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Perturbation of analytic semigroups and applications to partial differential equations

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Dedicated to Rainer Nagel on the occasion of his 75th birthday

Abstract. In a recent paper we presented a general perturbation result for generators of C_0-semigroups, c.f. Theorem 2.1 below. The aim of the present work is to replace, in case the unperturbed semigroup is analytic, the various admissibility conditions appearing in this result by simpler inclusion assumptions on the domain and the range of the perturbation. This is done in Theorem 2.4 and allows to apply our results also in situations which are only in part governed by analytic semigroups. The power of our approach to treat in a unified and systematic way wide classes of PDE's is illustrated by a generic example, a degenerate differential operator with generalized Wentzell boundary conditions, a reaction diffusion equation with unbounded delay and a perturbed Laplacian.

1. Introduction

Many linear partial differential equations can be rewritten as an *Abstract Cauchy Problem*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\, x(t) = Gx(t), \quad t \geq 0, \\ x(0) = x_0 \end{cases} \tag{ACP}$$

for an unbounded (differential) operator G on a Banach space X, cf. [9, Chap. VI]. It is well known that (ACP) is well posed if and only if G generates a C_0-semigroup on X, cf. [9, Sect. II.6]. Being concerned with the generator property of G one basically relies on two methods, the Lumer–Phillips theorem in the dissipative and the Hille–Yosida theorem in the general case. However, the latter is based on growth estimate of all powers of the resolvent of G and can be verified explicitly only in very special cases.

In order to check well posedness of (ACP) for (nondissipative) operators G where direct computations involving the resolvent are impossible to perform, one can try to split G into a sum "$G = A + P$" for a simpler generator A and a perturbation P

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and then use some kind of perturbation theory to conclude that also G generates a C_0-semigroup on X.

In [2] we presented an abstract result in this direction (see also Theorem 2.1 below), which can be interpreted as a purely operator-theoretic approach to former work by Weiss [24, Thms. 6.1 & 7.2] and Staffans [23, Thms. 7.1.2 & 7.4.5] on the well posedness of linear closed-loop systems. More precisely, we considered operators G of the form

$$G = (A_{-1} + BC)|_X$$

where $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(Z, U)$, cf. Sect. 2.1 for more details. Our former result was based on various "admissibility" conditions, cf. Notation 2.3 below, and unified previous perturbation results due to Desch–Schappacher, Miyadera–Voigt and Greiner.

In the present paper we replace these admissibility assumptions for generators A of *analytic* semigroups by simple inclusions concerning the range of B and the domain of C with respect to certain intermediate spaces. We emphasize that our approach is highly versatile and allows treating various classes of differential equations in a unified and systematic way. Moreover, we point out that in contrast to other well-known results on the perturbation of analytic semigroups (see, e.g., [9, Sect. III.2]) we do *not* assume that $P = BC$ is relatively bounded with respect to A. On the contrary, in most cases the perturbation P will change the domain of A, i.e., we may have $D(G) \neq D(A)$ as in the examples in Sect. 3.

This paper is organized as follows. In Sect. 2 we first recall briefly the perturbation result from [2] and then show how it can be simplified in the analytic case to obtain our main result, Theorem 2.4. For its proof we use a characterization of analytic semigroups of angle $\theta \in (0, \frac{\pi}{2}]$, see Lemma 2.6, which might be of its own interest. Then, in Sect. 3 we illustrate the power of our approach by four examples. First, we consider a generic example which significantly extends boundary perturbations considered by Greiner and Greiner–Kuhn, cf. Corollaries 3.6 and 3.7. Next, we apply our results to

- a degenerate second-order differential equation on $\mathrm{C}[0, 1]$ with generalized Wentzell boundary conditions,
- a reaction–diffusion equation on $\mathrm{L}^p[0, \pi]$ subject to Neumann boundary conditions with distributed unbounded delay, and
- a second-order differential equation on $\mathrm{L}^2(\Omega)$ with perturbed Robin-type boundary conditions on some domain $\Omega \subset \mathbb{R}^n$.

In the Appendix we collect some results which are useful in order to verify the assumptions of our main results.

We mention that this is the first in a series of papers dedicated to applications of our perturbation result from [2]. In forthcoming works we will treat, among other, perturbations of operator matrices, generators of cosine families, diffusion on networks, complete second-order and delay differential equations.

2. Perturbation of generators

2.1. The abstract perturbation result

In this subsection we briefly recall the Weiss–Staffans type perturbation result from [2].

On the Banach spaces X, U and Z we consider the operators

- $A : D(A) \subset X \to X$,
- $B \in \mathcal{L}(U, X_{-1}^A)$,
- $C \in \mathcal{L}(Z, U)$

and assume that A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Here X_1^A and X_{-1}^A denote the inter- and extrapolated Sobolev spaces with respect to A (cf. [9, Sect. II.5.a]). Moreover, $Z = D(C)$ is a Banach space such that

$$X_1^A \hookrightarrow Z \hookrightarrow X,$$

where "$\hookrightarrow$" denotes a continuous injection. For a triple (A, B, C) as above and $t > 0$ we indicate by

$$\mathcal{F}_t^{(A,B,C)} : \mathrm{L}^p\big([0,t],U\big) \to \mathrm{L}^p\big([0,t],U\big)$$

the associated *input–output map*, i.e.,

$$\big(\mathcal{F}_t^{(A,B,C)}u\big)(r) = C\int_0^r T_{-1}(r-s)Bu(s)\,\mathrm{d}s \quad \text{for } u \in \mathrm{W}_0^{2,p}\big([0,t],U\big) \text{ and } r \in [0,t],$$

where $(T_{-1}(t))_{t\geq 0}$ denotes the extrapolated semigroup generated by A_{-1} on X_{-1}^A. Here for $p \geq 1$, $k \in \mathbb{N}$, an interval $I \subseteq \mathbb{R}$ such that $0 \in I$ and a Banach space V we define

$$\mathrm{W}_0^{k,p}(I,V) := \big\{f \in \mathrm{W}^{k,p}\big(I,V\big) : f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0\big\}.$$

Finally, we denote by $\mathrm{rg}(T)$ the range of a linear operator T. Then by [2, Thm. 10] the following holds.

THEOREM 2.1. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on X, $B \in \mathcal{L}(U, X_{-1}^A)$ and $C \in \mathcal{L}(Z, Y)$. Moreover, assume that there exist $1 \leq p < +\infty$, $t > 0$ and $M \geq 0$ such that*

(i) $\mathrm{rg}\big(R(\lambda, A_{-1})B\big) \subseteq Z$ *for some* $\lambda \in \rho(A)$,

(ii) $\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \in X$ *for all* $u \in \mathrm{L}^p\big([0,t],U\big)$,

(iii) $\int_0^t \big\|CT(s)x\big\|_U^p\,\mathrm{d}s \leq M \cdot \|x\|_X^p$ *for all* $x \in D(A)$,

(iv) $\int_0^t \Big\|C\int_0^r T_{-1}(r-s)Bu(s)\,\mathrm{d}s\Big\|_U^p\,\mathrm{d}r \leq M \cdot \|u\|_p^p$ *for all* $u \in \mathrm{W}_0^{2,p}\big([0,t],U\big)$,

(v) $1 \in \rho(\mathcal{F}_t^{(A,B,C)})$.

Then the operator

$$A_{BC} := (A_{-1} + BC)|_X, \quad D(A_{BC}) = \{x \in Z : (A_{-1} + BC)x \in X\} \qquad (2.1)$$

generates a C_0-semigroup $(T_{BC}(t))_{t\geq 0}$ on the Banach space X. Moreover, the perturbed semigroup verifies the variation of parameters formula

$$T_{BC}(t)x = T(t)x + \int_0^t T_{-1}(t-s) \cdot BC \cdot T_{BC}(s)x \,\mathrm{d}s \quad \textit{for all } t \geq 0 \textit{ and } x \in D(A_{BC}).$$

REMARK 2.2. Using the closed graph theorem one can show that condition (ii) in the previous result is equivalent to the estimate

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \right\|_X \leq M \cdot \|u\|_p \quad \text{for all } u \in \mathrm{W}^{1,p}\big([0,t], U\big) \qquad (2.2)$$

for some $M \geq 0$, cf. [2, Rem. 2].

NOTATION 2.3. It is convenient to use the following notions for operators A, B and C as above. Consider the conditions (i–v) in Theorem 2.1. Then

- the triple (A, B, C) is called *compatible*, if (i) holds,
- the operator B is called *p-admissible control operator*, if (ii) holds,
- the operator C is called *p-admissible observation operator*, if (iii) holds,
- the pair (B, C) is called *jointly p-admissible*, if (i–iv) hold,
- the identity $Id_U \in \mathcal{L}(U)$ is called *p-admissible feedback operator*, if (v) holds.

2.2. Perturbation of analytic semigroups

While Theorem 2.1 is valid for arbitrary semigroups, the following result is tailored for the analytic case. It substitutes the compatibility and admissibility conditions by inclusion relations between the range of B, the domain of C and certain intermediate spaces.

In the sequel $\omega_0(A)$ denotes the growth bound of the semigroup generated by A, cf. [9, Def. I.5.6], F_α^A is the Favard space of A of order $\alpha \in \mathbb{R}$, see [9, Sect. II.5.b], and $(-A)^\gamma$ indicates the fractional power of order $\gamma \in \mathbb{R}$ of A as in [9, Sect. II.5.c]. Finally, for a linear operator T we write $[D(T)] := (D(T), \|\bullet\|_T)$ with the graph norm $\|\bullet\|_T$ given by $\|x\|_T := \|x\| + \|Tx\|$ for $x \in D(T)$.

THEOREM 2.4. *Let $(A, D(A))$ generate an analytic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on X and assume that $B \in \mathcal{L}(U, X_{-1}^A)$ and $C \in \mathcal{L}(Z, U)$. For $\beta \in [0,1]$ and $\gamma \in (0,1]$ consider the following conditions:*

(i) $\mathrm{rg}(R(\lambda, A_{-1})B) \subseteq \mathrm{F}_{1-\beta}^A$.
(ii) $[D((\lambda - A)^\gamma)] \hookrightarrow Z$ *for some* $\lambda > \omega_0(A)$.
(iii) $\beta + \gamma < 1$.

Then the following holds.

(a) If (i) holds, then B is a p-admissible control operator for all $p > \frac{1}{1-\beta}$, and, if $\beta = 0$, then also for $p = 1$.
(b) If (ii) holds, then C is a p-admissible observation operator for all $p < \frac{1}{\gamma}$.
(c) If (i–iii) hold, then
  (1) (A, B, C) is compatible.
  (2) (B, C) is jointly p-admissible for all $\frac{1}{1-\beta} < p < \frac{1}{\gamma}$, and, if $\beta = 0$, then also for $p = 1$.
  (3) For every $0 < \varepsilon < 1 - (\beta + \gamma)$ and $\frac{1}{1-\beta} \le p < \frac{1}{\gamma}$ there exists $M \ge 0$ such that

$$\left\| \mathcal{F}_t^{(A,B,C)} \right\|_p \le M \cdot t^{\varepsilon} \quad \text{for all } 0 < t \le 1.$$

  Hence, $Id_U \in \mathcal{L}(U)$ is a p-admissible feedback operator for the triple (A, B, C).

In conclusion, conditions (i–iii) imply that the operator $(A_{-1} + BC)|_X$ generates an analytic C_0-semigroup of angle θ on X.

In order to prove part (a) and (c.2) of the above result we need the following quite crippled version of Young's inequality which, however, perfectly fits our needs. Here, for two functions K and v on $(0, t_0]$ for some $t_0 > 0$ we define their convolution by

$$(K * v)(t) := \int_0^t K(t-s)v(s)\,\mathrm{d}s, \quad t \in (0, t_0]$$

and put $(K * v)(0) := 0$.

LEMMA 2.5. *Let $K : (0, 1] \to \mathcal{L}(Y, X)$ be strongly continuous. Moreover, assume that $1 \le p, q, r \le +\infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $k(\bullet) := ||K(\bullet)||_{\mathcal{L}(Y,X)} \in \mathrm{L}^q[0, 1]$ and $v \in \mathrm{C}([0, 1], Y)$, then $K * v \in \mathrm{L}^r([0, 1], X)$ and*

$$\|K * v\|_r \le \|k\|_q \cdot \|v\|_p. \tag{2.3}$$

Proof. We adapt the proof of [3, Prop. 1.3.5] where convolutions on $\mathbb{R}_+$ are considered and K is assumed to be strongly continuous on $\mathbb{R}_+$. Fix $0 < t \le 1$ and $v \in \mathrm{C}([0, 1], Y)$. Then $s \mapsto b(s) := K(t-s)v(s)$ is continuous on $(0, t)$, hence measurable. Note that $k(t - \bullet) \in \mathrm{L}^q[0, t] \subseteq \mathrm{L}^1[0, t]$, thus the estimate

$$\|b(s)\| = \|K(t-s)v(s)\| \le k(t-s) \cdot \|v\|_\infty$$

implies that $||b(\bullet)||$ is integrable on $[0, t]$. By Bochner's theorem (see [3, Thm. 1.1.4]) this shows that b is integrable and hence $(K * v)(t)$ exists for all $t \in [0, 1]$. Next we show that $t \mapsto (K * v)(t)$ is continuous on $[0, 1]$. Let $t \in [0, 1]$ and $h \in \mathbb{R}$ such that $t + h \in [0, 1]$. Since $k \in \mathrm{L}^q[0, 1] \subseteq \mathrm{L}^1[0, 1]$ and $v \in \mathrm{C}([0, 1], Y)$ is uniformly continuous, we conclude

$$
\begin{aligned}
&\left\|(K * v)(t+h)-(K * v)(t)\right\| \\
&\quad \leq \int_0^t k(s) \cdot \left\|v(t+h-s)-v(t-s)\right\| \mathrm{d}s \\
&\qquad + \left|\int_t^{t+h} k(s) \cdot \left\|v(s+h-s)\right\| \mathrm{d}s\right| \\
&\quad \leq \|k\|_1 \cdot \sup_{s,s+h\in[0,1]} \left\|v(s+h)-v(s)\right\| + \left|\int_t^{t+h} k(s)\,\mathrm{d}s\right| \cdot \|v\|_\infty \\
&\quad \to 0 \text{ as } h \to 0.
\end{aligned}
$$

This shows that $K * v \in \mathrm{C}([0,1], X) \subset \mathrm{L}^r([0,1], X)$ and by the scalar-valued version of Young's inequality (see [21, Sect. IX.4, Expl. 1]) we finally obtain

$$\|K * v\|_r \leq \left\|k * \|v(\bullet)\|_Y\right\|_r \leq \|k\|_q \cdot \|v\|_p$$

as claimed. □

Now we are well prepared to give the

Proof of Theorem 2.4. Note that for every $\lambda > \omega_0(A)$ we have $\mathrm{F}^A_{1-\beta} = \mathrm{F}^{A-\lambda}_{1-\beta}$. Hence, replacing if necessary A by $A-\lambda$ we can assume without loss of generality that $\omega_0(A) < 0$ and $\lambda = 0$.

(a) Since $A^{-1}_{-1}B \in \mathcal{L}(U, X)$ and $\mathrm{F}^A_{1-\beta} \hookrightarrow X$, assumption (i) and the closed graph theorem imply that $A^{-1}_{-1}B \in \mathcal{L}(U, \mathrm{F}^A_{1-\beta})$. Hence, for all $u \in \mathrm{C}([0,1], U) \subset \mathrm{L}^p\big([0,1], U\big)$

$$v := A^{-1}_{-1}Bu \in \mathrm{C}\big([0,1], \mathrm{F}^A_{1-\beta}\big) \subset \mathrm{L}^p\big([0,1], \mathrm{F}^A_{1-\beta}\big). \tag{2.4}$$

Since $\operatorname{rg}(T(t)) \subseteq D(A^\infty)$ for all $t > 0$, we can define

$$K : (0,1] \to \mathcal{L}\big(\mathrm{F}^A_{1-\beta}, X\big), \quad K(t) := AT(t).$$

Then K is strongly continuous on $(0,1]$, and by [9, Prop. II.5.13] there exists $M > 0$ such that

$$\left\|t^\beta K(t)x\right\|_X \leq \sup_{s\in(0,1]} \left\|s^\beta AT(s)x\right\|_X \leq M \cdot \|x\|_{\mathrm{F}^A_{1-\beta}} \quad \text{for all } x \in \mathrm{F}^A_{1-\beta}.$$

This implies that

$$k(t) := \|K(t)\|_{\mathcal{L}(\mathrm{F}^A_{1-\beta}, X)} \leq M \cdot t^{-\beta} \quad \text{for all } t \in (0,1]. \tag{2.5}$$

Hence, $k \in \mathrm{L}^q[0,1]$ if $\beta \cdot q < 1$, i.e.,

$$k \in \mathrm{L}^q[0,1] \quad \text{if} \quad \begin{cases} q < \frac{1}{\beta} & \text{and } \beta > 0, \text{ or} \\ q \geq 1 & \text{and } \beta = 0. \end{cases}$$

Next we choose $r = +\infty$ in Young's inequality from Lemma 2.5. Then $q = \frac{p}{p-1}$ and from (2.3) it follows that there exists $M \geq 0$ such that for all $u \in \mathrm{C}([0,1],U)$

$$\left\| \int_0^1 T_{-1}(1-s)Bu(s)\,\mathrm{d}s \right\|_X = \big\|(K * v)(1)\big\|_X \leq \|K * v\|_\infty \\ \leq M \cdot \|k\|_q \cdot \|u\|_p$$

provided

$$\begin{cases} \dfrac{p}{p-1} = q < \dfrac{1}{\beta} \text{ and } \beta > 0 & \iff \quad p > \dfrac{1}{1-\beta} \text{ and } \beta > 0, \text{ or} \\ \dfrac{p}{p-1} = q \geq 1 \text{ and } \beta = 0 & \iff \quad p \geq 1 \text{ and } \beta = 0. \end{cases}$$

Since $\mathrm{W}^{1,p}([0,1],U) \subset \mathrm{C}([0,1],U)$, the assertion follows from Remark 2.2.

(b) For all $t > 0$ we have by (ii)

$$\big\|CT(t)\big\|_{\mathcal{L}(X,U)} \leq \big\|C(-A)^{-\gamma}\big\|_{\mathcal{L}(X,U)} \cdot \big\|(-A)^\gamma T(t)\big\|_{\mathcal{L}(X)}.$$

Since by [22, Lem. 12.36] there exists $M \geq 0$ such that

$$\big\|(-A)^\gamma T(t)\big\|_{\mathcal{L}(X)} \leq M \cdot t^{-\gamma} \quad \text{for all } t \in (0,1], \tag{2.6}$$

we conclude that C is a p-admissible observation operator for all $p < \frac{1}{\gamma}$.

(c.1) By [9, Props. II.5.14 & 5.33] we have

$$D\big((-A)^\alpha\big) \hookrightarrow \mathrm{F}_\alpha^A \hookrightarrow D\big((-A)^\delta\big) \quad \text{for all } 1 > \alpha > \delta > 0. \tag{2.7}$$

Since by assumption (iii) we have $1 - \beta > \gamma$, (2.7) and (ii) imply

$$\mathrm{rg}\big(A_{-1}^{-1}B\big) \subseteq \mathrm{F}_{1-\beta}^A \subseteq D\big((-A)^\gamma\big) \subseteq Z = D(C),$$

i.e., the triple (A, B, C) is compatible.

(c.2) Since $\mathrm{rg}(T(t)) \subseteq D(A^\infty)$ we can define

$$L : (0,1] \to \mathcal{L}\big(\mathrm{F}_{1-\beta}^A, X\big), \quad L(t) := (-A)^{1+\gamma}T(t).$$

Then L is strongly continuous on $(0,1]$. Using (2.5) and (2.6) we obtain for $0 < t \leq 1$ and suitable $M \geq 0$

$$l(t) := \big\|L(t)\big\|_{\mathcal{L}(\mathrm{F}_{1-\beta}^A, X)} \leq \big\|(-A)^\gamma T\big(\tfrac{t}{2}\big)\big\|_{\mathcal{L}(X)} \cdot \big\|AT\big(\tfrac{t}{2}\big)\big\|_{\mathcal{L}(\mathrm{F}_{1-\beta}^A, X)} \\ \leq M \cdot t^{-(\beta+\gamma)}. \tag{2.8}$$

Now choose in Young's inequality from Lemma 2.5 $p = \frac{1}{1-\beta} \leq r < \frac{1}{\gamma}$. Then we obtain $\frac{1}{q} = \beta + \frac{1}{r} > \beta + \gamma$, and hence

$$q \cdot (\beta + \gamma) < 1$$

which by (2.8) implies that $l \in \mathrm{L}^q[0,1]$. Thus, by (2.3) there exists $M \geq 0$ such that the input–output map $\mathcal{F}_t := \mathcal{F}_t^{(A,B,C)}$ for all $0 < t \leq 1$ and $u \in \mathrm{C}([0,1],U)$ satisfies

$$\begin{aligned} \left\|\mathcal{F}_t u\right\|_r &\leq \left(\int_0^1 \left\|C\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_U^r \mathrm{d}t\right)^{\frac{1}{r}} \\ &\leq \left\|C(-A)^{-\gamma}\right\| \cdot \left(\int_0^1 \left\|\int_0^t (-A)^{1+\gamma}T_{-1}(t-s)\cdot A_{-1}^{-1}Bu(s)\,\mathrm{d}s\right\|_X^r \mathrm{d}t\right)^{\frac{1}{r}} \\ &\leq M\cdot\left\|L * v\right\|_r \\ &\leq M\cdot\|l\|_q\cdot\|u\|_{\frac{1}{1-\beta}} \end{aligned}$$

where $v \in \mathrm{C}\big([0,1],\mathrm{F}_{1-\beta}^A\big)$ is given by (2.4). This shows that for every $\frac{1}{1-\beta} \leq r < \frac{1}{\gamma}$ and $0 < t \leq 1$ the input–output map has a unique bounded extension

$$\mathcal{F}_t : \mathrm{L}^{\frac{1}{1-\beta}}\big([0,t],U\big) \to \mathrm{L}^r\big([0,t],U\big). \tag{2.9}$$

Since $\mathrm{L}^r\big([0,t],U\big) \hookrightarrow \mathrm{L}^{\frac{1}{1-\beta}}\big([0,t],U\big)$ for $r \geq \frac{1}{1-\beta}$, this together with (b) and (c) proves that the pair (B,C) is jointly p-admissible for all $p \in (\frac{1}{1-\beta},\frac{1}{\gamma})$ and in case $\beta = 0$ also for $p = 1$.

(c.3) By Jensen's inequality we have for all $1 \leq p \leq r < +\infty$ and $u \in \mathrm{L}^r\big([0,t],U\big) \subseteq \mathrm{L}^p\big([0,t],U\big)$

$$\|u\|_p \leq t^{\frac{1}{p}-\frac{1}{r}}\cdot\|u\|_r.$$

This combined with (2.9) gives for all $\frac{1}{1-\beta} \leq p \leq r < \frac{1}{\gamma}$ and $u \in \mathrm{L}^r\big([0,t],U\big)$ that

$$t^{-\frac{1}{p}+\frac{1}{r}}\cdot\|\mathcal{F}_t u\|_p \leq \|\mathcal{F}_t u\|_r \leq M\cdot\|u\|_{\frac{1}{1-\beta}} \leq M\cdot t^{1-\beta-\frac{1}{p}}\cdot\|u\|_p.$$

For given $0 < \varepsilon < 1-(\beta+\gamma)$ we take $r := \frac{1}{1-\beta-\varepsilon} \in \big(\frac{1}{1-\beta},\frac{1}{\gamma}\big)$ and obtain by density of $\mathrm{L}^r\big([0,t],U\big)$ in $\mathrm{L}^p\big([0,t],U\big)$ that

$$\|\mathcal{F}_t\|_p \leq M\cdot t^{1-\beta-\frac{1}{r}} \leq M\cdot t^{\varepsilon}$$

as claimed. Clearly (a–c) combined with Theorem 2.1 imply that $(A_{-1}+BC)|_X$ generates a C_0-semigroup. This semigroup is analytic of angle θ by the following two lemmas. More precisely, Lemma 2.7 allows us to repeat the above reasoning for A, B, C replaced by $e^{i\varphi}A$, $e^{i\varphi}B$, C to obtain that also $e^{i\varphi}A_{BC}$ is a generator of a C_0-semigroup on X for all $\varphi \in (-\theta,\theta)$. By Lemma 2.6 this implies the assertion. □

In the following for an operator A and $\varphi \in \mathbb{R}$ we use the notation

$$A_\varphi := e^{i\varphi}A.$$

LEMMA 2.6. *Let* $0 < \theta \leq \frac{\pi}{2}$. *Then* A *generates an analytic semigroup of angle* θ *if and only if* A_φ *generates a* C_0*-semigroup for every* $\varphi \in (-\theta,\theta)$.

Proof. Assume first that A generates an analytic semigroup $(T(z))_{z\in\Sigma_\theta\cup\{0\}}$ of angle θ where Σ_θ denotes the open sector

$$\Sigma_\theta := \big\{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \theta\big\}.$$

Then it is clear that for every $\varphi \in (-\theta, \theta)$ the operators $T_\varphi(t) := T(e^{i\varphi}t)$ define a strongly continuous semigroup $(T_\varphi(t))_{t\geq 0}$ with generator A_φ, cf. [3, Prop. 3.7.2.(c)].

Conversely, assume that A_φ generates a C_0-semigroup $(T_\varphi(t))_{t\geq 0}$ for every $\varphi \in (-\theta, \theta)$. Then by [9, Thm. II.4.6.(b)] the operator A generates an analytic semigroup $(T(z))_{z\in\Sigma_{\theta'}\cup\{0\}}$ of some angle $\theta' > 0$. If $\theta' \geq \theta$, we are done and hence assume that $\theta' < \theta$. Then we have to show that the map $z \mapsto T(z)$ can be extended analytically from $\Sigma_{\theta'}$ to the open sector Σ_θ.

To this end we fix some $\varphi \in (\theta', \theta)$ and consider the two projections on the complex plane $P_{\pm\varphi} : \mathbb{C} \to \mathbb{C}$ onto $e^{\pm i\varphi} \cdot \mathbb{R}$ along $e^{\mp\varphi i} \cdot \mathbb{R}$. Then for $z \in \overline{\Sigma}_\varphi$ we put $r_\pm(z) := e^{\mp i\varphi} \cdot P_{\pm\varphi} z \geq 0$. Since $P_{\pm\varphi} = 1 - P_{\mp\varphi}$, this implies $z = r_+(z) \cdot e^{i\varphi} + r_-(z) \cdot e^{-i\varphi}$. Using this representation of z we define

$$\widetilde{T} : \overline{\Sigma}_\varphi \to \mathcal{L}(X), \quad \widetilde{T}(z) := T_\varphi\big(r_+(z)\big) \cdot T_{-\varphi}\big(r_-(z)\big). \tag{2.10}$$

Since the resolvents of $A_{\pm\varphi}$ commute, also the semigroups $(T_{\pm\varphi}(t))_{t\geq 0}$ commute. Using this fact and the equations $r_\pm(z + w) = r_\pm(z) + r_\pm(w)$ it follows that

$$\widetilde{T}(z) \cdot \widetilde{T}(w) = \widetilde{T}(z + w) \quad \text{for all } z, w \in \overline{\Sigma}_\varphi.$$

Next we show that $(\widetilde{T}(z))_{z\in\overline{\Sigma}_\varphi}$ is strongly continuous on the closed sector $\overline{\Sigma}_\varphi$. To this end choose $M, \omega > 0$ such that $\|T_{\pm\varphi}(t)\| \leq M \cdot e^{\omega t}$ for all $t \geq 0$. Then from the continuity of $r_\pm(\bullet)$ and the fact that $r_\pm(z) \leq \|P_{\pm\varphi}\| \cdot |z|$ we obtain for $x \in X$ and $z, w \in \overline{\Sigma}_\varphi$

$$\begin{aligned}
\big\|\widetilde{T}(z)x - \widetilde{T}(w)x\big\| &\leq \big\|T_\varphi\big(r_+(z)\big) \cdot \big[T_{-\varphi}\big(r_-(z)\big) - T_{-\varphi}\big(r_-(w)\big)\big]x\big\| \\
&\quad + \big\|\big[T_\varphi\big(r_+(z)\big) - T_\varphi\big(r_+(w)\big)\big] \cdot T_{-\varphi}\big(r_-(w)\big)x\big\| \\
&\leq Me^{\omega\left(\|P_\varphi\|+\|P_{-\varphi}\|\right)\cdot|z|} \cdot \Big(\big\|T_{-\varphi}\big(r_-(z)\big)x - T_{-\varphi}\big(r_-(w)\big)x\big\| \\
&\quad + \big\|T_\varphi\big(r_+(z)\big)x - T_\varphi\big(r_+(w)\big)x\big\|\Big) \\
&\to 0 \text{ as } w \to z \text{ in } \overline{\Sigma}_\varphi.
\end{aligned}$$

Hence, $(\widetilde{T}(z))_{z\in\overline{\Sigma}_\varphi}$ is strongly continuous as claimed. This implies in particular that for every $\psi \in [-\varphi, \varphi]$ the restriction

$$\widetilde{T}_\psi(t) := \widetilde{T}\big(e^{i\psi}t\big), \quad t \geq 0$$

defines a C_0-semigroup on X. Next we compute its generator $\tilde{A}_\psi$. Let

$$r_\pm := r_\pm\big(e^{i\psi}\big), \quad \text{i.e.} \quad e^{i\psi} = r_+ \cdot e^{i\varphi} + r_- \cdot e^{-i\varphi}.$$

Then by definition of $\widetilde{T}(z)$ in (2.10) we have $\widetilde{T}_\psi(t) = T_\varphi(r_+t) \cdot T_{-\varphi}(r_-t)$. Hence, for $x \in D(A)$ we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{T}_\psi(t)x &= r_+A_\varphi \cdot T_\varphi(r_+t) \cdot T_{-\varphi}(r_-t)x + r_-A_{-\varphi} \cdot T_\varphi(r_+t) \cdot T_{-\varphi}(r_-t)x \\ &= \big(r_+e^{i\varphi}A + r_-e^{-i\varphi}A\big) \cdot \widetilde{T}(z)x = e^{i\psi}A \cdot \widetilde{T}(z)x.\end{aligned}$$

This implies $e^{i\psi}A \subseteq \tilde{A}_\psi$, and since $\rho(e^{i\psi}A) \cap \rho(\tilde{A}_\psi) \neq \emptyset$, we obtain $\tilde{A}_\psi = e^{i\psi}A = A_\psi$. Since a generator uniquely determines the generated semigroup, we conclude that

$$T(z) = \widetilde{T}(z) \quad \text{for all } z \in \Sigma_{\theta'},$$

i.e., $(\widetilde{T}(z))_{z\in\overline{\Sigma}_\varphi}$ is a strongly continuous extension of $(T(z))_{z\in\Sigma_{\theta'}\cup\{0\}}$. For this reason from now on we can drop the tilde and write $T(z) = \widetilde{T}(z)$ for all $z \in \overline{\Sigma}_\varphi$.

Summing up, we showed that A generates a semigroup $(T(z))_{z\in\overline{\Sigma}_\varphi}$ which is strongly continuous on $\overline{\Sigma}_\varphi$ and analytic on $\Sigma_{\theta'}$. It remains to show that $(T(z))_{z\in\overline{\Sigma}_\varphi}$ is analytic on Σ_φ. To this end note that for each $r > 0$

$$\begin{aligned}&z \mapsto T\big(re^{\pm i\varphi}\big) \cdot T(z) = T\big(re^{\pm i\varphi} + z\big) \quad \text{is analytic on } \Sigma_{\theta'} \quad \Longrightarrow \\ &z \mapsto T(z) \quad \text{is analytic on } re^{\pm i\varphi} + \Sigma_{\theta'}.\end{aligned}$$

Since

$$\Sigma_\varphi = \bigcup_{r>0} \Big(re^{\pm i\varphi} + \Sigma_{\theta'}\Big)$$

this implies that $(T(z))_{z\in\overline{\Sigma}_\varphi}$ is analytic on the whole open sector Σ_φ as claimed. Recall that $\varphi \in (\theta', \theta)$ was arbitrary. Thus, from

$$\Sigma_\theta = \bigcup_{\varphi\in(-\theta,\theta)} \Sigma_\varphi$$

we finally conclude that $(T(t))_{t\geq 0}$ can be extended to an analytic semigroup $(T(z))_{z\in\overline{\Sigma}_\theta\cup\{0\}}$, i.e., is analytic of angle (at least) θ. □

LEMMA 2.7. *Let A generate an analytic semigroup of angle $\theta \in (0, \frac{\pi}{2})$. Moreover, let $\varphi \in (-\theta, \theta)$ and $\lambda > 0$ such that $\omega_0(A-\lambda), \omega_0(A_\varphi-\lambda) < 0$. Then for all $\alpha \in (0, 1]$ one has*

$$D\big((\lambda - A)^\alpha\big) = D\big((\lambda - A_\varphi)^\alpha\big) \quad \text{and} \quad \mathrm{F}_\alpha^A = \mathrm{F}_\alpha^{A_\varphi}. \tag{2.11}$$

Proof. Note that by the previous result A_φ generates an analytic semigroup. Without loss of generality we assume that $\lambda = 0$.

To show the first equality in (2.11) fix some $\alpha \in (0, 1)$. Then by the definition of $(-A)^{-\alpha}$ (see, e.g., [9, Def. II.5.25]), the equality

$$R(\lambda, A_\varphi) = e^{-i\varphi}R\big(e^{-i\varphi}\lambda, A\big) \tag{2.12}$$

and Cauchy's integral theorem it follows that $(-A_\varphi)^{-\alpha} = e^{-i\varphi\alpha} \cdot (-A)^{-\alpha}$. This implies

$$D((-A)^\alpha) = \mathrm{rg}((-A)^{-\alpha}) = \mathrm{rg}((-A_\varphi)^{-\alpha}) = D((-A_\varphi)^\alpha)$$

for $\alpha \in (0, 1)$ while for $\alpha = 1$ it is obviously satisfied.

Next we show that $\mathrm{F}_\alpha^A \subseteq \mathrm{F}_\alpha^{A_\varphi}$ for $\alpha \in (0, 1]$. Let $x \in \mathrm{F}_\alpha^A$. Then by (2.12), the resolvent equation, the Hille–Yosida theorem for A_φ and [9, Prop. II.5.12] we conclude that

$$\begin{aligned}
\sup_{\lambda>0}\bigl\|\lambda^\alpha A_\varphi R(\lambda, A_\varphi)x\bigr\| &= \sup_{\lambda>0}\bigl\|\lambda^\alpha A\bigl(R(e^{-i\varphi}\lambda, A) - R(\lambda, A)\bigr)x + \lambda^\alpha AR(\lambda, A)x\bigr\| \\
&\le \sup_{\lambda>0}\bigl\|(1 - e^{-i\varphi})\lambda R(e^{-i\varphi}\lambda, A)\lambda^\alpha AR(\lambda, A)x\bigr\| \\
&\quad + \sup_{\lambda>0}\bigl\|\lambda^\alpha AR(\lambda, A)x\bigr\| \\
&\le \left(1 + \sup_{\lambda>0}\bigl\|(1 - e^{-i\varphi})\lambda R(\lambda, A_\varphi)\bigr\|\right) \cdot \sup_{\lambda>0}\bigl\|\lambda^\alpha AR(\lambda, A)x\bigr\| \\
&< +\infty.
\end{aligned}$$

Again by [9, Prop. II.5.12] this implies that $x \in \mathrm{F}_\alpha^{A_\varphi}$, hence $\mathrm{F}_\alpha^A \subseteq \mathrm{F}_\alpha^{A_\varphi}$. In order to show the converse inclusion note that $A = e^{-i\varphi}A_\varphi$. The assertion then follows as above by interchanging the roles of A and A_φ and substituting φ by $-\varphi$. □

3. Examples

3.1. The generic example

Many concrete examples fit into the following general framework which generalizes boundary perturbations in the sense of Greiner, cf. [12].

We start with a Banach space X and a linear "maximal operator"[1] $A_m : D(A_m) \subseteq X \to X$. In order to single out a restriction A of A_m we take a Banach space ∂X, called "space of boundary conditions," and a linear "boundary operator" $L : D(A_m) \to \partial X$ and define

$$A \subseteq A_m, \quad D(A) = \bigl\{x \in D(A_m) : Lx = 0\bigr\} = \ker(L). \tag{3.1}$$

Next we perturb A in the following way. For a Banach space Z satisfying $D(A_m) \subseteq Z$ and $X_1^A \hookrightarrow Z \hookrightarrow X$, and operators $P \in \mathcal{L}(Z, X)$ and $\Phi \in \mathcal{L}(Z, \partial X)$ we consider

$$G \subseteq A_m + P, \quad D(G) := \bigl\{x \in D(A_m) : Lx = \Phi x\bigr\} = \ker(L - \Phi). \tag{3.2}$$

cf. Fig. 1.

Hence, G can be considered as a twofold perturbation of A,

[1] "Maximal" in the sense of a "big" domain, e.g., a differential operator without boundary conditions.

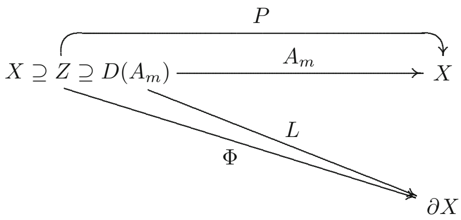

Figure 1. The operators defining G in (3.2)

- by the operator P to change its action, and
- by the operator Φ to change its domain.

We note that in [12] the operator $\Phi : X \to \partial X$ has to be bounded and $P = 0$. Below we will show that this example fits into our framework for unbounded operators Φ and P, too. To this end we first make the following

ASSUMPTIONS 3.1. (a) A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X.
(b) For some $\mu \in \mathbb{C}$ the restriction

$$L|_{\ker(\mu - A_m)} : \ker(\mu - A_m) \to \partial X$$

is invertible with bounded inverse

$$L_\mu := \big(L|_{\ker(\mu - A_m)}\big)^{-1} \in \mathcal{L}(\partial X, X).$$

Next we elaborate on the so-called abstract *Dirichlet operator* L_μ which plays a crucial role in this approach. Note that in contrast to the setting in [12, (1.13)] we do not assume that $D(A_m)$ equipped with some norm finer than $\|\bullet\|_X$ is complete.

PROPOSITION 3.2. *Let Assumption 3.1.(b) be satisfied. Then for all* $\lambda \in \rho(A)$

$$L|_{\ker(\lambda - A_m)} : \ker(\lambda - A_m) \to \partial X$$

is invertible with bounded inverse given by

$$L_\lambda = (\mu - A)R(\lambda, A)L_\mu \in \mathcal{L}(\partial X, X). \tag{3.3}$$

Proof. Let $\tilde{L}_\lambda \in \mathcal{L}(\partial X, X)$ be the operator defined by the right-hand side of (3.3). Then the identity $\tilde{L}_\lambda = \big(Id + (\mu - \lambda)R(\lambda, A)\big)L_\mu$ implies that $\mathrm{rg}(\tilde{L}_\lambda) \subseteq \ker(\mu - A_m) + D(A) \subseteq D(A_m)$ and $L\tilde{L}_\lambda = Id_{\partial X}$. Moreover, for $x \in \partial X$

$$\begin{aligned}(\lambda - A_m)\tilde{L}_\lambda x &= (\lambda - A_m)L_\mu x + (\mu - \lambda)(\lambda - A_m)R(\lambda, A)L_\mu x \\ &= (\lambda - \mu)L_\mu x + (\mu - \lambda)L_\mu x = 0,\end{aligned}$$

i.e., $\mathrm{rg}(\tilde{L}_\lambda) \subseteq \ker(\lambda - A_m)$. Summing up this proves that $L : \ker(\lambda - A_m) \to \partial X$ is surjective with right-inverse $\tilde{L}_\lambda$. To show injectivity assume that $x \in \ker(\lambda - A_m) \cap \ker(L)$. Then $x \in D(A)$ and $(\lambda - A)x = 0$ which implies $x = 0$ since $\lambda \in \rho(A)$. $\square$

Note that by the previous result $L_\lambda = R(\lambda, A_{-1})(\mu - A_{-1})L_\mu$; hence, the operator

$$L_A := (\mu - A_{-1})L_\mu = (\lambda - A_{-1})L_\lambda \in \mathcal{L}(\partial X, X_{-1}^A) \tag{3.4}$$

is independent of $\lambda \in \rho(A)$.

The following result gives sufficient conditions implying Assumption 3.1.(b). For a proof we refer to [12, Lem. 1.2] and [6, Lem. 2.2].

LEMMA 3.3. *If L is surjective and either*

- *A_m is closed and $L \in \mathcal{L}([D(A_m)], \partial X)$, or*
- *$\binom{A_m-\mu}{L} : D(A_m) \subset X \to X \times \partial X$ is closed for some (hence for all) $\mu \in \mathbb{C}$,*[2]

then for every $\lambda \in \rho(A)$, Assumption 3.1.(b) is satisfied.

Using the operator L_A from (3.4) we obtain the following representation of G in (3.2).

LEMMA 3.4. *We have*

$$G = \big(A_{-1} + P + L_A \cdot \Phi\big)\big|_X. \tag{3.5}$$

Proof. Denote by $\widetilde{G}$ the operator defined by the right-hand-side of (3.5) and fix some $\lambda \in \rho(A)$. Then for $x \in Z$ we have

$$\begin{aligned} x \in D(\widetilde{G}) &\iff (A_{-1} - \lambda)\big(Id - L_\lambda \Phi\big)x + (P + \lambda)x \in X \\ &\iff \big(Id - L_\lambda \Phi\big)x \in D(A) = \ker L \\ &\iff Lx = \Phi x \\ &\iff x \in D(G), \end{aligned} \tag{3.6}$$

where in (3.6) we used that $x = \big(Id - L_\lambda \Phi\big)x + L_\lambda \Phi x \in D(A) + \ker(\lambda - A_m) \subseteq D(A_m)$. Moreover, for $x \in D(G)$ we obtain

$$\begin{aligned} \widetilde{G}x &= (A_m - \lambda)\big(Id - L_\lambda \Phi\big)x + (P + \lambda)x \\ &= (A_m - \lambda)x + (P + \lambda)x \\ &= (A_m + P)x = Gx, \end{aligned}$$

hence $G = \widetilde{G}$ as claimed. □

In order to represent G given in (3.5) as A_{BC} like in (2.1), we define the product space

$$U := X \times \partial X \tag{3.7}$$

and the operators

$$B := \big(Id_X, L_A\big) \in \mathcal{L}(U, X_{-1}^A) \quad \text{and} \quad C := \binom{P}{\Phi} \in \mathcal{L}(Z, U). \tag{3.8}$$

Then a simple computation shows the following.

[2] In [6, Rem. 3.3], the authors present an example showing that $\binom{A_m}{L}$ is closed while A_m is not closed.

LEMMA 3.5. *The triple* (A, B, C) *given by* (3.1), (3.8) *is compatible. Moreover,* G *in* (3.2) *can be written as* $G = A_{BC}$.

By applying Theorem 2.1 to this situation we obtain the following result.

COROLLARY 3.6. *Assume that for some* $1 \le p < +\infty$ *the pairs* (L_A, P) *and* (L_A, Φ) *are jointly* p*-admissible for* A *and that there exists* $t > 0$ *such that* $1 \in \rho(\mathcal{F}_t^{(A,B,C)})$ *where*

$$\mathcal{F}_t^{(A,B,C)} = \begin{pmatrix} \mathcal{F}_t^{(A,Id_X,P)} & \mathcal{F}_t^{(A,L_A,P)} \\ \mathcal{F}_t^{(A,Id_X,\Phi)} & \mathcal{F}_t^{(A,L_A,\Phi)} \end{pmatrix} \in \mathcal{L}\big(X \times \partial X\big).$$

Then G *given by* (3.2) *generates a* C_0*-semigroup on* X*. Here the condition* $1 \in \rho(\mathcal{F}_t^{(A,B,C)})$ *is in particular satisfied if* $p > 1$ *and* $1 \in \rho(\mathcal{F}_t^{(A,L_A,\Phi)})$.

Proof. We only have to verify the assertion concerning the invertibility of $Id - \mathcal{F}_t^{(A,B,C)}$ for $p > 1$. This, however, follows immediately from Lemma A.3.(b) by the use of Schur complements (cf. [20, Lem. 2.1]). □

We note that Corollary 3.6 can be regarded as an operator-theoretic extension of the main result in [15] (Theorem 4.1) which is formulated in the language of systems theory and where $P = 0$ is assumed.

If in the above situation A generates an analytic semigroup, then from Theorem 2.4 and Lemma A.1 we obtain the following simplification.

COROLLARY 3.7. *Let* A *generate an analytic semigroup of angle* $\theta \in (0, \frac{\pi}{2}]$ *on* X*. If there exist* $\lambda > \omega_0(A)$, $\beta \ge 0$ *and* $\gamma > 0$ *such that*

(i) $\operatorname{rg}(L_\lambda) \subseteq \mathrm{F}^A_{1-\beta}$,
(ii) $[D((\lambda - A)^\gamma)] \hookrightarrow Z$,
(iii) $\beta + \gamma < 1$,

then G *given by* (3.2) *generates an analytic semigroup of angle* θ *on* X.

REMARK 3.8. The previous corollary improves [13, Thm. 2.6.(c)] (see [14,16,19] as well) where, by means of a resolvent estimate, a similar result in the context of abstract Hölder spaces is proved. In contrast to our approach, the approaches in the above references are not applicable to problems like Example 3.3 where only part of the system is governed by an analytic semigroup, but the semigroup associated with the whole system is not analytic, cf. Remark 3.13. We emphasize that nevertheless we can use the results in Theorem 2.4 to establish the admissibility of the related operators in Example 3.3.

3.2. A degenerate second-order differential operator on C[0, 1] with generalized Wentzell boundary conditions

As a first concrete application of our approach we prove the following generation result.

THEOREM 3.9. *Let* $a \in \mathrm{C}[0,1]$ *with* $a(s) > 0$ *for* $s \in (0,1)$ *such that* $\frac{1}{a} \in \mathrm{L}^1[0,1]$ *and* $[0,1] \ni s \mapsto \int_0^s \frac{1}{a(r)}\,\mathrm{d}r$ *is Hölder-continuous of exponent* $\delta \in (0,1]$. *Moreover, define* A_m *on* $\mathrm{C}[0,1]$ *by*

$$A_m f := a \cdot f'', \quad D(A_m) := \big\{ f \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) : a \cdot f'' \in \mathrm{C}[0,1] \big\}. \qquad (3.9)$$

Then $D(A_m) \subset \mathrm{C}^1[0,1]$ *and for all* $b, c \in \mathrm{C}[0,1]$ *and* $\Phi \in \mathcal{L}(\mathrm{C}^1[0,1], \mathbb{C}^2)$ *the operator*

$$Gf := A_m f + bf' + cf, \quad D(G) := \Big\{ f \in D(A_m) : \tbinom{(A_m f)(0)}{(A_m f)(1)} = \Phi f \Big\} \qquad (3.10)$$

generates a compact, analytic semigroup of angle $\frac{\pi}{2}$ *on* $\mathrm{C}[0,1]$.

Proof. Since $\frac{1}{a} \in \mathrm{L}^1[0,1]$, it follows for $f \in D(A_m)$ that $f'' = \frac{1}{a} \cdot A_m f \in \mathrm{L}^1[0,1]$. Hence, $f \in \mathrm{W}^{2,1}[0,1] \subset \mathrm{C}^1[0,1]$ and therefore $D(A_m) \subset \mathrm{C}^1[0,1]$. Next we define the operator $A \subset A_m$ with domain

$$D(A) := \Big\{ f \in D(A_m) : \tbinom{(A_m f)(0)}{(A_m f)(1)} = 0 \Big\}.$$

Moreover, we choose $X := \mathrm{C}[0,1]$, $\partial X := \mathbb{C}^2$, $Z := \mathrm{C}^1[0,1]$, $L := \binom{\delta_0 A_m}{\delta_1 A_m} : D(A_m) \to \partial X$ and $P := b(\bullet)\frac{\mathrm{d}}{\mathrm{d}s} + c(\bullet)$. Then the operators defined in (3.2) and (3.10) coincide.

We proceed by verifying the assumptions of Corollary 3.7. Firstly, by [5, Thm. 4.2] the operator A generates an analytic semigroup of angle $\frac{\pi}{2}$ on X. Hence, Assumption 3.1.(a) is satisfied.

(i) As shown in the proof of [8, Cor. 4.1, part (ii)] the operator $\tilde{A} \subset A_m$ with domain

$$D(\tilde{A}) := \Big\{ f \in D(A_m) : \tbinom{f(0)}{f(1)} = 0 \Big\}$$

is dissipative and invertible. Moreover, $[0, +\infty) \subset \rho(\tilde{A})$ and $\|\tilde{A}R(\lambda, \tilde{A})\| \le 2$ for all $\lambda \ge 0$. Next, let $\varepsilon_0(s) := 1 - s$ and $\varepsilon_1(s) := s$ for $s \in [0,1]$. Then $\tilde{L}_0 \in \mathcal{L}(\partial X, X)$ where $\tilde{L}_0 \binom{x_0}{x_1} := x_0 \cdot \varepsilon_0 + x_1 \cdot \varepsilon_1 \in \ker(A_m)$.[3] For $\lambda > 0$ define

$$L_\lambda := -\tfrac{1}{\lambda} \cdot \tilde{A}R(\lambda, \tilde{A})\tilde{L}_0 \in \mathcal{L}(\partial X, X).$$

Then essentially the same computations as in the proof of Proposition 3.2 show that L_λ is indeed the abstract Dirichlet operator for A_m and the (Wentzell-type) boundary operator $L = \binom{\delta_0 A_m}{\delta_1 A_m}$. Therefore, Assumption 3.1.(b) is satisfied. Moreover, $\|L_\lambda\| \le \frac{2}{\lambda} \cdot \|L_0\|$ for all $\lambda > 0$ and hence $\mathrm{rg}(L_\lambda) \subseteq \mathrm{F}_1^A$ by Lemma A.1.(a). This shows (i) for $\beta = 0$.

[3] $\tilde{L}_0$ is just the abstract Dirichlet operator for A_m and the (Dirichlet-type) boundary operator $\tilde{L} := \binom{\delta_0}{\delta_1}$.

(ii) As in the proof of [8, Cor. 4.1, part (iii)] Taylor's formula implies for $f \in D(A)$, $s \in [0,1]$ and $0 \neq \varepsilon \in (-1,1)$ such that $s+\varepsilon \in [0,1]$ the estimate

$$\begin{aligned} \left|f'(s)\right| &\leq \frac{2}{|\varepsilon|} \cdot \|f\|_\infty + \left| \int_s^{s+\varepsilon} \frac{\mathrm{d}r}{a(r)} \right| \cdot \|Af\|_\infty \\ &\leq \frac{2}{|\varepsilon|} \cdot \|f\|_\infty + M \cdot |\varepsilon|^\delta \cdot \|Af\|_\infty \end{aligned}$$

for some $M \geq 0$, where in the second inequality we used the assumption on the Hölder continuity of the antiderivative of $\frac{1}{a}$. Choosing $\rho := |\varepsilon|^{-(1+\delta)} > 1$ and $\alpha := \frac{1}{1+\delta} \in [\frac{1}{2}, 1)$ we obtain for $f \in D(A)$

$$\left\| \frac{\mathrm{d}}{\mathrm{d}s} f \right\|_\infty \leq (M+2) \cdot \left(\rho^\alpha \|f\|_\infty + \rho^{\alpha-1} \|Af\|_\infty\right)$$

for all $\rho > 1$. Since $(\frac{\mathrm{d}}{\mathrm{d}s}, \mathrm{C}^1[0,1])$ is closed, Lemma A.2 implies (ii) for all $\gamma \in (\alpha, 1) \neq \emptyset$.

(iii) follows since $\beta = 0$ and $\gamma < 1$.

Summing up, we verified all assumptions of Corollary 3.7 and hence G generates an analytic semigroup of angle $\frac{\pi}{2}$ on X. Finally, since at the beginning of the proof we showed that $D(A_m) \subset \mathrm{C}^1[0,1]$, we conclude by the closed graph and the Arzelà–Ascoli theorems that

$$X_1^G \hookrightarrow \mathrm{C}^1[0,1] \overset{\mathrm{c}}{\hookrightarrow} \mathrm{C}[0,1],$$

where "$\overset{\mathrm{c}}{\hookrightarrow}$" denotes a compact injection. Hence, $X_1^G \overset{\mathrm{c}}{\hookrightarrow} X$ and [9, Prop. II.4.25] implies that G has compact resolvent. By [9, Thm. II.4.29] it follows that the semigroup generated by G is compact. This completes the proof. □

COROLLARY 3.10. *Under the assumptions on* $a, b, c \in \mathrm{C}[0,1]$ *made in Theorem 3.9, the degenerate differential equation with generalized Wentzell boundary conditions given by*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t,s) = a(s)\,\dfrac{\mathrm{d}^2u}{\mathrm{d}s^2}(t,s) + b(s)\,\dfrac{\mathrm{d}u}{\mathrm{d}s}(t,s) + c(s)\,u(t,s), & 0 < s < 1,\ t \geq 0, \\ \dfrac{a\,\mathrm{d}^2u}{\mathrm{d}s^2}(t,j) = \varphi_j u(t), & j = 0,1,\ t \geq 0, \\ u(0,s) = f_0(s), & 0 \leq s \leq 1 \end{cases} \tag{DE}$$

is well posed on $\mathrm{C}[0,1]$ *for all functionals* $\varphi_0, \varphi_1 \in (\mathrm{C}^1[0,1])'$.

REMARK 3.11. Note that every function $a \in \mathrm{C}[0,1]$ of the form

$$a(s) = m(s) \cdot s^{\alpha_0}(1-s)^{\alpha_1},\ s \in [0,1]$$

for $\alpha_0, \alpha_1 \in [0,1)$ and a strictly positive $m \in \mathrm{C}[0,1]$ satisfies the assumption of Theorem 3.9. Hence, this result generalizes [8, Cor. 4.1] and [10, Thm. 3] where such a and less general boundary operators Φ were considered, respectively.

3.3. A reaction–diffusion equation on $\mathrm{L}^p[0,\pi]$ with nonlocal Neumann boundary conditions

For $1 \le p < +\infty$ define $X^p := \mathrm{L}^p[0,\pi]$. Then it is well known (or use [7, Thm. 2.2.(b)] and [3, Thm. 3.14.17]) that the operator $A \subset \frac{\mathrm{d}^2}{\mathrm{d}s^2}$ with domain

$$D(A) := \bigl\{ f \in \mathrm{W}^{2,p}[0,\pi] : f'(0) = 0 = f(\pi) \bigr\}$$

generates a bounded analytic semigroup on X^p. For $\gamma \in (0,1)$ we take the space

$$Z_\gamma := \bigl[D\bigl((-A)^\gamma\bigr) \bigr].$$

Moreover, let $Y^p := \mathrm{L}^p([-\pi,0], X^p)$ which by [4, Thm. A.6] is isometrically isomorphic to the space $L^p([-\pi,0]\times[0,\pi])$. For this reason in the sequel we will use the notation $v(r,s) := (v(r))(s)$ for $v \in Y^p$ and $r \in [-\pi,0]$, $s \in [0,\pi]$. Then the following holds.

THEOREM 3.12. *Let $p \in [1,+\infty)$ and $\gamma \in (0,\frac{1}{p})$. Then for every $P \in \mathcal{L}(Z_\gamma, X^p)$ and all functions $\mu : [-\pi,0] \to \mathbb{R}$ of bounded variation the operator*

$$\mathcal{G} := \begin{pmatrix} \frac{\partial^2}{\partial s^2} + P & 0 \\ 0 & \frac{\partial}{\partial r} \end{pmatrix},$$

$$D(\mathcal{G}) := \Biggl\{ \binom{f}{v} \in \mathrm{W}^{2,p}[0,\pi] \times \mathrm{W}^{1,p}\bigl([-\pi,0], X^p\bigr) : v(0) = f,\ f(\pi) = 0, \\ f'(0) = \int_0^\pi \int_{-\pi}^0 v(r,s)\,\mathrm{d}\mu(r)\,\mathrm{d}s \Biggr\}$$

generates a C_0-semigroup on $\mathcal{X}^p := X^p \times Y^p = \mathrm{L}^p[0,\pi] \times \mathrm{L}^p([-\pi,0], X^p)$.

REMARK 3.13. Note that the semigroup group generated by $\mathcal{G}$ will *not* be analytic. Nonetheless, Theorem 2.4 will be very helpful to deal with the analytic part in the first component. The whole matrix will then be treated by Corollary 3.6.

Proof. We first show how the operator $\mathcal{G}$ fits into the abstract framework from Sect. 3.1. To this end we introduce the following operators and spaces, where, for simplicity, in the sequel we put $X := X^p$, $Y := Y^p$ and $\mathcal{X} := \mathcal{X}^p$. Consider

- $A_m := \frac{\mathrm{d}^2}{\mathrm{d}s^2}$ with domain $D(A_m) = \{ f \in \mathrm{W}^{2,p}[0,\pi] : f(\pi) = 0\}$ on X,
- $L := \delta_0' : D(A_m) \to \partial X := \mathbb{C}$, i.e., $Lf = f'(0)$,
- $D_m := \frac{\mathrm{d}}{\mathrm{d}r}$ with domain $D(D_m) = \mathrm{W}^{1,p}([-\pi,0], X)$ on $Y = \mathrm{L}^p([-\pi,0], X)$,
- $K := \delta_0 : D(D_m) \to \partial Y := X = \mathrm{L}^p[0,\pi]$, i.e., $Kv = v(0)$,
- $A = A_m|_{\ker L}$, $D := D_m|_{\ker K}$.

Then, as mentioned above, A is the generator of an analytic semigroup $(T(t))_{t\ge0}$ on X while D generates the nilpotent left-shift semigroup $(S(t))_{t\ge0}$ on Y. Moreover, the associated Dirichlet operators exist for $\mu = 0$ and are given by

- $L_0 \in \mathcal{L}(\partial X, X) = \mathcal{L}(\mathbb{C}, \mathrm{L}^p[0,\pi])$, $(L_0 x)(s) = x \cdot (s - \pi)$ for $s \in [0,\pi]$,

- $K_0 \in \mathcal{L}(\partial Y, Y) = \mathcal{L}(\mathrm{L}^p[0,\pi], \mathrm{L}^p([-\pi,0], \mathrm{L}^p[0,\pi]))$, $(K_0 f)(r) := f$ for $r \in [-\pi, 0]$.

This shows that Assumption 3.1 is satisfied. Next, we define the spaces $\partial\mathcal{X} := \partial X \times \partial Y = \mathbb{C} \times \mathrm{L}^p[0,\pi]$ and $\mathcal{Z} := Z_\gamma \times [D(D_m)] = [D((-A)^\gamma)] \times \mathrm{W}^{1,p}([-\pi,0], X)$ and introduce the operator matrices

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : D(\mathcal{A}) := D(A) \times D(D) \subset \mathcal{X} \to \mathcal{X}, \quad \mathcal{L}_{\mathcal{A}} := \begin{pmatrix} L_A & 0 \\ 0 & K_D \end{pmatrix} : \mathcal{X} \to \mathcal{X}_{-1}^{\mathcal{A}},$$
$$\mathcal{P} := \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{Z} \to \mathcal{X}, \qquad \Phi := \begin{pmatrix} 0 & \varphi \\ Id & 0 \end{pmatrix} : \mathcal{Z} \to \partial\mathcal{X},$$

where $K_D := -D_{-1} K_0$ and $\varphi(v) := \int_0^\pi \int_{-\pi}^0 v(r,s)\,\mathrm{d}\mu(r)\,\mathrm{d}s$. Then as in Lemma 3.4 we can write $(\mathcal{G}, D(\mathcal{G}))$ as a perturbation of the form

$$\mathcal{G} = (\mathcal{A}_{-1} + \mathcal{P} + \mathcal{L}_{\mathcal{A}} \cdot \Phi)|_{\mathcal{X}}.$$

We proceed by verifying the conditions of Corollary 3.6. Since $\mathcal{A}$ is diagonal with diagonal domain, we can split the problem into three parts: We show that

(i) (L_A, P) is jointly p-admissible for A,
(ii) (K_D, φ) is jointly p-admissible for D,
(iii) $1 \in \rho(\mathcal{F}_t)$ for some $t > 0$ where

$$\mathcal{F}_t = \begin{pmatrix} \mathcal{F}_t^{(A,Id,P)} & \mathcal{F}_t^{(A,L_A,P)} & 0 \\ 0 & 0 & \mathcal{F}_t^{(D,K_D,\varphi)} \\ \mathcal{F}_t^{(A,Id,Id)} & \mathcal{F}_t^{(A,L_A,Id)} & 0 \end{pmatrix}. \tag{3.11}$$

As already mentioned, due to the nonanalytic part stemming from the left-shift semigroup $(S(t))_{t\geq 0}$ generated by D on Y, the operator matrix $\mathcal{G}$ will not generate an analytic semigroup on $\mathcal{X}$. Still, we use of our perturbation Theorem 2.4 to treat the analytic part (i) and also to prove (iii).

(i) First we use Lemma A.1 to show that $\mathrm{rg}(L_0) \subseteq \mathrm{Fav}^A_{\frac{p+1}{2p}}$. To this end note that for $c \in \partial X = \mathbb{C}$ we have for $\lambda > 0$

$$(L_\lambda c)(s) = c \cdot \frac{\sinh\big(\sqrt{\lambda}(s-\pi)\big)}{\sqrt{\lambda} \cdot \cosh(\pi\sqrt{\lambda})}, \quad s \in [0,\pi].$$

Using this representation and the estimate $\sinh(s) \leq \frac{e^s}{2}$ for $s \geq 0$ we obtain

$$\begin{aligned}\sup_{\lambda>0} \big\| \lambda^{\frac{p+1}{2p}} L_\lambda \big\| &= \sup_{\lambda>0} \lambda^{\frac{1}{2p}} \cdot \left(\int_0^\pi \left(\frac{\sinh\big(\sqrt{\lambda}(s-\pi)\big)}{\cosh(\pi\sqrt{\lambda})} \right)^p \mathrm{d}s \right)^{\frac{1}{p}} \\ &\leq \sup_{\lambda>0} \frac{e^{\pi\sqrt{\lambda}}}{2p^{\frac{1}{p}} \cdot \cosh(\pi\sqrt{\lambda})} \leq 1.\end{aligned}$$

This implies assumption (i) of Theorem 2.4 for $\beta = 1 - \frac{p+1}{2p} = \frac{p-1}{2p}$. Moreover, since $\gamma < \frac{1}{p}$,

$$\beta + \gamma < \frac{p-1}{2p} + \frac{1}{p} = \frac{p+1}{2p} \le 1,$$

Theorem 2.4 implies that (L_A, P) is jointly q-admissible for A for all $q \in (\frac{2p}{p+1}, \frac{1}{\gamma})$. If $p = 1$, then $\beta = 0$; hence, again by Theorem 2.4 we obtain the same conclusion for $q = 1$. For $p > 1$ we have $p \in (\frac{2p}{p+1}, \frac{1}{\gamma})$. This proves (i).

(ii) We first show that the functional φ is a p-admissible observation operator for D. In fact, for $v \in D(D) = \mathrm{W}_0^{1,p}([-\pi, 0], X)$ we have

$$\begin{aligned}
\int_0^\pi \left|\varphi\, S(t)v\right|^p \mathrm{d}t &= \int_0^\pi \left|\int_0^\pi \int_{-\pi}^{-t} v(r+t, s)\, \mathrm{d}\mu(r)\, \mathrm{d}s\right|^p \mathrm{d}t \\
&\le \int_0^\pi \left(\int_0^\pi \int_{-\pi}^{-t} \left|v(t+r, s)\right| \mathrm{d}|\mu|(r)\, \mathrm{d}s\right)^p \mathrm{d}t \\
&\le \int_0^\pi \big(\pi \cdot |\mu|[-\pi, -t]\big)^{p-1} \int_0^\pi \int_{-\pi}^{-t} \left|v(t+r, s)\right|^p \mathrm{d}|\mu|(r)\, \mathrm{d}s\, \mathrm{d}t \qquad (3.12)\\
&\le \big(\pi \cdot |\mu|[-\pi, 0]\big)^{p-1} \cdot \int_0^\pi \int_{-\pi}^{-t} \int_0^\pi \left|v(t+r, s)\right|^p \mathrm{d}s\, \mathrm{d}|\mu|(r)\, \mathrm{d}t \qquad (3.13)\\
&= \big(\pi \cdot |\mu|[-\pi, 0]\big)^{p-1} \cdot \int_0^\pi \int_{-\pi}^{-t} \|v(t+r)\|_X^p\, \mathrm{d}|\mu|(r)\, \mathrm{d}t \\
&= \big(\pi \cdot |\mu|[-\pi, 0]\big)^{p-1} \cdot \int_{-\pi}^0 \int_0^{-r} \|v(t)\|_X^p\, \mathrm{d}t\, \mathrm{d}|\mu|(r) \qquad (3.14)\\
&\le \pi^{p-1} \cdot \big(|\mu|[-\pi, 0]\big)^p \cdot \|v\|_p^p,
\end{aligned}$$

where in (3.12) we used Hölder's inequality twice and the Fubini–Tonelli theorem in (3.13), (3.14).

Next, as in the proof of [2, Cor. 25] we obtain for $u \in W_0^{1,p}([0, \pi], X)$ and $t \in (0, \pi]$ that

$$\int_0^t S_{-1}(t-r) K_D u(r)\, \mathrm{d}r = u\big(\max\{0, \bullet + t\}\big). \qquad (3.15)$$

Hence, the operator K_D is a p-admissible control operator for D.

Now we show that the pair (K_D, φ) is p-admissible. In fact, using (3.15) we obtain for $u \in \mathrm{W}^{1,p}([0, \pi], X)$ by essentially the same computations as above

$$\begin{aligned}
\int_0^\pi \left|\varphi \int_0^t S_{-1}(t-r) K_D u(r)\, \mathrm{d}r\right|^p \mathrm{d}t &= \int_0^\pi \left|\int_0^\pi \int_{-\pi}^0 u\big(\max\{0, r+t\}, s\big)\, \mathrm{d}\mu(r)\, \mathrm{d}s\right|^p \mathrm{d}t \\
&\le \int_0^\pi \left(\int_0^\pi \int_{-t}^0 \left|u(t+r, s)\right| \mathrm{d}|\mu|(r)\, \mathrm{d}s\right)^p \mathrm{d}t \\
&\le \big(\pi \cdot |\mu|[-\pi, 0]\big)^{p-1} \cdot \int_0^\pi \int_{-t}^0 \|u(t+r)\|_X^p\, \mathrm{d}|\mu|(r)\, \mathrm{d}t \\
&\le \pi^{p-1} \cdot \big(|\mu|[-\pi, 0]\big)^p \cdot \|u\|_p^p.
\end{aligned}$$

(iii) We first determine the input–output map of the complete system. Note that by (3.7) we should choose $\mathcal{U} = \mathcal{X} \times \partial\mathcal{X} = (X \times Y) \times (\partial X \times \partial Y)$. However, since the perturbation P only acts on X, we can cancel out the factor Y and choose

$$\mathcal{U} := X \times \partial\mathcal{X} = X \times (\partial X \times \partial Y) = \mathrm{L}^p[0,\pi] \times \mathbb{C} \times \mathrm{L}^p[0,\pi].$$

By this reduction and the diagonal structure of the generator $\mathcal{A}$ we obtain the simplified input–output map[4] $\mathcal{F}_t \in \mathcal{L}(\mathcal{U})$ given by (3.11). Note that by Theorem 2.4.(c.3) we have

$$\left\|\mathcal{F}_t^{(A,*,\star)}\right\| \to 0 \quad \text{as } t \to 0^+, \tag{3.16}$$

where "$*, \star$" indicates one of the pairs "Id, P", "L_A, P", "Id, Id" or "L_A, Id". In particular, $Id - \mathcal{F}_t^{(A,Id,P)}$ is invertible for $t > 0$ sufficiently small. Using Schur complements (cf. [20, Lem. 2.1]) the invertibility of $Id - \mathcal{F}_t$ is therefore equivalent to the invertibility of

$$\begin{aligned} Id - \left(\mathcal{F}_t^{(A,Id,Id)}\ \mathcal{F}_t^{(A,L_A,Id)}\right) \cdot \begin{pmatrix} \left(Id - \mathcal{F}_t^{(A,Id,P)}\right)^{-1} & \mathcal{F}_t^{(A,L_A,P)} \\ 0 & Id \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mathcal{F}_t^{(D,K_D,\varphi)} \end{pmatrix} \\ = Id - \mathcal{F}_t^{(A,Id,Id)} \cdot \mathcal{F}_t^{(A,L_A,P)} \cdot \mathcal{F}_t^{(D,K_D,\varphi)} - \mathcal{F}_t^{(A,L_A,Id)} \cdot \mathcal{F}_t^{(D,K_D,\varphi)}. \end{aligned}$$

Since by (3.16)

$$\left\|\mathcal{F}_t^{(A,Id,Id)} \cdot \mathcal{F}_t^{(A,L_A,P)} \cdot \mathcal{F}_t^{(D,K_D,\varphi)} + \mathcal{F}_t^{(A,L_A,Id)} \cdot \mathcal{F}_t^{(D,K_D,\varphi)}\right\| \to 0 \quad \text{as } t \to 0^+,$$

the assertion holds for $t > 0$ sufficiently small.

Summing up (i–iii), by Corollary 3.6 the matrix $\mathcal{G}$ generates a C_0-semigroup on $\mathcal{X}$. □

COROLLARY 3.14. *The reaction–diffusion equation subject to Neumann boundary conditions with distributed unbounded delay given by*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t,s) = \dfrac{\mathrm{d}^2u}{\mathrm{d}s^2}(t,s) + b(s)\,\dfrac{\mathrm{d}u}{\mathrm{d}s}(t,s) + c(s)\,u(t,s), & 0 < s < \pi,\ t \ge 0, \\ \dfrac{\mathrm{d}u}{\mathrm{d}s}(t,0) = \displaystyle\int_0^\pi \int_{-\pi}^0 u(t+r,s)\,\mathrm{d}\mu(r)\,\mathrm{d}s, & t \ge 0, \\ u(t,\pi) = 0, & t \ge 0, \\ u(r,s) = u_0(r,s), & 0 < s < \pi,\ r \in [-\pi,0], \\ u(0,s) = f_0(s), & 0 < s < \pi \end{cases} \tag{RDE}$$

is well posed on $\mathrm{L}^p[0,\pi]$ *for all* $p \in [1,2)$, $b, c \in \mathrm{L}^\infty[0,\pi]$ *and* μ *of bounded variation.*

[4] The real input–output map of the whole system is obtained from $\mathcal{F}_t$ by inserting at the second place a row and a column containing only zeros.

Proof. Let $P := b(\bullet)\frac{d}{ds} + c(\bullet)$ with domain $Z := W^{1,p}[0, \pi]$. By the previous result it suffices to prove that $Z_\gamma \hookrightarrow Z$ for $\gamma \in (\frac{1}{2}, \frac{1}{p}) \neq \emptyset$. To this end note that by [9, Expl. III.2.2] for each $f \in D(A)$ and $\varepsilon > 0$

$$\left\| f' \right\|_p \leq \tfrac{9}{\varepsilon} \cdot \|f\|_p + \varepsilon \cdot \|Af\|_p.$$

Setting $\rho := \varepsilon^{-2}$ the assertion follows from Lemma A.2. □

REMARK 3.15. Theorem 3.12 generalizes [15, Expl. 5.2] where only the exponent $p = 2$ and the perturbation $P = 0$ is considered.

3.4. The Laplacian with generalized Robin boundary conditions

For some open, bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ we consider on the Hilbert space $X := L^2(\Omega)$ the maximal operator

$$A_m f := \Delta f \qquad \text{with domain } D(A_m) := \left\{ f \in H^{\frac{3}{2}}(\Omega) : \Delta f \in L^2(\Omega) \right\}.$$

Next we choose the boundary space $\partial X := L^2(\partial\Omega)$ and the boundary operator $L : D(A_m) \subset X \to \partial X$, $Lf := \frac{\partial f}{\partial \nu}$[5] which is well-defined by [18, Chap. 2, Thm. 7.3]. Let $A := A_m|_{\ker(L)}$ whose domain is given by

$$D(A) = \left\{ f \in H^{\frac{3}{2}}(\Omega) : \Delta f \in L^2(\Omega),\ \tfrac{\partial f}{\partial \nu} = 0 \right\}.$$

Hence, A is the Neumann Laplacian on $L^2(\Omega)$ which is self-adjoint and dissipative, hence generates an analytic semigroup. Therefore, Assumption 3.1.(a) is satisfied.

To verify the existence of the abstract Dirichlet operators we use the fact that by [18, Chap. 2, Thm. 7.4] for $\mu > 0$ the operator[6]

$$\mathcal{P} := \binom{\Delta - \mu}{L} : \left\{ f \in H^{\frac{3}{2}}(\Omega) \,|\, (\Delta - \mu) f \in \Xi^{-\frac{1}{2}}(\Omega) \right\} \to H^{-\frac{1}{2}}(\Omega) \times L^2(\partial\Omega)$$

is an algebraic and topological isomorphism. Since $L^2(\Omega) \subset \Xi^{-\frac{1}{2}}(\Omega)$ this implies that for $g \in L^2(\partial\Omega)$ there exists a unique $f \in H^{\frac{3}{2}}(\Omega)$ such that

$$\mathcal{P} f = \begin{pmatrix} \Delta f - \mu f \\ Lf \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \iff \begin{cases} \Delta f = \mu f \in L^2(\Omega), \\ Lf = g. \end{cases}$$

Therefore, $f \in D(A_m)$. For $g \in \partial X$ define $L_\mu g := f \in X$. Then, by the continuity of the inverse $\mathcal{P}^{-1}$ we obtain $L_\mu \in \mathcal{L}(\partial X, X)$; hence, the Assumption 3.1.(b) is satisfied.

In the following result we investigate the generator property of perturbations of A.

5 Here $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative

6 For the definition of $\Xi^{-\frac{1}{2}}(\Omega)$ see [18, Chap. 2, Sect. 6.3].

THEOREM 3.16. *Let* $\alpha \in (0, \frac{3}{2})$. *Then for every operator* $\Phi \in \mathcal{L}(H^\alpha(\Omega), \mathrm{L}^2(\partial\Omega))$ *and* $P \in \mathcal{L}(H^\alpha(\Omega), \mathrm{L}^2(\Omega))$ *the operator*

$$G \subseteq \Delta + P,$$
$$D(G) := \left\{ f \in \mathrm{H}^{\frac{3}{2}}(\Omega) : \Delta f \in \mathrm{L}^2(\Omega),\ \tfrac{\partial f}{\partial \nu} = \Phi(f) \right\} = \ker(L - \Phi)$$

generates a compact, analytic semigroup of angle $\frac{\pi}{2}$ *on* $\mathrm{L}^2(\Omega)$.

Proof. In order to show that G generates an analytic semigroup we verify the conditions (i–iii) of Corollary 3.7. To this end we first choose $\delta \in (\alpha, \frac{3}{2})$ and $\lambda > 0$. By [11, Thm. 2] we have $D\big((\lambda - A)^{\frac{\vartheta}{2}}\big) = H^\vartheta(\Omega)$ for all $\vartheta \in (0, \frac{3}{2})$. Thus, the Dirichlet operators satisfy

$$\mathrm{rg}(L_\lambda) \subset D(A_m) \subset \mathrm{H}^{\frac{3}{2}}(\Omega) \subset \mathrm{H}^\delta(\Omega) = D\big((\lambda - A)^{\frac{\delta}{2}}\big) \subset \mathrm{F}^A_{\frac{\delta}{2}}$$

where the last inclusion follows from [9, Props. II.5.14 & 5.33]. This gives condition (i) for $\beta := 1 - \frac{\delta}{2}$. Using again [11, Thm. 2] we conclude that

$$Z := H^\alpha(\Omega) = \left[D\big((\lambda - A)^{\frac{\alpha}{2}}\big)\right]$$

which shows condition (ii) for $\gamma := \frac{\alpha}{2}$. Moreover, since $\delta > \alpha$

$$\beta + \gamma = 1 - \tfrac{\delta}{2} + \tfrac{\alpha}{2} < 1$$

which shows (iii). Summing up this implies that G generates an analytic semigroup.

To prove compactness of this semigroup first note that by [1, Thm. 6.2] we have the injections $[D(G)] \hookrightarrow \mathrm{H}^1(\Omega) \overset{c}{\hookrightarrow} \mathrm{L}^2(\Omega)$. Hence, [9, Prop. II.4.25] implies that G has compact resolvent and the assertion follows from [9, Thm. II.4.29]. □

REMARK 3.17. We note that the above result could be easily adapted to cover uniformly elliptic operators studied in [17, Thm. 1.1] by completely different methods.

We give a concrete application of Theorem 3.16.

COROLLARY 3.18. *Let* $1 < \alpha < \frac{3}{2}$. *Then the second-order differential equation with perturbed Robin boundary conditions given by*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,s) = \Delta u(t,s) + \displaystyle\sum_{i=1}^n b_i(s)\, \dfrac{\partial u}{\partial x_i}(t,s) + c(s)\, u(t,s), & s \in \Omega,\ t \geq 0, \\ \dfrac{\partial u}{\partial \nu}(t,z) = \displaystyle\sum_{j=1}^m \langle [\varphi_j u](t,\bullet),\ \omega_j\rangle_{\mathrm{L}^2(\partial\Omega)} \cdot g_j(z) + \eta(z) u(t,z), & z \in \partial\Omega,\ t \geq 0, \\ u(0,s) = f_0(s), & s \in \Omega \end{cases}$$

is well posed on $\mathrm{L}^2(\Omega)$ *for all* $\varphi_1, \ldots, \varphi_m \in \mathcal{L}(H^\alpha(\Omega), \mathrm{L}^2(\partial\Omega))$, $b_1, \ldots, b_n, c \in \mathrm{L}^\infty(\Omega)$, $\omega_1, \ldots, \omega_m \in \mathrm{L}^2(\partial\Omega)$, $g_1, \ldots, g_m \in \mathrm{L}^2(\partial\Omega)$ *and* $\eta \in \mathrm{L}^2(\partial\Omega)$.

Proof. The assertion follows immediately from Theorem 3.16 by choosing $P := \sum_{i=1}^n b_i(\bullet)\, \frac{\partial}{\partial x_i} + c(\bullet) \in \mathcal{L}(Z, X)$ and $\Phi := \sum_{j=1}^m \langle \varphi_j \bullet,\ \omega_j\rangle_{\mathrm{L}^2(\partial\Omega)} \cdot g_j + \eta \cdot \mathrm{tr}(\bullet) \in \mathcal{L}(Z, \partial X)$ for $Z := \mathrm{H}^\alpha(\Omega)$. □

4. Conclusions

We presented an approach to the perturbation of generators A of analytic semigroups which, under conditions on $\mathrm{rg}(B)$ and $D(C)$, gives that $(A_{-1} + BC)|_X$ generates an analytic semigroup as well. Our main results are Theorem 2.4 for the general case and Corollaries 3.6, 3.7 for our Generic Example 3.1. In contrast to the known literature on this subject, our results

- allow perturbations $P = BC$ which are not relatively A-bounded as, e.g., in [9, Sect. III.2],
- give the angle of analyticity of the perturbed semigroup, unlike [13, Thm. 2.6] in the situation of the generic example,
- are applicable also to coupled systems which are only in part governed by an analytic semigroup, cf. Remark 3.13.

Moreover, nevertheless being very general, our results applied to concrete situations recover or even improve generation results obtained by methods tailored for the specific case, cf. Remarks 3.8, 3.11, 3.15 and 3.17.

Appendix A.

In this appendix we collect some results which are quite helpful to check the hypotheses of Theorem 2.4 and Corollaries 3.6, 3.7 in applications. First we consider condition (i) in Corollary 3.7.

LEMMA A.1. *In the situation of Sect. 3.1, for* $\alpha \in (0, 1]$ *the following are equivalent.*

(a) There exists $\lambda_0 > \omega_0(A)$ *such that* $\sup_{\lambda>\lambda_0} \|\lambda^\alpha L_\lambda x\| < +\infty$ *for all* $x \in \partial X$.
(b) There exist $\lambda_0 > \omega_0(A)$ *and* $M > 0$ *such that* $\|Lx\| \geq \lambda^\alpha M \cdot \|x\|$ *for all* $\lambda \geq \lambda_0$ *and* $x \in \ker(\lambda - A_m)$.
(c) $\mathrm{rg}(L_\mu) = \ker(\mu - A_m) \subset \mathrm{F}^A_\alpha$ *for some* $\mu \in \rho(A)$.

Moreover, if $\alpha = 1$, *then (a–c) are also equivalent to*

(d) L_A *is a* 1*-admissible control operator for* A.
(d) For all $x \in D(A_m)$ *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset D(A)$ *such that* $\lim_{n\to+\infty} x_n = x$ *and* $\sup_{n\in\mathbb{N}} \|x_n\| < +\infty$.

Proof. The equivalence of (a) and (b) follows immediately from the definition of L_λ as the inverse of $L : \ker(\lambda - A) \to \partial X$. To show the equivalence of (a) and (c), note that from (3.3) we obtain for $x \in \partial X$ and fixed $\mu \in \rho(A)$

$$\begin{aligned}\sup_{\lambda>\lambda_0} \left\|\lambda^\alpha AR(\lambda, A)L_\mu x\right\| &\leq \sup_{\lambda>\lambda_0} \left\|\lambda^\alpha(\mu - A)R(\lambda, A)L_\mu x\right\| + \sup_{\lambda>\lambda_0} \left\|\mu\lambda^\alpha R(\lambda, A)L_\mu x\right\| \\ &= \sup_{\lambda>\lambda_0} \left\|\lambda^\alpha L_\lambda x\right\| + \sup_{\lambda>\lambda_0} \left\|\mu\lambda^\alpha R(\lambda, A)L_\mu x\right\|.\end{aligned}$$

Recall that $\alpha \leq 1$, hence by the Hille–Yosida theorem we have in any case

$$\sup_{\lambda > \lambda_0} \left\| \mu \lambda^\alpha R(\lambda, A) L_\mu x \right\| < +\infty.$$

Thus, we conclude that

$$\sup_{\lambda > \lambda_0} \left\| \lambda^\alpha A R(\lambda, A) L_\mu x \right\| < +\infty \quad \iff \quad \sup_{\lambda > \lambda_0} \left\| \lambda^\alpha L_\lambda x \right\| < +\infty.$$

By [9, Prop. II.5.12] the condition on the left-hand side is equivalent to $L_\mu x \in \mathrm{F}_\alpha^A$, and therefore we obtain (a) $\iff$ (c).

For the equivalence of (a–c) and (d–e) in case $\alpha = 1$ see [9, Proof of Cor. III.3.6], [9, Ex. III.3.8.(4)] and [9, Ex. II.5.23.(2)]. □

If one can represent a Banach space Z as the domain $[D(K)]$ of a closed operator K equipped with its graph norm, then the condition $[D((\lambda - A)^\gamma)] \hookrightarrow Z$ appearing in Theorem 2.4.(ii) and Corollary 3.7.(ii) can frequently be verified by the following result.

LEMMA A.2. *Let A be the generator of an analytic semigroup and let K be a closed linear operator such that $D(A) \subseteq Z = [D(K)]$. If for $\alpha \in (0, 1)$ and every $\rho \geq \rho_0 > 0$ we have*

$$\|Kx\| \leq M \cdot \left(\rho^\alpha \|x\| + \rho^{\alpha-1} \|Ax\|\right) \quad \textit{for all } x \in D(A)$$

and some constant $M \geq 0$, then $[D((\lambda - A)^\gamma)] \hookrightarrow Z$ for every $\gamma > \alpha$ and $\lambda > \omega_0(A)$.

Proof. By (the proof of) [22, Lem. 12.39] the operator $K(\lambda - A)^{-\gamma}$ is bounded which by the closed graph theorem implies that $D((\lambda - A)^\gamma) \subseteq D(K)$. Moreover, from the estimate

$$\|Kx\| \leq \left\| K(\lambda - A)^{-\gamma} \right\| \cdot \|(\lambda - A)^\gamma x\|$$

it follows that $[D((\lambda - A)^\gamma)] \hookrightarrow Z$. □

Finally, we consider pairs (B, C) where we assume that one of the operators B or C is bounded.

LEMMA A.3. *Let A be the generator of a C_0-semigroup and let $p \geq 1$.*

(a) If $B \in \mathcal{L}(U, X_{-1}^A)$ is a p-admissible control operator and $Z = X$, i.e., $C \in \mathcal{L}(X, U)$, then (A, B, C) is compatible, (B, C) is jointly p-admissible and there exists $M \geq 0$ and $t_0 > 0$ such that

$$\left\| \mathcal{F}_t^{(A,B,C)} \right\| \leq M \cdot t^{\frac{1}{p}} \quad \textit{for all } 0 < t \leq t_0.$$

In particular, Id_U is p-admissible for the triple (A, B, C).

(b) If $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(Z, U)$ is a p-admissible observation operator, then (A, B, C) is compatible, (B, C) is jointly p-admissible and there exists $M \geq 0$ and $t_0 > 0$ such that

$$\left\| \mathcal{F}_t^{(A,B,C)} \right\| \leq M \cdot t^{1-\frac{1}{p}} \quad \textit{for all } 0 < t \leq t_0.$$

In particular, if $p > 1$, then Id_U is p-admissible for the triple (A, B, C).

Proof. The assertions follow immediately from [2, Rems. 17 & 19]. □

REFERENCES

[1] R. A. Adams. *Sobolev Spaces, Pure and Applied Mathematics*, vol. 65. Academic Press, New York–London (1975).

[2] M. Adler, M. Bombieri, and K.-J. Engel. *On perturbations of generators of C_0-semigroups*. Abstr. Appl. Anal. (2014), Art. ID 213020. http://dx.doi.org/10.1155/2014/213020.

[3] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-Valued Laplace Transforms and Cauchy Problems*, *Monographs in Mathematics*, vol. 96. Birkhäuser Verlag, Basel (2001). http://www.springer.com/birkhauser/mathematics/book/978-3-0348-0086-0.

[4] A. Bátkai and S. Piazzera. *Semigroups for Delay Equations*, *Research Notes in Mathematics*, vol. 10. A K Peters Ltd., Wellesley, MA (2005). http://www.crcpress.com/product/isbn/9781568812434.

[5] M. Campiti and G. Metafune. *Ventcel's boundary conditions and analytic semigroups*. Arch. Math. (Basel) **70** (1998), 377–390. http://dx.doi.org/10.1007/s000130050210.

[6] V. Casarino, K.-J. Engel, R. Nagel, and G. Nickel. *A semigroup approach to boundary feedback systems*. Integral Equations Operator Theory **47** (2003), 289–306. http://dx.doi.org/10.1007/s00020-002-1163-2.

[7] R. Chill, V. Keyantuo, and M. Warma. *Generation of cosine families on $L^p(0, 1)$ by elliptic operators with Robin boundary conditions*. In: *Functional analysis and evolution equations*, pp. 113–130. Birkhäuser, Basel (2008). http://dx.doi.org/10.1007/978-3-7643-7794-6_7.

[8] K.-J. Engel and G. Fragnelli. *Analyticity of semigroups generated by operators with generalized Wentzell boundary conditions*. Adv. Differential Equations **10** (2005), 1301–1320. http://projecteuclid.org/euclid.ade/1355867753.

[9] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, *Graduate Texts in Mathematics*, vol. 194. Springer-Verlag, New York (2000). http://dx.doi.org/10.1007/b97696.

[10] A. Favini, G. R. Goldstein, J. A. Goldstein, and S. Romanelli. *General Wentzell boundary conditions, differential operators and analytic semigroups in $C[0, 1]$*. Bol. Soc. Parana. Mat. (3) **20** (2002), 93–103 (2003). http://dx.doi.org/10.5269/bspm.v20i1-2.7525.

[11] D. Fujiwara. *Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order*. Proc. Japan Acad. **43** (1967), 82–86. http://dx.doi.org/10.3792/pja/1195521686.

[12] G. Greiner. *Perturbing the boundary conditions of a generator*. Houston J. Math. **13** (1987), 213–229. http://dx.doi.org/10.1007/s00233-011-9361-3.

[13] G. Greiner and K. G. Kuhn. *Linear and semilinear boundary conditions: the analytic case*. In: *Semigroup Theory and Evolution Equations (Delft, 1989)*, *Lecture Notes in Pure and Appl. Math.*, vol. 135, pp. 193–211. Dekker, New York (1991).

[14] B. H. Haak, M. Haase, and P. C. Kunstmann. *Perturbation, interpolation, and maximal regularity*. Adv. Differential Equations **11** (2006), 201–240. http://projecteuclid.org/euclid.ade/1355867717.

[15] S. Hadd, R. Manzo, and A. Rhandi. *Unbounded perturbations of the generator domain*. Discrete Contin. Dyn. Syst. **35** (2015), 703–723. http://dx.doi.org/10.3934/dcds.2015.35.703.

[16] P. C. Kunstmann and L. Weis. *Perturbation theorems for maximal L_p-regularity*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **30** (2001), 415–435. http://www.numdam.org/item?id=ASNSP_2001_4_30_2_415_0.

[17] I. Lasiecka and R. Triggiani. *Feedback semigroups and cosine operators for boundary feedback parabolic and hyperbolic equations*. J. Differential Equations **47** (1983), 246–272. http://dx.doi.org/10.1016/0022-0396(83)90036-0.

[18] J.-L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications. Vol. I*. Springer-Verlag, New York-Heidelberg (1972). http://dx.doi.org/10.1007/978-3-642-65161-8. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[19] L. Maniar and R. Schnaubelt. *Robustness of Fredholm properties of parabolic evolution equations under boundary perturbations*. J. Lond. Math. Soc. (2) **77** (2008), 558–580. http://dx.doi.org/10.1112/jlms/jdn001.

[20] R. Nagel. *Towards a "matrix theory" for unbounded operator matrices*. Math. Z. **201** (1989), 57–68. http://dx.doi.org/10.1007/BF01161994.

[21] M. Reed and B. Simon. *Methods of Modern Mathematical Physics II. Fourier Analysis and Self-Adjointness*. Academic Press, New York (1975).

[22] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*, *Texts in Applied Mathematics*, vol. 13. Springer-Verlag, New York, 2nd edn. (2004). http://dx.doi.org/10.1007/b97427.

[23] O. Staffans. *Well-Posed Linear Systems*, *Encyclopedia of Mathematics and its Applications*, vol. 103. Cambridge University Press, Cambridge (2005). http://dx.doi.org/10.1017/CBO9780511543197.

[24] G. Weiss. *Regular linear systems with feedback*. Math. Control Signals Systems **7** (1994), 23–57. http://dx.doi.org/10.1007/BF01211484.

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