



## Addendum to the paper “On quasilinear parabolic evolution equations in weighted $L_p$ -spaces II”

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*Abstract.* This note is devoted to a small, but essential, extension of Theorem 2.1 of our recent paper (LeCrone et al. J Evolut Equ 14:509–533 2014). The improvement is explained in “The improvement” section and proved in “Proof of the main result” section. The importance of the extension is demonstrated in “Application to the Navier–Stokes equations” section with an application to the Navier–Stokes system in critical  $L_q$ -spaces.

### 1. The improvement

Let  $X_0, X_1$  be Banach spaces such that  $X_1$  embeds densely in  $X_0$ , let  $p \in (1, \infty)$  and  $1/p < \mu \leq 1$ . As in [6], we consider the following quasilinear parabolic evolution equation

$$\dot{u} + A(u)u = F_1(u) + F_2(u), \quad t > 0, \quad u(0) = u_1. \quad (1.1)$$

The space of initial data will be the real interpolation space  $X_{\gamma, \mu} = (X_0, X_1)_{\mu-1/p, p}$ , and the state space of the problem is  $X_\gamma = X_{\gamma, 1}$ . Let  $V_\mu \subset X_{\gamma, \mu}$  be open and  $u_1 \in V_\mu$ . Furthermore, let  $X_\beta = (X_0, X_1)_\beta$  denote the *complex interpolation spaces*. We will impose the following assumptions.

**(H1)**  $(A, F_1) \in C^{1-}(V_\mu; \mathcal{B}(X_1, X_0) \times X_0)$ .

**(H2)**  $F_2 : V_\mu \cap X_\beta \rightarrow X_0$  satisfies the estimate

$$|F_2(u_1) - F_2(u_2)|_{X_0} \leq C \sum_{j=1}^m (1 + |u_1|_{X_\beta}^{\rho_j} + |u_2|_{X_\beta}^{\rho_j}) |u_1 - u_2|_{X_{\beta_j}},$$

for some numbers  $m \in \mathbb{N}$ ,  $\rho_j \geq 0$ ,  $\beta \in (\mu - 1/p, 1)$ ,  $\beta_j \in [\mu - 1/p, \beta]$ , where  $C$  denotes a constant which may depend on  $|u_i|_{X_{\gamma, \mu}}$ .

**(H3)** For all  $j = 1, \dots, m$ , we have

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*Mathematics Subject Classification:* Primary 35K90, 35B30; Secondary 35B40

*Keywords:* Quasilinear parabolic evolution equations, Critical spaces, Navier–Stokes.

$$\rho_j(\beta - (\mu - 1/p)) + (\beta_j - (\mu - 1/p)) \leq 1 - (\mu - 1/p).$$

Allowing for equality in **(H3)** is not for free, there is no free lunch, in particular not in mathematics. We have to impose additionally the following structural **Condition (S)** on the Banach spaces  $X_0$  and  $X_1$ .

**(S)** The space  $X_0$  is of class UMD. The embedding

$$H_p^1(\mathbb{R}; X_0) \cap L_p(\mathbb{R}; X_1) \hookrightarrow H_p^{1-\beta}(\mathbb{R}; X_\beta),$$

is valid for each  $\beta \in (0, 1)$ ,  $p \in (1, \infty)$ .

REMARK 1.1. (i) By the *Mixed Derivative Theorem*, Condition **(S)** is valid if  $X_0$  is of class UMD, and if there is an operator  $A_\# \in \mathcal{H}^\infty(X_0)$ , with domain  $\mathcal{D}(A_\#) = X_1$ , and  $\mathcal{H}^\infty$ -angle  $\phi_{A_\#}^\infty < \pi/2$ . We refer to Prüss and Simonett [9], Sect. 4.5.

(ii) The assumption that  $X_0$  is a UMD space in Condition **(S)** cannot be skipped, since the maximal domain of  $(\frac{d}{dt})^\alpha$  in  $L_p(\mathbb{R}; X_0)$  is given by  $H_p^\alpha(\mathbb{R}; X_0)$  if  $X_0$  is a UMD space.

(iii) Condition **(S)** implies the embedding

$$\begin{aligned} {}_0\mathbb{E}_{1,\mu}(0, T) &:= {}_0H_{p,\mu}^1((0, T); X_0) \cap L_{p,\mu}((0, T); X_1) \\ &\hookrightarrow {}_0H_{p,\mu}^{1-\beta}((0, T); X_\beta). \end{aligned}$$

Indeed, if  $u \in {}_0\mathbb{E}_{1,\mu}(0, T)$ , then we first extend  $u$  to a function  $\tilde{u} \in {}_0\mathbb{E}_{1,\mu}(\mathbb{R}_+)$  by [7, Lemma 2.5]. This in turn is equivalent to the fact

$$[t \mapsto t^{1-\mu}\tilde{u}(t)] \in {}_0H_p^1(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; X_1).$$

In a next step, we extend  $v(t) := t^{1-\mu}\tilde{u}(t)$  by zero to  $\mathbb{R}_-$  to obtain

$$\tilde{v} \in H_p^1(\mathbb{R}; X_0) \cap L_p(\mathbb{R}; X_1).$$

By Condition **(S)**, it follows that  $\tilde{v} \in H_p^{1-\beta}(\mathbb{R}; X_\beta)$ , and hence,  $v \in {}_0H_p^{1-\beta}(\mathbb{R}_+; X_\beta)$  and therefore  $u \in {}_0H_{p,\mu}^{1-\beta}((0, T); X_\beta)$ .

The announced extension of Theorem 2.1 of [6] is the following result.

THEOREM 1.2. *Suppose that the structural assumption **(S)** holds, and assume that hypotheses **(H1)**, **(H2)**, **(H3)** are valid. Fix any  $u_0 \in V_\mu$  such that  $A_0 := A(u_0)$  has maximal  $L_p$ -regularity. Then there is  $T = T(u_0) > 0$  and  $\varepsilon = \varepsilon(u_0) > 0$  with  $\bar{B}_{X_{\gamma,\mu}}(u_0, \varepsilon) \subset V_\mu$  such that problem (1.1) admits a unique solution*

$$u(\cdot, u_1) \in H_{p,\mu}^1((0, T); X_0) \cap L_{p,\mu}((0, T); X_1) \cap C([0, T]; V_\mu),$$

for each initial value  $u_1 \in \bar{B}_{X_{\gamma,\mu}}(u_0, \varepsilon)$ . There is a constant  $c = c(u_0) > 0$  such that

$$|u(\cdot, u_1) - u(\cdot, u_2)|_{\mathbb{E}_{1,\mu}(0,T)} \leq c|u_1 - u_2|_{X_{\gamma,\mu}},$$

for all  $u_1, u_2 \in \bar{B}_{X_{\gamma,\mu}}(u_0, \varepsilon)$ .

We call  $j$  subcritical if in **(H3)** strict inequality holds, and critical otherwise. As  $\beta_j \leq \beta < 1$ , any  $j$  with  $\rho_j = 0$  is subcritical. Furthermore, **(H3)** is equivalent to  $\rho_j\beta + \beta_j - 1 \leq \rho_j(\mu - 1/p)$ ; hence, the minimal value of  $\mu$  is given by

$$\mu_{\text{crit}} = \frac{1}{p} + \beta - \min_j(1 - \beta_j)/\rho_j.$$

This number defines the *critical weight*. Theorem 1.2 shows that we have local well-posedness of (1.1) for initial values in the space  $X_{\gamma, \mu_{\text{crit}}}$ , provided **(H1)** holds for  $\mu = \mu_{\text{crit}}$ . Therefore, it makes sense to name this space the *critical space* for (1.1).

The main difference of Theorem 1.2 to our previous result, Theorem 2.1 in [6], is that here we may allow for equality in Condition **(H3)**, at the expense of assuming **(S)**. In the applications presented in [6], there was no need for this equality, as strict inequality had to be imposed to ensure **(H1)**. But meanwhile we realized that equality in **(H3)** is an important issue. To demonstrate this, we use Theorem 1.2 to study the Navier–Stokes equations and refer also to the recent paper Prüss [8] for an application to the quasi-geostrophic equations on compact surfaces without boundary in  $\mathbb{R}^3$ . We mention that the proofs of the remaining results in [6] remain valid without any changes.

## 2. Proof of the main result

We show how to extend the proof of Theorem 2.1 in [6] to the case of equality in **(H3)**. For this purpose, we fix any critical index  $j$ , i.e.,

$$\rho_j(\beta - (\mu - 1/p)) + \beta_j - (\mu - 1/p) = 1 - (\mu - 1/p),$$

and set

$$\frac{1}{r} = \frac{\beta_j - (\mu - 1/p)}{1 - (\mu - 1/p)}, \quad \frac{1}{r'} = \rho_j \frac{\beta - (\mu - 1/p)}{1 - (\mu - 1/p)}.$$

Then we have  $1/r < 1$ ,  $1/r' < \rho_j$ , and  $1/r + 1/r' = 1$ . Using the notation in the proof of Theorem 2.1 in [6], we start with equation (2.12) in [6]:

$$\begin{aligned} &|F_2(v) - F_2(u_0^*)|_{\mathbb{E}_{0,\mu}(0,T)} \\ &\leq C_{\varepsilon_0} \sum_{j=1}^m \left( \int_0^T (1 + |v(t)|_{X_\beta}^{\rho_j} + |u_0^*(t)|_{X_\beta}^{\rho_j})^p |v(t) - u_0^*(t)|_{X_{\beta_j}}^p t^{(1-\mu)p} dt \right)^{1/p}. \end{aligned}$$

We apply Hölders inequality to the result

$$\begin{aligned} &|F_2(v) - F_2(u_0^*)|_{\mathbb{E}_{0,\mu}(0,T)} \\ &\leq C_{\varepsilon_0} \sum_{j=1}^m \left( (\kappa(T) + |v|_{L_{\rho_j p r', \sigma'}(X_\beta)}^{\rho_j} + |u_0^*|_{L_{\rho_j p r', \sigma'}(X_\beta)}^{\rho_j}) |v - u_0^*|_{L_{p r, \sigma}(X_{\beta_j})} \right), \end{aligned}$$

where

$$\kappa(T) := \frac{1}{(p(1-\mu) + 1)^{1/(pr')}} T^{(p(1-\mu)+1)/(pr')} \rightarrow 0,$$

as  $T \rightarrow 0$  and  $1 - \mu = r(1 - \sigma) = \rho_j r'(1 - \sigma')$ . Note that  $\sigma, \sigma'$  are admissible, as

$$\sigma = 1 - \frac{1}{r} + \frac{\mu}{r} > \frac{1}{pr}, \quad \sigma' = 1 - \frac{1}{\rho_j r'} + \frac{\mu}{\rho_j r'} > \frac{1}{\rho_j pr'}.$$

Next we have by Condition (S), Sobolev embedding and Hardy's inequality

$$\begin{aligned} {}_0\mathbb{E}_{1,\mu}(0, T) &\hookrightarrow {}_0H_{p,\mu}^{1-\beta_j}((0, T); X_{\beta_j}) \hookrightarrow {}_0H_{pr,\mu}^{1-\beta_j-\frac{1}{pr'}}((0, T); X_{\beta_j}) \\ &\hookrightarrow L_{pr,\sigma}((0, T); X_{\beta_j}), \end{aligned}$$

as  $1/r + 1/r' = 1$  and

$$1 - \beta_j - \frac{1}{p} - (1 - \mu) = -\frac{1}{pr} - (1 - \sigma),$$

see, e.g., Prüss and Simonett [9], Sects. 4.5.5 and 3.4.6 or Meyries and Schnaubelt [7]. We emphasize that the embedding constants do not depend on  $T > 0$ . In the same way, we obtain the embedding

$${}_0\mathbb{E}_{1,\mu}(0, T) \hookrightarrow {}_0H_{p,\mu}^{1-\beta}((0, T); X_\beta) \hookrightarrow L_{\rho_j pr',\sigma'}((0, T); X_\beta),$$

as

$$1 - \beta - \frac{1}{p} - (1 - \mu) = -\frac{1}{\rho_j pr'} - (1 - \sigma').$$

The triangle inequality first yields

$$|v|_{L_{\rho_j pr',\sigma'}(X_\beta)} \leq |v - u_1^*|_{L_{\rho_j pr',\sigma'}(X_\beta)} + |u_1^*|_{L_{\rho_j pr',\sigma'}(X_\beta)},$$

where  $u_1^*(t) = e^{-A_0 t} u_1$ . This implies with  $v(0) = u_1$  the estimates

$$\begin{aligned} |v - u_1^*|_{L_{\rho_j pr',\sigma'}(X_\beta)} &\leq C |v - u_1^*|_{{}_0\mathbb{E}_{1,\mu}} \\ &\leq C (|v - u_0^*|_{{}_0\mathbb{E}_{1,\mu}} + |u_0^* - u_1^*|_{{}_0\mathbb{E}_{1,\mu}}) \leq C(r + |u_0 - u_1|_{X_{\gamma,\mu}}) \end{aligned}$$

and

$$\begin{aligned} |u_1^*|_{L_{\rho_j pr',\sigma'}(X_\beta)} &\leq |u_0^* - u_1^*|_{L_{\rho_j pr',\sigma'}(X_\beta)} + |u_0^*|_{L_{\rho_j pr',\sigma'}(X_\beta)} \\ &\leq C(|u_0 - u_1|_{X_{\gamma,\mu}} + \varphi(T)), \end{aligned}$$

with  $\varphi(T) = |u_0^*|_{L_{\rho_j pr',\sigma'}((0,T); X_\beta)}$ . Moreover, it holds that

$$\begin{aligned} |v - u_0^*|_{L_{pr,\sigma}(X_{\beta_j})} &\leq |v - u_1^*|_{L_{pr,\sigma}(X_{\beta_j})} + |u_1^* - u_0^*|_{L_{pr,\sigma}(X_{\beta_j})} \\ &\leq C(r + |u_0 - u_1|_{X_{\gamma,\mu}}). \end{aligned}$$

Therefore, choosing  $T > 0, r > 0$  and  $|u_0 - u_1|_{X_{\gamma,\mu}}$  small enough, we obtain the estimate  $|F_2(v)|_{\mathbb{E}_{0,\mu}} \leq r/3$ . In fact,  $\varphi(T) \rightarrow 0$  for  $T \rightarrow 0$ , as  $u_0^* \in L_{\rho_j p r', \sigma'}((0, T); X_\beta)$  by Proposition 3.4.3 of Prüss and Simonett [9].

A similar argument applies to the contraction estimate

$$|F_2(v_1) - F_2(v_2)|_{\mathbb{E}_{0,\mu}(0,T)} \leq C_{\varepsilon_0} \sum_{j=1}^m \left( (\kappa(T) + |v_1|_{L_{\rho_j p r', \sigma'}(X_\beta)}^{\rho_j} + |v_2|_{L_{\rho_j p r', \sigma'}(X_\beta)}^{\rho_j}) |v_1 - v_2|_{L_{pr,\sigma}(X_{\beta_j})} \right),$$

making use of  $|v_i| \leq |v_i - u_i^*| + |u_i^*|$  and

$$|v_1 - v_2|_{L_{pr,\sigma}(X_{\beta_j})} \leq |v_1 - v_2 - (u_1^* - u_2^*)|_{L_{pr,\sigma}(X_{\beta_j})} + |u_1^* - u_2^*|_{L_{pr,\sigma}(X_{\beta_j})}.$$

### 3. Application to the Navier–Stokes equations

Let  $\Omega \subset \mathbb{R}^n, n \geq 2$ , be a bounded domain with boundary  $\partial\Omega$  of class  $C^{3-}$  and consider the Navier–Stokes problem

$$\begin{aligned} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \pi &= 0, & \text{in } \Omega, \\ \operatorname{div} u &= 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \\ u(0) &= u_0, & \text{in } \Omega. \end{aligned} \tag{3.1}$$

Here  $u$  denotes the velocity field and  $\pi$  the pressure. We consider this problem in  $L_q(\Omega)^n$  with  $1 < q < \infty$ . Employing the Helmholtz projection  $P$  and the Stokes operator  $A$ , this problem can be reformulated as the abstract semilinear evolution equation

$$\dot{u} + Au = F(u), \quad t > 0, \quad u(0) = u_0, \tag{3.2}$$

in the Banach space  $X_0 = L_{q,\sigma}(\Omega) = PL_q(\Omega)^n$ , with bilinear nonlinearity  $F$  defined by

$$F(u) = G(u, u), \quad G(u_1, u_2) = -P(u_1 \cdot \nabla)u_2. \tag{3.3}$$

It is well known, see, e.g., Hieber and Saal [4] or Amann [1], that the Stokes operator  $A = -P\Delta$  with domain

$$X_1 = D(A) := \{u \in H_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega) : u = 0 \text{ on } \partial\Omega\}$$

is sectorial and admits a bounded  $\mathcal{H}^\infty$ -calculus with  $\mathcal{H}^\infty$ -angle equal to zero. Therefore,  $A$  has maximal  $L_p$ -regularity, even on the halfline, as 0 belongs to the resolvent set of  $A$ , and so  $-A$  generates an exponentially stable analytic  $C_0$ -semigroup in  $X_0$ . Thus with  $A(u) = A, F_1 = 0$ , and  $F_2 = F$ , problem (3.2) is of the form (1.1), and

Conditions **(H1)** and **(S)** are valid. To apply Theorem 1.2, we therefore have to estimate the nonlinearity in a proper way. This will be done as follows. As  $P$  is bounded in  $L_q(\Omega)^n$ , by Hölder’s inequality we obtain

$$|G(u_1, u_2)|_{X_0} \leq C|u_1 \cdot \nabla u_2|_{L_q} \leq C|u_1|_{L_{qr'}}|u_2|_{H_{qr}^1},$$

where  $r, r' > 1$  and  $1/r + 1/r' = 1$ . We choose  $r$  in such a way that the Sobolev indices of the spaces  $L_{qr'}(\Omega)$  and  $H_{qr}^1(\Omega)$  are equal, which means

$$1 - \frac{n}{qr} = -\frac{n}{qr'}, \quad \text{equivalently} \quad \frac{n}{qr} = \frac{1}{2} \left( 1 + \frac{n}{q} \right).$$

This is feasible if  $q \in (1, n)$ , we assume this in the sequel. Next we employ Sobolev embeddings to obtain

$$X_\beta \subset H_q^{2\beta}(\Omega)^n \hookrightarrow L_{qr'}(\Omega)^n \cap H_{qr}^1(\Omega)^n.$$

This requires for the Sobolev index  $2\beta - n/q$  of  $H_q^{2\beta}(\Omega)$

$$1 - \frac{n}{qr} = 2\beta - \frac{n}{q}, \quad \text{i.e.,} \quad \beta = \frac{1}{4} \left( 1 + \frac{n}{q} \right).$$

The condition  $\beta < 1$  is equivalent to  $n/q < 3$ , we assume this below. To meet Conditions **(H2)**, **(H3)**, we set  $m = 1, \rho_1 = 1, \beta_1 = \beta$ . Then **(H3)** requires

$$2\beta \leq 1 + \mu - 1/p,$$

hence the optimal choice  $\mu = \mu_{\text{crit}}$  is

$$\mu_{\text{crit}} - \frac{1}{p} = 2\beta - 1 = \frac{1}{2} \left( \frac{n}{q} - 1 \right).$$

Finally, the constraint  $\mu \leq 1$  requires that  $p$  should be chosen large enough, to be subject to the condition

$$\frac{2}{p} + \frac{n}{q} \leq 3.$$

Computing the space of admissible initial values then leads to

$$X_{\gamma, \mu} = {}_0B_{qp}^{n/q-1}(\Omega)^n \cap L_{q, \sigma}(\Omega).$$

Applying Theorem 1.2, this yields the following result on local well-posedness of the Navier–Stokes system (3.1) for initial values in these critical spaces.

**THEOREM 3.1.** *Let  $q \in (1, n), p \in (1, \infty)$  be such that  $2/p + n/q \leq 3$ , and suppose that  $\Omega \subset \mathbb{R}^n$  is bounded domain of class  $C^{3-}$ .*

Then, for each initial value  $u_0 \in {}_0B_{qp}^{n/q-1}(\Omega)^n \cap L_{q,\sigma}(\Omega)$ , the Navier–Stokes problem (3.1) admits a unique strong solution  $u$  in the class

$$u \in H_{p,\mu}^1((0, T); L_{q,\sigma}(\Omega)) \cap L_{p,\mu}((0, T); H_q^2(\Omega)^n),$$

for some  $T = T(u_0) > 0$ , with  $\mu = 1/p + n/2q - 1/2$ . The solution exists on a maximal interval  $(0, t_+(u_0))$  and depends continuously on  $u_0$ . In addition, we have

$$u \in C([0, t_+); B_{qp}^{n/q-1}(\Omega)^n \cap L_{q,\sigma}(\Omega)) \cap C((0, t_+); B_{qp}^{2(1-1/p)}(\Omega)^n),$$

i.e., it regularizes instantly if  $2/p + n/q < 3$ .

It is an easy consequence of this result that we also have global existence for initial values, which are small in one of the critical spaces.

**COROLLARY 3.2.** *Let the assumptions of Theorem 3.1 be valid. Then there exists  $r > 0$  such that the solution from Theorem 3.1 exists globally, provided  $|u_0|_{B_{qp}^{n/q-1}} < r$ .*

*Proof.* By the estimate of  $F(u)$  given above, it is easy to show via maximal regularity that  $v(t) = u(t) - e^{-At}u_0$  satisfies

$$|v|_{\mathbb{E}_{1,\mu}(0,T)} \leq C_1|u_0|_{X_{\gamma,\mu}}^2 + C_2|v|_{\mathbb{E}_{1,\mu}(0,T)}^2,$$

for each  $T < t_+(u_0)$ . Here  $C_1, C_2 > 0$  are constants independent of  $u_0$  and  $T$ . This inequality implies boundedness of  $|v|_{\mathbb{E}_{1,\mu}(0,T)}$  on  $[0, t_+)$ , hence global existence, provided  $|u_0|_{X_{\gamma,\mu}} < r := 1/2\sqrt{C_1C_2}$ . □

**REMARK 3.3.** (i) Consider the particular case  $n = 3, p = q = 2$ . Then we have

$$X_{\gamma,\mu} = {}_0H_2^{1/2}(\Omega)^3 \cap L_{2,\sigma}(\Omega), \quad X_\gamma = {}_0H_2^1(\Omega)^3 \cap L_{2,\sigma}(\Omega),$$

which yields the celebrated Fujita–Kato theorem, proved first in 1962 by means of the famous Fujita–Kato iteration, see [3].

(ii) In the general case, observe that the Sobolev index of the spaces  $B_{qp}^{n/q-1}(\Omega)$  equals  $-1$ , it is independent of  $q$ . These are the critical spaces for the Navier–Stokes equations in  $nD$ , and their homogeneous versions are known to be scaling invariant in  $\Omega = \mathbb{R}^n$ . We refer to Cannone [2] for the first results in this direction.

In a forthcoming paper, we will extend the range of  $q$  to  $[n, \infty)$ . Thus, by the embeddings

$$B_{q_1,p_1}^{n/q_1-1}(\Omega) \hookrightarrow B_{q_2,p_2}^{n/q_2-1}(\Omega), \quad 1 \leq q_1 < q_2 < \infty, \quad p_1, p_2 \in [1, \infty]$$

and

$$B_{q,p_1}^s(\Omega) \hookrightarrow B_{q,p_2}^s(\Omega), \quad 1 \leq p_1 \leq p_2 \leq \infty,$$

and by maximal  $L_p$ -regularity, this will cover the range  $1 \leq q < \infty, 1 \leq p \leq \infty$ .

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