2}$, and (1.1) with $p = k$ is called $\mathbf{L}^1$ or $\mathcal{F}\mathbf{L}^\infty$ critical since $\left\|u_{0,\mu}\right\|_{\mathbf{L}^1} = \|u_0\|_{\mathbf{L}^1}$ or $\left\|\mathcal{F}u_{0,\mu}\right\|_{\mathbf{L}^\infty} = \|\mathcal{F}u_0\|_{\mathbf{L}^\infty}$.

We now introduce *notation and function spaces*used in this paper. We denote the Lebesgue space by $\mathbf{L}^p = \left\{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\right\}$, where the norm $\|\phi\|_{\mathbf{L}^p} =$

$\left(\int |\phi(x)|^p \, dx\right)^{\frac{1}{p}}$ for $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbb{R}} |\phi(x)|$. The homogeneous Sobolev space is defined by

$$\overset{\circ}{\mathbf{H}}^{\alpha,1} = \left\{ \varphi \in \mathbf{S}'; \|\varphi\|_{\overset{\circ}{\mathbf{H}}^{\alpha,1}} = \left\| |\xi|^\alpha \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} < \infty \right\},$$

where $\mathcal{F}$ or $\widehat{\cdot}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ denotes its inverse one. Since

$$\widehat{u}_{0,\mu}(\xi) = \frac{1}{\sqrt{2\pi}} \int \mu^{\frac{k}{p}} u_0(\mu x) e^{-ix\xi} dx = \mu^{\frac{k}{p}-1} \widehat{u}_0\left(\frac{\xi}{\mu}\right)$$

by a direct computation, we find

$$\begin{aligned} \left\| |\xi|^\alpha \partial_\xi \widehat{u}_{0,\mu} \right\|_{\mathbf{L}^2}^2 &= \mu^{2\left(\frac{k}{p}-1\right)-2} \int \left| \left(\partial_\xi \widehat{u}_0\right)\left(\frac{\xi}{\mu}\right) \right|^2 |\xi|^{2\alpha} d\xi \\ &= \mu^{2\left(\frac{k}{p}-\frac{3}{2}+\alpha\right)} \int \left| \left(\partial_\xi \widehat{u}_0\right)(y) \right|^2 |y|^{2\alpha} dy = \left\| |\xi|^\alpha \partial_\xi \widehat{u}_0 \right\|_{\mathbf{L}^2}^2 \end{aligned}$$

if $\alpha = \frac{3}{2} - \frac{k}{p}$. Hence $\overset{\circ}{\mathbf{H}}^{\frac{3}{2}-\frac{k}{p},1}$ can be considered as a scaling invariant space. We now define the function space $\mathbf{X}^\alpha$ used in this paper as

$$\mathbf{X}^\alpha = \left\{ \varphi \in \mathbf{S}'; \|\varphi\|_{\mathbf{X}^\alpha} = \|\varphi\|_{\overset{\circ}{\mathbf{H}}^{\alpha,1}} + \|\widehat{\varphi}\|_{\mathbf{L}^\infty} < \infty \right\}.$$

Note that $\mathbf{X}^\alpha$ includes the Dirac delta function δ since its Fourier transform is equal to 1. Also we note that if $\alpha = \frac{1}{2}$ and $k = p$, then $\mathbf{X}^\alpha$ is a scaling invariant space. Let $\mathbf{C}(\mathbb{I}; \mathbf{B})$ be the space of continuous functions from an interval $\mathbb{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C.

To state our result, we define the space

$$\mathbf{Y}_T = \left\{ \mathcal{U}(-t) v \in \mathbf{C}\left([0,T]; \mathbf{X}^\alpha\right); \|v\|_{\mathbf{Y}_T} < \infty \right\}$$

with a norm

$$\|v\|_{\mathbf{Y}_T} = \sup_{t\in[0,T]} \|\varphi(t)\|_{\mathbf{X}^\alpha} = \sup_{t\in[0,T]} \left\| |\xi|^\alpha \partial_\xi \widehat{\varphi}(t) \right\|_{\mathbf{L}^2} + \sup_{t\in[0,T]} \|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty},$$

where $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t) v$, $\mathcal{U}(t) = e^{-it\frac{1}{2k}\left(-\partial_x^2\right)^k}$. We also define the closed ball $\mathbf{Y}_{T,\rho} = \left\{ v \in \mathbf{Y}_T; \|v\|_{\mathbf{Y}_T} \le \rho \right\}$. We introduce the operator $\mathcal{J} = x + it\partial_x \left(-\partial_x^2\right)^{k-1} = \mathcal{U}(t) x \mathcal{U}(-t)$, which commutes with the linear part $i\partial_t - \frac{1}{2k}\left(-\partial_x^2\right)^k$ of equation (1.1).

We prove the following result.

THEOREM 1.1. *Assume that* $u_0 \in \mathbf{X}^\alpha$ *with* $0 \le \alpha < \frac{1}{2}$ *and*

$$\frac{4k-1+2\alpha}{4(k-1)} < p < \frac{4k-1+2\alpha}{2(3-2\alpha)}. \tag{1.3}$$

Then there exists a time T such that the higher-order nonlinear Schrödinger equation (1.1) *has a unique solution $u \in \mathbf{Y}_T$. Moreover the following estimates are true*

$$\left\| \partial_x^j u(t) \right\|_{\mathbf{L}^\infty} \leq C t^{-\frac{2j+3-2\alpha}{4k}} \|u_0\|_{\mathbf{X}^\alpha} \tag{1.4}$$

for $0 < t \leq T$, $0 \leq j \leq k-1$ *and*

$$\left\| \partial_x^j u(t) \right\|_{\mathbf{L}^\infty(-R,R)} \leq C t^{-\frac{2j+3-2\alpha}{4k} - \frac{j+1-k}{2k(2k-1)}} R^{\frac{1+j-k}{2k-1}} \|u_0\|_{\mathbf{X}^\alpha} \tag{1.5}$$

for $0 < t \leq T$, $k \leq j \leq 2k-2$, $R > 0$.

REMARK 1.1. *Estimate (1.4) describes a global smoothing property of solutions and estimate (1.5) is concerned with a local smoothing property.*

REMARK 1.2. *We can see that our proof is also valid for fractional powers p such that $p \geq k-1$. Consider the following example $k = 3$, then by condition (1.3) we have $\frac{11+2\alpha}{8} < p < \frac{11+2\alpha}{2(3-2\alpha)}$. Therefore the higher-order Schrödinger equation*

$$i\partial_t u + \frac{1}{6}\partial_x^6 u = \lambda |u|^{2p} u$$

is acceptable in $\mathbf{X}^\alpha$, which is closely related to a scaling invariant space since $\mathbf{X}^{\frac{1}{2}}$ is the invariant space for $p = 3$.

We now introduce the factorization formulas for the higher-order nonlinear Schrödinger equation $i\partial_t u - \frac{1}{2k}\left(-\partial_x^2\right)^k u = 0$. We define the free evolution group

$$\mathcal{U}(t) = e^{-\frac{it}{2k}\left(-\partial_x^2\right)^k} = \mathcal{F}^{-1} E \mathcal{F},$$

where the multiplication factor $E(t,\xi) = e^{-\frac{it}{2k}\xi^{2k}}$. Then we write

$$\begin{aligned}
\mathcal{U}(t)\mathcal{F}^{-1}\phi &= \mathcal{F}^{-1}E\phi = \frac{1}{\sqrt{2\pi}}\int e^{ix\xi - \frac{it}{2k}\xi^{2k}}\phi(\xi)\,\mathrm{d}\xi \\
&= \frac{t^{\frac{1}{2k}}}{\sqrt{2\pi}}\mathcal{D}_t \int e^{it\left(x\xi - \frac{1}{2k}\xi^{2k}\right)}\phi(\xi)\,\mathrm{d}\xi = \mathcal{D}_t\mathcal{B}\frac{t^{\frac{1}{2k}}}{\sqrt{2\pi}}\int e^{it\left(x^{2k-1}\xi - \frac{1}{2k}\xi^{2k}\right)}\phi(\xi)\,\mathrm{d}\xi \\
&= \mathcal{D}_t\mathcal{B}M\frac{t^{\frac{1}{2k}}}{\sqrt{2\pi}}\int e^{-itS(x,\xi)}\phi(\xi)\,\mathrm{d}\xi = \mathcal{D}_t\mathcal{B}M\mathcal{V}\phi,
\end{aligned}$$

where the phase function $S(x,\xi) = \frac{2k-1}{2k}x^{2k} - x^{2k-1}\xi + \frac{1}{2k}\xi^{2k}$, $M(t,x) = e^{\frac{2k-1}{2k}itx^{2k}}$, the dilation operator $\mathcal{D}_t\phi = t^{-\frac{1}{2k}}\phi\left(xt^{-1}\right)$ and the operator

$$\mathcal{V}\phi = \frac{t^{\frac{1}{2k}}}{\sqrt{2\pi}}\int e^{-itS(x,\xi)}\phi(\xi)\,\mathrm{d}\xi$$

and we introduce the scaling operator $(\mathcal{B}\phi)(x) = \phi\left(x^{\frac{1}{2k-1}}\right)$ with the definition as $x^{\frac{1}{2k-1}} = x|x|^{-\frac{2k-2}{2k-1}}$. If we define the new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$, then we obtain

$$u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{D}_t\mathcal{B}M\mathcal{V}\widehat{\varphi}. \tag{1.6}$$

We also need the representation for the inverse evolution group $\mathcal{F}\mathcal{U}(-t)$

$$\begin{aligned}
\mathcal{F}\mathcal{U}(-t)\phi = \overline{E}\mathcal{F}\phi &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}x e^{\frac{it}{2k}\xi^{2k}-ix\xi}\phi(x)\\
&= \frac{t^{\frac{2k-1}{2k}}}{\sqrt{2\pi}}\int \mathrm{d}x e^{it\left(\frac{1}{2k}\xi^{2k}-x\xi\right)}\mathcal{D}_t^{-1}\phi(x)\\
&= \frac{(2k-1)t^{\frac{2k-1}{2k}}}{\sqrt{2\pi}}\int \mathrm{d}x x^{2k-2}e^{it\left(\frac{1}{2k}\xi^{2k}-x^{2k-1}\xi\right)}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi\\
&= \frac{(2k-1)t^{\frac{2k-1}{2k}}}{\sqrt{2\pi}}\int \mathrm{d}x x^{2k-2}e^{itS(x,\xi)}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi\\
&= \mathcal{V}^*\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi,
\end{aligned}$$

where $\mathcal{D}_t^{-1}\phi = t^{\frac{1}{2k}}\phi(xt)$, $\left(\mathcal{B}^{-1}\phi\right)(x) = \phi\left(x^{2k-1}\right)$ and the operator

$$\mathcal{V}^*\phi = \frac{(2k-1)t^{\frac{2k-1}{2k}}}{\sqrt{2\pi}}\int e^{itS(x,\xi)}\phi(x)x^{2k-2}\mathrm{d}x.$$

Next applying the operator $\mathcal{F}\mathcal{U}(-t)$ to Eq. (1.1) and using $\mathcal{F}\mathcal{U}(-t)\mathcal{L} = i\partial_t\mathcal{F}\mathcal{U}(-t)$, $\mathcal{L} = i\partial_t - \frac{1}{2k}\left(-\partial_x^2\right)^k$ and $u(t) = \mathcal{D}_t\mathcal{B}M\mathcal{V}\widehat{\varphi}$, we get

$$\begin{aligned}
i\partial_t\widehat{\varphi} &= \mathcal{F}\mathcal{U}(-t)\mathcal{L}u = \lambda\mathcal{F}\mathcal{U}(-t)\left(|u|^{2p}u\right)\\
&= \lambda\mathcal{V}^*\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(|\mathcal{D}_t\mathcal{B}M\mathcal{V}\widehat{\varphi}|^{2p}\mathcal{D}_t\mathcal{B}M\mathcal{V}\widehat{\varphi}\right)\\
&= \lambda t^{-\frac{p}{k}}\mathcal{V}^*\overline{M}\mathcal{B}^{-1}\left(|\mathcal{B}M\mathcal{V}\widehat{\varphi}|^{2p}\mathcal{B}M\mathcal{V}\widehat{\varphi}\right)\\
&= \lambda t^{-\frac{p}{k}}\mathcal{V}^*\overline{M}\left(|M\mathcal{V}\widehat{\varphi}|^{2p}M\mathcal{V}\widehat{\varphi}\right) = \lambda t^{-\frac{p}{k}}\mathcal{V}^*\left(|\mathcal{V}\widehat{\varphi}|^{2p}\mathcal{V}\widehat{\varphi}\right).
\end{aligned}$$

Thus we obtain the following equation

$$i\partial_t\widehat{\varphi} = \lambda t^{-\frac{p}{k}}\mathcal{V}^*\left(|\mathcal{V}\widehat{\varphi}|^{2p}\mathcal{V}\widehat{\varphi}\right) \tag{1.7}$$

for the new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$.

2. Preliminaries

Define the kernel

$$A_j(x) = \frac{1}{\sqrt{2\pi}}\int e^{-iS(x,\xi)}\xi^j\chi\left(\xi x^{-1}\right)\mathrm{d}\xi$$

for $j = 0, 1, 2, \ldots$ where the phase $S(x,\xi) = \frac{2k-1}{2k}x^{2k} - x^{2k-1}\xi + \frac{1}{2k}\xi^{2k}$, and the cutoff function $\chi(z) \in \mathbf{C}^2(\mathbb{R})$ is such that $\chi(z) = 0$ for $z \le \frac{1}{3}$ or $z \ge 3$ and $\chi(z) = 1$ for $\frac{2}{3} \le z \le \frac{3}{2}$. To compute the asymptotics of the kernel $A_j(x)$ for large x, we apply the stationary phase method (see [6], p. 163)

$$\int e^{irg(\eta)} f(\eta)\,\mathrm{d}\eta = e^{irg(\eta_0)} f(\eta_0)\sqrt{\frac{2\pi}{r\,|g''(\eta_0)|}}e^{i\frac{\pi}{4}\operatorname{sgn}g''(\eta_0)} + O\left(r^{-\frac{3}{2}}\right) \tag{2.1}$$

for $r \to +\infty$, where the stationary point η_0 is defined by $g'(\eta_0) = 0$. We change $\xi = x\eta$, and then, we get

$$A_j(x) = \frac{x^{1+j}}{\sqrt{2\pi}}\int e^{-ix^{2k}S(1,\eta)}\eta^j\chi(\eta)\,\mathrm{d}\eta.$$

By virtue of formula (2.1) with $r = x^{2k}$, $g(\eta) = -S(1,\eta) = -\left(\frac{2k-1}{2k} - \eta + \frac{1}{2k}\eta^{2k}\right)$, $f(\eta) = \eta^j\chi(\eta)$, $\eta_0 = 1$, we get

$$\begin{aligned} A_j(x) &= \frac{x^{1+j}}{\sqrt{2\pi}}\left(\sqrt{\frac{2\pi}{(2k-1)\,x^{2k}}}e^{-i\frac{\pi}{4}} + O\left(x^{-3k}\right)\right) \\ &= \frac{x^{j+1-k}}{\sqrt{2k-1}}e^{-i\frac{\pi}{4}} + O\left(\langle x\rangle^{j+1-3k}\right) \end{aligned}$$

for $x \to \infty$. In particular, we have the estimates $\left\|\langle x\rangle^{k-1-j}A_j\right\|_{\mathbf{L}^\infty} \le C$ for $j \ge 0$.

In the next lemma we obtain the estimates for the operator $\mathcal{V}$ in the uniform norm.

LEMMA 2.1. *Let* $0 \le j < 2k - \frac{3}{2} + \alpha$, $0 \le \alpha < 1$. *Then the estimate*

$$\begin{aligned} &\left\|\left\langle xt^{\frac{1}{2k}}\right\rangle^{\frac{3}{2}k+\alpha-\frac{3}{2}-j}\left(\left(\mathcal{V}\xi^j\phi\right)(x) - t^{-\frac{j}{2k}}A_j\left(xt^{\frac{1}{2k}}\right)\phi(x)\right)\right\|_{\mathbf{L}^\infty} \\ &\le Ct^{-\frac{j}{2k}}\left(\|\phi\|_{\mathbf{L}^\infty} + t^{-\frac{1-2\alpha}{4k}}\left\||\xi|^\alpha\partial_\xi\phi\right\|_{\mathbf{L}^2}\right) \end{aligned}$$

is valid for all $t > 0$.

Proof. We write

$$\begin{aligned} &\left(\mathcal{V}\xi^j\phi(\xi)\right)(x) - t^{-\frac{j}{2k}}A_j\left(xt^{\frac{1}{2k}}\right)\phi(x) \\ &= \frac{t^{\frac{1}{2k}}}{\sqrt{2\pi}}\int e^{-itS(x,\xi)}(\phi(\xi) - \phi(x))\,\chi\left(\xi x^{-1}\right)\xi^j\mathrm{d}\xi \\ &\quad + \frac{t^{\frac{1}{2k}}}{\sqrt{2\pi}}\int e^{-itS(x,\xi)}\phi(\xi)\left(1 - \chi\left(\xi x^{-1}\right)\right)\xi^j\mathrm{d}\xi = I_1(t,x) + I_2(t,x). \end{aligned}$$

In the first integral $I_1(t,x)$, we integrate by parts via the identity

$$e^{-itS(x,\xi)} = H_1\partial_\xi\left((\xi - x)\,e^{-itS(x,\xi)}\right) \tag{2.2}$$

with $H_1 = \left(1 - it\left(\xi - x\right)\left(\xi^{2k-1} - x^{2k-1}\right)\right)^{-1}$ to get

$$I_1 = Ct^{\frac{1}{2k}} \int e^{-itS(x,\xi)} \left(\phi\left(\xi\right) - \phi\left(x\right)\right)\left(\xi - x\right) \partial_\xi \left(H_1 \chi\left(\xi x^{-1}\right) \xi^j\right) \mathrm{d}\xi$$
$$+ Ct^{\frac{1}{2k}} \int e^{-itS(x,\xi)} \left(\xi - x\right) H_1 \chi\left(\xi x^{-1}\right) \xi^j \phi_\xi\left(\xi\right) \mathrm{d}\xi.$$

Using the estimates

$$\left|\phi\left(\xi\right) - \phi\left(x\right)\right| = \left|\int_x^\xi \phi'\left(y\right) \mathrm{d}y\right| \le C\left|x\right|^{-\alpha} \left|\xi - x\right|^{\frac{1}{2}} \left\| \left|\xi\right|^\alpha \partial_\xi \phi \right\|_{\mathbf{L}^2},$$
$$\left|\left(\xi - x\right) H_1 \chi\left(\xi x^{-1}\right) \xi^j\right| \le \frac{C\left|\xi - x\right|\left|x\right|^j}{1 + tx^{2k-2}\left(\xi - x\right)^2}$$

and

$$\left|\left(\xi - x\right) \partial_\xi \left(H_1 \chi\left(\xi x^{-1}\right) \xi^j\right)\right| \le \frac{C\left|x\right|^j}{1 + tx^{2k-2}\left(\xi - x\right)^2}$$

in the domain $0 < \frac{1}{3}x \le \xi \le 3x$ or $3x \le \xi \le \frac{1}{3}x < 0$, we obtain

$$\left|I_1\left(t, x\right)\right| \le Ct^{\frac{1}{2k}} \int_{\frac{1}{3}x}^{3x} \frac{\left|\phi\left(\xi\right) - \phi\left(x\right)\right|\left|x\right|^j \mathrm{d}\xi}{1 + tx^{2k-2}\left(\xi - x\right)^2}$$
$$+ Ct^{\frac{1}{2k}} \int_{\frac{1}{3}x}^{3x} \frac{\left|\xi - x\right|\left|x\right|^j \left|\xi\right|^{-\alpha} \left|\left|\xi\right|^\alpha \partial_\xi \phi\left(\xi\right)\right| \mathrm{d}\xi}{1 + tx^{2k-2}\left(\xi - x\right)^2}$$
$$\le Ct^{\frac{1}{2k}} \left\| \left|\xi\right|^\alpha \partial_\xi \phi \right\|_{\mathbf{L}^2}$$
$$\times \left(\int_{\frac{1}{3}x}^{3x} \frac{\left|\xi - x\right|^{\frac{1}{2}} \left|x\right|^{j-\alpha} \mathrm{d}\xi}{1 + tx^{2k-2}\left(\xi - x\right)^2} + \left(\int_{\frac{1}{3}x}^{3x} \frac{\left(\xi - x\right)^2 x^{2j-2\alpha} \mathrm{d}\xi}{\left(1 + tx^{2k-2}\left(\xi - x\right)^2\right)^2}\right)^{\frac{1}{2}}\right).$$

Changing $y = xt^{\frac{1}{2k}}$ and $\zeta = \xi t^{\frac{1}{2k}}$, we get

$$t^{\frac{1}{2k}} \int_{\frac{1}{3}x}^{3x} \frac{\left|\xi - x\right|^{\frac{1}{2}} \left|x\right|^{j-\alpha} \mathrm{d}\xi}{1 + tx^{2k-2}\left(\xi - x\right)^2}$$
$$\le Ct^{-\frac{j-\alpha}{2k} - \frac{1}{4k}} \left|y\right|^{j-\alpha} \int_{\frac{y}{3}}^{3y} \frac{\left|\zeta - y\right|^{\frac{1}{2}} \mathrm{d}\zeta}{1 + y^{2k-2}\left(\zeta - y\right)^2}$$
$$\le Ct^{-\frac{j-\alpha}{2k} - \frac{1}{4k}} \left|y\right|^{j+\frac{3}{2}-\alpha} \int_{\frac{1}{3}}^{3} \frac{\left|\zeta - 1\right|^{\frac{1}{2}} \mathrm{d}\zeta}{1 + y^{2k}\left(\zeta - 1\right)^2}$$
$$\le Ct^{-\frac{j-\alpha}{2k} - \frac{1}{4k}} \left|y\right|^{j+\frac{3}{2}-\alpha} \langle y\rangle^{-\frac{3}{2}k} \le Ct^{-\frac{j-\alpha}{2k} - \frac{1}{4k}} \langle y\rangle^{j-\frac{3}{2}(k-1)-\alpha}$$

and

$$
\begin{aligned}
& t^{\frac{1}{k}} \int_{\frac{1}{3}x}^{3x} \frac{(\xi - x)^2 x^{2j-2\alpha} \mathrm{d}\xi}{\left(1 + t x^{2k-2} (\xi - x)^2\right)^2} \\
& \leq C t^{-\frac{j-\alpha}{k} - \frac{1}{2k}} |y|^{2j-2\alpha} \int_{\frac{y}{3}}^{3y} \frac{(\zeta - y)^2 \mathrm{d}\zeta}{\left(1 + y^{2k-2} (\zeta - y)^2\right)^2} \\
& \leq C t^{-\frac{j-\alpha}{k} - \frac{1}{2k}} |y|^{2j-2\alpha+3} \int_{\frac{1}{3}}^{3} \frac{(\zeta - 1)^2 \mathrm{d}\zeta}{\left(1 + y^{2k} (\zeta - 1)^2\right)^2} \\
& \leq C t^{-\frac{j-\alpha}{k} - \frac{1}{2k}} |y|^{2j-2\alpha+3} \langle y \rangle^{-3k} \leq C t^{-\frac{j-\alpha}{k} - \frac{1}{2k}} \langle y \rangle^{2j-2\alpha-3(k-1)}.
\end{aligned}
$$

Thus we have

$$
\left\langle x t^{\frac{1}{2k}} \right\rangle^{\frac{3}{2}(k-1)-(j-\alpha)} |I_1(t,x)| \leq C t^{-\frac{j-\alpha}{2k} - \frac{1}{4k}} \left\| |\xi|^{\alpha} \partial_{\xi} \phi \right\|_{\mathbf{L}^2}
$$

for all $t > 0$. To estimate the second integral I_2, we integrate by parts via the identity

$$
e^{-itS(x,\xi)} = H_2 \partial_{\xi} \left(\xi e^{-itS(x,\xi)} \right) \tag{2.3}
$$

with $H_2 = \left(1 - it\xi\left(\xi^{2k-1} - x^{2k-1}\right)\right)^{-1}$ to get

$$
\begin{aligned}
I_2(t,x) = & -\frac{t^{\frac{1}{2k}}}{\sqrt{2\pi}} \int e^{-itS(x,\xi)} \phi(\xi) \xi \partial_{\xi} \left(H_2 \left(1 - \chi\left(\xi x^{-1}\right)\right) \xi^{j} \right) \mathrm{d}\xi \\
& - \frac{t^{\frac{1}{2k}}}{\sqrt{2\pi}} \int e^{-itS(x,\xi)} H_2 \left(1 - \chi\left(\xi x^{-1}\right)\right) \xi^{j+1} \phi_{\xi}(\xi) \mathrm{d}\xi.
\end{aligned}
$$

Using the estimates

$$
\left| H_2 \left(1 - \chi\left(\xi x^{-1}\right)\right) \xi^{j+1} \right| \leq \frac{C |\xi|^{j+1}}{1 + t|\xi| \left(|\xi|^{2k-1} + |x|^{2k-1}\right)}
$$

and

$$
\left| \xi \partial_{\xi} \left(H_2 \left(1 - \chi\left(\xi x^{-1}\right)\right) \xi^{j} \right) \right| \leq \frac{C |\xi|^{j}}{1 + t|\xi| \left(|\xi|^{2k-1} + |x|^{2k-1}\right)}
$$

in the domain $\xi \leq \frac{2}{3}x$, or $\xi \geq \frac{3}{2}x$, $x > 0$ or $\xi \geq \frac{2}{3}x$, or $\xi \leq \frac{3}{2}x$, $x < 0$, we obtain

$$
\begin{aligned}
|I_2| \leq & C t^{\frac{1}{2k}} \|\phi\|_{\mathbf{L}^{\infty}} \int \frac{|\xi|^{j} \mathrm{d}\xi}{1 + t|\xi| \left(|\xi|^{2k-1} + |x|^{2k-1}\right)} \\
& + C t^{\frac{1}{2k}} \int \frac{|\xi|^{1+j} \left|\partial_{\xi} \phi(\xi)\right| \mathrm{d}\xi}{1 + t|\xi| \left(|\xi|^{2k-1} + |x|^{2k-1}\right)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|I_2| \leq & C t^{\frac{1}{2k}} \|\phi\|_{\mathbf{L}^{\infty}} \int \frac{|\xi|^{j} \mathrm{d}\xi}{1 + t|\xi| \left(|\xi|^{2k-1} + |x|^{2k-1}\right)} \\
& + C t^{\frac{1}{2k}} \left\| |\xi|^{\alpha} \partial_{\xi} \phi \right\|_{\mathbf{L}^2} \left(\int \frac{|\xi|^{2+2(j-\alpha)} \mathrm{d}\xi}{\left(1 + t|\xi| \left(|\xi|^{2k-1} + |x|^{2k-1}\right)\right)^2} \right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence in the case of $t|x|^{2k}>1$ changing $\xi = xy$

$$
\begin{aligned}
|I_2(t,x)| &\le Ct^{\frac{1}{2k}}\|\phi\|_{\mathbf{L}^\infty}|x|^{1+j}\int\frac{|y|^j\,\mathrm{d}y}{1+t|x|^{2k}|y|\langle y\rangle^{2k-1}}\\
&\quad + Ct^{\frac{1}{2k}}|x|^{\frac{3}{2}+j-\alpha}\left\||\xi|^\alpha\partial_\xi\phi\right\|_{\mathbf{L}^2}\left(\int\frac{|y|^{2+2(j-\alpha)}\,\mathrm{d}y}{\left(1+t|x|^{2k}|y|\langle y\rangle^{2k-1}\right)^2}\right)^{\frac{1}{2}}\\
&\le Ct^{\frac{1}{2k}}\|\phi\|_{\mathbf{L}^\infty}|x|^{1+j}\left|xt^{\frac{1}{2k}}\right|^{\gamma-2k} + Ct^{\frac{1}{2k}}\left\||\xi|^\alpha\partial_\xi\phi\right\|_{\mathbf{L}^2}|x|^{\frac{3}{2}+j-\alpha}\left|xt^{\frac{1}{2k}}\right|^{-2k}\\
&\le Ct^{-\frac{j}{2k}}\|\phi\|_{\mathbf{L}^\infty}\left|xt^{\frac{1}{2k}}\right|^{\gamma-2k+j+1} + Ct^{-\frac{j-\alpha}{2k}-\frac{1}{4k}}\left\||\xi|^\alpha\partial_\xi\phi\right\|_{\mathbf{L}^2}\left|xt^{\frac{1}{2k}}\right|^{-2k+j+\frac{3}{2}-\alpha}
\end{aligned}
$$

if $j<2k-\frac{3}{2}+\alpha$, where $\gamma>0$ for $j=0$ and $\gamma=0$ for $j>0$. For $t|x|^{2k}<1$ by

$$
|x|^{1+j}\int\frac{|y|^j\,\mathrm{d}y}{1+t|x|^{2k}|y|\langle y\rangle^{2k-1}} \le C\int\frac{|y|^j\,\mathrm{d}y}{1+t|y|^{2k}} \le Ct^{-\frac{1}{2k}-\frac{j}{2k}}
$$

we have

$$
|I_2(t,x)| \le Ct^{-\frac{j}{2k}}\|\phi\|_{\mathbf{L}^\infty} + Ct^{-\frac{j-\alpha}{2k}-\frac{1}{4k}}\left\||\xi|^\alpha\partial_\xi\phi\right\|_{\mathbf{L}^2}.
$$

Hence

$$
\begin{aligned}
|I_2(t,x)| &\le Ct^{-\frac{j}{2k}}\|\phi\|_{\mathbf{L}^\infty}\left\langle xt^{\frac{1}{2k}}\right\rangle^{\gamma-2k+j+1}\\
&\quad + Ct^{-\frac{j-\alpha}{2k}-\frac{1}{4k}}\left\||\xi|^\alpha\partial_\xi\phi\right\|_{\mathbf{L}^2}\left\langle xt^{\frac{1}{2k}}\right\rangle^{-2k+j-\alpha+\frac{3}{2}}.
\end{aligned}
$$

Lemma 2.1 is proved. □

We introduce the norm

$$
\|\widehat{\varphi}\|_{\mathbf{W}} = \|\widehat{\varphi}\|_{\mathbf{L}^\infty} + t^{-\frac{1-2\alpha}{4k}}\left\||\xi|^\alpha\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2}.
$$

REMARK 2.1. *Let* $0\le j<2k-\frac{3}{2}+\alpha$, $0\le\alpha<1$. *Applying the estimate* $\left|A_j(x)\right|\le C\langle x\rangle^{j+1-k}$, *we get from the estimate of Lemma 2.1*

$$
\begin{aligned}
\left|\left(\mathcal{V}\xi^j\widehat{\varphi}\right)(x)\right| &\le \left|t^{-\frac{j}{2k}}A_j\left(xt^{\frac{1}{2k}}\right)\widehat{\varphi}(x)\right|\\
&\quad + Ct^{-\frac{j}{2k}}\left\langle xt^{\frac{1}{2k}}\right\rangle^{-\frac{3}{2}k-\alpha+\frac{3}{2}+j}\left(\|\widehat{\varphi}\|_{\mathbf{L}^\infty} + t^{-\frac{1-2\alpha}{4k}}\left\||\xi|^\alpha\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2}\right)\\
&\le Ct^{-\frac{j}{2k}}\left\langle xt^{\frac{1}{2k}}\right\rangle^{j+1-k}\|\widehat{\varphi}\|_{\mathbf{W}}. \qquad (2.4)
\end{aligned}
$$

Also by (1.6) we have for $u=\mathcal{D}_t\mathcal{B}M\mathcal{V}\widehat{\varphi}$, *if* $\widehat{\varphi}=\mathcal{F}\mathcal{U}(-t)u$

$$
\begin{aligned}
\left|\partial_x^j u(t,x)\right| &= Ct^{-\frac{1}{2k}}\left|\left(\mathcal{V}\xi^j\widehat{\varphi}\right)\left(t^{-\frac{1}{2k-1}}x^{\frac{1}{2k-1}}\right)\right|\\
&\le Ct^{-\frac{j+1}{2k}}\left\langle xt^{-\frac{1}{2k}}\right\rangle^{\frac{j+1-k}{2k-1}}\|\widehat{\varphi}\|_{\mathbf{W}}. \qquad (2.5)
\end{aligned}
$$

Since

$$\mathcal{V}^{*}\phi=\frac{(2k-1)\,t^{\frac{2k-1}{2k}}}{\sqrt{2\pi}}\int e^{itS(x,\xi)}\phi\left(x\right)x^{2k-2}\mathrm{d}x$$

we have

$$\left\|\mathcal{V}^{*}\phi\right\|_{\mathbf{L}^{\infty}}\leq Ct^{\frac{2k-1}{2k}}\left\|x^{2k-2}\phi\right\|_{\mathbf{L}^{1}}. \tag{2.6}$$

LEMMA 2.2. *Let* $p>\frac{k}{2k-2}$. *Then the estimate is true*

$$\left\|\mathcal{V}^{*}\left(\left|\mathcal{V}\widehat{\varphi}\right|^{2p}\mathcal{V}\widehat{\varphi}\right)\right\|_{\mathbf{L}^{\infty}}\leq C\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{2p+1}.$$

Proof. By (2.4) with $j=0$

$$\left\|\mathcal{V}\widehat{\varphi}\right\|_{\mathbf{L}^{\infty}}\leq C\left\langle xt^{\frac{1}{2k}}\right\rangle^{1-k}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}.$$

Therefore in view of (2.6) we have

$$\begin{aligned}\left\|\mathcal{V}^{*}\left(\left|\mathcal{V}\widehat{\varphi}\right|^{2p}\mathcal{V}\widehat{\varphi}\right)\right\|_{\mathbf{L}^{\infty}}&\leq Ct^{\frac{2k-1}{2k}}\left\|x^{2k-2}\left|\mathcal{V}\widehat{\varphi}\right|^{2p+1}\right\|_{\mathbf{L}^{1}}\\&\leq Ct^{\frac{2k-1}{2k}}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{2p+1}\int x^{2k-2}\left\langle xt^{\frac{1}{2k}}\right\rangle^{-(k-1)(2p+1)}\mathrm{d}x\\&\leq C\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{2p+1}\int\left\langle y\right\rangle^{-(k-1)(2p-1)}\mathrm{d}y\leq C\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{2p+1},\end{aligned}$$

since

$$\int\left\langle y\right\rangle^{-(k-1)(2p-1)}\mathrm{d}y<\infty$$

if $p>\frac{k}{2k-2}$. This completes the proof of the lemma. □

LEMMA 2.3. *Let* $p>1+\frac{3+2\alpha}{4(k-1)}$, $0\leq\alpha<\frac{1}{2}$. *Then we have*

$$\left\|\left|\partial_{x}\right|^{\alpha}\mathcal{J}\left|u\right|^{2p}u\right\|_{\mathbf{L}^{2}}\leq Ct^{-\frac{p}{k}}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{2p}\left\|\left|\partial_{x}\right|^{\alpha}\mathcal{J}u\right\|_{\mathbf{L}^{2}}+Ct^{\frac{1-2\alpha}{4k}-\frac{p}{k}}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^{2p+1}$$

for all $t>0$, *where* $\widehat{\varphi}=\mathcal{F}\mathcal{U}\left(-t\right)u$.

Proof. Applying the operator $\mathcal{J}=x+it\partial_{x}\left(-\partial_{x}^{2}\right)^{k-1}=\mathcal{U}\left(t\right)x\mathcal{U}\left(-t\right)$, by a direct computation we get

$$\left|\partial_{x}\right|^{\alpha}\mathcal{J}\left|u\right|^{2p}u=(p+1)\left|\partial_{x}\right|^{\alpha}\left(\left|u\right|^{2p}\mathcal{J}u\right)-p\left|\partial_{x}\right|^{\alpha}\left(\left|u\right|^{2p-2}u^{2}\overline{\mathcal{J}u}\right)+\left|\partial_{x}\right|^{\alpha}F,$$

where F consists of the terms

$$it\overline{u}^{2p-2}u\partial_{x}u\partial_{x}^{2k-2}u,\ it\overline{u}^{2p-2}\left(\partial_{x}u\right)^{2}\left(\partial_{x}^{2k-3}u\right),\ \text{etc.}$$

By Lemma 2.2 from [4] (see also [10]) we have

$$\left\|\left(-\Delta\right)^{\frac{\alpha}{2}}\left(uv\right)\right\|_{\mathbf{L}^{r}}\leq C\left\|\left(-\Delta\right)^{\frac{\alpha}{2}}u\right\|_{\mathbf{L}^{r_{1}}}\left\|v\right\|_{\mathbf{L}^{q_{1}}}+C\left\|u\right\|_{\mathbf{L}^{q_{2}}}\left\|\left(-\Delta\right)^{\frac{\alpha}{2}}v\right\|_{\mathbf{L}^{r_{2}}} \tag{2.7}$$

for $1 < r < \infty$, $1 < r_i < \infty$, $1 < q_i \le \infty$, such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_1} = \frac{1}{r_2} + \frac{1}{q_2}$. Then we get

$$\left\| |\partial_x|^{\alpha} \left(|u|^{2p} \mathcal{J}u \right) \right\|_{\mathbf{L}^2} \le C \left\| |u|^{2p-1} \right\|_{\mathbf{L}^{r_1}} \left\| |\partial_x|^{\alpha} u \right\|_{\mathbf{L}^{q_1}} \left\| \mathcal{J}u \right\|_{\mathbf{L}^{\frac{2}{1-2\alpha}}} + C \left\| u \right\|_{\mathbf{L}^{\infty}}^{2p} \left\| |\partial_x|^{\alpha} \mathcal{J}u \right\|_{\mathbf{L}^2}$$

where $\frac{1}{\alpha} = \frac{1}{r_1} + \frac{1}{q_1}$. By Sobolev's and Hölder's inequalities $\|\phi\|_{\mathbf{L}^{\frac{2}{1-2\alpha}}} \le \| |\partial_x|^{\alpha} \phi \|_{\mathbf{L}^2}$, $\frac{1}{q} = \frac{1}{2} - \alpha$, $\alpha \in \left[0, \frac{1}{2}\right)$. Hence

$$\left\| \mathcal{J}u \right\|_{\mathbf{L}^{\frac{2}{1-2\alpha}}} \le C \left\| |\partial_x|^{\alpha} \mathcal{J}u \right\|_{\mathbf{L}^2}$$

By using (2.5) we find $\left| \partial_x^j u(t,x) \right| \le C t^{-\frac{j+1}{2k}} \left\langle x t^{-\frac{1}{2k}} \right\rangle^{\frac{j+1-k}{2k-1}} \|\widehat{\varphi}\|_{\mathbf{W}}$. Hence

$$\begin{aligned} &\left\| |u|^{2p-1} \right\|_{\mathbf{L}^{r_1}} \left\| |\partial_x|^{\alpha} u \right\|_{\mathbf{L}^{q_1}} \\ &\le C t^{-\frac{2p-\alpha}{2k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p} \left\| \left\langle x t^{-\frac{1}{2k}} \right\rangle^{\frac{(1-k)(2p-1)}{2k-1}} \right\|_{\mathbf{L}^{r_1}} \left\| \left\langle x t^{-\frac{1}{2k}} \right\rangle^{\frac{\alpha+1-k}{2k-1}} \right\|_{\mathbf{L}^{q_1}} \\ &\le C t^{-\frac{p}{k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p}. \end{aligned}$$

Hence

$$\left\| |\partial_x|^{\alpha} \left(|u|^{2p} \mathcal{J}u \right) \right\|_{\mathbf{L}^2} \le C t^{-\frac{p}{k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p} \left\| |\partial_x|^{\alpha} \mathcal{J}u \right\|_{\mathbf{L}^2}.$$

In the same manner

$$\left\| |\partial_x|^{\alpha} \left(|u|^{2p-2} u^2 \overline{\mathcal{J}u} \right) \right\|_{\mathbf{L}^2} \le C t^{-\frac{p}{k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p} \left\| |\partial_x|^{\alpha} \mathcal{J}u \right\|_{\mathbf{L}^2}.$$

Next consider the term $|\partial_x|^{\alpha} \left(\overline{u}^{2p-2} u \left(\partial_x u\right) \partial_x^{2k-2} u \right)$ in $|\partial_x|^{\alpha} F$. Define the Littlewood–Paley decomposition. Let $\varphi_l \in C_0^{\infty}(\mathbb{R})$ be such that $\mathrm{supp}\varphi_0(\xi) \subset \{\xi : |\xi| < 2\}$, $\mathrm{supp}\varphi_l(\xi) \subset \left\{\xi : 2^{l-1} < |\xi| < 2^{l+1}\right\}$ for $l \ge 1$ and $\sum_{l=0}^{\infty} \varphi_l(\xi) = 1$ for all $\xi \in \mathbb{R}$. Then if we represent $u = \sum_{l=0}^{\infty} u_l$, where $u_l = \varphi_l u$, note that

$$\varphi_l(\xi) = \varphi_l(\xi) \left(\varphi_{l-1}(\xi) + \varphi_l(\xi) + \varphi_{l+1}(\xi) \right)$$

therefore, the estimate follows

$$\left\| |\partial_x|^{\alpha} \left(\overline{u}^{2p-2} u \left(\partial_x u\right) \partial_x^{2k-2} u \right) \right\|_{\mathbf{L}^2} \le C \sum_{l=0}^{\infty} \left\| |\partial_x|^{\alpha} \left(\overline{u_l}^{2p-2} u_l \left(\partial_x u\right) \partial_x^{2k-2} u_l \right) \right\|_{\mathbf{L}^2}.$$

Now applying estimate (2.7) we get

$$\begin{aligned} &\left\| |\partial_x|^{\alpha} \left(\overline{u_l}^{2p-2} u_l \left(\partial_x u\right) \partial_x^{2k-2} u_l \right) \right\|_{\mathbf{L}^2} \le C \|u_l\|_{\mathbf{L}^{\infty}}^{2p-1} \|\partial_x u_l\|_{\mathbf{L}^{\infty}} \left\| |\partial_x|^{\alpha} \partial_x^{2k-2} u_l \right\|_{\mathbf{L}^2} \\ &+ C \|u_l\|_{\mathbf{L}^{\infty}}^{2p-1} \left\| |\partial_x|^{\alpha} \partial_x u_l \right\|_{\mathbf{L}^2} \left\| \partial_x^{2k-2} u_l \right\|_{\mathbf{L}^{\infty}} \\ &+ C \|u_l\|_{\mathbf{L}^{\infty}}^{2p-2} \left\| |\partial_x|^{\alpha} u_l \right\|_{\mathbf{L}^2} \|\partial_x u_l\|_{\mathbf{L}^{\infty}} \left\| \partial_x^{2k-2} u_l \right\|_{\mathbf{L}^{\infty}}. \end{aligned}$$

By using (2.5) we find

$$\left\| \partial_x^j u_l \right\|_{\mathbf{L}^q} \le Ct^{-\frac{j+1}{2k}} 2^{\frac{l}{q}} \left\langle 2^l t^{-\frac{1}{2k}} \right\rangle^{\frac{j+1-k}{2k-1}} \|\widehat{\varphi}\|_{\mathbf{W}}.$$

Then we obtain

$$\begin{aligned} &\left\| |\partial_x|^\alpha \left(\overline{u_l}^{2p-2} u_l \left(\partial_x u\right) \partial_x^{2k-2} u_l \right) \right\|_{\mathbf{L}^2} \\ &\le Ct^{1-\frac{2p+2k+\alpha}{2k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p+1} 2^{\frac{l}{2}} \left\langle 2^l t^{-\frac{1}{2k}} \right\rangle^{\frac{k+\alpha-2p(k-1)}{2k-1}} \\ &\le Ct^{\frac{1}{4k}-\frac{2p+\alpha}{2k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p+1} \left(2^l t^{-\frac{1}{2k}}\right)^{\frac{1}{2}} \left\langle 2^l t^{-\frac{1}{2k}} \right\rangle^{\frac{k+\alpha-2p(k-1)}{2k-1}}. \end{aligned}$$

Therefore

$$\begin{aligned} &t \left\| |\partial_x|^\alpha \left(\overline{u}^{2p-2} u \left(\partial_x u\right) \partial_x^{2k-2} u \right) \right\|_{\mathbf{L}^2} \\ &\le Ct^{\frac{1}{4k}-\frac{2p+\alpha}{2k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p+1} \sum_{l=0}^{\infty} \left(2^l t^{-\frac{1}{2k}}\right)^{\frac{1}{2}} \left\langle 2^l t^{-\frac{1}{2k}} \right\rangle^{\frac{k+\alpha-2p(k-1)}{2k-1}} \\ &\le Ct^{\frac{1}{4k}-\frac{2p+\alpha}{2k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p+1}, \end{aligned}$$

since changing $y = \left(2^x t^{-\frac{1}{2k}}\right)^{\frac{1}{2}}$, we find

$$\begin{aligned} &\sum_{l=0}^{\infty} \left(2^l t^{-\frac{1}{2k}}\right)^{\frac{1}{2}} \left\langle 2^l t^{-\frac{1}{2k}} \right\rangle^{\frac{k+\alpha-2p(k-1)}{2k-1}} \\ &\le \int_0^\infty \left(2^x t^{-\frac{1}{2k}}\right)^{\frac{1}{2}} \left\langle 2^x t^{-\frac{1}{2k}} \right\rangle^{\frac{k+\alpha-2p(k-1)}{2k-1}} \mathrm{d}x \le \int_0^\infty \left\langle y^2 \right\rangle^{\frac{k+\alpha-2p(k-1)}{2k-1}} \mathrm{d}y \end{aligned}$$

if $\frac{k+\alpha-2p(k-1)}{2k-1} < -\frac{1}{2}$, i.e., $p > 1 + \frac{3+2\alpha}{4(k-1)}$. Lemma 2.3 is proved. □

3. Proof of Theorem

Let us consider the linearized version of (1.1) such that

$$\begin{cases} i\partial_t u - \frac{1}{2k}\left(-\partial_x^2\right)^k u = \lambda |v|^{2p} v, \ \ (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $v \in \mathbf{Y}_{T,\rho}$. Note that $\| |\partial_x|^\alpha \mathcal{J} u \|_{\mathbf{L}^2} = \left\| |\xi|^\alpha \partial_\xi \widehat{\varphi}(t) \right\|_{\mathbf{L}^2}$, so by Lemma 2.3 we find

$$\begin{aligned} \left\| |\partial_x|^\alpha \mathcal{J} |v|^{2p} v \right\|_{\mathbf{L}^2} &\le Ct^{-\frac{p}{k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p} \left\| |\partial_x|^\alpha \mathcal{J} u \right\|_{\mathbf{L}^2} + Ct^{\frac{1-2\alpha}{4k}-\frac{p}{k}} \|\widehat{\varphi}\|_{\mathbf{W}}^{2p+1} \\ &\le Ct^{-\frac{3p}{2k}+\frac{p}{k}\alpha} \|v\|_{\mathbf{X}^\alpha}^{2p+1} \end{aligned}$$

for $p > 1 + \frac{3+2\alpha}{4(k-1)}$, $0 \leq \alpha < \frac{1}{2}$. Then by the integral equation associated with (3.1) we have

$$\left\| |\partial_x|^{\alpha} \mathcal{J} u \right\|_{\mathbf{L}^2} \leq \|u_0\|_{\mathbf{X}^{\alpha}} + C \|v\|_{\mathbf{Y}_T}^{2p+1} \int_0^T t^{-\frac{3p}{2k}+\frac{p}{k}\alpha} dt$$
$$\leq \|u_0\|_{\mathbf{X}^{\alpha}} + C\rho^{2p+1} T^{1-\frac{3p}{2k}+\frac{p}{k}\alpha} \tag{3.2}$$

for $1 - \frac{3p}{2k} + \frac{p}{k}\alpha > 0$. In the same way as in the derivation of (1.7) we obtain

$$i\partial_t \mathcal{F}\mathcal{U}(-t) u(t) = \lambda t^{-\frac{p}{k}} \mathcal{V}^* \left(|\mathcal{V}\widehat{\varphi}|^{2p} \mathcal{V}\widehat{\varphi} \right),$$

where $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t) v(t)$. Then by Lemma 2.2

$$\|\mathcal{F}\mathcal{U}(-t) u(t)\|_{\mathbf{L}^{\infty}} \leq \|\widehat{u}_0\|_{\mathbf{L}^{\infty}} + C \|v\|_{\mathbf{Y}_T}^{2p+1} \int_0^T t^{-\frac{p}{k}-\frac{(1-2\alpha)}{4k}(2p+1)} dt$$
$$\leq \|\widehat{u}_0\|_{\mathbf{L}^{\infty}} + C\rho^{2p+1} T^{1-\frac{p}{k}-\frac{(1-2\alpha)}{4k}(2p+1)}$$

for $p > \frac{k}{2k-2}$ if $1 - \frac{p}{k} - \frac{(1-2\alpha)}{4k}(2p+1) > 0$. Therefore

$$\|u\|_{\mathbf{Y}_T} \leq \|u_0\|_{\mathbf{X}^{\alpha}} + C\rho^{2p+1} T^{1-\frac{p}{k}-\frac{1-2\alpha}{4k}(2p+1)}$$

for

$$1 + \frac{3+2\alpha}{4(k-1)} < p < \frac{4k-1+2\alpha}{2(3-2\alpha)}.$$

We may assume that $\|u_0\|_{\mathbf{X}^{\alpha}} \leq \frac{\rho}{2}$, and therefore, we find that there exists a time $T > 0$ such that $\|u\|_{\mathbf{Y}_T} \leq \rho$. This means that the mapping $\mathcal{S}$ defined by $u = \mathcal{S}v$ transforms $\mathbf{Y}_T$ into itself. In the same way, it is shown that there exists a time T such that $\|\mathcal{S}v_1 - \mathcal{S}v_2\|_{\mathbf{Y}_T} \leq \frac{1}{2} \|v_1 - v_2\|_{\mathbf{Y}_T}$. Hence we have the desired result by the contraction mapping principle. Smoothing properties of solutions (1.4) and (1.5) come from (2.5). This completes the proof of the theorem.

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Nakao Hayashi
Department of Mathematics,
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043
Japan
E-mail: nhayashi@math.sci.osaka-u.ac.jp

Pavel I. Naumkin
Centro de Ciencias Matemáticas
UNAM
Campus Morelia, AP 61-3 (Xangari)
CP 58089, Morelia, Michoacán
Mexico
E-mail: pavelni@matmor.unam.mx

Takayoshi Ogawa
Mathematical Institute
Tohoku University
Sendai 980-8578
Japan
E-mail: ogawa@math.tohoku.ac.jp